MODULES AND HOMOLOGICAL ALGEBRA

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by
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We certify that we have read this thesis and that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

It is proved that for every proper class $\mathcal{A}$ of short exact sequences of modules over an integral domain R the class $\hat{\mathcal{A}}=\{E \mid rE\in\mathcal{A}$, for some $0\neq r\in R\}$ is proper. For every proper class $\mathcal{A}$ containing the class $\hat{\mathcal{S}}_0$ of quasisplitting sequences and $r\in R$ the class $r\mathcal{A}$ is proper. In the case of $\mathbb{Z}$-modules $n\mathcal{A}$-projective and $n\mathcal{A}$-injective modules are studied.

ÖZET

Bir R tamlık bölgesi üzerindeki modüllerin kısa tam dizilerinden oluşan keyfi $\mathcal{A}$ öz sınıfı için $\hat{\mathcal{A}}$={E | rE∈$\mathcal{A}$, for some 0≠r∈R} sınıfının öz sınıf olduğu ispat edilmiştir. $\hat{\mathcal{S}}_0$ kuaziparçalanan diziler sınıfını içeren her $\mathcal{A}$ sınıfı ve r∈R için r$\mathcal{A}$'nın öz sınıf olduğu ispat edilmiştir. $\mathbb{Z}$-modüller için n$\mathcal{A}$-projektif ve n$\mathcal{A}$-injektif modüller incelenmiştir.

CONTENTS

CHAPTER ONE

INTRODUCTION

1.1. Introduction to Modules and Homological Algebra

Homological algebra first arose as an algebraic tool for the study of topological spaces, that is as a branch of algebraic topology. Subsequently applications to algebra (via non- abelian group theory, algebraic geometry etc) were found.

The best category for homological algebra is the category of modules, in particular $\mathbb{Z}$-modules, i.e. abelian groups. On the other hand homological methods are crucial in module theory and today owing to the application of homological methods, abelian group theory seems rather a part of module theory than of general group theory. Since D. Buchsbaum defined proper classes of short exact sequences in 1959. Relative homological algebra became one of the popular themes. It takes its origin from change of rings and purity of subgroups and studies homological notious and properties (such as injectivity, projectivity, homological dimension etc.) with respect to classes of short exact sequences satisfying Buchsbaum's axiom. There are some operations which give rise to new proper classes from given ones. In this thesis two of them will be studied.

In Chapter Two, some preliminary facts are given. In Section 2.1, a module structure on Hom(A,B) is defined. In Section 2.2, complexes of modules and their homology groups are given. For a short exact sequence of complexes the connecting homomorphisim is defined and the long exact sequence is given. In Section 2.3, describes the construction of derived functors, in particular functors $\text{Ext}^n(C,A)$. In Section 2.4, module structure on $\text{Ext}^n(A,B)$ is given and it is proved that multiplication by a scalar r in $Ext^n(A,B)$ is induced by multiplication by the same r in A. In Section 2.5, the definition of proper classes of short exact sequences and some relative notions are given. The equivalent conditions for a module to be $\mathcal{A}$-projective ($\mathcal{A}$-injective) are proved.

Main results are collected in Chapter Three. For a proper classes $\mathcal{A}$ let $\hat{\mathcal{A}}=\{E \mid rE\in\mathcal{A}$, for some $0\neq r\in R\}$ In case of $\mathbf{Z}$-modules the class $\hat{\mathcal{A}}$ was studied by Walker (Walker, 1964) for $\mathcal{A}=\mathcal{S}_0$, by Hart (Hart, 1974) for $\mathcal{A}=\mathcal{S}$ and $\mathcal{A}=\mathcal{D}$, the class of all torsion splitting short exact sequences and by R. Alizade (Alizade, 1986) for every $\mathcal{A}$. For arbitrary integral R, it is proved in Section 3.1, that $\hat{\mathcal{A}}$ is a proper class.

It is well known that the class of pure exact sequences of abelian groups is $\bigcap_{0\neq n\in Z} n\mathcal{A}bs$, $\mathcal{A}bs$ being the class of all short exact sequences and the class of neat exact sequences in $\bigcap_{p \text{ is prime}} p\mathcal{A}bs$. In Section 3.2, the class r $\mathcal{A}$ for $r\in R$ and every proper class $\mathcal{A}$ containing $\hat{\mathcal{S}}_o$ is investigated, $\hat{\mathcal{S}}_o$ being the class of quasisplitting sequences. It was shown that $\hat{\mathcal{A}}=\mathcal{A}$ for such classes. Using this fact it was proved that r$\mathcal{A}$ is aproper class for $\mathcal{A}$ containing $\hat{\mathcal{S}}_o$. As a corallary of this fact the class of neat exact sequences is proper.

In Section 3.3, n$\mathcal{A}$-projective and n$\mathcal{A}$-injective objects are studied in terms of n in the case R=$\mathbf{Z}$, i.e. in the case of abelian groups. In particular the complete description of n$\mathcal{A}bs$-projective and n$\mathcal{A}bs$-injective $\mathbf{Z}$-modules is given. All groups are considered as abelian and modules which are over an integral domain.

CHAPTER TWO

SOME FACTS ABOUT HOM , EXT AND PROPER CLASS

2.1 Hom(A,B) as a Module

Definition 2.1.1 Let us consider a set Hom(A,B)=set of all homomorphisms from A into B where A and B are R-modules and R be a commutative ring.

For f,g∈Hom(A,B) we define,

$$(f+g)(a)=f(a)+g(a) \quad a\in A$$
$$(rf)(x)=rf(x)$$

We want to show that Hom(A,B) is a left R-module.

$$R \times Hom(A,B) \to Hom(A,B)$$
$$(r,f) \to rf$$

Hom(A,B) is an abelian group under "+" ; O(a)=0, a∈A, O is the identity element; (-f)(a)=-f(a) , (-f) is the inverse element.

(1) Distributive laws;

$$r(f+g)=rf+rg$$

indeed;

$$(r(f+g))(x) = r(f+g)(x)$$
$$= r(f(x)+g(x))$$
$$= rf(x)+rg(x)$$
$$= (rf)(x)+(rg)(x)$$

$$(r_1+r_2)f = r_1f + r_2f$$

indeed;

$$((r_1+r_2)f)(x) = (r_1+r_2)f(x)$$
$$= r_1f(x) + r_2f(x)$$
$$= (r_1f)(x) + (r_2f)(x)$$

(2) "Associative law":

$$(r_1 r_2)f = r_1 (r_2f)$$

indeed;

$$((r_1 r_2)f)(x) = (r_1 r_2)f(x)$$
$$= r_1 (r_2f(x))$$
$$= r_1 ((r_2f)(x))$$

(3) Unitary law:

$$1 f = f$$

indeed;

$$(1f)(x) = 1f(x) = f(x)$$

so Hom(A,B) is a left R-module.

2.2 Complexes and Homology Groups

Definition 2.2.1 A complex **A** is a sequence of modules and maps

$$\ldots\ldots \to A_n \xrightarrow{d_n} A_{n-1} \to \ldots\ldots$$

$n \in Z$ with $d_n d_{n+1} = 0$ all n.

Definition 2.2.2 If **A** is a complex, then $d_n d_{n+1}=0$ implies Im $d_{n+1} \subset$ Ker d_n. The n^{th} homology group $H_n(A)$ is Ker d_n / Im d_{n+1} .

One writes Ker $d_n = Z_n(A) = Z_n$ and Im $d_{n+1} = B_n(A) = B_n$. Thus $H_n(A) = Z_n(A)/B_n(A)$.

Definition 2.2.3 A chain map $f: A \to A'$ is a family of homomorphisms $f_n: A_n \to A_n'$ making the following diagram commutative.

$$\begin{array}{ccccccc} \cdots\cdots \to & A_n & \xrightarrow{d_n} & A_{n-1} & \to \cdots\cdots \\ & \downarrow f_n & & \downarrow f_{n-1} & \\ \cdots\cdots \to & A_n' & \xrightarrow{d_n'} & A_{n-1}' & \to \cdots\cdots \end{array}$$

If $f: A \to A'$ is a chain map, let

$$H_n(f): H_n(A) \to H_n(A')$$

be given by

$$z_n + B_n \to f_n z_n + B_n'$$

Usually one writes f_{n*} or even f_* instead of $H_n(f)$.

Definition 2.2.4 Let $\{A_i\}$ $i \in I$ be a set of modules, $\{\alpha_i\}$ $i \in I$ be a set of homomorphisms with some index set I.

A sequence $\ldots A_{i-1} \to A_i \to A_{i+1} \to \ldots$ is called an exact sequence, if Im α_i=Ker α_{i+1} for every $i \in I$.

An exact sequence $0 \to A \to B \to C \to 0$ is called a short exact sequence.

Define $A' \xrightarrow{f} A \xrightarrow{g} A''$ to be exact if Ker $\mathbf{g}$ = Im $\mathbf{f}$. This is exact if and only if

$$A'_n \xrightarrow{f_n} A_n \xrightarrow{g_n} A''_n$$

is exact for each n.

Theorem 2.2.1 Let $0 \to A' \xrightarrow{i} A \xrightarrow{p} A'' \to 0$ be an exact sequence of complexes. For each n, there is a homomorphism

$$\delta_n : H_n(A'') \to H_{n-1}(A')$$

defined by

$$z'' + B_n(A'') \to i^{-1} d p^{-1} z'' + B_{n-1}(A')$$

In diagram;

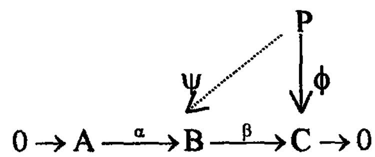
$$\begin{array}{ccccc} & & A_n & \xrightarrow{p} & A_n'' \to 0 \\ & & \downarrow d & & \\ 0 \to A_{n-1}' & \xrightarrow{i} & A_{n-1} & & \end{array}$$

Theorem 2.2.2 If $0 \to A' \xrightarrow{i} A \xrightarrow{p} A'' \to 0$ is an exact sequences of complexes then there is an exact sequence

$$\ldots \to H_n(A) \xrightarrow{p_*} H_n(A'') \xrightarrow{\delta_n} H_{n-1}(A') \xrightarrow{i_*} H_{n-1}(A) \to \ldots$$

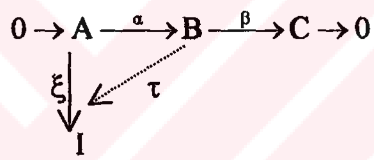
2.3 Derived Functors

Definition 2.3.1 Let P be a module. If every diagram



with an exact row can be completed by a suitable homomorphism $\psi: P \rightarrow B$, i.e. if there is a homomorphism $\psi: P \rightarrow B$ with $\beta \circ \psi = \phi$, for any given homomorphism $\phi: P \rightarrow C$, then P is called a projective module.

Definition 2.3.2 Let I be a module. If every diagram



with an exact row can be completed by a suitable homomorphism $\tau: B \rightarrow I$, i.e. if there is a homomorphism $\tau: B \rightarrow I$ with $\tau \circ \alpha = \xi$, for any given homomorphism $\xi: A \rightarrow I$, then I is called an injective module.

Definition 2.3.3 If $\ldots\ldots X_n \rightarrow X_{n-1} \rightarrow \ldots\ldots \rightarrow X_0 \xrightarrow{\varepsilon} A \rightarrow 0$ is an exact sequence and X_i is a projective module i=0,1,2,3.... then the sequence is called a projective resolution of a module A.

$\ldots. \rightarrow X_n \xrightarrow{d_n} X_{n-1} \xrightarrow{d_{n-1}} \ldots. \rightarrow X_0 \rightarrow 0$ is called deleted resolution for A.

An exact sequance $0 \rightarrow A \xrightarrow{\varepsilon} I^0 \xrightarrow{d^0} I' \xrightarrow{d'} \ldots. \rightarrow I^n \rightarrow \ldots..$ with injective modules I^k k=0,1,2....... is called an injective resolution of a module A. $0 \rightarrow I^0 \xrightarrow{d^0} I' \rightarrow \ldots..$ is called a deleted resolution for A.

Definition 2.3.4 If $\bar{f}: X_A \to X_{A'}$ is a chain map for which $f\varepsilon = \varepsilon'\bar{f}_0$, we say $\bar{f}$ is over f.

Given a functor T, we now describe its left derived functors L_nT (Rotman, 1979). For each module A, choose, once for all, a projective resolution of A, and let $\mathbf{P}_A$ be correspendig deleted complex. Next apply T to $\mathbf{P}_A$ to get the complex

$$....\mathbf{TP}_2 \to \mathbf{TP}_1 \to \mathbf{TP}_0 \to 0$$

Definition 2.3.5 For each module A, $(L_nT)A=H_n(T\mathbf{P}_A)=KerTd_n/ImTd_{n+1}$. To complete the definition of L_nT, we must describe its action on $f:A\to B$. There is a chain map $\bar{f}: \mathbf{P}_A \to \mathbf{P}_B$ over f. Define

$$f_*=(L_nT)f: L_nTA\to L_nTB$$

by

$$(L_nT)f=H_n(T\bar{f})$$

i.e if $z_n \in kertd_n$,then

$$z_n+ImTd_{n+1}\to(T\bar{f})\, z_n+ImT\,d'_{n+1}$$

Definition 2.3.6 For each module A, choose, once for all, an injective resolution

$$0 \to A \xrightarrow{d} E_0 \xrightarrow{d'} E_1 \to ...$$

and let $\mathbf{E}_A$ be the deleted resolution. If T is covariant, define the right derived functors R^nT on modules A by

$$R^nT(A)=H^n(T\mathbf{E}_A)=KerTd^n/ImTd^{n+1}.$$

Definition 2.3.7 If $T=Hom_R(C,)$, then $R^nT=Ext^n{}_R(C,)$. In particular,

$$Ext^n{}_R(C, A)=KerHom(C,d^n)/ImHom(C,d^{n-1})$$

Where $0 \to A \xrightarrow{d^0} E_0 \xrightarrow{d^1} E_1 \to .....$ is the choosen injective resolution of A.

Definition 2.3.8 If $T=Hom_R(,A)$, then $R^nT=Ext^n{}_R(,A)$. In particular,

$$Ext^n{}_R(C, A)=KerHom(d_{n+1},A)/ImHom(d_n,A)$$

Where $.... \to P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \to C$ is the choosen projective resolution of C.

Finally $Ext^n_R(A,B)=H_n(Hom_R(\mathbf{P}_A,B))=H^n(Hom_R(A,\mathbf{E}_B))$, where $\mathbf{P}_A$ is a deleted projective resolution of A and $\mathbf{E}_B$ is a deleted injective resolution of B.

If $0 \to B' \to B \to B'' \to 0$ is an exact sequence of modules, then there is an exact sequence

$$0 \to Hom(A,B') \to Hom(A,B) \to Hom(A,B'') \xrightarrow{\delta} Ext^1(A,B') \to \dots$$
$$\to Ext^n(A,B') \to Ext^n(A,B) \to Ext^n(A,B'') \to Ext^{n+1}(A,B') \to \dots$$

If $0 \to A' \to A \to A'' \to 0$ is an exact sequence of modules, then there is an exact sequence

$$0 \to Hom(A'',B) \to Hom(A,B) \to Hom(A',B) \xrightarrow{\delta} Ext^1(A'',B) \to \dots$$
$$\to Ext^n(A'',B) \to Ext^n(A,B) \to Ext^n(A',B) \to Ext^{n+1}(A'',B) \to \dots$$

2.4 Module Structure on $Ext^n(A,B)$

Theorem 2.4.1 If R is commutative, $Ext^n_R(A,B)$ is an R–module.

Proof We know that $Hom_R(\mathbf{P}_A, B)$ and $Hom_R(A,\mathbf{E}_B)$ are R modules. Hence $KerHom(d_{n+1},B)$, $ImHom(d_n,B)$ and $KerHom(A,d^n)$, $ImHom(A,d^{n-1})$ are R-modules.

So $Ext^n_R(A, B)=KerHom(d_{n+1},B)/ImHom(d_n,B)=KerHom(A,d^n)/ ImHom(A,d^{n-1})$ is an R-module.

Let R be commutative and let A be an R-module. If $r \in R$, then $\mu:A\to A$ defined by $a\to ra$ is an R-homomorphism, called multiplication by r.

Theorem 2.4.2 If $\mu:A\to A$ is multiplication by r, then $\mu^*:Ext^n_R(A, B)\to Ext^n_R(A,B)$ is also multiplication by r. If $\nu:B\to B$ is multiplication by r, then $\nu^*:Ext^n_R(A, B)\to Ext^n_R(A, B)$ is also multiplication by r.

Proof There is a diagram

$$\begin{array}{c} \ldots\ldots \to P_1 \to P_0 \to A \to 0 \\ \qquad\qquad\quad \downarrow \mu \\ \ldots\ldots \to P_1 \to P_0 \to A \to 0 \end{array}$$

where each row is a projective resolution of A. Recall the definition of $\mu^*:Ext^n_R(A, B)\to Ext^n_R(A, B)$; first fill in a chain map **g** over μ (so that $g_n:P_n\to P_n$) then apply the functor $Hom_R(,B)$ to the diagram, and

$$\mu^*(z_n+Imd_{n-1})=g_nz_n+ Imd_{n-1}.$$

We also know that any choice of chain map **g** over μ gives the same μ^*. In particular, if we define **g** by letting $g_n:P_n\to P_n$ be multiplication by r, then **g** is a chain map over μ, and

$$\mu^*(z_n+Imd_{n-1})=rz_n+ Imd_{n-1}=r(z_n+ Imd_{n-1})$$

The proof of second statement is dual.

2.5 Proper Classes of Short Exact Sequences

Definition 2.5.1 Let *A* be a class of short exact sequences of modules. If a short exact sequence

$$E:\ 0\to A\xrightarrow{\alpha}B\xrightarrow{\beta}C\to 0$$

belongs to *A*, α is said to be an *A*-monomorphism and β an *A*-epimorphism. A short exact sequence E is determined by each of the monomorphisim α and epimorphism β uniquely up to isomorphisim.

The class *A* is said to be proper, if it satisfies the following conditions (Buchsbaum, 1959), (MacLane, 1975), (Sklyarenko, 1978).

1) A long with any short exact sequence *A* contains every one isomorphic to it.
2) *A* contains all splitting short exact sequences.

3) The composite of two $\mathcal{A}$-monomorphisims is an $\mathcal{A}$-monomorphism if this composite is defined. The composite of two $\mathcal{A}$-epimorphisms is an $\mathcal{A}$-epimorphism if it is defined.
4) If β,α are monomorphisms and $\beta o \alpha$ is an $\mathcal{A}$-monomorphism, then α is an $\mathcal{A}$-monomorphism.If γ,δ are epimorphisms and $\delta o \gamma$ is an $\mathcal{A}$-epimorphism, then δ is an $\mathcal{A}$-epimorphism.

Examples 1) $\mathcal{S}_o$, the class of all splitting short exact sequences, is the smalest proper class.

2) *Abs*, the class of all short exact sequences, is the largest proper class.

3) $\mathcal{S}$, the class of al pure-exact short exact sequences.

Proposition 2.5.1 $Ext_{\mathcal{A}}(C,A)$ is a subgroup of $Ext_R(C,A)$ and if R is commutative then $Ext_{\mathcal{A}}(C,A)$ is a submodule of $Ext_R(C,A)$.

Proof It's obviously known that $Ext_{\mathcal{A}}(C,A)$ is a subgroup of $Ext_R(C,A)$. Let $E \in Ext_{\mathcal{A}}(C,A)$, $r \in R$ and $\mu: A \to A$ be the multiplication by r in A. Since $Ext_{\mathcal{A}}(C,A)$ is a subfunctor of $Ext_R(C,A)$ we have by Theorem 2.4.2 $rE=\mu^*(E) \in Ext_{\mathcal{A}}(C,A)$. So $Ext_{\mathcal{A}}(C,A)$ is a submodule.

Definition 2.5.2 Let $\mathcal{A}$ be class of short exact sequences. A module A said to be $\mathcal{A}$-projective ($\mathcal{A}$-injective), if for evey C ($\mathcal{A}$-monomorphism) $\sigma: B \to C$

$$\sigma_*: Hom(A,B) \to Hom(A,C) \quad (\sigma_*: Hom(C,A) \to Hom(B,A))$$

is an epimorphism.

A proper class $\mathcal{A}$ is said to be projective, if for every module A There is an $\mathcal{A}$-epimorphism from an $\mathcal{A}$-projective module P onto A. An injective class is defined dually.

Definition 2.5.3 The smallest proper class containing proper class $\mathcal{A}$, $\mathcal{C}$ is called the sum of proper classes $\mathcal{A}$ and $\mathcal{C}$ and denoted by $\mathcal{A}+\mathcal{C}$ (Pancar, 1997).

Since the intersection of any family of proper classes is proper, this definition is well- defined.

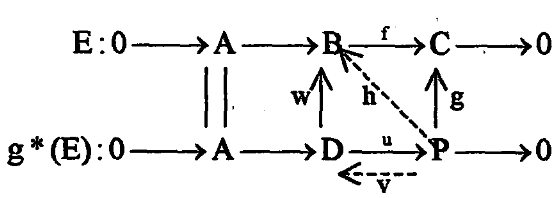
Theorem 2.5.1 Let $\mathcal{A}$ be a proper class. The following conditions are equivalent for a module P.

1) P is $\mathcal{A}$- projective.
2) For every short exact sequence $0\to A\to B\to C\to 0$ from $\mathcal{A}$ the sequence $0\to Hom(P,A)\to Hom(P,B)\to Hom(P,C)\to 0$ is an exact.
3) $Ext^1_{\mathcal{A}}(P,X)=0$ for every module X.

Proof 1)⇒2) Let $0\xrightarrow{\alpha}A\xrightarrow{\beta}B\xrightarrow{\gamma}C\xrightarrow{\delta}0$ be an exact sequence. We have to prove that $0\xrightarrow{\alpha_*}Hom(P,A)\xrightarrow{\beta_*}Hom(P,B)\xrightarrow{\gamma_*}Hom(P,C)\xrightarrow{\delta_*}0$ is an exact. In fact we must prove that $Im\gamma_*=Ker\delta_*$. In other words γ_* is an epimorphism. Let $f\in Hom(P,C)$. Since P is a projective, γ is an epimorphism and f is a homomorphism, there is a homomorphisim $h:P\to B$ (that is $h\in Hom(P,B)$) such that $f=\gamma_*(h)$, therefore γ_* is an epimorphisim.

2)⇒3) Let $0\longrightarrow X\longrightarrow Y\xrightarrow{f}P\longrightarrow 0$ be any short exact sequence from $\mathcal{A}$. Applying Hom(P,.) to this sequence we say by 2) that $f_*:Hom(P,Y)\to Hom(P,P)$ is an epimorphism. In particular there is a homomorphism $g:P\to Y$ such that $fog=f_*(g)=1_P$, i.e. the sequence E is splitting. So every element from $Ext_{\mathcal{A}}(P,X)$ is splitting.

3)⇒1) Let $E:0\longrightarrow A\longrightarrow B\xrightarrow{f}C\longrightarrow 0$ be any short exact sequence from $\mathcal{A}$ and $g:P\to C$ be any homomorphisim. Let $g^*(E):0\to A\to D\to P\to 0$. Then we have a commutative diagram with exact rows:



Since $E \in \mathcal{A}$, we have $g^*(E) \in \mathcal{A}$, i.e. $g^*(E) \in \mathrm{Ext}_{\mathcal{A}}(P, A)$. By 3) $g^*(E)$ is splitting, i.e. there is a homomorphism v: $P \rightarrow D$ with $u \circ v = 1_P$. Then for $h = w \circ v$: $P \rightarrow B$ we have $f \circ h = f \circ w \circ v = g \circ u \circ v = g \circ 1_P = g$. So P is $\mathcal{A}$-projective.

Dualy we can prove the following Theorem for $\mathcal{A}$-injective modules.

Theorem 2.5.2 Let $\mathcal{A}$ be a proper class. The following conditions are equivalent for a module I.

1) I is $\mathcal{A}$-injective.
2) For every short exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ from $\mathcal{A}$ the sequence $0 \rightarrow \mathrm{Hom}(C,I) \rightarrow \mathrm{Hom}(B,I) \rightarrow \mathrm{Hom}(A,I) \rightarrow 0$ is an exact.
3) $\mathrm{Ext}^1_{\mathcal{A}}(Y, I) = 0$ for every module Y.

CHAPTER THREE

MAIN RESULTS

3.1 Classes $\hat{A}$

Definition 3.1.1 For every proper class A of short exact sequences of modules over an integral domain R, we will let $\hat{A}$ denote the class of the short exact sequences E such that $rE \in A$ for some $0 \neq r \in R$. Thus;

$$\hat{A} = \{E \mid rE \in A, \text{ for some } 0 \neq r \in R\}$$

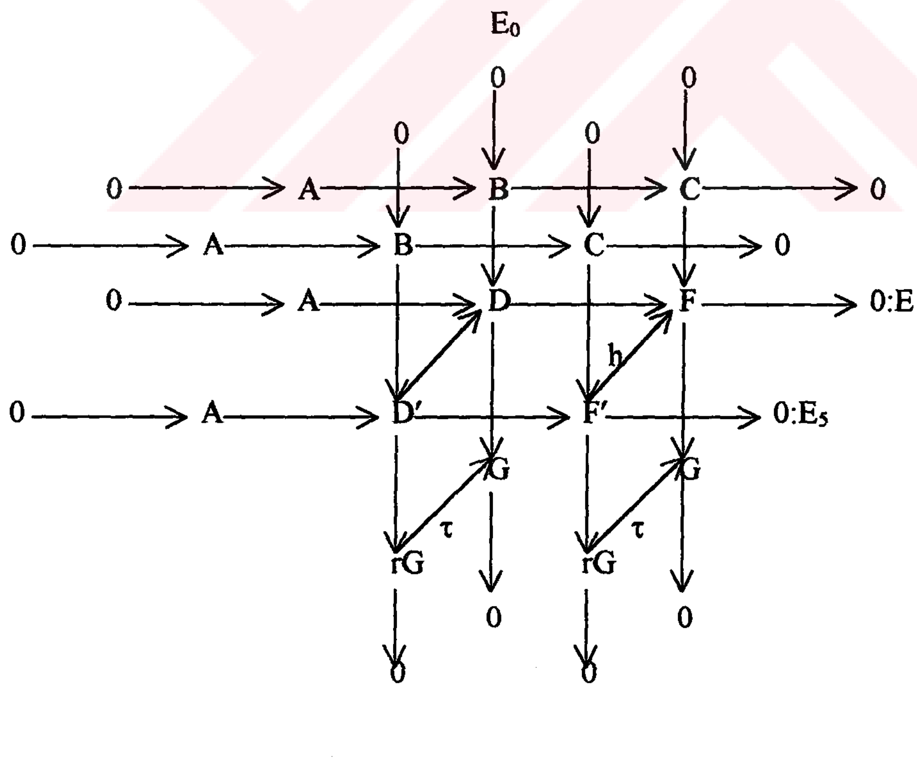
In case of **Z**-modules the class $\hat{A}$ was studied by Walker (Walker, 1964) for $A = S_0$, by Hart (Hart, 1974) for $A = S$ and $A = D$, the class of all torsion splitting short exact sequences and by R. Alizade (Alizade, 1986) for every A.

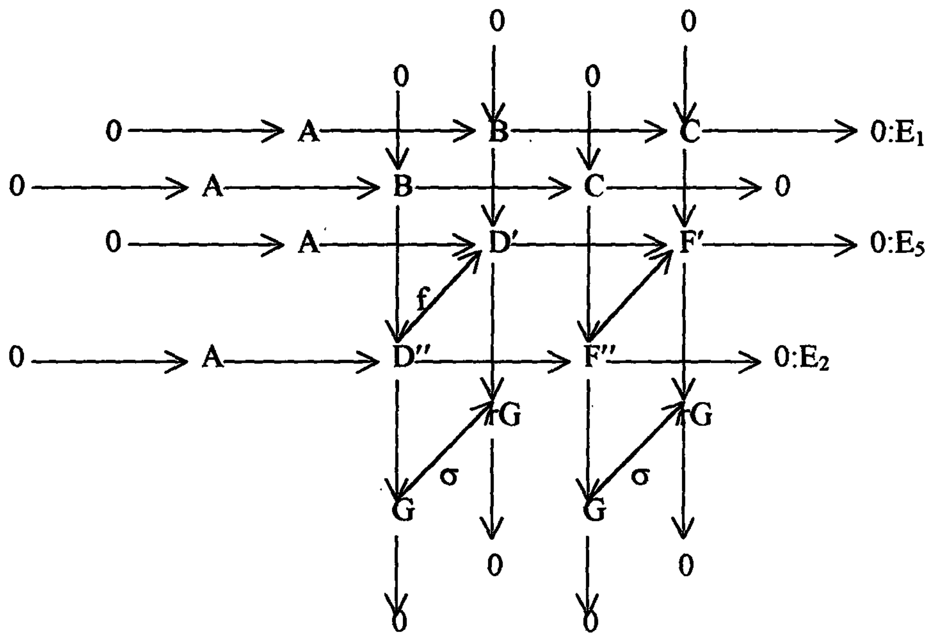
Theorem 3.1.1 $\hat{A}$ is a proper class for every proper class A.

Proof First, we will prove that $Ext_{\hat{A}}$ is an E-functor (Butler & Horrocks, 1961). We consider homomorphisms $f: A \to A'$ and $g: C' \to C$ and we suppose that $E \in Ext_{\hat{A}}(C,A)$. Then $rE \in Ext_A(C,A)$ for some $0 \neq r \in R$. Since Ext_A is a functor, we conclude that $r(f_* \circ g^*(E)) = f_*(g^*(rE)) \in Ext_A(C',A')$. This means that $f_* \circ g^*(E) \in Ext_{\hat{A}}(C',A')$. So we have shown that $Ext_{\hat{A}}$ is a functor. We will show that $Ext_{\hat{A}}(C,A)$ is a subgroup. We take arbitrary E' and E'' in $Ext_{\hat{A}}(C,A)$. Then r_1E', $rE'' \in Ext_A(C,A)$ for some nonzero r_1, $r \in R$. Since $Ext_A(C,A)$ is a subgroup, it follows that $rr_1(E' - E'') = r(r_1 E') - r_1 r(E'') \in Ext_A(C,A)$. $r \cdot r_1 \neq 0$ since r has no zero divisors, so $E' - E'' \in \hat{A}$.

This means that $E' - E'' \in Ext_{\hat{A}}(C,A)$. Now we want to show that $Ext_{\hat{A}}(C,A)$ is a submodule. Let $E \in Ext_{\hat{A}}(C,A)$, $r \in R$. Then $sE \in \mathcal{A}$ for some nonzero $s \in R$. Since $Ext_{\mathcal{A}}(C,A)$ is a submodule, $s(rE)=r(sE) \in \mathcal{A}$. Therefore $rE \in \hat{A}$. Hence $Ext_{\hat{A}}(C,A)$ is a submodule. Thus, $Ext_{\hat{A}}(C,A)$ is an E-functor. It will be enough to prove that the composition of two $\hat{A}$-monomorphisms $i:A \to B$ and $j:B \to D$ is an $\hat{A}$-monomorphism (Nunke, 1963).

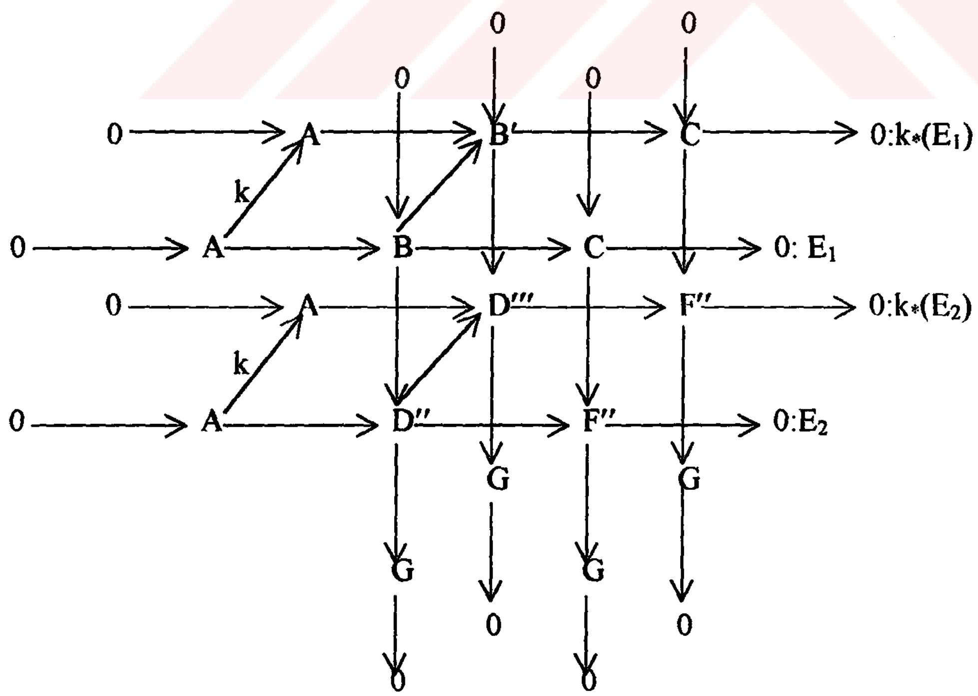
Since the short exact sequence E_0: $0 \to B \to D \to G \to 0$ belongs to $\hat{A}$, we have $r^*(E_0)=rE_0 \in \mathcal{A}$ for some $0 \neq r \in R$. Let us denote the homomorphism of multiplication by r by the same $r:G \to G$. We consider the epi-mono factorization of the homomorphism r: $r=\tau \circ \sigma$, where $\sigma:G \to rG$ is the standard epimorphism and $\tau:rG \to G$ the standard embedding. Then $r^*=\sigma^* \circ \tau^*$. We have the following commutative and exact diagrams:





Here $r^*(E_0)$: $0 \to B \to D'' \to G \to 0 \in \mathcal{A}$.

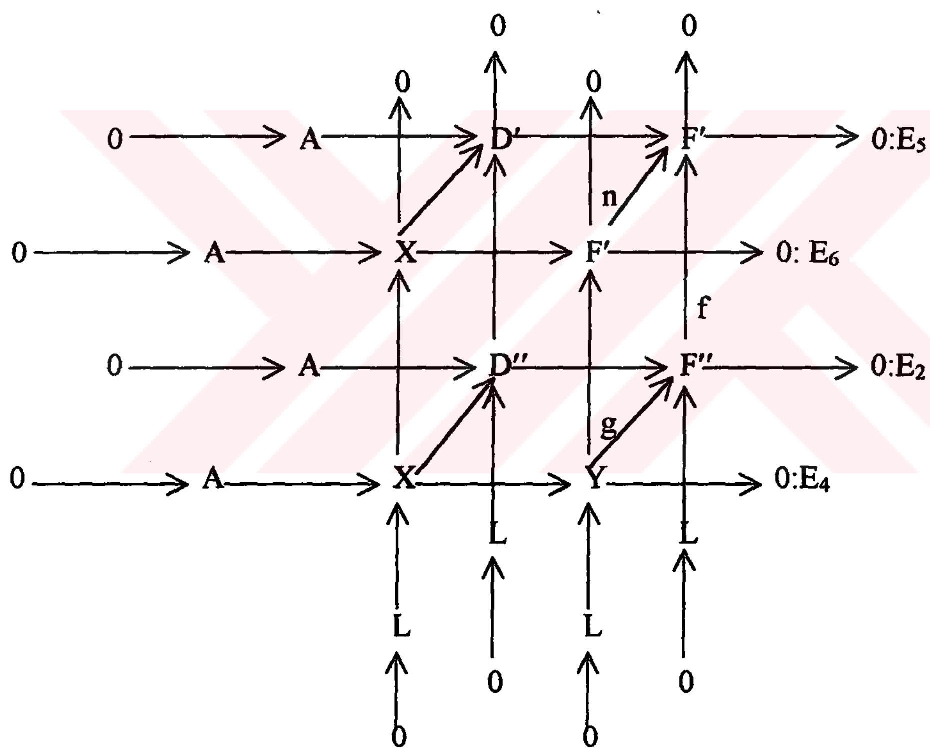
We will now show that $E_2 \in \hat{\mathcal{A}}$. Since $E_1 \in \hat{\mathcal{A}}$, it follows that $k_*(E_1) \in \mathcal{A}$ for some $k \neq 0$. We consider the following commutative and exact diagram.



Since Ext_A is a functor, we conclude that $0 \to B' \to D''' \to G \to 0$ belongs to A. Since $k_*(E_1) \in A$, the monomorphism η, being a composition of two A-monomorphisms, must be an A-monomorphism. Thus, $kE_2 = k_*(E_2) : 0 \to A \xrightarrow{\eta} D''' \to F'' \to 0 \in A$. Hence, $E_2 \in \hat{A}$.

Obviously, $L = \mathrm{Ker} f \cong \mathrm{Ker}\sigma = G[r]$. We consider the short exact sequence $E_3 : 0 \to L \to F'' \xrightarrow{f} F' \to 0$.

Since $rL=0$, $r_*(E_3) = rE_3 = r^*(E_3) : 0 \to L \to Y \to F' \to 0$ splits. We will consider the commutative and exact diagram



Since $Ext_{\hat{A}}$ is a functor, $g^*(E_2) = E_4 \in \hat{A}$. Since the short exact sequence $rE_3 = r_*(E_3) : 0 \to L \to Y \to F' \to 0$ splits, the functioriality of $Ext_{\hat{A}}$ implies that $E_6 \in \hat{A}$, i.e., $r_1 E_6 \in A$ for some $r_1 \neq 0$. But $E_6 = rE_5$, hence $(r_1 r) E_5 = r_1 E_6$. Therefore, $E_5 \in \hat{A}$.

We will prove that $r^*(E)\in \hat{A}$. We put $r=\varphi o\psi$, where $\psi:F\to rF$ and $\varphi:rF\to F$ are standard homomorphisms. Then $r^*=\psi^* o\varphi^*$. We will show that $\varphi^*(E)=\hat{A}$. It is easy to see that $F/Imh\cong G/rG$, hence $rF\subset Imh$. Then it is obvious that the embedding $\varphi:rF\to F$ factors through h, i.e., $\varphi=ho\xi$ for some homomorphism $\xi:rF\to F$. Hence, $\varphi^*(E)=\xi^* oh^*(E)$. But, $h^*(E):0\to A\to D'\to F'\to 0$: $E_5\in \hat{A}$. Thus, $\varphi^*(E)\in \hat{A}$, and, consequently, $E_5\in \hat{A}$. It is then obvious that $E\in \hat{A}$. This completes the proof of the theorem.

3.2 Classes r*A*

Definition 3.2.1 For every proper class A of short exact sequences of modules over an integral domain R, we will let rA denote the class of the short exact sequence rE such that $E\in A$, $r\in R$. Thus;

$$rA=\{rE|\ E\in A,\ r\in R\}.$$

Definition 3.2.2 Let S_0 be class of all splitting short exact sequences. Then $\hat{S}_0$ is the class of E such that rE is splitting short exact sequences for some nonzero $r\in R$. For **Z**-modules (i.e. abelian groups) the class $\hat{S}_0$ was studied in (Walker, 1964) where it was denoted by Text and short exact sequences from $\hat{S}_0$ are said to be torsion splitting (Fuchs, 1970).

Theorem 3.2.1 $A+\hat{S}_0=\hat{A}$.

Proof Denote $A+\hat{S}_0$ by B. Every $E\in A$ can be written as $E=1\cdot E$ $(r=1)$, therefore $E\in \hat{A}$ and then $A\subseteq \hat{A}$. On the other hand, $S_0\subseteq A$, therfore $\hat{S}_0\subseteq \hat{A}$. Thus $A+\hat{S}_0\subseteq \hat{A}$.

Conversely, let $E: 0\to A\to B\to C\to 0\in \hat{A}$. Then $rE\in A$ for some $r\neq 0$. Let us denote by r the endomorphism of multiplication by r on A. Then by Theorem 2.4.2 rE is the lower exact sequence in the following commutative diagram:

$$\begin{array}{l}
E:\ 0 \to A \to B \to C \to 0 \in \hat{A} \\
\qquad\ \ \ \downarrow r \\
rE:\ 0 \to A \to B' \to C \to 0 \in A
\end{array}$$

The endomorphism $r:A \to A$ can be represented in the natural way $r = \alpha \circ \sigma$, where $\sigma: A \to rA$ is epimorphism and $\alpha: rA \to A$ is inclusion map.

We have the following commutative diagrams with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & A[r] & & A[r] & & & & \\
 & & \downarrow & & \downarrow & & & & \\
E: 0 & \to & A & \longrightarrow & B & \xrightarrow{\theta} & C & \longrightarrow & 0 \in \hat{A} \\
 & & \sigma\downarrow & & \downarrow\gamma & & & & \\
E': 0 & \to & rA & \xrightarrow{\upsilon} & B_1 & \xrightarrow{\delta} & X & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
E': 0 & \longrightarrow & rA & \xrightarrow{\upsilon} & B_1 & \xrightarrow{\delta} & X & \longrightarrow & 0 \\
 & & \downarrow\alpha & & \downarrow\beta & & & & \\
rE: 0 & \longrightarrow & A & \xrightarrow{\mu} & B' & \longrightarrow & C & \longrightarrow & 0 \in A \\
 & & \downarrow & & \downarrow & & & & \\
 & & A/rA & & A/rA & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

A/rA is a bounded group. Therefore the short exact sequence $0\to rA\to A\to A/rA\to 0$ belongs to $\hat{S}_0$ and since $\hat{S}_0\subseteq B$, it belongs to B, i.e., α is a B-monomorphism. Since $rE\in A\subseteq B$, μ is a B-monomorphism. Therefore, by Definition 2.5.1, $\mu o\alpha=\beta o\nu$ is a B-monomorphisim. Hence, by Definition 2.5.1, ν is a B-monomorphism. Thus $E'\in B$.

A[r] is bounded. Hence the short exact sequence $0\to A[r]\to B\to B_1\to 0$ belongs to $\hat{S}_0$ and since $\hat{S}_0\subseteq B$, it belongs to B, i.e. γ is a B-epimorphism. Since $E'\in B$, δ is a B-epimorhism. By Definition 2.5.1 $\theta=\delta o\gamma$ is a B-epimorphism. Therefore $E\in B$ and we have $\hat{A}\subseteq B$.

Theorem 3.2.2 If $\hat{S}_0\subseteq A$ then $\hat{A}=A$.

Proof By Theorem 3.2.1 we know that $A+\hat{S}_0=\hat{A}$ and $\hat{S}_0\subseteq A$, hence $\hat{A}=A$.

Theorem 3.2.3 A short exact sequence E: $0\longrightarrow A\xrightarrow{\mu}B\xrightarrow{\upsilon}C\longrightarrow 0$ is divisible by nonzero $r\in R$ if and only if $r_1A=A\cap r_1B$, for all $r_1\backslash r$.

Proof If α is multiplication by r in A, then $\mathrm{Im}\alpha_*=r\mathrm{Ext}(C,A)$. $E\in\mathrm{Ext}(C,A)$ exactly if A/rA is a direct summand of B/rA by Theorem 53.1 in (Fuchs, 1970). This means that there is a p: $B/rA\to A/rA$ such that $p(a+ra)=a+rA$ and $A/rA\subseteq B/rA$.

We have to show that $r_1A=A\cap r_1B$. Let suppose that $x\in A\cap r_1B$; $x=r_1b$ $b+rA\in B/rA$ and $p(b+rA)=a+rA$. $x+rA=p(x+rA)=p(r_1b+rA)=r_1p(b+rA)=r_1a+rA$. Therefore $x-r_1a\in rA$; $x-r_1a=ra'=r_1a''$ hence $x=r_1(a+a'')\in r_1A$. So $A\cap r_1B\subseteq r_1A$. We know that $r_1A\subseteq A$, $r_1A\subseteq r_1B$, hence $r_1A\subseteq A\cap r_1B$. While the proof of the converse statement is a modification of Theorem 27.5 in (Fuchs, 1970).

Theorem 3.2.4 If $\hat{A} = A$ then rA is a proper class.

Proof First, we will prove that $\text{Ext}_{rA}(C, A)$ is an E-functor (Butler & Horrocks, 1961). We consider homomorphisms f:A→A′ and g:C′→C and we suppose that rE$\in \text{Ext}_{rA}(C, A)$. Then $E \in \text{Ext}_A(C, A)$, since Ext_A is a functor, we conclude that $f_*(g^*(rE))=r(f_*og^*(E))\in \text{Ext}_{rA}(C, A)$. So we have shown that Ext_{rA} is a functor.

We will show that Ext_{rA} is a subgroup. We take arbitrary rE′ and rE″ in $\text{Ext}_{rA}(C, A)$ with E′ and E″$\in \text{Ext}_A(C, A)$. Since $\text{Ext}_A(C, A)$ is a subgroup, it follows that rE′-rE″=r(E′-E″)=rE‴, E‴=E′-E″$\in \text{Ext}_A(C, A)$. This means that rE′-rE″ $\in \text{Ext}_{rA}(C, A)$. Now we have to show that $\text{Ext}_{rA}(C, A)$ is a submodule. Let E$\in \text{Ext}_{rA}(C, A)$, s$\in$R and μ:A→A be a multiplication by s. Therefore sE=μ*(E)$\in \text{Ext}_{rA}(C, A)$. Hence $\text{Ext}_{rA}(C, A)$ is a submodule.

Suppose that α:C→B and β:B→A are rA-monomorphisms. Without restriction of generality,we can assume that C⊆B⊆A and α, β are inclusion maps. We want to show that the inclusion map βoα:C→A is an rA-monomorphism.

Let E_1: 0→C→B→X→0∈rA; E_2: 0→B→A→Y→0∈rA with E_1=rE'_1; E_2=rE'_2 for some E'_1, E'_2∈A and by using Theorem 3.2.3, we know that for every r_1\r; $r_1C=C\cap r_1B$ and $r_1B=B\cap r_1A$. $r_1C=C\cap r_1B=C\cap(B\cap r_1A)=(C\cap B)\cap r_1A= C\cap r_1A$. This means that r_1\E: 0→C→A→Z→0 for every r_1\r,hence E=rE′. Now we have to show that E′∈A. Since α, β are rA-monomorphisms, therefore A-monomorphisms and A is a proper class, the composition βoα is an A-monomorphism, i.e. E∈A. Then E′∈ $\hat{A}$=A. So E=rE′∈rA. Thus rA is a proper class.

3.3 rA-Projective and rA-Injective Modules

Definition 3.3.1 A module I is injective with respect to a short exact sequence (or E-injective) E: 0→A→B→C→0 if Hom(E,I) is an exact. A module P is projective with respect to E (or E-projective) if Hom(P,E) is an exact.

Theorem 3.3.1 A direct product $I=\prod_{k\in K} I_k$ is E-injective if and only if I_k is E-injective for each $k\in K$. A direct sum $P=\oplus_{t\in T} P_t$ is E-projective if and only if P_t is E-projective for each $t\in T$.

Now we study rA-injective and rA-projective objects for **Z**-modules (i.e. abelian groups).

Corollary 3.3.1 Let $n=p_1^{k_1}\ldots\ldots p_m^{k_m}$ with p_i prime integers and E be a short exact sequence. Then for every $l\backslash n$, $\mathbf{Z}_l$ is E-injective (E-projective) if and only if $\mathbf{Z}_{p_t^s}$ is E-injective (E-projective) for every t=1,….,m and $0\leq s\leq k_t$.

Theorem 3.3.2 Let A be a proper class containing $\hat{S}_0$ and $n=p_1^{k_1}\ldots\ldots p_m^{k_m}$ be a positive integer. Then for a short exact sequence $E:0\to A\xrightarrow{f}B\to C\to 0$ the following conditions are equivalent:

1) $E\in nA$
2) $E\in A$ and $\mathbf{Z}_{p_t^s}$ is E-injective for every t=1,…,m, $0\leq s\leq k_t$.
3) $E\in A$ and $\mathbf{Z}_l$ is E-injective for every $l\backslash n$.
4) $E\in A$ and $\mathbf{Z}_l$ is E-projective for every $l\backslash n$.

Proof 1)⇒2) Let $E\in nA$, $t\in\{1,\ldots.,m\}$ and g. $A\to\mathbf{Z}_{p_t^s}$ be any homomorphism $0\leq s\leq k_t$. Let $g_*(E):0\to\mathbf{Z}_{p_t^s}\xrightarrow{h}D\to C\to 0$. i.e. we have a commutative diagram with exact rows:

$$\begin{array}{rcccccc} E: 0 \to & A & \xrightarrow{f} & B & \to & C & \to 0 \\ & {\scriptstyle g}\downarrow & & {\scriptstyle u}\downarrow & & \| & \\ g_*(E): 0 \to & \mathbf{Z}_{p_t^s} & \xrightarrow{h} & D & \to & C & \to 0 \end{array}$$

Then $g_*(E) \in nA$ since Ext_{nA} is a subfunctor, therefore $h(\mathbf{Z}_{p_t^s}) \cap p_t^s D = h(p_t^s \mathbf{Z}_{p_t^s}) = 0$ by Theorem 53.3 (Fuchs, 1970). Hence $h(\mathbf{Z}_{p_t^s})$ is a direct summand in D by proposition 27.1 (Fuchs, 1970), i.e. there is a homomorphisim h_1: D→ $\mathbf{Z}_{p_t^s}$ such that $h_1 o h = \mathbf{1}_{Z_{p_t^s}}$. Then for the homomorphism $v = h_1 o u$:B→ $\mathbf{Z}_{p_t^s}$ we have $vof = h_1 ouof = h_1 ohog = \mathbf{1}_{Z_{p_t^s}} og = g$. So $\mathbf{Z}_{p_t^s}$ is E-injective.

2)⇒3) folows from Corollary 3.3.1

3)⇒1) By Theorem 3.2.3, it is sufficient to show that f(mA)=f(A)∩mB for every m\n. Clearly f(mA)⊆f(A)∩mB. We have to show that f(mA)⊇f(A)∩mB. First of all A/mA is isomorphic to direct sum of groups $\mathbf{Z}_l$ with l\m and since A/mA is bounded and direct sum is a pure subgroup of direct product, A/mA is isomorphic to a direct summand of groups $\mathbf{Z}_l$ with l\m by Theorem 27.5 (Fuchs, 1970) and therefore it is E-injective. Then for canonical epimorphisim σ:A→A/mA we have a homamorphism g:B→A/mA such that gof=σ. Now let f(a) be any element from f(A)∩mB. Then f(a)=mb for some b∈B. Therefore σ(a)=gof(a)=g(mb)=mg(b)=0 since m(A/mA)=0. Then a∈Kerσ=mA and f(a)∈f(mA).

1)⇒4) Let E∈*nA* and x:$\mathbf{Z}_l$ →C be a homomorphism. We have a commutative diagram with exact rows:

$$\begin{array}{rcccccc} E: 0 \to & A & \longrightarrow & B & \xrightarrow{k} & C & \to 0 \\ & \| & & {\scriptstyle y}\uparrow & \nwarrow {\scriptstyle r} & \uparrow{\scriptstyle x} & \\ x^*(E): 0 \to & A & \xrightarrow{w} & F & \xrightarrow{z} & \mathbf{Z}_l & \to 0 \end{array}$$

Take any element b∈B with z(b)=l. Then z(lb)=lz(b)=0. i.e. lb∈Kerz=w(A) and lb=w(a) for some a∈A. Since x*(E)∈*nA*, w(lA)=w(A)∩lF by Theorem 53.3 in (Fuchs, 1970) , therefore w(a)∈w(lA), i.e. w(a)=w(la′). Since w is a monomorphism,

$a=la'$. Then for the element $b'=b-w(a')$ we have $z(b')=z(b)=l$ and $lb'=lb-lw(a')=w(a)-w(a)=0$, i.e. $o(b')\le l$. On the other hand $o(z(b'))=o(l)=l$, so $o(b')\ge l$. Thus $O(b')=l$, therefore $\langle b'\rangle\cong \mathbf{Z}_l$, so we can define a homomorphism z': $\mathbf{Z}_l\rightarrow F$ such that $z\circ z'=1_{Z_l}$. Then for the homomorphism $r=y\circ z'$ we have $k\circ r=k\circ y\circ z'=x\circ z\circ z'=x\circ 1_{Z_l}=x$. So $\mathbf{Z}_l$ is E-projective.

4)$\Rightarrow$1) Again by Theorem 3.2.3 it is sufficient to show that $f(A)\cap mB\subseteq f(mA)$ for every $m\backslash n$. Let $f(a)=mb\in f(A)\cap mB$. Then $mk(b)=k(mb)=k(f(a))=0$. Therefore a homomorphism $g:\mathbf{Z}_m\rightarrow C$ defined by $g(i)=ik(b)$ is well defined. Since $\mathbf{Z}_m$ is E-projective, there is a homomorphism $h:\mathbf{Z}_m\rightarrow B$ such that $k\circ h=g$. Then for the element $b'=b-h(1)\in B$ we have $k(b')=k(b)-k\circ h(1)=k(b)-g(1)=0$, therefore $b'\in Kerk=Imf$, i.e. $b'=f(a')$ for some $a'\in A$. On the other hand $f(ma')=mb'=mb-mh(1)=f(a)-h(m1)=f(a)-h(0)=f(a)$. Since f is a monomorphism $a=ma'$, i.e. $f(a)=f(ma')\in f(mA)$.

For the class *Abs* of all short exact sequences, we can describe all n*Abs*-injective and n*Abs*-projective groups.

Theorem 3.3.3 An abelian group I is n*Abs*-injective if and only if $I=D\oplus A$ where D is divisible and $nA=0$.

Proof Let I be n*Abs*-injective;there is a monomorphisim $f_1:I\rightarrow D'$ into a divisible group D′ by Theorem 24.1 (Fuchs, 1970). Let φ be the set of all possible homomorphism ϕ: $I\rightarrow M_\phi$ where $M_\phi=\mathbf{Z}_{p^k}$ for some $p^k\backslash n$. Denote $\prod_{\phi\in\varphi} M\phi$ by M and define a homomorphism $f_2:I\rightarrow M$ by $f_2(a)=(\ldots,\phi(a),\ldots)$.(i.e. ϕ^{th} coordinate of $f_2(a)$ is $\phi(a)$). Then the homomorphism $f:I\rightarrow D'\oplus M$ defined by $f(a)=(f_1(a),f_2(a))$ is a monomorphisim since f_1 is a monomorphism. On the other hand, for every homomorphism $g:I\rightarrow \mathbf{Z}_{p^k}$ with $p^k\backslash n$ we have $g=\phi$ and $M_\phi=\mathbf{Z}_{p^k}$ for some $\phi\in\varphi$. Therefore for the projection $P_\phi:D'\oplus M\rightarrow M_\phi$ onto ϕ^{th} coordinate we have $P_\phi\circ f=\phi=g$. So by Theorem 3.3.2 f is n*Abs*-monomorphisim. Since I is n*Abs*-injective. f is splitting, i.e. I is isomorphic to a direct summand of $D'\oplus M$, therefore it is also a direct sum of a divisible group and a group A with $nA=0$.

Let us suppose that $I=D\oplus A$ where D is divisible and $nA=0$. Therefore $Ext(C,nA)=0$. With respest to following diagram;

$$\begin{array}{ccc} \mathrm{Ext}_{Abs}(C,A) & \xrightarrow{n_*} & \mathrm{Ext}_{Abs}(C,A) \\ & f_* \searrow \quad \nearrow g_* & \\ & \mathrm{Ext}(C,nA)=\mathrm{Ext}(C,0)=0 & \end{array}$$

We have $f:A\to nA$ and $g:nA\to A$ $n_*=(gof)_*=g_*of_*=0$, since $Ext(C,nA)=0$, $n_*=0$. Hence $n\mathrm{Ext}_{Abs}(C,A)=\mathrm{Ext}_{nAbs}(C,A)=0$, by using Theorem 2.5.1 A is a $nAbs$ injective. Since every divisible group D is $nAbs$- injective so $I=D\oplus A$ is $nAbs$-injective.

Theorem 3.3.4 An abelian group P is $nAbs$-projective if and only if $P=F\oplus C$ where F is free and $nC=0$.

Proof There is an epimorphism $f_1:F'\to P$ from a free group F' onto P by Theorem33 (Rotman, 1979). Let φ be the set of all possible homomorphism $\phi:N_\phi\to P$ where $N_\phi=\mathbf{Z}_{p^k}$ for some $p^k\backslash n$. Denote $\oplus_{\phi\in\varphi} N_\phi$ by N and let $f_2:N\to P$ be defined by

$f_2\left(\sum_{\phi\in\varphi} n_\phi\right)=\sum_{\phi\in\varphi}\phi(n_\phi)$. Since $n_\phi=0$ for all but finete number of $\phi\in\varphi$, f_2 is well defined. The homomorphism $f:F'\oplus N\to P$ defined by $f(a,n)=f_1(a)+f_2(a)$ is an epimorphism since f_1 is an epimorphism. Now for every homomorphism g: $\mathbf{Z}_{p^k}\to P$ with $p^k\backslash n$, every $x\in\mathbf{Z}_{p^k}$ we have $foi_\phi(x)=f_1(0,i_\phi(x))=f_2(i_\phi(x))=\phi(x)=g(x)$.

Where $i_\phi:N_\phi\to\sum_{\psi\in\varphi}N_\psi$ is an inclusion map. So $foi_\phi=g$ and $\mathbf{Z}_{p^k}$ is projective with respect to the epimorphism f and by Theorem 3.3.2 f is an $nAbs$-epimorphism. Therefore f is splitting, i.e. P is a direct summand of $F'\oplus N$. Then the torsion part T(P) of P is bounded (recall that $nN=0$) and T(P) is a direct summand: $T(P)=F\oplus B$ by

Theorem 26b) and Theorem 27.5 (Fuchs, 1970). F is isomorphic to a subgroup of a free group F′ therefore it is free.

Let us suppose that P=F⊕C where F is free and nC=0. Therefore Ext(nC,A)=0. With respest to following diagram;

$$\begin{array}{ccc} \mathrm{Ext}_{Abs}(C,A) & \xrightarrow{n^*} & \mathrm{Ext}_{Abs}(C,A) \\ g^* \searrow & & \nearrow f^* \\ & \mathrm{Ext}(nC,A)=\mathrm{Ext}(0,A)=0 & \end{array}$$

We have f:C→nC and g:nC→C n*=(gof)*=g*of*=0, since Ext(nC,A)=0, n_*=0. Hence $n\mathrm{Ext}_{Abs}(C,A)=\mathrm{Ext}_{nAbs}(C,A)=0$, by using Theorem 2.5.1 A is a n*Abs*-projective. Since every fre group F is n*Abs*-projective so P=F⊕C is n*Abs*-projective.

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