DOKUZ EYLÜL UNIVERSITY
GRADUATE SCHOOL
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WEAKLY AND COFINITELY WEAK SUPPLEMENTED MODULES OVER DEDEKIND DOMAINS

by
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İZMİR

WEAKLY AND COFINITELY WEAK SUPPLEMENTED MODULES OVER DEDEKIND DOMAINS

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WEAKLY AND COFINITELY WEAK SUPPLEMENTED MODULES OVER DEDEKIND DOMAINS

ABSTRACT

The main purpose of this thesis is to study some classes of modules including supplemented, weakly supplemented, totally weak supplemented (briefly tws-modules) and cofinitely weak supplemented (briefly cws-modules) modules over commutative noetherian rings, in particular, over Dedekind domains. A module over a semilocal Dedekind domain is weakly supplemented if and only if it is a tws-module. If R is a non-semilocal Dedekind domain then an R-module is a tws-module exactly if it is torsion and weakly supplemented. Over a non-local Dedekind domain a module is supplemented if and only if it is torsion and a tws-module. Weakly supplemented modules and tws-modules coincide for finitely generated modules over Dedekind domains. If R is a local Dedekind domain and every weakly supplemented module is supplemented then R is complete. An integral domain R is one dimensional if and only if every R/I-module is supplemented, for every nonzero ideal I of R. The class of weakly supplemented modules is not closed under extensions. A module M is weakly supplemented if and only if U has a weak supplement in M, U and M/U are weakly supplemented, for some submodule U of M. An integral domain R is $h-semilocal$ if and only if every torsion R-module is a cws-module if and only if every torsion R-module with small radical is weakly supplemented.

Keywords: Noetherian ring, Dedekind domain, supplemented, weakly supplemented, totally weak supplemented, cofinite submodule, cofinitely weak supplemented, h−semilocal domain.

ZAYIF VE ZAYIF DUAL SONLU TÜMLENEN MODÜLLERİN DEDEKIND BÖLGELERİ ÜZERİNDE İNCELENMESİ

ÖZ

Bu tezde temel olarak tümlenen, zayıf tümlenen, tamamen zayıf tümlenen (kısaca tws), ve dual sonlu zayıf tümlenen (kısaca cws) modüllerini içeren bazı modül sınıflarını değişmeli noether halkalar, özel olarak Dedekind tamlık bölgeleri, üzerinde çalışılması amaçlanmaktadır. Yarı-yerel Dedekind tamlık bölgesi üzerinde bir M modülü zayıf tümlenendir ancak ve ancak M bir tws-modül dir. R yarı-yerel olmayan bir Dedekind tamlık bölgesi ise bir M modülü tws-modüldür ancak ve ancak M burulmalı ve zayıf tümlenendir. R yerel olmayan bir Dedekind tamlık bölgesi ise, bir M modülü tümlenendir ancak ve ancak M burulmalı ve tws-modüldür. Dedekind tamlık bölgesi üzerinde sonlu üretilmiş mödüller için zayıf tümlenen modüller sınıfı ile tws-modüller sınıfı çakışır. R yerel bir Dedekind tamlık bölgesi ve R üzerindeki her zayıf tümlenen modül tümlenen ise R tamdır. Bir R tamlık bölgesinin bir boyutlu olması için gerek ve yeter şart sıfırdan farklı her I ideali için her R/I-modülünün tümlenen olmasıdır. Zayıf tümlenen modüller sınıfı genişletme altında kapalı değildir. Bir M modülünün zayıf tümlenen olması için gerek ve yeter şart bir $U \subseteq M$ için, U'nun M'de bir zayıf tümleyene sahip olması, U ve M/U modüllerinin zayıf tümlenen olmasıdır.
Bir R tamlık bölgesi $h-yarıyerel$'dir ancak ve ancak her burulmalı R-modül cws-modül dir ancak ve ancak radikali küçük olan her burulmalı R-modül zayıf tümlenendir.

Anahtar Sözcükler: Noether halka, Dedekind tamlık bölgesi, tümlenen, zayıf tümlenen, tamamen zayıf tümlenen, dual sonlu alt modül, dual sonlu zayıf tümlenen, h−yarıyerel tamlık bölgesi.

CONTENTS

NOTATION

R	an associative ring with unit unless otherwise stated
$\mathbb{Z}$, $\mathbb{Z}^+$	the ring of integers, the set of all positive integers
$\mathbb{Q}$	the field of rational numbers
$\mathbb{Z}_{p^\infty}$	the Prüfer (divisible) group for the prime p
K	field of quotients of a (commutative) domain
R-module	*left* R-module
$\leqslant$	ideal
$\cong$	isomorphic
$\mathrm{Hom}_R(M, N)$	all R-module homomorphisms from M to N
$M \otimes_R N$	the tensor product of the *right* R-module M and the *left* R-module N
$\mathrm{Ker}(f)$	the kernel of the map f
$\mathrm{Im}(f)$	the image of the map f
$\mathrm{E}(M)$	the injective envelope (hull) of a module M
$\mathrm{T}(M)$	the torsion submodule of a module M
$\mathrm{Soc}(M)$	the socle of the R-module M
$\mathrm{Rad}(M)$	the radical of the R-module M
$\mathrm{J}(R)$	the Jacobson radical of the ring R
$P(M)$	the sum of all radical submodules of a module M
$(R, \mathfrak{m})$	R is a local ring with the unique maximal ideal $\mathfrak{m}$
Ω	the set of all maximal ideals of a ring
$\mathrm{u.dim}(M)$	the uniform dimension (=Goldie dimension) of M
$\mathrm{h.dim}(M)$	the hollow dimension (=dual Goldie dimension) of M
$\subseteq$	submodule
$\ll$	small (=superfluous) submodule
$\trianglelefteq$	essential submodule

CHAPTER ONE
INTRODUCTION

In module theory, the problem of decomposition of a module into a direct sum of its submodules is a fundamental one, and a wide area of module theory is related with this problem. It is well known that a submodule of a module need not be a direct summand. Moreover, we can not state that for every submodule U of a module M there is a minimal submodule V of M satisfying $U + V = M$. If this is the case (that is there is no submodule $\tilde{V}$ of V such that $\tilde{V} \subsetneqq V$ but still $U + \tilde{V} = M$), V is called a *supplement* of U. Minimality of V is equivalent to $U \cap V \ll V$. Reducing the last condition to $U \cap V \ll M$, we get the definition of a weak supplement. If every submodule of M has a supplement (weak supplement), we say that M is *supplemented* (respectively, *weakly supplemented*). M is called *totally supplemented* (*totally weak supplemented*) if every submodule of M is supplemented (respectively, weakly supplemented). A submodule U is called cofinite if M/U is finitely generated. M is called *cofinitely supplemented* (*cofinitely weak supplemented*) if every cofinite submodule of M has a supplement (weak supplement).

The classes of supplemented modules and weakly supplemented modules are well-studied in the literature. In a series of papers from 1974, H. Zöschinger considered the class of supplemented modules (Zöschinger, 1974a,b,c, 1979b, 1982a,b, 1986).
In recent years, the research on related concepts has regained interest; see Kuratomi (2003), Idelhadj & Tribak (2003b,a), Alizade & Büyükaşık (2003), Tuganbaev (2002), Keskin (2002b,a), Ganesan & Vanaja (2002), Keskin & Xue (2001), Alizade et al. (2001), Smith (2000), Keskin (2000b,a), Lomp (1999), Keskin et al.

(1999), Harmancı et al. (1999), Oshiro & Wisbauer (1995), Xin (1994), Vanaja (1993), Fieldhouse (1985), Oshiro (1984b,a), Inoue (1983), Hausen & Johnson (1983b,a), Hausen (1982), Mermut (2004).

In this thesis, we study the classes of supplemented modules, weakly supplemented modules, totally weak supplemented modules and cofinitely weak supplemented modules. We consider these modules over commutative rings, in particular over Dedekind domains, one dimensional domains, and more generally over noetherian rings.

Throughout this thesis all rings are associative and have an identity. If not stated otherwise, the symbol R, stands for a ring, and when R is a domain, K for its field of quotients.

In this chapter we introduce our basic terminology for rings and modules, as well as the fundamental results to be used in this thesis. In chapter 2, we shall investigate some well known results about the structure of supplemented modules over commutative noetherian rings which are mainly due to Zöschinger and his student Rudlof. In Chapter 3, we study weakly supplemented modules over commutative noetherian rings mainly over noetherian rings with finite Krull dimension. In Section 3.3, under a certain condition we prove that the class of weakly supplemented modules is closed under extensions. As a consequence of this we obtain some conditions equivalent to be weakly supplemented, for modules over Dedekind domains, one dimensional domains and commutative noetherian rings. In Chapter 4, we investigate totally weak supplemented modules. We characterize these modules over Dedekind domains and over semilocal noetherian rings. We also examine and determine the relation between totally weak supplemented, weakly supplemented and supplemented modules over Dedekind domains. In Chapter 5, we deal with cofinitely weak supplemented modules (briefly cws-modules). We give a characterization of cws-modules over (not necessarily commutative) noetherian rings. We characterize $h-semilocal$ domains in terms of cws-modules. For some certain modules over Dedekind domains we give some conditions equivalent to being a cws-module.

1.1 Noetherian rings

In this section we recall the definition of a noetherian ring and state some results. As a consequence of Krull's theorem we see that a commutative noetherian semilocal ring has finite Krull dimension (see Sharp (2000), Büyükaşık (2003)).

Definition 1.1.1. A ring R is said to be noetherian if it satisfies the following three equivalent conditions:

(i) Every non-empty set of ideals in R has a maximal element,

(ii) Every ascending chain of ideals in R is stationary,

(iii) Every ideal in R is finitely generated.

Definition 1.1.2. Let R be a commutative ring. A sequence,

$$P_0 \underset{\neq}{\subsetneq} P_1 \underset{\neq}{\subsetneq} \dots \underset{\neq}{\subsetneq} P_n$$

in which P_i is a prime ideal for all $1 \leqq i \leqq n$, is called a chain of prime ideals of R; n is the length of this chain.

The Krull dimension of R is defined to be

$$\sup\{n \in \mathbf{N} | \text{ there exists a chain of prime ideals of } R \text{ of length } n\}$$

if this suppremum exists, and ∞ otherwise.

Note that a ring has Krull dimension zero if and only if every prime ideal is a maximal ideal.

A ring R is called an *artinian ring* if every descending chain of ideals in R is stationary.

A relation between artinian, noetherian and Krull dimension of a commutative ring is given in the following theorem.

Theorem 1.1.3. *(by Atiyah & Macdonald (1994, Theorem 8.5)) Let R be commutative ring. Then R is artinian if and only if R is noetherian and Krull dimension of R is zero.*

Note that if the zero ideal of a commutative artinian ring R is prime then R must be a field. Therefore every commutative and artinian integral domain is a field.

The following theorem shows that any artinian ring is noetherian and also when the converse also holds.

Theorem 1.1.4. *(by Anderson & Fuller (1992, Theorem 15.20)) Let R be a ring with $\mathrm{Rad}(R) = J$. Then R is left artinian if and only if R is noetherian, J is nilpotent (i.e. $J^n = 0$ for some $n \in \mathbb{Z}^+$), and R/J is semisimple.*

Definition 1.1.5. Let P be a prime ideal of a commutative ring R. Then the height of P , denoted by $\mathrm{ht}(P)$ is defined to be the supremum of the lengths of the chains:

$$P_0 \subset P_1 \subset .... \subset P_n$$

of prime ideals of R for which $P_n = P$.

Let I be a proper ideal of a ring R. A prime ideal P in R is called a *minimal prime* ideal of I if $I \subseteq P$ and there is no prime ideal P' with $I \subsetneqq P' \subsetneqq P$.

Krull's Generalized Principal Ideal Theorem shows that in a commutative noetherian ring every ideal has finite height. Moreover, this height has an upper bound.

Theorem 1.1.6. *Sharp (2000, Theorem 15.4)(Krull's Generalized Principal Ideal Theorem) Let R be a commutative noetherian ring and let I be proper ideal of R which can be generated by n elements. Then $ht(P) \leqq n$ for each minimal prime ideal P of I.*

Remark 1.1.7. Let R be commutative noetherian semilocal ring (i.e. a ring with finitely many maximal ideals) with maximal ideals $P_1, ..., P_k$. Then by Theorem 1.1.6, $\mathrm{ht}(P_i) < \infty$ for every $1 \leqq i \leqq k$. Thus R has finite Krull dimension which is equal to $\sup\{\mathrm{ht}(P_i) | 1 \leqq i \leqq k\}$.

The following Lemma holds over an arbitrary commutative ring.

Lemma 1.1.8. *Sharp (2000, Lemma 3.55) Let P be a prime ideal of a commutative ring R, and let $I_1, ..., I_n$ be ideals in R. Then the following are equivalent.*

(i) $I_j \subseteq P$ *for some* j *with* $1 \leqq j \leqq n$,

(ii) $\bigcap_{i=1}^{n} I_i \subseteq P$,

(iii) $\prod_{i=1}^{n} I_i \subseteq P$.

1.2 Hollow and uniform modules

Definition 1.2.1. A submodule K of a module M is *essential* or *large* in M if $K \cap L \neq 0$ for all non-zero submodules L of M. We will denote essential submodules by $K \trianglelefteq M$. In this case M is called an *essential extension* of K.

In the following definition dual definitions for essential submodules and essential extension are introduced.

Definition 1.2.2. A submodule K of M is *small* in M provided $K + L \neq M$ holds for all proper submodules L of M. We will denote small submodules by $K \ll M$. The *radical* of a module M is the sum of all small submodules of M, equivalently intersection of all maximal submodules of M. We shall denote the radical of M by $\operatorname{Rad} M$ as usual. A module N is a *small cover* of a module M if there exists an epimorphism $f : N \to M$ such that $\operatorname{Ker}(f) \ll N$. N is called a *projective cover* of M if N is a small cover and N is projective.

A module is called *uniform* if every non-zero submodule of M is essential in M. Dually, M is called *hollow* if $M \neq 0$ and every proper submodule of M is small in M.

Definition 1.2.3. A module M is said to have *uniform dimension (or Goldie dimension)* n (written $\operatorname{u.dim} M = n$) if there is an essential submodule $V \subseteq M$ that is a direct sum of n uniform submodules. If no such an integer n exist, we write $\operatorname{u.dim} M = \infty$.

For a torsion-free abelian group G uniform dimension and (torsion-free) rank of G coincide (see, Fuchs (1970, Lemma 16.1, 16.2)). For any torsion-free module M over a domain, as in abelian groups, torsion-free rank of M is defined and this also coincides with uniform dimension of M.

Proposition 1.2.4. *(by Lam (1999, Exercise 6.10)) Let R be a domain with quotient field K. For any R-module M,*

$$\dim_K M \otimes_R K = \operatorname{u.dim}(M/T(M));$$

this number is called the "torsion-free rank" of M*. If* $\mathrm{u.dim}\, T(M) < \infty$*, then torsion-free rank of* M *is* $\mathrm{u.dim}\, M - \mathrm{u.dim}\, T(M)$*.*

We will use the uniform dimension only for torsion-free modules over domains. The following formula will be helpful for our task.

Proposition 1.2.5. *(by Lam (1999, Proposition 6.14)) Let* R *be a domain with quotient field* K*. For any torsion-free* R*-module* M*, we have*

$$\mathrm{u.dim}\, M = dim_K(M \otimes_R K).$$

Proof. Since M is torsion-free we may think of M as embedded in $M \otimes_R K$. First assume $\mathrm{u.dim}\, M = \infty$. In this case, M contains $U_1 \oplus U_2 \oplus \ldots$ with $U_i \neq 0$, so $M \otimes_R K$ contains $(U_1 \otimes_R K) \oplus (U_2 \otimes_R K) \ldots$; hence $\dim_K(M \otimes_R K) = \infty$. Now, assume that $\mathrm{u.dim}\, M < \infty$. Then same argument as above gives $\dim_K(M \otimes_R K) \geqslant n$. If this is a strict inequality, there would exist a direct sum

$$V_1 \oplus \ldots \oplus V_{n+1} \subseteq M \otimes_R K,$$

where the V_i's are nonzero K-subspaces. Then we see that the $M \cap V_i$'s are nonzero R-submodules of M, and

$$(M \cap V_1) \oplus \ldots \oplus (M \cap V_{n+1}) \subseteq M$$

gives a contradiction. Therefore $\dim_K(M \otimes_R K) = n$. □

From now on without any ambiguity, for torsion-free modules, we will use the uniform dimension and the torsion-free rank instead of each other.

1.3 Local and semilocal rings

A module is *local* if it has a greatest proper submodule. Equivalently, a module is local if and only if it is cyclic, non-zero, and has a unique maximal proper submodule.

Proposition 1.3.1. *(by Facchini (1998, Proposition 1.10)) The following conditions are equivalent for a ring R.*

(i) $R/\operatorname{Rad}(R)$ *is a division ring,*

(ii) ${}_RR$ *is a local module (that is, R has a unique maximal proper left ideal),*

(iii) the sum of two non-invertible elements of R is non-invertible,

(iv) $\operatorname{Rad}(R)$ *is a maximal left ideal,*

(v) $\operatorname{Rad}(R)$ *is the set of all non-invertible elements of R.*

If R is a ring satisfying the equivalent conditions of Proposition 1.3.1, R is called a *local ring.*

Definition 1.3.2. A ring R is said to be *semilocal* if $R/\operatorname{Rad}(R)$ is a semisimple ring.

Proposition 1.3.3. *(by Lam (2001, Proposition 20.2)) For a ring R, consider the following two conditions:*

(i) R is semilocal,

(ii) R has finitely many maximal ideals.

We have, in general, $(ii) \Rightarrow (i)$*. The converse holds if* $R/\operatorname{Rad}(R)$ *is commutative.*

From Proposition 1.3.3 we see that, a commutative ring R is semilocal if and only if R has finitely many maximal ideals.

Let R be a semilocal ring with Jacobson radical J. Then for every R-module M we have $\operatorname{Rad} M = JM$. Thus $M/\operatorname{Rad} M$ is a semisimple R/J-module. Hence $M/\operatorname{Rad} M$ is a semisimple R-module (see, Anderson & Fuller (1992)).

1.4 Coclosed submodules

A submodule N of a module M is said to be *closed* if N has no proper essential extension in M, i.e. if $N \trianglelefteq K$ for some $K \subseteq M$ then $K = N$. Dually we define:

Definition 1.4.1. A submodule N of a module M is *coclosed* in M, if whenever $N/K \ll M/K$ for some $K \subseteq M$ implies that $N = K$.

A submodule N of an R-module M is called a *complement* of a submodule L in M if it is maximal with respect to $N \cap L = 0$. By Zorn's Lemma every submodule has a complement. A submodule is a complement if and only if it is closed in M (see, Dung et al. (1994, pp.6)).

As a dual notion of complements we define notion of supplements.

Definition 1.4.2. Let N and L be submodules of M, we call N a *supplement* of L in M if N is minimal with respect to $N + L = M$. Equivalently N is a supplement of L if and only if $N + L = M$ and $N \cap L \ll N$ (see,Zöschinger (1974a)). A submodule is called a supplement in M if it is a supplement of some submodule in M. Following Zöschinger (1978) we call N a *weak supplement* of L in M if $N + L = M$ and $N \cap L \ll M$. N is called a weak supplement in M if N is a weak supplement of some submodule of M. It is clear from definitions that every supplement is a weak supplement.

The classes of complements and closed submodules are the same. The relation between supplements and coclosed submodules is given in the following proposition (see, Keskin (2000b, Lemma 1.1), or Lomp (1996, Proposition 1.2.1)).

Proposition 1.4.3. *Let N be a submodule of M. Consider the following statements:*

(i) N is a supplement in M,

(ii) N is coclosed in M,

(iii) for all $K \subseteq N$, $K \ll M$ implies $K \ll N$.

Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) holds and if N is a weak supplement in M, then (iii) $\Rightarrow$ (i) holds

Definition 1.4.4. An R-module M is said to be *supplemented* if every submodule has a supplement in M. M is called *amply supplemented* if for every submodules N and L of M with $N + L = M$, N contains a supplement of L in M. Clearly amply supplemented modules are supplemented.

1.5 Weak supplements

Following Zöschinger (1978) we say that M is *weakly supplemented* if every submodule of M has a weak supplement. A module is called *semilocal* if $M/\operatorname{Rad} M$ is semisimple.

Proposition 1.5.1. *(by Lomp (1999, Proposition 2.1)) Let M be an R-module and N a proper submodule of M. The following statements are equivalent.*

(i) M/N is semisimple,

(ii) *for every* $L \subseteq M$ *there exists a submodule* $K \subseteq M$ *such that* $L + K = M$ *and* $L \cap K \subseteq N$,

(iii) *there exists a decomposition* $M = M_1 \oplus M_2$ *such that* M_1 *is semisimple,* $N \trianglelefteq M_2$ *and* M_2/N *is semisimple.*

In the following Proposition we collect some results for weakly supplemented modules.

Proposition 1.5.2. *(by Lomp (1999, Proposition 2.2)) Assume* M *to be weakly supplemented. Then,*

(i) M *is semilocal,*

(ii) $M = M_1 \oplus M_2$ *with* M_1 *semisimple,* M_2 *semilocal and* $\operatorname{Rad} M \trianglelefteq M_2$

(iii) *every factor module of* M *is weakly supplemented,*

(iv) *any small cover of* M *is weakly supplemented,*

(v) *every supplement in* M *and every direct summand of* M *is weakly supplemented.*

1.6 Coatomic modules

A submodule N of a module M is said to be *radical* if $\operatorname{Rad} N = N$.
Let $P(M) = \sum\{N \subseteq M \mid \operatorname{Rad} N = N\}$. The module M is called *reduced* if $P(M) = 0$.

Definition 1.6.1. A module M is said to be *coatomic* if $\operatorname{Rad}(M/U) \neq M/U$ for every proper submodule U of M or equivalently every proper submodule of M is contained in a maximal submodule of M.

It is clear from the definition that every factor module of a coatomic module is coatomic. Finitely generated modules and semisimple modules are coatomic. Note that for every coatomic module M we have $\operatorname{Rad} M \ll M$.

In general the class of coatomic modules is not closed under submodules as the following example shows.

Example 1.6.2. Consider the ring,

$$R = \{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a,\, c \in \mathbb{Z},\, b \in \mathbb{Q} \}$$

Then ${}_R R$ is coatomic as it is finitely generated. Consider the submodule

$$M = \begin{pmatrix} 0 & \mathbb{Q} \\ 0 & 0 \end{pmatrix}$$

The left R-module structure of M is completely determined by the left $\mathbb{Z}$-module structure of $\mathbb{Q}$. Then M is not coatomic since ${}_{\mathbb{Z}}\mathbb{Q}$ is not coatomic: $\mathbb{Z} \subseteq \mathbb{Q}$ is a proper submodule and ${}_{\mathbb{Z}}\mathbb{Q}$ has no maximal submodule.

A module M is *injective* if for every monomorphism $f : A \to B$ and homomorphism $g : A \to M$ there exists a homomorphism $h : B \to M$ such that $h \circ f = g$.

We shall show that over a commutative noetherian ring every submodule of a coatomic module is coatomic. The proof of this is based on the following theorem due to Matlis.

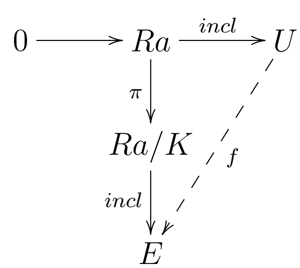
Theorem 1.6.3. *(by Matlis (1960, Proposition 3)) Let R be a commutative noetherian ring and M be an R-module. Then the following are equivalent.*

(i) M is artinian,

(ii) M is a submodule of $E_1 \oplus ... \oplus E_n$, where $E_i = E(R/P_i)$ with P_i a maximal ideal of R.

Lemma 1.6.4. *For every non-zero module U there exists a nonzero homomorphism $f : U \to E$, where E is the injective hull of a simple module.*

Proof. Let $0 \neq a \in U$. Then Ra has a maximal submodule, say K. Then Ra/K is a simple module. Consider the injective hull $E = E(Ra/K)$ of Ra/K. Since E is injective we have the following commutative diagram:



Observe that $f(a) \neq 0$.

□

Theorem 1.6.5. *(by Zöschinger (1980, Lemma 1.1)) Let M be a coatomic module over a commutative noetherian ring. Then every submodule of M is coatomic.*

Proof. Suppose U has a nonzero radical factor module,then by Lemma 1.6.4 there exists a nonzero homomorphism $f : U \to E$ with E injective hull of a simple module. Then $\mathrm{Im}(f)$ is also radical. By Theorem 1.6.3 E is artinian. Hence every coatomic submodule T of E is finitely generated (since $T/\operatorname{Rad} T$ is finitely generated and $\operatorname{Rad} T \ll T$). Since E is injective we have a homomorphism $g : M \to E$ with $\mathrm{Im}(f) \subseteq \mathrm{Im}(g)$. Then $\mathrm{Im}(g)$ is a coatomic submodule of E, hence $\mathrm{Im}(g)$ is finitely generated. But then $\mathrm{Im}(f)$ is finitely generated, contradiction. Hence U is a coatomic submodule of M. □

1.7 Dedekind domains

By an (integral) domain we will mean a commutative ring without zero divisors. Let R be such a ring and M be an R-module. The submodule $T(M) = \{m \in M \mid rm = 0 \text{ for some } 0 \neq r \in R\}$ of M is called *torsion submodule* of M. If $T(M) = M$ then M is said to be a torsion module, and if $T(M) = 0$ then M is said to be a *torsion-free* module. Let P be a prime ideal of R. The submodule $\{m \in M \mid P^n m = 0 \text{ for some } n \geq 1\}$ is said to be *P-primary part* of M. This submodule usually is denoted by $T_P(M)$.

By R_S, M_S we denote the localization of R and M respectively at the multiplicatively closed set $S \subseteq R$, (see Fuchs & Salce (2001) or Cohn (2002)).

A commutative ring R is a *valuation ring* if its ideals are totally ordered by inclusion. If, in addition, R is an integral domain, it is said to be a *valuation domain*. A noetherian valuation domain is said to be *discrete valuation ring (DVR)* . If R is a DVR then all its non-zero ideals are: $R > Rp > ... > Rp^n > ...$ for some $p \in R$, (see (Fuchs & Salce, 1985, Proposition 1.7 (b))).

A DVR is said to be *complete* if it is complete in its p-adic topology, see, for example, Kaplansky (1965) or Sharpe & Vamos (1972).

Let R be an integral domain, K be its field of fractions. An element of K is said to be *integral* over R if it is a root of a monic polynomial in $R[X]$. A commutative domain R is *integrally closed* if the elements of K which are integral over R are just the elements of R.

An integral domain satisfying any of the equivalent conditions in the following theorem is said to be a *Dedekind domain* .

Theorem 1.7.1. *(by Cohn (2002, Propositions 10.5.1,4,6)) For a commutative domain R, the following are equivalent:*

(i) Every ideal of R is projective,

(ii) R is Noetherian and R_P is a DVR for all maximal ideals P of R,

(iii) R is Noetherian, integrally closed and every nonzero prime ideal of R is maximal,

(iv) Every ideal of R can be expressed uniquely as a finite product of maximal ideals,

(v) Every ideal of R can be expressed as a finite product of prime ideals.

A module M over an integral domain is *divisible* if $rM = M$ for every $0 \neq r \in R$.

Every injective module is divisible. Over Dedekind domains these notions coincide, moreover we have the following (see, Alizade et al. (2001, Lemma 4.4) and Sharpe & Vamos (1972, Proposition 2.10)).

Lemma 1.7.2. *Let R be a Dedekind domain. For an R-module M the following are equivalent:*

(i) M is injective,

(ii) M is divisible,

(iii) $M = PM$ for every maximal ideal P of R,

(iv) M does not contain any maximal submodule.

Theorem 1.7.3. *(by Cohn (2002, Propositions 10.6.6)) A finitely generated module over a Dedekind domain is projective if and only if it is torsion-free.*

Theorem 1.7.4. *(by Cohn (2002, Propositions 10.6.7)) Let M be a finitely generated module over a Dedekind domain. Then $M = T(M) \oplus P$ where P is a torsion-free submodule of M and $T(M)$ is the torsion submodule.*

Theorem 1.7.5. *(by Cohn (2002, Propositions 10.6.8)) Any finitely generated torsion module over a Dedekind domain is a direct sum of cyclic modules.*

For a ring R let Ω be the set of all maximal ideals of R.

Theorem 1.7.6. *(by Cohn (2002, Propositions 10.6.9)) Any torsion module M over a Dedekind domain is a direct sum of its primary parts, in a unique way:*

$$M = \bigoplus_{P \in \Omega} T_P(M)$$

and when M is finitely generated, only finitely many terms on the right are different from zero.

1.8 Main results of the thesis

In chapter 2, we review some results for supplemented modules over commutative noetherian rings. The results which are obtained in this chapter are mainly based on a Theorem due to Rudlof (Theorem 2.3.1).

Zöschinger has proved that over a DVR a module M is supplemented if and only if $\operatorname{Rad} M$ is supplemented (Theorem 2.1.1). As a consequence of a Theorem due to Rudlof (Theorem 2.3.1) we show that Theorem 2.1.1 holds for all modules over commutative semilocal noetherian rings (Theorem 2.3.6). More generally, we obtain that a commutative semilocal noetherian ring R with Jacobson radical J is artinian if and only if $\operatorname{Rad} M$ is supplemented for every R-module M if and only if for every R-module M, $J^n M$ is supplemented for some $n \in \mathbb{Z}^+$.
If R is a Dedekind domain then an R-module M is supplemented if and only if $I^n M$ is supplemented for some nonzero ideal I of R and $n \geqq 0$ (Corollary 2.3.12).

It is well known that a ring R is left perfect if and only if every left R-module is supplemented (see, Mohamed & Müller (1990)). We obtain that a semilocal noetherian ring R with Jacobson radical J is artinian if and only if for every R-module M there exists $n \geqq 0$ such that $J^n M$ is supplemented (Corollary 2.3.7).

It is well known that supplemented modules are not closed under submodules (see, Zöschinger (1974a)).We obtain that a reduced module is supplemented if and only if every submodule is supplemented (Corollary 2.3.18).

In Chapter 3, we study weakly supplemented modules. In Section 3.1 we present a Theorem due to Zöschinger, in which characterization of weakly supplemented is given for modules over a Dedekind domain $R \neq K$ (Theorem 3.1.1). As a consequence of this we show that a torsion R-module is weakly supplemented

if and only if it is supplemented (Corollary 3.1.5).
In addition, if R is semilocal then an R-module M is weakly supplemented if and only if $T(M)$ and $M/T(M)$ are weakly supplemented (Corollary 3.1.6). If R is a DVR then an R-module is weakly supplemented if and only if it can be embedded in a supplemented module (Corollary 3.1.7).

In Section 3.3 we examine whether the class of weakly supplemented modules is closed under extensions i.e. if U and M/U are weakly supplemented for a module M and a submodule U of M then M is weakly supplemented. We give an example in order to show that this need not be true in general (Example 3.3.6). Under a certain condition we prove that the class of weakly supplemented modules is closed under extension (Theorem 3.3.1). Then Proposition 3.3.7, Corollary 3.3.8 and Proposition 3.3.10 are proved as a consequence of Theorem 3.3.1.

In Chapter 4, we investigate totally weak supplemented modules (or briefly tws-modules). In Section 4.1, we determine the structure of tws-modules over a Dedekind domain $R \neq K$. If R is semilocal, then a module is a tws-module if and only if it is weakly supplemented (Theorem 4.1.5). If R is non-semilocal then a module is a tws-module if and only if it is supplemented (Corollary 4.1.16).

In Theorem 4.1.21, we determine the explicit structure of a weakly supplemented module over a DVR.

Rudlof proved that if R is a complete DVR then every weakly supplemented module is supplemented. With the help of Theorem 4.1.21 we show that: If R is a DVR and every weakly supplemented R-module is supplemented then R is complete.

Over (non-semilocal) Dedekind domains the class of weakly supplemented is strictly larger than the class of tws-modules (see, Remark 4.1.6). We prove that a finitely generated module is weakly supplemented if and only it is a tws-module (Theorem 4.1.5, Corollary 4.1.34).

In Section 4.2, We give some sufficient conditions equivalent to being a tws-module over a semilocal noetherian ring (not necessarily commutative) (Theorem 4.2.2). For a module over a commutative semilocal noetherian ring some conditions equivalent to being a tws-module are proved (Theorem 4.2.3).

Lomp has proved that a ring R is semilocal if and only if every R-module with small radical is weakly supplemented (Lomp (1999, Theorem 3.5)). For a commutative noetherian ring R we show that R is semilocal if and only if every R-module with small radical is a tws-module (Corollary 4.2.11).

In Chapter 5, we study cofinitely weak supplemented modules (briefly cws-modules) over noetherian rings. In Section 5.2 we prove that a module M is a cws-module if and only if M/U is a cws-module for some submodule $U \subseteq \operatorname{Rad} M$ (Theorem 5.2.2). This leads to the characterization of cws-modules given in Corollary 5.2.4.

For a module M with $\operatorname{Rad} M \ll M$ we prove that M is finitely weak supplemented if and only if M is weakly supplemented if and only if every cyclic submodule of M has a weak supplement in M (Corollary 5.2.7).
If every cyclic submodule of $M/\operatorname{Rad} M$ is a direct summand then M is a finitely weak supplemented module (Proposition 5.2.8).

In Section 5.3, it is shown that a domain R is $h-semilocal$ if and only if every torsion R-module is a cws-module (Theorem 5.3.2). As a consequence, for some certain modules over Dedekind domains, we prove some conditions equivalent to being a cws-module (Corollary 5.3.4 and Corollary 5.3.5).

CHAPTER TWO
SUPPLEMENTED MODULES

In this chapter we will review some results on supplemented modules over some commutative rings including Dedekind domains and commutative noetherian rings. For more general results on supplemented modules we refer to Wisbauer (1991).

2.1 Supplemented modules over discrete valuation rings

In this section R is a discrete valuation ring (DVR), unless otherwise stated.

Theorem 2.1.1. *(by Zöschinger (1974a, Lemma 2.1)) For an $R-$module M, the following are equivalent:*

(i) *M has a small radical,*

(ii) *M is coatomic,*

(iii) *M is a direct sum of a finitely generated free submodule and a bounded submodule,*

(iv) *M is reduced and supplemented.*

Note that property (*iii*) is inherited by submodules. Hence a reduced module is supplemented if and only if every submodule is supplemented.

The following theorem gives the complete structure of a supplemented module. Recall that DVR is a local ring and its unique maximal ideal is of the form Rp for some $p \in R$.

Theorem 2.1.2. *(by Zöschinger (1974a, Theorem 2.4)) A module M is supplemented if and only if $M = M_1 \oplus M_2 \oplus M_3 \oplus M_4$ where $M_1 \cong R^{n_1}, M_2 \cong K^{n_2}, M_3 \cong (K/R)^{n_3}$ and $p^{n_4}M_4 = 0$ for some integer $n_i \geq 0$.*

Theorem 2.1.3. *(by Zöschinger (1974a, Lemma 2.2 and Lemma 2.5)) For a module M the following hold.*

(i) $\operatorname{Rad} M$ *is supplemented if and only if M is supplemented.*

(ii) M is supplemented if and only if $T(M)$ and $M/T(M)$ are supplemented.

Theorem 2.1.4. *(by Zöschinger (1974a, Satz 2.6)) For a module M, the following are equivalent:*

(i) M can be embedded in a supplemented module,

(ii) M is an extension of a supplemented module by a supplemented module,

(iii) Every $U \subseteq M$ with $U \subseteq \operatorname{Rad} M$ has a supplement in M,

(iv) The torsion part of M is supplemented, and $M/T(M)$ has finite rank.

Note that any supplemented module satisfies the equivalent conditions of Theorem 2.1.4. But the converse is not true, unless R is complete (see, (Zöschinger, 1974b, Theorem 2.2)).

In general the class of supplemented modules is not closed under submodules see (Zöschinger, 1974b, Satz 2.1 Bemerkung). If every submodule of a module M is supplemented M is said to be *totally supplemented* (see, Rudlof (1991)). Totally supplemented modules are characterized in the following theorem.

Theorem 2.1.5. *(by Zöschinger (1974a, Folgerung p.51)) For a DVR, the following are equivalent:*

(i) Every submodule of a supplemented module is supplemented,

(ii) Every extension of a supplemented module by a supplemented module is itself supplemented,

(iii) If every submodule of the radical of a module M has a supplement then M is supplemented,

(iv) Every torsion free reduced module with finite rank is free.

The conditions in 2.1.5 are also equivalent to R is a complete DVR, (see, Zöschinger (1974a)).

2.2 Supplemented modules over Dedekind domains

Throughout this section R is a Dedekind domain with quotient field $K \neq R$.

Note that Lemma 2.2.1, Lemma 2.2.2 and Lemma 2.2.4 are stated without proof in the proof of Theorem 3.1 in Zöschinger (1974a).

Lemma 2.2.1. *If the R-module ${}_RK$ is supplemented then it is hollow.*

Proof. Let U be a submodule of ${}_RK$ and V be a supplement of U. Since V is a supplement we have $\operatorname{Rad} V = V \cap \operatorname{Rad} K = V \cap K = V$ i.e. V does not contain any maximal submodule. Then by Lemma 1.7.2, V is injective. Therefore V is a direct summand of ${}_RK$. But ${}_RK$ is indecomposable, so $V =_R K$. Now suppose $U + L =_R K$ for some submodule L of ${}_RK$. Then by minimality of ${}_RK$ we must have $L =_R K$. Hence $U \ll_R K$. Therefore ${}_RK$ is hollow. □

Lemma 2.2.2. *If the R-module ${}_RK$ is hollow then R is local.*

Proof. Since hollow modules are closed under factor modules K/R is hollow. By Theorem 1.7.6 we have,

$$K/R = \bigoplus_{P\in\Omega} T_P(K/R).$$

As K/R is hollow we must have $|\Omega| = 1$, i.e. R is local. □

Remark 2.2.3. K/R is a cogenerator in the category of R-modules (see, Wisbauer (1991)).

Lemma 2.2.4. *Every supplement of a maximal submodule of a module M is a local module.*

Proof. Let K be a maximal submodule of M and L be a supplement of K in M i.e. $K + L = M$ and $K \cap L \ll L$. Then we have:

$$M/K \cong L/(K \cap L).$$

Since M/K is simple, $K \cap L$ is a maximal submodule of L. Then $Rx + K \cap L = L$ for some $x \in L$. Since $K \cap L \ll L$, we have $Rx = L$. If T is a maximal submodule of L then $K \cap L \subseteq T$ because $K \cap L \ll L$, so we must have $K \cap L = T$ i.e. L has a unique maximal submodule. Therefore L is a local module. □

Theorem 2.2.5. *(by Zöschinger (1974a, Theorem 3.1)) Let R be a non-local Dedekind domain. An $R-$module M is supplemented if and only if it is torsion and every primary part is supplemented.*

Proof. ($\Rightarrow$) Let R be a non-local Dedekind domain and ${}_RM$ be supplemented. Then the module $M/T(M)$ is radical: If U is a maximal submodule of M with $T(M) \subseteq U$, then U has a supplement V in M. Since U is maximal, V is a local module, hence it is cyclic i.e. $V \cong R/I$. On the other hand since R is non-local

$I \neq 0$ i.e. V is a torsion submodule so $V \subseteq T(M)$, a contradiction. Hence $M/T(M)$ is radical i.e. it has not any maximal submodule. Then by Lemma 1.7.2 $M/T(M)$ is injective, and since it is torsion-free, we have:

$$M/T(M) \cong K^{(J)}.$$

Then $K^{(J)}$ is also supplemented as it is isomorphic to a factor module of a supplemented module. Then K is also supplemented as a direct summand of a supplemented module. In this case by Lemma 2.2.1 and Lemma 2.2.2 we have $J = \emptyset$. Therefore M is torsion, and so

$$M = T(M) = \bigoplus_{P \in \Omega} T_P(M)$$

every primary component is a direct summand, hence supplemented.

($\Leftarrow$) Conversely suppose ${}_RM$ is torsion and every primary component M_P is supplemented. Then since M is torsion, M has the decomposition

$$M = \bigoplus_{P \in \Omega} M_P$$

and this decomposition has the property that

$$X = \bigoplus_{P \in \Omega} X \cap M_P$$

for every $X \subseteq M$. Let $U \subseteq M$. Then since M_P is supplemented for every $P \in \Omega$, $U \cap M_P$ has a supplement V_P in M_P. Then $\sum_{P \in \Omega} V_P$ is a supplement of U in M:

$$U = \bigoplus_{P \in \Omega} U \cap M_P, \quad V = \bigoplus_{P \in \Omega} V_P.$$

Then

$$U + V = \bigoplus_{P \in \Omega} (U \cap M_P + V_P) = \bigoplus_{P \in \Omega} M_P = M.$$

Suppose $U \cap V + X = V = \bigoplus_{P\in\Omega} V_P$ for some submodule X of V. Then we have

$$\bigoplus_{P\in\Omega} U \cap M_P + \bigoplus_{P\in\Omega} X_P = \bigoplus_{P\in\Omega} V_P$$

So we get

$$\bigoplus_{P\in\Omega} (U \cap M_P + X_P) = \bigoplus_{P\in\Omega} V_P.$$

Then

$$U \cap M_P + X_P = V_P.$$

Therefore $X_P = V_P$ for all $P \in \Omega$, which means that $X = \bigoplus_{P\in\Omega} V_P = V$. □

Theorem 2.2.6. *(by Zöschinger (1974a, Satz 3.4)) For a noetherian domain R, the following are equivalent.*

(i) Every coclosed module is closed,

(ii) Every closed module is coclosed,

(iii) R is Dedekind.

2.3 Supplemented modules over noetherian rings

Throughout this section R is a commutative noetherian ring.

Theorem 2.3.1. *(by Rudlof (1991, Proposition 2.6)) Let $U \subseteq M$ be a submodule such that M/U is reduced. The following statements are equivalent.*

(i) M is supplemented,

(ii) U and M/U are supplemented.

Since coatomic modules are closed under factor modules and submodules (over noetherian rings) we get the following corollary.

Corollary 2.3.2. *Let M a coatomic R-module and $U \subseteq M$. Then M is supplemented if and only if U and M/U are supplemented.*

Proof. Clearly M/U is coatomic. Then by Theorem 1.6.5 every submodule of M/U is coatomic. Therefore every submodule of M/U contains a maximal submodule i.e. $P(M/U) = 0$. This means that M/U is reduced. Now the proof is clear by Theorem 2.3.1. □

Definition 2.3.3. A subset A of a ring R is called *left T-nilpotent* if, for any sequence of elements $\{a_1, a_2, ...\} \subseteq A$, there exists an integer $n \geq 1$ such that $a_1 \cdot a_2 \dots a_n = 0$.

A ring R is a *left perfect* ring if $R/\operatorname{Rad}(R)$ is semisimple and $\operatorname{Rad}(R)$ is left T-nilpotent.

Theorem 2.3.4. *(by Mohamed & Müller (1990, Theorem 4.41)) The following are equivalent for a ring R.*

(i) R is left perfect,

(ii) Every R-module is supplemented,

(iii) Every free R-module is supplemented.

If R is a left perfect ring then by (Anderson & Fuller, 1992, Theorem 28.4) every non-zero left R-module contains a maximal submodule i.e. $P(M) = 0$ for every R-module and this implies that every R-module is reduced. Every left artinian ring is perfect (Anderson & Fuller, 1992, Corollary 28.8).

Lemma 2.3.5. *If R is a noetherian perfect ring then R is artinian.*

Proof. Let J be the Jacobson radical of R. Since R is noetherian, J is finitely generated, say $J = Ra_1 + Ra_2 + ... + Ra_n$ for some $a_i \in J$. Now since R is a perfect ring, for every a_i, $i = 1, ..., n$ there exist k_i such that $a_i^{k_i} = 0$. Therefore for sufficiently large $m \in \mathbb{Z}^+$ we have $J^m = 0$. Then by Anderson & Fuller (1992, Theorem 15.20) R is artinian. □

In Theorem 2.1.3 we have seen that over a DVR a module M is supplemented if and only if $\operatorname{Rad} M$ is supplemented. By using Theorem 2.3.1 we obtain the following.

Theorem 2.3.6. *Let R be a commutative noetherian semilocal ring with maximal ideals $P_1, \ldots, P_n$, Jacobson radical J and M be an R-module. The following are equivalent:*

(i) M *is supplemented.*

(ii) $P_1^{k_1} \ldots P_n^{k_n} M$ *is supplemented for some* $k_i \geqslant 0$.

(iii) $J^m M$ *is supplemented for some* $m \geqq 0$.

(iv) $\operatorname{Rad} M$ *is supplemented.*

Proof. (i)⇔(ii) The ring $R/P_1^{k_1} \ldots P_n^{k_n}$ is noetherian as R is noetherian, and every prime ideal is maximal: for if $P/P_1^{k_1} \ldots P_n^{k_n}$ is a prime ideal of $R/P_1^{k_1} \ldots P_n^{k_n}$ then P is a prime ideal of R and $P_1^{k_1} \ldots P_n^{k_n} \subseteq P$, hence by Lemma 1.1.8, $P_i \subseteq P$ for some $i = 1, ..., n$, i.e. P is a maximal ideal of R, hence $P/P_1^{k_1} \ldots P_n^{k_n}$ is a maximal ideal of $R/P_1^{k_1} \ldots P_n^{k_n}$. Therefore the ring $R/P_1^{k_1} \ldots P_n^{k_n}$ is artinian by Theorem 1.1.3. In this case $M/P_1^{k_1} \ldots P_n^{k_n} M$ is a reduced and supplemented R-module. Then Therorem 2.3.1 completes the proof.

(i)⇔(iii) The Jacobson radical of R is $J = P_1 \cap \ldots \cap P_n = P_1 \ldots P_n$ by Atiyah & Macdonald (1994, Proposition 1.10 (i)). Then $J^m = P_1^m \ldots P_n^m$ and from the proof of (i)⇔(ii) we get the ring R/J^m is artinian. Thus M/J^mM is reduced and supplemented. Then Theorem 2.3.1 completes the proof.

(i)⇔(iv) Since R is semilocal, R/J is semisimple and $\operatorname{Rad} M = JM$. Then M/JM is a semisimple and hence a supplemented R-module. Then we are done by Theorem 2.3.1. □

Corollary 2.3.7. *Let R be a semilocal ring with Jacobson radical J. Then the following are equivalent.*

(i) R is an artinian ring,

(ii) for every R-module M there exists $n \in \mathbb{N}$ such that J^nM is supplemented,

(iii) for every R-module M, $\operatorname{Rad} M$ is supplemented,

(iv) every R-module is supplemented.

Proof. (i)⇒(ii) Clear, since R is a perfect ring.

(ii)⇒(iii) By Theorem 2.3.6.

(iii)⇒(i) Let M be an R-module. Then by Theorem 2.3.6 M is supplemented. Therefore R is a perfect ring. Hence R is artinian, by Lemma 2.3.5.

(iv)⇔(i) By Theorem 2.3.4 □

To prove Corollary 2.3.7, for an arbitrary ring we shall assume $\operatorname{Rad} S = R$ for some R-module S. Rings with this property are considered in Generalov (1983).

Corollary 2.3.8. *Let R be a ring. Suppose* $\operatorname{Rad} S = R$ *for some R-module S. Then the following statements are equivalent.*

(i) R is artinian,

(ii) $\operatorname{Rad} M$ *is supplemented for every R-module M.*

Proof. (i)⇒(ii) By Theorem 2.3.4.

(ii)⇒(i) By hypothesis R is supplemented. Then R is a semiperfect ring (see, Wisbauer (1991, 42.6)). Therefore R is semilocal. Then R is an artinian ring by Corollary 2.3.7. □

We shall call the domain R *one dimensional* if R is not a field and R/I is artinian for every non-zero ideal I of R. The domain R is *h-semilocal* if R/I is a semilocal ring for every nonzero ideal of R.

Lemma 2.3.9. *A domain R is one dimensional if and only if R is noetherian and every non-zero prime ideal of R is maximal.*

Proof. (⇒) Let

$$I_1 \subseteq I_2 \subseteq \dots$$

be an ascending chain of ideals of R. If $I_i = 0$ for every $i \in \mathbb{Z}^+$, then there is nothing to prove. Suppose $I_i \neq 0$ for some $i \in \mathbb{Z}^+$. Since R is one dimensional R/I_i is artinian, hence R/I_i is noetherian by Theorem 1.1.3. Therefore the chain of ideals

$$I_{i+1}/I_i \subseteq I_{i+2}/I_i \subseteq ...$$

of R/I_i is stationary, which implies that the chain

$$I_1 \subseteq I_2 \subseteq \dots$$

is stationary. Hence R is noetherian.

If P is a non-zero prime ideal of R then R/P is an integral domain. By hypothesis

R/P is an artinian ring, then by Theorem 1.1.3 R/P is a field i.e. P is a maximal ideal.

($\Leftarrow$) Let I be a nonzero ideal of R. Then R/I is noetherian and every prime ideal in R/I is maximal i.e. Krull dimension of R/I is zero. Then by Theorem 1.1.3 R/I is artinian. □

Note that by Theorem 1.7.1 and Lemma 2.3.9 Dedekind domains are one dimensional.

Theorem 2.3.10. *Let R be a noetherian domain. Consider the following statements:*

(i) R is one dimensional,

(ii) for every non-zero ideal I of R every R/I-module is supplemented,

(iii) for every non-zero ideal I of R, and for every R/I-module M, $\operatorname{Rad} M$ is supplemented.

Then (i)$\Leftrightarrow$(ii)$\Rightarrow$(iii), and if R is $h-semilocal$ (iii)$\Rightarrow$(ii) hold.

Proof. (i)$\Rightarrow$(ii) Let I be a nonzero ideal of R. Then R/I is an artinian ring because R is one dimensional. By Theorem 2.3.4 every R/I-module is supplemented.

(ii)$\Rightarrow$(i) Let I be a nonzero ideal of R. By hypothesis every R/I-module is supplemented. Then R/I is a perfect ring by Theorem 2.3.4, so by Lemma 2.3.5 R/I is artinian. Hence R is one dimensional.

(ii)$\Rightarrow$(iii) Let I be a nonzero ideal of R. By hypothesis R/I is supplemented. Then R/I is a semiperfect ring (see, Wisbauer (1991) 42.6). Hence R/I is semilocal. Therefore by Theorem 2.3.6 Rad M is supplemented for every R/I-module M.

(iii)⇒(ii) If R is h-semilocal and I is a nonzero ideal of R then R/I is semilocal. Let M be an R/I-module. Then by hypothesis Rad M is supplemented. Therefore by Theorem 2.3.6 M is supplemented. □

Proposition 2.3.11. *Let R be a one dimensional domain and M be an R-module. Suppose $K \subseteq M$ and $r^n M \subseteq K$ for some $0 \neq r \in R$ and $n \geqq 0$. Then K is supplemented if and only if $r^n M$ is supplemented.*

Proof. Clearly $(Rr^n)M \subseteq K$. Since R is a domain $r^n \neq 0$ for every $n > 0$, and by hypothesis R/Rr^n is artinian. Hence $K/(Rr^n)M$ is a reduced and supplemented R-module. Then the proof is clear by Theorem 2.3.1. □

Corollary 2.3.12. *Let R be a one dimensional domain and M be an R-module. Then M is supplemented if and only if $r^n M$ is supplemented for some $0 \neq r \in R$ and $n \geqq 0$.*

Corollary 2.3.13. *Let R be a one dimensional and I be a nonzero ideal of R. The following are equivalent.*

(i) M is supplemented,

(ii) every submodule $K \subseteq M$ with $I^n M \subseteq K$ for some $n \geqq 0$ is supplemented.

In a Dedekind domain every nonzero ideal is a product of finitely many maximal ideals (see, Theorem 1.7.1). Hence we have the following corollary.

Corollary 2.3.14. *Let R be a Dedekind domain and M be an R-module. Then M is supplemented if and only if $r^n M$ is supplemented for some $0 \neq r \in R$ and $n \geqq 0$.*

Proposition 2.3.15. *Let M be a semilocal module. Then M is supplemented if and only if* $\operatorname{Rad} M$ *is supplemented.*

Proof. Since M is semilocal $M/\operatorname{Rad} M$ is semisimple. Semisimple modules are reduced and supplemented. So the proof is clear by Theorem 2.3.1. □

Weakly supplemented modules are semilocal. So we obtain the following corollary.

Corollary 2.3.16. *Let M be a weakly supplemented module then M is supplemented if and only if* $\operatorname{Rad} M$ *is supplemented.*

Theorem 2.3.17. *(by Zöschinger (1982b, Lemma 2.6)) If M is a supplemented module then every submodule X of M with $P(M) \subseteq X \subseteq M$ is supplemented.*

As a consequence of Theorem 2.1.1 we have seen that over a DVR every submodule of a reduced and supplemented module M is supplemented, i.e. M is totally supplemented. Using Theorem 2.3.17 we have the following corollary.

Corollary 2.3.18. *Let M be a reduced module (i.e $P(M) = 0$). Then the following are equivalent:*

(i) M is supplemented,

(ii) M is totally supplemented.

A module M is said to be a *max* module if every submodule of M contains a maximal submodule or equivalently $\operatorname{Rad} U \subsetneqq U$ for every submodule $U \subseteq M$. Therefore for a max module M we have $P(M) = 0$. By definition coatomic modules contain maximal submodules. Over a noetherian ring every submodule of a coatomic module is coatomic (see, Zöschinger (1980, Lemma 1.1)). Hence coatomic modules are max modules.

Corollary 2.3.19. *Let M be a max module. Then M is supplemented if and only if M is totally supplemented.*

The following is a consequence of Theorem 2.3.1 and Corollary 2.3.19.

Corollary 2.3.20. *For a coatomic module M the following are equivalent.*

(i) M is supplemented,

(ii) M is totally supplemented,

(iii) U and M/U are supplemented for some submodule U of M.

CHAPTER THREE
WEAKLY SUPPLEMENTED MODULES

In this chapter, some results on weakly supplemented modules are reviewed and some new results are proved. Mainly we shall consider weakly supplemented modules over Dedekind domains and commutative noetherian rings. For more general properties of weakly supplemented modules we refer to Rudlof (1991), Lomp (1996) and Lomp (1999). Section 3.1, we remind a structure theorem on weakly supplemented modules over Dedekind domains which is due to Zöschinger (Theorem 3.1.1) . Section 3.2, mainly include some review of the results on weakly supplemented modules. In Section 3.3, we give an example in order to show that the class of weakly supplemented modules need not be closed under extension. Then we give a sufficient condition under which weakly supplemented modules are closed under extension.

3.1 Weakly supplemented modules over Dedekind domains

Let R be Dedekind domain which is not a field. In this section we present a theorem due to Zöschinger, in which characterization of weakly supplemented is given. As a consequence we show that, a torsion R-module is weakly supplemented if and only if it is supplemented. If R is semilocal then an R-module M is weakly supplemented if and only if $T(M)$ and $M/T(M)$ are weakly supplemented. If R is a DVR then an R-module is weakly supplemented if and only if it can be embedded in a supplemented module.

In the following theorem Zöschinger characterizes weakly supplemented over Dedekind domains (see, Zöshinger (1986)).

Theorem 3.1.1. *Let R be a Dedekind domain and M an R-module. Then M is weakly supplemented if and only if*

(i) $M/\operatorname{Rad} M$ *is semisimple,*

(ii) $M/T(M)$ *has a finite Goldie dimension (finite rank),*

(iii) $T_P(M)$ *is a direct sum of an Artinian and a bounded submodule for every* $P \in \Omega$.

Lemma 3.1.2. *(by Lam (1999, Exercise 6.34)) Let R be a domain and M be an R-module. Then the torsion submodule $T(M)$ of M is closed in M.*

Proof. Suppose $T(M) \trianglelefteq K$ for some $K \subseteq M$. Let $k \in K$, then since $T(M)$ is essential in K we have $0 \neq rk \in T(M)$ for some $r \in R$. Then $srk = 0$ for some $0 \neq s \in R$. Since R is a domain $sr \neq 0$. Therefore $k \in T(M)$ i.e. $K = T(M)$. Hence $T(M)$ is closed in M. □

From Theorem 3.1.1 and Lemma 3.1.2 we get the following theorem:

Theorem 3.1.3. *Let R be a Dedekind domain and M be an R-module. Then M is weakly supplemented if and only if*

(i) $M/\operatorname{Rad} M$ *is semisimple,*

(ii) $T(M)$ *and* $M/T(M)$ *are weakly supplemented.*

Proof. (⇒) (i). By Theorem 3.1.1, $M/\operatorname{Rad} M$ is semisimple.
(ii). $M/T(M)$ is weakly supplemented as a factor module of a weakly supplemented module. By Lemma 3.1.2, $T(M)$ is a closed submodule of M. Then by

Theorem 2.2.6 $T(M)$ is a coclosed submodule of M. By (Lomp, 1996, Proposition 1.2.1) $T(M)$ is a supplement in M. Every supplement in M is weakly supplemented by (Lomp, 1999, Proposition 2.2). Hence $T(M)$ is weakly supplemented.

($\Leftarrow$) $M/T(M)$ is weakly supplemented so it has finite rank by Theorem 3.1.1. Since $T(M)$ is weakly supplemented, $T_P(M)$ is a direct sum of an artinian and a bounded submodule for every $P \in \Omega$. Now by Theorem 3.1.1, M is weakly supplemented. $\square$

Over a semilocal ring $M/\operatorname{Rad} M$ is semisimple for every R-module. So a torsion-free module over such a ring is weakly supplemented if and only if it has finite rank.

Now the following corollary is clear by Theorem 3.1.1

Corollary 3.1.4. *Let R be a semilocal Dedekind domain. Then an R-module is weakly supplemented if and only if*

(i) $M/T(M)$ has finite rank,

(ii) $T_P(M)$ is a direct sum of an artinian and a bounded submodule for every $P \in \Omega$.

Let R be a Dedekind domain. Suppose R is not a complete DVR. Then the class of weakly supplemented modules is strictly larger than the class of supplemented modules. For torsion modules one has the following Corollary.

Corollary 3.1.5. *Let R be a Dedekind domain and M be a torsion R-module. Then M is weakly supplemented if and only if it is supplemented.*

Corollary 3.1.6. *Suppose R is semilocal. Then an R-module M is weakly supplemented if and only if $T(M)$ and $M/T(M)$ are weakly supplemented.*

Proof. $M/\operatorname{Rad} M$ is semisimple as R is semilocal. Then the proof is clear by Theorem 3.1.3. □

From Theorem 2.1.4, Corollary 3.1.4 and Corollary 3.1.5 we get the following:

Corollary 3.1.7. *Let R be a DVR and M be an R-module. Then the following are equivelent.*

(i) M is weakly supplemented,

(ii) M can be embedded in a supplemented module,

(iii) M is an extension of a supplemented module by a supplemented module,

(iv) Every $U \subseteq M$ with $U \subseteq \operatorname{Rad} M$ has a supplement in M,

(v) The torsion part of M is supplemented, and $M/T(M)$ has finite rank.

Note that (*ii*) is also inherited by submodules. Therefore over a DVR the class of weakly supplemented modules is closed under submodules.

3.2 Weakly supplemented modules over noetherian rings

Throughout this section R is a noetherian ring, unless otherwise stated. In Section 3.1 we have seen some characterization of weakly supplemented modules over Dedekind domains. Dedekind domains are one-dimensional. Rudlof extended this characterization to an arbitrary noetherian ring with finite Krull dimension (see, Rudlof (1991) and Rudlof (1992)).

A module M is *semiartinian* if every factor module of M contains a minimal submodule (see, Dung et al. (1994)). Over a noetherian ring a module M is semiartinian if and only if every finitely generated submodule of M is artinian. M is a *minimax* module if there exists a finitely generated submodule U of M with M/U is artinian.
Note that supplemented is called complemented in Rudlof (1991).

Theorem 3.2.1. *(by Rudlof (1991, Theorem 3.1)) Let* $(R, \mathfrak{m})$ *be a local ring and* M *be an* R*-module. Then the following are equivalent:*

(i) M *is weakly supplemented,*

(ii) In every submodule V *of* M*,* $\mathfrak{m}V$ *has a supplement in* V*,*

(iii) M *is a small cover of a supplemented module,*

(iv) Every semiartinian factor module of M *is supplemented,*

(v) M *is an extension of a supplemented module by a supplemented module,*

(vi) $M = A + B$ *with a minimax module* A *and a discrete module* B*.*

Note that the condition Theorem 3.2.1 (*ii*) is inherited by submodules. Therefore, over a local ring R, the class of weakly supplemented modules is closed under submodules.

Lemma 3.2.2. *Let* R *be one dimensional. Then every torsion* R*-module is semiartinian.*

Proof. Let M be a torsion R-module and U be a finitely generated submodule of M. Then $U = Ru_1 + ... + Ru_n$ for some $u_i \in R$. We may assume that each u_i is non-zero. Then $Ru_i \cong R/I_i$ for some $0 \neq I_i \leq R$. Since R is one dimensional, Ru_i is artinian. Hence U is artinian i.e M is semiartinian. □

In Corollary 3.1.4, we have seen that, over a Dedekind domain, every torsion and weakly supplemented module is supplemented. This fact has the following generalization to one dimensional domains.

Proposition 3.2.3. *(by Rudlof (1991, Proposion 3.3 (b))) If M is semiartinian and weakly supplemented then M is supplemented.*

From Lemma 3.2.2 and Proposition 3.2.3 we obtain the following Corollary.

Corollary 3.2.4. *Let R be a one dimensional domain and M be a torsion R-module. Then M is weakly supplemented if and only if M is supplemented.*

Proposition 3.2.5. *(by Rudlof (1991, Proposition 3.4)) For a radical module M, the following are equivalent,*

(i) *M is weakly supplemented,*

(ii) *M is a small cover of a supplemented module,*

(iii) *M is an extension of a coatomic module by a semiartinian supplemented module,*

(iv) *M_P is an R_P minimax module for all $P \in \Omega$,*

(v) *M_P is a weakly supplemented R_P-module for all $P \in \Omega$.*

Theorem 3.2.6. *(by Rudlof (1991, Theorem 3.5)) Let R be a ring with finite Krull dimension. Then the following statements are equivalent for an R-module M:*

(i) *M is weakly supplemented,*

(ii) *M is a small cover of a supplemented module,*

(iii) *M has a small, coatomic submodule U such that M/U is semiartinian and supplemented.*

(iv) $M/\operatorname{Rad}(M)$ *is semisimple and every every semiartinian factormodule of* M *is supplemented.*

(v) $M/\operatorname{Rad}(M)$ *is semisimple and the* R_P*-module* M_P *is weakly supplemented for all* $P \in \Omega$*.*

3.3 Extension of weakly supplemented modules

A class $\mathfrak{M}$ of modules is closed under extension if whenever U, $M/U \in \mathfrak{M}$, M is also contained in $\mathfrak{M}$. In general the class of weakly supplemented modules need not closed under extension.

Theorem 3.3.1. *Let* $0 \to L \to M \to N \to 0$ *be a short exact sequence. If* L *and* N *are weakly supplemented and* L *has a weak supplement in* M *then* M *is weakly supplemented.*

If L *is coclosed then the converse holds; that is if* M *is weakly supplemented then* L *and* N *are weakly supplemented.*

Proof. Without restriction of generality we will assume that $L \subseteq M$. Let S be a weak supplement of L in M i.e. $L + S = M$ and $L \cap S \ll M$. Then we have,

$$M/L \cap S \cong L/L \cap S \oplus S/L \cap S$$

$L/L \cap S$ is weakly supplemented as a factor module of L which is weakly supplemented. On the other hand $S/L \cap S \cong M/L \cong N$ is weakly supplemented. Then $M/L \cap S$ is weakly supplemented as a sum of weakly supplemented modules. Therefore M is weakly supplemented by Lomp (1999, Proposition 2.2 (4)).

If L is coclosed then $L \cap S \ll L$ by Lomp (1996, Proposition 1.2.1) i.e L is a supplement of S in M. Then by Proposition 1.5.2 L and N are weakly supplemented. □

We will give an example of a module M for which there exists a submodule U such that U and M/U are weakly supplemented but M is not weakly supplemented. First we need some results which are necessary to present these example.

Lemma 3.3.2. *(by Santa-Clara & Smith (2004, Lemma 1.)) Let P be a finitely generated ideal of a commutative ring R and let an R-module $M = \prod_{i \in I} M_i$ be the direct product of R-modules $M_i (i \in I)$. Then $PM = \prod_{i \in I}(PM_i)$.*

Proof. It is clear that $PM \subseteq \prod_{i \in I}(PM_i)$. Conversely, let $m \in \prod_{i \in I}(PM_i)$. Then $m = (m_i)$, where $m_i \in PM_i (i \in I)$. There exist a positive integer k and elements $p_i \in P (1 \leqq i \leqq k)$ such that $P = Rp_1 + ... + Rp_k$. For each $i \in I$, $m_i \in PM_i = p_1 M_i + ... + p_k M_i$, so that $m_i = p_1 m_{i1} + ... + p_k m_{ik}$, for some elements $m_{ij} \in M_i (1 \leqq j \leqq k)$. It follows that $m = p_1(m_{i1}) + \ldots + p_k(m_{ik}) \in PM$. □

Proposition 3.3.3. *(by Santa-Clara & Smith (2004, Proposition 2.)) Let R be commutative domain which contains an infinite collection of distinct finitely generated maximal ideals $P_i (i \in I)$ such that $\bigcap_{i \in J} P_i = 0$, for every infinite subset J of I. Let the R-module $M = \prod_{i \in I}(R/P_i)$ be the direct product of the simple R-modules $R/P_i (i \in I)$. Then the torsion submodule of M is not a direct summand of M.*

Proof. Let $M_i = R/P_i (i \in I)$. Clearly the torsion submodule T of M is the direct sum of the modules R/P_i $(i \in I)$, i.e., $T = \bigoplus_{i \in I}(R/P_i)$. Suppose that $M = T \oplus M'$, for some submodule M' of M. Clearly $M' \neq 0$. Let $0 \neq x \in M'$. Then $x = (x_i)$, where $x_i \in M_i (i \in I)$ and $x_j \neq 0$, for some $j \in I$. Note that $P_j M_i = M_i$, for all $i \in I \setminus \{j\}$, and hence, by Lemma 3.3.2, $P_j M \cong \prod_{i \in I \setminus \{j\}} M_i$. In particular, note that $M/P_j M \cong M_j$, so $M/P_j M$ is simple. But $M/P_j M \cong (T/P_j T) \oplus (M'/P_j M')$ and $T \neq P_j T$. Thus $M' = P_j M'$ and in particular, we have $x = (x_j) \in P_j M'$, so that $x_j = 0$, a contradiction. Thus T is not a direct summand of M. □

Lemma 3.3.4. *Let R be a Dedekind domain and let $\{P_i\}_{i \in I}$ be an infinite set of distinct maximal ideals of R. Then $\bigcap_{i \in I} P_i = 0$.*

Proof. This is clear because in a Dedekind domain every non-zero ideal is a product of finitely many maximal ideals in a unique way (see, Theorem 1.7.1). □

Proposition 3.3.5. *Let R be a Dedekind domain and $\{P_i\}_{i\in I}$ be an infinite collection of distinct maximal ideals of R. Let $M = \prod_{i\in I}(R/P_i)$ be the direct product of the simple R-modules R/P_i and T be the torsion submodule of M. Then the following hold,*

(i) M/T *is divisible, i.e.* $M/T \cong K^{(J)}$ *for some index set* J,

(ii) $\operatorname{Rad} M = 0$.

Proof. (i) Let P be a maximal ideal of R. Then $P(M/T) = (PM+T)/T$. Now if P is not one of the ideals $\{P_i\}_{i\in I}$ then $PM + T = M$ and so $P(M/T) = M/T$. Suppose $P \in \{P_i\}_{i\in I}$, say $P = P_j$ for some $j \in I$, then $PM = \prod_{i\in I\setminus\{j\}}(R/P_i)$. In this case we also have $PM + T = M$ and hence $P(M/T) = M/T$. Therefore by Lemma 1.7.2 M/T is divisible.

(ii) Clearly $M_j = \prod_{i\in I\setminus\{j\}}(R/P_i)$ is a maximal submodule of M for every $j \in I$. Then

$$\operatorname{Rad} M \subseteq \bigcap_{j\in I} M_j = 0.$$

Hence $\operatorname{Rad} M = 0$. □

Now, we give an example in order to prove that the class of weakly supplemented modules need not be closed under extensions.

Example 3.3.6. Let R and M be as in Proposition 3.3.5 and T be the torsion submodule of M. Let N be a submodule of M such that $N/T \cong K$. Then T is weakly supplemented because it is semisimple, also N/T is weakly supplemented by Theorem 3.1.1. But N is not weakly supplemented: $\operatorname{Rad} N = 0$ by Propoposition 3.3.5 and N is not semisimple since $N/T \cong K$ is not semisimple, now by Lomp (1999, Corollary 2.3), N is not weakly supplemented.

Proposition 3.3.7. *Let R be a Dedekind domain and M be an R-module. If $T(M)$ has a weak supplement in M then M is weakly supplemented if and only if $T(M)$ and $M/T(M)$ are weakly supplemented.*

Proof. ($\Rightarrow$) By Theorem 3.1.3.

($\Leftarrow$) By Theorem 3.3.1. □

Corollary 3.3.8. *If $\operatorname{Rad} T(M) \ll T(M)$ then M is weakly supplemented if and only if $T(M)$ has a weak supplement in M and $M/T(M)$ is weakly supplemented.*

Proof. By Corollary 4.1.2 $T(M)/\operatorname{Rad} T(M)$ is semisimple. Then $T(M)$ is weakly supplemented by Proposition 1.5.2. Now the proof is clear by Proposition 3.3.7. □

Lemma 3.3.9. *Let R be a ring, $I \leqslant R$ and M be an R-module. If IM has a weak supplement K in M, then K is a weak supplement of $I^n M$ in M for every $n \geqq 1$.*

Proof. By hypothesis $IM + K = M$. Then we have $I^2M + IK = IM$, so $I^2M + IK + K = IM + K$ which gives $I^2M + K = M$. Continuing in this way we get:

$$I^n M + K = M \text{ and } I^n M \cap K \subseteq IM \cap K \ll M.$$

This means that K is a weak supplement of $I^n M$ in M. □

Proposition 3.3.10. *Let R be a one dimensional domain and M be an R-module. Suppose I is a nonzero ideal of R. Then M is weakly supplemented if $I^n M$ is weakly supplemented and $I^k M$ has a weak supplement in M for some $k \leqq n$.*

Proof. Since R is a domain and $I \neq 0$, then $I^n \neq 0$. So R/I^n is artinian ring. Then M/I^nM is a (weakly) supplemented R/I^n-module, hence it is also a supplemented R-module. By Lemma 3.3.9, I^nM has a weak supplement in M. Therefore by Theorem 3.3.1, M is weakly supplemented. □

CHAPTER FOUR
TOTALLY WEAK SUPPLEMENTED MODULES

In this Chapter, we investigate totally weak supplemented modules and study these modules over Dedekind domains and commutative noetherian rings. In Section 4.1, we characterize totally weak supplemented modules over Dedekind domains. The relation between totally supplemented, supplemented and weakly supplemented modules is determined for modules over Dedekind domains. An example is given to show that a weakly supplemented module need not be totally weak supplemented over a Dedekind domain. On the other hand for finitely generated modules we obtain that totally weak supplemented and weakly supplemented modules coincide over Dedekind domains. In Section 4.2, some results are proved for totally weak supplemented modules over commutative noetherian rings. It is shown that a commutative noetherian ring R is semilocal if and only if every R-module M with $\operatorname{Rad} M \ll M$ is totally weak supplemented.

4.1 Totally weak supplemented modules over Dedekind domains

A module M is said to be *totally weak supplemented* (briefly tws-module) if every submodule of M is weakly supplemented. In this section we determine the structure of tws-modules over Dedekind domains. It is shown that over a semilocal Dedekind domain a module is a tws-module if and only if it is weakly supplemented, and over a non-semilocal Dedekind domain a module is a tws-module if and only if it is supplemented. A finitely generated module over a

Dedekind domain is a tws-module if and only if it is weakly supplemented.

By Ω we denote the set of all maximal ideals of a ring R.

We begin with the following lemma.

Lemma 4.1.1. *Let R be a domain and P a maximal ideal of R. Then for every P-primary R-module M, $M/\operatorname{Rad} M$ is semisimple.*

Proof. Since R is commutative we have

$$\operatorname{Rad} M = \bigcap_{Q\in\Omega} QM.$$

First we will show that $QM = M$ for every $Q \in \Omega \setminus \{P\}$.
Let $x \in M$, then $P^n x = 0$ for some $n \in \mathbb{N}$. Since $P^n + Q = R$, we have $1 = p + q$ for some $p \in P^n$ and $q \in Q$. So we get $x = xp + xq = xq \in QM$, hence $M = QM$.
Therefore

$$\operatorname{Rad} M = \bigcap_{Q\in\Omega} QM = PM.$$

Then since R/P is a field $M/\operatorname{Rad} M = M/PM$ is a semisimple R/P-module, and so it is semisimple as an R-module. □

Corollary 4.1.2. *Let R be a Dedekind domain and M be a torsion R-module, then $M/\operatorname{Rad} M$ is semisimple.*

Proof. Since R is a Dedekind domain and M is a torsion R-module, we have

$$M = \bigoplus_{P\in\Omega} T_P(M).$$

Then

$$M/\operatorname{Rad} M = [\bigoplus_{P\in\Omega} T_P(M)]/[\bigoplus_{P\in\Omega} \operatorname{Rad} T_P(M)]$$
$$\cong \bigoplus_{P\in\Omega}[T_P(M)/\operatorname{Rad} T_P(M)]$$

is semisimple by Lemma 4.1.1. □

Lemma 4.1.3. *Let R be a Dedekind domain and M be P-primary for some $P \in \Omega$. Then M is divisible if and only if $M = PM$.*

Proof. (⇒) By Alizade et al. (2001, Lemma 4.4.), $PM = M$.

(⇐) From the proof of Lemma 4.1.1 we have $M = QM$ for every $Q \in \Omega\setminus\{P\}$. Therefore by Alizade et al. (2001, Lemma 4.4.) M is divisible. □

Lemma 4.1.4. *Let R be a Dedekind domain and M be a P-primary R-module. Suppose M is a direct sum of an artinian submodule and a bounded submodule. Then every submodule of M is a direct sum of an artinian submodule and a bounded submodule.*

Proof. Suppose

$$M = A \oplus B$$

with A an artinian and B a bounded submodule of M.
Let U be a submodule of M and D the divisible part of U. Then $U = D \oplus C$ where C is a reduced submodule of U.
Let

$$\pi : A \oplus B \to B$$

be the canonical projection, then $\pi(D)$ is a divisible submodule of B as a homomorphic image of the divisible submodule D. Since B is bounded it has no nonzero divisible submodule i.e. $\pi(D) = 0$. Therefore $D \subseteq A$, and hence D is

artinian.
Since B is bounded then $P^nB = 0$ for some $n \in \mathbb{N}$. Then

$$P^nC \subseteq P^nM = P^nA,$$

so P^nC is artinian. Then for the descending chain

$$P^nC \supseteq P^{n+1}C \supseteq ... \supseteq P^{n+k}C \supseteq ...$$

there exists $t \in \mathbb{N}$ such that $P^{n+t}C = P^{n+t+1}C$. Then by Lemma 4.1.3 $P^{n+t}C$ is a divisible submodule of C, but C is reduced, so we must have $P^{n+t}C = 0$, which shows that C is bounded. □

Now we are able to give a characterization of *tws*-modules over semilocal Dedekind domains.

Theorem 4.1.5. *Let R be a semilocal Dedekind domain and M be an R-module. The following are equivalent.*

(i) M is a tws-module,

(ii) M is weakly supplemented,

(iii) $M/T(M)$ has finite Goldie dimension and $T_P(M)$ is a direct sum of an artinian submodule and a bounded submodule for every $P \in \Omega$.

Proof. (i)⇒(ii) Clear.

(ii)⇒ (iii) By Theorem 3.1.1.

(iii)⇒ (i) Let U be a submodule of M. Since R is semilocal, then $U/\operatorname{Rad} U$ is semisimple.
$[U+T(M)]/T(M)$ has finite Goldie dimension as a submodule of $M/T(M)$, then

$$U/T(U) \cong [U+T(M)]/T(M)$$

also has finite Goldie dimension.
By Lemma 4.1.4 $T_P(U)$ is a direct sum of an Artinian submodule and a bounded submodule. Therefore by Theorem 3.1.1 U is weakly supplemented, hence M is a *tws*-module. □

Remark 4.1.6. If R is a non-semilocal Dedekind domain then a weakly supplemented module need not be a *tws*-module, e.g. the $\mathbb{Z}$-module $\mathbb{Q}$ is weakly supplemented by Theorem 3.1.1, but it is not a *tws*-module since the submodule $\mathbb{Z}$ of $\mathbb{Q}$ is not weakly supplemented.

Corollary 4.1.7. *Let R be a semilocal Dedekind domain and M be an R-module. The following are equivalent.*

(i) M is a tws-module,

(ii) M is weakly supplemented,

(iii) For some nonzero ideal of R, $I^n M$ is weakly supplemented and has a weak supplement in M.

Proof. By Proposition 3.3.10 and Theorem 4.1.5. □

A weakly supplemented module over a non-semilocal domain need not be torsion (see Remark, 4.1.6). The following proposition shows that over such a domain a *tws*-module is necessarily torsion.

Proposition 4.1.8. *Let R be a non-semilocal domain and M be an R-module. If M is a tws-module then M is torsion.*

Proof. Suppose $Rm \cong_R R$ for some $m \in M$. Since Rm is weakly supplemented, ${}_RR$ is also weakly supplemented. Then by Lomp (1999, Corollary 3.2), R is a semilocal ring, a contradiction. Hence M is a torsion module. □

In the following theorem we give a characterization of tws-modules over non-semilocal Dedekind domains.

Theorem 4.1.9. *Let R be a non-semilocal Dedekind domain. Then an R-module M is a tws-module if and only if M is torsion and $T_P(M)$ is a direct sum of an artinian submodule and a bounded submodule for every $P \in \Omega$.*

Proof. ($\Rightarrow$) By Proposition 4.1.8 M is a torsion module. Since M is weakly supplemented, then by Theorem 3.1.1 $T_P(M)$ is a direct sum of an Artinian submodule and a bounded submodule for every $P \in \Omega$.

($\Leftarrow$) Let U be a submodule of M. Then by Lemma 4.1.2 $U/\operatorname{Rad} U$ is semisimple. Since M is torsion we have $U = T(U)$ and so $U/T(U)$ has finite Goldie dimension. By Lemma 4.1.4 $T_P(U)$ is a direct sum of an Artinian submodule and a bounded submodule. Then by Theorem 3.1.1 U is weakly supplemented. Therefore M is a tws-module. □

Corollary 4.1.10. *Let R be a Dedekind domain and M be a torsion R-module. Then the following are equivalent.*

(i) *M is weakly supplemented,*

(ii) *M is a tws-module,*

(iii) *$T_P(M)$ is a direct sum of an Artinian submodule and a bounded submodule for every $P \in \Omega$.*

Proof. If R is semilocal then this follows by Theorem 4.1.5.
If R is non-semilocal, Theorem 3.1.1 and Theorem 4.1.9 complete the proof. □

Any small cover of a weakly supplemented module is weakly supplemented (see, Lomp (1999)). The following example shows that a small cover of a tws-module need not be a tws-module.

Example 4.1.11. The $\mathbb{Z}$-module $\mathbb{Q}$ is a small cover of $\mathbb{Q}/\mathbb{Z}$. Since $\mathbb{Q}$ is weakly supplemented, $\mathbb{Q}/\mathbb{Z}$ is also weakly supplemented as a factor module of $\mathbb{Q}$. Then $\mathbb{Q}/\mathbb{Z}$ is a tws-module by Corollary 4.1.10. But $\mathbb{Q}$ is not a tws-module by remark 4.1.6.

Now, we obtain the following:

Proposition 4.1.12. *Let R be a Dedekind domain and M be a torsion module. Suppose M/L is weakly supplemented for some $L \ll M$. Then M is a tws-module.*

Proof. M is weakly supplemented because M is a small cover of M/L with the canonical epimorphism $f : M \to M/L$. Hence by Corollary 4.1.10 M is a tws-module. □

An immediate consequence of Proposition 4.1.12 is the following:

Corollary 4.1.13. *Let R be a Dedekind domain and M be a torsion R-module with $\operatorname{Rad} M \ll M$. Then M is a tws-module.*

Proof. By Corollary 4.1.2 $M/\operatorname{Rad} M$ is semisimple, so it is weakly supplemented. Then M is weakly supplemented since M is a small cover of $M/\operatorname{Rad} M$. Therefore by Proposition 4.1.12 M is a $tws-$module. □

Now we determine the relation between supplemented modules and tws-modules. First we state the following observation:

Corollary 4.1.14. *Let R be a non-local Dedekind domain. Then an R-module M is supplemented if and only if M is totally supplemented.*

Proof. (⇒) Let U be submodule of M. By Theorem 2.2.5 M is torsion and every primary component is a direct sum of an artinian submodule and a bounded submodule. Then U is torsion as a submodule of a torsion module, and by Lemma 4.1.4 every primary component of U is a direct sum of an artinian submodule and a bounded submodule. Hence U is supplemented by Theorem 2.2.5. Therefore M is totally supplemented.

(⇐) Clear. □

Corollary 4.1.15. *Let R be a non-local Dedekind domain and M be a torsion R-module. Then M is supplemented if and only if M is a tws-module.*

Proof. By Corollary 4.1.10 and Theorem 2.2.5. □

Corollary 4.1.16. *Let R be a non-semilocal Dedekind domain and M be an R-module. Then the following are equivalent.*

(i) M is supplemented,

(ii) M is totally supplemented,

(iii) M is a tws-module.

Proof. By Theorem 2.2.5, Theorem 4.1.9 and Corollary 4.1.14. □

The following is an immediate consequence of Corollary 3.2.4.

Corollary 4.1.17. *Let R be a one dimensional domain and M be a torsion R-module. Then M is a tws-module if and only if M is totally supplemented.*

Proposition 4.1.18. *Let R be a Dedekind domain and M be an R-module. Suppose either R is semilocal or M is torsion. The following are equivalent.*

(i) $\operatorname{Rad} M$ *is weakly supplemented and has a weak supplement in* M,

(ii) M *is weakly supplemented,*

(iii) M *is a tws-module.*

Proof. (i)⇒(ii) In both cases $M/\operatorname{Rad} M$ is semisimple and so weakly supplemented. Then by Theorem 3.3.1 M is weakly supplemented.

(ii)⇔(iii) By Theorem 4.1.5 if R is semilocal. By Corollary 4.1.10 if R is non-semilocal.

(iii)⇒(i) Clear. □

Now we describe the structure of weakly supplemented modules over DVR and determine when a weakly supplemented module is supplemented over a DVR.

From Theorem 2.1.2, we see that over a DVR a reduced torsion-free module is supplemented if and only if it is free.

Lemma 4.1.19. *Let R be a DVR and M be a torsion R-module. Then M is weakly supplemented if and only if it is supplemented.*

Proof. (⇐) Clear.

(⇒) By Theorem 3.2.1, M is an extension of a supplemented module by a supplemented module i.e there exists a supplemented submodule U of M such that M/U is supplemented. Then by Zöschinger (1974a, Lemma 2.5) M is supplemented. □

To determine the structure of weakly supplemented modules over DVR, we use the following lemma.

Lemma 4.1.20. *Let R be a DVR and M torsion R-module. Then the following are equivalent.*

(i) M is supplemented,

(ii) M is totally supplemented,

(iii) M is weakly supplemented,

(iv) M is a tws-module,

(v) The divisible part of M is artinian and the reduced part is bounded.

Proof. (i) $\Leftrightarrow$ (ii) By Zöschinger (1974a, Lemma 2.5).

(i) $\Leftrightarrow$ (iii) By Lemma 4.1.19.

(iii) $\Leftrightarrow$ (iv) By Theorem 4.1.5.

(i) $\Leftrightarrow$ (v) By Theorem 2.1.2. $\square$

Let R be a domain and M be an R-module. A submodule U of M is called *pure* (in the sense of Kaplansky) in M if $rU = U \cap rM$ for every $r \in R$. Clearly the torsion submodule of M is a pure submodule of M. Over a DVR a bounded and pure submodule is a direct summand (see, Kaplansky (1965)).

Theorem 4.1.21. *Let R be a DVR and M be an R-module. Then M is weakly supplemented if and only if $M = K^{n_1} \oplus (K/R)^{n_2} \oplus B \oplus N$, where B is bounded, N is reduced torsion-free with finite rank and $n_i \geqslant 0$.*

Proof. (⇒) For the module M we always has the decomposition

$$M = K^{(I)} \oplus (K/R)^{(J)} \oplus L$$

with L is reduced. In this decomposition all three summands are weakly supplemented as a direct summand of M since M is weakly supplemented. Therefore by theorem 3.1.1, the torsion free module $K^{(I)}$ has finite rank i.e. I is finite, and by Lemma 4.1.20 the divisible submodule $(K/R)^{(J)}$ is artinian i.e. J is finite. By Theorem 4.1.5 M is a tws-module, hence the torsion submodule $T(L)$ is weakly supplemented. Then by Lemma 4.1.20, $T(L)$ is bounded. Now, $T(L)$ is a bounded and pure submodule of L, then

$$L = T(L) \oplus N$$

(see, Kaplansky (1965)), where N is reduced and torsion free. Hence we have the desired decomposition for M.

(⇐) If

$$M = K^{n_1} \oplus (K/R)^{n_2} \oplus B \oplus N,$$

then $K^{n_1} \oplus (K/R)^{n_2} \oplus B$ is supplemented by Theorem 2.1.2, hence weakly supplemented, and by Theorem 3.1.1 N is weakly supplemented. Therefore M is weakly supplemented as a direct sum of finitely many weakly supplemented modules. □

The following corollary is an immediate consequence of Theorem 2.1.2 and Theorem 4.1.21.

Corollary 4.1.22. *Let R be a DVR and M a module whose reduced part is torsion. Then M is weakly supplemented if and only if M is supplemented.*

Corollary 4.1.23. *(by Rudlof (1991, Corollary 3.2))Let R be complete DVR and M be an R-module. Then the following are equivalent.*

(i) M is totally supplemented,

(ii) M is supplemented,

(iii) M is weakly supplemented,

(iv) M is a tws-module.

Now we want to determine, when a weakly supplemented module is supplemented over DVR. By using Theorem 2.1.2, Theorem 4.1.21 and 4.1.22 we see that it is sufficient to determine when a reduced torsion-free module of finite rank is free. If R is a DVR then each rank one torsion-free module, which is not isomorphic to K, is isomorphic to an ideal of R (see, Fuchs & Salce (1985)). In a DVR it is well known that every ideal is cyclic. Therefore every rank one reduced torsion-free module is free.

In Matlis (1966), a domain is called a *D-domain* if every torsion-free module of finite rank is a direct sum of modules of rank one. Since every torsion-free divisible module is a direct sum of modules of rank one, R is a D-domain if and only if every reduced torsion-free module of finite rank is a direct sum of modules of rank one.

A DVR is a D-domain if and only if it is complete (see, Fuchs & Salce (2001, pp. 497 and pp. 501).

Now our question is reduced to determine when a reduced torsion-free module with finite rank is free. From the remarks above we have the following Corollary:

Corollary 4.1.24. *Let R be a DVR. Then R is complete if and only if every weakly supplemented R-module is supplemented.*

More generally from Theorem 2.1.5 and Corollary 4.1.24 we have the following observation:

Corollary 4.1.25. *Let R be a DVR. Then the following are equivalent.*

(i) R is complete,

(ii) every weakly supplemented module is supplemented,

(iii) every supplemented module is totally supplemented.

Corollary 4.1.26. *If R is a Dedekind domain and M a torsion R-module then the following are equivalent.*

(i) M is supplemented,

(ii) M is weakly supplemented,

(iii) M is a tws-module,

(iv) M has a small submodule U such that M/U is weakly supplemented.

Proof. (i)⇒(ii) Clear.

(ii)⇒(iii) By Corollary 4.1.10.

(iii)⇒(i) By Lemma 4.1.20 and Corollary 4.1.15.

(ii)⇔(iv) By Lomp (1999, Proposition 2.2). □

Now we mention some consequences of Corollary 4.1.26.

Corollary 4.1.27. *Let R be a non-local Dedekind domain and M be an R-module. Then M is supplemented if and only if M is torsion and M/L is weakly supplemented for some $L \ll M$.*

Proof. Suppose first that M is supplemented. Then by Theorem 2.2.5 M is torsion, and clearly M/L is weakly supplemented.

Converse is by Corollary 4.1.26. □

In any module the zero submodule is a small submodule. Therefore we have the following corollary.

Corollary 4.1.28. *Let R be a non-local Dedekind domain and M be an R-module. Then M is supplemented if and only if M is torsion and weakly supplemented.*

We have seen that over a semilocal Dedekind domain a module is a *tws*-module if and only if it is weakly supplemented. But a weakly supplemented module over a non-semilocal Dedekind domain need not be a *tws*-module (see, Remark 4.1.6). We prove that every finitely generated weakly supplemented module is a *tws*-module.

We begin with the following lemma.

Lemma 4.1.29. *Let R be a domain with $R \neq K$ and M be a torsion-free weakly supplemented module with $\operatorname{Rad} M = 0$. Then $M = 0$.*

Proof. By Lomp (1999, Corollary 2.3) M is semisimple. Then $M = \bigoplus_{i \in I} S_i$ where S_i is a simple module for every $i \in I$. Then $S_i \cong R/P_i$ for some maximal ideal P_i of R. Since M is torsion-free either $P_i = 0$ or $S_i = 0$. If $P_i = 0$ then R is a field, a contradiction. Hence we must have $S_i = 0$ for every $i \in I$, and so $M = 0$. □

Let R be a ring with Jacobson radical J and let P be a projective R-module. Then $\operatorname{Rad} P = JP$ (see, Anderson & Fuller (1992)). Therefore we have that $\operatorname{Rad} P = 0$ for every projective module over a non-semilocal Dedekind domain R.

Corollary 4.1.30. *Let R be a non-semilocal Dedekind domain. If P is a projective and weakly supplemented R-module then $P = 0$.*

Proof. Since R is a non-semilocal Dedekind domain we have $\mathrm{J}(R) = 0$, and so $\operatorname{Rad} P = 0$. Then $P = 0$ by Lemma 4.1.29. □

Theorem 4.1.31. *Let R be a non-semilocal Dedekind domain and M a finitely generated R-module. If M is weakly supplemented then it is torsion.*

Proof. Let M be a finitely generated R-module. Then $M/T(M)$ is finitely generated and torsion-free. Therefore $M/T(M)$ is projective as R is a Dedekind domain. So

$$M = T(M) \bigoplus M/T(M).$$

Now, since M is weakly supplemented then $M/T(M)$ is also weakly supplemented. By Corollary 4.1.30 we have $M/T(M) = 0$. Hence M is torsion. □

Corollary 4.1.32. *Let R be a non-semilocal Dedekind domain and M be a finitely generated R-module. Then the following are equivalent,*

(i) M is supplemented

(ii) M is a tws-module

(iii) M is weakly supplemented

(iv) M is torsion

Proof. (i)⇔(ii) By Corollary 4.1.16.

(ii)⇒(iii) Clear.

(iii)⇒(iv) By Theorem 4.1.31.

(iv)⇒(i) Let $M = Rm_1 + ...Rm_k$ for some $m_i \in M$ and $k \in \mathbf{N}$. Then $Rm_i \cong R/I_i (1 \leqq i \leqq k)$ for some $0 \neq I_i \subseteq R$. Since R is a Dedekind domain R/I_i is artinian and hence supplemented R-module. Then M is supplemented. □

The following example shows that if R is a semilocal Dedekind domain, then a finitely generated tws-module need not be supplemented.

Example 4.1.33. Let $\mathbb{Z}$ be the ring of integers and p, q be two different primes in $\mathbb{Z}$. Let R be the localization of $\mathbb{Z}$ at $S = \mathbb{Z} \backslash (p\mathbb{Z} \cup qZ)$. Then R is a semilocal Dedekind domain. ${}_RR$ is weakly supplemented by Lomp (1999, Corollary 2.2), hence it is a tws-module by Theorem 4.1.5. Note that ${}_RR$ is not torsion, hence it is not supplemented by Theorem 2.2.5.

The following is a consequence of Theorem 4.1.5 and Corollary 4.1.32.

Corollary 4.1.34. *Let R be a Dedekind domain and M be a finitely generated R-module. Then M is weakly supplemented if and only if M is a tws-module.*

4.2 Totally weak supplemented modules over noetherian rings

In Sec 3.2, as a consequence of Theorem 3.2.1, we have seen that over a local noetherian ring a module is weakly supplemented if and only if it is a tws-module. We will see that this is also the case for modules over semilocal noetherian rings.

Recall that a noetherian semilocal ring has finite Krull dimension (see Remark 1.1.7).

Theorem 4.2.1. *Let R be a noetherian semilocal ring. Then for an $R-$module M the followings are equivalent:*

(i) M is totally weak supplemented,

(ii) M is weakly supplemented,

(iii) the R_P-module M_P is weakly supplemented for every maximal ideal P of R.

Proof. (i)⇒(ii) Clear.

(ii)⇒(iii) Since R is semilocal then $M/\operatorname{Rad} M$ is semisimple, and by Theorem 3.2.6 the R_P-module M_P is weakly supplemented for every maximal ideal P of R.

(iii)⇒ (i) Let U be a submodule of M. Then $U/\operatorname{Rad} U$ is semisimple. Since R_P is a local ring, the R_P-module M_P is totally weak supplemented by Theorem 3.2.1. So U_P is weakly supplemented as an R_P-module, hence by Theorem 3.2.6, U is weakly supplemented. Therefore M is totally weak supplemented. □

Theorem 4.2.2. *Let R be a semilocal ring (not necessarily commutative) with Jacobson radical J and M be an R-module. Then the following hold:*

(i) If JM is weakly supplemented and has a weak supplement in M then M is weakly supplemented.

If in addition, R is noetherian then

(ii) If JM has a weak supplement in M and J^nM is weakly supplemented for some $n \geqq 0$ then M is weakly supplemented.

(iii) Suppose either $J^nM \ll M$ or J^nM is weakly supplemented and has a weak supplement in M. Then M is weakly supplemented.

Proof. (i) Since M/JM is semisimple it is weakly supplemented. Then by Theorem 3.3.1, M is weakly supplemented.

(ii) Now if R is noetherian then R/J^n is artinian. Therefore M/J^nM is a weakly supplemented R-module. On the other hand by Lemma 3.3.9 J^nM has a weak supplement in M. Then M is weakly supplemented by Theorem 3.3.1.

(iii) Suppose $J^nM \ll M$. From the proof of (ii), M/J^nM is weakly supplemented. Then by Proposition 1.5.2, M is weakly supplemented. The other part is clear by Theorem 3.3.1. □

Theorem 4.2.3. *Let R be a commutative semilocal noetherian ring with Jacobson radical J and M be an R-module. The following are equivalent:*

(i) M is weakly supplemented,

(ii) M is a tws-module,

(iii) J^nM is weakly supplemented for some $n \geqq 0$, and has a weak supplement in M.

Proof. (i)⇔(ii) By Theorem 4.2.1.

(ii)⇒(iii) Obvius.

(iii)⇒(i) By Theorem 3.3.1. □

Now we prove some results for coatomic modules. Coatomic modules have small radical, but the converse need not be true in general. For example the $\mathbb{Z}$-module $\mathbb{Z}^{(\mathbb{N})}$ has zero (small) radical but it is not coatomic, because $\mathbb{Z}^{(\mathbb{N})}/N \cong_{\mathbb{Z}} \mathbb{Q}$

for some proper submodule N of $\mathbb{Z}^{(\mathbb{N})}$ (see, Güngöroğlu (1998)). For semilocal modules we have the following.

Lemma 4.2.4. *Let M be a semilocal module. Then M is coatomic if and only if* $\operatorname{Rad} M \ll M$.

Proof. ($\Rightarrow$) Clear.

($\Leftarrow$) Let U be a proper submodule of M. Since $\operatorname{Rad} M \ll M$, we have $U + \operatorname{Rad} M \neq M$. Since $M/\operatorname{Rad} M$ is semisimple $(U+\operatorname{Rad} M)/\operatorname{Rad} M$ is contained in a maximal submodule $K/\operatorname{Rad} M$ of $M/\operatorname{Rad} M$. Then K is a maximal submodule of M contaning U. Hence M is coatomic. □

Corollary 4.2.5. *Let M be a semilocal module. Suppose* $\operatorname{Rad} M$ *is coatomic. Then M is coatomic.*

Proof. We have $\operatorname{Rad} M \ll M$. Then by Lemma 4.2.4 M is coatomic. □

The following is an immediate consequence of Lemma 4.2.4 and Lemma 1.6.5.

Corollary 4.2.6. *Let R be a commutative noetherian ring and M be a semilocal R-module. The following are equivalent.*

(i) $\operatorname{Rad} M$ *is coatomic,*

(ii) M is coatomic,

(iii) Every submodule of M is coatomic.

Corollary 4.2.7. *Let R be a commutative noetherian semilocal ring with Jacobson radical J and M be an R-module. The following are equivalent.*

(i) *$J^n M$ is coatomic for some $n \geqslant 1$,*

(ii) *M is coatomic,*

(iii) *Every submodule of M is coatomic.*

Proof. (i)⇒(ii) Since R is a semilocal ring, $\operatorname{Rad}(J^{n-1}M) = J(J^{n-1}M) = J^n M$, and $J^{n-1}M/J^n M$ is semisimple, then by Corollary 4.2.5, $J^{n-1}M$ is coatomic. Continuing in this way we get $\operatorname{Rad} M = JM$ is coatomic, therefore again by Corollary 4.2.5 M is coatomic.

(ii)⇒(iii) By Lemma 1.6.5.

(iii)⇒(i) Clear. □

Over one dimensional domains the Jacobson radical J in Corollary 4.2.7 can be replaced by any nonzero ideal, as the following theorem shows.

Theorem 4.2.8. *Let R be a one-dimensional domain, I a nonzero ideal of R and M be an R-module. The following are equivalent.*

(i) *$I^n M$ is coatomic for some $n \in \mathbb{Z}^+$,*

(ii) *M is coatomic.*

Proof. (i)⇒(ii) Since R is one dimensional the ring R/I^n is artinian for every $n \in \mathbb{Z}^+$, so R/I^n is a perfect ring. Then $M/I^n M$ is a coatomic R/I^n-module, hence $M/I^n M$ is a coatomic R-module. Then by Güngöroğlu (1998, Lemma 3(2)), M is a coatomic module.

(ii)⇒(i) R is a commutative noetherian ring. Then by Lemma 1.6.5 $I^n M$ is coatomic for every $n \in \mathbb{Z}^+$. □

It has been proved in Lomp (1999, Theorem 3.5) that a ring R is semilocal if and only if every R-module M with small radical is weakly supplemented. The following example shows that if R is a semilocal ring and M is an R-module with small radical i.e weakly supplemented, then M need not be a tws-module.

Example 4.2.9. Let R be a Dedekind domain but not a field and P be a maximal ideal in R. Consider the ring R_P obtained by localization of R at P. Then R_P is a DVR, so it is weakly supplemented. If K is the field of fractions of R_P and I an infinite index set then the R_P-module $K^{(I)}$ is not weakly supplemented by Theorem 4.1.21. Now consider the ring:

$$S = \begin{pmatrix} R_P & K^{(I)} \\ 0 & R_P \end{pmatrix}$$

Then ${}_S S$ is a semilocal ring, so it is weakly supplemented, and has small radical as it is finitely generated. Consider the submodule:

$$M = \begin{pmatrix} 0 & K^{(I)} \\ 0 & 0 \end{pmatrix}$$

Then M is not a weakly supplemented S-module, because the structure of M as an R_P-module and as an S-module is same. Thus ${}_S S$ is not a tws-module.

We will show that over a commutative noetherian semilocal ring R, every R-module with small radical is a tws-module.

Theorem 4.2.10. *Let R be a commutative noetherian ring and M an $R-$module. The following are equivalent:*

(i) $\operatorname{Rad} M \ll M$ *and* $M/\operatorname{Rad} M$ *is semisimple,*

(ii) M *is a coatomic module and* $M/\operatorname{Rad} M$ *is semisimple,*

(iii) $\operatorname{Rad} M$ *is a coatomic submodule of* M *and* $M/\operatorname{Rad} M$ *is semisimple,*

(iv) M *is weakly supplemented and* $\operatorname{Rad} M \ll M$.

Proof. (i) $\Rightarrow$ (ii) We must show that $\operatorname{Rad}(M/U) \neq M/U$ for every proper submodule U of M or equivalently every proper submodule U of M is contained in a maximal submodule of M. Let U be a proper submodule of M, then since $\operatorname{Rad} M \ll M$ we have $U + \operatorname{Rad} M \neq M$. Then $(U + \operatorname{Rad} M)/\operatorname{Rad} M$ is proper a submodule of $M/\operatorname{Rad} M$. Hence $(U+\operatorname{Rad} M)/\operatorname{Rad} M$ is contained in a maximal submodule of $M/\operatorname{Rad} M$, say $K/\operatorname{Rad} M$. Then K is a maximal submodule of M containing U. Therefore M is a coatomic module.

(ii)$\Rightarrow$(iii) By Lemma 1.6.5 $\operatorname{Rad} M$ is a coatomic submodule of M.

(iii)$\Rightarrow$(iv) Let $N = \operatorname{Rad} M$. Suppose $N + K = M$ for some proper submodule K of M. Then

$$M/K \cong N/N \cap K.$$

Since $N \cap K$ is a proper submodule of U, it is contained in some maximal submodule of U. Thus K is contained in a maximal submodule T of M. On the other hand, by definition N is also contained in T. We get,

$$M = N + K \subseteq T,$$

a contradiction. Thus $\operatorname{Rad} M \ll M$.
Now $M/\operatorname{Rad} M$ is semisimple, so it is weakly supplemented. Thus M is weakly supplemented because it is a small cover of $M/\operatorname{Rad} M$.

(iv)$\Rightarrow$(i) Clear. □

Corollary 4.2.11. *Let R be a commutative noetherian ring. Then the following are equivalent.*

(i) *R is a semilocal ring,*

(ii) *Every R-module M with $\operatorname{Rad} M \ll M$ is weakly supplemented,*

(iii) *Every R-module M with $\operatorname{Rad} M \ll M$ is totally weak supplemented,*

(iv) *Every coatomic R-module is weakly supplemented,*

(v) *Every coatomic R-module is totally weak supplemented,*

(vi) *Every projective coatomic R-module is weakly supplemented,*

(vii) *Every projective coatomic R-module is totally weak supplemented.*

Proof. (i) $\Leftrightarrow$ (ii) By Lomp (1999, Theorem 3.5).

(ii)$\Rightarrow$(iii) Let M be an R-module with $\operatorname{Rad} M \ll M$. By hypothesis M is weakly supplemented. Then by Theorem 4.2.10 M is coatomic and hence every submodule of M is coatomic by Theorem 1.6.5. Therefore for every submodule U of M, $\operatorname{Rad} U \ll U$. Now by hypothesis U is weakly supplemented. Hence M is totally weak supplemented.

(iii)$\Rightarrow$(iv) Coatomic modules have small radicals.

(iv)$\Rightarrow$(v) By Theorem 1.6.5 every submodule of a coatomic module is coatomic. Hence by the hypothesis coatomic modules are totally weak supplemented.

(v) $\Rightarrow$(vi) Clear.

(vi)$\Rightarrow$(i) ${}_RR$ is projective, and coatomic since it is finitely generated. Hence by hypothesis ${}_RR$ is weakly supplemented. Hence R is semilocal by Lomp (1999, Corollary 3.2).

(vii)⇒(vi) Clear.

(iv)⇒(vii) Clear. □

CHAPTER FIVE
COFINITELY WEAK SUPPLEMENTED MODULES

Let R be a ring and M be an R-module. A submodule U of M is called *cofinite* if M/U is finitely generated. A module M is called *cofinitely supplemented* if every cofinite submodule of M has a supplement in M (see, Bilhan (1999), Alizade et al. (2001)).

A module M is called *cofinitely weak supplemented* (briely cws-module) if every cofinite submodule of M has a weak supplement in M (see, Büyükaşık (2001), Alizade & Büyükaşık (2003)). In section 5.1 we shall summarize main properties of cws-modules, and we state the results which we shall use in this chapter.

In section 5.2 we prove some equivalent conditions for a module to be a cws-module over a noetherian ring (not necessarily commutative).

In section 5.3, we study cws-modules over some commutative domains. We characterize $h-semilocal$ domains as the domains over which every torsion R-module is a cws-module. We also give a sufficient condition for an arbitrary module to be a cws-module over an $h-semilocal$ domain.

5.1 Cofinitely weak supplemented modules

Proposition 5.1.1. *(by Alizade & Büyükaşık (2003)) Let R be a ring and M be a cws-module. Then the following holds.*

(i) Every homomorphic image of M is a cws-module,

(ii) Any small cover of M is a cws-module,

(iii) Every cofinite submodule of $M/\operatorname{Rad} M$ is a direct summand,

(iv) Any M-generated module is a cws-module.

Arbitrary sum of *cws*-modules is a *cws*-module. A ring R is semilocal if and only if every R-module is a *cws*-module (see, Alizade & Büyükaşık (2003)).

Theorem 5.1.2. *Let $0 \to L \to M \to N \to 0$ be a short exact sequence. If L and N are cws-modules and L has a weak supplement in M then M is a cws-module.*

Proof. Let S be a weak supplement of L in M i.e. $L + S = M$ and $L \cap S \ll M$. Then we have,

$$M/L \cap S \cong L/L \cap S \oplus S/L \cap S$$

$L/L \cap S$ is *cws* as a factor module of L which is a *cws*-module. On the other hand

$$S/L \cap S \cong M/L \cong N$$

is a *cws*-module. Then $M/L \cap S$ is *cws* as a sum of *cws*-modules. Therefore M is a *cws*-module, because M is a small cover of $M/L \cap S$. □

For a module N, let Γ be the set of all submodules K such that K is a weak supplement for some maximal submodule of N and let $cws(N)$ denote the sum of all submodules from Γ. As usual $cws(N) = 0$ if $\Gamma = \emptyset$.

Theorem 5.1.3. *(by Alizade & Büyükaşık (2003, Theorem 2.16)) For a module N, the following statements are equivalent.*

(i) N is a cws-module,

(ii) Every maximal submodule of N has a weak supplement,

(iii) $N/cws(N)$ has no maximal submodules.

Theorem 5.1.4. *(by Alizade & Büyükaşık (2003, Theorem 2.21)) Let M be an R-module with $\operatorname{Rad} M \ll M$. Then the following statements are equivalent.*

(i) M is a cws-module,

(ii) $M/\operatorname{Rad} M$ is a cws-module,

(iii) Every cofinite submodule of $M/\operatorname{Rad} M$ is a direct summand,

(iv) Every maximal submodule of $M/\operatorname{Rad} M$ is a direct summand,

(v) Every maximal submodule of $M/\operatorname{Rad} M$ is a weak supplement,

(vi) Every maximal submodule of M is a weak supplement.

5.2 Cofinitely weak supplemented modules over noetherian rings

Throughout this section R is a noetherian ring, unless otherwise stated.

Lemma 5.2.1. *Let M be a module and U be a finitely generated submodule of M contained in $\operatorname{Rad} M$. Then U is small in M.*

Proof. Let $U = Ru_1 + Ru_2 + ... + Ru_n$ and $\operatorname{Rad} M = \sum_{i \in I} S_i$, where I is a some index set and $S_i \ll M$ for every $i \in I$. Since $U \subseteq \operatorname{Rad} M$, $u_k \in \sum_{i \in F_k} S_i$ for some finite subset $F_k \subseteq I$, for every $k \in \{1, \ldots, n\}$. Then $Ru_k \subseteq \sum_{i \in F_k} S_i$, hence $U \subseteq \sum_{i \in F} S_i$, where $F = \bigcup_{i=1}^{n} F_k$ is a finite subset of I. Then $\sum_{i \in F} S_i \ll M$ and $U \ll M$ by (Anderson & Fuller, 1992, Corollary 15.18). □

Theorem 5.2.2. *Let M be an R-module and $X \subseteq \operatorname{Rad} M$. Then M is a cws-module if and only if M/X is a cws-module.*

Proof. $\Rightarrow$ Clear, since factor modules of *cws*-modules are *cws*.

$\Leftarrow$ Let K be a maximal submodule of M, then by hypothesis $X \subseteq \operatorname{Rad} M \subseteq K$, so K/X is a maximal submodule of M/X. Then K/X has a weak supplement N/X in M/X i.e.

$$K/X + N/X = M/X \text{ and } (K \cap N)/X \subseteq \operatorname{Rad}(M/X) = \operatorname{Rad} M/X$$

(the last equality follows from the fact that $X \subseteq \operatorname{Rad} M$) and so $K \cap N \subseteq \operatorname{Rad} M$. By (Alizade & Büyükaşık (2003, Lemma 2.1)) without loss of generality we may suppose N/X is cyclic. Then $N = Rn + X$ for some $n \in N$.
We get

$$M = K + N = K + Rn + X = K + Rn \text{ and } K \cap Rn \leqslant K \cap N \leqslant \operatorname{Rad} M.$$

Now since R is noetherian, Rn is noetherian so $K \cap Rn$ is finitely generated. Therefore $K \cap Rn \ll M$ by Lemma 5.2.1. Hence Rn is a weak supplement of K in M. Thus M is a *cws*-module by Theorem 5.1.3. $\square$

Theorem 5.2.3. *Let M be an R-module. Then M is a cws-module if and only if every maximal submodule of $M/\operatorname{Rad} M$ is a direct summand.*

Proof. ($\Rightarrow$) By Proposition 5.1.1.

($\Leftarrow$) We will show that every maximal submodule of M has a weak supplement. Let K be a maximal submodule of M. Then $K/\operatorname{Rad} M$ is a maximal submodule of $M/\operatorname{Rad} M$, hence by hypothesis

$$K/\operatorname{Rad} M \oplus S/\operatorname{Rad} M = M/\operatorname{Rad} M$$

for some submodule $S/\operatorname{Rad} M \subseteq M/\operatorname{Rad} M$. Then since K is a maximal submodule of M, $M/K \cong S/\operatorname{Rad} M$ is a simple module, therefore $\operatorname{Rad} M$ is a maximal submodule of S, and so $Rs + \operatorname{Rad} M = S$ for some $s \in S \setminus \operatorname{Rad} M$. We get

$$S/\operatorname{Rad} M = (Rs + \operatorname{Rad} M)/\operatorname{Rad} M \cong Rs/(Rs \cap \operatorname{Rad} M)$$

which is a simple module; hence $Rs \cap \operatorname{Rad} M$ is a maximal submodule of Rs.
Clearly,

$$M = K + S = K + \operatorname{Rad} M + Rs = K + Rs.$$

Then we get

$$M/K = (Rs + K)/K \cong Rs/(Rs \cap K),$$

so $(Rs \cap K)$ is a maximal submodule of Rs.
Since $Rs \cap \operatorname{Rad} M \subseteq Rs \cap K$ and both are maximal submodules of Rs we get $Rs \cap \operatorname{Rad} M = Rs \cap K$.
Now, since R is noetherian, Rs is noetherian, then $Rs \cap K$ is a finitely generated submodule of M. Hence by Lemma 5.2.1 $Rs \cap K \ll M$. Therefore Rs is a weak supplement of K in M. Hence M is a *cws*-module by Theorem 5.1.3. □

As a consequence we obtain the following. Note that the following corollary is proved in Alizade & Büyükaşık (2003, Theorem 2.21) for a module M with $\operatorname{Rad} M \ll M$.

Corollary 5.2.4. *For an R-module M, the following are equivalent.*

(i) M is a cws-module,

(ii) M/X is a cws-module for some $X \subseteq \operatorname{Rad} M$,

(iii) $M/\operatorname{Rad} M$ is a cws-module,

(iv) Every cofinite submodule of $M/\operatorname{Rad} M$ is a direct summand,

(v) Every maximal submodule of $M/\operatorname{Rad} M$ is a direct summand,

(vi) Every maximal submodule of $M/\operatorname{Rad} M$ is a weak supplement,

(vii) Every maximal submodule of M is a weak supplement.

Proof. (i)$\Leftrightarrow$(ii)$\Leftrightarrow$(iii) by Theorem 5.2.2 and Theorem 5.2.3.

(iii)$\Rightarrow$(iv) is obvious since $\operatorname{Rad}(M/\operatorname{Rad} M) = 0$.

(iv)$\Rightarrow$(v) maximal submodules are cofinite.

(v)$\Leftrightarrow$(vi) is obvious.

(v)$\Rightarrow$(iii) By Theorem 5.2.3.

(vii)$\Leftrightarrow$(i) By Theorem 5.1.3 □

Corollary 5.2.4 has the following consequence.

Corollary 5.2.5. *Let M be an R-module. If M is semilocal, then M is a cws-module.*

Proof. Since M is semilocal $M/\operatorname{Rad} M$ is semisimple. Then every submodule of $M/\operatorname{Rad} M$ is a direct summand. Therefore by Corollary 5.2.4 M is a *cws*-module. □

In Alizade & Büyükaşık (2003), a module M is called *finitely weak supplemented* (briefly fws-module) if every finitely generated submodule of M has a weak supplement in M. The class of fws-modules is properly contained in the class of weakly supplemented modules (see, Alizade & Büyükaşık (2003, Proposion 3.9). We are going to prove that over a noetherian ring, every fws-module with small radical is weakly supplemented. First we need the following lemma.

Lemma 5.2.6. *Let M be an R-module. If every finitely generated submodule of M is a direct summand of M then M is semisimple.*

Proof. Let $K = \operatorname{Soc} M$. We want to show that $K = M$. Suppose $K \subsetneqq M$. Then there exist $m \in M \setminus K$. Now, since Rm is noetherian $Rm \cap K$ is a finitely generated submodule of M, and hence by hypothesis

$$M = (Rm \cap K) \oplus N.$$

Then by modular law we get $Rm = (Rm \cap K) \oplus T$ for some nonzero submodule T of M. T is noetherian as a submodule of Rm, so there is a maximal submodule L of T, and by hypothesis $M = L \oplus S$ for some $S \subseteq M$. Then by modular law we get $T = L \oplus T \cap S$. Since L is maximal in T, $T \cap S$ is a nonzero simple submodule of T. Then

$$T \cap S \subseteq T \cap K = T \cap (Rm \cap K) = 0,$$

contradiction. Thus $\operatorname{Soc} M = K = M$, i.e. M is semisimple. □

Corollary 5.2.7. *Let M be an R-module with $\operatorname{Rad} M \ll M$. Then the following are equivalent.*

(i) M is weakly supplemented,

(ii) Every cyclic submodule of M has (is) a weak supplement,

(iii) Every finitely generated submodule of $M/\operatorname{Rad} M$ is a direct summand,

(iv) M is a fws-module,

(v) M is semilocal.

Proof. (i)⇔(ii) Clear.

(ii)⇔(iii)⇔ (iv) by Alizade & Büyükaşık (2003, Thorem 3.8).

(iii)$\Leftrightarrow$(v) $\operatorname{Rad}(M/\operatorname{Rad} M) = 0$. Now Lemma 5.2.6 gives the proof.

(v)$\Rightarrow$(i) By Lomp (1999, Thorem 2.7). □

Proposition 5.2.8. *Let M be an R-module. Suppose every cyclic submodule of $M/\operatorname{Rad} M$ is a direct summand. Then M is a fws-module.*

Proof. Let K be a finitely generated submodule of M. By Lemma 5.2.6 $M/\operatorname{Rad} M$ is semisimple. Then

$$(K + \operatorname{Rad} M)/\operatorname{Rad} M \oplus N/\operatorname{Rad} M = M/\operatorname{Rad} M.$$

We get

$$M = K + \operatorname{Rad} M + N = K + N \text{ and } K \cap N \subseteq \operatorname{Rad} M.$$

Since K is finitely generated then by hypothesis $K \cap N$ is also finitely generated, so by Lemma 5.2.1 $K \cap N \ll M$. Hence N is a weak supplement of K in M. Thus M is a fws-module. □

Note that any semilocal module satisfies the hypothesis in Proposition 5.2.8.

Corollary 5.2.9. *Every semilocal module is an fws-module. supplemented.*

5.3 Cofinitely weak supplemented modules over commutative domains

In Lomp (1999) it has been proved that every finite sum of weakly supplemented modules is weakly supplemented. But arbitrary sum of weakly supplemented modules need not be weakly supplemented (see, Alizade & Büyükaşık (2003, Example 2.14)).

Proposition 5.3.1. *Let* $M = \sum_{i \in I} M_i$*, where* M_i *is weakly supplemented for every* $i \in I$*. If* $\operatorname{Rad} M \ll M$ *then* M *is weakly supplemented.*

Proof. Let $N = \bigoplus_{i \in I} M_i$. Then we have an epimorphism $f : N \to M$, and $N/\operatorname{Rad} N = \bigoplus_{i \in I} M_i/\operatorname{Rad} M_i$ is semisimple i.e. N is semilocal. Therefore M is semilocal, and hence M is weakly supplemented by Lomp (1999, Theorem 2.7) □

In Matlis (1966), a domain S is called *h-local* if every non-zero ideal of S is contained in only finite number of maximal ideals and S/P is a local ring for every non-zero prime ideal P of S. A domain S is *h-semilocal* if every non-zero ideal I of S is contained in only finitely many maximal ideals of S i.e. S/I is a semilocal ring. Clearly Dedekind domains and h-local domains are h-semilocal domains.

In Alizade et al. (2001), it is proved that a commutative domain R is h-local if and only if every torsion $R-$module is cofinitely supplemented. We prove the following theorem.

Theorem 5.3.2. *The following statements are equivalent for a commutative domain R.*

(i) R is h-semilocal,

(ii) Every cyclic torsion R-module is weakly supplemented,

(iii) Every torsion R-module M with $\mathrm{Rad}(M) \ll M$ *is weakly supplemented,*

(iv) For every torsion R-module M, every maximal submodule of M has a weak supplement in M,

(v) Every torsion R-module is a cws-module.

Proof. (i)$\Rightarrow$ (ii) Let $M \cong R/I$ for some $0 \neq I \leqslant R$. Then R/I is a semilocal R/I-module, so R/I is a semilocal R-module, hence weakly supplemented by Lomp (1999, Corollary 3.2). Therefore M is weakly supplemented.

(ii)$\Rightarrow$(iii) Let M be torsion R-module with $\mathrm{Rad}\, M \ll M$. By hypothesis Rm is weakly supplemented, hence semilocal for every $m \in M$. Since $M = \sum_{m \in M} Rm$, then by Proposition 5.3.1, M is weakly supplemented.

(iii)$\Rightarrow$(v) Let M be a torsion module. Then Rm is torsion and $\mathrm{Rad}\, Rm \ll Rm$ by Anderson & Fuller (1992, Theorem 10.4), so by hypothesis Rm is weakly supplemented. Therefore $M = \sum_{m \in M} Rm$ is a cws-module by Alizade & Büyükaşık (2003, Proposition 2.12).

(iv)$\Leftrightarrow$(v) By Alizade & Büyükaşık (2003, Theorem 2.16).

(v)$\Rightarrow$(i) Let $0 \neq I \leqslant R$. Then R/I is a torsion R-module and hence it is a weakly supplemented R-module. Then R/I is a weakly supplemented R/I-module. Therefore by Lomp (1999, Corollary 3.2), R/I is semilocal. Then by

Proposition 1.3.3 I is contained only in finitely many maximal ideals. Hence R is h-semilocal. □

Corollary 5.3.3. *Let R be an h-semilocal domain, M be an R-module and $T(M)$ be the torsion submodule of M. Suppose $T(M)$ has a weak supplement in M and $M/T(M)$ is a cws-module. Then M is a cws-module.*

Proof. By Theorem 5.3.2, $T(M)$ is a *cws*-module. Then by Theorem 5.1.2 M is a *cws*-module. □

Let R be a Dedekind domain and M be an R-module. Note that R is noetherian. Denote by $D(M)$ the divisible part of M. Then $D(M)$ is injective by (Sharpe & Vamos (1972, Proposition 2.10), hence $M = D(M) \oplus N$ for some submodule N of M. In this case N is called *reduced part* of M. By Alizade et al. (2001, Lemma 4.4), $D(M)$ has no maximal submodule and hence $D(M)$ is the only cofinite submodule of $D(M)$. Thus $D(M)$ is a *cws*-module. In this case, from Alizade & Büyükaşık (2003, Proposition 2.5 and Proposition 2.12) we get, M is a *cws*-module if and only if the reduced part of M is a *cws*-module. Hence using Corollary 5.2.4 we have the following corollary.

Corollary 5.3.4. *Let R be a Dedekind domain, M be an R-module and N be the reduced part of M. Then the following are equivalent.*

(i) M is a cws-module,

(ii) N is a cws-module,

(iii) every maximal submodule of $N/\operatorname{Rad} N$ is a direct summand.

Corollary 5.3.5. *Let R be a Dedekind domain and M be an R-module . Suppose the reduced part N of M is torsion. Then M is a cws-module.*

Proof. Since R is h-semilocal, N is a cws-module by Theorem 5.3.2. Therefore by Corollary 5.3.4 M is a cws-module. □

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