DOKUZ EYLÜL UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

NUMERICAL SOLUTIONS OF LINEAR AND NONLINEAR EIGENVALUE PROBLEMS USING TAYLOR'S DECOMPOSITION METHOD

by
Meltem ADIYAMAN

June, 2009
İZMİR

NUMERICAL SOLUTIONS OF LINEAR AND NONLINEAR EIGENVALUE PROBLEMS USING TAYLOR'S DECOMPOSITION METHOD

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Ph.D. THESIS EXAMINATION RESULT FORM

We have read the thesis entitled **"NUMERICAL SOLUTIONS OF LINEAR AND NONLINEAR EIGENVALUE PROBLEMS USING TAYLOR'S DECOMPOSITION METHOD"** completed by **MELTEM ADIYAMAN** under supervision of **PROF. DR. ŞENNUR SOMALI** and we certify that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Doctor of Philosophy.

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ABSTRACT

The main purpose of this thesis is to solve regular Sturm-Liouville eigenvalue problems and some special nonlinear eigenvalue problems numerically using Taylor's decomposition method. The numerical scheme is based on the application of the Taylor's decomposition to the corresponding first order differential equation system. The technique is illustrated with three problems, regular Sturm-Liouville eigenvalue problems, Bratu problem and Euler buckling problem. The results show that the method converges rapidly and hence approximates the exact solution very accurately for relatively large step-sizes.

Keywords: Taylor's decomposition, regular Sturm-Liouville eigenvalue problems, nonlinear eigenvalue problems, Bratu problem, Euler Buckling problem.

TAYLOR AYRIŞMA METODU KULLANIMI İLE DOĞRUSAL VE DOĞRUSAL OLMAYAN ÖZDEĞER PROBLEMLERİNİN SAYISAL ÇÖZÜMLERİ

ÖZ

Bu tezin temel amacı homojen ve periyodik Sturm-Liouville özdeğer problemlerini ve bazı doğrusal olmayan özdeğer problemlerini sayısal olarak Taylor ayrışma metodu kullanımı ile çözmektir. Sayısal düzen Taylor ayrışımının incelenen problemlere karşılık gelen birinci mertebeden diferansiyel denklem sistemine uygulanmasına dayanmaktadır. Teknik üç problemle örneklendirilmiştir, homojen ve periyodik Sturm-Liouville özdeğer problemleri, Bratu problemi ve Euler burkulma problemi. Sonuçlar metodun hızla yakınsadığını ve böylece daha geniş adım aralıkları için gerçek çözüme çok iyi bir doğrulukla yaklaştığını göstermiştir.

Anahtar sözcükler: Taylor ayrışımı, homojen ve periyodik Sturm-Liouville özdeğer problemleri, doğrusal olmayan özdeğer problemleri, Bratu Problemi, Euler burkulma problemi .

CONTENTS

CHAPTER ONE
INTRODUCTION

1.1 Introduction

Investigation of the exact and numerical solutions of eigenvalue problems have been focused by some researchers for many years. We refer to Dijksma, & Langer (1996), Fulton (1977) and Walter (1973) and their reference lists which give this long history. Typical topics studied have been on existence and location of the eigenvalues, oscillation, comparison of the eigenfunctions, their completeness, asymptotics, and applications to physics and engineering. Linear eigenvalue problems are well studied in comparison with nonlinear eigenvalue problems, since the nonlinear eigenvalue problems share several difficulties caused from nonlinearity. Some examples of the numerical and analytic treatment of nonlinear eigenvalue problems are given in Anderssen, & de Hoog (1984), Andrew (1988), Andrew (1988), Andrew, & Paine (1986), Belford (1969), Binding, Browne, & Watson (2000), Binding, & Volmer (1996), Bujurke, Salimath, & Shiralashetti (2008), Buckmire (2004), Busca, & Quaas (2004), Euler (1744), Everitt, et al. (1983), Khuri (2004), Dijksma, & Langer (1996), Fulton (1977), Gentry, & Travis (1976), Griffel (1981), Kreiss (1972), Lou, Nie, & Wan (2004), Makin, & Thompson (2004), Odejide, & Aregbesola (2006), Pimbley (1962), Romeiras, & Rowlands (1986), Rynne (1999), Shibita (2002) and Shibita (1996), Somali, & Gokmen (2007), Somali, & Oger (2004), Walter (1973) and Wazwaz (2005). This thesis is concerned with the numerical solutions of regular Sturm-Liouville eigenvalue problems and two nonlinear eigenvalue problems; Bratu problem and Euler buckling problem.

We investigate the computation of eigenvalues of regular Sturm-Liouville eigenvalue problems

$$\begin{aligned} -y''(x)+r(x)y(x) &= \lambda y(x), \qquad 0 \leqslant x_0 < x < x_n \\ y(x_0) &= y(x_n) = 0, \end{aligned}$$

where $r(x) \in C^{p+q}[x_0, x_n]$. There have been a number of papers (see Anderssen and de Hoog, 1984; Andrew, 1988, 1988, 1989; Andrew and Paine, 1986) dealing with the same problem with different boundary conditions using various methods. A survey paper related to this problem can be found in Andrew, (1994). Andrew (Andrew, 1989) used the approach to improve finite difference eigenvalue estimates of periodic Sturm-Liouville eigenvalue problems. It is well known that when finite difference methods are used to approximate the eigenvalues, $\lambda_1 < \lambda_2 < \lambda_3 < \ldots$, of Sturm-Liouville eigenvalue problems, the error in approximation for λ_k is known to increase rapidly with k. In this thesis, we used Taylor's decomposition method to find eigenvalues and corresponding eigenfunctions. The properties and some examples of regular Sturm-Liouville eigenvalue problems are given in chapter 2.

The "Bratu problem" or "Bratu's problem" is defined as $\Delta y + \lambda e^y = 0$ with zero on the boundary. The Bratu problem in 1-dimensional planar coordinates,

$$y'' + \lambda e^y = 0,$$

$$y(0) = y(1) = 0$$

has two known bifurcated exact solutions for values of $\lambda < \lambda_c$, unique solution for $\lambda = \lambda_c$ and no solutions for $\lambda > \lambda_c$. The value of λ_c is simply $8(\alpha^2 - 1)$ where α is the fixed point of the hyperbolic cotangent function $\coth x$. Bratu problem is a nonlinear eigenvalue problem that appears in a number of applications, from the fuel ignition model found in thermal combustion theory (Frank-Kamenetski, 1955) to the Chandrasekhar model for the expansion of the universe (Chandrasekhar, 1957). The exact solution and some applications in science are given in chapter 3.

Another nonlinear eigenvalue problem

$$y'' + \lambda \sin y = 0,$$

$$y'(0) = y'(1) = 0$$

is called Euler buckling problem. This problem concerns the buckling of elastic rods, extensively studied in the last few years (Domokos, & Holmes, 1993, Griffel, 1981, Jones, 2006 and Stakgold, 1971). Stakgold (1971) tells the physical meaning of Euler buckling problem as follows.

The buckling of a thin rod under compression is perhaps the simplest and oldest physical example to illustrate branching.

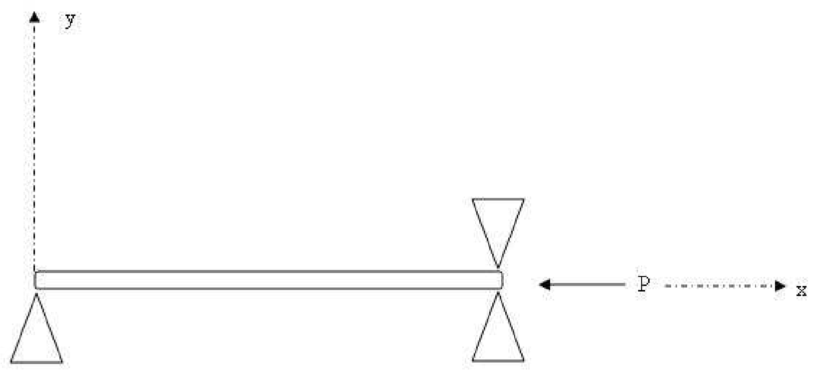

Figure 1.1 Thin rod and horizontal load P.

Figure 1.1 shows a homogeneous, thin rod whose ends are pinned, the left end being fixed, the right end free to move along the x-axis. In its unloaded state the rod coincides with the portion of the x-axis between 0 and 1. Under a compressive load P, a possible state for the rod is that of pure compression, but experience shows that, for sufficiently large values of P, transverse deflection can occur. Assuming that this buckling takes place in x, y-plane, we investigate the equilibrium of forces on a portion of the rod including its left end. The forces and moments are taken positive as drawn in the Figure 1.2. Let X be the original x-coordinate of a material point along the rod. This point is moved after buckling to $(x+u, v)$. We let y be the angle between the tangent to the buckled rod and the x-axis, and s the arclength measured from the left end and $\lambda = P/EI$, E and I are fixed positive, physical constants. The detailed properties of Euler buckling problem are given in chapter 4.

In general, numerical techniques for solving initial value problems for ordinary differential equations are more highly developed than techniques for solving boundary value problems. It is therefore reasonable to be able to reduce to a boundary value problem to the problem of solving one or more initial value problems. In fact, one

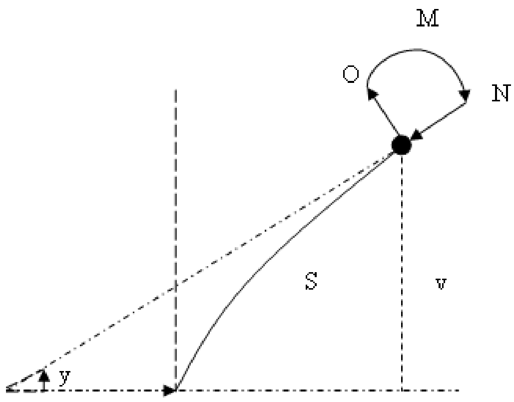

Figure 1.2 Thin rod and horizontal load P.

of the standard methods for solving boundary value problems for linear differential equations involves just such a reduction. In the case of nonlinear equations, the situation is not so straightforward. Now, an eigenvalue problem can be thought of as basically a boundary value problem, but with an additional difficulty on a parameter in the equation that must be simultaneously determined. In this thesis, regular Sturm-Liouville eigenvalue problem the Bratu and Euler buckling problems are solved numerically by converting them into a differential equation systems with initial conditions as recommended above.

The eigenvalues and corresponding eigenfunctions of regular Sturm-Liouville problems, Bratu problem and Euler buckling problem are found approximately by considering the Taylor's decomposition method on two points which is the application of the following theorem given in Ashyralyev, & Sobolevskii (2004)

Theorem 1.1.1. *Let the function* $v(t) \quad (0 \leqslant t \leqslant T)$ *have a* $(p+q+1)$*-th continuous derivative and* $t_{k-1}, t_k \in [0,T]_\tau$*, where*

$$[0,T]_\tau = \{t_k = k\tau, k = 0,1,\ldots,N, N_\tau = T\}. \tag{1.1.1}$$

Then the following relation holds:

$$v(t_k) - v(t_{k-1}) + \sum_{j=1}^{p} \alpha_j v^{(j)}(t_k)\tau^j - \sum_{j=1}^{q} \beta_j v^{(j)}(t_{k-1})\tau^j \tag{1.1.2}$$

$$= \frac{(-1)^p}{(p+q)!} \int_{t_{k-1}}^{t_k} (t_k - s)^q (s - t_{k-1})^p v^{(p+q+1)}(s) ds,$$

where

$$\begin{cases} \alpha_j = \dfrac{(p+q-j)!p!(-1)^j}{(p+q)!j!(p-j)!} & \textit{for any } j, \quad 1 \leqslant j \leqslant p, \\ \\ \beta_j = \dfrac{(p+q-j)!q!}{(p+q)!j!(q-j)!} & \textit{for any } j, \quad 1 \leqslant j \leqslant q. \end{cases} \tag{1.1.3}$$

The main advantage of the method, which computes eigenvalues and corresponding eigenfunctions approximately with high order accuracy for relatively large step sizes, is that it can be applied directly for all types of differential equations, linear or nonlinear, homogeneous or inhomogeneous, with constant or with variable coefficients.

In chapter 2, a method for finding eigenvalues and the corresponding eigenfunctions for regular Sturm-Liouville eigenvalue problems using Taylor's decomposition method is developed. Further error analysis and numerical results of the method are given by comparing the results of other methods. In chapter 3, the application of the Taylor's decomposition method for the nonlinear initial value problem corresponding to the Bratu problem, error analysis and numerical results of the method are discussed. In chapter 4, the eigenvalues and eigenfunctions of Euler buckling problem are found approximately using Taylor's decomposition method. In the conclusion, we summarize the study and present our suggestions regarding future works. The theory of existence, uniqueness and spectral properties of nonlinear eigenvalue problems is given in Appendix A. The theoretical properties, development and some applications of Taylor's decomposition are given in Appendix B.

CHAPTER TWO
TAYLOR'S DECOMPOSITION ON TWO POINTS FOR REGULAR STURM-LIOUVILLE EIGENVALUE PROBLEMS

2.1 Sturm-Liouville Eigenvalue Problems

Sturm-Liouville eigenvalue problems are important in applied mathematics. In recent years there has been a considerable renewal of interest in the Sturm-Liouville eigenvalue problems, from the point of view of both mathematics and their applications to physics and engineering. For many important applications in science and engineering it is required to determine the eigenvalues as well as the corresponding eigenfunctions. In fact, the general theory of eigenvalues and eigenfunctions is one of the deepest and richest part of mathematical physics. In applications, for instance, involving vibration and stability of deformable bodies the vital piece of information required is the smallest eigenvalue (Brunt, 2003 and Frederick, 1995). Engineers are often interested in the location of the smallest eigenvalue since this gives potentially the most visual structure of dynamic systems. The seismic damage to a structure can be catastrophic if its fundamental frequency (related in some way to the smallest eigenvalue) is of the same order as the frequency of the earth quake (Brunt, 2003). The eigenvalues are also crucial in finding the stability region of solutions of Sturm-Liouville eigenvalue problems (Bender, & Orszag, 1987). Generally, finding the eigenvalues and corresponding nontrivial solutions poses a formidable task.

Keller gives the mathematical structure of Sturm-Liouville eigenvalue problems in Keller (2006). If the coefficients of the equation and/or of the boundary conditions depend upon a parameter, it is frequently of interest to determine the value or values of the parameter for which nontrivial solutions exist. These special parameter values are called eigenvalues, and the corresponding nontrivial solutions are called eigenfunctions, and the problems described above are called eigenvalue problems. All along a great deal of interest has been focused on the exact and numerical solutions of the special case of eigenvalue problems, that is, Sturm-Liouville eigenvalue problems

$$\begin{aligned} Ly+\lambda r(x)y=(p(x)y')'-q(x)y+\lambda r(x)y &= 0, \\ a_0y(a)-a_1p(a)y'(a) &= 0, \\ b_0y(b)+b_1p(b)y'(b) &= 0, \end{aligned}$$

where $p(x)>0$, $r(x)>0$ and $q(x)\geqslant 0$ while $p'(x)$, $q(x)$ and $r(x)$ are continuous on $[a,b]$. The constants a_0, a_1, b_0 and b_1 are nonnegative and at least one of each pair does not vanish. It is known that for such problems there exists an infinite sequence of nonnegative eigenvalues

$$0\leqslant\lambda_1<\lambda_2\leqslant\lambda_3\ldots.$$

In addition there exist corresponding eigenfunctions, $y_n(x)$ which are twice continuously differentiable and satisfy the orthogonal relations:

$$\int_a^b y_n(x)y_m(x)r(x)\,dx=\delta_{nm}, \qquad n,\,m=1,2,\ldots.$$

The reader can find much information about Sturm-Liouville eigenvalue problems in Keller (2006).

In section 2, we consider the regular Sturm-Liouville eigenvalue problems

$$\begin{aligned} -y''(x)+r(x)y(x) &= \lambda y(x), \qquad 0\leqslant x_0<x<x_n \\ y(x_0) &= y(x_n)=0, \end{aligned}$$

where $r(x)\in C^{p+q}[x_0,x_n]$. The behavior of eigenvalues and corresponding eigenfunctions are obtained by Taylor's decomposition method. In section 3, a bound of the error between the exact solution and approximate solution of regular Sturm-Liouville eigenvalue problem and the convergence of the method for constant functions $r(x)$ are given. The technique is illustrated with two examples and the numerical results are given by comparing the results of other methods in section 4.

2.2 Taylor's Decomposition on Two Points For Regular Sturm-Liouville Eigenvalue Problems

We consider the regular Sturm-Liouville eigenvalue problem

$$\begin{aligned} -y''(x)+r(x)y(x) &= \lambda y(x), \qquad 0 \leqslant x_0 < x < x_n, \\ y(x_0) &= y(x_n)=0, \end{aligned} \tag{2.2.1}$$

where $r(x) \in C^{p+q}[x_0,x_n]$. Introducing a new depending variable $y'(x)=z(x)$, (2.2.1) can be written as

$$\begin{bmatrix} y'(x) \\ z'(x) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ r(x)-\lambda & 0 \end{bmatrix} \begin{bmatrix} y(x) \\ z(x) \end{bmatrix},$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y(x_0) \\ z(x_0) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y(x_n) \\ z(x_n) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Defining $Y(x) = \begin{bmatrix} y(x) \\ z(x) \end{bmatrix}$, $A(x) = \begin{bmatrix} 0 & 1 \\ r(x)-\lambda & 0 \end{bmatrix}$, $C_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $C_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, we have

$$\begin{aligned} Y'(x) &= A(x)Y(x) \\ C_0Y(x_0)+C_1Y(x_n) &= \mathbf{0}. \end{aligned} \tag{2.2.2}$$

Following Ashyralyev, & Sobolevskii (2004), we will consider the application of Taylor's decomposition of function $Y(x)$ on two points. We need to find $Y^{(j)}(x)$ for any $1 \leqslant j \leqslant p$ and q. Using the equation $Y'(x)=A(x)Y(x)$, we get

$$Y^{(j)}(x) = A_j(x)Y(x), \tag{2.2.3}$$

where

$$A_0(x) = I,$$

$$A_1(x) = A(x),$$

$$A_j(x) = A'_{j-1}(x) + A_{j-1}(x)A(x), \quad 2 \leqslant j \leqslant p,$$

where I is the 2×2 identity matrix. By using the structure of the matrix $A(x)$, we obtain the entries of the matrix of $A_j(x) = \begin{bmatrix} a_{j(1,1)}(\lambda;x) & a_{j(1,2)}(\lambda;x) \\ a_{j(2,1)}(\lambda;x) & a_{j(2,2)}(\lambda;x) \end{bmatrix}$ as in the following formulas

$$\begin{aligned}
a_{j(1,1)}(\lambda;x) &= \frac{\partial a_{j-1(1,1)}(\lambda;x)}{\partial x} + (r(x)-\lambda)a_{j-2(2,2)}(\lambda;x) \\
&= a_{j-1(2,1)}(\lambda;x) \\
a_{j(2,2)}(\lambda;x) &= \frac{\partial a_{j-1(2,2)}(\lambda;x)}{\partial x} + a_{j(1,1)}(\lambda;x) \\
a_{j(1,2)}(\lambda;x) &= a_{j-1(2,2)}(\lambda;x) \\
a_{j(2,1)}(\lambda;x) &= -\frac{\partial a_{j(2,2)}(\lambda;x)}{\partial x} + a_{j+1(2,2)}(\lambda;x)
\end{aligned} \tag{2.2.4}$$

for $2 \leqslant j \leqslant p$, where

$$\begin{aligned}
&a_{0(1,1)}(\lambda;x) = 1, \quad a_{1(1,1)}(\lambda;x) = 0, \\
&a_{0(1,2)}(\lambda;x) = 0, \quad a_{1(1,2)}(\lambda;x) = 1, \\
&a_{0(2,1)}(\lambda;x) = 0, \quad a_{1(2,1)}(\lambda;x) = r(x)-\lambda, \\
&a_{0(2,2)}(\lambda;x) = 1, \quad a_{1(2,2)}(\lambda;x) = 0.
\end{aligned}$$

From the Theorem 1.1.1, we have the following relation

$$Y(x_k) - Y(x_{k-1}) + \sum_{j=1}^{p} \alpha_j Y^{(j)}(x_k)h^j - \sum_{j=1}^{q} \beta_j Y^{(j)}(x_{k-1})h^j$$

$$= \frac{(-1)^p}{(p+q)!} \int_{x_{k-1}}^{x_k} (x_k - s)^q (s - x_{k-1})^p Y^{(p+q+1)}(s)ds \tag{2.2.5}$$

on the uniform grid

$$[x_0, x_n]_h = \{x_k = x_0 + kh, k = 0,1,\ldots,n,\ nh = x_n - x_0, n \in N\},$$

where

$$\begin{aligned}
\alpha_j &= \frac{(p+q-j)!p!(-1)^j}{(p+q)!j!(p-j)!}, \qquad 1 \leqslant j \leqslant p, \\
\beta_j &= \frac{(p+q-j)!q!}{(p+q)!j!(q-j)!}, \qquad 1 \leqslant j \leqslant q.
\end{aligned} \tag{2.2.6}$$

Rewriting the formula (2.2.5) by neglecting the last term we obtain a one step difference scheme of $(p+q)$-order of accuracy for the approximate solution of problem (2.2.2)

$$Y_k - Y_{k-1} + \sum_{j=1}^{p} \alpha_j A_j(x_k) Y_k h^j - \sum_{j=1}^{q} \beta_j A_j(x_{k-1}) Y_{k-1} h^j = 0, \qquad (2.2.7)$$

where $Y_k = \begin{bmatrix} y_k \\ z_k \end{bmatrix}$ is the approximate value of $Y(x_k)$. For a simple computation, let $p = q$, then we have

$$\left(I + \sum_{j=1}^{p} \alpha_j A_j(x_k) h^j\right) Y_k = \left(I + \sum_{j=1}^{p} (-1)^j \alpha_j A_j(x_{k-1}) h^j\right) Y_{k-1},$$

where

$$\begin{aligned} \alpha_j &= \frac{(2p-j)!p!(-1)^j}{(2p)!j!(p-j)!}, \\ \beta_j &= \frac{(2p-j)!p!}{(2p)!j!(p-j)!} = (-1)^j \alpha_j. \end{aligned}$$

Letting $M(x_k) = \left(I + \sum_{j=1}^{p} \alpha_j A_j(x_k) h^j\right)$ and $N(x_{k-1}) = \left(I + \sum_{j=1}^{p} (-1)^j \alpha_j A_j(x_{k-1}) h^j\right)$ we write

$$Y_k = M^{-1}(x_k) N(x_{k-1}) Y_{k-1}. \qquad (2.2.8)$$

Since the accuracy and convergence of the method is independent of h, taking $h = x_n - x_0$ gives

$$Y_1 = M^{-1}(x_n) N(x_0) Y_0$$

and substituting the boundary condition of (2.2.1), we get

$$\left(C_1 M^{-1}(x_n) N(x_0) + C_0\right) Y_0 = \mathbf{0}.$$

To obtain a nontrivial solution Y_0, we must have the following equation

$$\det(C_1 M^{-1}(x_n) N(x_0) + C_0) = 0.$$

Defining $M(x_n) = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$ and $N(x_0) = \begin{bmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{bmatrix}$, we have

$$\left(C_1 M^{-1}(x_n)N(x_0) + C_0\right) =$$

$$= \frac{1}{\det(M)} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} m_{22} & -m_{12} \\ -m_{21} & m_{11} \end{bmatrix} \begin{bmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ \dfrac{m_{22}n_{11} - m_{12}n_{21}}{detM} & \dfrac{m_{22}n_{12} - m_{12}n_{22}}{detM} \end{bmatrix}.$$

For $\det(C_1 M^{-1}(x_n)N(x_0) + C_0) = \dfrac{m_{22}n_{12} - m_{12}n_{22}}{detM} = 0$, we must have the following statement

$$m_{22}n_{12} - m_{12}n_{22} = 0. \tag{2.2.9}$$

Since

$$M(x_n) = \left(I + \sum_{j=1}^{p} \alpha_j A_j(x_n) h^j\right) =$$

$$\begin{bmatrix} 1 + \sum\limits_{j=1}^{p} \alpha_j a_{j(1,1)}(\lambda; x_n) h^j & \sum\limits_{j=1}^{p} \alpha_j a_{j(1,2)}(\lambda; x_n) h^j \\ \sum\limits_{j=1}^{p} \alpha_j a_{j(2,1)}(\lambda; x_n) h^j & 1 + \sum\limits_{j=1}^{p} \alpha_j a_{j(2,2)}(\lambda; x_n) h^j \end{bmatrix},$$

$$N(x_0) = \left(I + \sum_{j=1}^{p} (-1)^j \alpha_j A_j(x_0) h^j\right) =$$

$$\begin{bmatrix} 1 + \sum\limits_{j=1}^{p} (-1)^j \alpha_j a_{j(1,1)}(\lambda; x_0) h^j & \sum\limits_{j=1}^{p} (-1)^j \alpha_j a_{j(1,2)}(\lambda; x_0) h^j \\ \sum\limits_{j=1}^{p} (-1)^j \alpha_j a_{j(2,1)}(\lambda; x_0) h^j & 1 + \sum\limits_{j=1}^{p} (-1)^j \alpha_j a_{j(2,2)}(\lambda; x_0) h^j \end{bmatrix},$$

using the entries m_{12}, m_{22}, n_{12} and n_{22} of the above matrices we obtain (2.2.9) interms of λ,

$$m_{22}n_{12} - m_{12}n_{22} = \left(1 + \sum_{j=1}^{p} \alpha_j a_{j(2,2)}(\lambda; x_n) h^j\right) \left(\sum_{j=1}^{p} (-1)^j \alpha_j a_{j(1,2)}(\lambda; x_0) h^j\right)$$

$$-\left(\sum_{j=1}^{p}\alpha_j a_{j(1,2)}(\lambda;x_n)h^j\right)\left(1+\sum_{j=1}^{p}(-1)^j\alpha_j a_{j(2,2)}(\lambda;x_0)h^j\right).$$

Since the entries of $A_j(x)$ are defined interms of diagonal entries in (2.2.4), we write

$$\begin{aligned} F(\lambda) &= m_{22}n_{12}-m_{12}n_{22} \\ &= \left(1+\sum_{j=1}^{p}\alpha_j a_{j(2,2)}(\lambda;x_n)h^j\right)\left(\sum_{j=1}^{p}(-1)^j\alpha_j a_{j-1(2,2)}(\lambda;x_0)h^j\right) \end{aligned}$$

$$-\left(\sum_{j=1}^{p}\alpha_j a_{j-1(2,2)}(\lambda;x_n)h^j\right)\left(1+\sum_{j=1}^{p}(-1)^j\alpha_j a_{j(2,2)}(\lambda;x_0)h^j\right). \tag{2.2.10}$$

Solving the nonlinear equation $F(\lambda)=0$ by Newton's method, we find the approximate eigenvalues.

To find the corresponding eigenfunctions of the regular Sturm-Liouville eigenvalue problem (2.2.1), we substitute the eigenvalue to (2.2.1) and we solve the obtained boundary value problem by Taylor's decomposition method on two points x_{k-1} and x_k with the uniform grid

$$[x_0,x_n]_h=\{x_k=x_0+kh, k=0,1,\ldots,n,\ nh=x_n-x_0, n\in N\}$$

for $p=q$. Then we get a homogeneous linear equation system of $2n$ equations with $2n$ unknowns $z_0,\ y_1,\ z_1,\ y_2,\ z_2\ldots,y_{n-1},\ z_{n-1},z_n$ which are the approximated values of $y'(x_0),\ y(x_1),\ y'(x_1)\ y(x_2)\ y'(x_2),\ldots,y(x_{n-1}),\ y'(x_{n-1}),\ y'(x_n)$ respectively. Solving the $2n\times 2n$ homogeneous system, we obtain approximate values of the eigenfunction and the derivative of (2.2.1) at the point $x=x_k$.

2.3 Error Analysis

In this section we will show the convergence of the method for eigenfunctions with the constant function $r(x)=c$ by obtaining approximate value of eigenfunction at a point $\bar{x} \in [x_0, x_n]$ of the problem

$$\begin{aligned} -y''(x) + cy(x) &= \lambda y(x), \qquad 0 \leqslant x_0 < x < x_n, \\ y(0) = 0,\ y(x_n) &= 0. \end{aligned} \tag{2.3.1}$$

We know from the theory of the Sturm-Liouville eigenvalue problems the eigenvalues $k^2\pi^2 - c > 0$ of (2.3.1) are positive. Without loss of generality, we may choose $r(x) = 0$ then $A_j(x) = A_j$, that is, $a_{j(2,2)}(\lambda; x_n) = a_{j(2,2)}(\lambda; 0) = a_{j(2,2)}(\lambda)$. Using (2.2.4), we can find explicit values of $a_{j(1,1)}$, $a_{j(2,2)}$ as follows

$$\begin{aligned} a_{2j(1,1)} &= (-1)^j \lambda^j, \\ a_{2j(2,2)} &= (-1)^j \lambda^j, \\ a_{2j+1(1,1)} &= 0, \\ a_{2j+1(2,2)} &= 0, \qquad j \geqslant 0. \end{aligned}$$

This yields

$$\begin{aligned} m_{22} &= 1 + \sum_{j=1}^{\lfloor p/2 \rfloor} \alpha_{2j} a_{2j(2,2)} h^{2j} + \sum_{j=0}^{\lfloor p/2 \rfloor} \alpha_{2j+1} a_{2j+1(2,2)} h^{2j+1} \\ &= 1 + \sum_{j=1}^{\lfloor p/2 \rfloor} \alpha_{2j} (-1)^j \lambda^j h^{2j}, \\ n_{22} &= 1 + \sum_{j=1}^{\lfloor p/2 \rfloor} (-1)^{2j} \alpha_{2j} a_{2j(2,2)} h^{2j} + \sum_{j=0}^{\lfloor p/2 \rfloor} (-1)^{2j+1} \alpha_{2j+1} a_{2j+1(2,2)} h^{2j+1} \\ &= 1 + \sum_{j=1}^{\lfloor p/2 \rfloor} (-1)^j \alpha_{2j} \lambda^j h^{2j} \\ &= m_{22}, \\ m_{12} &= \sum_{j=1}^{p} \alpha_j a_{j-1(2,2)} h^j = \sum_{j=0}^{p-1} \alpha_{j+1} a_{j(2,2)} h^{j+1} \\ &= \sum_{j=0}^{\lfloor \frac{p-1}{2} \rfloor} \alpha_{2j+1} a_{2j(2,2)} h^{2j+1} + \sum_{j=0}^{\lfloor \frac{p-1}{2} \rfloor} \alpha_{2j+2} a_{2j+1(2,2)} h^{2j+2} \\ &= \sum_{j=0}^{\lfloor \frac{p-1}{2} \rfloor} \alpha_{2j+1} (-1)^j \lambda^j h^{2j+1}, \end{aligned}$$

$$\begin{aligned}
n_{12} &= \sum_{j=1}^{p}(-1)^j\alpha_j a_{j-1(2,2)}h^j = \sum_{j=0}^{p-1}(-1)^{j+1}\alpha_{j+1}a_{j(2,2)}h^{j+1} \\
&= \sum_{j=0}^{\lfloor\frac{p-1}{2}\rfloor}(-1)^{2j+1}\alpha_{2j+1}a_{2j(2,2)}h^{2j+1} + \sum_{j=0}^{\lfloor\frac{p-1}{2}\rfloor}(-1)^{2j+2}\alpha_{2j+2}a_{2j+1(2,2)}h^{2j+2} \\
&= \sum_{j=0}^{\lfloor\frac{p-1}{2}\rfloor}(-1)^{j+1}\alpha_{2j+1}\lambda^j h^{2j+1} \\
&= -m_{12}, \\
m_{11} &= 1+\sum_{j=1}^{p}\alpha_j a_{j(1,1)}h^j = 1+\sum_{j=1}^{p}\alpha_j a_{j(2,2)}h^j \\
&= m_{22}, \\
m_{21} &= \sum_{j=1}^{p}\alpha_j a_{j(2,1)}h^j = \sum_{j=1}^{p}\alpha_j a_{j+1(2,2)}h^j \\
&= -\lambda\sum_{j=1}^{p}\alpha_j a_{j+2(1,2)}h^j \\
&= -\lambda m_{12}.
\end{aligned}$$

Using (2.2.8) for $k=1$, we have

$$Y_1 = M^{-1}(\bar{x})N(x_0)Y_0, \tag{2.3.2}$$

where Y_0 and Y_1 are the approximated values of $Y(x_0)$ and $Y(\bar{x})$ respectively with the stepsize $h=\bar{x}-x_0$.

$$\begin{aligned}
Y_1 &= \frac{1}{\det(M)}\begin{bmatrix} m_{22} & -m_{12} \\ -m_{21} & m_{22}\end{bmatrix}\begin{bmatrix} n_{11} & n_{12} \\ n_{21} & n_{22}\end{bmatrix}\begin{bmatrix} 0 \\ z_0\end{bmatrix} \\
&= \frac{z_0}{\det(M)}\begin{bmatrix} -2m_{22}m_{12} \\ -\lambda(m_{12})^2+(m_{22})^2\end{bmatrix}.
\end{aligned} \tag{2.3.3}$$

The first component of the above vector (2.3.3) gives the approximate eigenfunction y_1 and the second component of the above vector (2.3.3) gives the derivative of the approximate eigenfunction z_1 of the regular Sturm-Liouville problem (2.3.1) at $\bar{x}$. Now we will show that y_1 and z_1 converge to exact functions $y(\bar{x})$ and $y'(\bar{x})$ respectively as $p\to\infty$.

Using the Stirling's Formula $n! \approx \sqrt{2\pi}\, n^{\frac{n+1}{2}}\ e^{-n}$ for α_j in (2.2.6), we obtain

$$
\begin{aligned}
\alpha_j &= \frac{(2p-j)!p!(-1)^j}{(2p)!j!(p-j)!} \\
&\approx (-1)^j \frac{(2p-j)^{\frac{2p-j+1}{2}} \sqrt{2\,\pi}\, e^{-(2p-j)} p^{\frac{p+1}{2}} \sqrt{2\,\pi}\, e^{-p}}{j!\,(2p)^{\frac{2p+1}{2}} \sqrt{2\pi}\, e^{-2p} (p-j)^{\frac{p-j+1}{2}} \sqrt{2\,\pi}\, e^{-(p-j)}} \\
&\approx (-1)^j \frac{1}{j!} \frac{2^{\frac{2p-j+1}{2}} (p-\frac{j}{2})^{\frac{2p-j+1}{2}}}{2^{\frac{2p+1}{2}} p^{\frac{p}{2}} (p-j)^{\frac{p-j+1}{2}}} \\
&\approx (-1)^j \frac{1}{j!} \frac{1}{2^j} \left(\frac{p-\frac{j}{2}}{p-j}\right)^{\frac{p-j+1}{2}} \left(\frac{p-\frac{j}{2}}{p}\right)^{\frac{p}{2}}.
\end{aligned}
$$

This gives,

$$\lim_{p\to\infty} \alpha_j = (-1)^j \frac{1}{j!}\frac{1}{2^j}.$$

Thus,

$$
\begin{aligned}
\lim_{p\to\infty} m_{22} &= \lim_{p\to\infty} \left(1 + \sum_{j=1}^{\lfloor \frac{p}{2} \rfloor} \alpha_{2j} (-1)^j \lambda^j\, h^{2j}\right) \\
&= 1 + \sum_{j=1}^{\infty} \frac{1}{(2j)!} \frac{1}{2^{2j}} (-1)^j \lambda^j\, h^{2j} \\
&= \sum_{j=0}^{\infty} (-1)^j \left(\frac{\sqrt{\lambda}\, h}{2}\right)^{2j} \frac{1}{(2j)!} \\
&= \cos(\sqrt{\lambda}) \frac{h}{2}.
\end{aligned}
\tag{2.3.4}
$$

Using the same idea we obtain

$$
\begin{aligned}
\lim_{p\to\infty} m_{12} &= \lim_{p\to\infty} \left(\sum_{j=0}^{\lfloor \frac{p-1}{2} \rfloor} (-1)^j \alpha_{2j+1} (\lambda)^j\, h^{2j+1}\right) \\
&= \frac{1}{\sqrt{\lambda}} \sin(\sqrt{\lambda})\, \frac{h}{2}.
\end{aligned}
\tag{2.3.5}
$$

It follows from (2.3.4) and (2.3.5) that

$$\begin{aligned}
\lim_{p\to\infty} \det(M) &= m_{22}^2 + \lambda m_{12}^2 \\
&= \cos^2(\sqrt{\lambda})\frac{h}{2} + \lambda\left(\frac{1}{\sqrt{\lambda}}\sin(\sqrt{\lambda})\,\frac{h}{2}\right)^2 \\
&= 1.
\end{aligned}$$

Hence the approximate eigenfunction of (2.3.1) to the corresponding eigenvalue λ converges to exact eigenfunction

$$\begin{aligned}
\lim_{p\to\infty} y_1 &= 2\frac{z_0}{\det(M)}\frac{1}{\sqrt{\lambda}}(\cos(\sqrt{\lambda}\,\frac{h}{2}))(\sin(\sqrt{\lambda}\,\frac{h}{2})) \\
&= \frac{z_0}{\sqrt{\lambda}}\sin(\sqrt{\lambda}\,(\bar{x}-x_0)).
\end{aligned}$$

Since we have $z(x) = y'(x)$, the derivative of approximate eigenfunction of (2.3.1) to the corresponding eigenvalue λ converges to derivative of the exact solution

$$\begin{aligned}
\lim_{p\to\infty} z_1 &= \frac{z_0}{\det(M)}\left((-\lambda)\frac{1}{\lambda}\sin^2(\sqrt{\lambda}\,\frac{h}{2}) + \cos^2(\sqrt{\lambda}\,\frac{h}{2})\right) \\
&= z_0\,\cos(\sqrt{\lambda}\,(\bar{x}-x_0)),
\end{aligned}$$

where $\lambda = k^2\pi^2$, $k = 1,2,\ldots$.

2.4 Numerical Results for Regular Sturm-Liouville Eigenvalue Problems

We consider two regular Sturm-Liouville eigenvalue problems, one of them has polynomial coefficients and the other has periodic coefficients taken from Bujurke, Salimath, & Shiralashetti (2008).

Example 1: Consider the Titchmarch equation

$$\begin{aligned}
y'' + (\lambda - x^{2n})y(x) &= 0, \\
y(0) = y(1) &= 0,
\end{aligned}$$

where n is a nonnegative integer. We obtain the numerical solutions taking $n = 0, 2$. The accuracy of the method is tested by comparing with the exact solution which exists when $n = 0$ and Finite Difference Method (FDM) solution when $n = 2$.

Tables 2.1 and 2.2 give computed eigenvalues and the solution $y(x)$ of Titchmarch problem using Taylor's decomposition method (TDM), Haar wavelet series method (HWSM) and FDM for $p = 16$ and $n = 0, 2$, the integer parameter in Titchmarch problem.

Example 2: Consider the Mathieu's equation

$$\begin{aligned} y'' + (\lambda - 2\theta\cos(2x))y &= 0, \\ y(0) = y(\pi) &= 0. \end{aligned}$$

We will solve these two problems approximately using Taylor's decomposition method (TDM) and we will compare our results with the results in Bujurke, Salimath, & Shiralashetti (2008). Bujurke, Salimath, & Shiralashetti (2008) that solve Example 1 and Example 2 approximately using Haar wavelets. So they transform the interval $[0, \pi]$ to $[0, 1]$ because of the properties of Haar wavelets. So, to compare the results we normalize the interval $[0, \pi]$ by using $x = \pi t$, the Mathieu's equation in Example 2 transformed into

$$\begin{aligned} y'' + (\pi^2\lambda - 2\pi^2\theta\cos(2\pi t))y &= 0, \\ y(0) = y(1) &= 0. \end{aligned}$$

The estimation of the eigenvalues for this problem is more complicated to the problems discussed Example 1. We obtain eigenpairs corresponding to a fixed value of $\theta = 5$, demonstrating the fact that the first eigenvalue can even be negative, a distinguishing feature of Mathieu's equation. We also demonstrate graphically the fact that the first eigenfunction has no zeros in $(0, 1)$ and the nth eigenfunction has $n - 1$ zeros in $(0, 1)$ Binding, & Volmer (1996) and Everitt, et al. (1983) (see Fig. 2.1). Table 2.3 gives the asymptotic behavior of higher eigenvalues of Mathieu's equation and these eigenvalues are $\lambda_n = n^2 + O(1)$, which is consistent with the classical theorem on asymptoticity of the eigenvalues $\lim_{n\to\infty} \lambda_n^{1/2}/n = 1$ from Brunt (2003). Shifting symmetry of solutions for selected values of parameter is displayed in

Figure 2.2.

The numerical calculations and all figures in this work are performed using Mathematica.

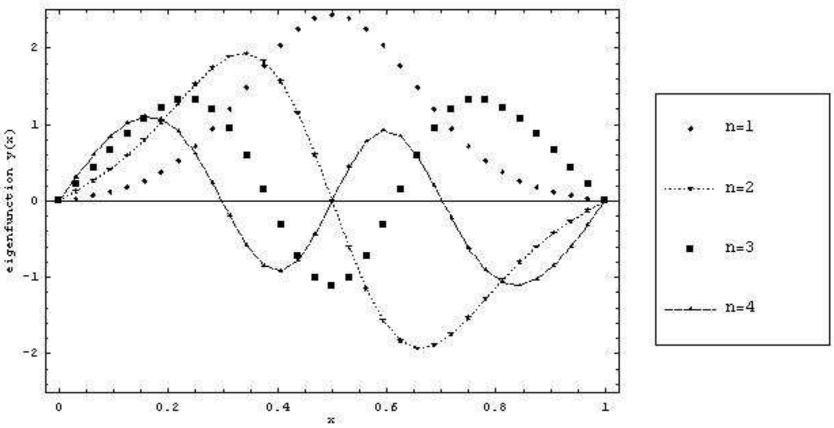

Figure 2.1 Higher eigenfunctions of Mathieu's equation for a fixed parameter $\theta = 5$.

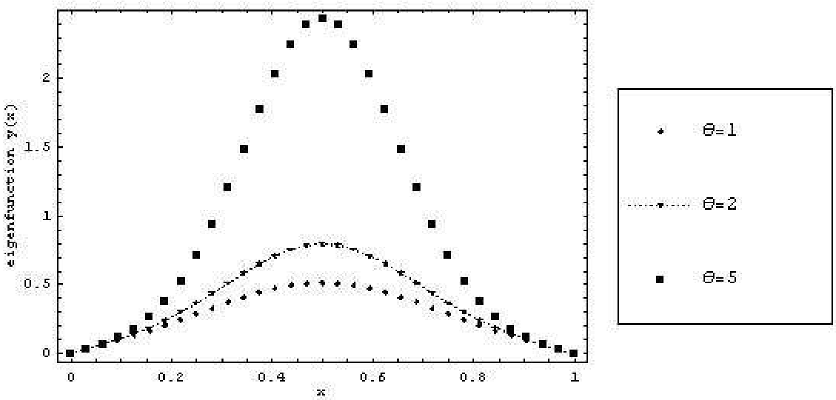

Figure 2.2 Solutions of Mathieu's equation for different parameters of θ.

Table 2.1 Comparison of the first eigenvalue and solutions of Example 1 using Taylor's decomposition method, exact values and the Table 4 from Bujurke, Salimath, & Shiralashetti (2008), when $p = 16$, $n = 0$ and $h = 0.0625$.

x	HWSM	FDM	TDM	Exact
0	0	0	0	0
0.0625	0.27521	0.278599	0.275899	0.275899
0.125	0.54181	0.541196	0.541196	0.541196
0.1875	0.78549	0.785695	0.785695	0.785695
0.25	1.00482	1	1	1
0.3125	1.17851	1.17588	1.17588	1.17588
0.375	1.31285	1.30656	1.30656	1.30656
0.4375	1.38376	1.38704	1.38704	1.38704
0.5	1.41103	1.41421	1.41421	1.41421
0.5625	1.38376	1.38704	1.38704	1.38704
0.625	1.31285	1.30656	1.30656	1.30656
0.6875	1.17851	1.17588	1.17588	1.17588
0.75	1.00482	1	1	1
0.8125	0.78549	0.785695	0.785695	0.785695
0.875	0.54181	0.541196	0.541191	0.541191
0.9375	0.27521	0.275899	0.275899	0.275899
1	0	0	0	0

$\lambda_1 = 10.9334$ (HWSM), 10.8379 (FDM), 10.8696 (TDM), 10.8696 (Exact)

In the Table 2.4 the observed orders $ord(h)$ are computed using the following formula

$$ord(h) = \frac{\log \frac{y_{4h} - y_{2h}}{y_{2h} - y_h}}{\log 2}$$

where y_{4h}, y_{2h} and y_h are the approximated value of eigenfunctions at x_k to the corresponding eigenvalue λ when the problems are solved with stepsizes $4h$, $2h$ and h respectively. The observed orders given in the following tables are well confirm the theoretical results. That is the order of Taylor's decomposition method is order of $2p$.

Table 2.2 Comparison of the first eigenvalue and solutions of Example 1 using Taylor's decomposition method and the Table 4 from Bujurke, Salimath, & Shiralashetti (2008), when $p = 16$, $n = 2$ and $h = 0.0625$.

x	HWSM	FDM	TDM
0	0	0	0
0.0625	0.27521	0.27756	0.277563
0.125	0.54181	0.54434	0.544337
0.1875	0.78949	0.78996	0.789953
0.25	1.00485	1.00488	1.00487
0.3125	1.18153	1.18075	1.18074
0.375	1.31286	1.31082	1.31076
0.4375	1.39372	1.38996	1.38994
0.5	1.42102	1.41527	1.41529
0.5625	1.39371	1.38591	1.38598
0.625	1.31285	1.30323	1.30334
0.6875	1.18154	1.18066	1.17081
0.75	1.0048	0.99361	0.993792
0.8125	0.77949	0.77917	0.779357
0.875	0.53481	0.53577	0.535934
0.9375	0.27726	0.27277	0.272878
1	0	0	0

$\lambda_1 = 10.3452$ (HWSM), 9.95067 (FDM), 9.98317 (TDM).

Table 2.3 Comparison of higher eigenvalues for Mathieu's equation obtained from FDM, HWSM and TDM corresponding to $\theta = 5$.

n	n^2	λ_n(FDM)	λ_n (HWSM)	λ_n (TDM)
1	1	-57311	-5.4665	-5.79008
2	4	2.0992	2.6161	2.09946
3	9	9.2365	9.4227	9.23633
4	16	16.648	16.3707	16.6482
5	25	25.511	24.1471	25.5108
6	36	36.359	36.6577	36.3589

$\lambda_1 = -5.46653$ (HWSM), -5.73115 (FDM), -5.79008 (TDM).

Table 2.4 Comparison of the solutions corresponding to the first eigenvalue for different step-sizes and observed orders of Example 1 for $n = 2$ at $x = 1/2$ using Taylor's decomposition method.

	$p = 2$	$p = 3$	$p = 4$
$n = 2$	-1.6205094696443345	-1.9314644751012906	-2.0235744281581614
$n = 4$	-2.0166390991925601	-2.038800266323615	-2.0392696315435903
$n = 8$	-2.037712615075868	-2.039247754178566	-2.0392599136760863
$n = 16$	-2.0391623932478797	-2.039259748585679	-2.0392599481929437
$n = 32$	-2.039253807660177	-2.0392599452145523	-2.0392599483660714
Observed Orders			
$ord(1/8)$	4.23247	7.90607	10.6574
$ord(1/16)$	3.86153	5.22141	8.13719
$ord(1/32)$	3.98674	5.93074	7.63932

CHAPTER THREE
BRATU PROBLEM

The boundary problem

$$\begin{aligned} y'' + \lambda e^{y} = 0, \qquad 0 < x < 1, \\ y(0) = 0 \quad \text{and} \quad y(1) = 0 \end{aligned} \tag{3.0.1}$$

is referred to as the Bratu Problem in 1-dimensional planar coordinates. It is nonlinear eigenvalue problem with two known bifurcated solutions for $\lambda < \lambda_c$ and no solutions for $\lambda > \lambda_c$, and a unique solution when $\lambda = \lambda_c$. The classical Bratu problem (see Buckmire, 2003);

$$\begin{aligned} \triangle u + \lambda e^{u} = 0 \quad &\text{on} \quad \Omega : \{(x,y) \in 0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant 1\}, \\ u = 0 \quad &\text{on} \quad \partial\Omega \end{aligned}$$

is a nonlinear elliptical partial differential equation that appears in a number of applications, from the fuel ignition model found in thermal combustion theory (Frank-Kamenetski, 1955) to the Chandrasekhar model for the expansion of the universe (Chandrasekhar, 1957). It is also a nonlinear eigenvalue problem that is often used as a benchmarking tool for numerical methods (Abbott, 1978 and Ascher, Mattheji, & Russell, 1995) due to the bifurcation nature of the solution for $\lambda < \lambda_c$. In Jacobsen, & Schmitt (2002), Jacobsen and Schmitt provide an excellent summary of the significance and history of Bratu problem. Several numerical techniques, such as Mickens Finite difference scheme (Buckmire, 2004), weighed residual method (Odejide, & Aregbesola, 2006), Adomian decomposition method (Wazwaz, 2005) and Laplace transform decomposition numerical algorithm (Khuri,2004) have been implemented independently to handle the Bratu model numerically.

The exact solution to (3.0.1) is given in Buckmire (2003), Khuri (2004) and Wazwaz (2005) and represented here as

$$y(x) = -2\ln\left[\frac{\cosh\left((x-\frac{1}{2})\frac{\theta}{2}\right)}{\cosh(\frac{\theta}{4})}\right] \tag{3.0.2}$$

where θ solves

$$\theta = \sqrt{2\lambda}\cosh\left(\frac{\theta}{4}\right). \tag{3.0.3}$$

There are two solutions to (3.0.3) for values $0 < \lambda < \lambda_c$. For $\lambda > \lambda_c$ there is no solution. The solution (3.0.2) is unique only for a critical value of $\lambda = \lambda_c$ which solves

$$1 = \sqrt{2\lambda_c}\sinh\left(\frac{\theta_c}{4}\right)\frac{1}{4}. \tag{3.0.4}$$

The critical value θ_c is

$$\theta_c = 4.79871456.$$

The exact value of θ_c can therefore be used in (3.0.4) to obtain the exact value of λ_c

$$\lambda_c = \frac{8}{\sinh^2\left(\frac{\theta_c}{4}\right)} = 3.513830719.$$

This chapter presents Taylor's decomposition method for solving the nonlinear 1-dimensional Bratu problem (3.0.1). The algorithm illustrates how the Taylor's decomposition technique (Ashyralyev, & Sobolevskii, 2004) can be efficiently manipulated to approximate the solution of this non-linear boundary value problem. In section 2, the computation of the eigenvalues of the problem by using Taylor's decomposition method is given. In section 3, the application and error analysis of the method for the nonlinear initial value problem corresponding to the Bratu problem are discussed. The last section demonstrates numerically accurate solutions to 1-dimensional Bratu problem for some $\lambda \leqslant \lambda_c$ eigenvalues.

3.1 Computation of Eigenvalues and Eigenfunctions by Taylor's Decomposition Method

For convenience we introduce the following notations as in chapter 2;

$$\begin{aligned} Y'(x) &= F(Y(x)), \qquad 0 < x < 1, \\ A_0Y(0) + A_1Y(1) &= \mathbf{0}, \end{aligned} \tag{3.1.1}$$

where

$$Y(x) = \begin{bmatrix} y(x) \\ z(x) \end{bmatrix},\ F(Y(x)) = \begin{bmatrix} f_1^{(0)}(y,z) \\ f_2^{(0)}(y,z) \end{bmatrix},$$

$$A_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\ A_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},\ f_1^{(0)}(y,z) = z \text{ and } f_2^{(0)}(y,z) = -\lambda e^y.$$

Defining the following recurrence relations for $j = 1,\dots,2p$,

$$f_1^{(j)}(y,z) = z\,\frac{\partial f_1^{(j-1)}(y,z)}{\partial y} - \lambda e^y\,\frac{\partial f_1^{(j-1)}(y,z)}{\partial z}, \tag{3.1.2}$$

and

$$f_2^{(j)}(y,z) = z\,\frac{\partial f_2^{(j-1)}(y,z)}{\partial y} - \lambda e^y\,\frac{\partial f_2^{(j-1)}(y,z)}{\partial z} \tag{3.1.3}$$

we get

$$f_1^{(j)}(y,z) = f_2^{(j-1)}(y,z) \quad \text{for} \quad j = 1,\dots,2p, \tag{3.1.4}$$

$$Y^{(j)}(x) = \begin{bmatrix} f_1^{(j-1)}(y,z) \\ f_2^{(j-1)}(y,z) \end{bmatrix} \quad \text{for} \quad j = 1,\dots,2p+1. \tag{3.1.5}$$

We first give the following lemma which defines $f_2^{(j-1)}(y,z)$ explicitly.

Lemma 3.1.1. *For $j = 1,\dots,2p$, let $f_2^{(j)}(y,z)$ satisfies the recurrence relation (3.1.3) with $f_2^{(0)}(y,z) = -\lambda e^y$. Then*

$$f_2^{(j-1)}(y,z) = \sum_{i=0}^{b_j} (-1)^{i+1} a_{j,i} \lambda^{i+1} (e^y)^{i+1} z^{j-2i-1}, \tag{3.1.6}$$

where $b_j = \lfloor \frac{j-1}{2} \rfloor$ and

$$a_{j,i} = \begin{cases} 1, & i = 0, \\ (i+1)a_{j-1,i} + (j-2i)a_{j-1,i-1}, & 1 \leqslant i \leqslant b_{j-1}, \\ 0, & else, \end{cases}$$

for $j = 1,\dots,2p+1$.

Proof. The proof follows induction argument based on the equation (3.1.3). Let $j = 1$, we get

$$
\begin{aligned}
f_2^{(1)}(y,z) &= z\frac{\partial f_2^{(0)}(y,z)}{\partial y} - \lambda e^y \ \frac{\partial f_2^{(0)}(y,z)}{\partial z} \\
&= z(-\lambda e^y) \\
&= -\lambda z e^y
\end{aligned}
$$

and for $j = 2$ with $a_{0,2} = 1$, right hand side of (3.1.6) becomes

$$
\begin{aligned}
\sum_{i=0}^{0}(-1)^{i+1}a_{i,2}\lambda^{i+1}(e^y)^{i+1}z^{2-2i-1} &= -\lambda e^y z \\
&= f_2^{(1)}(y,z).
\end{aligned}
$$

Suppose it is true for $j = k$ that is

$$
f_2^{(k-1)}(y,z) = \sum_{i=0}^{b_k}(-1)^{i+1}a_{k,i}\lambda^{i+1}(e^y)^{i+1}z^{k-2i-1}
$$

where $b_k = \lfloor \frac{k-1}{2} \rfloor$.
Now we will show that (3.1.6) is true for $j = k+1$.

$$
\begin{aligned}
f_2^{(k)}(y,z) &= z\frac{\partial f_2^{(k-1)}(y,z)}{\partial y} - \lambda e^y \frac{\partial f_2^{(k-1)}(y,z)}{\partial z} \\
&= z\sum_{i=0}^{b_k}(-1)^{i+1}a_{k,i}\lambda^{i+1}(i+1)(e^y)^{i+1}z^{k-2i-1} \\
&\quad -\lambda e^y \sum_{i=0}^{b_k}(-1)^{i+1}a_{k,i}\lambda^{i+1}(e^y)^{i+1}(k-2i-1)z^{k-2i-2} \\
&= \sum_{i=0}^{b_k}(-1)^{i+1}a_{k,i}\lambda^{i+1}(i+1)(e^y)^{i+1}z^{(k+1)-2i-1} \\
&\quad + \sum_{i=0}^{b_k}(-1)^{i+2}a_{k,i}\lambda^{i+2}(e^y)^{i+2}(k-2i-1)z^{k-2i-2} \\
&= \sum_{i=0}^{b_k}(-1)^{i+1}(i+1)a_{k,i}\lambda^{i+1}(e^y)^{i+1}z^{(k+1)-2i-1} \\
&\quad + \sum_{i=1}^{b_k+1}(-1)^{i+1}a_{k,i-1}\lambda^{i+1}(e^y)^{i+1}(k+1-2i)z^{k-2i}
\end{aligned}
$$

Case i. Let k is odd that is $k = 2c-1$, $c = 1,\ldots,p$, so $b_k = \lfloor \frac{2c-1-1}{2} \rfloor = c-1 =$

$\lfloor \dfrac{2c-1}{2} \rfloor = b_{k+1}$. Then

$$\begin{aligned} f^{(k)}(y,z) &= \sum_{i=0}^{c-1} (-1)^{i+1}(i+1)a_{2c-1,i}\lambda^{i+1}(e^y)^{i+1}z^{2c-2i-1} \\ &+ \sum_{i=1}^{c} (-1)^{i+1}a_{2c-1,i-1}\lambda^{i+1}(e^y)^{i+1}(2c-1-2i+1)z^{2c-1-2i} \\ &= \sum_{i=0}^{c-1} (-1)^{i+1}(i+1)a_{2c-1,i}\lambda^{i+1}(e^y)^{i+1}z^{2c-2i-1} \\ &+ \sum_{i=0}^{c-1} (-1)^{i+1}a_{2c-1,i-1}\lambda^{i+1}(e^y)^{i+1}(2c-1-2i+1)z^{2c-1-2i} \end{aligned}$$

since $a_{2c-1,-1} = 0$. Hence

$$\begin{aligned} f^{(k)}(y,z) &= \sum_{i=0}^{c-1} (-1)^{i+1}[(i+1)a_{2c-1,i} + (2c-2i)a_{2c-1,i-1}]\lambda^{i+1}(e^y)^{i+1}z^{2c-2i-1} \\ &= \sum_{i=0}^{b_{k+1}} (-1)^{i+1}[(i+1)a_{k,i} + (k+1-2i)a_{k,i-1}]\lambda^{i+1}(e^y)^{i+1}z^{(k+1)-2i-1} \\ &= \sum_{i=0}^{b_{k+1}} (-1)^{i+1}a_{k+1,i}\lambda^{i+1}(e^y)^{i+1}z^{(k+1)-2i-1} \end{aligned}$$

Case ii. Let k is even that is $k = 2c$, $c = 1,\dots,p$, so $b_k = \lfloor \dfrac{2c-1}{2} \rfloor = c-1$ and $b_{k+1} = \lfloor \frac{2c+1-1}{2} \rfloor = c$. Then

$$\begin{aligned} f^{(k)}(y,z) &= \sum_{i=0}^{c-1} (-1)^{i+1}(i+1)a_{2c,i}\lambda^{i+1}(e^y)^{i+1}z^{2c+1-2i-1} \\ &+ \sum_{i=1}^{c} (-1)^{i+1}a_{2c,i-1}\lambda^{i+1}(e^y)^{i+1}(2c-2i+1)z^{2c+1-1-2i} \\ &= \sum_{i=0}^{c} (-1)^{i+1}(i+1)a_{2c,i}\lambda^{i+1}(e^y)^{i+1}z^{(2c+1)-2i-1} \\ &+ \sum_{i=0}^{c} (-1)^{i+1}a_{2c,i-1}\lambda^{i+1}(e^y)^{i+1}(2c+1-2i)z^{(2c+1)-1-2i} \end{aligned}$$

since $a_{2c,c} = 0$ and $a_{2c,-1} = 0$. Hence

$$\begin{aligned} f^{(k)}(y,z) &= \sum_{i=0}^{c}(-1)^{i+1}[(i+1)a_{2c,i} + (2c+1-2i)a_{2c,i-1}]\lambda^{i+1}(e^y)^{i+1}z^{(2c+1)-2i-1} \\ &= \sum_{i=0}^{b_{k+1}}(-1)^{i+1}[(i+1)a_{k,i} + (k+1-2i)a_{k,i-1}]\lambda^{i+1}(e^y)^{i+1}z^{(k+1)-2i-1} \\ &= \sum_{i=0}^{b_{k+1}}(-1)^{i+1}a_{k+1,i}\lambda^{i+1}(e^y)^{i+1}z^{(k+1)-2i-1}. \end{aligned}$$

Theorem 3.1.1. *If $f_1^{(j)}(y,z)$ and $f_2^{(j)}(y,z)$ are sufficiently smooth and satisfy (3.1.4) and (3.1.6) respectively then for $j = 2,\ldots,2p+1$, the following relations hold:*

$$\begin{aligned} &\text{a) } (-1)^j f_1^{(j-1)}(0,z_1) - f_1^{(j-1)}(0,z_0) \\ &= (z_1+z_0)(-1)^j \sum_{i=0}^{b_{j-1}}(-1)^{i+1}a_{j-1,i}\lambda^{i+1}\left[\sum_{n=0}^{j-2i-3}(-1)^n z_1^{j-2i-3-n}z_0^n\right], \end{aligned} \qquad (3.1.7)$$

for any fixed z_0, z_1,

$$\begin{aligned} \text{b) } (-1)^j f_2^{(j-1)}(0,z_1) - f_2^{(j-1)}(0,z_0) &= -2\sum_{i=0}^{b_j}(-1)^{i+1}\lambda^{i+1}a_{j,i}z_0^{j-2i-1} \\ &= -2\, f_2^{(j-1)}(0,z_0), \end{aligned} \qquad (3.1.8)$$

for $z_1 = -z_0$.

Proof. We split the proof (a) into two cases.

a) Case i. Let $j = 2k+1$ for $k = 1,\ldots,p$,

$$(-1)^{2k+1}f_1^{(2k)}(0,z_1) - f_1^{(2k)}(0,z_0) = -f_2^{(2k-1)}(0,z_1) - f_2^{(2k-1)}(0,z_0).$$

Using (3.1.6) from Lemma 3.1.1, we have

$$
\begin{aligned}
(-1)^{2k+1} f_1^{(2k)}(0,z_1) - f_1^{(2k)}(0,z_0) &= -f_2^{(2k-1)}(0,z_1) - f_2^{(2k-1)}(0,z_0) \\
&= -\sum_{i=0}^{k-1} (-1)^{i+1} a_{2k,i} \lambda^{i+1} z_1^{2k-2i-1} - \sum_{i=0}^{k-1} (-1)^{i+1} a_{2k,i} \lambda^{i+1} z_0^{2k-2i-1} \\
&= \sum_{i=0}^{k-1} (-1)^{i+2} a_{2k,i} \lambda^{i+1} \left[z_1^{2k-2i-1} + z_0^{2k-2i-1} \right] \\
&= (z_1 + z_0) \left(-\sum_{i=0}^{k-1} (-1)^{i+1} a_{2k,i} \lambda^{i+1} \left[\sum_{n=0}^{2k-2i-2} (-1)^n z_1^{2k-2i-2-n} z_0^n \right] \right).
\end{aligned}
$$

Case ii. Let $j = 2k$ for $k = 1, 2, \dots, p$. The proof is analogous to the case i.
b) Case i. Let $j = 2k+1$ for $k = 1, \dots, p$, using Lemma 3.1.1,

$$
\begin{aligned}
&(-1)^{2k+1} f_2^{(2k)}(0,z_1) - f_2^{(2k)}(0,z_0) \\
&= -\sum_{i=0}^{k} (-1)^{i+1} a_{2k+1,i} \lambda^{i+1} z_1^{2k+1-2i-1} - \sum_{i=0}^{k} (-1)^{i+1} a_{2k+1,i} \lambda^{i+1} z_0^{2k+1-2i-1} \\
&= \sum_{i=0}^{k} (-1)^{i+1} a_{2k+1,i} \lambda^{i+1} \left[-(-z_0)^{2k-2i} - z_0^{2k-2i} \right] \\
&= -2 \sum_{i=0}^{k} (-1)^{i+1} a_{2k+1,i} \lambda^{i+1} z_0^{2k-2i}.
\end{aligned}
$$

Case ii. Let $j = 2k$ for $k = 1, 2, \dots, p$. The proof is analogous to the case i.

We consider the application of Taylor's decomposition Ashyralyev, & Sobolevskii (2004) of solution to (3.1.1) on two points x_k and x_{k-1}:

$$
Y(x_k) - Y(x_{k-1}) + \sum_{j=1}^{p} \alpha_j Y^{(j)}(x_k) h^j - \sum_{j=1}^{q} \beta_j Y^{(j)}(x_{k-1}) h^j = \tau_k, \tag{3.1.9}
$$

where

$$
\tau_k = \frac{(-1)^p}{(p+q)!} \int_{x_{k-1}}^{x_k} (x_k - s)^q (s - x_{k-1})^p \, Y^{(p+q+1)}(s) ds, \tag{3.1.10}
$$

and $x_k = kh$, $k = 0, \ldots, n$, $nh = 1$, $n \in N$ with the stepsize h,

$$\alpha_j = \frac{(p+q-j)!p!(-1)^j}{(p+q)!j!(p-j)!} \quad , \quad 1 \leqslant j \leqslant p,$$

$$\beta_j = \frac{(p+q-j)!q!}{(p+q)!j!(q-j)!} \quad , \quad 1 \leqslant j \leqslant q\,.$$

Neglecting the last term of (3.1.9), we obtain single-step difference schemes of $(p+q)$-order of accuracy for the approximate solution to the problem (3.1.1)

$$Y_k - Y_{k-1} + \sum_{j=1}^{p} \alpha_j Y_k^{(j)} h^j - \sum_{j=1}^{q} \beta_j Y_{k-1}^{(j)} h^j = 0, \tag{3.1.11}$$

where $Y_k^{(j)} = \begin{bmatrix} y_k^{(j)} \\ z_k^{(j)} \end{bmatrix}$ is the approximate value of $Y^{(j)}(x_k)$. For the computation of the eigenvalues of (3.0.1), putting $h = 1$ and $p = q$, the approximation (3.1.11) gives

$$Y_1 - Y_0 + \sum_{j=1}^{p} (-1)^j \beta_j Y_1^{(j)} - \sum_{j=1}^{p} \beta_j Y_0^{(j)} = 0, \tag{3.1.12}$$

where $\alpha_j = (-1)^j \beta_j$. Writing (3.1.12) with respect to the components and imposing the boundary conditions $y_0 = y(0) = 0$ and $y_1 = y(1) = 0$, we have the following equations

$$\sum_{j=1}^{p} \beta_j \left[(-1)^j f_1^{(j-1)}(0, z_1) - f_1^{(j-1)}(0, z_0) \right] = 0 \tag{3.1.13}$$

and

$$z_1 - z_0 + \sum_{j=1}^{p} \beta_j \left[(-1)^j f_2^{(j-1)}(0, z_1) - f_2^{(j-1)}(0, z_0) \right] = 0. \tag{3.1.14}$$

Using Theorem 3.1.1.a, the equation (3.1.13) becomes

$$(z_1 + z_0) \left[\sum_{j=2}^{p} \beta_j \sum_{i=0}^{b_{j-1}} (-1)^{j+i+1} a_{j-1,i} \lambda^{i+1} \left[\sum_{n=0}^{j-2i-3} (-1)^n z_1^{j-2i-3-n} z_0^n \right] \right] = 0. \tag{3.1.15}$$

It is clear that for $z_1 = -z_0$, that is, $y'(1) = -y'(0)$, (3.1.15) is satisfied. Thus, taking

$z_1 = -z_0$ and using Theorem 3.1.1.b, the equation (3.1.14) will be

$$G(z_0,\lambda) = z_0 + \sum_{j=1}^{p} \beta_j f_2^{(j-1)}(0,z_0) = 0. \tag{3.1.16}$$

As Figure 3.1 demonstrates, we observed that λ has a maximum value. To find the

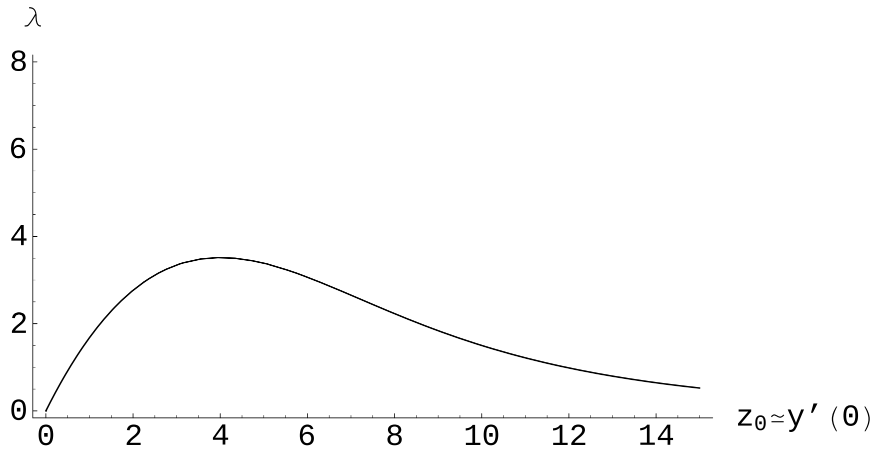

Figure 3.1 Graph of the equation $G(\lambda, z_0) = 0$.

maximum value, it is necessary to satisfy

$$\frac{d\lambda}{dz_0} = -\frac{\partial G/\partial z_0}{\partial G/\partial \lambda} = 0, \text{ that is } \frac{\partial G}{\partial z_0} = 0.$$

Solving nonlinear equations

$$G(z_0,\lambda) = 0 \text{ and } \frac{\partial G}{\partial z_0} = 0$$

by Newton's method, we find critical eigenvalue $\lambda_c = 3.5138307192516$ and the corresponding initial value $z_0 \simeq y'(0)$.

Figure 3.1 and Table 3.1 show that there is no z_0 for $\lambda > \lambda_c$ and a unique solution corresponding to the initial value $z_{1,0} = z_0$ for $\lambda = \lambda_c$, and there are two solutions corresponding to the initial values $z_{1,0}$ and $z_{2,0}$ for $\lambda < \lambda_c$. It is conclude that, the numerical results obtained using Taylor's decomposition method agree with the exact results of Bratu problem given in Khuri (2004).

Table 3.1 Corresponding to the initial values $z_{1,0}$ and $z_{2,0}$ for various $\lambda \leqslant \lambda_c$ obtained from (3.1.16)

λ	$z_{1,0}$	$z_{2,0}$
0.5	0.261277	12.9998
1	0.549353	10.8469
2	1.24822	8.26876
3	2.3196	6.10338
3.5138307192516	4.	–

3.2 Error Analysis of the Approximate Solution

Now we find an approximate solution to the initial value problem

$$\begin{aligned} Y'(x) &= F(Y(x)) \\ Y(0) &= Y_0 \end{aligned} \tag{3.2.1}$$

that corresponds to Bratu Problem (3.0.1) for an eigenvalue $\lambda \leqslant \lambda_c$ and the initial value z_0. Using Taylor's decomposition on two points x_{k-1}, x_k on the uniform grid

$$[0,1]_h = \{x_k = kh,\ k = 0,1,\ldots,n,\ nh = 1,\ n \in N\},$$

for $p = q$, we get

$$Y_k - Y_{k-1} + \sum_{j=1}^{p} (-1)^j \beta_j Y_k^{(j)} h^j - \sum_{j=1}^{p} \beta_j Y_{k-1}^{(j)} h^j = 0 \quad \text{and} \quad Y_0 = \begin{bmatrix} y_0 \\ z_0 \end{bmatrix}, \tag{3.2.2}$$

where $y_0 = y(0)$, $z_0 \simeq z(0)$. Solving the nonlinear system (3.2.2) by Newton's method, we obtain the approximate value y_k of the eigenfunction $y(x)$ at $x = x_k$ with $O(h^{2p})$.

Lemma 3.2.1. *Let $Y(x)$ has $(2p+1)$ continuous derivatives on $[0,1]$, then the truncation error τ_k at x_k for the Taylor's decomposition method (3.2.2) is*

$$\|\tau_k\| \leqslant \text{const.} \frac{\xi\, h^{2p+1} M^{p+1}}{(2p)!} \tag{3.2.3}$$

where $M = \max\limits_{(y,z)\in D} \{|f_1^{(0)}(y,z)|,\ |f_2^{(0)}(y,z)|\}$, D is 2-dimensional box in R^2, $\xi = \max\{a_{j,i}\}$, $j = 1,\ldots,2p+1$, $i = 0,\ldots,p$, and const. *is a constant independent of h and p, and $\|\cdot\|$ denotes $\|\cdot\|_\infty$.*

Proof. From (3.1.6) in Lemma 3.1.1, we have

$$
\begin{aligned}
|f_2^{(j)}(y,z)| &\leqslant \sum_{i=0}^{b_{j+1}} a_{j+1,i}\lambda^{i+1}(e^y)^{i+1}|z|^{j-2i} \\
&\leqslant \sum_{i=0}^{b_{j+1}} a_{j+1}(\lambda e^y)^{i+1}|z|^{j-2i} \\
&\leqslant \sum_{i=0}^{b_{j+1}} a_{j+1}|f_2^{(0)}(y,z)|^{i+1}|f_1^{(0)}(y,z)|^{j-2i} \\
&\leqslant \xi \sum_{i=0}^{b_{j+1}} M^{i+1}M^{j-2i} \\
&\leqslant \xi M^{j+1} \sum_{i=0}^{b_{j+1}} \left(\frac{1}{M}\right)^{i} \\
&\leqslant \xi M^{i+1} \left| \frac{1-(\frac{1}{M})^{b_{j+1}+1}}{1-\frac{1}{M}} \right| \\
&\leqslant \xi M^{j+1-b_{j+1}} \frac{1}{|M-1|},
\end{aligned}
$$

and hence

$$
|f_2^{(j)}(y,z)| \leqslant \xi M^{j+1-b_{j+1}} \frac{1}{|M-1|}, \ \ M \neq 1. \tag{3.2.4}
$$

Since $f_1^{(2p)}(y,z) = f_2^{(2p-1)}(y,z)$, using (3.2.4) with $b_j = \lfloor \frac{j-1}{2} \rfloor$ for $j = 2p-1$ and $j = 2p$, we obtain

$$
\begin{aligned}
||Y^{(2p+1)}(s)|| &\leqslant \max\left\{ \left|f_2^{(2p-1)}(y(s),z(s))\right|, \left|f_2^{(2p)}(y(s),z(s))\right| \right\} \\
&\leqslant \max\left\{ \frac{\xi M^{2p-b_{2p}}}{|M-1|}, \frac{\xi M^{2p+1-b_{2p+1}}}{|M-1|} \right\} \\
&\leqslant \xi \frac{M^{p+1}}{|M-1|}.
\end{aligned}
$$

From (3.1.10), we obtain

$$\begin{aligned}
||\tau_k|| &\leqslant \frac{1}{(2p)!}\int_{x_{k-1}}^{x_k} \|(x_k-s)^p(s-x_{k-1})^p\, Y^{(2p+1)}(s)\|\, ds \\
&\leqslant \frac{1}{(2p)!}h^{2p}\int_{x_{k-1}}^{x_k} ||Y^{(2p+1)}(s)||\, ds \\
&\leqslant \frac{1}{(2p)!}h^{2p}\,\xi\,\frac{M^{p+1}}{|M-1|}\left|\int_{x_{k-1}}^{x_k} ds\right|
\end{aligned}$$

$$\|\tau_k\| \leqslant \text{const.}\frac{h^{2p+1}}{(2p)!}\,\xi\, M^{p+1},$$

where *const.* is independent of h and p which proves the assertion.

Lemma 3.2.2. *Let $f_2^{(0)}(y,z)$ be Lipshitz in y with constants K in 2-dimensional box D and let*

$$d_{i,j} = \max_{(y,z)\in D}\left|\frac{\partial f_i^{(j)}(y,z)}{\partial y}\right|,\ \ s_{i,j} = \max_{(y,z)\in D}\left|\frac{\partial f_i^{(j)}(y,z)}{\partial z}\right|,\ \ i=1,2, \tag{3.2.5}$$

for all $j=1,2,\dots,2p$. Then $F^{(j)}(Y(x))$ is Lipshitz in Y on D with constant L where $L = \max_{1\leqslant j\leqslant p}\{l_{1,j},\ l_{2,j}\}$ with $l_{1,j} = \max_{1\leqslant j\leqslant p}\{d_{1,j},\ Ks_{1,j}\}$, $l_{2,j} = \max_{1\leqslant j\leqslant p}\{d_{2,j},\ Ks_{2,j}\}$.

Proof. Using recurrence relation (3.1.2), we get

$$\begin{aligned}
|f_1^{(j)}(y,z) - f_1^{(j)}(\tilde{y},\tilde{z})| &\leqslant \left| z\frac{\partial f_1^{(j-1)}(y,z)}{\partial y} - \tilde{z}\frac{\partial f_1^{(j-1)}(\tilde{y},\tilde{z})}{\partial y}\right| \\
&\quad + \left| f_2^{(0)}(y,z)\frac{\partial f_1^{(j-1)}(y,z)}{\partial z} + f_2^{(0)}(\tilde{y},\tilde{z})\frac{\partial f_1^{(j-1)}(\tilde{y},\tilde{z})}{\partial z}\right| \\
&\leqslant d_{1,j}|z-\tilde{z}| + s_{1,j}|f_2^{(0)}(y,z) - f_2^{(0)}(\tilde{y},\tilde{z})|.
\end{aligned}$$

Since $f_2^{(0)}(y,z)$ is Lipschitz in y with the constant K, we obtain from (3.2.5) that

$$\begin{aligned}
|f_1^{(j)}(y,z) - f_1^{(j)}(\tilde{y},\tilde{z})| &\leqslant d_{1,j}|z-\tilde{z}| + s_{1,j}K|y-\tilde{y}| \\
&\leqslant l_{1,j}\left(|z-\tilde{z}| + |y-\tilde{y}|\right).
\end{aligned}$$

For $f_1^{(j)}(y,z)$, using a similar technique, we get

$$|f_2^{(j)}(y,z) - f_2^{(j)}(\tilde{y},\tilde{z})| \leqslant l_{2,j}\left(|z-\tilde{z}| + |y-\tilde{y}|\right).$$

Therefore, we obtain

$$||F^{(j)}(Y) - F^{(j)}(\tilde{Y})|| \leqslant L||Y-\tilde{Y}||, \;\; \forall\, Y,\, \tilde{Y} \in D.$$

Theorem 3.2.1. *If F is Lipschitz in Y with constant L and if the local error at each step satisfies Lemma 3.2.1, then the global error for (3.1.11) is bounded by*

$$\|Y(x_k) - Y_k\| \leqslant C_0\|Y(0) - Y_0\| + C_1 \frac{\xi h^{2p} M^{p+1}}{(2p)!},$$

where $C_0 = e^{\overline{x}\frac{2LB(h)}{1-LhB(h)}}$, $C_1 = \text{const.}\dfrac{C_0}{L}\dfrac{1}{1+\frac{\beta_2}{\beta_1}\,h+\cdots+\frac{\beta_p}{\beta_1}\,h^{p-1}}$ *for some* $\overline{x} > 0$.

Proof. Subtracting the equation (3.1.11) from (3.1.9) and taking the norms yields

$$\begin{aligned}
\|Y(x_k) - Y_k\| &\leqslant \|Y(x_{k-1}) - Y_{k-1}\| + \sum_{j=1}^{p} \beta_j \|Y^{(j)}(x_k) - Y_k^{(j)}\| h^j \\
&+ \sum_{j=1}^{p} \beta_j \|Y^{(j)}(x_{k-1}) - Y_{k-1}^{(j)}\| h^j + \|\tau_k\| \\
&\leqslant \|Y(x_{k-1}) - Y_{k-1}\| + \sum_{j=1}^{p} \beta_j h^j L \|Y(x_k) - Y_k\| \\
&+ \sum_{j=1}^{p} \beta_j h^j L \|Y(x_{k-1}) - Y_{k-1}\| + \|\tau_k\|.
\end{aligned}$$

Using Lemma 3.2.2 and the fact that $Y^{(j)}(x) = F^{(j-1)}(Y(x))$, we have

$$E_k \leqslant E_{k-1} + \sum_{j=1}^{p} \beta_j h^j L E_k + \sum_{j=1}^{p} \beta_j h^j L E_{k-1} + \|\tau_k\|,$$

where $E_k = \|Y(x_k) - Y_k\|$. It follows that

$$E_k \leqslant \frac{1+\,h\,B(h)}{1-\,h\,B(h)} E_{k-1} + \frac{1}{1-\,h\,B(h)}\|\tau_k\|,$$

where $B(h) = L\sum_{j=1}^{p} \beta_j h^{j-1}$. Since $\beta_1 = 1/2$ and $\beta_j/\beta_1 \leqslant 1$, $j = 1,\ldots,p$ and $\frac{\beta_2}{\beta_1}\,h + \cdots + \frac{\beta_p}{\beta_1}\,h^{p-1} < 1$ for large p, $1 - hB(h)$ is positive and thus

$$\begin{aligned} E_k &\leqslant e^{x_k \frac{2B(h)}{1-h\,B(h)}} E_0 + \frac{1}{2hB(h)}\,\|\tau_k\|\,\left(e^{x_k \frac{2B(h)}{1-hB(h)}} - 1\right) \\ &\leqslant e^{x_k \frac{2B(h)}{1-h\,B(h)}} E_0 + \frac{1}{2Lh\beta_1}\;\frac{1}{1 + \frac{\beta_2}{\beta_1}\,h + \ldots + \frac{\beta_p}{\beta_1}\,h^{p-1}}\,\|\tau_k\|\,\left(e^{x_k \frac{2B(h)}{1-hB(h)}}\right). \end{aligned}$$

Since $\frac{1}{1+\frac{\beta_2}{\beta_1}\,h+\ldots+\frac{\beta_p}{\beta_1}\,h^{p-1}} = O(1)$ we obtain from the bound of local error (3.2.3) we therefore have that

$$E_k \leqslant C_0 E_0 + C_1 \frac{\xi h^{2p} M^{p+1}}{(2p)!}.$$

3.3 Numerical Results for Bratu Problem

The Taylor's decomposition method described in previous sections is applied to the one dimensional Bratu Problem. The following graphs show the exact solutions and the approximate solutions of Bratu problem for $\lambda = 1$, the corresponding initial values $z_{1,0} = 0.549353$, $z_{2,0} = 10.8469$ for $p = 2$.

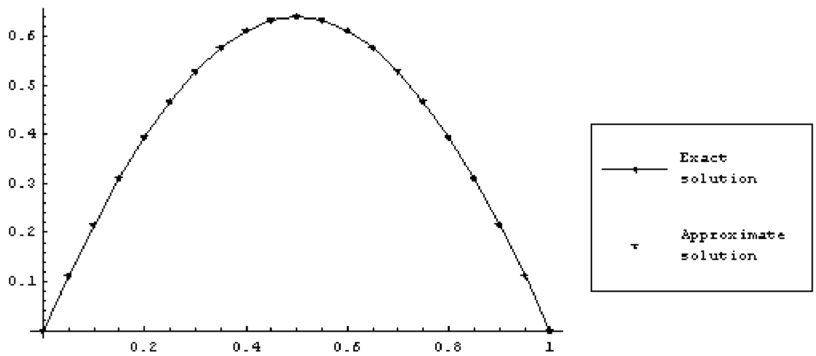

Figure 3.2 Lower solution for $z_{1,0} = 0.549353$ when $\lambda = 1$.

The numerical results using constant stepsize and observed orders of convergence for the solution $y(x)$ are listed in the Tables 3.2, 3.3, 3.4, 3.5 and 3.6 for different

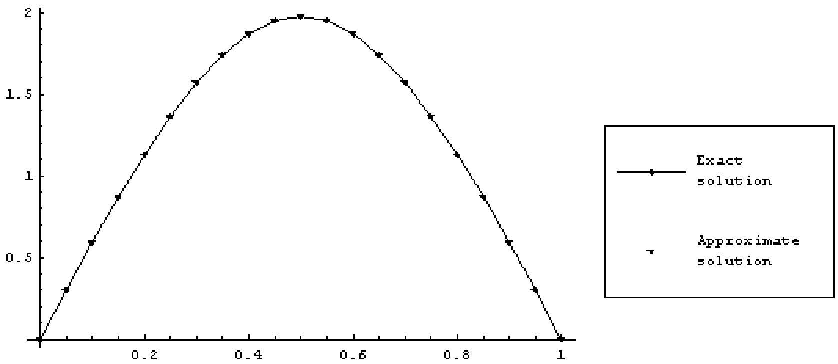

Figure 3.3 Upper solution for $z_{2,0} = 10.8469$ when $\lambda = 1$.

eigenvalues λ. The observed orders are computed using

$$2p = \frac{\log \frac{e_h}{e_{h/2}}}{\log 2},$$

where e_h and $e_{h/2}$ are the maximum error moduli of the global errors when the problem is solved with stepsize h and $h/2$ respectively.

Table 3.2 Maximum error moduli and observed errors for the solution $y(x)$ for $\lambda = 1$ and $z_0 = 0.549353$.

p	2	3	4
$e_{1/10}$	2.9042(-7)	1.52083(-10)	6.86108(-14)
$e_{1/20}$	1.81472(-8)	2.373398(-12)	4.02456(-16)
Observed Orders			
	4.00033	6.00141	7.41346

Table 3.3 Maximum error moduli and observed errors for the solution $y(x)$ for $\lambda = 1$ and $z_0 = 10.8469$.

p	2	3	4
$e_{1/10}$	1.4734(-3)	1.5602(-5)	2.7229(-7)
$e_{1/20}$	9.5390(-5)	2.8821(-7)	1.4438(-9)
Observed Orders			
	3.94914	5.7584	7.55922

A comparison of the errors generated using Taylor's decomposition for different stepsize $h = \frac{1}{n}$ are illustrated in Figure 3.4 and 3.5.

Table 3.4 Maximum error moduli and observed errors for the solution $y(x)$ for $\lambda = 3$ and $z_0 = 2.3196$.

p	2	3	4
$e_{1/10}$	1.33602(-5)	2.1112(-8)	5.02158(-11)
$e_{1/20}$	8.38644(-7)	3.30479(-10)	1.95621(-13)
Observed Orders			
	3.99374	5.99736	8.00394

Table 3.5 Maximum error moduli and observed errors for the solution $y(x)$ for $\lambda = 3$ and $z_0 = 6.10338$.

p	2	3	4
$e_{1/10}$	2.3498(-4)	9.2964(-7)	5.6874(-9)
$e_{1/20}$	1.4808(-5)	1.4977(-8)	2.3545(-11)
Observed Orders			
	3.99809	5.95585	7.9162

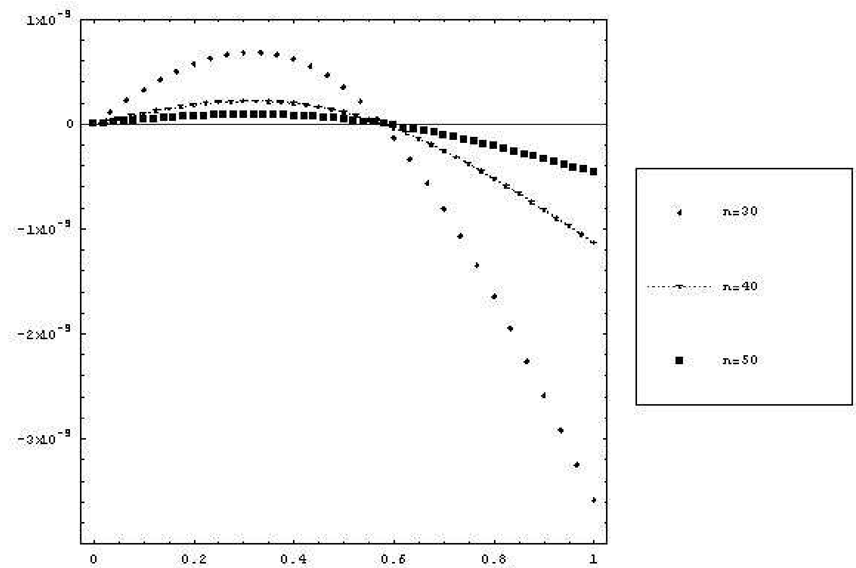

Figure 3.4 Errors for lower solution when $\lambda = 1$.

Table 3.6 Maximum error moduli and observed errors for the solution $y(x)$ for $\lambda = 3.513807192516$ and $z_0 = 4$.

p	2	3	4
$e_{1/10}$	5.4628(-5)	1.7783(-7)	5.5514(-10)
$e_{1/20}$	3.7633(-6)	2.805(-9)	2.2216(-12)
Observed Orders			
	3.85957	5.98635	7.96511

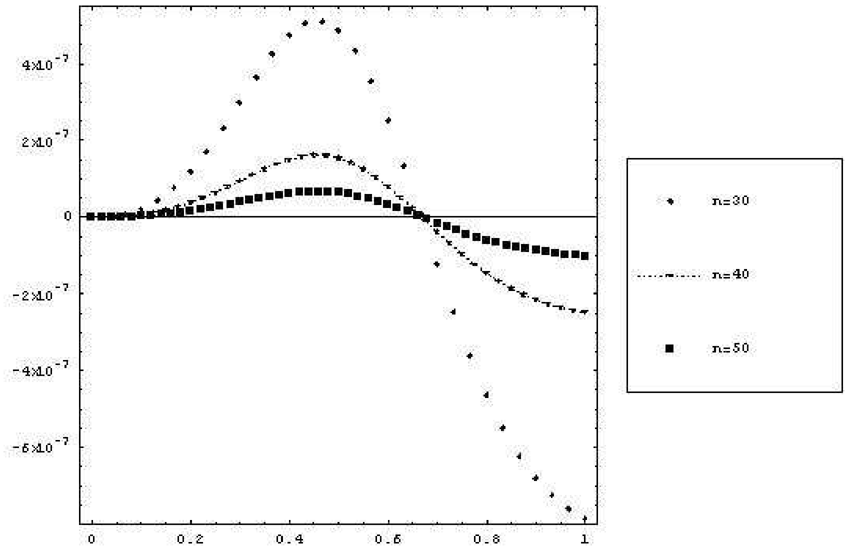

Figure 3.5 Errors for $\lambda_c = 3.513807192516$.

As seen in Figures 3.4 and 3.5, the discretization errors for $p = 2$ and different step sizes $h = \dfrac{1}{n}$ $(n = 30,\ 40,\ 50)$ are bounded by -4 x 10^{-9}. It is worth noting that in the study Buckmire (2003) the Mickens discretization and standard discretization errors are bounded by 1.5 x 10^{-6} for $n = 100$. As a result, tables and figures demonstrate the power of the current study.

CHAPTER FOUR
EULER BUCKLING PROBLEM

4.1 Euler Buckling Problem

We examine an elementary, classical problem- buckling of an end-loaded rod- which possesses a completely soluble continuous model in the form of a nonlinear, second order boundary value problem which is described as in Stakgold (1971), Jones (2006), Domokos, & Holmes (1993), Griffel (1981). An essential complete analysis of this problem was provided by Euler (1744).

The classic simple example of a nonlinear eigenvalue problem is the problem of an elastic rod under compression with its ends clamped; the angular displacement y of the rod under a compressive load λ satisfies the following equation as given in Griffel (1981) with $f(x,y) = \sin y$

$$\begin{aligned} &y'' + \lambda \sin y = 0 \\ &y'(0) = 0 \text{ and } y'(1) = 0. \end{aligned} \tag{4.1.1}$$

The solution $y \equiv 0$ of (4.1.1) corresponds to the rod being straight. For small loads, that is, for small values of λ, we expect that the trivial solution is the only solution. But when the load λ is increased, we expect that the rod buckles at the some stage corresponding to the appearance of nonzero solutions of (4.1.1). It is well known that the linear case, where $f(x,y) = y$, we know that if λ is not an eigenvalue, then the zero solution is the only solution of

$$\begin{aligned} &y'' + \lambda y = 0 \\ &y'(0) = 0 \text{ and } y'(1) = 0. \end{aligned}$$

In the case which λ is an eigenvalue, there are infinitely many solutions since any multiple of an eigenfunction is an eigenfunction. For the nonlinear eigenvalue problem (4.1.1) one finds that for small λ the only solution is zero solution as the linear case. But the eigenvalue λ increases as it reaches a critical value λ_1 at which a nonzero solution appears corresponding to buckling of the rod. For $\lambda > \lambda_1$ the nonlinear problem behaves quite differently from the linear problem: For a range of values $\lambda_1 < \lambda < \lambda_2$ there is exactly one nonzero solution of (4.1.1) for each λ, and when λ exceeds λ_2 a second nonzero solution appears; similarly there is a value λ_3 beyond which there are three nonzero solutions, and so on. Namely, one may establish inductively

$$\begin{array}{ll} 0 \leqslant \lambda \leqslant \pi^2, & \text{only the trivial solution,} \\ \pi^2 < \lambda \leqslant 4\pi^2 & \text{one nontrivial solution,} \\ n^2\pi^2 < \lambda \leqslant (n+1)^2\pi^2, & n \text{ nontrivial solutions,} \end{array}$$

as it is given in Stakgold (1971). This behavior is a simple example of the phenomenon of bifurcation or branching; it occurs in many different areas of applied mathematics. In mechanics there are many situations where sudden jumps from one kind of behavior to another (analogous to the buckling rod) occur as some parameter (analogous to the compressive load on the rod) is continuously varied; such problems are described by nonlinear eigenvalue equation similar to (4.1.1).

After reviewing background of Euler Buckling problem, in section 2, we establish lemmas and a theorem, and then we give the application of Taylor's decomposition method to the Euler Buckling problem. The last section demonstrates the numerical results accompanying the theoretical results and the behavior of solution of Euler Buckling problem.

4.2 Application of Taylor's Decomposition Method to the Euler Buckling Problem

For convenience we introduce the following notations as in chapter 2:

$$Y(x) = \begin{bmatrix} y(x) \\ z(x) \end{bmatrix}, \; F(Y(x)) = \begin{bmatrix} f_1^{(0)}(y,z) \\ f_2^{(0)}(y,z) \end{bmatrix},$$

$$A_0 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \; A_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \; f_1^{(0)}(y,z) = z \text{ and } f_2^{(0)}(y,z) = -\lambda \sin y.$$

Thus, the Euler Buckling Problem (4.1.1) can be written in the form

$$\begin{aligned} Y'(x) &= F(Y(x)), \qquad 0<x<1, \\ A_0Y(0)+A_1Y(1) &= \mathbf{0}, \end{aligned} \tag{4.2.1}$$

Defining the following recurrence relations for $j = 1,\ldots,2p$,

$$f_i^{(j)}(y,z) = z\,\frac{\partial f_i^{(j-1)}(y,z)}{\partial y} - \lambda \sin y\,\frac{\partial f_i^{(j-1)}(y,z)}{\partial z}, \qquad i = 1,\ 2 \tag{4.2.2}$$

we obtain

$$Y^{(j)}(x) = \begin{bmatrix} f_1^{(j-1)}(y,z) \\ f_2^{(j-1)}(y,z) \end{bmatrix} = \begin{bmatrix} f_2^{(j-2)}(y,z) \\ f_2^{(j-1)}(y,z) \end{bmatrix} \quad \text{for} \quad j = 2,\ldots,2p+1. \tag{4.2.3}$$

We first give the following lemma which defines $f_2^{(j-1)}(y,z)$ explicitly.

Lemma 4.2.1. *Let $a_{2m+2,i,k}$ and $a_{2m+1,i,k}$ be defined as follows*

$$a_{2m+2,i,k} = \begin{cases} 1, & i=0,\ k=0, \\ (2k+1)a_{2m+1,i,k} + (i-2-2k)a_{2m+1,i,k-1} & \\ \qquad +(2m+2-2i)a_{2m+1,i-1,k-1}, & 1 \leqslant i \leqslant m,\ 0 \leqslant k \leqslant \frac{i}{2}, \\ 0, & else, \end{cases} \tag{4.2.4}$$

for $m = 0,\ldots,p-1$ and

$$a_{2m+1,i,k} = \begin{cases} 1, & i=0,\ k=0, \\ (2k+2)a_{2m,i,k+1} + (i+1-2k)a_{2m,i,k} & \\ \qquad +(2m+1-2i)a_{2m,i-1,k}, & 1 \leqslant i \leqslant m,\ 0 \leqslant k \leqslant \frac{i}{2}, \\ 0, & else, \end{cases} \tag{4.2.5}$$

for $m = 0,\ldots,p$, then the following equations hold

$$\begin{aligned} \text{a)} \quad & \sum_{k=0}^{\lfloor \frac{i+1}{2} \rfloor} (-1)^{m-k}\,(2k-(i+1))\,a_{2m,i,k}(\cos y)^{i-2k}(\sin y)^{2k+1} \\ & = \sum_{k=0}^{\lfloor \frac{i}{2} \rfloor} (-1)^{m-k}\,(2k-(i+1))\,a_{2m,i,k}(\cos y)^{i-2k}(\sin y)^{2k+1} \end{aligned} \tag{4.2.6}$$

for $i = 0,\dots,m-1,\ m = 0,\dots,p.$

$$\text{b)}\quad \sum_{k=0}^{\lfloor\frac{i+1}{2}\rfloor}(-1)^{m-k}\,2k\,a_{2m,i,k}\,(\cos y)^{i+2-2k}(\sin y)^{2k-1} = \sum_{k=0}^{\lfloor\frac{i}{2}\rfloor}(-1)^{m+1-k}(2k+2)a_{2m,i,k+1}(\cos y)^{i-2k}(\sin y)^{2k+1} \tag{4.2.7}$$

for $i = 0,\dots,m-1,\ m = 0,\dots,p.$

$$\text{c)}\quad \sum_{i=0}^{m-1}\lambda^{i+2}\,(2m-2i-1)z^{2m-2i-2}\sum_{k=0}^{\lfloor\frac{i+1}{2}\rfloor}(-1)^{m+1-k}\,a_{2m,i,k}\,(\cos y)^{i+1-2k}(\sin y)^{2k+1} = \sum_{i=0}^{m}\lambda^{i+1}\,(2m-2i+1)z^{2m-2i}\sum_{k=0}^{\lfloor\frac{i}{2}\rfloor}(-1)^{m+1-k}\,a_{2m,i-1,k}\,(\cos y)^{i-2k}(\sin y)^{2k+1} \tag{4.2.8}$$

for $i = 0,\dots,m-1,\ m = 0,\dots,p.$

$$\text{d)}\quad \sum_{k=0}^{\lfloor\frac{i}{2}\rfloor}(-1)^{m+1-k}\,(2k-i)\,a_{2m+1,i,k}(\cos y)^{i-2k-1}(\sin y)^{2k+2} = \sum_{k=0}^{\lfloor\frac{i+1}{2}\rfloor}(-1)^{m-k}\,(2k-i-2)\,a_{2m+1,i,k-1}(\cos y)^{i+1-2k}(\sin y)^{2k} \tag{4.2.9}$$

for $i = 0,\dots,m-1,\ m = 0,\dots,p-1.$

$$\text{e)}\quad \sum_{k=0}^{\lfloor\frac{i}{2}\rfloor}(-1)^{m+1-k}\,(2k+1)\,a_{2m+1,i,k}(\cos y)^{i+1-2k}(\sin y)^{2k} = \sum_{k=0}^{\lfloor\frac{i+1}{2}\rfloor}(-1)^{m+1-k}\,(2k+1)\,a_{2m+1,i,k}(\cos y)^{i+1-2k}(\sin y)^{2k} \tag{4.2.10}$$

for $i = 0,\dots,m-1,\ m = 0,\dots,p-1.$

$$\text{f)}\quad \sum_{i=0}^{m}\lambda^{i+2}\,(2m-2i)z^{2m-2i-1}\sum_{k=0}^{\lfloor\frac{i}{2}\rfloor}(-1)^{m+2-k}\,a_{2m+1,i,k}\,(\cos y)^{i-2k}(\sin y)^{2k+2} = \sum_{i=0}^{m}\lambda^{i+1}\,(2m-2i+2)z^{2m-2i+1}\sum_{k=0}^{\lfloor\frac{i+1}{2}\rfloor}(-1)^{m+1-k}\,a_{2m+1,i-1,k-1} \times(\cos y)^{i+1-2k}(\sin y)^{2k} \tag{4.2.11}$$

for $i = 0,\dots,m-1$, $m = 0,\dots,p-1$.

Proof. a) Let i be odd, i.e. $i = 2c+1$, then $\lfloor \frac{i+1}{2} \rfloor = \lfloor \frac{2c+2}{2} \rfloor = c+1 = \frac{i+1}{2}$. In this case the term $2k-(i+1)$ becomes 0 for $k = \lfloor \frac{i+1}{2} \rfloor$, since $2(c+1)-(2c+1+1) = 2c+2-2c-2 = 0$. Therefore it is sufficient to write the sum on the left hand side of (4.2.6) from 0 to $\lfloor \frac{i}{2} \rfloor$, that is,

$$\sum_{k=0}^{\lfloor \frac{i}{2} \rfloor} (-1)^{m-k} a_{2m,i,k} \, (2k-(i+1)) \, (\cos y)^{i-2k} (\sin y)^{2k+1}.$$

For the case $i = 2c$, we have $\lfloor \frac{i+1}{2} \rfloor = \lfloor \frac{2c+1}{2} \rfloor = c = \lfloor \frac{i}{2} \rfloor$. So we can write $\lfloor \frac{i}{2} \rfloor$ instead of writing $\lfloor \frac{i+1}{2} \rfloor$. Hence we have

$$\begin{aligned}
&\sum_{k=0}^{\lfloor \frac{i+1}{2} \rfloor} (-1)^{m-k} a_{2m,i,k} \, (2k-(i+1)) \, (\cos y)^{i-2k} (\sin y)^{2k+1} \\
&\quad = \sum_{k=0}^{\lfloor \frac{i}{2} \rfloor} (-1)^{m-k} a_{2m,i,k} \, (2k-(i+1)) \, (\cos y)^{i-2k} (\sin y)^{2k+1}.
\end{aligned}$$

for $i = 1,\dots,m-1$, $m = 0,\dots,p$.

b) It is clear that the term $2k$ is 0 for $k = 0$, so we can rewrite the sum on the left of (4.2.7) as

$$\begin{aligned}
&\sum_{k=1}^{\lfloor \frac{i+1}{2} \rfloor} (-1)^{m-k} \, (2k) \, a_{2m,i,k} \, (\cos y)^{i+2-2k} (\sin y)^{2k-1} \\
&\quad = \sum_{k=0}^{\lfloor \frac{i+1}{2} \rfloor - 1} (-1)^{m-k-1} \, (2k+2) \, a_{2m,i,k+1} \, (\cos y)^{i-2k} (\sin y)^{2k+1} \\
&\quad = \sum_{k=0}^{\lfloor \frac{i-1}{2} \rfloor} (-1)^{m-k+1} \, (2k+2) \, a_{2m,i,k+1} \, (\cos y)^{i-2k} (\sin y)^{2k+1}
\end{aligned}$$

For $i = 2c$, $\lfloor \frac{i}{2} \rfloor = \lfloor \frac{2c}{2} \rfloor = c = \frac{i}{2}$, so we have $a_{2m,i,\frac{i}{2}+1}$ in the last sum and it equals to 0 by the definition of $a_{2m,i,k}$. For $i = 2c+1$ we have $\lfloor \frac{i}{2} \rfloor = \lfloor \frac{2c+1}{2} \rfloor = c = \lfloor \frac{i-1}{2} \rfloor$. Thus we rewrite the last sum from 0 to $\lfloor \frac{i}{2} \rfloor$, since the sum will not change. Hence we get the following equation:

$$\sum_{k=0}^{\lfloor \frac{i+1}{2} \rfloor} (-1)^{m-k}\,(2k)\,a_{2m,i,k}\,(\cos y)^{i+2-2k}(\sin y)^{2k-1}$$
$$= \sum_{k=0}^{\lfloor \frac{i}{2} \rfloor} (-1)^{m+1-k}(2k+2)a_{2m,i,k+1}\,(\cos y)^{i-2k}(\sin y)^{2k+1}$$

for $i = 0,\dots,m-1$, $m = 0,\dots,p$.

c) Shifting the sum on the left hand side of (4.2.8) to $i = 0$ and remembering $a_{2m,-1,k} = 0$, we have

$$\sum_{i=1}^{m} \lambda^{i+1}\,(2m-2i+1)\,z^{2m-2i} \sum_{k=0}^{\lfloor \frac{i}{2} \rfloor} (-1)^{m+1-k}\,a_{2m,i-1,k}\,(\cos y)^{i-2k}(\sin y)^{2k+1}$$
$$= \sum_{i=0}^{m} \lambda^{i+1}(2m-2i+1)z^{2m-2i} \sum_{k=0}^{\lfloor \frac{i}{2} \rfloor} (-1)^{m+1-k}\,a_{2m,i-1,k}(\cos y)^{i-2k}(\sin y)^{2k+1}.$$

for $m = 0,\dots,p$.

d) Shifting the sum on the left hand side of (4.2.9) from $k = 0$ to $k = 1$ and remembering $a_{2m+1,i,-1} = 0$, we get

$$\sum_{k=1}^{\lfloor \frac{i}{2} \rfloor+1} (-1)^{m+1-k+1}\,a_{2m+1,i,k-1}\,(2k-2-i)(\cos y)^{i-2k+1}(\sin y)^{2k}$$
$$= \sum_{k=0}^{\lfloor \frac{i+2}{2} \rfloor} (-1)^{m-k}\,a_{2m+1,i,k-1}\,(2k-2-i)(\cos y)^{i+1-2k}(\sin y)^{2k}$$

For $i = 2c$, $\lfloor \frac{i+2}{2} \rfloor = \lfloor \frac{2c+2}{2} \rfloor = c+1$, then the term $(2k-2-i)$ becomes 0 for $k = \lfloor \frac{i+2}{2} \rfloor$, since $(2c+2)-2-2c = 0$. Therefore it is sufficient to write the last sum form 0 to $\lfloor \frac{i+1}{2} \rfloor$, that is,

$$\sum_{k=0}^{\lfloor \frac{i+1}{2} \rfloor} (-1)^{m-k} a_{2m+1,i,k-1}(2k-2-i)(\cos y)^{i+1-2k}(\sin y)^{2k}. \qquad (4.2.12)$$

For the case $i = 2c+1$, we have $\lfloor \frac{2c+3}{2} \rfloor = c+1 = \lfloor \frac{2c+2}{2} \rfloor = \lfloor \frac{i+1}{2} \rfloor$. So we can write

$\lfloor \frac{i+1}{2} \rfloor$ instead of writing $\lfloor \frac{i+2}{2} \rfloor$ in the sum (4.2.12). Hence we get

$$\sum_{k=0}^{\lfloor \frac{i}{2} \rfloor} (-1)^{m+1-k} \, (2k-i) \, a_{2m+1,i,k} (\cos y)^{i-2k-1} (\sin y)^{2k+2}$$
$$= \sum_{k=0}^{\lfloor \frac{i+1}{2} \rfloor} (-1)^{m-k} \, (2k-2-i) \, a_{2m+1,i,k-1} (\cos y)^{i+1-2k} (\sin y)^{2k}$$

for $i = 0, \ldots, m$, $m = 0, \ldots, p-1$.
e) For $i = 2c$, $\lfloor \frac{i+1}{2} \rfloor = \lfloor \frac{2c+1}{2} \rfloor = c = \lfloor \frac{i}{2} \rfloor$. So we can write $\lfloor \frac{i+1}{2} \rfloor$ instead of writing $\lfloor \frac{i}{2} \rfloor$ in the sum on the left hand side of (4.2.10). For $i = 2c+1$, we get $\lfloor \frac{i+1}{2} \rfloor = \lfloor \frac{2c+2}{2} \rfloor = c+1 = \frac{i+1}{2}$. In this case, by definition of $a_{2m+1,i,k}$ we have $a_{2m+1,i,\frac{i}{2}+\frac{1}{2}} = 0$ for $k = \frac{i+1}{2}$. Hence we obtain the following equation for $i = 0, \ldots, m$, $m = 0, \ldots, p-1$

$$\sum_{k=0}^{\lfloor \frac{i}{2} \rfloor} (-1)^{m+1-k} \, (2k+1) \, a_{2m+1,i,k} \, (\cos y)^{i+1-2k} (\sin y)^{2k}$$
$$= \sum_{k=0}^{\lfloor \frac{i+1}{2} \rfloor} (-1)^{m+1-k} \, (2k+1) \, a_{2m+1,i,k} \, (\cos y)^{i+1-2k} (\sin y)^{2k}.$$

f) It is clear that the term $(2m-2i)$ is 0 for $i = m$, so we can rewrite the sum on the left hand side of (4.2.11) as

$$\sum_{i=0}^{m-1} \lambda^{i+2} z^{2m-2i-1} (2m-2i) \sum_{k=0}^{\lfloor \frac{i}{2} \rfloor} (-1)^{m-k+2} a_{2m+1,i,k} (\cos y)^{i-2k} (\sin y)^{2k+2}$$
$$= \sum_{i=1}^{m} \lambda^{i+1} z^{2m-2i+1} (2m-2i+2) \sum_{k=0}^{\lfloor \frac{i-1}{2} \rfloor} (-1)^{m-k} a_{2m+1,i-1,k}$$
$$\times (\cos y)^{i-1-2k} (\sin y)^{2k+2}$$
$$= \sum_{i=1}^{m} \lambda^{i+1} z^{2m-2i+1} (2m-2i+2) \sum_{k=1}^{\lfloor \frac{i-1}{2} \rfloor + 1} (-1)^{m+1-k} a_{2m+1,i-1,k-1}$$
$$\times (\cos y)^{i+1-2k} (\sin y)^{2k}$$
$$= \sum_{i=1}^{m} \lambda^{i+1} z^{2m-2i+1} (2m-2i+2) \sum_{k=1}^{\lfloor \frac{i+1}{2} \rfloor} (-1)^{m+1-k} a_{2m+1,i-1,k-1}$$
$$\times (\cos y)^{i+1-2k} (\sin y)^{2k}$$

By definition of $a_{2m+1,i-1,k-1}$, we have $a_{2m+1,-1,k-1} = 0$ and $a_{2m+1,i-1,-1} = 0$ so we can rewrite the last sum as follows:

$$\sum_{i=0}^{m}\lambda^{i+1}z^{2m-2i+1}\sum_{k=0}^{\lfloor\frac{i+1}{2}\rfloor}(-1)^{m+1-k}\ (2m-2i+2)\ a_{2m+1,i-1,k-1}(\cos y)^{i+1-2k}(\sin y)^{2k}$$

for $m=0,\ldots,p-1$. This completes the proof.

Lemma 4.2.2. *Let $f_2^{(j)}(y,z)$, $a_{2m+2,i,k}$ and $a_{2m+1,i,k}$ satisfy the recurrence relations (4.2.2), (4.2.4) and (4.2.5) respectively with $f_2^{(0)}(y,z)=-\lambda\sin y$ for $j=0,\ldots,2p$. Then*

$$f_2^{(2m)}(y,z)=\sum_{i=0}^{m}\lambda^{i+1}z^{2m-2i}\sum_{k=0}^{\lfloor i/2\rfloor}(-1)^{m+1-k}a_{2m+1,i,k}(\cos y)^{i-2k}(\sin y)^{2k+1} \quad (4.2.13)$$

for $m=0,\ldots,p$ and

$$f_2^{(2m+1)}(y,z)=\sum_{i=0}^{m}\lambda^{i+1}z^{2m+1-2i}\sum_{k=0}^{\lfloor(i+1)/2\rfloor}(-1)^{m+1-k}a_{2m+2,i,k}(\cos y)^{i+1-2k}(\sin y)^{2k} \quad (4.2.14)$$

for $m=0,\ldots,p-1$.

Proof. The proof follows induction argument based on the equation (4.2.2). Since $f_2^{(0)}(y,z)=-\lambda\sin y$, $f_2^{(j)}(y,z)=z\dfrac{\partial f_2^{(j-1)}(y,z)}{\partial y}-\lambda\ \sin y\dfrac{\partial f_2^{(j-1)}(y,z)}{\partial z}$ and $\dfrac{\partial f_2^{(0)}(y,z)}{\partial z}=0$; we get the function $f_2^{(1)}(y,z)=z(-\lambda\cos y)=-\lambda z\cos y$.
For $j=1$ with $a_{2,0,0}=1$, we have

$$\begin{aligned}&\sum_{i=0}^{0}\lambda^{i+1}z^{0-2i+1}\sum_{k=0}^{\lfloor\frac{i+1}{2}\rfloor}(-1)^{1-k}a_{2,i,k}(\cos y)^{i+1-2k}(\sin y)^{2k}\\&=\lambda z(-1)a_{2,0,0}\cos y=-\lambda z\cos y.\end{aligned}$$

For $j=2$ with $a_{3,0,0}=1$, we have

$$\begin{aligned}f_2^{(2)}(y,z)&=z\frac{\partial f_2^{(1)}(y,z)}{\partial y}-\lambda\ \sin y\frac{\partial f_2^{(1)}(y,z)}{\partial z}\\&=z(\lambda\ z\ \sin y)-\lambda\sin y(-\lambda\cos y)\\&=\lambda z^2\sin y+\lambda^2\sin y\cos y.\end{aligned}$$

The summation becomes

$$\sum_{i=0}^{1}\lambda^{i+1}z^{2-2i}\sum_{k=0}^{\lfloor\frac{i}{2}\rfloor}(-1)^{2-k}a_{3,i,k}(\cos y)^{i-2k}(\sin y)^{2k+1}$$
$$=\lambda z^2(-1)a_{3,0,0}\sin y+\lambda^2 a_{3,0,0}\cos y\sin y$$
$$=\lambda z^2\sin y+\lambda^2\cos y\sin y.$$

Suppose it is true for $j=2m-1$, that is,

$$f_2^{(2m-1)}(y,z)=\sum_{i=0}^{m-1}\lambda^{i+1}z^{2m-2i-1}\sum_{k=0}^{\lfloor\frac{i+1}{2}\rfloor}(-1)^{m-k}a_{2m,i,k}(\cos y)^{i+1-2k}(\sin y)^{2k}.$$

Now we will show that (4.2.13) is true for $j=2m$

$$f_2^{(2m)}(y,z)=z\,\frac{\partial f_2^{(2m-1)}(y,z)}{\partial y}-\lambda\,\sin y\,\frac{\partial f_2^{(2m-1)}(y,z)}{\partial z}$$
$$=z\sum_{i=0}^{m-1}\lambda^{i+1}z^{2m-2i-1}\sum_{k=0}^{\lfloor\frac{i+1}{2}\rfloor}(-1)^{m-k}a_{2m,i,k}\,[-(i+1-2k)$$
$$\times(\cos y)^{i-2k}\sin y\,(\sin y)^{2k}+2k\cos y\,(\cos y)^{i+1-2k}(\sin y)^{2k-1}\Big]$$
$$-\lambda\sin y\sum_{i=0}^{m-1}\lambda^{i+1}(2m-2i-1)z^{2m-2i-2}\sum_{k=0}^{\lfloor\frac{i+1}{2}\rfloor}(-1)^{m-k}a_{2m,i,k}$$
$$\times(\cos y)^{i+1-2k}(\sin y)^{2k}$$
$$=\sum_{i=0}^{m-1}\lambda^{i+1}z^{2m-2i}\left[\sum_{k=0}^{\lfloor\frac{i+1}{2}\rfloor}(-1)^{m-k}a_{2m,i,k}\,(2k-(i+1))\,(\cos y)^{i-2k}(\sin y)^{2k+1}\right.$$
$$\left.+\sum_{k=0}^{\lfloor\frac{i+1}{2}\rfloor}(-1)^{m-k}\,2k\,a_{2m,i,k}\,(\cos y)^{i+2-2k}(\sin y)^{2k-1}\right]+\sum_{i=0}^{m-1}\lambda^{i+2}(2m-2i-1)$$
$$\times z^{2m-2i-2}\sum_{k=0}^{\lfloor\frac{i+1}{2}\rfloor}(-1)^{m+1-k}a_{2m,i,k}(\cos y)^{i+1-2k}(\sin y)^{2k+1}$$

Substituting equations (4.2.6), (4.2.7) and (4.2.8) into the last observation, we have

$$\begin{aligned}
f_2^{(2m)}(y,z) &= \sum_{i=0}^{m-1} \lambda^{i+1} z^{2m-2i} \left[\sum_{k=0}^{\lfloor \frac{i}{2} \rfloor} (-1)^{m-k} (2k-(i+1)) a_{2m,i,k} (\cos y)^{i-2k} \right. \\
&\quad \left. \times (\sin y)^{2k+1} + \sum_{k=0}^{\lfloor \frac{i}{2} \rfloor} (-1)^{m+1-k} (2k+2) a_{2m,i,k+1} (\cos y)^{i-2k} (\sin y)^{2k+1} \right] \\
&\quad + \sum_{i=0}^{m} \lambda^{i+1} (2m-2i+1) z^{2m-2i} \sum_{k=0}^{\lfloor \frac{i}{2} \rfloor} (-1)^{m+1-k} a_{2m,i-1,k} \\
&\quad \times (\cos y)^{i-2k} (\sin y)^{2k+1}.
\end{aligned}$$

Rewrite the first summation in the last equation from 0 to m, since $a_{2m,m,k} = 0$ and $a_{2m,m,k+1} = 0$

$$\begin{aligned}
f_2^{(2)}(y,z) &= \sum_{i=0}^{m} \lambda^{i+1} z^{2m-2i} \sum_{k=0}^{\lfloor \frac{i}{2} \rfloor} (-1)^{m+1-k} \left[(i+1-2k) a_{2m,i,k} + (2k+2) a_{2m,i,k+1} \right] \\
&\quad \times (\cos y)^{i-2k} (\sin y)^{2k+1} + \sum_{i=0}^{m} \lambda^{i+1} z^{2m-2i} \\
&\quad \times \sum_{k=0}^{\lfloor \frac{i}{2} \rfloor} (-1)^{m+1-k} (2m-2i+1) a_{2m,i-1,k} (\cos y)^{i-2k} (\sin y)^{2k+1} \\
&= \sum_{i=0}^{m} \lambda^{i+1} z^{2m-2i} \sum_{k=0}^{\lfloor \frac{i}{2} \rfloor} (-1)^{m+1-k} \left[(i+1-2k) a_{2m,i,k} + (2k+2) a_{2m,i,k+1} \right. \\
&\quad \left. + (2m-2i+1) a_{2m-i-1,k} \right] (\cos y)^{i-2k} (\sin y)^{2k+1} \\
&= \sum_{i=0}^{m} \lambda^{i+1} z^{2m-2i} \sum_{k=0}^{\lfloor \frac{i}{2} \rfloor} (-1)^{m+1-k} a_{2m+1,i,k} (\cos y)^{i-2k} (\sin y)^{2k+1}.
\end{aligned}$$

After these, now we will find $f_2^{(2m+1)}(y,z)$;

$$\begin{aligned}
f_2^{(2m+1)}(y,z) &= z\frac{\partial f_2^{(2m)}(y,z)}{\partial y} - \lambda \sin y \frac{\partial f_2^{(2m)}(y,z)}{\partial z} \\
&= z\sum_{i=0}^{m} \lambda^{i+1} z^{2m-2i} \sum_{k=0}^{\lfloor \frac{i}{2} \rfloor} (-1)^{m+1-k} a_{2m+1,i,k} \Big[(2k-i)(\cos y)^{i-1-2k} \\
&\times (\sin y)^{2k+2} + (2k+1)(\cos y)^{i-2k+1}(\sin y)^{2k}\Big] \\
&-\lambda \sin y \sum_{i=0}^{m} \lambda^{i+1}(2m-2i) z^{2m-2i-1} \\
&\times \sum_{k=0}^{\lfloor \frac{i}{2} \rfloor} (-1)^{m+1-k} a_{2m+1,i,k} (\cos y)^{i-2k} (\sin y)^{2k+1} \\
&= \sum_{i=0}^{m} \lambda^{i+1} z^{2m-2i+1} \left[\sum_{k=0}^{\lfloor \frac{i}{2} \rfloor} (-1)^{m+1-k}\ (2k-i)\ a_{2m+1,i,k} (\cos y)^{i-1-2k} \right. \\
&\left. \times (\sin y)^{2k+2} + \sum_{k=0}^{\lfloor \frac{i}{2} \rfloor} (-1)^{m+1-k} (2k+1) a_{2m+1,i,k} (\cos y)^{i-2k+1} (\sin y)^{2k} \right] \\
&+ \sum_{i=0}^{m} \lambda^{i+2}\ (2m-2i)\ z^{2m-2i-1} \sum_{k=0}^{\lfloor \frac{i}{2} \rfloor} (-1)^{m+2-k} a_{2m+1,i,k} \\
&\times (\cos y)^{i-2k} (\sin y)^{2k+2}.
\end{aligned}$$

Substituting equations (4.2.9), (4.2.10) and (4.2.11) into the last equation, we have

$$\begin{aligned}
& f_2^{(2m+1)}(y,z) = \\
&= \sum_{i=0}^{m} \lambda^{i+1} z^{2m-2i+1} \left[\sum_{k=0}^{\lfloor \frac{i+1}{2} \rfloor} (-1)^{m-k}\ (2k-i-2)\ a_{2m+1,i,k-1} (\cos y)^{i+1-2k} (\sin y)^{2k} \right. \\
&\left. + \sum_{k=0}^{\lfloor \frac{i+1}{2} \rfloor} (-1)^{m+1-k}\ (2k+1)\ a_{2m+1,i,k} (\cos y)^{i+1-2k} (\sin y)^{2k} \right] + \sum_{i=0}^{m} \lambda^{i+1} z^{2m-2i+1} \\
&\times \sum_{k=0}^{\lfloor \frac{i+1}{2} \rfloor} (-1)^{m+1-k}\ (2m-2i+2)\ a_{2m+1,i-1,k-1} (\cos y)^{i+1-2k} (\sin y)^{2k} \\
&= \sum_{i=0}^{m} \lambda^{i+1} z^{2m-2i+1} \sum_{k=0}^{\lfloor \frac{i+1}{2} \rfloor} (-1)^{m+1-k} \big[(i-2-2k) a_{2m+1,i,k-1} \\
&+ (2k+1) a_{2m+1,i,k} + (2m-2i+2) a_{2m+1,i-1,k-1}\big] (\cos y)^{i+1-2k} (\sin y)^{2k}.
\end{aligned}$$

Hence we get,

$$f_2^{(2m+1)} = \sum_{i=0}^{m} \lambda^{i+1} z^{2m+1-2i} \sum_{k=0}^{\lfloor \frac{i+1}{2} \rfloor} (-1)^{m+1-k} a_{2m+2,i,k} (\cos y)^{i+1-2k} (\sin y)^{2k}.$$

Theorem 4.2.1. *If $f_1^{(j)}(y,z)$ and $f_2^{(j)}(y,z)$ are sufficiently smooth and satisfy (4.2.3), (4.2.13) and (4.2.14) then the following relations hold for $j = 2, \ldots, 2p+1$:*

$$\begin{aligned} &\text{a) } y_1 - y_0 + \sum_{j=1}^{p} \beta_j \left[(-1)^j f_1^{(j-1)}(y_1, 0) - f_1^{(j-1)}(y_0, 0) \right] \\ &\qquad = -2 \left(y_0 + \sum_{j=1}^{p} \beta_j f_1^{(j-1)}(y_0, 0) \right) \quad \textit{and} \\ &\sum_{j=1}^{p} \beta_j \left[(-1)^j f_2^{(j-1)}(y_1, 0) - f_2^{(j-1)}(y_0, 0) \right] = 0 \end{aligned} \tag{4.2.15}$$

for $y_1 = -y_0$,

$$\begin{aligned} &\text{b) } \sum_{j=1}^{p} \beta_j \left[(-1)^j f_2^{(j-1)}(y_1, 0) - f_2^{(j-1)}(y_0, 0) \right] = -2 \sum_{j=1}^{p} \beta_j f_2^{(j-1)}(y_0, 0) \quad \textit{and} \\ &y_1 - y_0 + \sum_{j=1}^{p} \beta_j \left[(-1)^j f_1^{(j-1)}(y_1, 0) - f_1^{(j-1)}(y_0, 0) \right] = 0 \end{aligned} \tag{4.2.16}$$

for $y_1 = y_0$.

Proof. Let $j = 2m+1$ for $m = 0, \ldots, p-1$, then $f_1^{(2m+1)}(y,z)$ becomes

$$\begin{aligned} f_1^{(2m)}(y,z) &= f_2^{(2m-1)}(y,z) \\ &= \sum_{i=0}^{m-1} \lambda^{i+1} z^{2m-2i-1} \sum_{k=0}^{\lfloor (i+1)/2 \rfloor} (-1)^{m-k} a_{2m,i,k} (\cos y)^{i+1-2k} (\sin y)^{2k} \end{aligned} \tag{4.2.17}$$

by Lemma (4.2.2). Since all terms of previous sum contain z,

$f_1^{(2m)}(y,0) = f_2^{(2m-1)}(y,0) = 0$ for $m = 1, \ldots, p$. Hence we get the following equations

$$\begin{aligned} y_1 - y_0 + \sum_{j=1}^{p} \beta_j \left[(-1)^j f_1^{(j-1)}(y_1,0) - f_1^{(j-1)}(y_0,0) \right] \\ = y_1 - y_0 + \sum_{j=1}^{p} \beta_j \left[f_1^{(j-1)}(y_1,0) - f_1^{(j-1)}(y_0,0) \right]. \end{aligned} \tag{4.2.18}$$

and

$$\sum_{j=1}^{p} \beta_j \left[(-1)^j f_2^{(j-1)}(y_1,0) - f_2^{(j-1)}(y_0,0) \right] = \sum_{j=1}^{p} \beta_j \left[-f_2^{(j-1)}(y_1,0) - f_2^{(j-1)}(y_0,0) \right]. \tag{4.2.19}$$

Letting $j = 2m+2$ for $m = 0, \ldots, p-1$ and using (4.2.13), we get

$$\begin{aligned} f_1^{(2m+2-1)}(y,z) &= f_2^{(2m)}(y,z) \\ &= \sum_{i=0}^{m} \lambda^{i+1} z^{2m-2i} \sum_{k=0}^{\lfloor i/2 \rfloor} (-1)^{m+1-k} a_{2m+1,i,k} (\cos y)^{i-2k} (\sin y)^{2k+1} \end{aligned} \tag{4.2.20}$$

Substituting the value $z = 0$ into (4.2.20), we obtain

$$\begin{aligned} f_1^{(2m+1)}(-y_0,0) &= f_2^{(2m)}(-y_0,0) \\ &= \lambda^{m+1} \sum_{k=0}^{\lfloor m/2 \rfloor} (-1)^{m+1-k} a_{2m+1,m,k} (\cos y_0)^{i-2k} \left[-(\sin y_0)^{2k+1} \right] \\ &= -f_2^{(2m)}(y_0,0) \\ &= -f_1^{(2m+1)}(y_0,0) \end{aligned}$$

which gives the following relations

$$\begin{aligned} f_1^{(2m+1)}(-y_0,0) = -f_1^{(2m+1)}(y_0,0), \\ f_2^{(2m)}(-y_0,0) = -f_2^{(2m)}(y_0,0). \end{aligned} \tag{4.2.21}$$

Using (4.2.21) for $y_1 = -y_0$, we obtain the following relations

$$\begin{aligned} y_1 - y_0 + \sum_{j=1}^{p} \beta_j \left[(-1)^j f_1^{(j-1)}(y_1, 0) - f_1^{(j-1)}(y_0, 0)\right] \\ = -y_0 - y_0 + \sum_{j=1}^{p} \beta_j \left[f_1^{(j-1)}(-y_0, 0) - f_1^{(j-1)}(y_0, 0)\right] \\ = -2\left(y_0 + \sum_{j=1}^{p} \beta_j f_1^{(j-1)}(y_0, 0)\right), \end{aligned}$$

$$\begin{aligned} \sum_{j=1}^{p} \beta_j \left[(-1)^j f_2^{(j-1)}(y_1, 0) - f_2^{(j-1)}(y_0, 0)\right] \\ = \sum_{j=1}^{p} \beta_j \left[-f_2^{(j-1)}(-y_0, 0) - f_2^{(j-1)}(y_0, 0)\right] \\ = \sum_{j=1}^{p} \beta_j \left[f_2^{(j-1)}(y_0, 0) - f_2^{(j-1)}(y_0, 0)\right] = 0 \end{aligned}$$

Similarly for $y_1 = y_0$ using (4.2.21), we observe that

$$\begin{aligned} y_1 - y_0 + \sum_{j=1}^{p} \beta_j \left[(-1)^j f_1^{(j-1)}(y_1, 0) - f_1^{(j-1)}(y_0, 0)\right] \\ = y_0 - y_0 + \sum_{j=1}^{p} \beta_j \left[f_1^{(j-1)}(y_0, 0) - f_1^{(j-1)}(y_0, 0)\right] = 0. \end{aligned}$$

and

$$\begin{aligned} \sum_{j=1}^{p} \beta_j \left[(-1)^j f_2^{(j-1)}(y_1, 0) - f_2^{(j-1)}(y_0, 0)\right] \\ = \sum_{j=1}^{p} \beta_j \left[-f_2^{(j-1)}(y_0, 0) - f_2^{(j-1)}(y_0, 0)\right] \\ = -2\sum_{j=1}^{p} \beta_j f_2^{(j-1)}(y_0, 0). \end{aligned}$$

So the assertions (a) and (b) are proved.

We consider the application of Taylor's decomposition Ashyralyev, & Sobolevskii (2004) of solution to (4.2.1) on two points x_k and x_{k-1}

$$Y(x_k) - Y(x_{k-1}) + \sum_{j=1}^{p} \alpha_j Y^{(j)}(x_k) h^j - \sum_{j=1}^{q} \beta_j Y^{(j)}(x_{k-1}) h^j = \tau_k, \qquad (4.2.22)$$

where

$$\tau_k = \frac{(-1)^p}{(p+q)!}\int_{x_{k-1}}^{x_k}(x_k-s)^q(s-x_{k-1})^p\,Y^{(p+q+1)}(s)ds, \tag{4.2.23}$$

and $x_k = kh$, $k = 0,\dots,n$, $nh = 1$, $n \in N$ with the stepsize h,

$$\alpha_j = \frac{(p+q-j)!p!(-1)^j}{(p+q)!j!(p-j)!} \quad , \quad 1 \leqslant j \leqslant p,$$

$$\beta_j = \frac{(p+q-j)!q!}{(p+q)!j!(q-j)!} \quad , \quad 1 \leqslant j \leqslant q\,.$$

Neglecting the last term of (4.2.22), we obtain single-step difference schemes of $(p+q)$-order of accuracy for the approximate solution to the problem (4.2.1)

$$Y_k - Y_{k-1} + \sum_{j=1}^{p}\alpha_j Y_k^{(j)}h^j - \sum_{j=1}^{q}\beta_j Y_{k-1}^{(j)}h^j = 0, \tag{4.2.24}$$

where $Y_k^{(j)}$ is the approximate value of $Y_k^{(j)}(x_k)$. For the computation of the eigenvalues of (4.1.1), putting $h = 1$ and $p = q$, the approximation (4.2.24) gives

$$Y_1 - Y_0 + \sum_{j=1}^{p}(-1)^j\beta_j Y_1^{(j)} - \sum_{j=1}^{p}\beta_j Y_0^{(j)} = 0, \tag{4.2.25}$$

where $\alpha_j = (-1)^j\beta_j$. Writing (4.2.25) with respect to the components and imposing the boundary conditions $z_0 = z(0) = y'(0) = 0$ and $z_1 = z(1) = y'(1) = 0$, we have the following equations

$$y_1 - y_0 + \sum_{j=1}^{p}\beta_j\left[(-1)^j f_1^{(j-1)}(y_1,0) - f_1^{(j-1)}(y_0,0)\right] = 0 \tag{4.2.26}$$

and

$$\sum_{j=1}^{p}\beta_j\left[(-1)^j f_2^{(j-1)}(y_1,0) - f_2^{(j-1)}(y_0,0)\right] = 0. \tag{4.2.27}$$

Using Theorem 4.2.1.a for $y_1 = -y_0$, the equation (4.2.26) becomes

$$G_1(y_0,\lambda) = -2\left(y_0 + \sum_{j=1}^{p}\beta_j f_1^{(j-1)}(y_0,0)\right) = 0 \tag{4.2.28}$$

and (4.2.27) is satisfied. And for $y_1 = y_0$, (4.2.26) is satisfied by Theorem (4.2.1).b and the equation (4.2.27) becomes

$$G_2(y_0,\lambda) = -2\sum_{j=1}^{p} \beta_j f_2^{(j-1)}(y_0,0) = 0. \tag{4.2.29}$$

The following graph, Figure 4.1 gives the relation between the initial values y_0 and the eigenvalues λ of the nonlinear eigenvalue problem (4.1.1).

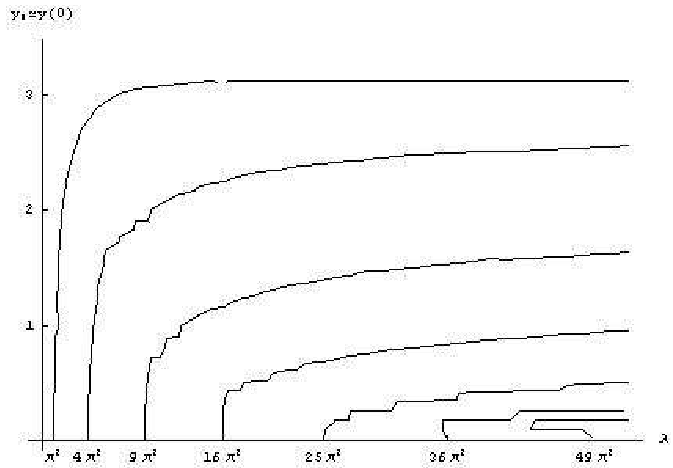

Figure 4.1 Bifurcation diagram obtained from (4.2.28) and (4.2.29).

Table 4.1 Corresponding to the initial values $y_{1,0}$, $y_{2,0}$, $y_{3,0}$ and $y_{4,0}$ for various λ obtained from (4.2.28) and (4.2.29).

λ	$y_{1,0}$	$y_{2,0}$	$y_{3,0}$	$y_{4,0}$
$15 > \pi^2$	1.7471	–	–	–
$45 > 4\pi^2$	2.8578	1,0092	–	–
$90 > 9\pi^2$	3.0718	2.3413	0.3236	–
$160 > 16\pi^2$	3.1272	2.7999	2.0239	0.3771

From Figure 4.1 and Table 4.1 we observe that; there is only trivial initial condition for $0 \leqslant \lambda \leqslant \pi^2$, there is one nontrivial initial condition from (4.2.28) for $\pi^2 < \lambda \leqslant 4\pi^2$, there are n nontrivial initial conditions for $n^2\pi^2 < \lambda \leqslant (n+1)^2\pi^2$. These results show that, the numerical results obtained using Taylor's decomposition method agree with the theoretical results of Euler Buckling problem given in Stakgold (1971).

Now we find an approximate solution to the corresponding initial problem

$$\begin{aligned} Y'(x) &= F(Y(x)) \\ Y(0) &= Y_0 \end{aligned} \tag{4.2.30}$$

that corresponds to Euler Buckling Problem (4.1.1) for an eigenvalue λ and the initial value y_0. Using Taylor's Decomposition on two points x_{k-1}, x_k on the uniform grid

$$[0,1]_h = \{x_k = kh,\ k = 0,1,\dots,n,\ nh = 1,\ n \in N\},$$

for $p = q$, we get

$$Y_k - Y_{k-1} + \sum_{j=1}^{p} (-1)^j \beta_j Y_k^{(j)} h^j - \sum_{j=1}^{p} \beta_j Y_{k-1}^{(j)} h^j = 0 \quad \text{and} \quad Y_0 = \begin{bmatrix} y_0 \\ z_0 \end{bmatrix}, \tag{4.2.31}$$

where $y_0 \simeq y(0)$, $z_0 = z(0)$. Solving the nonlinear system (4.2.31) by Newton's method, we obtain the approximate value y_k of the eigenfunction $y(x)$ at $x = x_k$ with $O(h^{2p})$.

It is clear that $f_2^{(0)}(y,z) = \sin y$ is Lipschitz in y in 2-dimensional box D. Using the results Lemma (3.2.1) and Theorem (3.2.1), the global error for (4.2.24) is bounded by

$$||Y(x_k) - Y_k|| \leqslant C_0 ||Y(0) - Y_0|| + C_1 \frac{\xi h^{2p} M^{p+1}}{(2p)!}$$

where $C_0 = e^{\bar{x}\frac{2LB(h)}{1-LB(h)}}$, $C_1 = const.\dfrac{C_0}{L}\dfrac{1}{1+\frac{\beta_2}{\beta_1}h+\ldots+\frac{\beta_p}{\beta_1}h^{p-1}}$, D is 2-dimensional box in R^2, $M = \max\limits_{(y,z)\in D}\{|f_1^{(0)}(y,z)|,\ |f_2^{(0)}(y,z)|\}$, $\xi = \max\{\sum\limits_{k=0}^{\lfloor\frac{i+1}{2}\rfloor} a_{j,i,k}\}$, $j = 1,\dots,2p$, $i = 0,\dots,p$, and *const.* is a constant independent of h and p, $\|\cdot\|$ denotes $\|\cdot\|_\infty$, $L = \max\limits_{1\leqslant j\leqslant p}\{l_{1,j}, l_{2,j}\}$ with $l_{1,j} = \max\limits_{1\leqslant j\leqslant p}\{d_{1,j}, s_{1,j}\}$, $l_{2,j} = \max\limits_{1\leqslant j\leqslant p}\{d_{2,j}, s_{2,j}\}$, $d_{k,j} = \max\limits_{(y,z)\in D}\left|\dfrac{\partial f_k^{(j)}(y,z)}{\partial y}\right|$, $s_{k,j} = \max\limits_{(y,z)\in D}\left|\dfrac{\partial f_k^{(j)}(y,z)}{\partial z}\right|$, $k = 1,2$ and $B(h) = L\sum\limits_{j=1}^{p}\beta_j h^{j-1}$ for some $\bar{x} > 0$.

4.3 Numerical Results for Euler Buckling Problem

The Taylor's decomposition method described in the previous sections is applied to Euler Buckling Problem. The approximate solutions of Euler buckling problem for $\lambda = 15$, $\lambda = 45$, $\lambda = 90$ and $\lambda = 160$ generated using Taylor's decomposition method for step-size $h = \dfrac{1}{20}$ are illustrated in Figure 4.2, Figure 4.3, Figure 4.4 and Figure 4.5 respectively.

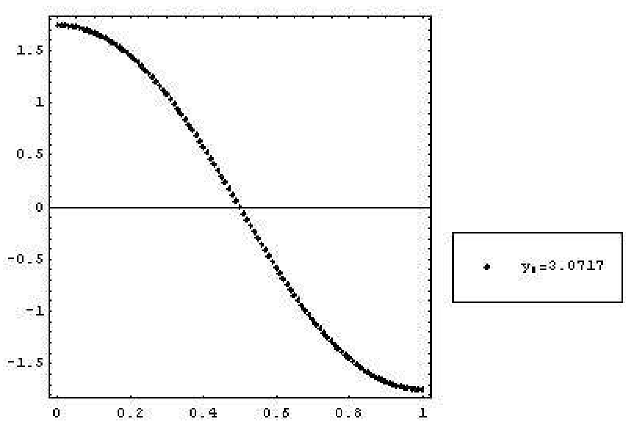

Figure 4.2 Solution of (4.1.1) corresponding to the initial value y_0 for $\pi^2 \leqslant \lambda = 15 < 4\pi^2$.

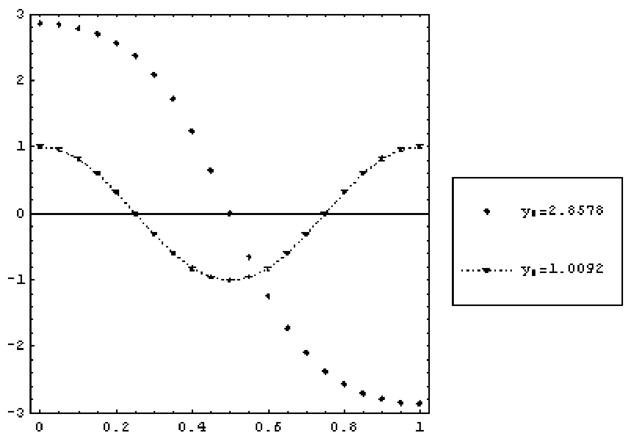

Figure 4.3 Solution of (4.1.1) corresponding to the initial value y_0 for $4\pi^2 \leqslant \lambda = 45 < 9\pi^2$.

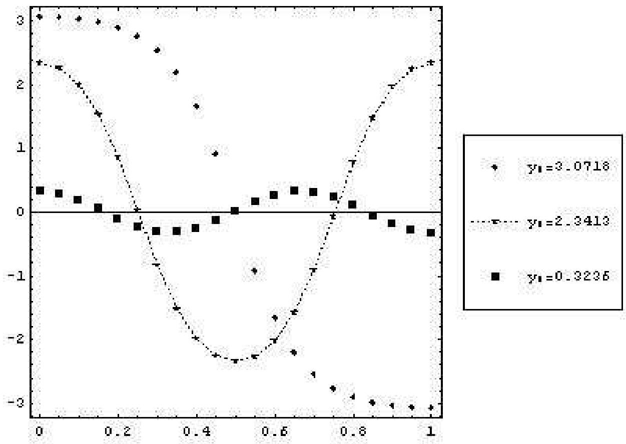

Figure 4.4 Solution of (4.1.1) corresponding to the initial value y_0 for $9\pi^2 \leqslant \lambda = 90 < 16\pi^2$.

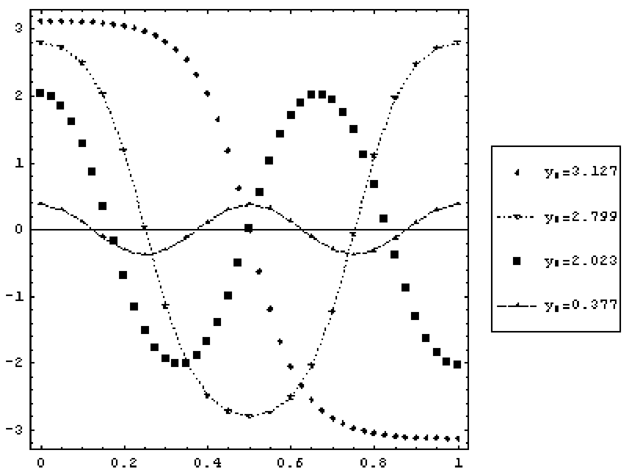

Figure 4.5 Solution of (4.1.1) corresponding to the initial value y_0 for $16\pi^2 \leqslant \lambda = 160 < 25\pi^2$.

CHAPTER SIX
CONCLUSIONS

Taylor's decomposition method is applied to solve regular Sturm-Liouville eigenvalue problem, one-dimensional Bratu problem and Euler buckling problem by converting them into a system of differential equation with initial conditions. As a result, we obtained the behavior of eigenvalues and the corresponding eigenfunctions with high order accuracy for relatively large step sizes and the observed orders are in good agrement with the predicted ones in the theorem. This method can be extended to some nonlinear eigenvalue problem to investigate the behavior of the eigenvalues and eigenfunction. Higher order accuracy difference schemes generated by Taylor's decomposition on three points for the boundary value problem of elliptic equations with the operator acting in an arbitrary Banach space were presented in Ashyralyev, & Sobolevskii (2004). Hence the Taylor's decomposition for nonlinear elliptic eigenvalue problem is an open problem which may be worth investigating.

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APPENDIX A
EXISTENCE, UNIQUENESS, AND SPECTRAL PROPERTIES OF NONLINEAR EIGENVALUE PROBLEMS

We consider the following nonlinear eigenvalue problem:

$$(p(x)u')' + \lambda f(x,u) = 0, \quad 0 \leqslant x \leqslant 1, \tag{6.0.1}$$

$$a_0u(0) - a_1u'(0) = 0, \qquad |a_0| + |a_1| \neq 0, \tag{6.0.2}$$

$$b_0u(1) - b_1u'(1) = 0, \qquad |b_0| + |b_1| \neq 0. \tag{6.0.3}$$

We suppose that $p(x) > 0$ and $p'(x)$ is continuous on $0 \leqslant x \leqslant 1$ and that $f(x,u)$ satisfies the following conditions:

H-1: $f(x,u)$ is continuously differentiable in D :
$0 \leqslant x \leqslant 1, \quad -\infty < u < \infty.$

H-2: $0 < f_u(x,u) < \rho(x)$ on $D,$ where $\rho(x) > 0$ in $0 \leqslant x \leqslant 1,$

H-3: $f(x,0) \neq 0$ on $0 \leqslant x \leqslant 1.$

Our main result is the

Theorem 6.0.1. *Let $f(x,u)$ satisfy H-1, 2, 3, and let the constants a_i, b_i satisfy*

$$a_i \geqslant 0,\ b_i \geqslant 0, \quad (i = 0,1),\ a_0 + b_0 > 0.$$

Then, there exists a unique solution of (6.0.1) , (6.0.2), (6.0.3) for all λ in $0 < \lambda < \mu_1\{\rho\}$, where $\mu_1\{\rho\}$ is the principal (i.e., least) eigenvalue of

$$(p(x)u')' + \mu\rho(x)u = 0, \quad 0 \leqslant\leqslant 1, \tag{6.0.4}$$

$$a_0u(0) - a_1u'(0) = 0, \tag{6.0.5}$$

$$b_0u(1) + b_1u'(1) = 0. \tag{6.0.6}$$

Proof. We outline the proof which is based on the technique used recently by Keller (1966). The initial value problem

$$\begin{aligned}&(p(x)y')' + \lambda f(x,y) = 0,\\&a_y(0) - a_1 y'(0) = 0,\\&c_0 y(0) - c_1 y'(0) = s, \qquad a_1 c_0 - a_0 c_1 = 1,\end{aligned}$$

has the unique solution $y(s;x)$. The problem (6.0.1), (6.0.2), (6.0.3) has as many solutions as there are real roots, $s*$, of

$$\phi(s) \equiv b_0 y(s;1) + b_1 y'(s;1) = 0.$$

We shall show that $\phi' s$ is positive and bounded away from zero, from which it follows that $\phi(s) = 0$ always has and only one root.

Since $y(s;x)$ is continuously differentiable with respect to s, the derivative $\omega(x) = \partial y(s;x)/\partial s$ satisfies the variational problem

$$\begin{aligned}(p(x)\omega')' + \lambda f_u(x,y)\omega =&\ 0,\\a_0\omega(0) - a_1\omega'(0) =&\ 0,\\c_0\omega(0) - c_1\omega'(0) =&\ 1.\end{aligned}$$

Clearly we must show that $\phi'(s) \equiv b_0\omega(1) + b_1\omega'(1)$ is positive and bounded away from zero. To do this we consider the linear problem

$$\begin{aligned}(p(x)v')' + \lambda\rho(x)v =&\ 0,\\a_v(0) - a_1 v'(0) =&\ 0,\\c_0 v(0) - c_1 v'(0) =&\ 0.\end{aligned} \tag{6.0.7}$$

For a fixed $\lambda \equiv \lambda_1$, say, let l be the first value of $x > 0$ at which $b_0 v(l) + b_1 v'(l) = 0$. (That such an l exists will be clear from the formulation of the problem 6.0.8.) Then the unique solution $v_1(x)$ of (6.0.7) also satisfies

$$\begin{aligned}(p(x)v')' + \lambda_1\rho(x)v_1 =&\ 0,\\a_0 v_1(0) - a_1 v_1'(0) =&\ 0,\\b_0 v_1(l) + b_1 v_1'(l) =&\ 0\end{aligned} \tag{6.0.8}$$

where $\lambda = \lambda_1(l)$ is the principle eigenvalue of (6.0.8) and $v_1(x)$ is the corresponding eigenfunction normalized so that it satisfies the third equation in (6.0.7). We now show that $b_0\omega(x) + b_1\omega'(x) > 0$ on $0 < x < l$. We do this by contradiction. If $b_0\omega(x) + b_1\omega'(x) = 0$ for some κ in $0z\kappa < l$, then $\omega(x)$ would satisfy

$$\begin{aligned} (p(x)\omega')' + \lambda_1 f_u(x,y)\omega &= 0, \\ a_0\omega(0) - a_1\omega'(0) &= 0, \\ b_0\omega(\kappa) + b_1\omega'(\kappa) = 0. \end{aligned} \tag{6.0.9}$$

Now, from the usual variational characterization Courant, & Hilbert (1953) of the principle eigenvalue of problems of the form of (6.0.8), we know that as the coefficients $\rho(x)$ varies in one sense, the eigenvalue λ_1 varies in the opposite sense, and as the length of the interval varies in one sense, the eigenvalue λ_1 varies in the opposite sense. Thus, for fixed $\lambda = \lambda_1$, since $f_u(x,y) < \rho(x)$, the third equation in (6.0.9) can not hold for $\kappa < l$. Hence, we conclude that $b_0\omega(x) + b_1\omega'(x) > 0$ on $0 < x < l$.

Finally, by once again using the fact that $\lambda_(l)$ varies in the opposite sense from l, we conclude that if $\lambda < \lambda_1 \equiv \mu_1\rho$, then $l > 1$. Therefore, $\phi'(s) \equiv b_0\omega(1) + b_1\omega'(1) > 0$.

REMARK. Actually, condition H-3 is not necessary for our proof. However, if $f(x,y) \equiv 0$, the unique solution will be the trivial one. If $f(x,y) = 0$, then the problem is closely related to one treated thoroughly by Pimbley (1962) . Pimbley's Theorem 1 gives uniqueness in the same range of λ. The extension to the case $f(x,0) \neq 0$ is by no means trivial, however, and the consequences of this condition are pointed out in some detail in Keller, & Cohen (1967).

APPENDIX B
TAYLOR'S DECOMPOSITION METHOD

7.1 Taylor's Decomposition Method

This section is from the book of Ashyralyev, & Sobolevskii (2004).

We consider the initial-value problem

$$y'(t) + a(t)y(t) = f(t), \qquad 0 < t \leqslant T, \quad y(0) = y_0 \tag{7.1.1}$$

assuming $a(t)$ and $f(t)$ to be such that problem (7.1.1) has a unique smooth solution defined on $[0,T]$. The usage of Taylor's decomposition on two points in the construction of the single-step difference schemes of a high order of accuracy for approximate solutions of problem (7.1.1) is based on the following theorem.

Theorem 7.1.1. *Let the function* $v(t) \quad (0 \leqslant t \leqslant T)$ *have a* $(p+q+1)$*-th continuous derivative and* $t_{k-1}, t_k \in [0,T]_\tau$, *where*

$$[0,T]_\tau = \{t_k = k\tau, k = 0,1,\ldots,N, N_\tau = T\}. \tag{7.1.2}$$

Then the following relation holds:

$$v(t_k) - v(t_{k-1}) + \sum_{j=1}^{p} \alpha_j v^{(j)}(t_k)\tau^j - \sum_{j=1}^{q} \beta_j v^{(j)}(t_{k-1})\tau^j \tag{7.1.3}$$

$$= \frac{(-1)^p}{(p+q)!} \int_{t_{k-1}}^{t_k} (t_k - s)^q (s - t_{k-1})^p v^{(p+q+1)}(s)\,ds,$$

where

$$\begin{cases} \alpha_j = \dfrac{(p+q-j)!p!(-1)^j}{(p+q)!j!(p-j)!} & \text{for any } j, \quad 1 \leqslant j \leqslant p, \\[2ex] \beta_j = \dfrac{(p+q-j)!q!}{(p+q)!j!(q-j)!} & \text{for any } j, \quad 1 \leqslant j \leqslant q. \end{cases} \tag{7.1.4}$$

Proof. Using the formula for integration by parts, we obtain the representation

$$\int_{t_{k-1}}^{t_k} (t_k - s)^q (s - t_{k-1})^p v^{(p+q+1)}(s)\, ds$$
$$= \sum_{\gamma=0}^{p+q} (-1)^{p+q-\gamma}((t_k - s)^q (s - t_{k-1})^p)^{(p+q-\gamma)} v^{\gamma}(s)]_{t_{k-1}}^{t_k}.$$

We will calculate the expressions

$$(-1)^{p+q-j}[(t_k - s)^q (s - t_{k-1})^p]^{(p+q-j)}|_{s=t_k},$$

and

$$(-1)^{p+q-j}[(t_k - s)^q (s - t_{k-1})^p]^{(p+q-j)}|_{s=t_{k-1}}.$$

If $j \geqslant p+1$, then $p+q-j \leqslant q-1$. Therefore

$$(-1)^{p+q-j}[(t_k - s)^q (s - t_{k-1})^p]^{(p+q-j)}|_{s=t_k} = 0.$$

If $0 \leqslant j \leqslant p$, then

$$(-1)^{p+q-j}[(t_k - s)^q (s - t_{k-1})^p]^{(p+q-j)}|_{s=t_k}$$
$$= (-1)^{p+q-j} \sum_{i=0}^{p+q-j} \frac{(p+q-j)!}{i!(p+q-j-i)!} ((t_k - s)^q)^{(p+q-j-i)} ((s - t_{k-1})^p)^{(i)}|_{s=t_k}$$
$$= (-1)^{p+q-j} \sum_{i=0}^{p+q-j} \frac{(p+q-j)!}{i!(p+q-j-i)!} \frac{q!}{(j+i-p)!}$$
$$\times (-1)^{(p+q-j-i)} (t_k - s)^{j+i-p} \frac{p!}{(p-i)!} (s - t_{k-1})^{p-i}|_{s=t_k}$$
$$= (-1)^{p+q-j} \frac{(p+q-j)!}{(p-j)!q!} q!(-1)^q \frac{p!}{j!} \tau^j$$
$$= \frac{(-1)^{p-j}}{(p+q)!} \frac{(p+q)!(p+q-j)!}{(p-j)!j!} p!\tau^j = (-1)^p (p+q)!\alpha_j$$

So,

$$(-1)^{p+q-j}[(t_k - s)^q (s - t_{k-1})^p]^{(p+q-j)}|_{s=t_k} = (-1)^p (p+q)!\alpha_j.$$

If $j \geqslant q+1$, then $p+q-j \leqslant p-1$. Therefore

$$(-1)^{p+q-j}[(t_k - s)^q (s - t_{k-1})^p]^{(p+q-j)}|_{s=t_{k-1}} = 0.$$

If $0 \leqslant j \leqslant q$, then

$$
\begin{aligned}
&(-1)^{p+q-j}[(t_k-s)^q(s-t_{k-1})^p]^{(p+q-j)}|_{s=t_{k-1}} \\
&= (-1)^{p+q-j}\sum_{i=0}^{p+q-j}\frac{(p+q-j)!}{i!(p+q-j-i)!}\,((t_k-s)^q)^{(p+q-j-i)}((s-t_{k-1})^p)^{(i)}|_{s=t_{k-1}} \\
&= (-1)^{p+q-j}\sum_{i=0}^{p+q-j}\frac{(p+q-j)!}{i!(p+q-j-i)!}\frac{q!}{(j+i-p)!} \\
&\times(-1)^{(p+q-j-i)}(t_k-s)^{j+i-p}\frac{p!}{(p-i)!}(s-t_{k-1})^{p-i}|_{s=t_{k-1}} \\
&= (-1)^{p+q-j}\frac{(p+q-j)!}{p!(q-j)!j!}q!(-1)^{q-j}\frac{p!}{0!}\tau^j \\
&= \frac{(-1)^{p-j}}{(p+q)!}\,\frac{(p+q)!(p+q-j)!q!}{(q-j)!j!}\tau^j = (-1)^p(p+q)!\beta_j
\end{aligned}
$$

So,

$$(-1)^{p+q-j}[(t_k-s)^q(s-t_{k-1})^p]^{(p+q-j)}|_{s=t_{k-1}} = (-1)^p(p+q)!\beta_j.$$

Theorem (7.1.1) is proved.

Note that relation (7.1.3) is called Taylor's decomposition of function $v(t)$ on two points.

Now, we will consider applications of Taylor's decomposition of function on two points. From (7.1.3) it is clear that for the approximate solution of problem (7.1.1) it is necessary to find $y^{(j)}(t_k)$ for any j, $1 \leqslant j \leqslant p$ and $y^{(j)}(t_{k-1})$ for any j, $1 \leqslant j \leqslant q$. Using the equation

$$y'(t) = -a(t)y(t) + f(t),$$

we obtain

$$y^{(j)}(t) = a_j(t)y(t) + f_j(t), \tag{7.1.5}$$

where

$$
\begin{cases}
a_1(t) = -a(t),\ f_1(t) = f(t), \\
a_j(t) = a'_{j-1}(t) - a_{j-1}(t)a(t),\ f_j(t) = f'_{j-1}(t) + a_{j-1}(t)f(t), \\
\text{for any } j,\ 2 \leqslant j \leqslant p.
\end{cases} \tag{7.1.6}
$$

Replacing $y^{(j)}(t)$ by (7.1.5) and neglecting the last term, we obtain the single-step difference schemes of $(p+q)$-order of accuracy for the approximate solution of problem (7.1.1)

$$\frac{u_k-u_{k-1}}{\tau}+\sum_{j=1}^{p}\alpha_j a_j(t_k)\tau^{j-1}u_k-\sum_{j=1}^{q}\beta_j a_j(t_{k-1})\tau^{j-1}u_{k-1}=\varphi_k^{p,q}, \tag{7.1.7}$$

$$\varphi_k^{p,q}=-\sum_{j=1}^{p}\alpha_j f_j(t_k)\tau^{j-1}+\sum_{j=1}^{q}\beta_j f_j(t_{k-1})\tau^{j-1},\ 1\leqslant k\leqslant N,\ u_0=y_0.$$

Now let us give some examples for the constructed difference schemes.

In the case $p+q=1$ from formulas (7.1.4), (7.1.6) and (7.1.7) it follows that

$$\frac{u_k-u_{k-1}}{\tau}+a(t_{k-1})u_{k-1}=f(t_{k-1}),\ 1\leqslant k\leqslant N,\ u_0=y_0$$

(an explicit Euler 's difference scheme of first order of accuracy for the initial-value problem (7.1.1)), and that

$$\frac{u_k-u_{k-1}}{\tau}+a(t_k)u_k=f(t_k),\ 1\leqslant k\leqslant N,\ u_0=y_0$$

(an implicit Euler 's difference scheme of first order of accuracy for the initial-value problem (7.1.1)).

In the case $p+q=2$ from formulas (7.1.4),(7.1.6) and (7.1.7) it follows that

$$\frac{u_k-u_{k-1}}{\tau}+(a(t_{k-1})+\frac{\tau}{2}(a'(t_{k-1})-a^2(t_{k-1})))u_{k-1}$$

$$=f(t_{k-1})+\frac{\tau}{2}(f'(t_{k-1})-a(t_{k-1})f(t_{k-1})),\ 1\leqslant k\leqslant N,\ u_0=y_0$$

(an explicit difference scheme of second order of accuracy for the initial-value problem (7.1.1)); and that

$$\frac{u_k-u_{k-1}}{\tau}+\frac{\tau}{2}(a(t_k)u_k+a(t_{k-1})u_{k-1})=\frac{1}{2}(f(t_k)-f(t_{k-1})),\ 1\leqslant k\leqslant N,\ u_0=y_0$$

(a Crank-Nicolson difference scheme of second order accuracy); and that

$$\frac{u_k - u_{k-1}}{\tau} + (a(t_k) + \frac{\tau}{2}(-a'(t_k) + a^2(t_k)))u_k$$

$$= f(t_k) - \frac{\tau}{2}(f'(t_k) - a(t_k)f(t_k)),\ 1 \leqslant k \leqslant N,\ u_0 = y_0$$

(an implicit difference scheme of second order of accuracy for the initial-value problem (7.1.1)).

Now, we consider the initial-value problem

$$y'(t) + a(t)y(t) = f(t, y(t)), \qquad 0 < t \leqslant T, \quad y(0) = y_0 \tag{7.1.8}$$

assuming $a(t)$ and $f(t, y(t))$ to be such that problem (7.1.8) has a unique smooth solution defined on $[0, T]$. Using the equation

$$y'(t) = -a(t)y(t) + f(t, y(t)),$$

we obtain

$$y^{(j)}(t) = a_j(t)y(t) + f_j(t, y(t)), \tag{7.1.9}$$

where

$$\begin{cases} f_\lambda(t, y(t)) = f(t, y(t)), \qquad \lambda = 1, \\ \\ f_{\lambda+1}(t, y(t)) = \dfrac{\partial^\lambda}{\partial t^\lambda} f(t, y(t)) \\ \\ + \displaystyle\sum_{i=1}^{\lambda} \binom{\lambda}{i} \frac{\partial^\lambda}{\partial t^{\lambda-i} \partial y^i} f(t, y(t)) f_i(t, y(t)), \quad 0 \leqslant \lambda \leqslant p-1. \end{cases} \tag{7.1.10}$$

Replacing $y^{(j)}(t)$ by (7.1.9) and neglecting the last term, we obtain the single-step difference schemes of $(p+q)$-order of accuracy for an approximate solution of problem (7.1.8)

$$\frac{u_k - u_{k-1}}{\tau} + \sum_{j=1}^{p} \alpha_j a_j(t_k) \tau^{j-1} u_k - \sum_{j=1}^{q} \beta_j a_j(t_{k-1}) \tau^{j-1} u_{k-1}$$

$$= \varphi_k^{p,q}(u_k, u_{k-1}), \varphi_k^{p,q}(u_k, u_{k-1}) = -\sum_{j=1}^{p} \alpha_j f_j(t_k, u_k)\tau^{j-1}$$

$$+\sum_{j=1}^{q} \beta_j f_j(t_{k-1}u_{k-1})\tau^{j-1},\ 1 \leqslant k \leqslant N,\ u_0 = y_0.$$

Now, we consider the initial-value problem

$$\mathbf{y}'(t) + A(t)\mathbf{y}(t) = \mathbf{f}(t), \quad 0 < t \leqslant T, \quad \mathbf{y}(0) = \mathbf{y}_0 \tag{7.1.11}$$

where $\mathbf{f}(t)$ be a column vector whose components are known vector function of t. Using the equation

$$\mathbf{y}'(t) = -A(t)\mathbf{y}(t) + \mathbf{f}(t),$$

we obtain

$$\mathbf{y}^{(j)}(t) = A_j(t)\mathbf{y}(t) + \mathbf{F}_j(t), \tag{7.1.12}$$

where

$$\begin{cases} A_1(t) = -A(t), \mathbf{F}_1(t) = \mathbf{f}(t), \\ A_j(t) = A_{j-1}^{(1)}(t) - A_{j-1}(t)A(t),\ \mathbf{F}_j(t) = \mathbf{F}'_{j-1}(t) + A_{j-1}(t)\mathbf{f}(t), \\ \text{for any } j,\ 2 \leqslant j \leqslant p. \end{cases}$$

Replacing $y^{(j)}(t)$ by (7.1.12) and neglecting the last term, we obtain the single-step difference schemes of $(p+q)$-order of accuracy for an approximate solution of problem (7.1.11)

$$\frac{u_k - u_{k-1}}{\tau} + \sum_{j=1}^{p} \alpha_j A_j(t_k)\tau^{j-1}u_k - \sum_{j=1}^{p} \beta_j A_j(t_{k-1})\tau^{j-1}u_{k-1} = \varphi_k^{p,q},$$

$$\varphi_k^{p,q} = -\sum_{j=1}^{p} \alpha_j F_j(t_k)\tau^{j-1} + \sum_{j=1}^{q} \beta_j F_j(t_{k-1})\tau^{j-1}, \quad 1 \leqslant k \leqslant N, \quad u_0 = y_0.$$

Note that using Taylor's decomposition on two points, we can extend our discussion to construct the difference schemes of an arbitrary high order of accuracy for approximate solutions of the initial-value problem for the first order nonlinear system of ordinary differential equations

$$\mathbf{y}'(t) + A(t)\mathbf{y}(t) = \mathbf{f}(t, \mathbf{y}(t)), \quad 0 < t \leqslant T, \quad \mathbf{y}(0) = \mathbf{y}_0.$$

where $\mathbf{f}(t, \mathbf{y}(t))$ is a column vector and $A(t)$ be an $m \times m$ matrix function of t.