ON THE KINEMATICS OF SPATIAL MOTION

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İZMİR

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I want to dedicate this thesis to my wife Sedef and my sister Canan who were so helpful and understanding.

ABSTRACT

In this thesis, the special points; the acceleration centers, the Bresse surfaces and the inflection points of the dual spherical motion $\hat{X} = \hat{A}\hat{x}$ are discussed. Based upon a canonical frame field the dual spherical motion is studied by using the dual Darboux vector.

ÖZET

Bu çalışmada, dual küresel harekete, $\hat{X} = \hat{A}\hat{x}$, ait özel noktalar; ivme merkezleri, Bresse yüzeyleri, büküm noktaları tartışılmıştır. Kanonik bir üçyüzlü yardımıyla, dual Darboux vektörü de kullanılarak, dual küresel hareket üzerinde çalışılmıştır.

CONTENTS

Page

Contents....VII

List of Figures....IV

Chapter One

INTRODUCTION

1.1 The Algebra of Dual Numbers....2

1.2 Dual Numbers....2

1.3 Dual Vectors....7

1.3.1 Addition of Dual Vectors....7

1.3.2 Multiplication of a Dual Vector by a dual number....7

1.3.3 Dual Scalar Product of Dual Vectors....7

1.3.4 Dual Vector Product of Dual Vectors....7

1.3.5 Norm of a Dual Vector....8

1.4 The Unit Dual Vector....8

1.5 Dual Functions....9

1.5.1 Series Expansion of Dual Functions....9

Chapter Two

THE SPATIAL MOTION

2.1 The E. Study Principle....13

2.2 The Dual Angle....15

2.3 The Line Complex....20

2.4 The Line Congruence.....21
2.5 The Ruled Surface.....21

Chapter Three
STRAIGHT LINE
ADJOINT TO A RULED SURFACE

3.1 The Acceleration Center.....28
3.2 The Bresse Hyperboloid.....31
3.3 The Inflection Surfaces.....35

Chapter Four
A CANONICAL FRAME FIELD

4.1 The Acceleration Center.....43
4.2 The Bresse Congruence.....44
4.3 The Inflection Surfaces.....47

References.....49

LIST OF FIGURES

		Page
Figure 2.1	The direction and the moment of a straight line	14
Figure 2.2	The geometric meaning of the dual angle	17
Figure 2.3	The ruled surface of $\vec{X}(t) = \vec{x}(t) + \varepsilon\vec{x}^{*}(t)$	19
Figure 3.1	The straight line adjoint to a ruled surface	26
Figure 3.2	The dual curve on the D.U.S.	27
Figure 3.3	The ruled surface adjoint to a ruled surface	32
Figure 4.1	The Frenet frame of a ruled surface	37
Figure 4.2	The canonical frame field	38

CHAPTER ONE
INTRODUCTION

The usage of dual numbers in spatial kinematics can not be neglected. E. Study principle is the basic tool in this concept.

In the preliminaries, we study the dual numbers algebra, the dual vectors, the dual functions and their series expansions (Köse 1974). Then, one to one correspondence between the straight lines in R^3 and the points of the Dual Unit Sphere , known as the Study Theorem, is given. The angle between the dual vectors on the Dual Unit Sphere gives the relation between the corresponding straight lines in R^3. These contents are given in Chapter I and Chapter II (Köse 1974 & Müller 1963 & Bottema et al. 1979).

In Chapter III, relative motion of a straight line adjoint to a ruled surface R is given. The acceleration center, the Bresse hyperboloid and the inflection surface of this relative motion are determined by the help of the construction parameters of the ruled surface R.

In Chapter IV, considering the dual spherical motion K/K' and the associated spatial motion H/H', the kinematic properties, the acceleration center, the Bresse congruence and the inflection surface are discussed.

It is believed that the contents of Chapter III and Chapter IV are original.

1.1 The Algebra of Dual Numbers

The algebra of dual numbers was developed by Clifford in the midnineteenth century. And systematically applied to the kinematics by Study [1891,1903] and Kotel'nikov[1895].

The dual number system is a "complex" system with two units just as the ordinary complex number system. In complex number system, an element is denoted by $a+ib$, where a and b are real numbers and $i^2=-1$. An element in dual number system is denoted by $a+\varepsilon b$, where a and b are real numbers and $\varepsilon^2=0$.

1.2 Dual Numbers

A dual number A can be defined as an ordered pair $A=(a,a^*)$ of real numbers a and a^*, with operations addition and multiplication defined below. The real numbers a and a^* are called the real part and the dual part of A, respectively. We can write simply,

$\operatorname{Re} A = a$ and $DuA = a^*$.

Let us define the set of dual numbers by $D=\{(a,a^*);\, a,a^*\in R\}$. Two dual numbers $A=(a,a^*)$ and $B=(b,b^*)$ are equal whenever they have the same real parts and same dual parts. Hence,

$(a,a^*)=(b,b^*)$ iff $a=b$ and $a^*=b^*$.

The addition operation, $\oplus$, and the multiplication operation, $\otimes$, are defined for the dual numbers $A=(a,a^*)$ and $B=(b,b^*)$, for all $A,B\in D$ as follows ;

$$(a,a^*)\oplus(b,b^*)=(a+b,a^*+b^*) \qquad (1.2.1)$$

$$(a,a^*)\otimes(b,b^*)=(ab,ab^*+a^*b) \tag{1.2.2}$$

In particular,

$(a,a^*)=(a,0)\oplus(0,a^*)$ and $(0,1)\otimes(a^*,0)=(0,a^*)$.

Hence $(a,a^*)=(a,0)\oplus(0,1)\otimes(a^*,0)$ (1.2.3)

We will investigate later that $(a,0)$ is to be identified as the real number a. Taking the ordered pairs $(a,0)$ and $(a^*,0)$ as the real numbers a, a^* respectively and denoting by ε the dual number $(0,1)$ ($(0,1)$ is the dual unit in D), we can write (1.2.3) as

$$(a,a^*) = a+\varepsilon a^* .$$

Also we can note that,

$\varepsilon^2=(0,1)\otimes(0,1)=(0,0)$ (the zero element in D). That is ,

$\varepsilon^2=0$ and it is clear that $\varepsilon^2=\varepsilon^3=\dots\dots\dots\dots=\varepsilon^n=\dots\dots=0$.

Defining the subtraction as the inverse operation of addition , one can easily obtain that addition, subtraction and multiplication exist for any pair in D and they are commutative, associative and distributive.

Theorem 1.2.1 The set of dual numbers with respect to addition and multiplication, $<D,\oplus,\otimes>$, is a commutative ring with identity.

Proof We prove the theorem in two steps,

i) $<D,\oplus>$ is an abelian group.

ii) Multiplication is associative and it has distributive property over addition and (1,0) is the multiplicative identity.

i) $< D,\oplus >$ is an abelian group.

R1) It is clear that addition is closed on D. For all $A,B \in D$ we have $A \oplus B \in D$.

R2) For all $A=(a,a^*), B=(b,b^*), C=(c,c^*) \in D$ addition is associative,

$$(A \oplus B) \oplus C = ((a,a^*) \oplus (b,b^*)) \oplus (c,c^*) = (a+b, a^* + b^*) \oplus (c,c^*)$$
$$= ((a+b)+c, (a^*+b^*)+c^*) = (a+(b+c), a^* + (b^* + c^*))$$
$$= (a,a^*) \oplus (b+c, b^* + c^*) = A \oplus (B \oplus C).$$

R3) $0=(0,0) \in D$ is the additive identity in D . $\forall (a,a^*) \in D$ we have the requirement $(a,a^*) \oplus (0,0) = (a+0, a^*+0) = (a,a^*)$.

R4) $(-a,-a^*) \in D$ is the additive inverse of $(a,a^*) \in D$. That is, $(a,a^*) \oplus (-a,-a^*) = (a+(-a), a^* + (-a^*)) = (0,0)$. If $A=(a,a^*) \in D$ then we denote $(-a,-a^*) \in D$ by $-A$.

Hence $< D,\oplus >$ is a group.

R5) Moreover for all $A,B \in D$ we have $A \oplus B = B \oplus A$. That is, $(a,a^*) \oplus (b,b^*) = (a+b, a^*+b^*) = (b+a, b^*+a^*) = (b,b^*) \oplus (a,a^*)$.

So we can say that $< D,\oplus >$ is an abelian group.

ii) We can easily check the conditions of second step ;

R6) It is clear that multiplication is closed on D. For all $A,B \in D$, we have $A \otimes B \in D$.

R7) Multiplication is associative. That is, for all $A,B,C \in D$

$$(A \otimes B) \otimes C = ((a,a^*) \otimes (b,b^*)) \otimes (c,c^*) = (ab, ab^* + a^*b) \otimes (c,c^*) =$$
$$(abc, abc^* + ab^*c + a^*bc) = (a,a^*) \otimes (bc, bc^* + b^*c) = A \otimes (B \otimes C) .$$

R8) Multiplication is distributive over addition. That is,

$(A \oplus B) \otimes C = ((a,a^*) \oplus (b,b^*)) \otimes (c,c^*) = (a+b, a^* + b^*) \otimes (c,c^*) =$
$= ((a+b)c, (a^* + b^*)c + (a+b)c^*) = (ac + bc, a^*c + ac^* + b^*c + bc^*) =$
$(ac, a^*c + ac^*) \oplus (bc, b^*c + bc^*) = A \otimes C \oplus B \otimes C$ for all $A,B,C \in D$, the right distributive property holds. Similarly $A \otimes (B \oplus C) = A \otimes B \oplus A \otimes C$ for all $A,B,C \in D$, the left distributive property holds.

Hence $< D,\oplus,\otimes >$ is a ring.

Moreover

R9) For all $A,B \in D$, we have $A \otimes B = (a,a^*) \otimes (b,b^*) = (ab, ab^* + a^*b) =$
$(ba, ba^* + b^*a) = B \otimes A$. Multiplication is commutative. And,

R10) $(1,0) \in D$ is the identity element with respect to multiplication;

$$(a,a^*) \otimes (1,0) = (1,0) \otimes (a,a^*) = (a,a^*), \ \forall A \in D$$

Then $< D,\oplus,\otimes >$ is a commutative ring with identity .

Theorem 1.2.2 $< D,\oplus,\otimes >$ is not a field.

Proof It is enough to show that one of the field properties is violated. Let us discuss the inverse of an element $(a,a^*) \in D$ with respect to multiplication.

$$A \otimes X = X \otimes A = (1,0)$$

$(a,a^*)\otimes(x,x^*)=(1,0) \quad \Rightarrow \quad (ax,ax^*+a^*x)=(1,0) \quad \Rightarrow \quad (x,x^*)=(\frac{1}{a},-\frac{a^*}{a^2})$,

this result is meaningful when a is different from zero. In other words, a dual number of the type $(0,a^*)\in D$ has no inverse. So every element in D does not have an inverse in D . Then $<D,\oplus,\otimes>$ is not a field .

One can also check that the multiplication of non-zero dual numbers may give the zero element. For the case $A=(0,a^*)$, $B=(0,b^*)\in D$, we have $A\otimes B=(0,a^*)\otimes(0,b^*)=(0,0)$.

Theorem 1.2.3 The set of real numbers is isomorphic to a subset of D which is consist of the elements with only real parts.

Proof Let us define a function $f:R\rightarrow D'\subset D$ such that $f:a\rightarrow(a,0)$. Then f is an isomorphism.

i) f is linear : $\forall a,b\in R$,

$f(a+b)=(a+b,0)=(a,0)\oplus(b,0)=f(a)\oplus f(b)$.

With respect to multiplication

$f(a.b)=(a.b,0)=(a,0)\otimes(b,0)=f(a)\otimes f(b)$.

ii) f is one to one : If $a\neq b$ then $(a,0)\neq(b,0)$ implies $f(a)\neq f(b)$.

iii) f is onto : $\forall(a,0)\in D'$, $\exists a\in R$ such that $f(a)=(a,0)$.

Hence we have proved that R is isomorphic to $D'\subset D$. And also we have seen that D' consists of the elements of type $A=(a,0)$ where $\operatorname{Re} A$ is the set of real numbers and DuA is only zero.

Using the theorem 1.2.3, a real number a can be defined by $(a,0)$ in the dual number system.

1.3 Dual Vectors

If $\vec{v},\vec{v}^* \in R^3$ then we can define a dual vector $\vec{V}$ in three dimensional dual space, D^3, by $\vec{V} = \vec{v} + \varepsilon\vec{v}^*$. The set D^3 is defined by $D^3 = \{\vec{a} + \varepsilon\vec{a}^* : \vec{a},\vec{a}^* \in R^3\}$.

Let $\vec{V},\vec{W} \in D^3$ and $d \in D$, where $\vec{V} = \vec{v} + \varepsilon\vec{v}^*$, $\vec{W} = \vec{w} + \varepsilon\vec{w}^*$ and $d = d_1 + \varepsilon d_1^*$ with $\vec{v},\vec{v}^*,\vec{w},\vec{w}^* \in R^3$, $d_1, d_1^* \in R$.

Then we mention the followings:

1.3.1 Addition of Dual Vectors

$$\vec{V} + \vec{W} = (\vec{v} + \vec{w}) + \varepsilon(\vec{v}^* + \vec{w}^*)$$

1.3.2 Multiplication of a Dual Vector by a Dual Number

$$d.\vec{V} = (d_1 + \varepsilon d_1^*)(\vec{v} + \varepsilon\vec{v}^*) = d_1\vec{v} + \varepsilon(d_1\vec{v}^* + d_1^*\vec{v})$$

1.3.3 Dual Scalar(Dot) Product of Dual Vectors

$$\vec{V}.\vec{W} = (\vec{v} + \varepsilon\vec{v}^*)(\vec{w} + \varepsilon\vec{w}^*) = \vec{v}\vec{w} + \varepsilon(\vec{v}\vec{w}^* + \vec{v}^*\vec{w}) = \vec{W}.\vec{V}$$

(Scalar product is commutative)

1.3.4 Dual Vector(Cross) Product of Dual Vectors

$$\vec{V} \times \vec{W} = (\vec{v} + \varepsilon\vec{v}^*) \times (\vec{w} + \varepsilon\vec{w}^*) = \vec{v} \times \vec{w} + \varepsilon(\vec{v} \times \vec{w}^* + \vec{v}^* \times \vec{w}) \neq \vec{W} \times \vec{V}$$

(Cross product is not commutative).

1.3.5 The Norm of a Dual Vector

$$\|\vec{V}\| = (\vec{V}.\vec{V})^{1/2} = \left[(\vec{v}+\varepsilon\vec{v}^*)(\vec{v}+\varepsilon\vec{v}^*)\right]^{1/2} = (\vec{v}.\vec{v}+2\varepsilon\vec{v}\vec{v}^*)^{1/2}$$

$$= (\|\vec{v}\|^2 + 2\varepsilon\vec{v}\vec{v}^*)^{1/2} = \|\vec{v}\| + \varepsilon\frac{\vec{v}\vec{v}^*}{\|\vec{v}\|}$$ (See also (1.5.5) for the proof).

After defining the norm of a dual vector, now we can discuss the unit dual vector and normalization of any dual vector.

1.4 The Unit Dual Vector

If the norm of a dual vector is (1,0) then we call that vector the unit dual vector.

If $\vec{V} = \vec{v} + \varepsilon\vec{v}^*$ is a unit dual vector then $\|\vec{V}\| = \|\vec{v}\| + \varepsilon\frac{\vec{v}\vec{v}^*}{\|\vec{v}\|} = (\|\vec{v}\|, \frac{\vec{v}\vec{v}^*}{\|\vec{v}\|}) = (1,0)$

and which implies $\|\vec{v}\| = 1$ and $\vec{v}.\vec{v}^* = 0$.

As in the real case, we can make any dual vector a unit dual vector by dividing that vector to its norm. Hence taking any $\vec{A} \in D^3$ with $\|\vec{A}\| \neq (0,0)$, $\vec{U} = \frac{\vec{A}}{\|\vec{A}\|}$ is a dual unit vector.

It is not hard to adapt the properties of real vectors to the dual vectors. For example, triple product, linear dependence of dual vectors, basis in three dimensional dual space, orthogonality of dual vectors, Gram-Schimidt process of orthogonalization, dual matrices and so on. As in the complex numbers system one can discuss everything done in real numbers also in dual numbers.

We now consider the functions of dual variables.

1.5 Dual Functions

Let F be a set of dual numbers. A function f defined on F is a rule which assigns to each d in F a dual number $A \in D$. The dual number A is called the value of f at d and denoted by $f(d)$; that is $A = f(d)$. The set F is called the domain of definition of f.

Suppose that $A = a + \varepsilon a^*$ is the value of the function f at $d = x + \varepsilon x^*$; that is

$$a + \varepsilon a^* = f(x + \varepsilon x^*).$$

Each of the real numbers a and a^* depends on the real variables x and x^*. If, for instance, $f(d) = d^2$, then $f(d) = (x + \varepsilon x^*) = x^2 + \varepsilon 2xx^*$ hence $a = x^2$ and $a^* = 2xx^*$.

This simple sketch illustrates that a function of a dual variable can be expressed in terms of a pair of real valued functions of real variables x and x^*,

$$f(d) = a(x, x^*) + \varepsilon a^*(x, x^*).$$

Since $\varepsilon^n = 0$, $n > 1$, it is convenient to obtain a Taylor series expansion with ε variable. Hence

$$f(x + \varepsilon x^*) = f(x) + \varepsilon x^* f'(x).$$

1.5.1 Series expansion of Dual Functions As we have mentioned above the Taylor series expansion of a dual function is

$$f(x + \varepsilon x^*) = f(x) + \varepsilon x^* f'(x).$$

We can obtain this result by the Taylor series expansion of $f(d)$ at $d_0 = 0$. Similar to the real case :

$$f(d) = f(d_0) + \frac{(d-d_0)}{1!} f'(d_0) + \dots\dots\dots\dots + \frac{(d-d_0)^n}{n!} f^{(n)}(d_0) + \dots\dots$$

If we write d as $d = x + \varepsilon x^*$ and make Taylor series expansion at $d_0 = 0$ then,

$$f(x+\varepsilon x^*) = f(0) + \frac{x+\varepsilon x^*}{1!} f'(0) + \dots\dots\dots + \frac{x^n + n\varepsilon x^{n-1} x^*}{n!} f^{(n)}(0) + \dots\dots\dots\dots =$$

$$\{f(0) + \frac{x}{1!} f'(0) + \dots\dots\dots + \frac{x^{(n)}}{n!} f^{(n)}(0) + \dots\} + \varepsilon x^* \{f'(0) + \dots\dots + \frac{x^{n-1}}{(n-1)!} f^{(n)}(0) + \dots\}$$

where the first part of the expression is the Taylor series expansion of $f(x)$ and the second part is the Taylor series expansion of $f'(x)$. Hence we get,

$$f(x+\varepsilon x^*) = f(x) + \varepsilon x^* f'(x).$$

In particular, we have ;

$\cos(x+\varepsilon x^*) = \cos x - \varepsilon x^* \sin x$

$\sin(x+\varepsilon x^*) = \sin x + \varepsilon x^* \cos x$

$\tan(x+\varepsilon x^*) = \tan x + \varepsilon x^* (1+\tan^2 x)$

$\cot(x+\varepsilon x^*) = \cot x - \varepsilon x^* \operatorname{cosec}^2 x$

and consider

$$(1+d)^m = 1 + md + \frac{m(m-1)}{2!} d^2 + \dots\dots\dots + \frac{m(m-1)\dots\dots(m-n+1)}{n!} d^n + \dots\dots ,$$

again if we put $x + \varepsilon x^*$ instead of d then we have

$$(1+d)^m = 1+m(x+\varepsilon x^*)+\frac{m(m-1)}{2!}(x+\varepsilon x^*)^2+......+\frac{m(m-1)...(m-n+1)}{n!}(x+\varepsilon x^*)^n+...$$
$$=\{1+mx+\frac{m(m-1)}{2!}x^2+.......+\frac{m(m-1)...(m-n+1)}{n!}x^n+.......\}+mx^*\varepsilon\{1+\frac{(m-1)}{1!}x+......$$
$$.....+\frac{(m-1)...(m-n+1)}{(n-1)!}x^{n-1}+....\} .$$

Then it is clear that,

$$(1+d)^m = (1+x)^m + \varepsilon m x^*(1+x)^{m-1} \quad . \tag{1.5.1}$$

If we take $m=-1$ in (1.5.1) then

$$(1+d)^{-1} = \frac{1}{1+d} = \frac{1}{(1+x)} - \varepsilon\frac{x^*}{(1+x)^2} \quad . \tag{1.5.2}$$

If we take $d=d^2$ in (1.5.2) then

$$\frac{1}{1+d^2} = \frac{1}{1+x^2} - \varepsilon\frac{2xx^*}{(1+x^2)^2} \quad . \tag{1.5.3}$$

If we take $m=-\frac{1}{2}$ in (1.5.2) then

$$\frac{1}{(1+d)^{1/2}} = \frac{1}{(1+x)^{1/2}} - \varepsilon\frac{x^*}{2(1+x)^{3/2}} \ , \tag{1.5.4}$$

and a special case of (1.5.4) is

$$\frac{1}{(1+2\varepsilon k)^{1/2}} = 1-\varepsilon k \quad .$$

Depending on these expansions, we can give a simple sketch for the norm of a dual vector $\vec{A} = \vec{a} + \varepsilon\vec{a}^*$:

$$\|\vec{A}\| = (\vec{A}.\vec{A})^{1/2} = \{(\vec{a} + \varepsilon\vec{a}^*)(\vec{a} + \varepsilon\vec{a}^*)\}^{1/2} = (\vec{a}.\vec{a} + 2\varepsilon\vec{a}.\vec{a}^*)^{1/2} = (\|\vec{a}\|^2 + 2\varepsilon\vec{a}.\vec{a}^*)^{1/2}$$

$$= \|\vec{a}\|(1 + 2\varepsilon\frac{\vec{a}.\vec{a}^*}{\|\vec{a}\|^2})^{1/2} = \|\vec{a}\|(1 + \varepsilon\frac{\vec{a}.\vec{a}^*}{\|\vec{a}\|^2}) = \|\vec{a}\| + \varepsilon\frac{\vec{a}.\vec{a}^*}{\|\vec{a}\|} = (\|\vec{a}\|, \frac{\vec{a}.\vec{a}^*}{\|\vec{a}\|}) \qquad (1.5.5)$$

CHAPTER TWO

THE SPATIAL MOTION

Definition (Dual Unit Sphere) 2.1.1 The set of the dual points $\{\vec{X} = \vec{x} + \varepsilon\vec{x}^* : \|\vec{X}\| = (1,0); \vec{x}, \vec{x}^* \in R^3\}$ is called the dual unit sphere (D.U.S) in D.

2.1 The E. Study Principle

The dual points on the D.U.S represents the straight lines in R^3, and vice versa.

Theorem (E. Study) 2.1.2 There is a one to one correspondence between the straight lines in R^3 and the dual points (not the pure duals, s.t. $(0, \vec{a}^*)$) of the D.U.S.

Proof To define a straight line in R^3 we need a point , say $\vec{m} \in R^3$, and the direction vector , say $\vec{d} \in R^3$. So the vectorial equation of the straight line is $(\vec{x} - \vec{m}) \times \vec{d} = 0$. Here we use $\vec{x}$ to define the arbitrary points of the straight line and we take $\vec{d}$ as a unit vector. Then the vectorial equation, $(\vec{x} - \vec{m}) \times \vec{d} = 0$, implies $\vec{x} \times \vec{d} = \vec{m} \times \vec{d} = \vec{d}_0^*$ (let us denote $\vec{x} \times \vec{d}$ and $\vec{m} \times \vec{d}$ by the vector $\vec{d}_0^*$). The result $\vec{d}_0^*$ has a physical meaning that it is the vectorial moment of $\vec{d}$ with respect to the origin (hence the moment of the straight line).

Taking the norm of $\vec{d}_0^*$ we have,

$$\|\vec{d}_0^*\| = \|\vec{m} \times \vec{d}\| = \|\vec{m}\|\|\vec{d}\| \sin\varphi = \|\vec{m}\| \sin\varphi = \delta \ ,$$

which is the smallest distance between the straight line and the origin.

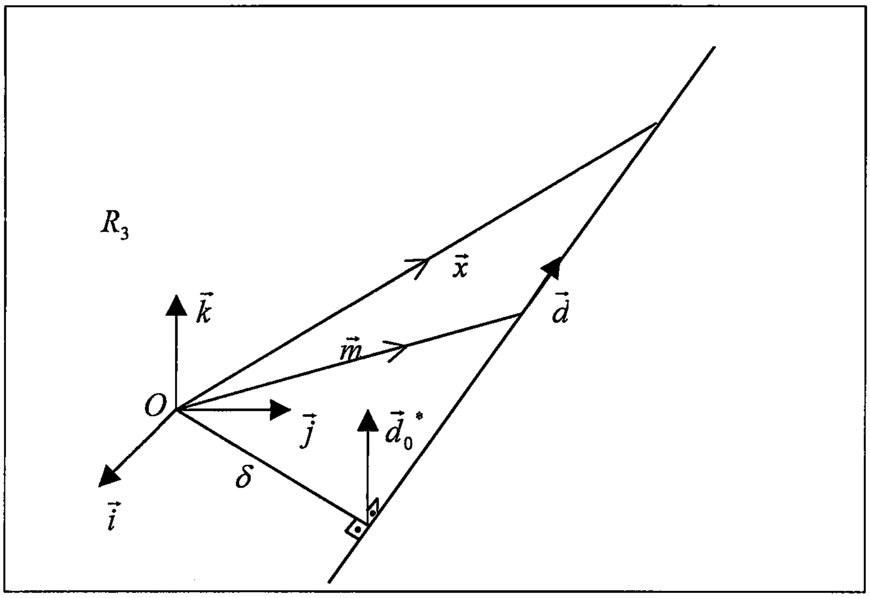

Figure 2.1 The direction and the moment of a straight line

Our claim is that $(\vec{d},\vec{d}_0^{\,*}) \in$ D.U.S defines a unique straight line in R^3. Since $\vec{d}_0^{\,*} = \vec{x} \times \vec{d}$, $\vec{d}_0^{*} \perp \vec{x}$ and $\vec{d}_0^{*} \perp \vec{d}$. If we take a plane in R^3, passing through origin and perpendicular to $\vec{d}_0^{*}$, and draw a circle of radius δ which lies in this plane then the diameter perpendicular to $\vec{d}$ intersects this circle at two distinct points. If we draw two straight lines tangent to the circle at these points then we get the straight lines corresponding to the vector couples $(\vec{d},\vec{d}_0^{*})$ and $(\vec{d},-\vec{d}_0^{*})$. These two straight lines are called the spears, in kinematics. Thus we get a unique straight line corresponds to the couple $(\vec{d},\vec{d}_0^{*})$.

Since the dual vector $(\vec{d},\vec{d}_0^{*}) = \vec{d} + \varepsilon\vec{d}_0^{*}$ has the properties, $\vec{d}.\vec{d} = 1$ and $\vec{d}.\vec{d}_0^{*} = 0$, $(\vec{d},\vec{d}_0^{*})$ is a dual vector on D.U.S.

If we define the vector $\vec{A} = \vec{d} + \varepsilon\vec{d}_0^{*}$ then $\vec{A}$ corresponds to a point on the D.U.S. Since $\vec{d}.\vec{d} = 1$, instead of $(\vec{d},-\vec{d}_0^{*})$ we can take $(-\vec{d},-\vec{d}_0^{*})$. This is the case of taking the direction of the straight line $-\vec{d}$ instead of $\vec{d}$. Defining the vector $-\vec{A} = -\vec{d} - \varepsilon\vec{d}_0^{*}$ on D.U.S., one can easily observe that the opposite spears correspond to the diametrical points $(\vec{A},-\vec{A})$ of D.U.S.

Since we have restricted ourselves not to take a dual point of $(0, a^*)$ type (this is the case that the moment is zero), any dual point $(\vec{a}, \vec{a}^*)$ on D.U.S. represents a straight line with direction $\vec{a}$ and vectorial moment $\vec{a}^*$ (with respect to the origin) in R^3. Thus any dual point of the D.U.S. is the image of a spear.

The six coordinates of the real vectors $\vec{d}$ and $\vec{d}_0^*$ in an orthogonal coordinate system are called the Plücker coordinates of the straight line, and the vectors $\vec{d}$ and $\vec{d}_0^*$ satisfy the conditions $\vec{d}.\vec{d} = 1$ and $\vec{d}.\vec{d}_0^* = 0$.

2.2 The Dual Angle

The angle between the dual vectors is called a dual angle. Similar to the dual numbers, a dual angle has a real part and a dual part. Let us denote the angle between the dual unit vectors $\vec{A} = \vec{a} + \varepsilon\vec{a}^*$ and $\vec{B} = \vec{b} + \varepsilon\vec{b}^*$ by $\Phi = \varphi + \varepsilon\varphi^*$. Now we will investigate the geometric notions of φ and φ^* (the real part and the dual part of Φ, respectively).

By, (1.3.3) and (1.5.1), the inner product of $\vec{A}$ and $\vec{B}$ is ;

$$\vec{A}.\vec{B} = \vec{a}.\vec{b} + \varepsilon(\vec{a}.\vec{b}^* + \vec{a}^*.\vec{b}) \quad \text{and}$$

$$\vec{A}.\vec{B} = \|\vec{A}\|.\|\vec{B}\|.\cos\Phi = \cos\Phi = \cos\varphi - \varepsilon\varphi^*\sin\varphi \ .$$

Then $\quad \vec{a}.\vec{b} + \varepsilon(\vec{a}.\vec{b}^* + \vec{a}^*.\vec{b}) = \cos\varphi - \varepsilon\varphi^*\sin\varphi \qquad (2.2.1)$

The real part of the (2.2.1) gives ;

$$\vec{a}.\vec{b} = \cos\varphi \ .$$

Taking the angle between the real unit vectors $\vec{a}$ and $\vec{b}$ as φ, and using the fact that the inner product of two real vectors can be expressed by the cosine of the angle between them, we have

$$\vec{a}.\vec{b} = \|\vec{a}\|.\|\vec{b}\|.\cos\varphi = \cos\varphi \quad .$$

Now we will investigate the dual part , φ^* , of the dual angle Φ . We know that the dual unit vectors $\vec{A}$ and $\vec{B}$ represent two straight lines , say l_1 and l_2 respectively. If we take a unit vector which is perpendicular to both l_1 and l_2 then we can denote it by

$$\vec{n} = \mp \frac{\vec{a}\times\vec{b}}{\|\vec{a}\times\vec{b}\|} \quad .$$

A straight line passing through the shortest distance between the straight lines l_1 and l_2 intersects these lines at two points, say $\vec{x}$ and $\vec{y}$, respectively. On the other hand, the vectorial moments of the lines l_1 and l_2 with respect to the origin are $\vec{a}^* = \vec{x}\times\vec{a}$ and $\vec{b}^* = \vec{y}\times\vec{b}$, respectively. So

$$\vec{a}^*.\vec{b} = (\vec{x}\times\vec{a}).\vec{b} = \vec{x}.(\vec{a}\times\vec{b}) \quad (2.2.2)$$

$$\vec{a}.\vec{b}^* = \vec{a}.(\vec{y}\times\vec{b}) = -\vec{y}.(\vec{a}\times\vec{b}) . \quad (2.2.3)$$

The sum of (2.2.2) and (2.2.3) gives ;

$$\vec{a}^*.\vec{b} + \vec{a}.\vec{b}^* = (\vec{x}-\vec{y})(\vec{a}\times\vec{b}) \ . \quad (2.2.4)$$

If the shortest distance between the lines l_1 and l_2 is measured as γ, then it is clear that

$$\vec{x}-\vec{y} = \gamma(\mp\vec{n}) = \mp\gamma\frac{\vec{a}\times\vec{b}}{\|\vec{a}\times\vec{b}\|} \quad . \quad (2.2.5)$$

From (2.2.4) and (2.2.5), we get

$$\vec{a}.\vec{b}^* + \vec{a}^*.\vec{b} = \mp\gamma \frac{(\vec{a}\times\vec{b})^2}{\left\|\vec{a}\times\vec{b}\right\|} = \mp\gamma\left\|\vec{a}\times\vec{b}\right\| = \mp\gamma\sin\varphi \qquad (2.2.6)$$

From (2.2.1) and (2.2.6) we get

$$-\varphi^*\sin\varphi = \mp\gamma\sin\varphi \qquad (2.2.7)$$

Taking the suitable sign at the right hand side of (2.2.7), we obtain that the shortest distance, γ, is equal to φ^*.

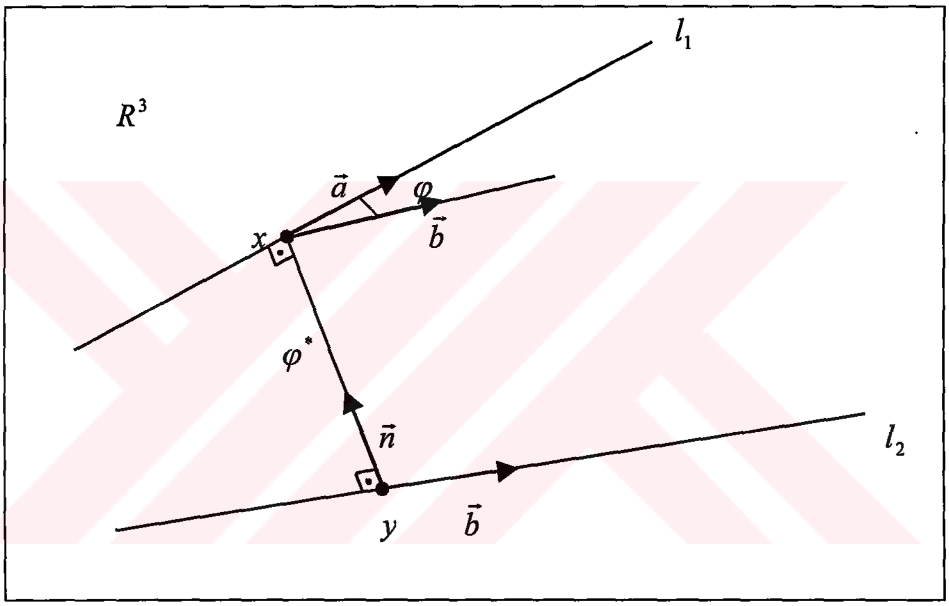

Figure 2.2 The geometric meaning of the dual angle

As a summary, if we denote the dual angle between the dual vectors of D.U.S. by $\Phi = \varphi + \varepsilon\varphi^*$ then φ is the angle between the straight lines and φ^* is the shortest distance between these lines.

As a consequence of the last statements, we have the following cases:

1) If $\vec{A}.\vec{B} = 0$ then $\vec{A}$ and $\vec{B}$ represent perpendicular intersecting straight lines in R^3.

2) If $\vec{A}.\vec{B}$ gives a pure dual number or $\vec{a}.\vec{b}=0$ then $\vec{A}$ and $\vec{B}$ represent skew straight lines in R^3.

3) If $\vec{A}.\vec{B}$ has a dual part equal to zero, i.e. $\vec{a}^*\vec{b}+\vec{a}.\vec{b}^*=0$ then $\vec{A}$ and $\vec{B}$ represent intersecting straight lines.

4) If $\vec{A}.\vec{B}$ has a real part equal to +1 or -1 and dual part different from zero then $\vec{A}$ and $\vec{B}$ represent parallel lines in R^3.

5) If $\vec{A}.\vec{B}$ has only a real part equal to +1 or -1 (dual part is equal to zero) then $\vec{A}$ and $\vec{B}$ represent coincident two lines in R^3.

Since there is a one to one correspondence between the straight lines in R^3 and the dual unit vectors of the D.U.S., one parameter motion on D.U.S. (this is a curve on D.U.S.) represents a ruled surface in R^3.

The motion of a point on D.U.S. is the motion of a dual unit vector oriented at the origin. If we define the motion on the D.U.S. by the equation $\vec{X}(t)=\vec{x}(t)+\varepsilon\vec{x}^*(t)$, then at each t, $\vec{X}(t)$ represents a straight line passing through the point $\vec{x}(t)\times\vec{x}^*(t)$ with direction $\vec{x}(t)$. The continuous change of the point $\vec{X}(t)$ on the D.U.S. draws a curve on D.U.S. and causes a continuous change of the represented straight line in R^3. This is nothing but the definition of the ruled surface denoted by $\vec{m}(t,u)=\vec{p}(t)+u\vec{x}(t)$. Here $\vec{p}(t)$ is the base curve and $\vec{x}(t)$ is the generator of the ruled surface. Since the moment of the straight line is independent from the choice of the point on the line, $\vec{x}^*(t)=\vec{p}(t)\times\vec{x}(t)$. Where by the rule of vectorial division,

$$\vec{p}(t)=\frac{\vec{x}^*(t)\times\vec{x}(t)}{\|\vec{x}(t)\|}+\lambda\vec{x}(t)=\vec{x}^*(t)\times\vec{x}(t)+\lambda\vec{x}(t)$$

(where λ is the parameter and $\|\vec{x}(t)\|=1$).

Hence the equation of the ruled surface becomes,

$$\vec{m}(t,u) = \vec{p}(t) + u\vec{x}(t),$$

$$\vec{m}(t,u) = \vec{x}^*(t) \times \vec{x}(t) + \lambda\vec{x}(t) + u\vec{x}(t),$$

$$= \vec{x}^*(t) \times \vec{x}(t) + (\lambda + u)\vec{x}(t).$$

By using α instead of $\lambda + u$, we have

$$\vec{m}(t,u) = \vec{x}^*(t) \times \vec{x}(t) + \alpha\vec{x}(t),$$

which is the equation of the ruled surface in R^3 represented by the curve $\vec{X}(t) = \vec{x}(t) + \varepsilon\vec{x}^*(t)$ on the D.U.S.

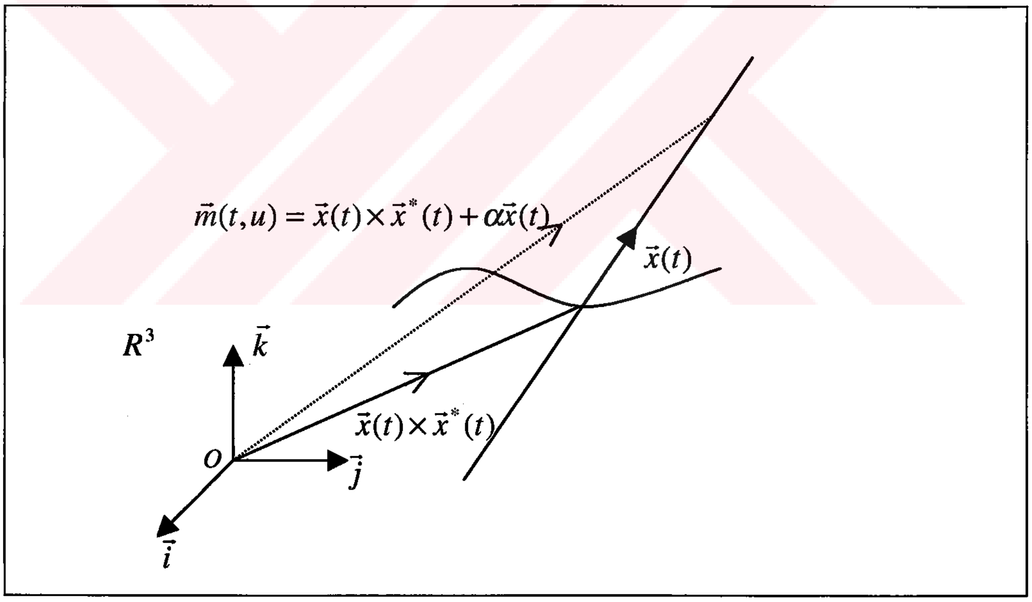

Figure 2.3 The ruled surface of $\vec{X}(t) = \vec{x}(t) + \varepsilon\vec{x}^*(t)$

On the other hand, if we take a ruled surface with the equation $\vec{m}(t,u) = \vec{p}(t) + u\vec{x}(t)$ then, it is clear that this ruled surface is represented by the curve $\vec{X}(t) = \vec{x}(t) + \varepsilon\vec{p}(t) \times \vec{x}(t)$ on the D.U.S.

Let us fix the variable t and take $t = t_0$. Hence $\vec{m}(t_0,u) = \vec{p}(t_0) + u\vec{x}(t_0)$ is a straight line passing through the point $\vec{p}(t_0)$ with the direction vector $\vec{x}(t_0)$, and u is the parameter of this line. Then the moment of this line is $\vec{x}^*(t_0) = \vec{p}(t_0) \times \vec{x}(t_0)$. We can determine this line by the dual vector $(\vec{x}(t_0), \vec{x}^*(t_0))$ on D.U.S (Since $\|\vec{x}(t_0)\| = 1$ and $\vec{x}(t_0).\vec{x}^*(t_0) = 0$). Here $(\vec{x}(t_0), \vec{x}^*(t_0)) = \vec{x}(t_0) + \varepsilon\vec{x}^*(t_0)$. The fixed variable t_0 defines a point on D.U.S. but the change of the free variable t causes the motion of the straight line in R^3 hence the motion on the D.U.S. By this motion, we get the representative curve on D.U.S. And the equation of this curve is then $\vec{X}(t) = \vec{x}(t) + \varepsilon\vec{x}^*(t)$.

2.3 The Line Complex

We proved that any dual vector $\vec{X} = \vec{x} + \varepsilon\vec{x}^* = (x_1, x_2, x_3) + \varepsilon(x_1^*, x_2^*, x_3^*)$ on the D.U.S. determines a straight line in R^3. Since $\vec{X}$ is a vector on the D.U.S. then the following conditions are satisfied.

1) $\vec{x}.\vec{x} = x_1^2 + x_2^2 + x_3^2 = 1$
2) $\vec{x}.\vec{x}^* = x_1x_1^* + x_2x_2^* + x_3x_3^* = 0$.

In addition if there exists another independent condition including the Plücker coordinates such that,

3) $F(x_1, x_2, x_3; x_1^*, x_2^*, x_3^*) = 0$

then we have 6 unknowns, the Plücker coordinates $(x_1, x_2, x_3, x_1^*, x_2^*, x_3^*)$, and three equations 1), 2), 3), hence the straight line $\vec{X}$ has three free parameters.

Any set of straight lines depending on three free parameters, i.e. ∞^3 number of straight lines in R^3, is said to be a line complex in R^3.

If u,v,w denote these parameters of the line complex then $\vec{X}$ can be written by $\vec{X}=\vec{x}(u,v,w)+\varepsilon\vec{x}^*(u,v,w)$.

2.4 The Line Congruence

If a line complex with three independent conditions

1) $x_1^2+x_2^2+x_3^2=1$,

2) $x_1x_1^*+x_2x_2^*+x_3x_3^*=0$,

3) $F(x_1,x_2,x_3;x_1^*,x_2^*,x_3^*)=0$,

has another independent condition such that

4) $G(x_1,x_2,x_3;x_1^*,x_2^*,x_3^*)=0$,

then we have a set of straight lines with two free parameters, i.e. ∞^2 number of straight lines, in R^3. Any set of straight lines depending on two free parameters is called a line congruence in R^3.

If u,v denote these parameters of the line congruence then the unit dual vector $\vec{X}$ can be written by $\vec{X}=\vec{x}(u,v)+\varepsilon\vec{x}^*(u,v)$.

2.5 The Ruled Surface

Including the natural conditions

1) $x_1^{\,2} + x_2^{\,2} + x_3^{\,2} = 1$,

2) $x_1 x_1^* + x_2 x_2^* + x_3 x_3^* = 0$

of the dual unit vector $\vec{X} = \vec{x} + \varepsilon \vec{x}^*$, if there exists three other independent conditions

3) $F(x_1, x_2, x_3; x_1^*, x_2^*, x_3^*) = 0$,

4) $G(x_1, x_2, x_3; x_1^*, x_2^*, x_3^*) = 0$,

5) $H(x_1, x_2, x_3; x_1^*, x_2^*, x_3^*) = 0$,

then the straight line has one free parameter. There are ∞^1 number of straight lines in this case. Any set of straight lines depending on one free parameter is called a ruled surface in R^3. If t denotes the free parameter of the ruled surface then the unit dual vector $\vec{X}$ is represented by $\vec{X} = \vec{x}(t) + \varepsilon \vec{x}^*(t)$.

As it is discussed before, the continuous change of t causes a motion of the straight line in R^3 i.e., a ruled surface. On the other hand, the change of t causes the motion of the dual unit vector $\vec{X}(t)$ on the D.U.S. and draws a dual curve on the D.U.S. The dual curve $\vec{X}$ is the dual spherical image of the ruled surface $\vec{X}(t)$.

On the other hand, we can define $d\phi = d\varphi + \varepsilon d\varphi^*$ as the dual arc-element of the curve $\vec{X} = \vec{X}(t)$. For the small increments, it is clear that the angular change is equal to the vectorial change on the sphere. Hence,

$$d\phi . d\phi = d\vec{X} . d\vec{X} \ ,$$

$$d\phi . d\phi = d\varphi . d\varphi + 2\varepsilon d\varphi . d\varphi^* \qquad \text{and}$$

$$d\vec{X} . d\vec{X} = d\vec{x} . d\vec{x} + 2\varepsilon d\vec{x} . d\vec{x}^* \qquad \text{imply}$$

$$d\varphi^2 = d\vec{x}^2 \quad \text{and} \quad d\varphi.d\varphi^* = d\vec{x}.d\vec{x}^* \;.$$

Here $d\phi$ denotes the angle between the neighbour vectors $\vec{X}(t)$ and $\vec{X}(t+dt)$. Also $d\phi$ measures the distance between the end points of $\vec{X}(t)$ and $\vec{X}(t+dt)$ on the D.U.S. Depending on the previous discussions, the real part $d\varphi$ and the dual part $d\varphi^*$ of $d\phi$ represent the angle and the distance between the neighbour straight lines represented by $\vec{X}(t)$ and $\vec{X}(t+dt)$ on the ruled surface, respectively.

The inner product $d\vec{X}.d\vec{X} = d\vec{x}.d\vec{x} + 2\varepsilon d\vec{x}.d\vec{x}^*$ is invariant under the transformations. Then the ratio of the quantities $d\vec{x}.d\vec{x}^*$ and $d\vec{x}.d\vec{x}$ is also invariant under the transformations. This ratio is the differential invariant of the ruled surface. We denote it by

$$\frac{1}{d} = \frac{d\vec{x}.d\vec{x}^*}{d\vec{x}.d\vec{x}} = \frac{d\varphi^*}{d\varphi}$$

and $\frac{1}{d}$ is called the distribution parameter of the ruled surface. (Hereinafter hat over an alphabet will define a dual vector).

Any motion on a sphere can be represented by a rotation. This can be thought as the rotation of a moving sphere over a fixed reference sphere. Let us denote the moving sphere by K, the fixed sphere by K' and the corresponding reference frames by R and E, respectively. Then we say that R moves with respect to E and we may interpret this as, the D.U.S. K rigidly connected with R moves over the D.U.S. K' rigidly connected with E. The motion is called a dual spherical motion and will be denoted by K/K'. If $\hat{x}$ is a point on K coinciding at the instant t with the point $\hat{X}$ on K', we have;

$$\hat{X} = \hat{A}\hat{x} \;, \qquad \text{where} \qquad \hat{A} = (\hat{\alpha}_{ik})$$

is the transformation matrix at the said instant R onto E. $\hat{A}$ is a function of t. Infact, since $E=\{\hat{e}_1,\hat{e}_2,\hat{e}_3\}$ and $R=\{\hat{r}_1,\hat{r}_2,\hat{r}_3\}$ are two orthonormal right handed trihedra defined on K' and K, respectively, any point on the D.U.S. can be written unambiguously as a linear combination of $\hat{e}_1,\hat{e}_2,\hat{e}_3$ as well as of $\hat{r}_1,\hat{r}_2,\hat{r}_3$. Therefore $\hat{X}\in K'$ and $\hat{x}\in K$ can be written as

$$\sum_{i=1}^{3}\hat{X}_i\hat{e}_i=\sum_{i=1}^{3}\hat{x}_i\hat{r}_i \tag{2.5.1}$$

The components $(\hat{X}_1,\hat{X}_2,\hat{X}_3)$ and $(\hat{x}_1,\hat{x}_2,\hat{x}_3)$ are the position vectors of a point $\hat{X}$ with respect to E and R, respevtively. We obtain from (2.5.1) that $\hat{X}_i=(\hat{e}_i\hat{r}_1)\hat{x}_1+(\hat{e}_i\hat{r}_2)\hat{x}_2+(\hat{e}_i\hat{r}_3)\hat{x}_3$, $(i=1,2,3)$ and putting $\hat{e}_i\hat{r}_k=\hat{\alpha}_{ik}$ yields the dual matrix

$$\hat{A}=(\hat{\alpha}_{ik})=(\alpha_{ik})+\varepsilon(\alpha_{ik})^*=A+\varepsilon A^* \ .$$

Then we see that

$\hat{X}=\hat{A}\hat{x}$ and $\hat{A}$ is an othogonal dual matrix. Then $\hat{A}\hat{A}^T=I$. Differentiating both sides of $\hat{A}\hat{A}^T=I$ with respect to t (we put dot over a symbol to denote the differentiation) gives

$$\dot{\hat{A}}\hat{A}^T+\hat{A}\dot{\hat{A}}^T=0 \ . \tag{2.5.2}$$

(2.5.2) implies $\dot{\hat{A}}\hat{A}^T=-\hat{A}\dot{\hat{A}}^T$. Hence $\dot{\hat{A}}\hat{A}^T$ is a skew-stmmetric matrix. So we define $\dot{\hat{A}}\hat{A}^T$ as

$$\dot{\hat{A}}\hat{A}^T = \begin{bmatrix} 0 & -\hat{w}_3 & \hat{w}_2 \\ \hat{w}_3 & 0 & -\hat{w}_1 \\ -\hat{w}_2 & \hat{w}_1 & 0 \end{bmatrix} = \hat{\Omega} \quad . \qquad (2.5.3)$$

The dual velocity of the point $\hat{x}$ on K is defined as $\hat{v} = \dot{\hat{x}} = \dot{\hat{A}}\hat{x}$, therefore

$$\hat{v} = \dot{\hat{A}}\hat{A}^T\hat{A}\hat{x} = \dot{\hat{A}}\hat{A}^T\hat{X} = \hat{\Omega}\hat{X} \qquad \text{(Veldkamp 1967).}$$

Introducing the vector $\hat{w}$ given by $\hat{w}^T = (\hat{w}_1, \hat{w}_2, \hat{w}_3)$, we may write

$$\hat{v} = \dot{\hat{X}} = \hat{w} \times \hat{X} \quad .$$

The dual vector $\hat{w}$ is called dual angular velocity (the dual Darboux vector) of the motion K/K'. If the D.U.S. K and K' correspond to the line spaces H and H', respectively, Then the dual spherical motion K/K' corresponds to the spatial motion in 3-space denoted by H/H'.

CHAPTER THREE
STRAIGHT LINE
ADJOINT TO A RULED SURFACE

A straight line (or a located vector) $d(\sigma)$ adjoint to a ruled surface R in H, defines a ruled surface depending on the Frenet frame $\{r, E_1, E_2, E_3\}$ of R. We can derive the Frenet frame or the natural trihedron of the ruled surface R as

$$E_1 = L(\sigma) \quad , \quad E_2 = \frac{dL}{d\sigma} \quad , \quad E_3 = E_1 \times E_2 \tag{3.1.1}$$

where $L(\sigma)$ is the unit direction vector of the instantaneous screw axis of R, and the origin of this Frenet frame is at the center (or sitriction point) of R, represented by $r(\sigma)$.

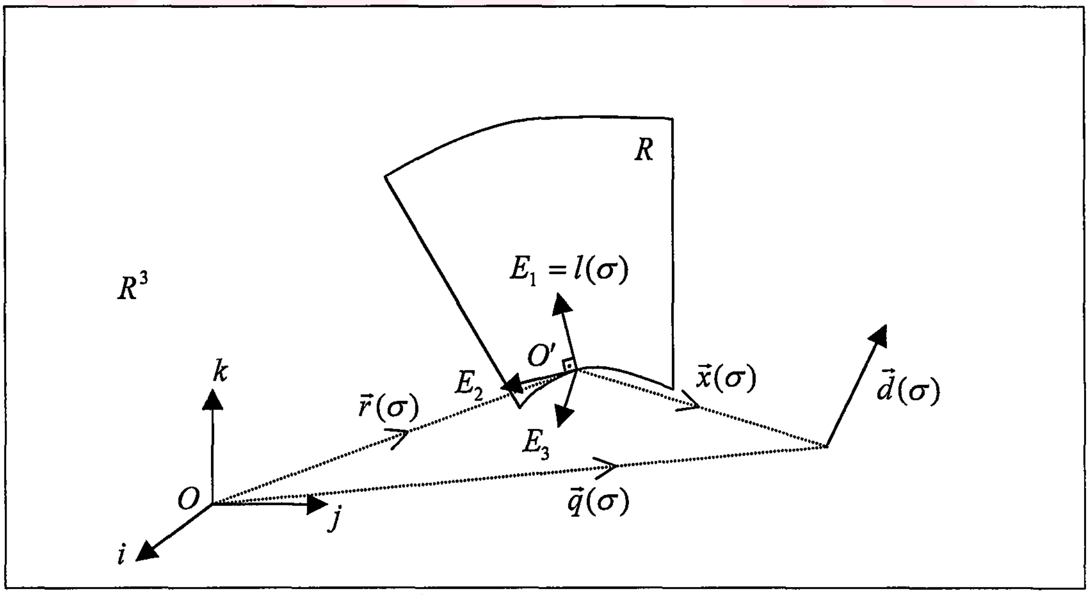

Figure 3.1 The straight line adjoint to a ruled surface

Then we have the well-known differential formulas of the Frenet frame ;

$$\frac{dr}{d\sigma} = \alpha E_1 + \gamma E_3,$$
$$\frac{dE_1}{d\sigma} = E_2, \qquad (3.1.2)$$
$$\frac{dE_2}{d\sigma} = -E_1 + \beta E_3,$$
$$\frac{dE_3}{d\sigma} = -\beta E_2,$$

where the coefficients α , β and γ are called the construction parameters or the curvature functions of R .

Since $d(\sigma)$ draws a ruled surface relative to $\{E_1, E_2, E_3\}$, the representative curve of this ruled surface on the D.U.S. can be obtained. Let us denote this curve by $\hat{x}(\sigma_0)$, where σ_0 is the arc-length parameter on the D.U.S., then we have

$$\vec{\hat{x}}(\sigma_0) = (\vec{d}(\sigma_0), \vec{d}(\sigma_0) \times \vec{q}(\sigma_0)),$$
$$\vec{\hat{x}}(\sigma_0) = (\vec{d}(\sigma_0), \vec{d}(\sigma_0) \times (\vec{r}(\sigma_0) + \vec{x}(\sigma_0))),$$
$$\vec{\hat{x}}(\sigma_0) = \vec{d}(\sigma_0) + \varepsilon\{\vec{d}(\sigma_0) \times (\vec{r}(\sigma_0) + \vec{x}(\sigma_0))\}.$$

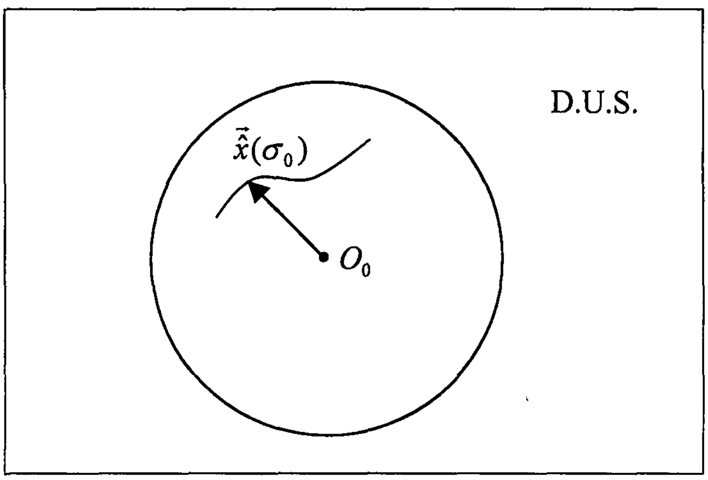

Figure 3.2 The dual curve on the D.U.S.

Let $\dfrac{d\sigma}{d\sigma_0} = s$ then

$$\frac{d\vec{\hat{x}}}{d\sigma_0} = \frac{d\vec{\hat{x}}}{d\sigma} \cdot \frac{d\sigma}{d\sigma_0} = \vec{\hat{x}}'.s \ ,$$

$$\frac{d^2\vec{\hat{x}}}{d\sigma_0^{\ 2}} = \frac{d^2\vec{\hat{x}}}{d\sigma^2} \cdot \left(\frac{d\sigma}{d\sigma_0}\right)^2 + \frac{d\vec{\hat{x}}}{d\sigma} \cdot \frac{d^2\sigma}{d\sigma_0^{\ 2}} = \vec{\hat{x}}''.s^2 + \vec{\hat{x}}' \cdot \frac{d^2\sigma}{d\sigma_0^{\ 2}}$$

and

$$\frac{d\vec{\hat{x}}}{d\sigma_0} = \vec{d}.s + \varepsilon\{\vec{d}' \times (\vec{r} + \vec{x}) + \vec{d} \times (\vec{r}' + \vec{x}')\}.s \ ,$$

$$\frac{d^2\vec{\hat{x}}}{d\sigma_0^{\ 2}} = \{\vec{d}''.s^2 + \vec{d} \cdot \frac{d^2\sigma}{d\sigma_0^{\ 2}}\} + \varepsilon\{(\vec{d}'' \times (\vec{r} + \vec{x}) + 2.\vec{d}' \times (\vec{r}' + \vec{x}') + \vec{d} \times (\vec{r}'' + \vec{x}'')).s^2$$
$$+ (\vec{d}' \times (\vec{r} + \vec{x}) + \vec{d} \times (\vec{r}' + \vec{x}')) \cdot \frac{d^2\sigma}{d\sigma_0^{\ 2}}\} .$$

3.1 The Acceleration Center : We want to determine the points that sasisfy the condition $\dfrac{d^2\vec{\hat{x}}}{d\sigma_0^{\ 2}} = 0$.

The real part of the equation is

$$\vec{d}''.s^2 + \vec{d}' \cdot \frac{d^2\sigma}{d\sigma_0^{\ 2}} = 0 \ . \qquad (3.1.3)$$

(3.1.3) implies $\vec{d}(\sigma)$ is constant for every σ, σ_0 parameters .

The dual part of the equation ; since $\vec{d}(\sigma)$ is constant, from (3.1.3), $\vec{d}'(\sigma) = \vec{d}''(\sigma) = 0$ and

$$\frac{d^2\sigma}{d\sigma_0{}^2} \cdot \{\vec{d} \times (\vec{r}' + \vec{x}')\} + s^2 . \{\vec{d} \times (\vec{r}'' + \vec{x}'')\} = 0 \qquad \Rightarrow$$

$$\vec{d} \times \{\frac{d^2\sigma}{d\sigma_0{}^2} \cdot (\vec{r}' + \vec{x}') + s^2 . (\vec{r}'' + \vec{x}'')\} = 0 \qquad \Rightarrow$$

$$\vec{d} \, // \, \frac{d^2\sigma}{d\sigma_0{}^2} \cdot (\vec{r}' + \vec{x}') + s^2 . (\vec{r}'' + \vec{x}'') \qquad \text{or}$$

$$\frac{d^2\sigma}{d\sigma_0{}^2} \cdot (\vec{r}' + \vec{x}') + s^2 . (\vec{r}'' + \vec{x}'') = 0 \ ,$$

then we can write,

$$k_0 . \vec{d} = \frac{d^2\sigma}{d\sigma_0{}^2} \cdot (\vec{r}' + \vec{x}') + s^2 . (\vec{r}'' + \vec{x}'') \qquad , \qquad \forall \, k_0 \in R \ , \tag{3.1.4}$$

where

$$\vec{r}' = \alpha E_1 + \gamma E_3 \ ,$$

$$\vec{r}'' = \alpha' E_1 + (\alpha - \gamma\beta) E_2 + \gamma' E_3$$

and

$$\vec{x} = x_1 E_1 + x_2 E_2 + x_3 E_3 \ ,$$

$$\vec{x}' = (x_1{}' - x_2) E_1 + (x_2{}' - \beta x_3 + x_1) E_2 + (x_3{}' + \beta x_2) E_3 \ ,$$

$$\vec{x}'' = (x_1{}'' - 2x_2{}' - x_1 + \beta x_3) E_1 + (x_2{}'' + 2x_1{}' - 2\beta x_3{}' - (1 + \beta^2) x_2 - \beta'' x_3) E_2$$
$$+ (x_3{}'' + 2\beta x_2{}' + \beta x_1 + \beta'' x_2 - \beta^2 x_3) E_3 \ .$$

Then from (3.1.4), we get

$$\vec{x}'' = \frac{k_0}{s^2}\cdot\vec{d} - \frac{d^2\sigma}{d\sigma_0{}^2}\cdot\frac{1}{s^2}(\vec{r}' + \vec{x}') - \vec{r}'' . \tag{3.1.5}$$

Taking $\frac{k_0}{s^2} = k$ and $-\frac{d^2\sigma}{d\sigma_0{}^2}\cdot\frac{1}{s^2} = p$ and from the expansions of $\vec{r}', \vec{x}', \vec{r}''$ and $\vec{x}''$, we get ;

$$\vec{x}'' = k.\vec{d} + p.(\vec{r}' + \vec{x}') - \vec{r}'' ,$$

$$(x_1'' - 2x_2' - x_1 + \beta x_3, x_2'' + 2x_1' - 2\beta x_3' - (1+\beta^2)x_2 - \beta' x_3, x_3'' + 2\beta x_2' + \beta x_1 + \beta' x_2 - \beta^2 x_3) =$$
$$k.(d_1, d_2, d_3) \;+\; p.(\beta, 0, \gamma) \;+\; p.(x_1' - x_2, x_2' - \beta x_3 + x_1, x_3' + \beta x_2) \;-\; (\alpha', \alpha - \gamma\beta, \gamma'),$$

$$\begin{aligned}
x_1'' - 2x_2' - x_1 + \beta x_3 &= kd_1 + p\alpha + p(x_1' - x_2) - \alpha' , \\
x_2'' + 2x_1' - 2\beta x_3' - (1+\beta^2)x_2 - \beta' x_3 &= kd_2 + p(x_2' - \beta x_3 + x_1) - \alpha + \gamma\beta , \\
x_3'' + 2\beta x_2' + \beta x_1 + \beta' x_2 - \beta^2 x_3 &= kd_3 + p\gamma + p(x_3' + \beta x_2) - \gamma' ,
\end{aligned}$$

$$\begin{aligned}
x_1'' &= px_1' + 2x_2' + x_1 - px_2 - \beta x_3 + (kd_1 + p\alpha - \alpha') , \\
x_2'' &= -2x_1' + px_2' + 2\beta x_3' + px_1 + (1+\beta^2)x_2 + (\beta' - p\beta)x_3 + (kd_2 - \alpha + \gamma\beta) , \\
x_3'' &= -2\beta x_2' + px_3' - \beta x_1 + (p\beta - \beta')x_2 + \beta^2 x_3 + (kd_3 + p\gamma - \gamma') ,
\end{aligned}$$

and in matrix form, we have

$$\begin{pmatrix} x_1'' \\ x_2'' \\ x_3'' \end{pmatrix} = \begin{pmatrix} p & 2 & 0 \\ -2 & p & 2\beta \\ 0 & -2\beta & p \end{pmatrix}\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} + \begin{pmatrix} 1 & -p & -\beta \\ p & 1+\beta^2 & -(p\beta - \beta') \\ \beta & p\beta - \beta' & \beta^2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + C$$

or

$$X'' = AX' + BX + C \ , \qquad \text{where } C = \begin{pmatrix} kd_1 + p\alpha - \alpha' \\ kd_2 + \gamma\beta - \alpha \\ kd_3 + p\gamma - \gamma' \end{pmatrix}.$$

The solution can be obtained by transforming the second order system to a first order system.

Let $X(\sigma) = Y_1(\sigma)$ and $Y_1'(\sigma) = Y_2(\sigma)$ then

$$Y_1'(\sigma) = Y_2(\sigma) \ ,$$
$$Y_2'(\sigma) = A(\sigma)Y_2(\sigma) + B(\sigma)Y_1(\sigma) + C(\sigma)$$

or

$$\begin{pmatrix} Y_1'(\sigma) \\ Y_2(\sigma) \end{pmatrix} = \begin{pmatrix} 0 & I \\ B(\sigma) & A(\sigma) \end{pmatrix} \begin{pmatrix} Y_1(\sigma) \\ Y_2(\sigma) \end{pmatrix} + \begin{pmatrix} 0 \\ C(\sigma) \end{pmatrix}$$

or simply

$$Z'(\sigma) = M_{6\times6}(\sigma) \cdot Z(\sigma) + D(\sigma) \ .$$

The solution set of this system of differential equations will give the position of $d(\sigma)$ with respect to the frame field $\{r, E_1, E_2, E_3\}$.

3.2 The Bresse Hyperboloid : The points that satisfy, $\vec{a}.\vec{v} = 0$ or $\dfrac{d^2\vec{x}}{d\sigma_0^{\ 2}} \cdot \dfrac{d\vec{x}}{d\sigma_0} = 0$.

$$\frac{d\vec{x}}{d\sigma_0}\cdot\frac{d^2\vec{x}}{d\sigma_0{}^2} = s\vec{d}'(\vec{d}''s^2+\vec{d}'\frac{d^2\sigma}{d\sigma_0{}^2})+\varepsilon\{s^2(\vec{d}'(\vec{d}\times(\vec{r}''+\vec{x}''))+\vec{d}''\times(\vec{r}+\vec{x})))+$$

$$s^3(\vec{d}''(\vec{d}\times(\vec{r}+\vec{x})+\vec{d}'\times(\vec{r}+\vec{x})))+\frac{d^2\sigma}{d\sigma_0{}^2}(\vec{d}'(\vec{d}\times(\vec{r}'+\vec{x}')))+s\frac{d^2\sigma}{d\sigma_0{}^2}(\vec{d}'(\vec{d}\times(\vec{r}'+\vec{x}')))\}=0$$

Since $\|d(\sigma)\|=1$, $\vec{d}.\vec{d}=1$ and $\vec{d}.\vec{d}'=0$ $(where\ \vec{d}\perp\vec{d}')$. Taking

$$\vec{d}(\sigma)=R_1,\quad \vec{d}'(\sigma)=R_2\quad and\quad R_1\times R_2=R_3$$

we have a frame field $\{R_1,R_2,R_3\}$, where

$$\begin{aligned} R_1' &= R_2, \\ R_2' &= -R_1+\tilde{\beta}R_3, \\ R_3' &= -\tilde{\beta}R_2, \end{aligned}$$

and $\tilde{\alpha},\tilde{\beta},\tilde{\gamma}$ are the construction parameters of the ruled surface determined by $\{x(\sigma),R_1,R_2,R_3\}$.

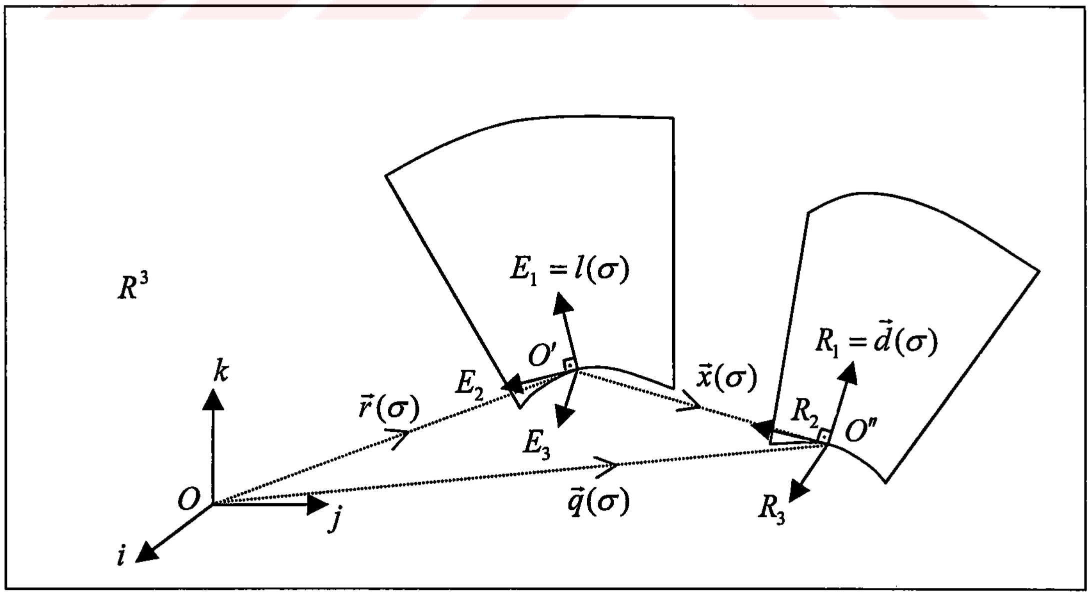

Figure 3.3 The ruled surface adjoint to a ruled surface

The real part of the equation is

$$s\vec{d}'(\vec{d}''s^2 + \vec{d}'\frac{d^2\sigma}{d\sigma_0{}^2}) = 0 \ , \tag{3.2.1}$$

then (3.2.1) implies

$$sR_2(R_2{}'s^2 + R_2\frac{d^2\sigma}{d\sigma_0{}^2}) = 0 \qquad \Rightarrow$$

$$sR_2((-R_1 + \tilde{\beta}R_3)s^2 + R_2\frac{d^2\sigma}{d\sigma_0{}^2}) = 0 \ \Rightarrow$$

$$s\frac{d^2\sigma}{d\sigma_0{}^2} = \frac{d\sigma}{d\sigma_0}\cdot\frac{d^2\sigma}{d\sigma_0{}^2} = 0 \qquad \Rightarrow$$

$$\frac{d\sigma}{d\sigma_0} = \text{constant}\,.$$

The dual part is simply

$$s^2\{\vec{d}'(\vec{d}\times(\vec{r}+\vec{x}) + \vec{d}''\times(\vec{r}+\vec{x}))\} + s^3\{\vec{d}''(\vec{d}\times(\vec{r}+\vec{x}) + \vec{d}'\times(\vec{r}+\vec{x}))\} = 0 \tag{3.2.2}$$

Making the following replacements

$$\vec{d} = R_1$$

$$\vec{d}' = R_2$$

$$\vec{d}'' = R_2{}' = -R_1 + \tilde{\beta}R_3$$

$$\vec{x} = x_1R_1 + x_2R_2 + x_3R_3$$

$$\vec{x}' = \tilde{\alpha}R_1 + \tilde{\gamma}R_3$$

$$\vec{x}'' = \tilde{\alpha}'R_1 + (\tilde{\alpha} - \tilde{\beta}\tilde{\gamma})R_2 + \tilde{\gamma}'R_3 \qquad \text{into} \qquad (3.2.2)$$

and using the relation $r_1E_1 + r_2E_2 + r_3E_3 = \tilde{r}_1R_1 + \tilde{r}_2R_2 + \tilde{r}_3R_3$, we get

$$\begin{pmatrix}\tilde{r}_1\\ \tilde{r}_2\\ \tilde{r}_3\end{pmatrix}=\begin{pmatrix}E_1.R_1 & E_2.R_1 & E_3.R_1\\ E_1.R_2 & E_2.R_2 & E_3.R_2\\ E_1.R_3 & E_2.R_3 & E_3.R_3\end{pmatrix}\begin{pmatrix}r_1\\ r_2\\ r_3\end{pmatrix} \qquad \text{or} \qquad \tilde{r}=Br\,.$$

Then $Br'=\tilde{r}'$, $Br''=\tilde{r}''$ and $\tilde{\alpha},\tilde{\beta},\tilde{\gamma}$ are the construction parameters of the ruled surface determined by $\{\vec{x},R_1,R_2R_3\}$.

Without lost of generality we can take $\|R_1\|=\|R_2\|=\|R_3\|=1$. Hence

$$\vec{d}'(\vec{d}\times(\vec{r}''+\vec{x}''))=-R_3Br''-\tilde{\gamma}'\,,$$

$$\vec{d}'(\vec{d}''\times(\vec{r}+\vec{x}))=R_3Br+\tilde{\beta}R_1Br+x_3+\tilde{\beta}x_1\,,$$

$$\vec{d}''(\vec{d}\times(\vec{r}+\vec{x}))=\tilde{\beta}R_2Br+\tilde{\beta}x_2\,,$$

$$\vec{d}''(\vec{d}'\times(\vec{r}+\vec{x}))=-R_3Br-\tilde{\beta}R_1Br-x_3-\tilde{\beta}x_1\,.$$

So (3.2.2) yields ,

$$\{-R_3Br''-\tilde{\gamma}'+R_3Br+\tilde{\beta}R_1Br+x_3+\tilde{\beta}x_1\}+s.\{\tilde{\beta}R_2Br+\tilde{\beta}x_2-R_3Br-\tilde{\beta}R_1Br-x_3-\tilde{\beta}x_1\}=0$$

or

$$x_1\tilde{\beta}(1-s)+x_2s\tilde{\beta}+x_3(1-s)=-\tilde{r}_3''+\tilde{\gamma}'+(s-1)\tilde{r}_3+(s-1)\tilde{\beta}\tilde{r}_1-s\tilde{\beta}\tilde{r}_2\,.$$

Taking x_2 and x_3 as free parameters we can define a line congruence ;

$$x_1=\frac{1}{\tilde{\beta}(1-s)}\{-s\tilde{\beta}x_2+(s-1)x_3+(s-1)\tilde{\beta}\tilde{r}_1+-s\tilde{\beta}\tilde{r}_2+(s-1)\tilde{r}_3-\tilde{r}_3''+\tilde{\gamma}'\}\,,$$

$$x_2=x_2\,,$$

$$x_3=x_3\,,$$

under the conditions $s=\dfrac{d\sigma}{d\sigma_0}$ is a constant different from 1 and $\tilde{\beta}\neq 0$.

3.3 The Inflection Surfaces : The points that satisfy $\vec{a}\times\vec{v}=0$ or $\dfrac{d\vec{x}}{d\sigma_0}\times\dfrac{d^2\vec{x}}{d{\sigma_0}^2}=0$.

$\dfrac{d\vec{x}}{d\sigma_0}\times\dfrac{d^2\vec{x}}{d{\sigma_0}^2}=0$ implies

$$s\vec{d}'\times(s^2\vec{d}''+\frac{d^2\sigma}{d{\sigma_0}^2}\vec{d}') + \varepsilon\{s^2(\vec{d}'\times(\vec{d}\times(\vec{r}''+\vec{x}'')))+s^3((\vec{d}\times(\vec{r}'+\vec{x}'))\times\vec{d}'') + \frac{d^2\sigma}{d{\sigma_0}^2}(\vec{d}'\times(\vec{d}\times(\vec{r}'+\vec{x}')))+s\frac{d^2\sigma}{d{\sigma_0}^2}((\vec{d}\times(\vec{r}'+\vec{x}'))\times\vec{d}')=0. \quad (3.3.1)$$

From the real part of (3.3.1) , we have

$$s\vec{d}'\times s^2\vec{d}''+s\vec{d}'\times\frac{d^2\sigma}{d{\sigma_0}^2}\vec{d}'=0 \quad\Rightarrow\quad s^3\vec{d}'\times\vec{d}''=0 \quad\Rightarrow\quad \vec{d}'\,//\,\vec{d}'' .$$

Since $\vec{d}\cdot\vec{d}'=0$ and $\vec{d}'\,//\,\vec{d}''$, $\vec{d}\cdot\vec{d}''=0$.

And from the dual part of (3.3.1),

$\vec{d}'\times(\vec{d}\times\vec{r}'')=(\vec{d}'.\vec{r}'').\vec{d}$,

$\vec{d}'\times(\vec{d}\times\vec{x}'')=(\vec{d}'.\vec{x}'').\vec{d}$,

$(\vec{d}\times\vec{r}')\times\vec{d}''=-(\vec{d}''.\vec{r}').\vec{d}$,

$(\vec{d}\times\vec{x}')\times\vec{d}''=-(\vec{d}''.\vec{x}').\vec{d}$,

$\vec{d}'\times(\vec{d}\times\vec{r}')=(\vec{d}'.\vec{r}').\vec{d}$,

$\vec{d}'\times(\vec{d}\times\vec{x}')=(\vec{d}'.\vec{x}').\vec{d}$,

$(\vec{d}\times\vec{r}')\times\vec{d}'=-(\vec{d}'.\vec{r}').\vec{d}$,

$(\vec{d}\times\vec{x}')\times\vec{d}'=-(\vec{d}'.\vec{x}').\vec{d}$, then the dual part yields

$$\{s^2\vec{d}'(\vec{r}+\vec{x})'' - s^3\vec{d}''(\vec{r}+\vec{x})' + \frac{d^2\sigma}{d{\sigma_0}^2}\vec{d}'(\vec{r}+\vec{x})' - s\frac{d^2\sigma}{d{\sigma_0}^2}\vec{d}'(\vec{r}+\vec{x})'\}\vec{d} = 0$$

or

$$\{s^2\vec{d}'(\vec{r}+\vec{x})'' - s^3\vec{d}''(\vec{r}+\vec{x})' + (1-s)\frac{d^2\sigma}{d{\sigma_0}^2}\vec{d}'(\vec{r}+\vec{x})'\}\vec{d} = 0$$

and which implies the condition that the coefficient is zero,

$$s^2\vec{d}'(\vec{r}+\vec{x})'' - s^3\vec{d}''(\vec{r}+\vec{x})' + (1-s)\frac{d^2\sigma}{d{\sigma_0}^2}\vec{d}'(\vec{r}+\vec{x})' = 0\,.$$

CHAPTER FOUR

A CANONICAL FRAME FIELD

Let E be the fixed natural frame field of the D.U.S. Consider a moving orthonormal frame field ,

$$\hat{r}_1 = \vec{L} + \varepsilon \frac{1}{a}\frac{d\vec{L}}{d\sigma},$$

$$\hat{r}_2 = \frac{1}{a}\frac{d\vec{L}}{d\sigma} - \varepsilon\vec{L},$$

$$\hat{r}_3 = \vec{L} \times \frac{1}{a}\frac{d\vec{L}}{d\sigma},$$

as a canonical frame field which is obtained from the following ruled surface

$$m(\sigma,u) = \vec{r}(\sigma) + u\vec{L}(\sigma) \quad , \qquad \text{where } \left\|\vec{L}(\sigma)\right\| = 1 \quad \text{and} \quad \left\|\frac{d\vec{L}}{d\sigma}\right\| = a.$$

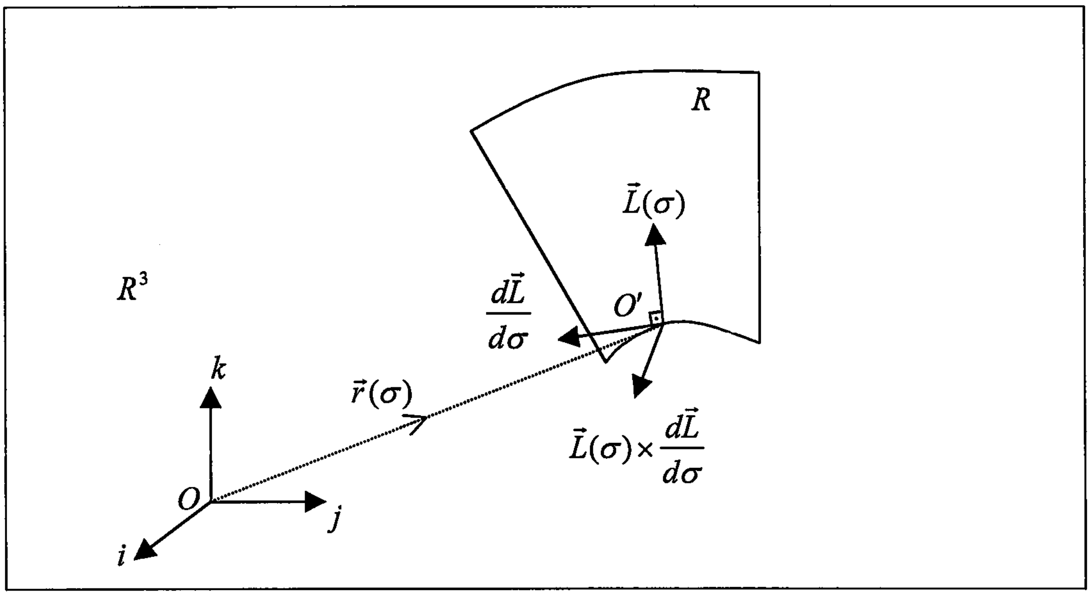

Figure 4.1 The Frenet frame of a ruled surface

Any point $\hat{P}$ on the D.U.S. can be defined by two of the frame fields $\hat{E} = \{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$ and $\hat{R} = \{\hat{r}_1, \hat{r}_2, \hat{r}_3\}$. That is $\hat{P} = \sum \hat{X}_i \hat{e}_i = \sum \hat{x}_i \hat{r}_i$. Then we get $\hat{X}_i = (\hat{e}_i \hat{r}_1)\hat{x}_1 + (\hat{e}_i \hat{r}_2)\hat{x}_2 + (\hat{e}_i \hat{r}_3)\hat{x}_3$.

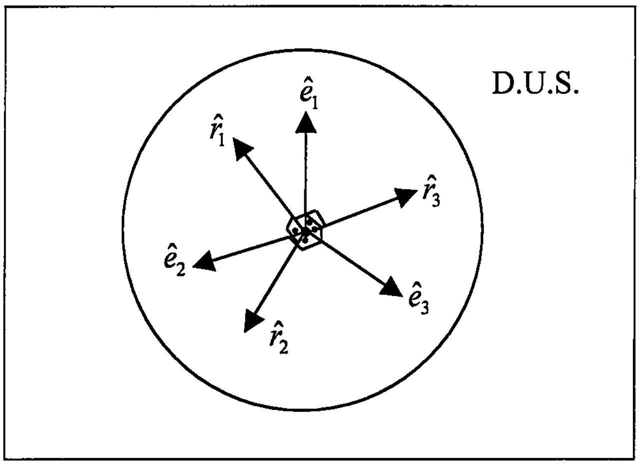

Figure 4.2 The canonical frame field

Let $\hat{\alpha}_{ik} = \hat{e}_i \cdot \hat{r}_k$ then $\hat{A} = (\hat{\alpha}_{ik})$ where $\hat{A}$ is an orthogonal matrix i.e. $\hat{A}\hat{A}^T = I$, then

$$\hat{X} = \hat{A}\hat{x} \quad , \quad \hat{R} = \hat{A}\hat{E} \quad , \quad \hat{E} = \hat{A}^T \hat{R} \ .$$

Let us compute the entries of $\hat{A}$

$$\hat{e}_1 \cdot \hat{r}_1 = (l_1 + \varepsilon \frac{1}{a}\frac{dl_1}{d\sigma}),$$

$$\hat{e}_1 \cdot \hat{r}_2 = \frac{1}{a}\frac{dl_1}{d\sigma} + \varepsilon(-l_1),$$

$$\hat{e}_1 \cdot \hat{r}_3 = \frac{1}{a}(l_2 \frac{dl_3}{d\sigma} - l_3 \frac{dl_2}{d\sigma}),$$

$$\hat{e}_2 \cdot \hat{r}_1 = l_2 + \varepsilon \frac{1}{a}\frac{dl_2}{d\sigma},$$

$$\hat{e}_2 \cdot \hat{r}_2 = \frac{1}{a}\frac{dl_2}{d\sigma} + \varepsilon(-l_2),$$

$$\hat{e}_2 \cdot \hat{r}_3 = \frac{1}{a}(l_3 \frac{dl_1}{d\sigma} - l_1 \frac{dl_3}{d\sigma}),$$

$$\hat{e}_3 \cdot \hat{r}_1 = l_3 + \varepsilon\frac{1}{a}\frac{dl_3}{d\sigma},$$

$$\hat{e}_3 \cdot \hat{r}_2 = \frac{1}{a}\frac{dl_3}{d\sigma} + \varepsilon(-l_3),$$

$$\hat{e}_3 \cdot \hat{r}_3 = \frac{1}{a}(l_1 \frac{dl_2}{d\sigma} - l_2 \frac{dl_1}{d\sigma}) \ . \text{ Then}$$

$$\hat{A} = \begin{pmatrix} l_1 + \varepsilon\frac{1}{a}\frac{dl_1}{d\sigma} & \frac{1}{a}\frac{dl_1}{d\sigma} - \varepsilon l_1 & \frac{1}{a}(l_2 \frac{dl_3}{d\sigma} - l_3 \frac{dl_2}{d\sigma}) \\ l_2 + \varepsilon\frac{1}{a}\frac{dl_2}{d\sigma} & \frac{1}{a}\frac{dl_2}{d\sigma} - \varepsilon l_2 & \frac{1}{a}(l_3 \frac{dl_1}{d\sigma} - l_1 \frac{dl_3}{d\sigma}) \\ l_3 + \varepsilon\frac{1}{a}\frac{dl_3}{d\sigma} & \frac{1}{a}\frac{dl_3}{d\sigma} - \varepsilon l_3 & \frac{1}{a}(l_1 \frac{dl_2}{d\sigma} - l_2 \frac{dl_1}{d\sigma}) \end{pmatrix},$$

where the first coulumn of $\hat{A}$ is $\vec{L} + \varepsilon\frac{1}{a}\frac{d\vec{L}}{d\sigma}$,

the second coulumn is $\frac{1}{a}\frac{d\vec{L}}{d\sigma} - \varepsilon\vec{L}$ and

the third coulumn is $\frac{1}{a}\vec{L} \times \frac{d\vec{L}}{d\sigma}$.

Since the transformation is orthogonal, the matrix representation $\hat{A}$ of the transformation is a dual orthogonal matrix, i.e.

$$\hat{A}^T \hat{A} = \hat{A}\hat{A}^T = I \ . \tag{4.1.1}$$

Taking the derivative of (4.1.1), we get

$$\dot{\hat{A}}^T \hat{A} + \hat{A}^T \dot{\hat{A}} = 0 \qquad \text{or} \qquad \dot{\hat{A}}^T \hat{A} = -\hat{A}^T \dot{\hat{A}} = -(\dot{\hat{A}}^T \hat{A})^T .$$

Therefore $\dot{\hat{A}}^T\hat{A}$ is a skew-symmetric matrix. Let us denote $\dot{\hat{A}}^T\hat{A}$ by $\hat{\Omega}$.

R moves with respect to E, we may interpret this as follows : The D.U.S K rigidly connected with R moves over the D.U.S. K' rigidly connected with E. For sake of simplicity we call K' the fixed and K the moving sphere.

On the other hand, any point $\hat{X}$ on K' draws a path $\hat{x}$ on K and $\hat{x} = \hat{A}^T\hat{X}$. Then the velocity of $\hat{x}$ is,

$$\hat{v} = \dot{\hat{x}} = \dot{\hat{A}}^T\hat{X} = \dot{\hat{A}}^T I\hat{X} = \dot{\hat{A}}^T\hat{A}\hat{A}^T\hat{X} = \dot{\hat{A}}^T\hat{A}\hat{x}\ , \qquad \text{hence} \qquad \dot{\hat{x}} = \dot{\hat{A}}^T\hat{A}\hat{x}\ .$$

Let $\hat{w}^T = (\hat{w}_1, \hat{w}_2, \hat{w}_3)$ then $\hat{v} = \dot{\hat{x}} = \hat{\Omega}\hat{x} = \hat{w} \times \hat{x}$.

$\hat{w}$ is the dual angular velocity (the Dual Darboux vector) :

$$\hat{\Omega} = \dot{\hat{A}}^T\hat{A} = \begin{pmatrix} 0 & -\hat{w}_3 & \hat{w}_2 \\ \hat{w}_3 & 0 & -\hat{w}_1 \\ -\hat{w}_2 & \hat{w}_1 & 0 \end{pmatrix}.$$

Let us denote $\dfrac{d\sigma}{d\sigma_0}$ by b, where σ_0 is the arc-length parameter on the D.U.S. Then

$$\dot{\hat{A}} = \begin{pmatrix} b(\frac{dl_1}{d\sigma} + \varepsilon\frac{1}{a}\frac{d^2l_1}{d\sigma^2}) & b(\frac{1}{a}\frac{d^2l_1}{d\sigma^2} - \varepsilon\frac{dl_1}{d\sigma}) & \frac{b}{a}(l_2\frac{d^2l_3}{d\sigma^2} - l_3\frac{d^2l_2}{d\sigma^2}) \\ b(\frac{dl_2}{d\sigma} + \varepsilon\frac{1}{a}\frac{d^2l_2}{d\sigma^2}) & b(\frac{1}{a}\frac{d^2l_2}{d\sigma^2} - \varepsilon\frac{dl_2}{d\sigma}) & \frac{b}{a}(l_3\frac{d^2l_1}{d\sigma^2} - l_1\frac{d^2l_3}{d\sigma^2}) \\ b(\frac{dl_3}{d\sigma} + \varepsilon\frac{1}{a}\frac{d^2l_3}{d\sigma^2}) & b(\frac{1}{a}\frac{d^2l_3}{d\sigma^2} - \varepsilon\frac{dl_3}{d\sigma}) & \frac{b}{a}(l_1\frac{d^2l_2}{d\sigma^2} - l_2\frac{d^2l_1}{d\sigma^2}) \end{pmatrix},$$

hence

$$\hat{\Omega} = \dot{\hat{A}}^T \hat{A} = \begin{pmatrix} \hat{\Omega}_{11} & \hat{\Omega}_{12} & \hat{\Omega}_{13} \\ \hat{\Omega}_{21} & \hat{\Omega}_{22} & \hat{\Omega}_{23} \\ \hat{\Omega}_{31} & \hat{\Omega}_{32} & \hat{\Omega}_{33} \end{pmatrix} \quad .$$

The multiplication of matrices $\dot{\hat{A}}^T$ and $\hat{A}$ yields ;

$$\hat{\Omega}_{11} = b(\frac{d\vec{L}}{d\sigma} + \varepsilon \frac{1}{a}\frac{d^2\vec{L}}{d\sigma^2})(\vec{L} + \varepsilon \frac{1}{a}\frac{d\vec{L}}{d\sigma}) = 0 \ ,$$

$$\hat{\Omega}_{12} = b(\frac{d\vec{L}}{d\sigma} + \varepsilon \frac{1}{a}\frac{d^2\vec{L}}{d\sigma^2})(\frac{1}{a}\frac{d\vec{L}}{d\sigma} - \varepsilon\vec{L}) = ba \ ,$$

$$\hat{\Omega}_{13} = b(\frac{d\vec{L}}{d\sigma} + \varepsilon \frac{1}{a}\frac{d^2\vec{L}}{d\sigma^2})(\vec{L} \times \frac{1}{a}\frac{d\vec{L}}{d\sigma}) = \varepsilon \frac{b}{a^2}\frac{d^2\vec{L}}{d\sigma^2}(\vec{L} \times \frac{d\vec{L}}{d\sigma}) \ ,$$

$$\hat{\Omega}_{21} = b(\frac{1}{a}\frac{d^2\vec{L}}{d\sigma^2} - \varepsilon \frac{d\vec{L}}{d\sigma})(\vec{L} + \varepsilon \frac{1}{a}\frac{d\vec{L}}{d\sigma}) = -ba \ ,$$

$$\hat{\Omega}_{22} = b(\frac{1}{a}\frac{d^2\vec{L}}{d\sigma^2} - \varepsilon \frac{d\vec{L}}{d\sigma})(\frac{1}{a}\frac{d\vec{L}}{d\sigma} - \varepsilon\vec{L}) = 0 \ ,$$

$$\hat{\Omega}_{23} = b(\frac{1}{a}\frac{d^2\vec{L}}{d\sigma^2} - \varepsilon \frac{d\vec{L}}{d\sigma})(\vec{L} \times \frac{1}{a}\frac{d\vec{L}}{d\sigma}) = \frac{b}{a^2}\frac{d^2\vec{L}}{d\sigma^2}(\vec{L} \times \frac{d\vec{L}}{d\sigma}) \ ,$$

$$\hat{\Omega}_{31} = \frac{b}{a}(\vec{L} \times \frac{d^2\vec{L}}{d\sigma^2})(\vec{L} + \varepsilon \frac{1}{a}\frac{d\vec{L}}{d\sigma}) = -\varepsilon \frac{b}{a^2}\frac{d^2\vec{L}}{d\sigma^2}(\vec{L} \times \frac{d\vec{L}}{d\sigma}) \ ,$$

$$\hat{\Omega}_{32} = \frac{b}{a}(\vec{L} \times \frac{d^2\vec{L}}{d\sigma^2})(\frac{1}{a}\frac{d\vec{L}}{d\sigma} - \varepsilon\vec{L}) = -\frac{b}{a^2}\frac{d^2\vec{L}}{d\sigma^2}(\vec{L} \times \frac{d\vec{L}}{d\sigma}) \ ,$$

$$\hat{\Omega}_{33} = \frac{b}{a}(\vec{L} \times \frac{d^2\vec{L}}{d\sigma^2})(\vec{L} \times \frac{1}{a}\frac{d\vec{L}}{d\sigma}) = 0 \quad .$$

Hence

$$\hat{\Omega} = \begin{pmatrix} 0 & ba & \varepsilon \frac{b}{a^2}\frac{d^2\vec{L}}{d\sigma^2}(\vec{L} \times \frac{d\vec{L}}{d\sigma}) \\ -ba & 0 & \frac{b}{a^2}\frac{d^2\vec{L}}{d\sigma^2}(\vec{L} \times \frac{d\vec{L}}{d\sigma}) \\ -\varepsilon \frac{b}{a^2}\frac{d^2\vec{L}}{d\sigma^2}(\vec{L} \times \frac{d\vec{L}}{d\sigma}) & -\frac{b}{a^2}\frac{d^2\vec{L}}{d\sigma^2}(\vec{L} \times \frac{d\vec{L}}{d\sigma}) & 0 \end{pmatrix}$$

and

$$\hat{w}_1 = -\frac{b}{a^2}\frac{d^2\vec{L}}{d\sigma^2}(\vec{L}\times\frac{d\vec{L}}{d\sigma}) \ ,$$

$$\hat{w}_2 = \varepsilon\frac{b}{a^2}\frac{d^2\vec{L}}{d\sigma^2}(\vec{L}\times\frac{d\vec{L}}{d\sigma}) \ ,$$

$$\hat{w}_3 = -ba \ .$$

Then $\hat{v} = \hat{w}\times\hat{x}$ implies

$$\vec{\hat{v}} = \{ x_2 ba + \varepsilon(x_2^{*} ba + x_3\frac{b}{a^2}\frac{d^2\vec{L}}{d\sigma^2}(\vec{L}\times\frac{d\vec{L}}{d\sigma})) ,$$
$$-x_1 ba + x_3\frac{b}{a^2}\frac{d^2\vec{L}}{d\sigma^2}(\vec{L}\times\frac{d\vec{L}}{d\sigma}) + \varepsilon(-x_1^{*} ba + x_3^{*}\frac{b}{a^2}\frac{d^2\vec{L}}{d\sigma^2}(\vec{L}\times\frac{d\vec{L}}{d\sigma})) ,$$
$$-x_2\frac{b}{a^2}\frac{d^2\vec{L}}{d\sigma^2}(\vec{L}\times\frac{d\vec{L}}{d\sigma}) + \varepsilon(x_2^{*}\frac{b}{a^2}\frac{d^2\vec{L}}{d\sigma^2}(\vec{L}\times\frac{d\vec{L}}{d\sigma}) - x_1\frac{b}{a^2}\frac{d^2\vec{L}}{d\sigma^2}(\vec{L}\times\frac{d\vec{L}}{d\sigma}))\}$$

where $\hat{x} = (x_1 + \varepsilon x_1^{*}, x_2 + \varepsilon x_2^{*}, x_3 + \varepsilon x_3^{*})$.

Let $\frac{d^2\vec{L}}{d\sigma^2}(\vec{L}\times\frac{d\vec{L}}{d\sigma}) = l_1$ then

$$\hat{v} = \{x_2 ba + \varepsilon(x_2^{*} ba + x_3\frac{b}{a^2}l_1) , -x_1 ba + x_3\frac{b}{a^2}l_1 + \varepsilon(-x_1^{*} ba + x_3^{*}\frac{b}{a^2}l_1),$$
$$-x_2\frac{b}{a^2}l_1 + \varepsilon(x_2^{*}\frac{b}{a^2}l_1 - x_1\frac{b}{a^2}l_1)\}$$

The acceleration can be computed by the formula

$$\vec{\hat{a}} = \frac{d\vec{\hat{v}}}{d\sigma_0}, \text{ that is, } \vec{\hat{a}} = \frac{d\hat{w}}{d\sigma_0}\times\hat{x} + \hat{w}\times\frac{d\hat{x}}{d\sigma_0} = \frac{d\hat{w}}{d\sigma_0}\times\hat{x} + \hat{w}\times\hat{v} = \frac{d\hat{w}}{d\sigma_0}\times\hat{x} + \hat{w}\times(\hat{w}\times\hat{x})$$

where $\frac{d\hat{w}}{d\sigma_0} = \frac{d\hat{w}}{d\sigma}\cdot\frac{d\sigma}{d\sigma_0} = \frac{d\hat{w}}{d\sigma}\cdot b$.

After some computations and letting $\frac{d^3\vec{L}}{d\sigma^3}(\vec{L}\times\frac{d\vec{L}}{d\sigma}) = l_2$, we obtain

$$\vec{\hat{a}} = \{x_3 \frac{b^2}{a} l_1 - x_1 b^2 a^2 + \varepsilon(-x_2 \frac{b^2}{a^4} l_1^{\,2} - x_1^{\,*} b^2 a^2 + x_3^{\,*} \frac{b^2}{a} l_1 + x_3 \frac{b^2}{a^2} l_2),$$

$$-x_2 (b^2 a^2 + \frac{b^2}{a^4} l_1^{\,2}) + x_3 \frac{b^2}{a^2} l_2 + \varepsilon(-x_3 \frac{b^2}{a} l_1 - x_1 \frac{b^2}{a^4} l_1^{\,2} - x_2^{\,*} (b^2 a^2 + \frac{b^2}{a^4} l_1^{\,2}) + x_3^{\,*} \frac{b^2}{a^2} l_2),$$

$$(x_1 \frac{b^2}{a} l_1 - x_3 \frac{b^2\,{}^{2}}{a^4 l_1} - x_2 \frac{b^2}{a^2} l_2) + \varepsilon((-x_2 + x_1^{\,*}) \frac{b^2}{a} l_1 - x_3^{\,*} \frac{b^2}{a^4} l_1^{\,2} - (x_1 + x_2^{\,*}) \frac{b^2}{a^2} l_2)\} \;.$$

4.1 The Acceleration Center : The poins that satisfy, $\vec{\hat{a}} = 0$.

The real part of the acceleration center gives the equations

$$x_3 \frac{b^2}{a} l_1 - x_1 b^2 a^2 = 0 \;,$$

$$-x_2 (b^2 a^2 + \frac{b^2}{a^4} l_1^{\,2}) + x_3 \frac{b^2}{a^2} l_2 = 0 \;,$$

$$(x_1 \frac{b^2}{a} l_1 - x_3 \frac{b^2\,{}^{2}}{a^4 l_1} - x_2 \frac{b^2}{a^2} l_2) = 0 \;,$$

and

$$x_1 = x_3 \frac{l_1}{a^3} \;, \quad x_2 = \pm x_3 \sqrt{\frac{(l_1 - l_1^{\,2})}{(a^6 + l_1^{\,2})}} \;, \quad x_3 = x_3 \;.$$

Since $\hat{x} = (x_1, x_2, x_3) + \varepsilon(x_1^{\,*}, x_2^{\,*}, x_3^{\,*})$ on the D.U.S. satisfies the condition that

$$x_1^{\,2} + x_2^{\,2} + x_3^{\,2} = 1,$$

we impose the condition $l_2 = \pm \dfrac{\sqrt{(l_1 - l_1^{\,2})(a^6 + l_1^{\,2})}}{a^2}$.

Simplifying the dual part of the acceleration center, we get

$$-\frac{b^2 l_2^{\,2}}{(a^6 + l_1^{\,2})} x_3^{\,*} = 0,$$

(where a, b, l_1, and l_2 are all different from zero) then $x_3^{\,*} = 0$,

$$x_1^* = -\frac{l_1^2}{a^6}x_2 + \frac{l_2}{a^4}x_3 ,$$

$$x_2^* = -\frac{a^3 l_1}{a^6 + l_1^2}x_3 - \frac{l_1^2}{a^6 + l_1^2}x_1 .$$

Using the values of x_1, x_2, x_3 from the real part, we obtain

$$x_1^* = \pm x_3\sqrt{\frac{(l_1 - l_1^2)}{(a^6 + l_1^2)}} , \quad x_2^* = -\frac{l_1}{a^3} , \quad x_3^* = 0 ,$$

where this result naturaly satisfies the condition;

$$x_1 x_1^* + x_2 x_2^* + x_3 x_3^* = 0 .$$

Setting $x_3 = u$, the acceleration center defines a ruled surface in R^3 such that ;

$$\vec{x}(u,\mu) = \vec{x} \times \vec{x}^* + \mu\vec{x}$$

$$= (u\frac{l_1}{a^3}, \pm u^2\sqrt{\frac{l_1 - l_1^2}{a^6 + l_1^2}}, -u\frac{l_1^2}{a^6} - u^2\frac{l_1 - l_1^2}{a^6 + l_1^2}) + \mu(u\frac{l_1}{a^3}, \pm u\sqrt{\frac{l_1 - l_1^2}{a^6 + l_1^2}}, u)$$

$$= \{(\mu + 1)u\frac{l_1}{a^3}, \pm(u + \mu)u\sqrt{\frac{l_1 - l_1^2}{a^6 + l_1^2}}, (\mu - \frac{l_1^2}{a^6})u - \frac{l_1 - l_1^2}{a^6 + l_1^2}u^2\},$$

$$-1 \le l_1 \le 1 .$$

4.2 The Bresse Congruence : The points that satisfy, $\vec{\tilde{a}} \cdot \vec{\tilde{v}} = 0$. (4.2.1)

The real part of (4.2.1) implies

$$-x_1 x_3\frac{b^3}{a}l_2 + x_3^2\frac{b^3}{a^4}l_1 l_2 + x_2^2\frac{b^3}{a^4}l_1 l_2 = 0 \qquad (4.2.2)$$

Since $x_1^2 + x_2^2 + x_3^2 = 1$, (4.2.2) yields

$$-x_1 x_3 \frac{b^3}{a} l_2 + (1-x^2)\frac{b^3}{a^4} l_1 l_2 = 0 ,$$

hence

$$\begin{aligned} x_1 &= \frac{-x_3 a^3 \pm \sqrt{x_3^2 a^6 + 4l_1^2}}{2l_1} , \\ x_2 &= \pm\sqrt{x_3^2(-2a^6-1)+1+4l_1^2 \pm 2x_3 a^3 \sqrt{x_3^2 a^6 + 4l_1^2}} , \\ x_3 &= x_3 . \end{aligned} \qquad (4.2.3)$$

After the cancellations of the dual part of (4.2.1), we get

$$-x_1 x_2 b^3 a^3 + x_2 x_3 \frac{b^3}{a} l_2 + x_1 x_2^* b^3 a^3 - x_1 x_3^* \frac{b^3}{a} l_2 - x_1^* x_3 \frac{b^3}{a} l_2 + 2x_3 x_3^* \frac{b^3}{a^4} l_1 l_2 +$$
$$2x_2 x_2^* \frac{b^3}{a^4} l_1 l_2 + 2x_1 x_2 \frac{b^3}{a^4} l_1 l_2 = 0.$$

Let $x_1 = h_1(x_3)$, $x_2 = h_2(x_3)$ at (4.2.3) and let $x_3^* = x_3^*$.

On the other hand , $x_1 x_1^* + x_2 x_2^* + x_3 x_3^* = 0$ implies , $x_1^* = -\dfrac{x_2 x_2^* + x_3 x_3^*}{x_1}$.

Then the dual part is ,

$$h_1(x_3)h_2(x_3)(2\frac{b^3}{a^4} l_1 l_2 - b^3 a^3) + h_2(x_3)x_3 \frac{b^3}{a} - h_1(x_3)x_3^* \frac{b^3}{a} l_2 + \frac{h_2(x_3)x_2^* + x_3 x_3^*}{h_1(x_3)} x_3 \frac{b^3}{a} l_2 +$$
$$h_1(x_3)x_2^* b^3 a^3 + 2\frac{b^3}{a^4} l_1 l_2 (h_2(x_3)x_2^* + x_3 x_3^*) = 0 ,$$

where $h_1(x_3) \neq 0$ (since $l_1 \neq 0$).

Let

$$\frac{h_2(x_3)x_3b^3l_2}{h_1(x_3)a}+h_1(x_3)b^3a^3+\frac{2b^3l_1l_2h_2(x_3)}{a^4}=d(x_3) \quad ,$$

$$h_1(x_3)h_2(x_3)(\frac{2b^3}{a^4}l_1l_2-b^3a^3)+h_2(x_3)x_3\frac{b^3}{a}-h_1(x_3)x_3^*\frac{b^3}{a}l_2+\frac{x_3{}^2x_3^*b^3l_2}{h_1(x_3)a}+$$
$$+\frac{2b^3l_1l_2x_3x_3^*}{a^4}=-c(x_3,x_3^*),$$

then

$$x_1^*=-\frac{h_2(x_3)c(x_3,x_3^*)+d(x_3)x_3x_3^*}{d(x_3)h_1(x_3)},$$

$$x_2^*=\frac{c(x_3,x_3^*)}{d(x_3)},$$

$$x_3^*=x_3^*$$

is the solution to the dual part.

Setting $x_3=u$ and $x_3^*=v$, the solution defines a line congruence;

$$\vec{x}(u,v,\mu)=\vec{x}\times\vec{x}^*+\mu\vec{x}$$

$$=(h_2(u)v-u\frac{c(u,v)}{d(u)},-u\frac{h_2(u)c(u,v)+d(u)uv}{d(u)h_1(u)}-h_1(u)v,h_1(u)\frac{c(u,v)}{d(u)}+h_2(u)\frac{h_2(u)c(u,v)+d(u)uv}{d(u)h_1(u)}$$
$$+\mu(h_1(u),h_2(u),u)$$

$$= \{h_2(u)v - u\frac{c(u,v)}{d(u)} + \mu h_1(u), -\frac{h_2(u)c(u,v)u + d(u)u^2 v}{d(u)h_1(u)} - h_1(u)v + \mu h_2(u),$$
$$h_1(u)\frac{c(u,v)}{d(u)} + \frac{h_2(u)}{d(u)h_1(u)}(h_2(u)c(u,v) + d(u)uv) + \mu u\},$$

4.3 The Inflection Surface : The points that satisfy the condition, $\vec{a} \times \vec{v} = 0$ (4.3.1)

The real part of (4.3.1) is equal to zero implies three equations,

$$x_1^2 b^3 l_1 + x_3^2 \frac{b^3}{a^6} l_1^3 - 2x_1 x_3 \frac{b^3}{a^3} l_1^2 = 0, \quad (4.3.2)$$

$$-x_2^2 \frac{b^3}{a} l_2 = 0, \quad (4.3.3)$$

$$-2x_1 x_3 b^3 l_1 + x_1^2 b^3 a^3 + x_3^2 \frac{b^3}{a^3} l_1^2 = 0, \quad (4.3.4)$$

from (4.3.3) (since $b \neq 0$ and $l_2 \neq 0$) $x_2 = 0$ and we get the following system,

$$x_2 = 0 \;, \quad (4.3.5)$$

$$x_1^2 + x_3^2 = 1 \;, \quad (4.3.6)$$

$$-2x_1 x_3 l_1 + x_1^2 a^3 + x_3^2 \frac{l_1^2}{a^3} = 0, \quad (4.3.7)$$

and from (4.3.6) $x_1 = \pm\sqrt{1 - x_3^2}$, which together with (4.3.7) imply

$$x_3^4 (a^{12} + l_1^4) - 2a^{12} x_3^2 + a^{12} = 0.$$

Let $x_3{}^2 = u$ then $u_{1,2} = \dfrac{2a^{12} \pm \sqrt{-4a^{12}l_1{}^4}}{2(a^{12} + l_1{}^4)}$ gives a complex root. This comlex root means that we don't have a real solution to the real part. On the other hand, since the solution to the dual part depends on the solution of the real part, we also don't have a real solution to the dual part. Consequently, we have no solution for inflection points (See also Köse et al. 2003 p.4).

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