DOKUZ EYLÜL UNIVERSITY

GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

MEAN RESIDUAL LIFE FUNCTION OF SYSTEMS

by

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September, 2006

İZMİR

MEAN RESIDUAL LIFE FUNCTION OF SYSTEMS

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by
Selma GÜRLER

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Ph.D. THESIS EXAMINATION RESULT FORM

We have read the thesis entitled **"MEAN RESIDUAL LIFE FUNCTION OF SYSTEMS"** completed by **SELMA GÜRLER** under supervision of **PROF. DR. İSMİHAN BAYRAMOĞLU** and we certify that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Doctor of Philosophy.

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THE MEAN RESIDUAL LIFE FUNCTION OF SYSTEMS

ABSTRACT

Most of the fault-tolerant systems such as parallel and k-out-of-n consist of components having independent and nonidentically distributed lifetimes. These types of system structures have founded wide applications in both industrial and technical areas during the past several decades. For the improvement of the reliability of the operation of such complex technical systems, the implementation of the structural redundancy is widely used.

In this thesis, we consider the mean residual life (MRL) function of a parallel and k-out-of-n systems consisting of n components having independent and nonidentically distributed lifetimes. Numerical results are introduced to study the effect of increasing the system level and various parameters on the mean residual life of the systems. Further, the relation between the mean residual life of the system and the mean residual life of its components is investigated.

Keywords : Mean residual life function, Parallel systems, k-out-of-n systems, Symmetric functions, Permanents

SİSTEMLERİN ORTALAMA GERİYE KALAN YAŞAM FONKSİYONU

ÖZ

Paralel ve *n*-taneden-*k*-tane sistemleri gibi birçok hata önleme sistemi bağımsız ve aynı olmayan yaşam zamanı dağılımına sahip bileşenlerden oluşur. Geçen birkaç on yıl süresince sistem yapılarının bu tipleri endüstriyel ve teknik alanlarda geniş uygulama alanı bulmuşlardır. Bu çeşit karmaşık teknik sistemlerin işlemlerinin güvenilirliğinin geliştirilmesi için yapısal yedeklemenin uygulanması yaygın olarak kullanılmaktadır.

Bu tezde *n* tane bağımsız ve aynı olmayan yaşam zamanı dağılımına sahip bileşenlerden oluşan paralel ve *n*-taneden-*k*-tane sistemlerinin ortalama geriye kalan yaşam fonksiyonu ele alınmıştır. Sistem düzeyinin ve çeşitli parametrelerin, sistemin ortalama geriye kalan yaşamı üzerindeki etkisini incelemek için sayısal sonuçlar verilmiştir. Ayrıca sistemlerin ortalama geriye kalan yaşamı ile bileşenlerinin ortalama geriye kalan yaşamı arasındaki ilişki incelenmiştir.

Anahtar sözcükler : Ortalama geriye kalan yaşam fonksiyonu, Paralel sistemler, *n*-taneden-*k*-tane sistemler, Simetrik fonksiyonlar, Permanents

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CHAPTER ONE
INTRODUCTION

Determination of mean residual life (MRL) function of a system is an important problem in statistical theory of reliability. Given that a unit is of age t, the remaining life after time t is random. The expected value of this random residual life is called the mean residual life at time t. The mean residual life function is a helpful tool in model building. It is also used to characterize some special statistical probability distributions.

The concept of mean residual life is based on conditional expectations and has been of much interest in the actuarial science, survival studies and reliability theory. Reliability engineers, statisticians, and others have shown intensified interest in the MRL and derived many useful results. In biomedical sciences, researchers analyze survivorship studies by MRL. Actuaries apply MRL to setting rates and benefits of life insurance. In economics, MRL is applied for investigating landholding. In industrial reliability studies of repair and replacement strategies, the mean residual life function may be more relevant than the hazard function.

In the last two decades, the mean residual life has gathered considerable interest and many useful results are derived. The MRL has been employed in life length studies by various authors. Bryson & Siddiqui (1969) use a decreasing MRL function as one of several possible criteria for aging and develop a chain of implications for the various criteria. Hollander & Proschan (1975) develop a test statistic for a decreasing MRL function. Hall & Wellner (1981) and Oakes & Dasu (1990) characterize the class of distributions with linear mean residual life. Tang, Lu & Chew (1999) characterize the general behaviors of the MRL for both continuous and discrete lifetime distributions, with respect to their failure rates. Nair & Nair (1989) have extended the concept of the bivariate case, and derived relationship between the reliability and mean residual life function.

In the next sections, we provide a literature review and some fundamental results about the mean residual life function of lifetime of a component.

1.1 The Mean Residual Life (MRL) Function of a Continuous Random Variable

Let *X* be a random variable representing the life length. Then, *X* is a nonnegative continuous random variable, and *F*(*x*) is the cumulative distribution function of *X*. The survival probability of a unit corresponding to a mission of duration *x* is

$$\overline{F}(x) = 1 - F(x). \tag{1.1}$$

and let $f(x) = F^{'}(x)$ be the density function.

The corresponding conditional survival function (or reliability) of $X - t$, the residual lifetime of a unit, at age *t* is given by,

$$\overline{F}(x|t) = \frac{\overline{F}(t+x)}{\overline{F}(t)}, \qquad \text{if } \overline{F}(t) > 0. \tag{1.2}$$

The random variable $X \geq 0$ is a continuous random variable with the reliability function $\overline{F}(x)$, and finite expectation μ. The mean residual life function ψ_F of a component, with life distribution function *F* pertaining to a life length *X*, is defined by the following conditional expectation of *X-t* given $X > t$:

$$\psi_F(t) = E(X - t | X > t) \tag{1.3}$$

This means that $\psi_F(t)$ is the expected remaining life given survival at age *t*. The MRL function in Equation (1.3) can also be expressed as in Equation (1.4).

$$\psi_F(t) = E(X | X > t) - t \tag{1.4}$$

This function is also interpreted as the conditional expectation of residual life length of *X*, given $X > t$. Let the random variable $Y = (X \mid X > t)$, and its probability density function be as below:

$$f_Y(x) = \begin{cases} \dfrac{f(x)}{\overline{F}(t)} & x \geq t \\ 0 & x < t \end{cases} \tag{1.5}$$

The following elementary equalities yield the MRL function. Equation (1.4) can be written as

$$\psi_F(t) = \frac{1}{\overline{F}(t)} \int_t^{\infty} x dF(x) - t \,. \tag{1.6}$$

The integral expression in (1.6) is computed in the following steps.

$$\int_t^{\infty} x dF(x) = -\int_t^{\infty} x d(1 - F(x))$$

$$= -x(1 - F(x))\Big|_t^{\infty} + \int_t^{\infty} (1 - F(x)) dx \,. \tag{1.7}$$

Since $\lim_{x \to \infty} x(1 - F(x))$ =0, then

$$\int_t^{\infty} x dF(x) = t\,\overline{F}(t) + \int_t^{\infty} \overline{F}(x) dx \,. \tag{1.8}$$

After all computations, the MRL function is obtained as below.

$$\psi_F(t) = \frac{1}{\overline{F}(t)} \left[t\overline{F}(t) + \int_t^{\infty} \overline{F}(x) dx \right] - t$$

$$\psi_F(t) = \frac{1}{\overline{F}(t)} \int_t^\infty \overline{F}(x)dx \tag{1.9}$$

Here ψ_F (.) is nonnegative and ψ_F (0)=$E(X)$, i.e. the MRL function at the time origin is equal to the ordinary expectation.

It is known that $\overline{F}(x)$ can be recovered from $\psi_F(t)$ by the inversion formula (Cox, 1962). Using Equation (1.9), the survival function can be obtained by the inversion formula. The process is followed by rewriting (1.9) as follows,

$$\psi_F(t)\overline{F}(t) = \int_t^\infty \overline{F}(x)dx \, . \tag{1.10}$$

Derivation of the Equation (1.10) according to *t* is

$$\psi_F^{'}(t)\overline{F}(t) + \psi_F(t)(\overline{F}(t))^{'} = -\overline{F}(t)$$

$$\psi^{'}{}_F(t) - \psi_F(t)\frac{f(t)}{\overline{F}(t)} = -\frac{\overline{F}(t)}{\overline{F}(t)}$$

$$\frac{f(t)}{\overline{F}(t)} = \frac{\psi_F^{'}(t)+1}{\psi_F(t)} \, . \tag{1.11}$$

Hence in order to obtain the survival function $\overline{F}(x)$, we integrate both sides of Equation (1.11) on $[0, x]$:

$$\int_0^x \frac{d}{dt}(\ln \overline{F}(t)) = -\int_0^x \frac{\psi_F^{'}(t)+1}{\psi_F(t)} dt$$

$$\ln \overline{F}(x) = -\int_0^x \frac{\psi_F^{'}(t)+1}{\psi_F(t)} dt$$

$$\overline{F}(x) = \exp\left\{-\int_0^x \frac{\psi_F'(t)}{\psi_F(t)}dt - \int_0^x \frac{1}{\psi_F(t)}dt\right\}$$

$$\overline{F}(x) = \exp\left\{-\int_0^x \frac{d}{dt}\ln\psi_F(t) - \int_0^x \frac{1}{\psi_F(t)}dt\right\}.$$

Hence the inversion formula is obtained as follows,

$$\overline{F}(x) = \frac{\psi_F(0)}{\psi_F(x)}\exp\left\{-\int_0^x \frac{1}{\psi_F(t)}dt\right\}. \tag{1.12}$$

1.2 Failure Rate (FR)

A basic quantity, fundamental in survival analysis is the hazard function. This function is also known as the conditional failure rate in reliability, the force of mortality in demography, the age-specific failure rate in epidemiology.

The failure rate, which is defined as the probability that a device will fail in the next time unit given that it has been working properly up to time *t*, is

$$r(t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} P(t \le T < t + \Delta t | T \ge t) \tag{1.13}$$

$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \frac{P(t \le T < t + \Delta t, T \ge t)}{P(T \ge t)}.$$

The conditional failure rate at time *t* is obtained as below,

$$r(t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \frac{F(t + \Delta t) - F(t)}{\overline{F}(t)}$$

$$= \frac{f(t)}{\overline{F}(t)} \tag{1.14}$$

when $f(t)$ exists and $\overline{F}(t)>0$ (Barlow & Proschan, 1975).

The failure rate is a non-negative function. It tells us how quickly individuals of a given age are experiencing the event of interest. This function is particularly useful in determining the appropriate failure distributions, and for describing the way in which the chance of experiencing the event changes with time. There are many general shapes for the failure rate. Some types of failure rates are increasing, decreasing, constant and bath-tube shaped. Most often a bath-tube shaped failure is appropriate in populations followed from birth (Høyland & Rausand, 1994).

Life function and failure rate function are useful identities for application. If $r(t)$ is known $\overline{F}(x)$ can be determined. This useful identity is obtained by integrating both sides of (1.14) which is given as below,

$$\int_0^x r(t)dt = -\int_0^x \frac{d}{dt}\ln \overline{F}(t) .$$

A related quantity is the cumulative failure rate function defined by,

$$\int_0^x r(t)dt = -\ln \overline{F}(x) .$$

For continuous lifetimes, the following relationship exists,

$$\overline{F}(x) = \exp\left[-\int_0^x r(t)dt \right]. \tag{1.15}$$

This implies that the failure rate function and the mean residual life function are characterizing the distribution function.

$$\overline{F}(x) = \exp\left[-\int_0^x r(t)dt\right] = \frac{\psi_F(0)}{\psi_F(x)}\exp\left\{-\int_0^x \frac{1}{\psi_F(t)}dt\right\}$$

Both failure rate and mean residual life function are conditioned on survival to time *t*. While the failure rate function at *t* provides information about a small interval after time *t*, the mean residual life function at *t* considers information about the whole interval after *t*. When both $\psi_F(t)$ and *r*(*t*) exist, a relationship

$$\psi_F'(t) = \psi_F(t)r(t) - 1$$

between two functions holds.

1.3 Aging Properties of the MRL Function

Modeling of the aging process of a component or a system can be performed in various ways. Some helpful tools commonly used for such modeling are the failure rate function and the mean residual life function, as well as the reliability function.

The set of all lifetime distribution functions has important connections by the notion of aging. Monotone aging models are very useful and important in reliability applications. For example, the Gamma and Weibull with a shape parameter greater than 1 is an IFR (increasing failure rate) model-adverse aging. The Gamma and Weibull with a shape parameter that is between 0 and 1 is a DFR (decreasing failure rate) model-beneficial aging. Another important subclass is the set of those, which have bathtub-shaped (or upside down bathtub-shaped) functions. Bathtub-shaped failure rate functions and their corresponding mean residual life functions are faced frequently in many practical situations. Such types of life distributions include IDFR (increasing decreasing FR), DIFR (decreasing increasing FR), DIMRL (decreasing increasing MRL) and IDMRL (increasing decreasing MRL), among others. Guess, Hollander & Proschan (1986) define the IDMRL and DIMRL classes and propose a testing procedure. The lognormal distributions, used for repair times as well as lifetimes, are in the IDFR class. A life distribution with upside down bathtub shaped mean residual life is in the IDMRL class. Human life length can be modeled well by this class.

Although many parametric models have monotone failure rate or mean residual life (Gamma and Weibull with a shape parameter greater than 1), there are life distributions that exhibit non-monotone properties of failure rate and mean residual life. Esary & Proschan (1963) discuss the system failure rate and component failure rate associations. Glaser (1980) discusses the relation between the density function and trend change in its failure rate. Mudholkar & Srivastava (1993) suggest an exponentiated-Weibull distribution which can be DIFR or IDFR depending on the parameter values. Lim & Park (1995) discuss the trend change in mean residual life. Ghai & Mi (1999) and Xie, Goh & Tang (2004) focus on the underlying associations between the mean residual life and failure rate function.

If $\psi_F(t)$ is nonincreasing in t, the life distribution F is said to have decreasing mean residual life (DMRL). The DMRL class models aging that is adverse. Barlow, Marshall & Proschan (1963), note that the DMRL class is a natural one in reliability theory and they have studied some properties of this class. The older a DMRL unit is, the shorter is the remaining life on the average. If $r(t)$ increases monotonically over time, the distribution is said to have increasing failure rate (IFR). For IFR class, the aging has an adverse effect on its failure rate. If $r(t)$ decreases monotonically, we have decreasing failure rate (DFR). For this class, the aging is beneficial to the system. The IFR property is characteristic of devices that consistently deteriorate with age, whereas the DFR property is characteristic of devices that consistently improve with age. A common description, which is appropriate for modeling human lifetimes, shows three phases: an initial phase during which the failure rate decreases, followed by a middle phase during which the failure rate is essentially constant, concluded by a final phase during which the failure rate increases. Such failure rates are usually termed bathtub shaped. More often, the life distributions exhibit such failure rates, and are more realistic models than the monotone failure rate models in many practical situations. The following classes of life distributions are defined which show a trend change in its failure rate.

Definition 1.1 A life distribution function F is said to have a DIFR if there exists a point t_0 such that $r'(t)<0$ for $t<t_0$, $r'(t_0)=0$, and $r'(t)>0$ for $t>t_0$.

Definition 1.2 A life distribution function F is said to have an IDFR if there exists a point t_0 such that $r'(t)>0$ for $t<t_0$, $r'(t_0)=0$, and $r'(t)<0$ for $t>t_0$.

If $t_0=0$, then DIFR and IDFR are equivalent to IFR and DFR, respectively. Examples of these cases are illustrated in Figure 1.1.

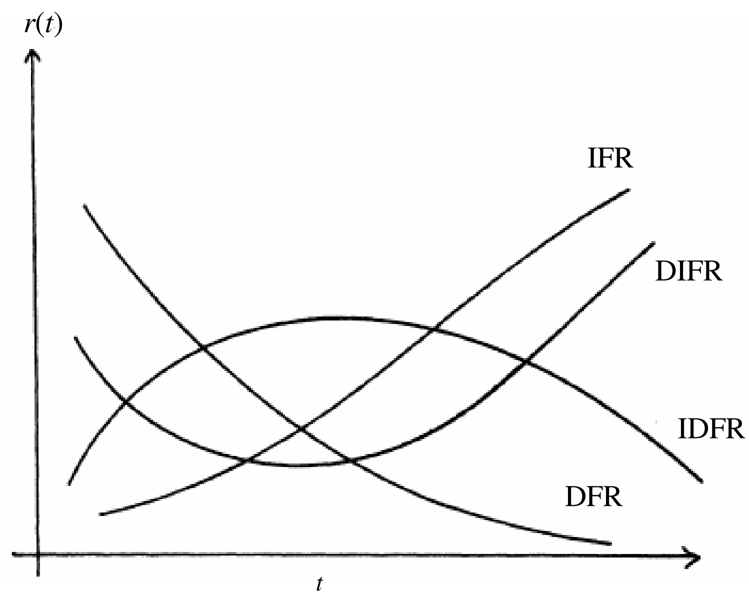


Figure1.1 Examples of the trend change in failure rate

There also exists relationship between FR and MRL as:

- If $r(t)$ is increasing, then $\psi(t)$ is decreasing
- If $r(t)$ is decreasing, then $\psi(t)$ is increasing
- If $r(t)$ is a constant function (i.e, F is an exponential distribution) if and only if $\psi(t)$ is a constant.

Definition 1.3 A life distribution F is said to have an IDMRL if there exists a point t_0 such that $\psi'(t)>0$ for $t<t_0$, $\psi'(t_0)=0$, and $\psi'(t)<0$ for $t>t_0$.

Definition 1.4 A life distribution F is said to have an DIMRL if there exists a point t_0 such that $\psi'(t)<0$ for $t<t_0$, $\psi'(t_0)=0$, and $\psi'(t)>0$ for $t>t_0$.

If $t_0=0$ IDMRL and DIMRL are equivalent to DMRL and IMRL, respectively.

In the next sections, several well known parametric families of life distributions in reliability applications are presented.

1.3.1 Exponential Distribution

The simplest and most important distribution in survival studies is the exponential distribution. In the late 1940's researchers chose the exponential distribution to describe the life pattern of electronic systems. It is famous for its unique "lack of memory" property which requires that the age of the animals or the individual does not affect future survival (Lee, 1992).

The exponential distribution is characterized by a constant hazard rate λ, the only parameter. The hazard rate is both IFR and DFR. A large λ indicates high risk and short survival while a small λ indicates low risk and long survival. When $\lambda=1$, the distribution is often referred to as the unit exponential distribution.

The distribution function $F(t)$, failure rate $r(t)$ and mean residual life $\psi_F(t)$ functions are respectively,

$$F(t)=1-e^{-\lambda t} \text{ for } t\geq 0,\ \lambda>0,$$

$$r(t)=\frac{f(t)}{\overline{F}(t)}=\lambda$$

$$\psi_F(t)=\frac{1}{\overline{F}(t)}\int_t^{\infty}\overline{F}(x)dx=\frac{1}{e^{-\lambda t}}\int_t^{\infty}e^{-\lambda x}dx=\frac{1}{\lambda}.$$

1.3.2 Weibull Distribution

The Weibull distribution is a generalization of the exponential distribution. However, unlike the exponential distribution, it does not assume a constant hazard rate and therefore has broader application. The distribution is characterized by two parameters, λ and α. The value of α determines the shape of the distribution curve and the value of λ determines its scaling. The distribution function is given by

$$F(t) = 1 - e^{-(\lambda t)^{\alpha}} \text{ for } t \geq 0 \text{ where } \lambda, \alpha > 0.$$

The failure rate and mean residual life functions are respectively,

$$r(t) = \frac{f(t)}{\overline{F}(t)} = \alpha\lambda(\lambda t)^{\alpha-1} \text{ for } t>0$$

$$\psi_F(t) = \frac{1}{\overline{F}(t)}\int_t^{\infty}\overline{F}(x)dx = \frac{1}{e^{-(\lambda t)^{\alpha}}}\int_t^{\infty} e^{-(\lambda x)^{\alpha}}dx.$$

The Weibull distribution F_α is IFR and DMRL for $\alpha \geq 1$ and DFR and IMRL for $0 < \alpha \leq 1$; for $\alpha = 1$, $F_\alpha(t) = 1 - e^{-\lambda t}$, the exponential distribution which is both DFR and IFR as t increases. The failure rate and mean residual life functions of the Weibull are plotted in Figure 1.2 and Figure 1.3. The parameter α is called the shape parameter; as α increases the failure rate function rises more steeply and the probability density becomes more peaked.

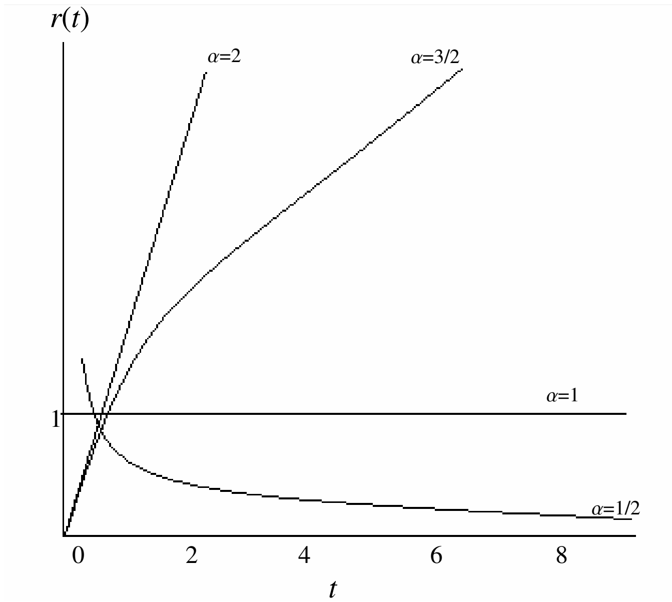


Figure 1.2 Failure rate curves of the Weibull distribution for $\lambda=1$

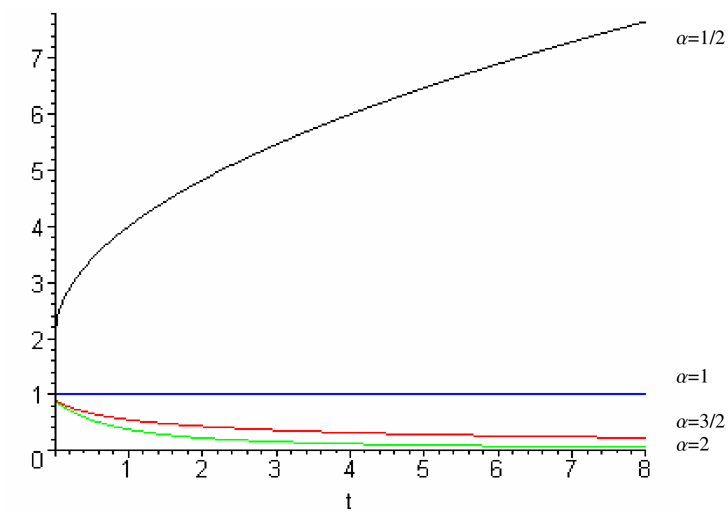


Figure 1.3 MRL curves of the Weibull distribution for $\lambda=1$.

1.3.3 Gamma Distribution

The gamma distribution, which includes the exponential and chi-square distribution, has been used by Brown & Flood, in 1947, to describe the life of glass tumblers circulating in a cafeteria. Since then, this distribution has been used as a model for industrial reliability problems. The gamma distribution with integer parameter α is the distribution of the sum of n independent exponential random variables, each with failure rate λ.

$$f_{\lambda,\alpha}(t) = \frac{\lambda}{\Gamma(\alpha)}(\lambda t)^{\alpha-1} e^{-\lambda t} \text{ for } t \geq 0 \text{ where } \lambda, \alpha > 0.$$

The failure rate and mean residual life functions for the gamma distribution are,

$$r(t) = \frac{\frac{\lambda}{\Gamma(\alpha)}(\lambda t)^{\alpha-1} e^{-\lambda t}}{1 - \int_0^t \frac{\lambda}{\Gamma(\alpha)}(\lambda x)^{\alpha-1} e^{-\lambda x} dx}, \quad t \geq 0,$$

$$\psi_F(t) = \frac{1}{1 - \int_0^t \frac{\lambda}{\Gamma(\alpha)}(\lambda x)^{\alpha-1} e^{-\lambda x} dx} \int_t^\infty \left(1 - \int_0^x \frac{\lambda}{\Gamma(\alpha)}(\lambda y)^{\alpha-1} e^{-\lambda y} dy \right) dx.$$

The failure rate and mean residual life functions of the gamma are plotted in Figure 1.4 and Figure 1.5. When $0<\alpha\leq 1$ the failure rate decreases monotonically from infinity to λ as time increases from zero to infinity. So there are negative aging and IMRL. When $\alpha\geq 1$ the failure rate increases monotonically from zero to λ as time increases. There is positive aging and DMRL. For $\alpha=1$, $F_\alpha(t)$, the exponential distribution which is both DFR and IFR. Thus, the gamma distribution describes a different type of survival pattern where the hazard rate is decreasing or increasing to a constant value as time approaches infinity.

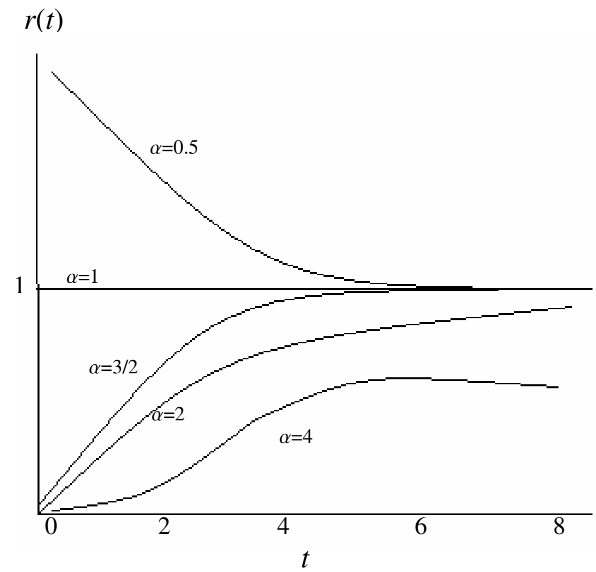


Figure 1.4 Failure rate curves of the gamma distribution for $\lambda=1$

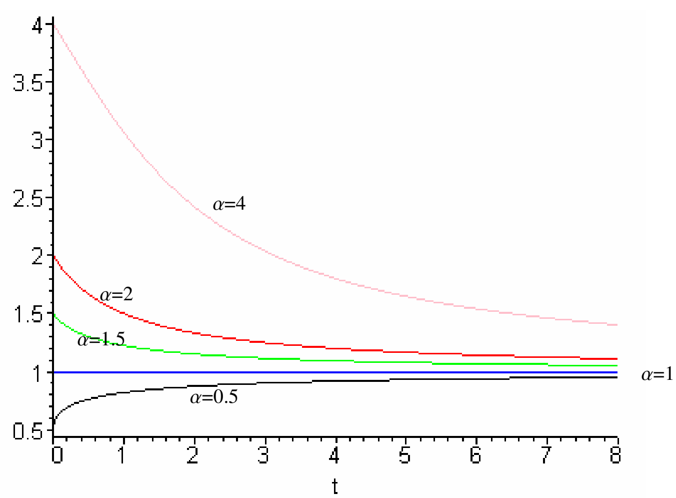


Figure 1.5 MRL curves of the gamma distribution for $\lambda=1$

Table 1.1 summarizes the aging properties of MRL and failure rate for various distributions.

Table 1.1 MRL for some distributions

Distribution	FR				MRL			
	IDFR	IFR	DFR	CFR	IMRL	DMRL	CMRL	DIMRL
Exponential				yes			yes	
Weibull		yes $\alpha > 1$	yes $\alpha < 1$	yes $\alpha = 1$	yes $\alpha < 1$	yes $\alpha > 1$	yes $\alpha = 1$	
Gamma		yes $\alpha > 1$	yes $\alpha < 1$	yes $\alpha = 1$	yes $\alpha < 1$	yes $\alpha > 1$	yes $\alpha = 1$	

1.4 System Structures of Independent Components

Applicable system configurations include combinations of series, parallel and *k*-out-of-*n*. The individual components are assumed to fail independently of one another and the lifetimes of the components are continuous.

1.4.1 The Series System

Assume that system *A* has a series structure; that is the system functions if and only if each component functions. A series structure is shown in Figure 1.6.

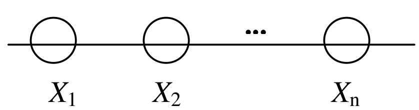


Figure1.6 Series structure

To indicate the state of the *i*th component, it is assigned a binary indicator variable x_i to component *i*,

$$x_i = \begin{cases} 1 & \text{if component } i \text{ is functioning,} \\ 0 & \text{if component } i \text{ is failed,} \end{cases}$$

for i=1,…,n, where n is the number of components in the system. Similarly, the binary variable ϕ indicates the state of the system (Barlow & Proschan, 1975):

$$\phi = \begin{cases} 1 & \text{if system is functioning,} \\ 0 & \text{if system is failed.} \end{cases}$$

Since the state of the system is determined completely by the states of the components, the structure function of the system is written

$$\phi = \phi(\mathbf{x}), \text{ where } \mathbf{x} = (x_1, \dots, x_n).$$

The series structure function is given by

$$\phi(\mathbf{x}) = \prod_{i=1}^{n} x_i = \min(x_1 \dots x_n).$$

The survival probability of such a system A corresponding to a mission of duration x is

$$S(x) = P(X_{1:n} > x) = \overline{F}^{n}(x).$$

The corresponding conditional reliability of system having non-failure element at time t is

$$S(x|t) = P(X_{1:n} > t + x | X_{1:n} > t) = \left[\frac{\overline{F}(t+x)}{\overline{F}(t)}\right]^{n}, \text{ if } \overline{F}(t) > 0. \tag{1.16}$$

The mean residual life function of a system A with series structure is defined by the conditional expectation of residual life length

$$\psi_n(t) = E(X_{1:n} - t | X_{1:n} > t),$$

given $X_{1:n} > t$ (all components of A functioning at time t). Now the survival function is,

$$S(x) = \overline{F}^{\,n}(x) = \frac{\psi_n(0)}{\psi_n(x)} \exp\left\{ -\int_0^x \frac{1}{\psi_n(t)} dt \right\}$$

and

$$\overline{F}(x) = \left[\frac{\psi_n(0)}{\psi_n(x)} \exp\left\{ -\int_0^x \frac{1}{\psi_n(t)} dt \right\} \right]^{1/n}, \qquad (1.17)$$

that is, $\psi_n(t)$ defines F for some n.

1.4.2 The Parallel System

Assume that the system A has a parallel structure; that is, the system goes out of service when all of its components fails. A parallel structure is given in Figure 1.7.

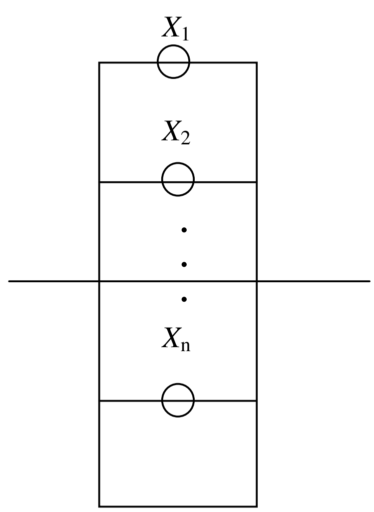


Figure 1.7 Parallel structure

The structure function is given by

$$\phi(\mathbf{x}) = \coprod_{i=1}^{n} x_i = \max(x_1 \ldots x_n),$$

where

$$\coprod_{i=1}^{n} x_i \equiv 1-\prod_{i=1}^{n}(1-x_i),$$

$$x_1 \vee x_2 = 1-(1-x_1)(1-x_2)\ .$$

Note that $\coprod$ and $\vee$ bear the same relation to each other as $\sum$ and + (Barlow & Proschan, 1975).

The survival probability of system corresponding to a mission of duration x is

$$S(x) = P(X_{n:n} > x) = 1 - F^n(x)$$

The conditional probability of system's failing in the interval (t, t+x], with no failing components at time t is

$$S(x|t) = P(X_{n:n} \le t + x | X_{1:n} > t)$$

$$S(x|t) = \left[1-\frac{\overline{F}(t+x)}{\overline{F}(t)}\right]^n, \text{ if } \overline{F}(t)>0.$$

The conditional expectation of residual life length of the system A having parallel structure

$$\psi_n(t) = E(X_{n:n} - t | X_{1:n} > t)$$

given $X_{1:n}>t$ is called the mean residual life function of parallel system (Bairamov, Ahsanullah & Akhundov, 2002).

1.4.3 The k-out-of-n System

A k-out-of-n structure functions if and only if at least k of the n components function. The structure function is given by

$$\phi(x_1, x_2, \ldots, x_n) = \begin{cases} 1 \quad \text{if } \sum_{i=1}^{n} x_i \geq k\ , \\ \\ 0 \quad \text{if } \sum_{i=1}^{n} x_i < k\ , \end{cases}$$

or equivalently,

$$\phi(x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} x_i \quad \text{for } k{=}n,$$

while

$$\phi(x_1, x_2, \ldots, x_n) = (x_1 \ldots x_k) \vee (x_1 \ldots x_{k-1} x_{k+1}) \vee \ldots \vee (x_{n-k+1} \ldots x_n)$$

$$\equiv \max\{(x_1 \ldots x_k), (x_1 \ldots x_{k-1} x_{k+1}), \ldots, (x_{n-k+1} \ldots x_n)\}$$

for $1 \leq k \leq n$, where every choice of k out of the n x's appears once exactly. It is clear that a series structure is an n-out-of-n structure and a parallel structure is 1-out-of-n structure.

1.5 Thesis Outline

This thesis consists of six chapters that investigate methodologies of system mean residual life function, important properties and modeling some well known distribution functions. We first present a general formulation of the problem. Using this framework, a review of relevant papers available in the literature is presented, followed by a more detailed problem statement. The remainder of the thesis presents the methodology and results. Chapter 2 presents mean residual life theory existing in the literature of parallel system and k-out-of-n system consisting of n identical and independent components and their application in some lifetime distribution functions with their aging modeling. In Chapter 3 and 4, we present the mean residual life function of parallel and k-out-of-n systems consisting of n components having independent and nonidentically distributed lifetimes and we establish new representations of the MRL function for such systems. We give some examples related to some lifetime distribution functions. Chapter 5 presents some real problem examples and numerical results for evaluating mean residual life of the k-out-of-n system with different system level. Finally Chapter 6 gives conclusions of this thesis and describes further research issues.

The most important results interesting to determining the mean residual life function of parallel and k-out-of-n systems, consisting of n components having independent and identically distributed lifetimes, are studied by Bairamov & et al. (2002) and Asadi & Bairamov (2005), (2006). The contribution of this thesis is the new representation of mean residual life function for parallel system consisting of n components with independent lifetimes having distribution functions (F_i), i=1, 2,…, n, respectively. Parallel system of n nonidentical components with exponential and power distributed lifetimes is considered and its mean residual life curves under the different conditions are examined. A recurrence relation which expresses the mean residual life function of n components in terms of mean residual life function of $n-1$ components is investigated. Another contribution of this study is that the mean residual life function of k-out-of-n system consisting of n components having independent and nonidentically distributed lifetimes is derived. Finally, the Weibull parametric model is examined to show how one can utilize the derived results to calculate the mean residual life for practical problems. And the relation between the mean residual life of the system and the mean residual life of its components is investigated.

CHAPTER TWO
THE MEAN RESIDUAL LIFE FUNCTION OF SYSTEMS

The reliability of a system or component is the probability that an item will perform satisfactorily, for a given period, under specified conditions. It is seen that reliability is the probability of survival and that reliability can be expressed mathematically throughout the entire life of a component. Reliability testing falls into two main types: firstly that of one-shot devices, or cases where success is defined in qualitative terms, for example a fuse or a parachute cases where the device either functions successfully when required to or else it does not; secondly the quantitative parameter case where some continuous variable, such as time to failure is being measured, and reliability is defined in terms of this variable. But reliability is not confined to single component.

A technical system will normally compromise a number of components that are interconnected in such a way that the system is able to perform a set of required functions. Determination of mean residual life function of a system is an important problem of statistical theory of reliability. To calculate system mean residual life, we must have a knowledge about the life distribution functions of those components which can cause the system to fail.

The mean residual life function ψ_F of a component, with life distribution function F pertaining to a life length X, is defined by the following conditional expectation of X-t given $X > t$:

$$\psi_F(t) = E(X - t | X > t) . \qquad (2.1)$$

It is assumed that a system have n components. Let X_i, i=1, 2,…, n be the survival time of ith component, such that $X_1, X_2, \ldots, X_n$ are independent and identically distributed random variables with continuous distribution function F. Let also $X_{1:n} \le X_{2:n} \le \cdots \le X_{n:n}$ be the ordered lifetimes of the components

This chapter is devoted to presenting necessary definitions and preliminary results on the mean residual life function of parallel system and k-out-of-n system. We present the MRL function of systems consisting of n identical and independent components with life lengths being distributed as well known distributions.

2.1 The Mean Residual Life Function of a Simple Parallel System

If high reliability is required for a system, the components must be designed in a parallel structure. A parallel system functions, if and only if at least one component functions. Lifetime of the last element of a parallel system, that is the component which have largest lifetime, is represented as $X_{n:n}$. Assume that at time t, t>0, the residual lifetime of a parallel system consisting of n identical and independent components is $X_{n:n} - t \mid X_{n:n} > t$. If S denotes the survival function of this conditional random variable then, it can be shown that, for x>0:

$$S(x|t) = P(X_{n:n} > x + t \mid X_{n:n} > t)$$

$$= \frac{P(X_{n:n} > x + t, X_{n:n} > t)}{P(X_{n:n} > t)}$$

$$= \frac{1 - P(X_{n:n} < x + t)}{1 - P(X_{n:n} < t)}$$

$$= \frac{1}{1 - F^{n}(t)} \left[1 - F^{n}(x + t)\right]. \tag{2.2}$$

Definition 2.1 The mean residual life function of a system having parallel structure given $X_{n:n} > t$ (last element of the system functions at time t) is

$$\psi_{n}(t) = E(X_{n:n} - t \mid X_{n:n} > t). \tag{2.3}$$

Using the survival function in (2.2) the mean residual life function of the system defined in (2.3) is given by

$$\psi_n(t) = \int_0^\infty S(x|t)dx = \frac{1}{1-F^n(t)}\int_0^\infty \left[1-F^n(x+t)\right]dx$$

$$= \frac{1}{1-F^n(t)}\int_t^\infty \left[1-F^n(x)\right]dx \,. \qquad (2.4)$$

Example 2.1 Let $F(x)$ be the exponential distribution function;

$$F(x) = 1-\exp(-\lambda x), \quad x \geq 0, \ \lambda > 0.$$

Figure 2.1 is plotted to examine the changes of the MRL, several choices of number of components n (2, 4) and λ (0.5, 1, 2). MRL function of the system is nonincreasing function of t. When λ's increase then the MRL of the system decreases. As the number of components increase then the MRL increases as expected.

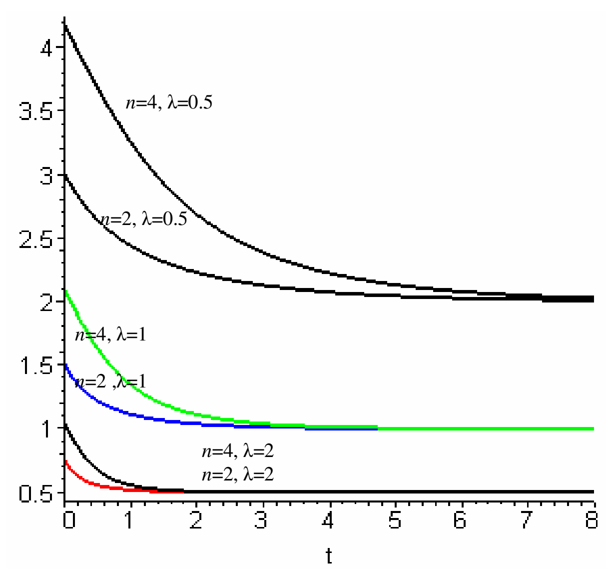


Figure 2.1 MRL of a parallel system consisting of n (2,4) components having exponential distributed lifetimes.

Example 2.2 Let the lifetimes of the components be the Weibull distributed random variables. The distribution function of a component is

$$F(x) = 1 - e^{-(\lambda x)^{\alpha}} \text{ for } x \geq 0 \text{ where } \lambda,\ \alpha > 0.$$

In Figure 2.2, MRL of the system consisting of n (2, 4) components are presented. The lifetimes of the components are distributed as Weibull distribution with α (0.5, 1, 2). It is assumed that the scale parameter λ is 1. The system has nondecreasing MRL for $0<\alpha<1$ and nonincreasing MRL for $\alpha \geq 1$.

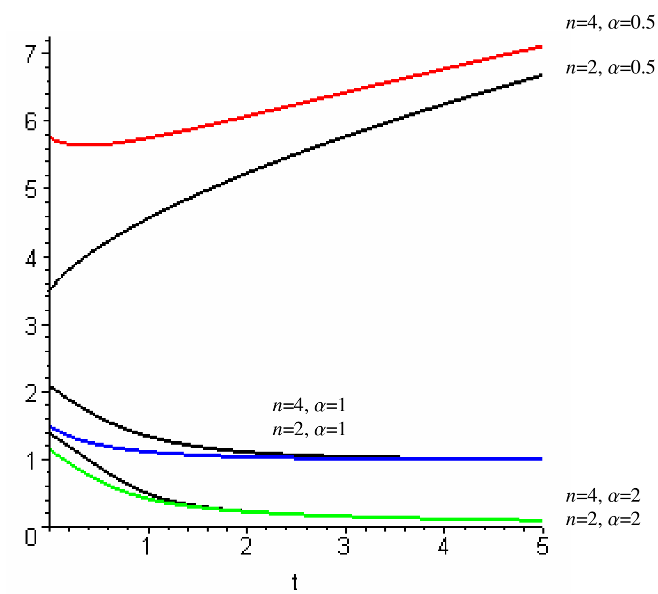


Figure 2.2 MRL of a parallel system consisting of n (2,4) components having Weibull distributed lifetimes (λ=1).

Example 2.3 Let the lifetimes of the components be the Gamma distributed random variables. The probability distribution function is

$$f_{\lambda,\alpha}(x) = \frac{\lambda}{\Gamma(\alpha)}(\lambda x)^{\alpha-1} e^{-\lambda x} \text{ for } x \geq 0 \text{ where } \lambda, \alpha > 0.$$

Figure 2.3 is plotted to present the changes of the MRL, for several choices of number of components n (2, 4) and α (0.5, 1, 2) assuming the scale parameter λ is 1.

For n=2 and 0<α<1, the MRL function is nondecreasing function of t. For 0<α<1 and larger n, the mean residual life function is nonincreasing. It is seen that when $\alpha \geq 1$ there is positive aging and the MRL function decreases monotonically as time increases. It is clear that the increasing of number of components n is a negative aging factor on MRL of system.

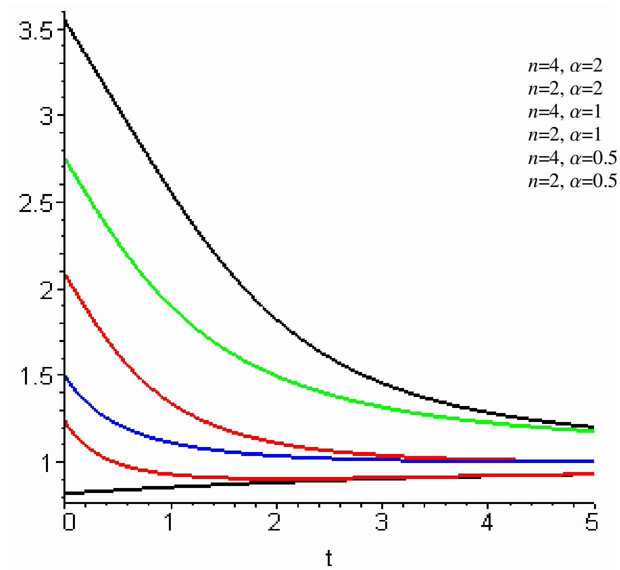


Figure 2.3 MRL of a parallel system consisting of n (2,4) components having Gamma distributed lifetimes (λ=1).

Parameters written on the top of the figure represent each line respectively.

2.2 The Mean Residual Life Function of the Parallel System with Components All Alive at Time *t*

Let $X_1, X_2, \ldots, X_n$ denote the lifetimes of n components connected in a system with parallel structure. It is assumed that X_i's are continuous, independent and identically distributed random variables with common distribution function F and survival function $\bar{F} = 1 - F$. Let also $X_{i:n}$ i=1, 2,…, n, be the ith smallest among $X_1, X_2, \ldots, X_n$, so that the lifetimes of the components are ordered, i.e.,

$X_{1:n} \le X_{2:n} \le \cdots \le X_{n:n}$. If we denote the survival function of the system at time t by $S(t)$, we have

$$S(t) = P(X_{n:n} > t)$$
$$= 1 - F^n(t), \quad t > 0. \tag{2.5}$$

The conditional probability of survival of system defined as parallel structure having no failing component at time t is

$$S(x|t) = P(X_{n:n} > t + x | X_{1:n} > t). \tag{2.6}$$

The conditional probability of system's failing in the interval $(t, t+x]$, with no failing component at time t is

$$F(x|t) = P(X_{n:n} \le t + x | X_{1:n} > t)$$

$$= \frac{1}{\overline{F}^n(t)} P(X_1 < t + x, X_2 < t + x, \ldots, X_n < t + x, X_1 > t, \ldots, X_n > t)$$

$$= \left[\frac{\overline{F}(t) - \overline{F}(t+x)}{\overline{F}(t)} \right]^n$$

$$= \left[1 - \frac{\overline{F}(t+x)}{\overline{F}(t)} \right]^n, \quad \text{if} \quad \overline{F}(t) > 0. \tag{2.7}$$

From this point Bairamov & et al. (2002) have given a proposition for exponential distribution.

Proposition 2.1 Let $F(x)$ be the exponential distribution function;

$$F(x) = 1 - \exp(-\lambda x), \quad x \ge 0, \; \lambda > 0.$$

Using the lack of memory property $\overline{F}(t+x)=\overline{F}(t)\overline{F}(x)$ it can be obtain from (2.7)

$$F(x|t) = F^n(x) = P(X_{n:n} \le x),$$

and the conditional survival function,

$$S(x|t) = 1 - F^n(x) = P(X_{n:n} > x). \tag{2.8}$$

As it is shown in Equation (2.8) for the exponential distribution, it is true that

$$S(x|t) = P(X_{n:n} > t + x | X_{1:n} > t)$$

$$= P(X_{n:n} > x) = S(x).$$

It is clear that the exponential distribution is the only one satisfying (2.8).

Then from Equation (2.7) it can be observed that F satisfying Equation (2.8) must be exponential distribution.

Definition 2.2 The conditional expectation of residual life length of a system having parallel structure

$$\phi_n(t) = E(X_{n:n} - t | X_{1:n} > t) \tag{2.9}$$

given $X_{1:n}$>t (all elements of system function at time t) is called the mean residual life function of parallel system (Bairamov & et al., 2002).

The distribution function of the random variable $Y = \left(X_{n:n} | X_{1:n} > t\right)$,

$$P(X_{n:n} \le x \;|X_{1:n} > t) = \frac{P(X_1 \le x, \ldots X_n \le x, X_1 > t, \ldots, X_n > t)}{P(X_1 > t, \ldots, X_n > t)}$$

$$= \frac{P(t < X_1 < x, \ldots, t < X_n < x)}{P(X_1 > t) \ldots P(X_n > t)}$$

$$= \frac{\left[F(x) - F(t)\right]^n}{\overline{F}^n(t)} . \tag{2.10}$$

Differentiating (2.10) with respect to *x* we obtain

$$\frac{d}{dx} P(X_{n:n} \le x | X_{1:n} > t) = \frac{n}{\overline{F}^n(t)} f(x) \left[F(x) - F(t)\right]^{n-1} . \tag{2.11}$$

Using the result in (2.11), the mean residual life function of the parallel system is obtained as given below.

$$\phi_n(t) = E(X_{n:n} - t | X_{1:n} > t) = E(X_{n:n} | X_{1:n} > t) - t$$

$$= \frac{n}{\overline{F}^n(t)} \int_t^\infty x\left(F(x) - F(t)\right)^{n-1} f(x)dx \text{ -}t \tag{2.12}$$

Example 2.4 Let $F(x)$ be the exponential distribution function;

$$F(x) = 1 - \exp(-\lambda x), \qquad x \ge 0, \;\; \lambda > 0.$$

From Equation (2.12), it can be obtained that the mean residual life function of a parallel system consisting of three identical and independent components with exponential distribution function. The mean residual life function of the system for exponentially distributed lifetimes is

$$\phi_3(t) = \frac{11}{6\lambda} \quad \lambda > 0.$$

Figure 2.4 is plotted to examine the changes of the MRL for several choices of number of components n (2, 3) and λ (0.5, 1, 2). It is clear that the MRL function of the system consisting of n components does not depend on t. The MRL of the system decreases when λ's increase and the number of components decrease. When the number of components increases, the MRL increases as it is expected.

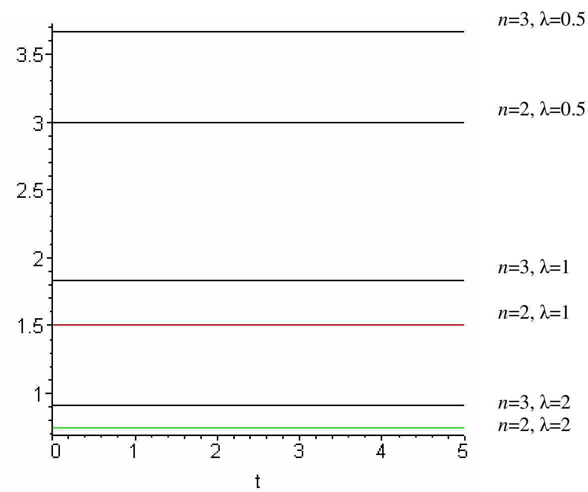


Figure 2.4 MRL of parallel system consisting of n(2,3) components having exponential distributed lifetimes.

Example 2.5 Let the lifetimes of the components be the Weibull distributed random variables. The distribution function of a component is

$$F(x) = 1 - e^{-(\lambda x)^{\alpha}} \text{ for } x \geq 0 \text{ where } \lambda, \alpha > 0.$$

In Figure 2.5, the MRL function of the system is plotted for several choices of number of components n (2, 3) and α (0.5, 1, 2) assuming the scale parameter λ is 1. The system has increasing MRL for $0<\alpha<1$ and decreasing MRL for $\alpha>1$. For $\alpha=1$ the system has constant MRL function, which is given as in Example 2.4.

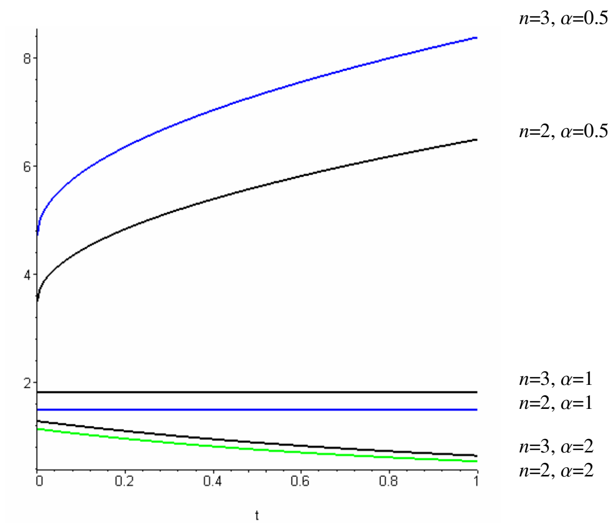


Figure 2.5 MRL of a parallel system consisting of n (2,3) components having Weibull distributed lifetimes (λ=1).

Example 2.6 Let the lifetimes of the components be the Gamma distributed random variables. The probability distribution function is

$$f_{\lambda,\alpha}(x) = \frac{\lambda}{\Gamma(\alpha)}(\lambda x)^{\alpha-1}e^{-\lambda x} \text{ for } x \geq 0 \text{ where } \lambda, \alpha > 0 .$$

Figure 2.6 is plotted to examine the changes of the MRL, several choices of number of components n (2, 3) and α (0.5, 1, 2) assuming the scale parameter λ is 1. When $0<\alpha<1$ there is negative aging and the MRL function increases monotonically as time increases. When $\alpha>1$ there is positive aging and the MRL function decreases monotonically as time increases. It is clear that the increasing of number of components n is a negative aging factor on MRL of system.

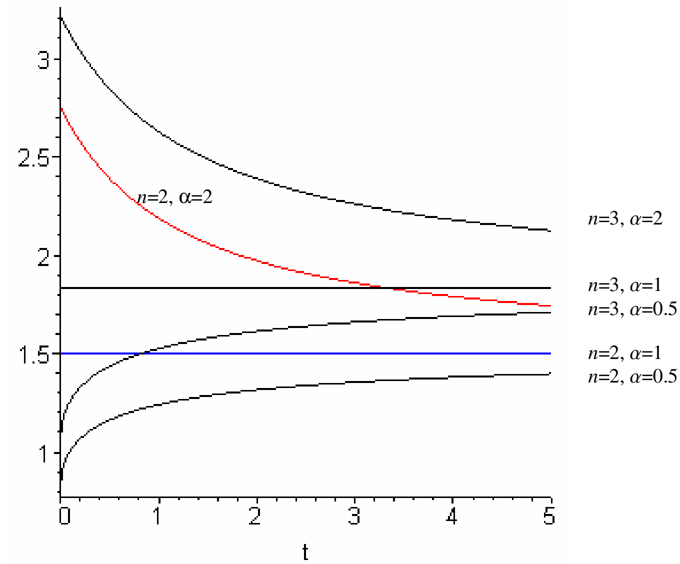


Figure 2.6 MRL of a parallel system consisting of n (2,3) components having Gamma distributed lifetimes (λ=1).

Bairamov & et al. (2002) obtained that the survival function of a component in terms of the mean residual life function of the system which is defined in (2.12).

Theorem 2.1 Let $\phi_n(t)$ be the mean residual function of a system having a parallel structure and consisting of n identical and mutually independent components with continuous life distribution function F. Then the following identity holds

$$\overline{F}(x) = \exp\left\{-\frac{1}{n}\int_0^x \frac{\phi_n^{'}(t)+1}{\phi_n(t)-\phi_{n-1}(t)}dt\right\}, \tag{2.13}$$

where $\phi_{n-1}(t)$ is the mean residual life function of similar system having (n-1) components.

Proof. From (2.12), the MRL function for the system having (n-1) components we have

$$\phi_{n-1}(t) = \frac{n-1}{(\overline{F}(t))^{n-1}} \int_t^\infty x\big(F(x) - F(t)\big)^{n-2} f(x)dx - t \,. \tag{2.14}$$

Also from (2.12) one can write

$$\left[\phi_n(t) + t\right] \overline{F}^{\,n}(t) = n \int_t^\infty x\big(F(x) - F(t)\big)^{n-1} f(x)dx \,. \tag{2.15}$$

Differentiating Equation (2.15) with respect to t it is obtained

$$\left[\phi_n^{'}(t) + 1\right] \overline{F}^{\,n}(t) - n\left[\phi_n(t) + t\right] \overline{F}^{\,n-1}(t) f(t)$$

$$= -nf(t)(n-1)\int_t^\infty x\big(F(x) - F(t)\big)^{n-2} f(x)dx \text{ - } n\, xf(x) \big(F(x) \text{ - } F(t)\big)^{n-1}\Big|_{x=t} \,.$$

Using the Equation (2.14), the Equation (2.15) may be rewritten as follows

$$\left[\phi_n(t) + 1\right] \overline{F}^{\,n}(t) - n\left[\phi_n(t) + t\right] \overline{F}^{\,n-1}(t) f(t) = -nf(t)\left[\phi_{n-1}(t) + t\right] \overline{F}^{\,n-1}(t).$$

After some derivation, one can obtain

$$\frac{f(t)}{\overline{F}(t)} = \frac{1}{n} \frac{\phi_n^{'}(t) + 1}{\phi_n(t) - \phi_{n-1}(t)} \tag{2.16}$$

and

$$\frac{d}{dt}(\ln \overline{F}(t)) = -\frac{1}{n} \frac{\phi_n^{'}(t) + 1}{\phi_n(t) - \phi_{n-1}(t)}.$$

Integrating (2.16) over $[0, x]$ and using $\overline{F}(0) = 1,$ it is obtained

$$\overline{F}(x) = \exp\left\{-\frac{1}{n}\int_0^x \frac{\phi_n^{'}(t)+1}{\phi_n(t)-\phi_{n-1}(t)}\,dt\right\}$$

and this completes the proof.

Asadi & Bayramoglu (2005) extended the definition of the MRL function proposed by Bairamov & et al. (2002) and explored its properties. They defined the MRL function of a system, under the condition that $X_{r:n}>t$, i.e., $(n-r+1)$, $r=1,2,\ldots,n$, components of the system are still working. Also they have showed that when the components of the system have a common increasing failure rate distribution then $M_n^r(t)$ is decreasing in t.

Definition 2.3 The MRL function of a parallel system, under the condition that $X_{r:n}>t$, i.e., $(n-r+1)$ components of the system are still working.

$$M_n^r(t) = E(X_{n:n} - t \mid X_{r:n} > t)\,,\quad r = 1,2,\ldots,n \tag{2.17}$$

It is assumed that the lifetime of the components of the system is independent and identically distributed with common distribution function F. A representation formula for $M_n^r(t)$ is given in the following theorem.

Theorem 2.2 If $M_n^r(t)$ is the MRL of the parallel system defined as (2.17), then for $\overline{F}(t)>0$

$$M_n^r(t) = \frac{\sum_{i=0}^{r-1}\binom{n}{i}\Phi^i(t)\sum_{j=1}^{n-i}(-1)^{j+1}\binom{n-i}{j}M_j(t)}{\sum_{i=0}^{r-1}\binom{n}{i}\Phi^i(t)}\qquad r=1,2,\ldots,n\quad t>0, \tag{2.18}$$

where $M_j(t) = \dfrac{\int_t^\infty \overline{F}^j(x)dx}{\overline{F}^j(t)}$ and $\Phi(t) = \dfrac{F(t)}{\overline{F}(t)}$.

Proof. If $S(x|t)$ denotes the conditional survival of $X_{n:n}$ at $x+t$ given that $X_{r:n}$ is greater than *t*, then

$$S(x|t) = P(X_{n:n} > x+t | X_{r:n} > t)$$

$$= \frac{P(X_{n:n} > x+t, X_{r:n} > t)}{P(X_{r:n} > t)}$$

$$= \frac{\sum_{i=0}^{r-1}\binom{n}{i}F^{i}(t)\sum_{j=1}^{n-i}(-1)^{j+1}\binom{n-i}{j}\overline{F}^{n-i-j}(t)\overline{F}^{j}(x+t)}{\sum_{i=0}^{r-1}\binom{n}{i}F^{i}(t)\overline{F}^{n-i}(t)} . \qquad (2.19)$$

Therefore

$$M_n^r(t) = \int_0^\infty S(x|t)dx$$

$$= \frac{\sum_{i=0}^{r-1}\binom{n}{i}F^{i}(t)\sum_{j=1}^{n-i}(-1)^{j+1}\binom{n-i}{j}\overline{F}^{n-i-j}(t)\int_0^\infty \overline{F}^{j}(x+t)dx}{\sum_{i=0}^{r-1}\binom{n}{i}F^{i}(t)\overline{F}^{n-i}(t)}$$

Hence, on taking $M_j(t) = \dfrac{\int_t^\infty \overline{F}^{j}(x)dx}{\overline{F}^{j}(t)}$ and $\Phi(t) = \dfrac{F(t)}{\overline{F}(t)}$,

$$M_n^r(t) = \frac{\sum_{i=0}^{r-1}\binom{n}{i}\Phi^{i}(t)\sum_{j=1}^{n-i}(-1)^{j+1}\binom{n-i}{j}M_j(t)}{\sum_{i=0}^{r-1}\binom{n}{i}\Phi^{i}(t)} .$$

This is end of the proof.

In Theorem 2.2, if r=1 then,

$$M_n^r(t) = E(X_{n:n} - t \mid X_{1:n} > t) = \sum_{i=1}^{n} (-1)^{i+1} \binom{n}{i} M_i(t).$$

Therefore, $M_n^r(t)$ can be written as

$$M_n^r(t) = \frac{\sum_{i=0}^{r-1} \binom{n}{i} \Phi^i(t) M_{(n-i)}^1(t)}{\sum_{i=0}^{r-1} \binom{n}{i} \Phi^i(t)}.$$

$M_n^r(t)$ is a convex combination of $M_{(n-i)}^1(t)$, i=0,1,…,r-1.

2.3 The Mean Residual Life Function of the *k*-out-of-*n* System

An important method for improving the reliability of a system is to build redundancy into it. A common structure of redundancy is the *k*-out-of-*n* system. A system consists of *n* components that work (or is good) if and only if at least *k* of the *n* components work, is called a *k*-out-of-*n*:*G* system. A system consists of *n* components that fails if and only if at least *k* of the *n* components fail is called a *k*-out-of-*n*:*F* system. Based on these definitions, a *k*-out-of-*n*:*G* system is equivalent to an (*n*-*k*+1)-out-of-*n*:*F* system. The term *k*-out-of-*n* system is often used to indicate a *G* system. Since the value of *n* is usually larger than the value of *k*, redundancy generally built into a *k*-out-of-*n* system. Both parallel and series systems are special cases of the *k*-out-of-*n* system. A series system is equivalent to a *n*-out-of-*n* system while a parallel system is equivalent to an 1-out-of-*n* system.

The *k*-out-of-*n* system structure is a very popular type of redundancy in fault-tolerant systems. It finds wide applications in both industrial and military systems. Fault tolerant systems include the multidisplay system in a cockpit, the multiengine system in an airplane, and the multipump system in a hydraulic control system. For example, it may be possible to drive a car with a V8 engine if only four cylinders are firing. However, if less than four cylinders fire, then the automobile cannot be driven. Thus, the functioning of the engine may be represented by a 4-out-of-8 system. The system is tolerant of failures of up to four cylinders for minimal functioning of the engine. In a data processing system with five video displays, a minimum of three displays operable may be sufficient for full data display. In this case the display subsystem behaves as a 3-out-of-5 system. In the case of an automobile with four tires, for example, usually one additional spare tire is equipped on the vehicle. Thus, the vehicle can be driven as long as at least 4-out-of-5 tires are good condition. Among applications of the k-out-of-n system model, the design of electronic circuits such as very large scale integrated and the automatic repairs of faults in an on-line system would be the most conspicuous. This type of system demonstrates what is called the voting redundancy. In such a system, several parallel outputs are channeled through a decision making device that provides the required system function as long as at least a predetermined number k of n parallel outputs are in agreement.

There are many papers related to the k-out-of-n system. Arulmozhi (2003) studied on simple and efficient computational method for determining the reliability unequal and equal reliabilities for components. Barlow & Heidtmann (1984) have presented methods to get expressions for reliability methods. Sarhan & Abouammoh (2001) have investigated the reliability of nonrepairable k-out-of-n systems with nonidentical components subjected to independent and common shocks and the relationship between the failure rate of the system and that of its components. Li & Chen (2004) studied the aging properties of the residual life length of a k-out-of-n system with independent (not necessarily identical) components given that the $(n-k)$th failure has occurred at time $t \geq 0$. Belzunce, Franco & Ruiz (1999) define new aging classes and provide characterizations for a nonparametric class of life distributions based on aging, and variability orderings of the residual life of k-out-of-n systems. Li & Zuo (2002) studied on behaviors of aging properties based upon the residual life. They also paid special attention to the residual life of a 1-out-of-n (parallel) system given that the $(n-r)$th failure occurs at time $t \geq 0$.

In this section, a detailed coverage on mean residual life evaluation of the k-out-of-n system is provided. It is assumed that components independent of one another and the lifetimes of components are identically distributed.

If X_i represents the lifetime to the ith component, i=1,2,…, n, the survival function of the k-out-of-n system is the same as that of the (n-k+1)th order statistic $X_{n-k+1:n}$ from this set of n random variables. The results obtained for order statistics hold for k-out-of-n systems, so the study of order statistics plays an important role in reliability theory. Recently, Asadi & Bayramoglu (2006) proposed a new definition for the mean residual life function of the system, and obtain several properties of that system.

Let $X_1, X_2, \ldots, X_n$ denote the lifetimes of n components connected in a system with a k-out-of-n system. Assume that X_i are independent and identically distributed random variables with common continuous distribution function F, and survival function (reliability function) $\overline{F} = 1 - F$. Let also $X_{1:n} \le X_{2:n} \le \cdots \le X_{n:n}$ be the ordered lifetimes of the components. Then $X_{k:n}$, k=1,2,…,n, represents the lifetime of the (n-k+1)-out-of-n system. If we denote the survival function of the system, at time t, by $S(t)$, we have

$$S(t) = P(X_{k:n} \ge t)$$

$$= \sum_{i=0}^{k-1} P(exactly\ (n-i)\ of\ X_1, X_2, \ldots, X_n \ge t)$$

$$= \sum_{i=0}^{k-1} \binom{n}{i} F^i(t) \overline{F}^{n-i}(t), \quad t > 0. \tag{2.20}$$

The lifetime of a (n-k+1)-out-of-n system is $X_{k:n}$, the MRL function of a system is equal to

$$E(X_{k:n} - t \mid X_{k:n} > t) = \frac{\int_t^\infty \overline{F}_{k:n}(x)dx}{\overline{F}_{k:n}(t)}, \tag{2.21}$$

where $\overline{F}_{k:n}$ denotes the survival function of $X_{k:n}$.

Asadi & Bayramoglu (2006) have given MRL function of the *k*-out-of-*n* system assuming that at time *t* all the components are working, i.e. $X_{1:n} > t$. The residual lifetime of the system is $X_{k:n} - t \mid X_{1:n} > t$. If *S* denotes the survival function of this conditional random variable, then it can be shown that, for *x*>0,

$$S(x|t) = P(X_{k:n} > x + t \mid X_{1:n} > t)$$

$$= \frac{P(X_{k:n} > x + t, X_{1:n} > t)}{P(X_{1:n} > t)}$$

$$= \sum_{i=0}^{k-1} \binom{n}{i} \frac{\overline{F}(x+t)^{n-i} \left(F(x+t) - F(t)\right)^i}{\overline{F}^n(t)}$$

$$= \sum_{i=0}^{k-1} \binom{n}{i} \left(\frac{\overline{F}(x+t)}{\overline{F}(t)} \right)^{n-i} \left(1 - \frac{\overline{F}(x+t)}{\overline{F}(t)} \right)^i. \tag{2.22}$$

The MRL function of the system, given that all components of the system are working at time *t*, is

$$H_n^k(t) = E(X_{k:n} - t \mid X_{1:n} > t)$$

$$= \int_0^\infty S(x|t)dx$$

$$= \sum_{i=0}^{k-1} \binom{n}{i} \int_0^{\infty} \left(\frac{\overline{F}(x+t)}{\overline{F}(t)} \right)^{n-i} \left(1 - \frac{\overline{F}(x+t)}{\overline{F}(t)} \right)^{i} dx$$

$$= \sum_{i=0}^{k-1} \binom{n}{i} \int_t^{\infty} \left(\frac{\overline{F}(x)}{\overline{F}(t)} \right)^{n-i} \left(1 - \frac{\overline{F}(x)}{\overline{F}(t)} \right)^{i} dx \,. \qquad (2.23)$$

$H_n^k(t)$ is the MRL of $X_{k:n}$ at the system level.

Example 2.7 Let $F(x)$ be the exponential distribution function;

$$F(x) = 1 - \exp(-\lambda x), \qquad x \geq 0, \;\; \lambda > 0.$$

From (2.23) it can be obtained that the mean residual life function of a 2-out-of-3 system consisting of three identical and independent components with exponential distribution function

$$H_3^2(t) = \frac{5}{6\lambda}\,.$$

The MRL of the system having independent exponential components does not depend on *t*.

Asadi & Bayramoglu (2006) have shown that when the distribution function F is absolutely continuous, then it can be uniquely determined by $H_n^k(t)$ and $H_{n-1}^{k-1}(t)$.

Theorem 2.3 Let the components of the system have a common absolutely continuous distribution function F. Let also f and $\overline{F}$ denote the density, and survival functions corresponding to F, respectively. Then the survival function $\overline{F}$ can be represented in terms of $H_n^k(t)$ and $H_{n-1}^{k-1}(t)$ as

$$\overline{F}(t) = e^{-\frac{1}{n}\int_0^t \eta(x)dx}, \quad t>0,\ k=1,\dots,n, \tag{2.24}$$

where $\eta(x) = \left(1 + \dfrac{dH_n^k(x)}{dx}\right) / \left(H_n^k(x) - H_{n-1}^{k-1}(x)\right)$ and $H_{n-1}^0(x) = 0$.

Proof. It is taken $\theta_t(x) = \dfrac{\overline{F}(x)}{\overline{F}(t)}$, for $x>t$. Then

$$H_n^k(t) = \int_t^\infty \sum_{i=0}^{k-1} \binom{n}{i} (\theta_t(x))^{n-i} (1-\theta_t(x))^i \, dx$$

$$H_n^k(t) = \int_t^\infty \left[1 - \sum_{i=k}^{n} \binom{n}{i} (\theta_t(x))^{n-i} (1-\theta_t(x))^i \right] dx .$$

On taking the derivative of $H_n^k(t)$ with respect to t,

$$\frac{dH_n^k(t)}{dt} = -1 - r(t) \int_t^\infty \left(\sum_{i=k}^{n} \binom{n}{i} (n-i)(\theta_t(x))^{n-i} (1-\theta_t(x))^i - i(\theta_t(x))^{n-i+1} (1-\theta_t(x))^{i-1} \right) dx$$

$$= -1 - nr(t) \left[\sum_{i=k}^{n} \int_t^\infty \binom{n}{i} (\theta_t(x))^{n-i} (1-\theta_t(x))^i \, dx \right.$$

$$\left. - \sum_{i=k-1}^{n-1} \int_t^\infty \binom{n-1}{i} (\theta_t(x))^{n-i+1} (1-\theta_t(x))^{i-1} \, dx \right]$$

$$= -1 + nr(t)\left[H_n^k(t) - H_{n-1}^{k-1}(t)\right], \tag{2.25}$$

where $r(t) = \dfrac{f(t)}{\overline{F}(t)}$ denotes the hazard rate of F. Hence $r(t) = \eta(t)/n$, the proof is completed.

In following theorem Asadi & Bayramoglu (2006) have proved a result showing that, when the components have a common IFR (DFR) distribution, then $H_n^k(t)$ is decreasing (increasing) in t.

Theorem 2.4 If the components of the system have a IFR (DFR) distribution function F, then $H_n^k(t)$ is decreasing (increasing) in t.

Proof. Let $r(t)$ denote the hazard rate of F. Then, $r(t)$ is increasing (decreasing) if and only if for x, t>0, $\frac{\overline{F}(x+t)}{\overline{F}(t)}$ is decreasing (increasing) in t. From this result, it can be easily seen that the survival function $S(x|t)$ defined in (2.22) is decreasing (increasing) in t. This is turn implies that $H_n^k(t)$ is decreasing (increasing) in t, and the proof is completed.

A comparison between two systems based on their MRL can be given considering their hazard rates. Let the components of system 1 (S_1) and system 2 (S_2) have the distribution function F and G; survival functions $\overline{F}$ and $\overline{G}$; and hazard rates $r_F(t)$ $r_G(t)$, respectively. If, for t>0,

$$r_F(t) \le r_G(t) \text{, then } H_n^{1k}(t) \ge H_n^{2k}(t), \tag{2.26}$$

where $H_n^k(t)$ and $H_n^{2k}(t)$ denote the mean residual life of S_1 and S_2, respectively. From the assumption that $r_F(t) \le r_G(t)$ for t>0, it is true that

$$\frac{\overline{F}(x+t)}{\overline{F}(t)} \ge \frac{\overline{G}(t+x)}{\overline{G}(t)}. \tag{2.27}$$

From the equation (2.27), comparison of the systems based on their MRL functions in (2.26) is proved.

CHAPTER THREE

THE MEAN RESIDUAL LIFE FUNCTION OF PARALLEL SYSTEMS WITH NONIDENTICAL COMPONENTS

In the study of reliability of the technical systems and subsystems, parallel systems play an important role. In this chapter, we consider the mean residual life function of parallel system consisting of n components having independent and nonidentically distributed lifetimes. We establish new representations of the MRL function for such a system. An effective representation using permanents has been provided. Examples utilizing some distribution functions are presented.

3.1 Introduction

A parallel system, consisting of n components, is a system which functions if and only if at least one of its n components functions. Assume that $X_1, X_2, \ldots, X_n$ are independent, but not identically distributed random variables with distribution function F_i and survival function $\overline{F}_i = 1 - F_i$. Let also $X_{1:n} \le X_{2:n} \le \cdots \le X_{n:n}$ be the ordered lifetimes of the components. If we denote the survival function of the system at time t by $\overline{F}_{(n)}(t)$, we have

$$\begin{aligned}\overline{F}_{(n)}(t) &= P(X_{n:n} > t) \\ &= 1 - \prod_{i=1}^{n} F_i(t), \ t > 0.\end{aligned} \tag{3.1}$$

Assuming that each component of the system has survived up to time t, the survival function of $X_i - t$ given that $X_i > t$, $i = 1, \ldots, n$, is

$$S(x \mid t) = \frac{\overline{F}_i(t+x)}{\overline{F}_i(t)}. \tag{3.2}$$

The mean residual life function of each component is

$$\psi(t) = E(X_i - t \mid X_i > t) = \int_0^\infty S(x \mid t)dx = \frac{1}{\overline{F}(t)}\int_t^\infty \overline{F}(x)dx \;\; i = 1,\ldots,n. \qquad (3.3)$$

The mean residual life function for the nonidentical case can be expressed in terms of so-called symmetric functions and permanents. We describe shortly below the definition of symmetric functions and permanents and provide some of their useful properties.

Definition 3.1 Let $\mathbf{x} = (x_1,\ldots,x_n) \in \Re^n$, $(n \ge 1)$, the rth $(1 \le r \le n)$ elementary symmetric function denoted by $\sigma_r(x_1,\ldots,x_n)$, is the sum of all products of r distinct variables chosen from n variables (see MacDonald, 1979). That is

$$\sigma_r(\mathbf{x}) = \sigma_r(x_1,\ldots,x_n) = \sum_{1 \le i_1 < \cdots < i_r \le n} x_{i_1} \ldots x_{i_r}.$$

It is convenient to define $\sigma_r(\mathbf{x}) = 0$ for $r < 0$ and $r > n.$ And $\sigma_r(\mathbf{x}) = 1$, when $r = 0$.

The generating function $G(x)$ for the elementary symmetric function is,

$$G(x) = \prod_{i=1}^{n}(1 - x_i x) = \sum_{r=0}^{n}(-1)^r \sigma_r(\mathbf{x})x^r. \qquad (3.4)$$

The function $G(x)$ may be interpreted as the generating function for the subsets of the set $\{x_1,\ldots,x_n\}$ and $\sigma_r(\mathbf{x})$ is all $r-$element subsets. A recurrence relation for the symmetric functions can be obtained from (3.4) as (see Oruç & Akmaz, 2004)

$$\sigma_r(x_1,\ldots,x_n) = \sigma_r(x_1,\ldots,x_{n-1}) + x_n\sigma_{r-1}(x_1,\ldots,x_{n-1}). \qquad (3.5)$$

When the variables are independent but not assumed to be identically distributed, the usage of the permanents provides an effective technique to handle the case of order statistics from nonidentical parents.

Consider S_n as the set of permutations of $1,2,\ldots,n$. If A is an $n\times n$ matrix and a_{ij} its entries. The permanent of A, denoted by $PerA$, is defined as:

$$PerA=\sum_{\pi\in S_n}\prod_{i=1}^{n}a_{i\pi(i)} \;=\sum_{\pi\in s_n}a_{1\pi(1)}a_{2\pi(2)}\ldots a_{n\pi(n)}$$

where the summation extends over all permutations of $\{1,2,\ldots,n\}$. And π is the element of set of permutation.

The permanent does not change when the rows or columns of the matrix are permuted. And also, the permanent admits a Laplace expansion along any row or column of the matrix. Thus if we denote by $A(i,j)$ the matrix obtained by deleting row i and column j of the $n\times n$ matrix A, then for $i,j=1,2,\ldots,n$

$$PerA=\sum_{j=1}^{n}a_{ij}PerA(i,j)$$

and

$$PerA=\sum_{i=1}^{n}a_{ij}PerA(i,j).$$

If $a_1,a_2,\ldots,$ are column vectors, then

$$[\underbrace{a_1}_{i_1},\underbrace{a_2}_{i_2},\ldots,]$$

will denote the matrix obtained by taking i_1 copies of a_1, i_2 copies of a_2 and so on (Bapat & Beg, 1989).

Vaughan & Venables (1972) have shown that the density of $X_{r:n}$ is conveniently expressed in terms of permanents, when $X_1, X_2, \ldots, X_n$ are order statistics of the independent random variables with absolutely continuous distribution functions $F_1, F_2, \ldots, F_n$ and densities $f_1, f_2, \ldots, f_n$ respectively. The distribution function of $X_{r:n}$ $(1 \le r \le n)$ is given by Bapat & Beg (1989)

$$P(X_{r:n} \le x) = \sum_{i=r}^{n} \frac{1}{i!(n-i)!} Per \begin{bmatrix} F_1(x) & 1-F_1(x) \\ \vdots & \vdots \\ \underbrace{F_n(x)}_{i} & \underbrace{1-F_n(x)}_{n-i} \end{bmatrix}, \ -\infty < x < \infty$$

where $PerA$ denotes the permanent of a square matrix A; the permanent is defined just like the determinant, except that all signs in the expansion are positive. A simple argument shows (David, 1981) that

$$\begin{aligned} P(X_{r:n} \le x) &= \sum_{i=r}^{n} P(exactly \ \ i \ \ variables \ from \ X_1, \ldots, X_n \ are \le x) \\ &= \sum_{i=r}^{n} \sum_{j_1, \ldots j_n} \prod_{l=1}^{i} F_{j_l}(x) \prod_{l=i+1}^{n} \left[1 - F_{j_l}(x)\right], \end{aligned}$$

where the summation extends over all permutations $j_1, \ldots, j_n$ of $1, \ldots, n$ for which $j_1 < \cdots < j_i$ and $j_{i+1} < \cdots < j_n$. That is the summation is over all $\binom{n}{r}$ distinct combinations of the integers $\{1, 2, \ldots, n\}$ taken i at a time such that exactly i of the X_i's greater than t (and remaining X_i's are less than or equal to t), $i \ge r$.

3.2 The MRL Function of Simple Parallel System with Nonidentical Components

Assume that at time t, $t>0$ the residual lifetime of the system is $X_{n:n}-t \mid X_{n:n}>t$. If S denotes the survival function of this conditional random variable then, it can be shown that, for $x>0$,

$$S(x \mid t) = P(X_{n:n} > x+t \mid X_{n:n} > t)$$

$$= \frac{1}{1-\prod_{i=1}^{n} F_i(t)} \left[1-\prod_{i=1}^{n} F_i(x+t)\right].$$

The mean residual life function is

$$\psi_n(t) = E(X_{n:n} - t \mid X_{n:n} > t)$$

$$= \int_0^{\infty} S(x \mid t)dx = \frac{1}{1-\prod_{i=1}^{n} F_i(t)} \int_t^{\infty} \left[1-\prod_{i=1}^{n} F_i(x)\right]dx \tag{3.6}$$

for $i=1,\ldots,n.$ The mean residual life function (3.6) can be expressed in terms of generating function (3.4) taking $x=1$. Let us rewrite the product part of Equation (3.6) as

$$\prod_{i=1}^{n}\left[1-\overline{F}_i\right] = \sum_{r=0}^{n}(-1)^r \sum_{1\le i_1<\ldots<i_r\le n} \overline{F}_{i_1}\ldots\overline{F}_{i_r}$$

$$= \sum_{r=0}^{n}(-1)^r \sigma_r(\overline{F}_1,\ldots,\overline{F}_n),$$

where $\sigma_r(\overline{F}_1,\ldots,\overline{F}_n) = \sum_{1\le i_1<\ldots<i_r\le n} \overline{F}_{i_1}\ldots\overline{F}_{i_r}$. Then,

$$\psi_n(t) = \frac{1}{1 - \sum_{r=0}^{n} (-1)^r \sigma_r(\overline{F}_1, \ldots, \overline{F}_n)} \int_t^{\infty} \left[1 - \sum_{r=0}^{n} (-1)^r \sigma_r(\overline{F}_1, \ldots, \overline{F}_n) \right] dx.$$

Then by Definition 3.1 $\sigma_0(x_1, \ldots, x_n) = 1$, the mean residual life function can be written as follows:

$$\psi_n(t) = \frac{1}{\sum_{r=1}^{n} (-1)^{r+1} \sigma_r(\overline{F}_1, \ldots, \overline{F}_n)} \sum_{r=1}^{n} (-1)^{r+1} \int_t^{\infty} \sigma_r(\overline{F}_1, \ldots, \overline{F}_n) dx. \qquad (3.7)$$

For the case when F_i's are identical, it is easy to see that

$$\sigma_r(\overline{F}_1, \ldots, \overline{F}_n) = \binom{n}{r} \overline{F}^r. \qquad (3.8)$$

Hence, $\psi_n(t)$ can be written as

$$\psi_n(t) = \frac{1}{1 - F^n(t)} \int_t^{\infty} \left[1 - F^n(x) \right] dx. \qquad (3.9)$$

The following examples illustrate this concept.

Example 3.1 An important life distribution is the exponential distribution. Let

$$F_i(x) = \begin{cases} 1 - e^{-\lambda_i x}, & x \geq 0, \qquad \lambda_i > 0, \quad i = 1, \ldots, n \\ 0, & x < 0. \end{cases}$$

The MRL function of such a system containing three components has the following form,

$$\psi_3(t) = \frac{\sum_{r=1}^{3}(-1)^{r+1}\sum_{1\le i_1<\cdots<i_r\le 3}\frac{1}{\sum_{k=1}^{r}\lambda_{i_k}}\prod_{k=1}^{r}e^{-\lambda_{i_k}t}}{\sum_{r=1}^{3}(-1)^{r+1}\sum_{1\le i_1<\cdots<i_r\le 3}\prod_{k=1}^{r}e^{-\lambda_{i_k}t}},$$

where

$$\sum_{r=1}^{3}(-1)^{r+1}\sum_{1\le i_1<\ldots<i_r\le 3}\prod_{k=1}^{r}e^{-\lambda_{i_k}t} = \sum_{j=1}^{3}e^{-\lambda_j t} - e^{-(\lambda_1+\lambda_2)t} - e^{-(\lambda_1+\lambda_3)t} - e^{-(\lambda_2+\lambda_3)t} + e^{-(\lambda_1+\lambda_2+\lambda_3)t}.$$

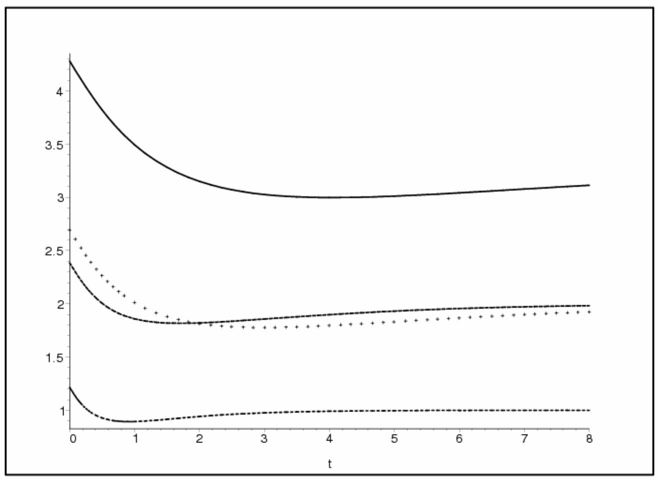


Figure 3.1 The MRL of a parallel system consisting of n=3 components having exponential distributed lifetimes.

In Figure 3.1, we have presented the graphs of the MRL function of a system containing three components in which the lifetime of the components are assumed to be exponential distribution function with different parameters.

The MRL function of a system containing n components has the form

$$\psi_n(t) = \frac{\sum_{r=1}^{n}(-1)^{r+1}\sum_{1\le i_1<\cdots<i_r\le n}\frac{1}{\sum_{k=1}^{r}\lambda_{i_k}}\prod_{k=1}^{r}e^{-\lambda_{i_k}t}}{\sum_{r=1}^{n}(-1)^{r+1}\sum_{1\le i_1<\cdots<i_r\le n}\prod_{k=1}^{r}e^{-\lambda_{i_k}t}}. \tag{3.10}$$

For the large values of t, the MRL function is,

$$\lim_{t\to\infty}\psi_n(t)=\frac{1}{\min(\lambda_1,\lambda_2,\ldots,\lambda_n)}. \tag{3.11}$$

For identically distributed components, i.e., when $\lambda_1=\lambda_2=\cdots=\lambda_n=\lambda,$ the mean residual life function of the system in Equation (3.10) is reduced to expression given below:

$$\psi_n(t)=\frac{\sum_{r=1}^{n}(-1)^{r+1}\binom{n}{r}\frac{1}{r\lambda}e^{-r\lambda t}}{\sum_{r=1}^{n}(-1)^{r+1}\binom{n}{r}e^{-r\lambda t}}. \tag{3.12}$$

For this case, the mean residual life function for large values of t is

$$\lim_{t\to\infty}\psi_n(t)=\frac{1}{\lambda}. \tag{3.13}$$

Example 3.2 Another important class of life distributions is the power distribution. Let

$$F_i(x)=\begin{cases}1-(1-x)^{\theta_i}, & 0<x<1,\ i=1,\ldots,n\\ 0, & otherwise\end{cases}$$

The MRL function of the system consisting of three components is plotted for selected values of parameter θ_i in Figure 3.2. When θ_i's increase then the MRL of the system decreases.

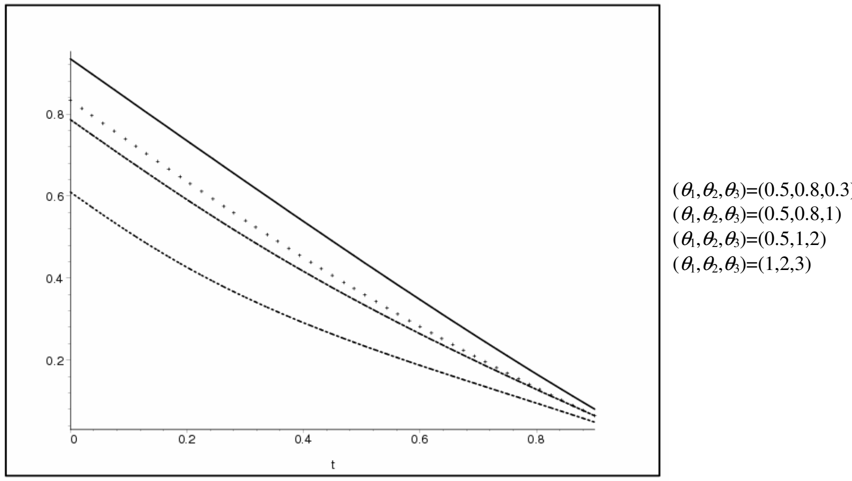


Figure 3.2 The MRL of a parallel system consisting of n=3 components having power distributed lifetimes.

In the graph above the curve on the top corresponds to $(\theta_1, \theta_2, \theta_3) = (0.5, 0.8, 0.3)$.

The MRL function of a system containing n components is given as below.

$$\psi_n(t) = \frac{(1-t)\sum_{r=1}^{n}(-1)^{r+1}\sum_{1\le i_1<\cdots<i_r\le n}\frac{1}{\sum_{k=1}^{r}\theta_{i_k}+1}\prod_{k=1}^{r}(1-t)^{\theta_{i_k}}}{\sum_{r=1}^{n}(-1)^{r+1}\sum_{1\le i_1<\cdots<i_r\le n}\prod_{k=1}^{r}(1-t)^{\theta_{i_k}}} \tag{3.14}$$

In the following theorem we present a recurrence formula for the mean residual life function defined in Equation (3.7).

Theorem 3.1 Let $\psi_{n-1}(t)$ be the mean residual life function of a system having a parallel structure and consisting of $(n-1)$ independent and nonidentical components with distribution function F_i, $i = 1, 2, \ldots, n-1$. Then,

$$\psi_n(t) = \omega\psi_{n-1}(t) + \delta, \quad t > 0 \tag{3.15}$$

where,

$$\omega = \frac{A_{n-1}(t)}{A_{n-1}(t) + \overline{F}_n(t)\left[1 - A_{n-1}(t)\right]}, \quad \delta = \frac{\int_t^\infty \overline{F}_n(x)\left[1 - A_{n-1}(x)\right]dx}{A_{n-1}(t) + \overline{F}_n(t)\left[1 - A_{n-1}(t)\right]}$$

and

$$A_n(t) = \sum_{r=1}^{n} (-1)^{r+1} \sigma_r(\overline{F}_1, \ldots, \overline{F}_n). \tag{3.16}$$

Proof. The mean residual life function of a system consisting of $(n-1)$ nonidentical components is

$$\psi_{n-1}(t) = \frac{1}{\sum_{r=1}^{n-1}(-1)^{r+1}\sigma_r(\overline{F}_1, \ldots, \overline{F}_{n-1})} \sum_{r=1}^{n-1}(-1)^{r+1}\int_t^\infty \sigma_r(\overline{F}_1, \ldots, \overline{F}_{n-1})dx. \tag{3.17}$$

We have,

$$\sum_{r=1}^{n-1}(-1)^{r+1}\sigma_r(\overline{F}_1, \ldots, \overline{F}_{n-1})\psi_{n-1}(t) = \sum_{r=1}^{n-1}(-1)^{r+1}\int_t^\infty \sigma_r(\overline{F}_1, \ldots, \overline{F}_{n-1})dx. \tag{3.18}$$

From (3.16), $A_{n-1}(t) = \sum_{r=1}^{n-1}(-1)^{r+1}\sigma_r(\overline{F}_1, \ldots, \overline{F}_{n-1})$ where $\overline{F_i}(t) = 1 - F_i(t)$.

Using (3.5) one can write $A_n(t)$ in terms of $A_{n-1}(t)$ as follows:

$$A_n(t) = \sum_{r=1}^{n}(-1)^{r+1}\left[\sigma_r(\overline{F}_1, \ldots, \overline{F}_{n-1}) + \overline{F}_n \sigma_{r-1}(\overline{F}_1, \ldots, \overline{F}_{n-1})\right]$$

$$= \sum_{r=1}^{n-1}(-1)^{r+1}\sigma_r(\overline{F}_1, \ldots, \overline{F}_{n-1}) + \overline{F}_n \sum_{r=1}^{n}(-1)^{r+1}\sigma_{r-1}(\overline{F}_1, \ldots, \overline{F}_{n-1}).$$

Therefore,

$$A_n(t) = A_{n-1}(t) + \overline{F}_n(t)\left[1 - A_{n-1}(t)\right]. \tag{3.19}$$

The mean residual life function of a system consisting of n nonidentical components is,

$$\psi_n(t) = \frac{1}{A_n(t)} \sum_{r=1}^{n} (-1)^{r+1} \int_t^{\infty} \sigma_r(\overline{F}_1, \ldots, \overline{F}_n)\,dx. \tag{3.20}$$

Hence, we can use the recurrence relations (3.5) and (3.19) in (3.20).

$$\psi_n(t) = \frac{\sum_{r=1}^{n} (-1)^{r+1} \int_t^{\infty} \left[\sigma_r(\overline{F}_1, \ldots, \overline{F}_{n-1}) + \overline{F}_n(x)\sigma_{r-1}(\overline{F}_1, \ldots, \overline{F}_{n-1})\right] dx}{A_{n-1}(t) + \overline{F}_n(t)\left[1 - A_{n-1}(t)\right]}$$

$$= \frac{\sum_{r=1}^{n} (-1)^{r+1} \int_t^{\infty} \sigma_r(\overline{F}_1, \ldots, \overline{F}_{n-1})\,dx + \sum_{r=1}^{n} (-1)^{r+1} \int_t^{\infty} \overline{F}_n(x)\sigma_{r-1}(\overline{F}_1, \ldots, \overline{F}_{n-1})\,dx}{A_{n-1}(t) + \overline{F}_n(t)\left[1 - A_{n-1}(t)\right]}$$

It can be shown that the first part of the integral is equal to Equation (3.18). So we have,

$$\psi_n(t) = \frac{\left(A_{n-1}(t)\psi_{n-1}(t) + \sum_{r=0}^{n-1} (-1)^r \int_t^{\infty} \overline{F}_n(x)\sigma_r(\overline{F}_1, \ldots, \overline{F}_{n-1})\,dx\right)}{A_{n-1}(t) + \overline{F}_n(t)\left[1 - A_{n-1}(t)\right]}.$$

Since

$$\sum_{r=0}^{n-1} (-1)^r \sigma_r(\overline{F}_1, \ldots, \overline{F}_{n-1}) = 1 - A_{n-1}(x),$$

we obtain

$$\psi_n(t) = \frac{A_{n-1}(t)}{A_{n-1}(t) + \overline{F}_n(t)[1 - A_{n-1}(t)]}\psi_{n-1}(t) + \frac{\int_t^\infty \overline{F}_n(x)[1 - A_{n-1}(x)]dx}{A_{n-1}(t) + \overline{F}_n(t)[1 - A_{n-1}(t)]}.$$

Thus the theorem is proved. Recurrence relation in Equation (3.15) expresses the mean residual life function of n components in terms of mean residual life function of $n-1$ components.

Remark 3.1 For the case of a system having independent and identically distributed lifetimes with distribution function $F(x)$ from Theorem 3.1, it follows that

$$\psi_n(t) = \frac{1 - F^{n-1}(t)}{1 - F^n(t)}\psi_{n-1}(t) + \frac{\int_t^\infty \overline{F}(x)F^{n-1}(x)dx}{1 - F^n(t)} \tag{3.21}$$

For $F(x) = 1 - e^{-\lambda x}$, $x > 0$, $\lambda > 0$, (3.21) becomes

$$\psi_n(t) = \frac{\sum_{r=1}^{n-1}(-1)^{r+1}\binom{n-1}{r}e^{-r\lambda t}}{\sum_{r=1}^{n}(-1)^{r+1}\binom{n}{r}e^{-r\lambda t}}\psi_{n-1}(t) + \frac{1}{\lambda n}.$$

For large values of t, $\psi_n(t)$ is

$$\lim_{t\to\infty}\psi_n(t) = \frac{1}{\lambda}, \tag{3.22}$$

where

$$\lim_{t\to\infty}\frac{\sum_{r=1}^{n-1}(-1)^{r+1}\binom{n-1}{r}e^{-r\lambda t}}{\sum_{r=1}^{n}(-1)^{r+1}\binom{n}{r}e^{-r\lambda t}} = \frac{n-1}{n}.$$

3.3 The MRL Function of a System Having n Components All Alive at Time t

Consider a parallel system with independent and nonidentically distributed components each following the distribution function F_i and survival function (reliability function) $\overline{F}_i = 1 - F_i$, $i = 1, 2, \ldots, n.$ When the system is put into operation at time t, all components are working. Let also $X_{1:n} \le X_{2:n} \le \cdots \le X_{n:n}$ be the ordered lifetimes of the components. The consideration of the mean residual life function of this system leads us to the following definition.

Definition 3.2 The MRL function of a system under the condition all components alive at time t, i.e., $X_{1:n} > t$, is

$$\phi_n(t) = E(X_{n:n} - t \mid X_{1:n} > t) = E(X_{n:n} \mid X_{1:n} > t) - t. \qquad (3.23)$$

In Theorem 3.2 we obtain a representation formula for mean residual life function of a parallel system under condition that all components are survived.

Theorem 3.2 Let $\phi_n(t)$ be the mean residual life function of a system having a parallel structure and consisting of n independent and nonidentically distributed components with distribution function F_i, $i = 1, 2, \ldots, n$, respectively. Given that all components of the system are working at time t then,

$$\phi_n(t) = \frac{1}{\prod_{i=1}^{n} \overline{F}_i(t)} \frac{1}{(n-1)!} \int_t^{\infty} x Per \left[\underbrace{f(x)}_{1} \;\; \underbrace{F(x) - F(t)}_{n-1} \right] dx - t \quad t > 0. \qquad (3.24)$$

Proof. We have,

$$P(X_{n:n} \le x \mid X_{1:n} > t) = \frac{P(X_1 \le x, \ldots, X_n \le x, X_1 > t, \ldots, X_n > t)}{P(X_1 > t, \ldots, X_n > t)}. \qquad (3.25)$$

From (3.25) we get

$$P(X_{n:n} \le x \mid X_{1:n} > t) = \frac{\prod_{i=1}^{n} \left[F_i(x) - F_i(t)\right]}{\prod_{i=1}^{n} \overline{F}_i(t)}. \tag{3.26}$$

Differentiating (3.26) with respect to x we obtain the probability density function of conditional random variable $(X_{n:n} \mid X_{1:n} > t)$ as

$$\frac{1}{\prod_{i=1}^{n} \overline{F}_i(t)} \sum_{k=1}^{n} f_k(x) \prod_{i \ne k} \left[F_i(x) - F_i(t)\right]. \tag{3.27}$$

Using the expression (3.27), the mean residual life function given in Definition 3.2 can be written as,

$$\psi_n(t) = \frac{1}{\prod_{i=1}^{n} \overline{F}_i(t)} \int_t^{\infty} x \sum_{k=1}^{n} f_k(x) \prod_{i \ne k} \left[F_i(x) - F_i(t)\right] dx - t.$$

An argument shows that

$$\sum_{k=1}^{n} f_k(x) \prod_{i \ne k} \left[F_i(x) - F_i(t)\right] = \sum_{j_1, \ldots, j_n} f_{j_1}(x) \prod_{l=2}^{n} \left[F_{j_l}(x) - F_{j_l}(t)\right],$$

where the summation extends over all permutations $j_1, \ldots j_n$ of $1, \ldots, n$ for which j_1 and $j_2 < \cdots < j_n$. The result now follows from the definition of the permanent:

$$\sum_{j_1, \ldots, j_n} f_{j_1}(x) \prod_{l=2}^{n} \left[F_{j_l}(x) - F_{j_l}(t)\right] = \frac{1}{(n-1)!} Per \begin{bmatrix} f_1(x) & F_1(x) - F_1(t) \\ \vdots & \vdots \\ \underbrace{f_n(x)}_{1} & \underbrace{F_n(x) - F_n(t)}_{n-1} \end{bmatrix}. \tag{3.28}$$

Given that all components of the system are working at time t, we obtain

$$\phi_n(t) = \frac{1}{\prod_{i=1}^{n} \overline{F}_i(t)} \frac{1}{(n-1)!} \int_t^\infty x Per\left[\underbrace{f(x)}_{1} \; \underbrace{F(x) - F(t)}_{n-1} \right] dx - t \quad t > 0.$$

Thus the proof is completed.

Two examples are given for lifetimes which are distributed as exponential and power distribution functions.

Example 3.3 Let $F_i(x)$, $i = 1, 2, \ldots, n$ be the exponential distribution function;

$$F_i(x) = \begin{cases} 1 - e^{-\lambda_i x} & x \geq 0, \lambda_i > 0 \\ 0, & x < 0 \end{cases}$$

Then, using Equation (3.24) one can show that for $i = 1, \ldots, n$, the MRL function of a system containing three components has the following form:

$$\phi_3(t) = \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} - \frac{1}{\lambda_1 + \lambda_2} - \frac{1}{\lambda_1 + \lambda_3} - \frac{1}{\lambda_2 + \lambda_3} + \frac{1}{\lambda_1 + \lambda_2 + \lambda_3}.$$

Note that the MRL of a system having independent and nonidentical exponential components does not depend on t. When the values of $\lambda_i \geq 1$ then the MRL of the system decreases. The MRL function of a system containing n components has the form

$$\phi_n(t) = \sum_{i=1}^{n} (-1)^{i+1} \sum_{1 \leq j_1 < \cdots < j_i \leq n} \frac{1}{\sum_{k=1}^{i} \lambda_{j_k}}. \qquad (3.29)$$

Example 3.4 Let $F_i(x)$, $i = 1,2,\ldots,n$, be the power distribution function;

$$F_i(x) = \begin{cases} 1-(1-x)^{\theta_i}, & 0 < x < 1 \\ 0, & otherwise \end{cases}$$

Then, the MRL function of the system containing three components has the following form:

$$\phi_3(t) = (1-t)\left[1 - \frac{\theta_1}{\theta_1+1} - \frac{\theta_2}{\theta_2+1} - \frac{\theta_3}{\theta_3+1} + \frac{\theta_1+\theta_2}{\theta_1+\theta_2+1} + \frac{\theta_1+\theta_3}{\theta_1+\theta_3+1} \right.$$
$$\left. + \frac{\theta_2+\theta_3}{\theta_2+\theta_3+1} - \frac{\theta_1+\theta_2+\theta_3}{\theta_1+\theta_2+\theta_3+1}\right], \quad t > 0.$$

In Figure 3.3, we have presented the graph of the MRL function of the system containing three components in which the lifetime of the components are assumed to be power distribution with different parameter values. It is seen that the MRL function is a decreasing function of parameters θ_1, θ_2 and θ_3.

As a result, the MRL function of a system containing n components is,

$$\phi_n(t) = (1-t)\left[1 - \left(\sum_{i=1}^{n}(-1)^{i+1} \sum_{1\le j_1<\cdots<j_i\le n} \sum_{k=1}^{i} \frac{\theta_{j_k}}{\theta_{j_k}+1}\right)\right] \quad t > 0. \tag{3.30}$$

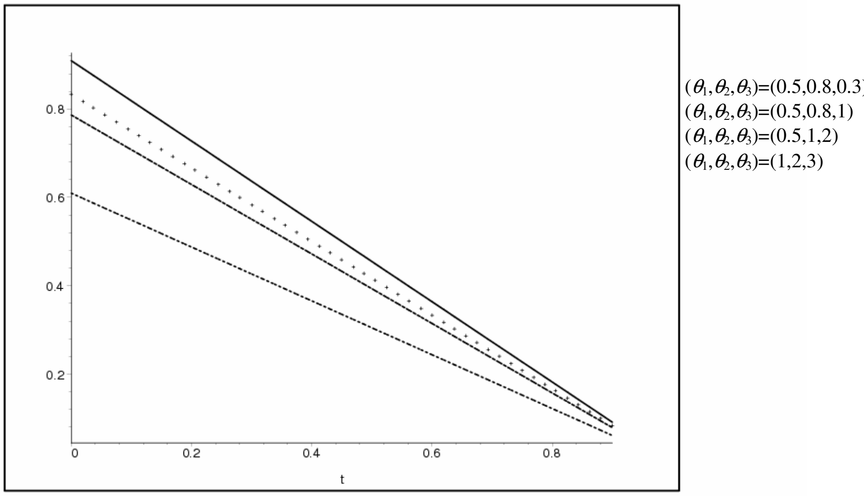


Figure 3.3 The MRL of a parallel system consisting of *n*=3 components having power distributed lifetimes.

Asadi and Bayramoglu (2005) have given an extension of the $\phi_n(t)$ as assuming that $X_1, X_2, \ldots, X_n$ are independent, identically distributed random variables with distribution function F and survival function $\overline{F} = 1 - F$. They defined the MRL function of a system, under the condition that $X_{r:n} > t$, i.e., $(n-r+1)$, $r = 1, 2, \ldots, n$, components of the system are still working as

$$M_n^r(t) = E(X_{n:n} - t \mid X_{r:n} > t),\ r = 1, 2, \ldots, n. \tag{3.31}$$

Then for $\overline{F}(t) > 0$,

$$M_n^r(t) = \frac{\sum_{i=0}^{r-1} \binom{n}{i} \Phi^i(t) \sum_{j=1}^{n-i} (-1)^{j+1} \binom{n-i}{j} M_j(t)}{\sum_{i=0}^{r-1} \binom{n}{i} \Phi^i(t)} \quad r=1,2,\ldots,n \ \ t>0, \tag{3.32}$$

where $M_j(t) = \dfrac{\int_t^\infty \overline{F}^{\,j}(x)dx}{\overline{F}^{\,j}(t)}$, $\Phi(t) = \dfrac{F(t)}{\overline{F}(t)}$.

In the following theorem we define the MRL function of a parallel system, assuming that $X_1, X_2, \ldots, X_n$ are independent but nonidentically distributed random variables with distribution function F_i and survival function $\overline{F_i} = 1 - F_i$, $i = 1,2,\ldots,n$, under the condition that $X_{r:n} > t$, i.e., $(n-r+1)$, $r = 1,2,\ldots,n$, components of the system are still working.

Theorem 3.3 Let $M_n^r(t)$ be the mean residual life function of a parallel system consisting of n independent and nonidentically distributed components. Then for $t > 0$ and $\overline{F}(t) > 0$,

$$M_n^r(t) = \frac{\sum_{i=0}^{r-1} \sum_{j_1,\ldots,j_n} \prod_{l=1}^{i} F_{j_l}(t) \sum_{k=1}^{n-i} (-1)^{k+1} \prod_{l=k+1}^{n-i} \overline{F}_{j_{(l+i)}}(t) \int_t^\infty \prod_{l=1}^{k} \overline{F}_{j_{(l+i)}}(x)dx}{\sum_{i=0}^{r-1} \frac{1}{i!(n-i)!} Per\left[\underbrace{F(t)}_{i} \ \underbrace{\overline{F}(t)}_{n-i} \right]}. \tag{3.33}$$

Proof. It is clear that

$$S(x \mid t) = P(X_{n:n} > x+t \mid X_{r:n} > t) = \frac{P(X_{n:n} > x+t, X_{r:n} > t)}{P(X_{r:n} > t)}$$

$$= \frac{\sum_{i=0}^{r-1} \sum_{j_1,\ldots,j_n} \prod_{l=1}^{i} F_{j_l}(t) \sum_{k=1}^{n-i} (-1)^{k+1} \prod_{l=1}^{k} \overline{F}_{j_{(l+i)}}(x+t) \prod_{l=k+1}^{n-i} \overline{F}_{j_{(l+i)}}(t)}{\sum_{i=0}^{r-1} \sum_{j_1,\ldots,j_n} \prod_{l=1}^{i} F_{j_l}(t) \prod_{l=i+1}^{n} \left[1 - F_{j_l}(t)\right]}. \tag{3.34}$$

The full sum in the denominator is recognizable as the permanent of a matrix, so $S(x \mid t)$ has the form

$$S(x|t)=\frac{\sum_{i=0}^{r-1}\sum_{j_1,\ldots,j_n}\prod_{l=1}^{i}F_{j_l}(t)\sum_{k=1}^{n-i}(-1)^{k+1}\prod_{l=k+1}^{n-i}\overline{F}_{j_{(l+i)}}(t)\prod_{l=1}^{k}\overline{F}_{j_{(l+i)}}(x+t)}{\sum_{i=0}^{r-1}\frac{1}{i!(n-i)!}Per\left[\underbrace{F(t)}_{i}\ \underbrace{\overline{F}(t)}_{n-i}\right]}.$$

For $r=1,2,\ldots,n$ and $t>0$,

$$M_{(n)}^{r}(t)=\int_{0}^{\infty}S(x\mid t)dx$$

$$=\frac{\sum_{i=0}^{r-1}\sum_{j_1,\ldots,j_n}\prod_{l=1}^{i}F_{j_l}(t)\sum_{k=1}^{n-i}(-1)^{k+1}\prod_{l=k+1}^{n-i}\overline{F}_{j_{(l+i)}}(t)\int_{t}^{\infty}\prod_{l=1}^{k}\overline{F}_{j_{(l+i)}}(x)dx}{\sum_{i=0}^{r-1}\frac{1}{i!(n-i)!}Per\left[\underbrace{F(t)}_{i}\ \underbrace{\overline{F}(t)}_{n-i}\right]}. \qquad (3.35)$$

Therefore the proof is completed.

CHAPTER FOUR
THE MEAN RESIDUAL LIFE FUNCTION OF *k*-OUT-OF-*n* SYSTEM WITH NONIDENTICAL COMPONENTS

4.1 Introduction

Many technical systems or subsystems have k-out-of-n system structure. The entire system is working if at least k of its n components are operating. It fails if $n-k+1$ or more components fail. Hence, a k-out-of-n system breaks down at the time of the $(n-k+1)$th component failure. Since all components start working at the same time, this approach leads to a kind of redundancy called active redundancy of $n-k$ components. Important particular cases of k-out-of-n systems are parallel and series systems corresponding to k=1 and k=n, respectively.

In this chapter, we provide the results on mean residual life function for k-out-of-n systems consisting of n independent and nonidentical distributed components. Parallel and k-out-of-n systems consist of nonidentical components find wide applications in both industrial and technical areas. For example, many of the air traffic control (ATC) communication systems are multi channel systems with identical and nonidentical elements. In a communication system with three transmitters, the average message load may be such that at least two transmitters must be operational at all times or critical messages may be lost. Thus, the transmission subsystem functions as a 2-out-of-3 system. For the improvement of the reliability of the operation of such complex technical systems the implementation of the structural redundancy is widely used by the method of the k-out-of-n reservation.

4.2 The MRL Function of the *k*-out-of-*n* System

Asadi & Bayramoglu (2006) have studied the MRL function of *k*-out-of-*n* system under the condition that at time t all the components are working, i.e. $X_{1:n} > t$. In the following theorem we propose the MRL function assuming that $X_1, X_2, \ldots, X_n$ are independent but nonidentically distributed random variables with distribution function F_i and survival function $\overline{F_i} = 1 - F_i$. Let also $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$ be the ordered lifetimes of the components. $X_{k:n}$, $k = 1, 2, \ldots, n$, represents the lifetime of $(n-k+1)$-out-of-n system.

Definition 4.1 The mean residual life function of the *k*-out-of-*n* system under the condition that all components alive at time t, is

$$H_{(n)}^{k}(t) = E(X_{k:n} - t \mid X_{1:n} > t),\ k = 1, 2, \ldots, n. \tag{4.1}$$

Theorem 4.1 If $H_{(n)}^{k}(t)$ is the MRL of the parallel system defined as (4.1), then for $\overline{F}(t) > 0$,

$$H_{(n)}^{k}(t) = \frac{\sum_{i=0}^{k-1} \binom{n}{i}}{n! \prod_{i=1}^{n} \overline{F}_i(t)} \int_t^\infty Per\left[\underbrace{\overline{F}(t) - \overline{F}(x)}_{i} \ \underbrace{\overline{F}(x)}_{n-i} \right] dx,\ k = 1, 2, \ldots, n,\ t > 0. \tag{4.2}$$

Proof. If S denotes the survival function of conditional random variable $X_{k:n} - t \mid X_{1:n} > t$ then for $x > 0$,

$$S(x \mid t) = P\big(X_{k:n} - t \mid X_{1:n} > t\big)$$

$$= \frac{P(X_{k:n} > x + t, X_{1:n} > t)}{P(X_{1:n} > t)}$$

$$= \frac{\sum_{i=0}^{k-1} \sum_{j_1,\ldots,j_n} \prod_{l=1}^{i} \left[F_{j_l}(x+t) - F_{j_l}(t)\right] \prod_{l=i+1}^{n} \overline{F}_{j_l}(x+t)}{\prod_{i=1}^{n} \overline{F}_i(t)} \tag{4.3}$$

$$= \frac{\sum_{i=0}^{k-1} \sum_{j_1,\ldots,j_n} \prod_{l=1}^{i} \left[\overline{F}_{j_l}(t) - \overline{F}_{j_l}(x+t)\right] \prod_{l=i+1}^{n} \overline{F}_{j_l}(x+t)}{\prod_{i=1}^{n} \overline{F}_i(t)}$$

Hence the full sum is recognizable as the permanent of a matrix, so $S(x \mid t)$ has the expression as follows.

$$\frac{\sum_{i=0}^{k-1} \frac{1}{i!(n-i)!} Per\left[\underbrace{\overline{F}(t) - \overline{F}(x+t)}_{i} \quad \underbrace{\overline{F}(x+t)}_{n-i}\right]}{\prod_{i=1}^{n} \overline{F}_i(t)} \tag{4.4}$$

Given that all the components of the system are working at time t, the MRL function of the system is

$$H_{(n)}^{k}(t) = \int_{0}^{\infty} S(x \mid t)\,dx$$

$$= \frac{\sum_{i=0}^{k-1} \binom{n}{i} \int_{t}^{\infty} Per\left[\underbrace{\overline{F}(t) - \overline{F}(x)}_{i} \quad \underbrace{\overline{F}(x)}_{n-i}\right] dx}{n! \prod_{i=1}^{n} \overline{F}_i(t)}, \quad k = 1, 2, \ldots, n, \; t > 0.$$

Thus the proof is completed.

The motivation for this structure can be given as an example of the high priority freight train, which is structured as a 3-out-of-4 system consisting of four locomotives (Nelson, 1982). The train is delayed only if two or more locomotives fail. It is assumed that the four locomotives in a train fail independently and times to failure for locomotives are distributed as nonidentical exponential distribution. The MRL of such a system is,

$$H_{(4)}^{2}(t)=\frac{1}{\lambda_1+\lambda_2+\lambda_3}+\frac{1}{\lambda_1+\lambda_3+\lambda_4}+\frac{1}{\lambda_1+\lambda_2+\lambda_4}+\frac{1}{\lambda_2+\lambda_3+\lambda_4}-\frac{3}{\lambda_1+\lambda_2+\lambda_3+\lambda_4}.$$

It is clear that the MRL of the system is a decreasing function of failure rates $(\lambda_1,\lambda_2,\lambda_3,\lambda_4)$ as expected.

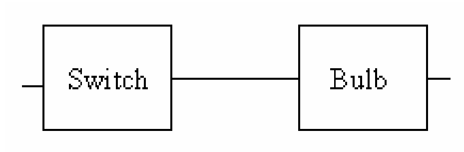
CHAPTER FIVE
APPLICATION

Many applications in various areas involve the modeling of lifetime data. In these applications the outcome of interest is the time X, until some event occurs. This event may be death, the appearance of a tumor, the development of some disease, recurrence of a disease, failing of an equipment, and so forth. Life data can be lifetimes of products in the marketplace, such as the time the product operated successfully or the time the product operated before it failed. These lifetimes can be measured in hours, miles, cycles-to-failure, stress cycles or any other metric with which the life or exposure of a product can be measured. All such data of product lifetimes can be encompassed in the term life data or, more specifically, product life data. The subsequent analysis and prediction are described as life data analysis. We will limit our examples and discussions to lifetimes of inanimate objects, such as equipment, components and systems as they apply to reliability engineering.

Before performing life data analysis, the failure mode and the life units (hours, cycles, miles, etc.) must be specified and clearly defined. Further, it is quite necessary to define exactly what constitutes a failure. In other words, before performing the analysis it must be clear when the product is considered to have actually failed. This may seem rather obvious, but it is not uncommon for problems with failure definitions or time unit discrepancies to completely invalidate the results of expensive and time consuming life testing and analysis.

It is important to define what is considered to be failure. There are light bulbs designed for interior use. The useful life (reliability) of an interior bulb is significantly decreased when used outside because it is not designed for the temperature fluctuations and moisture levels of the outdoor environment. It is being utilized outside its prescribed operating conditions.

Consider a system consisting of a light switch and a light bulb. There are two components in this system both of which must function in order for the system to function. This is an example of a serial system.



As the number of components is increased, the reliability of the serial system decreases. If we provide a redundant switch and a redundant light bulb, the system will provide light if either the primary or backup components function. Providing redundant components or systems as back-ups to the primary components or systems in complex systems has supplied high reliabilities.

In this chapter, it is provided some real problem examples and numerical results for evaluating mean residual life of the *k*-out-of-*n* system with different system level. Further, the relation between the mean residual life for the system and the mean residual life of its components is investigated.

5.1 The *k*-out-of-*n* Parallel Configuration and Examples

The *k*-out-of-*n* configuration is a special case of parallel redundancy. This type of configuration requires that at leas*t* *k* components succeed out of the total *n* parallel components for the system to succeed. For example, consider an airplane that has four engines. Furthermore, suppose that the design of the aircraft is such that at least two engines are required to function for the aircraft to remain airborne. This means that the engines are reliability-wise in a *k*-out-of-*n* configuration, where $k = 2$ and $n = 4$. More specifically, they are in a 2-out-of-4 configuration.

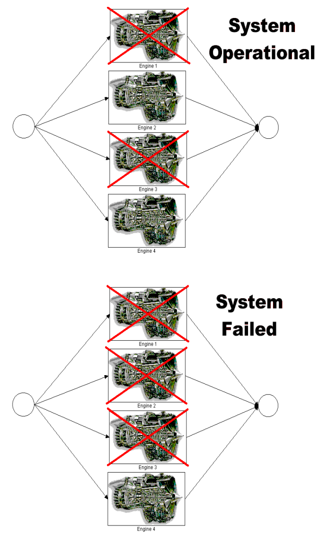


Figure 5.1 A 2-out-of-4 configuration.

Even though we classified the *k*-out-of-*n* configuration as a special case of parallel redundancy, it can also be viewed as a general configuration type. If the number of units required is equal to the number of units in the system, it is a series system. In other words, a series system of statistically independent components is an *n*-out-of-*n* system and a parallel system of statistically independent components is a 1-out-of-*n* system.

In the following, it is given some examples to illustrate the applications of the *k*-out-of-*n* systems in various engineering areas.

Example 1

Three hard discs in a computer system are configured reliability-wise in parallel. At least two of them must function in order for the computer to work properly. Each hard disc is of the same size and speed, but they are made by different manufacturers and have different reliabilities. Since at least two hard discs must be functioning at all times, only one failure is allowed. This is a 2-out-of-3 configuration.

The following operational combinations are possible for a system success:

1. All 3 hard discs operate.

2. HD #1 fails, while HDs #2 and #3 continue to operate.
3. HD #2 fails, while HDs #1 and #3 continue to operate.
4. HD #3 fails, while HDs #1 and #2 continue to operate.

Example 2

Communication is one of the most important domains of air traffic control (ATC) system from flight safety point of view. The most fundamental and difficult problem is providing reliability and fault tolerance of such systems. Many of the ATC communication systems are multichannel systems with identical homogeneous elements. The examples of such systems are the multichannel radio centers which one-type radio stations supply the communication on the various frequency channels, the multichannel radio relay lines which several linear paths (channels) supply the transmission of the various information flows, the multichannel radio transponders and radars, the satellite communication systems with the multistation approach using the opportunity of the simultaneous calls of several on-ground station to one satellite transceiver and others.

For the improvement of the reliability of the operation of such kind complex technical systems, the implementation of the structural redundancy is widely used by the method of the *k*-out-of-*n* reservation.

Example 3

An example is the Active Phased Array Radar (APAR) at the Royal Netherlands Navy. This radar has a cubical shape. On each of the four sides, it has a so-called face, consisting of thousands of transmit and receive elements. Each face covers a quarter of a circle, and together they cover the whole space around the ship of which it is a part. The elements on a face are identical and are partly redundant. A certain percentage of the total number of elements per face is allowed to fail, without losing the function of the specific radar face. Say that this percentage is 10% and that the total number of elements is 3000, then we have a 2700-out-of-3000 system (de Smidt-Destombes, van der Heijden & van Harten, 2004).

Example 4

Another example for which a similar trade-off applies is the active Towed Array Sonar (ATAS) for searching mines and submarines. The ATAS consists of several tens of hydrophones, let us assume 64 pieces. Say that 10% failed components is acceptable for full operation, then we model the ATAS as a 58-out-of-64 system. A smaller example is the frigate communication system (say, a 6-out-of-8 system). (de Smidt-Destombes & et al., 2004).

Example 5

Taking a simple telecom application and assuming a constant 8-hour load, it is standard practice to divide the required capacity between two or more parallel strings. In this case there is no redundancy, but string failures cause the system output to be reduced, rather than failing completely. If the reduced output is tolerated then the system behave statistically as a k-out-of-n system, where k is the minimum acceptable number of functioning strings and n is the total number of strings.

Example 6

A high-priority freight train required three locomotives for a one day run. If such a train arrived late, the railroad had a pay a large penalty. To assess its risks, the railroad needed to know the reliability of such trains. Experience indicated that times to failure for such locomotives could be approximated with an exponential distribution with a mean of 43.3 days. It was assumed that the three locomotives in a train fail independently. To reduce the chance of delay, the railroad used trains with four locomotives. Then the train was delayed only if two or more locomotives failed (Nelson, 1982). Then it can be modeled the freight train as a 3-out-of-4 system.

5.2 Numerical Results- A Case Study

This section introduces a numerical example for the various value of k of n components of parallel system. Also the relationship between the mean residual life for the system and the mean residual life of its components is investigated.

Let us consider an airplane that has three engines. Furthermore, suppose that the design of the aircraft is such that at least two engines are required to function for the aircraft to remain airborne. This means that the engines are reliability-wise in a k-out-of-n configuration, where k = 2 and n = 3. More specifically, they are in a 2-out-of-3 configuration. It is assumed that the times to failure of a component X_i, i=1, 2, 3, are Weibull distributed random variables with parameters (α_i, λ_i), respectively. This assumption is coming from a previous study provided by Petit & Turnbull (2001), which is performed in order to improve the safety and reliability for the next generation of General Aviation Aircraft System. They have indicated that data of an engine is distributed as the Weibull distribution function.

Also this distribution was selected based on the common usage in engineering, versatility and to reduce the complexity of the data analysis. The two parameter Weibull distribution is a time dependent distribution that is also one of the most useful probability distributions in reliability. It can be used to model both increasing, and decreasing failure rates. α is referred to as the shape parameter. If α<1, the mean residual life function is increasing over time. If α>1, the mean residual life function is decreasing over time. If α=1, the mean residual life function is constant over time, that is the exponential distribution. The MRL curves of the Weibull distribution for various shape parameter values and λ=1 are shown in Figure 5.2.

The time to failure X of an engine is said to be Weibull distributed with parameters α_i>0 and λ_i>0 for i=1, 2, 3 if the distribution function is given by

$$F(t)=\begin{cases}1-e^{-(\lambda_i t)^{\alpha_i}} & t>0\\ 0 & otherwise\end{cases}. \quad (5.1)$$

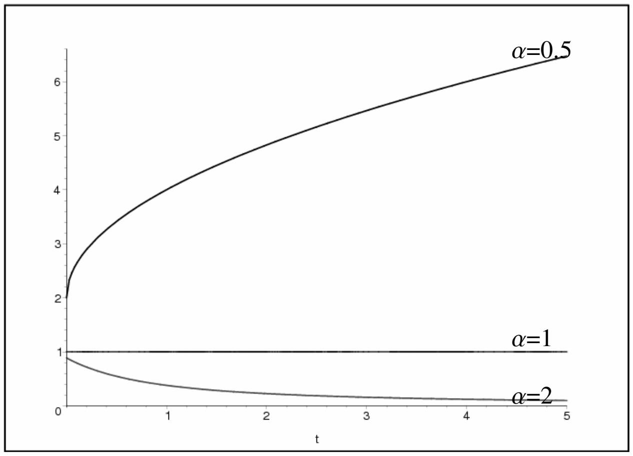


Figure 5.2 MRL curves of the Weibull distribution for $\lambda=1$

It is assumed that the scale parameter λ is identical for all components lifetimes. It is also assumed that the working of the components is independent of one another. The mean residual life of an engine at age *t* is the average remaining life among those engines which have survived until time *t*. The mean residual life function of the 2-out-of-3 system under the condition that all components alive at time t is

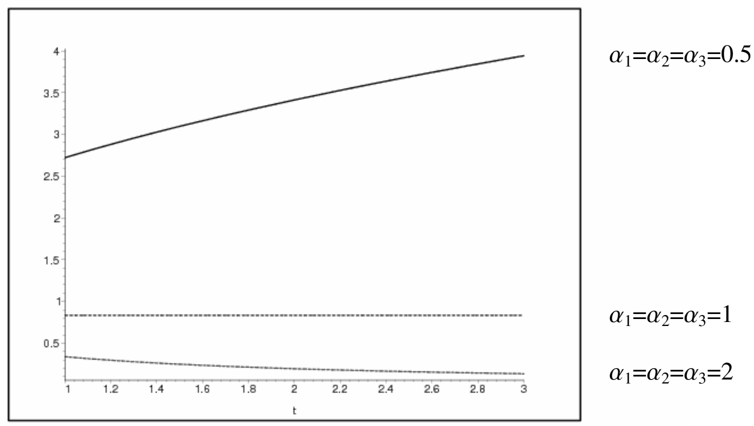
$$H^2_{(3)}(t)=\frac{\sum_{i=0}^{1}\binom{3}{i}}{3!\prod_{i=1}^{3}\overline{F}_i(t)}\int_t^\infty Per\left[\underbrace{e^{-(\lambda t)^{\alpha_i}}-e^{-(\lambda x)^{\alpha_i}}}_{i}\ \underbrace{e^{-(\lambda x)^{\alpha_i}}}_{3-i}\right]dx,\ \ t>0. \quad (5.2)$$

Figure 5.3a-e shows the mean residual life function of this system with different shape parameter α_i, *i*=1, 2, 3 of lifetime distribution of components when:

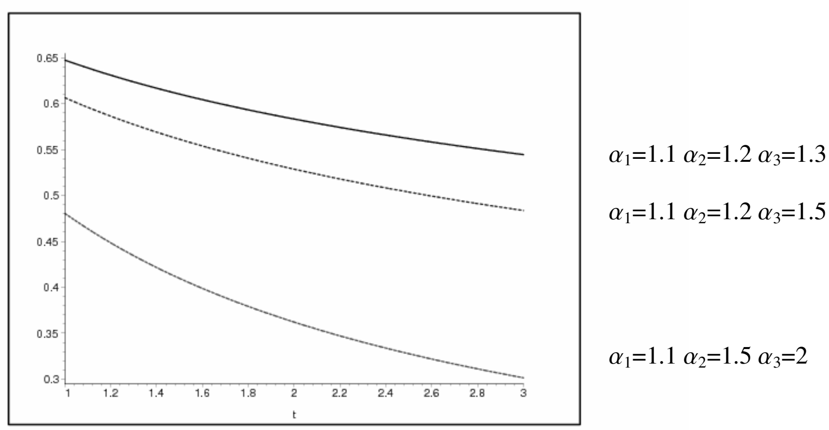
a. All components have identically distributed i.e. $\alpha_1=\alpha_2=\alpha_3$.
b. All components have a linear decreasing mean residual life function i.e. $\alpha_i>1$.
c. All components have a linear increasing mean residual life function i.e. $\alpha_i<1$.

d. The first component has an increasing mean residual life function while the rest two components have a decreasing mean residual life function.

e. The first and second components have an increasing mean residual life function while the third one has a decreasing mean residual life function.



(a)



(b)

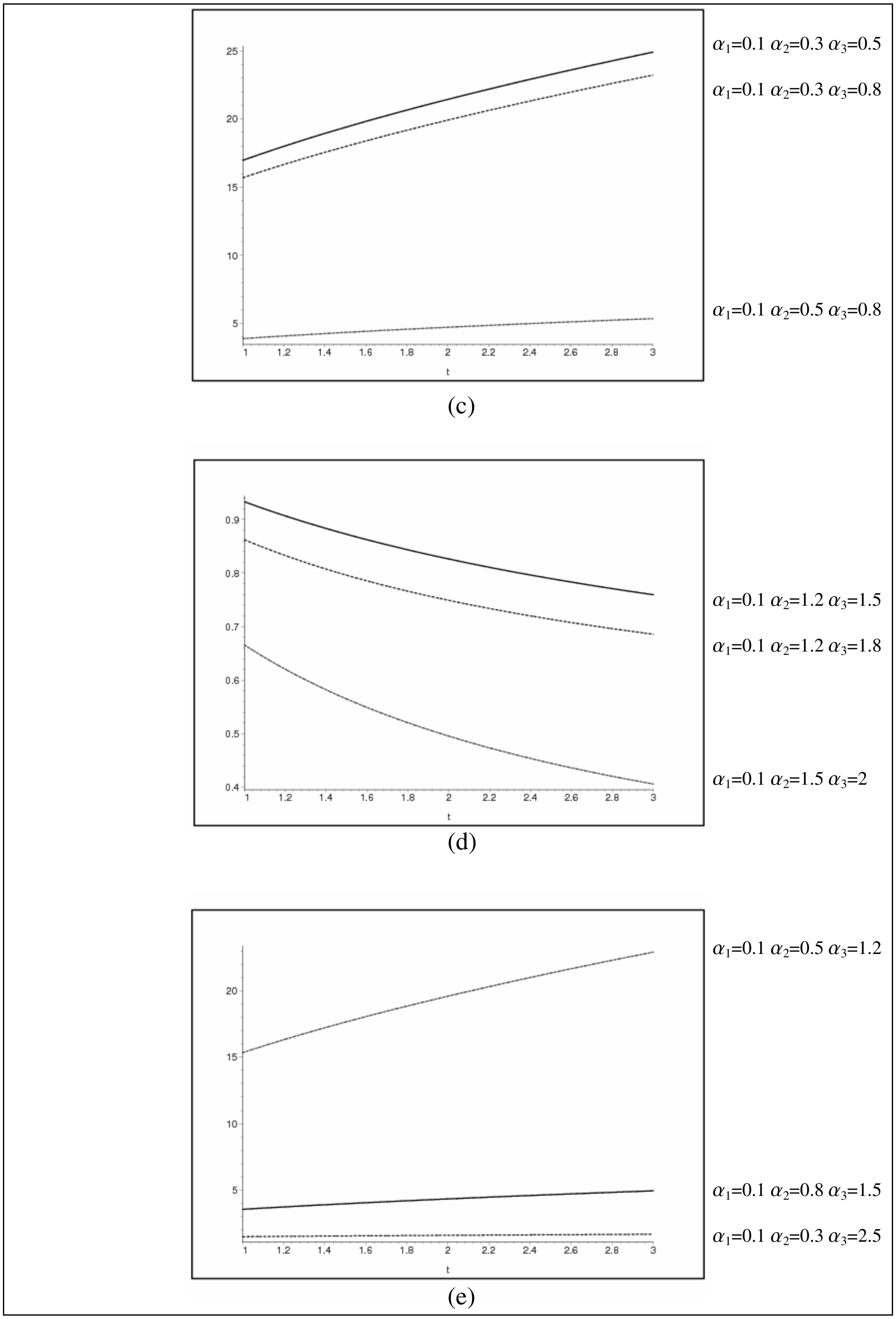


Figure 5.3 The MRL curves of the 2-out-of-3 system with Weibull distributed components (λ =1).

Based on the Figure 5.3 the following conclusions are possible. It is seen from Figure 5.3(a) that the components having constant mean residual life functions, i.e. α_1=α_2=α_3=1, the mean residual life function of the 2-out-of-3 system is constant. When all components identically distributed and α_i>1, the system has decreasing MRL function, otherwise the system has increasing MRL function. When either all the components have a linear decreasing mean residual life function (b), i.e. α_i>1, or two components have linear decreasing mean residual life function (d), the system has a linear decreasing. As the values of α_i get larger, the values of mean residual life decrease (b). Either all components have linear increasing mean residual life function (c), i.e. α_i<1, or two components have linear increasing mean residual life function (e), the system has increasing mean residual life function.

In Table 5.1, a particular case with n=3 and k=1, 2, 3 is analyzed numerically to study the effect of increasing the system level and various parameters on the mean residual life of the system. The mean residual life of the k-out-of-3 configuration was calculated versus different parameters of required units. All the computations were done using Maple 5.1.

A parallel system is equivalent to a 1-out-of-3 system, i.e. the k is equal to 1, while a series system is equivalent to a 3-out-of-3 system, i.e., the k is equal to 3. The system structure changes from a parallel structure to a 2-out-of-3 structure, then to a series structure. In other words, the system structure changes from strong to weak as the system level increases. So it is necessary to provide redundant equipment in a parallel structure, in cases where the failure of the system is not acceptable. When the components have constant mean residual life functions, i.e. coming from the exponential distribution; the mean residual life function of the system is constant in all system levels. Since the system structure changes from strong to weak as the system level increases, the mean residual life decreases for all parameters.

Table 5.1 The mean residual life of different system level.

n=3	$\alpha_1, \alpha_2, \alpha_3$	t=1	t=2	t=3	t=4	t=5
k=1	1.00, 1.00, 1.00	1.83	1.83	1.83	1.83	1.83
	0.50, 0.50, 0.50	8.39	9.91	11.07	12.06	12.92
	2.00, 2.00, 2.00	0.65	0.40	0.00	0.00	0.00
	1.10, 1.20, 1.30	1.38	1.28	1.21	1.16	1.13
	1.10, 1.50, 2.00	1.12	0.99	0.92	0.88	0.75
	0.30, 0.50, 0.80	25.82	31.04	35.07	38.48	41.52
	0.50, 1.20, 1.50	4.23	4.98	5.58	6.09	6.55
	0.50, 1.50, 2.00	4.12	4.89	5.50	6.02	6.39
	0.50, 0.80, 1.20	4.63	5.41	6.02	6.54	7.00
	0.50, 0.80, 2.50	4.55	5.35	5.98	6.51	6.98
k=2	1.00, 1.00, 1.00	0.83	0.83	0.83	0.83	0.83
	0.50, 0.50, 0.50	2.72	3.41	3.94	4.39	4.78
	2.00, 2.00, 2.00	0.34	0.19	0.00	0.00	0.00
	1.10, 1.20, 1.30	0.65	0.58	0.55	0.52	0.50
	1.10, 1.50, 2.00	0.48	0.36	0.30	0.26	0.24
	0.30, 0.50, 0.80	2.78	3.45	3.97	4.41	4.80
	0.50, 1.20, 1.50	0.78	0.72	0.68	0.64	0.62
	0.50, 1.50, 2.00	0.59	0.45	0.38	0.33	0.30
	0.50, 0.80, 1.20	1.18	1.29	1.37	1.43	1.48
	0.50, 0.80, 2.50	0.98	1.08	1.18	1.26	1.33
k=3	1.00, 1.00, 1.00	0.33	0.33	0.33	0.33	0.33
	0.50, 0.50, 0.50	0.89	1.17	1.38	1.56	1.71
	2.00, 2.00, 2.00	0.15	0.08	0.00	0.00	0.00
	1.10, 1.20, 1.30	0.27	0.24	0.22	0.21	0.20
	1.10, 1.50, 2.00	0.20	0.13	0.10	0.08	0.07
	0.30, 0.50, 0.80	0.75	0.91	1.03	1.11	1.19
	0.50, 1.20, 1.50	0.30	0.25	0.22	0.20	0.19
	0.50, 1.50, 2.00	0.22	0.15	0.11	0.09	0.07
	0.50, 0.80, 1.20	0.40	0.41	0.41	0.41	0.40
	0.50, 0.80, 2.50	0.22	0.12	0.07	0.05	0.03

CHAPTER SIX
CONCLUSIONS AND RECOMMENDATION FOR FUTURE RESEARCH

6.1 Conclusions

The concept of mean residual life of a system has been of much interest during the past several decades. Predicting the MRL functions of systems so that intervention prevents intolerable decay or outright failure is important in modern life.

In this thesis, we consider the problem of determining the mean residual life function in parallel and k-out-of-n systems consisting of n components having independent and nonidentically distributed lifetimes. We establish new representations, identities and a recurrence relation of mean residual life function for parallel structures consisting of n components with independent lifetimes having distribution functions (F_i) and probability density functions (f_i), i=1, 2,…, n, respectively. Some examples which are satisfied with the results of several lifetime distribution functions are given. The results for the MRL function of k-out-of-n system consisting of n ordered lifetimes of the components are obtained. We examined the Weibull parametric model that shows the mean residual life function of k-out-of-n system. We compute the values of mean residual life function and provide graphical comparisons for the shape of function for selected combinations of parameter values.

The results for MRL function of concerning structures are derived in general and we apply these results to some parametric models such as exponential, power, Weibull, gamma distributions. It is concluded that when the components have constant mean residual life functions, i.e., distributed as the exponential distribution; the mean residual life function of the parallel system is constant in all the system level under the condition that all components alive at time t. As the structure of system changes from strong to weak, then the mean residual life decreases.

Redundant equipment in a parallel structure is an important point when the consequence of a failure is high and the probability of a failure is not acceptable.

6.2 Recommendations for Future Research

In reliability theory, the lifetime of a *k*-out-of-*n* system is usually described by the (*n*-*k*+1)th order statistics $X_{n-k+1:n}$ from the sample $X_1,\ldots,X_n$, where the random variable X_i represents the lifetime or failure time of the *i*th component of the system, $1 \le i \le n$. In the conventional modeling of these structures, the component lifetimes are supposed to be independent and identically distributed random variables. It reflects the assumption that the failure of any component does not affect the remaining ones. However, in some systems, a component failure will more or less strongly influence the remaining parts of the system. Thus, a more flexible model, which is therefore more applicable to practical situations, must take some dependence among the system components into account. After each failure the remaining components possess a possibly different failure rate than before. That is, the underlying failure rate of the remaining components is adjusted according to the number of preceding failures.

Sequential order statistics have been introduced in Kamps (1995) as an extension of ordinary order statistics in order to model sequential *k*-out-of-*n* systems, where the failures of components possibly affect remaining ones. The model of sequential order statistics is flexible in the sense that, after the failure of some component, the distribution of the residual lifetime of the components at work may change. In order to distinguish this concept from the ordinary approach, the respective *k*-out-of-*n* system is called sequential *k*-out-of-*n* system. For a more detailed discussion Cramer & Kamps (1996) can be referred. At this point it may be concerned that the aging properties of sequential *k*-out-of-*n* systems consisting of *n* components based on their mean residual life.

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