ASYMPTOTIC FORMULAS FOR THE EIGENVALUES OF A SHRÖDINGER OPERATOR

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ABSTRACT

The multidimensional Schrödinger operator is a fundamental operator of quantum mechanics.

We consider the Schrödinger operator

$$Lu = -\Delta u + q(x)u$$

in the parallelepiped F with Neumann boundary condition

$$\left.\frac{\partial u}{\partial n}\right|_{\partial F} = 0$$

where $q(x)$ is a periodic function of $W_2^l(F)$. First we study the general property of L and its connection with other boundary conditions. Then we obtain asymptotic formulas for the eigenvalues of the operator L.

ÖZET

Çok boyutlu Schrödinger operatörü, Kuantum Mekaniğinin temel operatörlerindendir.

$$Lu = -\Delta u + q(x)u$$

şeklinde, Neumann sınır değer koşulu

$$\left.\frac{\partial u}{\partial n}\right|_{\partial F} = 0,$$

ile verilmiş olan Schrödinger operatörü üzerinde çalıştık. Burada F n-boyutlu paralel yüzlü olup, $q(x)$ de $W_2^l(F)$'de tanımlı periyodik bir fonksiyondur. Öncelikle L operatörünün genel özellikleri ve diğer sınır değer koşulları ile bağlantısı üzerinde çalıştık. Sonra da L operatörüne ait özdeğerlerin asimtotik formüllerini elde ettik.

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CHAPTER ONE
INTRODUCTION

We study the Schrödinger operator

$$L = -\Delta + q(x)$$

in an n-dimensonal parallelepiped with Neumann boundary conditions, where $q(x)$ is a periodic function. The aim of this thesis is to find an asymptotic formula for the eigenvalues of the operator L.

First asymptotic formula for the eigenvalues of Schrödinger operator in the parallelepiped with quasiperiodic boundary conditions is obtained in papers (Veliev, 1987), (Veliev, 1988). The other asymptotic formulas for quasiperiodic boundary conditions in two and three dimensional cases are obtained in (Feldman, 1990), (Feldman, 1991), (Karpeshina, 1990), (Karpeshina, 1992). The asymptotic formula for Dirichlet boundary conditions in two dimensions is obtained in (Hald & McLaughlin, 1996).

We use the method of papers (Veliev, 1987), (Veliev, 1988) to find the asymptotic formula for the Neumann boundary conditions in n-dimensions.

In chapter one, some properties of multidimensional periodic functions and boundary value problems in n-dimensional parallelepiped are studied.

In chapter two, we obtain asymptotic formulas for the eigenvalues of the operator L.

1.1. On n-dimensional periodic functions

Periodicity of $q(x)$, where $x \in R^n$, means that, see (Eastham, 1973), (Reed & Simon, 1978), there are n linearly independent vectors $w_1, w_2, ..., w_n$ such that

$$q(x + w_k) = q(x) \quad , \qquad k = 1,2,...,n.$$

Clearly for every linear combination $m_1 w_1 + m_2 w_2 + ... + m_n w_n$ of the vectors $w_1, w_2, ..., w_n$ with integer coefficients $m_1, m_2, ..., m_n$ we have

$$q(m_1 w_1 + m_2 w_2 + ... + m_n w_n) = q(x).$$

In other words for every $w \in \Omega \equiv \{m_1 w_1 + m_2 w_2 + ... + m_n w_n : m_1, m_2, ..., m_n \in Z\}$ the relation

$$q(x + w) = q(x) \qquad 1.1$$

holds. The set Ω is called lattice generated by $w_1, w_2, ..., w_n$. There is a unique n-dimensional parallelogram F which has the origin in R^n as one corner and the w_k forming the sides which meet at the corner. We call F "the fundamental domain of the lattice Ω". The function $q(x)$ satisfying (1.1) is said to be periodic with respect to the lattice Ω and F is called the period parallelogram of $q(x)$. To find the functions which are periodic with respect to the lattice Ω, we need to determine the lattice Γ dual to Ω. The dual lattice Γ is determined as following:

For each integer k $(1 \le k \le n)$, there is a unique vector γ^k in R^n such that

$$\langle \gamma^k, w_j \rangle = \delta_{k,j}, \qquad 1.2$$

where $\langle .,. \rangle$ is the dot product in R^n.

Explicitly, in fact,

$$\gamma^k = (F^T)^{-1} e_k \ , \qquad 1.3$$

where F is the matrix $(w_1 w_2 ... w_n)$, the superscript T denotes the transpose and $e_1 = (1,0,...,0), e_2 = (0,1,...,0),...,e_n = (0,0,...,1)$ are coordinate vectors. The γ^k are clearly linearly independent and are called the reciprocal vectors of w_j. The lattice

$$\Theta \equiv \{m_1\gamma^1 + m_2\gamma^2 + ... + m_n\gamma^n : m_1 m_2,...,m_n \in Z\} \qquad 1.4$$

is called reciprocal lattice of Ω and the lattice defined by

$$\Gamma = 2\pi\Theta \qquad 1.5$$

is the dual lattice of Ω.

Let B denote the n-dimensional parallelogram which has the origin in R^n as one corner and the γ^k forming the sides which meet at that corner. Also, let

$$B = (\gamma^1\gamma^2 ... \gamma^n).$$

By (1.3),

$$B = (F^T)^{-1} \qquad 1.6$$

and there follows from this, a relation between the n-dimensional volumes $|F|$ and $|B|$ of F and B. Let e denote the vector with every component unity. Then

$$|F| = \det\begin{pmatrix} 0 & 1 \\ F^T & e \end{pmatrix}$$

and, considering the transposed determinant,

$$|B| = \det\begin{pmatrix} 0 & B \\ 1 & e^T \end{pmatrix}.$$

Hence, if E denotes the $n \times n$ matrix with every entry unity, (1.6) gives

$$|F||B| = \det\left\{\begin{pmatrix} 0 & 1 \\ F^T & e \end{pmatrix}\begin{pmatrix} 0 & B \\ 1 & e^T \end{pmatrix}\right\}$$

$$= \det\begin{pmatrix} 1 & e^T \\ e & \mathrm{I}+\mathrm{E} \end{pmatrix} = \det\begin{pmatrix} 1 & e^T \\ 0 & \mathrm{I} \end{pmatrix} = 1.$$

Thus

$$|B| = |F|^{-1}.$$

The functions $e^{i\langle\gamma,x\rangle}$ for $\gamma\in\Gamma$ is periodic with respect to Ω. Indeed

$$e^{i\langle\gamma,x+w\rangle}=e^{i\langle\gamma,x\rangle}e^{i\langle\gamma,w\rangle}=e^{i\langle\gamma,x\rangle}e^{i2\pi k}=e^{i\langle\gamma,x\rangle}$$

If $w_1,w_2,...,w_n$ are an orthonormal basis, say $w_1=(a_1,0,...,0)$, $w_2=(0,a_2,0,...,0)$, ... , $w_n=(0,0,...,a_n)$ in the n-dimensional Euclidean space R^n, then the vectors $\gamma^1=(\frac{2\pi}{a_1},0,...,0)$, $\gamma^2=(0,\frac{2\pi}{a_2},0,...,0)$, ... ,$\gamma^n=(0,0,..,\frac{2\pi}{a_n})$ are biorthoganal to the vectors $w_1,w_2,...,w_n$, i.e., $\langle w_i,\gamma^k\rangle=2\pi\delta_{i,k}$, where $\langle .,.\rangle$ is the dot product in R^n.

The set $\Omega=\{m_1w_1+m_2w_2+...+m_nw_n : m_i\in Z, i=1,2,...,n\}$ is a lattice in R^n with the reduced orthonormal basis $\{w_i\}_{i=1}^n$ and $\Gamma=\{m_1\gamma^1+m_2\gamma^2+...+m_n\gamma^n : m_i\in Z, i=1,2,...,n\}$ is the dual lattice of Ω .

We consider the periodic potential $q(x)$ of $W_2^l(F)$, where $W_2^l(F)$ is the set of functions such that $D^\alpha f\in L_2(F)$ for all $|\alpha|\le l$, $\alpha=(\alpha_1,\alpha_2,...,\alpha_n)$, $|\alpha|=|\alpha_1|+|\alpha_2|+...+|\alpha_n|$, $D^\alpha=\frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}...\partial x_n^{\alpha_n}}$. The fourier coefficient of $q(x)$ with respect to the orthonormal basis $\{e^{i\langle\gamma,x\rangle}\}_{\gamma\in\Gamma}$, $\gamma=(\gamma_1,\gamma_2,...,\gamma_n)$, in $L_2(F)$ is

$$q_\gamma=(q(x),e^{i\langle\gamma,x\rangle})=\int_F q(x)\overline{e^{i\langle\gamma,x\rangle}}dx=\int_0^{a_1}\int_0^{a_2}...\int_0^{a_n}q(x)e^{-i\gamma_1x_1}e^{-i\gamma_2x_2}...e^{-i\gamma_nx_n}dx.$$

Integrating by part with respect to x_1, we have

$$q_\gamma=\frac{1}{i\gamma_1}\int_0^{a_2}...\int_0^{a_n}e^{-i\gamma_2x_2}...e^{-i\gamma_nx_n}(q(x)e^{-i\gamma_1x_1}\Big|_0^{a_1})dx+\frac{1}{i\gamma_1}\int_0^{a_1}\int_0^{a_2}...\int_0^{a_n}\frac{\partial q(x)}{\partial x_1}e^{-i\langle\gamma,x\rangle}dx,$$

since $e^{i\langle\gamma,x\rangle}$ is periodic with respect to Ω, $q(x)e^{-i\gamma_1x_1}\Big|_0^{a_1}=0$. Hence

$$q_\gamma=\frac{1}{i\gamma_1}\int_0^{a_1}\int_0^{a_2}...\int_0^{a_n}\frac{\partial q(x)}{\partial x_1}e^{-i\langle\gamma,x\rangle}dx.$$

Again integrating by part with respect to x_1 and using the periodicity of $e^{i\langle\gamma,x\rangle}$, we get

$$q_\gamma = -\frac{1}{(i\gamma_1)^2}\int_0^{a_1}\int_0^{a_2}\dots\int_0^{a_n}\frac{\partial^2 q(x)}{\partial x_1^{\,2}}e^{-i\langle\gamma,x\rangle}dx.$$

In this way, integrating by part with respect to x_1, l times, we obtain

$$q_\gamma = \frac{(-1)^l}{(i\gamma_1)^l}\int_0^{a_1}\int_0^{a_2}\dots\int_0^{a_n}\frac{\partial^l q(x)}{\partial x_1^{\,l}}e^{-i\langle\gamma,x\rangle}dx,$$

where the integral $a_\gamma^1 = \int_0^{a_1}\int_0^{a_2}\dots\int_0^{a_n}\frac{\partial^l q(x)}{\partial x_1^{\,l}}e^{-i\langle\gamma,x\rangle}dx$ is the fourier coefficient of the function $\frac{\partial^l q(x)}{\partial x_1^{\,l}} \in L_2(F)$. Hence we have the following relation

$$|q_\gamma| \le \frac{1}{|\gamma_1|^l}|a_\gamma^1| \le \frac{1}{|\gamma|^l}|a_\gamma^1|, \qquad \sum_{\gamma\in\Gamma}|a_\gamma^1|^2 < \infty.$$

Similarly,

$$|q_\gamma| \le \frac{1}{|\gamma_i|^l}|a_\gamma^i| \le \frac{1}{|\gamma|^l}|a_\gamma^i|, \qquad \sum_{\gamma\in\Gamma}|a_\gamma^i| < \infty, \qquad \forall i = 1,2,\dots,n.$$

It is not hard to see that these relations imply the relation

$$\sum_{\gamma\in\Gamma}(q(x), e^{i\langle\gamma,x\rangle})^2(1+|\gamma|^{2l}) < \infty .$$

So it is convinient to define a periodic function $q(x)$ in $W_2^l(F)$ as a function satisfying the last relation.

1.2. On boundary value problems in an n-dimensional parallelpiped

Now we consider the Schrödinger operator in $L_2(F)$, defined by the differential expression

$$lu = -\Delta u + q(x)u \qquad 1.7$$

and the Neumann boundry condition

$$\left.\frac{\partial u}{\partial n}\right|_{\partial F} = 0 \qquad 1.8$$

where $F=[0,a_1)\times[0,a_2)\times...\times[0,a_n)=R^n/\Omega$ is the fundamental domain of the lattice Ω, ∂F is the boundry of F, $x=(x_1,x_2,...,x_n)\in R^n$, $\Delta=\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}$ is the laplace operator in R^n, $\frac{\partial}{\partial n}$ denotes differentiation along the outward normal and $q(x)\in W_2^l(F)$ is real valued , periodic (with respect to lattice Ω) and l times differentiable potential. We denote this operator , defined by the differential expression (1.7) and the Neumann boundry condition (1.8) , by L; the eigenfunction and the eigenvalue of the operator L by $\Psi_N(x)$ and Λ_N, respectively.

Let us denote by L^0 the operator defined by the differential expression (1.7) in the case when $q(x)=0$ and the boundry condition (1.8).

The quasiperiodic or t-periodic problem comprises (1.7), considered to hold in F, and the boundary conditions

$$u(x+w_j)=u(x)e^{i2\pi t_j},\ j=1,2,...,n.$$

Where $w_1,w_2,...,w_n$ is a basis of the lattice Ω, $t_1,t_2,...,t_n$ are real parameters in $[0,1]$, $t=\sum_{i=1}^{n}\gamma^i t_i$, $\gamma^1,\gamma^2,...,\gamma^n$ is the basis of the dual lattice. These boundary condition, in brief, can be written as

$$u(x+w)=u(x)e^{i\langle w,t\rangle},\quad w\in\Omega \qquad 1.9$$

We denote the eigenvalues and the eigenfunctions of the quasiperiodic problem as λ_N and $\varphi_N(x)$, respectively.

The Dirichlet problem is the one for which (1.7) holds in F with the boundary condition

$$u(x)\big|_{\partial F}=0 \qquad 1.10$$

The eigenvalues and the eigenfunctions of the Dirichlet problem are denoted by ν_N and $\chi_N(x)$.

There is the following relation between the eigenvalues of Neumann, Dirichlet and the quasiperiodic boundary conditions (see; Eastham, 1973):

Let F denote the set of all complex-valued functions $f(x)$ which are continuous in F and have piecewise continuous first-order partial derivatives in F. Then the Dirichlet integral $J(f,g)$ in n-dimension is defined by

$$J(f,g) = \int_F \{ grad f(x).grad\overline{g(x)} + q(x)f(x)\overline{g(x)} \} dx \qquad 1.11$$

for $f(x)$ and $g(x)$ in F. Here,

$$grad f(x) = \frac{\partial f}{\partial x_1} e_1 + \frac{\partial f}{\partial x_2} e_2 + \ldots + \frac{\partial f}{\partial x_n} e_n$$

If, in (1.9), $g(x)$ also has piecewise continuous second-order partial derivatives in F, then Green's theorem gives

$$J(f,g) = -\int_F f(x).\{ \Delta\overline{g(x)} - q(x)f(x)\overline{g(x)} \} dx + \int_{\partial F} f \frac{\partial g}{\partial n} dS \qquad 1.12$$

where dS denotes an element of surface area ∂F.

We consider $J(f,g)$ as applied to the quasiperiodic problem first. If $f(x)$ and $g(x)$ satisfy the boundary conditions (1.9), the integral over ∂F in (1.12) is zero because the integrals over oposite faces of ∂F cancel out. In particular, when $g(x) = \varphi_N(x)$, (1.12) gives

$$J(f,g) = \lambda_N f_N \qquad 1.13$$

where $f_N = \int_F f(x)\overline{\varphi_N(x)} dx$ is the fourier coefficient, and we have used the fact that $\varphi_N(x)$ is the eigenfunction of the quasiperiodic problem with the corresponding eigenvalue λ_N. A particular case of (1.13) is

$$J(\varphi_N, \varphi_M) = \begin{cases} \lambda_N & M = N \\ 0 & M \neq N \end{cases} \qquad 1.14$$

It now follows that

$$\sum_{N=0}^{\infty} \lambda_N |f_N|^2 \leq J(f, f) \qquad 1.15$$

for all $f(x) \in \mathsf{F}$ which satisfy (1.9).

From (1.15) we obtain the following

$$\lambda_0 = \min \left(J(f,f) \Big/ \int_F |f(x)|^2 dx \right), \qquad 1.16$$

the minimum being taken over all $f(x)$ $(\neq 0)$ in F which satisfy (1.9).Furthermore, the minimum in (1.16) is attained only when $f_N = 0$ for all N such that $\lambda_N > \lambda_0$, i.e., only when $f(x)$ is an eigenfunction corresponding to λ_0. In the case of the quasiperiodic problems, the eigenfunctions are real valued and therefore, for these problems, $f(x)$ can be confined to being real valued in (1.16).

The results (1.13)-(1.16) followed from (1.12) because of the vanishig of the integral over ∂F when $g(x)$ is an eigenfunction. Corresponding results hold for the Dirichlet and the Neumann problems. In the first problem, the integral over ∂F in (1.12) vanishes if $f(x) = 0$ on ∂F. In the second problem, the integral vanishes without any boundary condition on $f(x)$ since, by (1.8), $\partial g / \partial n = 0$ on ∂F when $g(x)$ is an eigenfunction. Thus, for the Neumann problem, we have

$$\sum_{N=0}^{\infty} \Lambda_N |f_N|^2 \leq J(f, f) \qquad 1.17$$

for all $f(x) \in \mathsf{F}$, where $f_N = \int_F f(x) \Psi_N(x) dx$, while the corresponding result holds for the Dirichlet problem if $f(x)$ is in F and $f(x) = 0$ on ∂F.

In the following theorem, these results concerning $J(f,f)$ are used.

Theorem 1.1: For $N \geq 0$,

$$\Lambda_N \leq \lambda_N \leq \nu_N \,. \qquad 1.18$$

Proof: To prove the left-hand inequality, we first take $f(x)=\varphi_0(x)$ in (1.17). Since $\lambda_0 = J(f,f)$ by (1.14), we obtain

$$\lambda_0 \geq \sum_{N=0}^{\infty} \Lambda_N |f_N|^2 \geq \Lambda_0 \sum_{N=0}^{\infty} |f_N|^2 \,.$$

By the Parseval formula,

$$\sum_{N=0}^{\infty} |f_N|^2 = \int_F |f(x)|^2 dx = 1 \,.$$

Hence

$$\lambda_0 \geq \Lambda_0 \,.$$

Next we take

$$f(x) = c_0\varphi_0(x) + c_1\varphi_1(x) \,,$$

where c_o and c_1 are constants such that $|c_0|^2 + |c_1|^2 = 1$ and

$$c_0 \int_F \varphi_0(x)\Psi_0(x)dx + c_1 \int_F \varphi_1(x)\Psi_0(x)dx = 0 \,.$$

Such a choice of c_0 and c_1 is always possible. The first condition makes $\int_F |f(x)|^2 dx = 1$ and the second makes $f_0 = 0$.

By (1.14),

$$J(f,f) = \lambda_0 |c_0|^2 + \lambda_1 |c_1|^2 \leq \lambda_1 (|c_0|^2 + |c_1|^2) = \lambda_1 \,.$$

Also, by (1.17) and the fact that $f_0 = 0$,

$$J(f,f) \geq \sum_{N=1}^{\infty} \Lambda_N |f_N|^2 \geq \Lambda_1 \sum_{N=1}^{\infty} |f_N|^2 = \Lambda_1 \int_F |f_N|^2 dx = \Lambda_1 \,.$$

Hence

$$\lambda_1 \geq \Lambda_1 \,.$$

The argument can be extended to the general case N. We consider

$$f(x) = c_0\varphi_0(x) + c_1\varphi_1(x) + \ldots + c_N\varphi_N(x)$$

where c_i are constants such that $|c_0|^2 + |c_1|^2 + \ldots + |c_N|^2 = 1$ and $f_i = 0$ for $0 \le i \le N-1$. The latter conditions are N linear algebraic equations to be satisfied by $N+1$ numbers $c_0, c_1, \ldots, c_N$, and such numbers always exist. The proof of the theorem for general N now follows the same as the proof for $N = 1$.

The proof of the right-hand side of the inequality (1.18) is similar and we use (1.15) instead of (1.17).

□

CHAPTER TWO

ASYMPTOTIC FORMULAS FOR THE EIGENVALUES OF A SCHRÖDINGER OPERATOR

To obtain the asymptotic formulas for the eigenvalues of the operator L we first consider the eigenvalues and eigenfunctions of unperturbed operator L_0.

2.1 On the unperturbed operator L_0

Let γ be an element of the lattice $\frac{\Gamma}{2}$, where Γ is defined as in section 1.1. We consider the function

$$\sum_{\alpha\in A_\gamma} e^{i\langle\alpha,x\rangle},$$

where

$$A_\gamma=\{\alpha=(\alpha_1,\alpha_2,\ldots,\alpha_n)\in R^n:|\alpha_i|=|\gamma_i|,i=1,2,\ldots,n\} \qquad 2.1$$

The norm of the function $\sum_{\alpha\in A_\gamma} e^{i\langle\alpha,x\rangle}$ in $L_2(F)$ is

$$\left\|\sum_{\alpha\in A_\gamma} e^{i\langle\alpha,x\rangle}\right\|^2=(\sum_{\alpha\in A_\gamma} e^{i\langle\alpha,x\rangle},\sum_{\alpha\in A_\gamma} e^{i\langle\alpha,x\rangle})=\int_F(\sum_{\alpha\in A_\gamma} e^{i\langle\alpha,x\rangle})^2dx=2^n a_1a_2\ldots a_n .$$

By (2.1)

$$(\sum_{\alpha\in A_\gamma} e^{i\langle\alpha,x\rangle})^2=\sum_{\alpha\in A_{2\gamma}} e^{i\langle\alpha,x\rangle}+\sum_{k=1}^{n-1}\sum_{\alpha\in A_{(2\gamma_1,2\gamma_1,\ldots,2\gamma_k,0,\ldots,0)}} e^{i\langle\alpha,x\rangle}+2^n,$$

taking the integral over F, we get

$$\left\|\sum_{\alpha\in A_\gamma} e^{i\langle\alpha,x\rangle}\right\|^2 = \int_F \Big(\sum_{\alpha\in A_\gamma} e^{i\langle\alpha,x\rangle}\Big)^2 dx = 2^n a_1 a_2 \ldots a_n .$$

Without loss of generality, we can take $\left\|\sum_{\alpha\in A_\gamma} e^{i\langle\alpha,x\rangle}\right\| = 1$.

Lemma 2.1: The normalized eigenfunction and the eigenvalue of the operator L^0 are $u_\gamma = \sum_{\alpha\in A_\gamma} e^{i\langle\alpha,x\rangle}$ and $|\gamma|^2$ respectively, where $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n) \in \frac{\Gamma}{2}$.

Proof: To prove this lemma we must show that the function u_γ satisfies the equation

$$-\Delta u_\gamma = |\gamma|^2 u_\gamma \qquad 2.2$$

and the boundry conditions (1.8). Differentiating the function

$$u_\gamma = \sum_{\alpha\in A_\gamma} e^{i\langle\alpha,x\rangle} = \sum_{\alpha\in A_\gamma} e^{i(\alpha_1 x_1 + \ldots + \alpha_n x_n)}$$

by x_j, we get

$$\frac{\partial u_\gamma}{\partial x_j} = \sum_{\alpha\in A_\gamma} \frac{\partial}{\partial x_j} e^{i(\alpha_1 x_1 + \ldots + \alpha_n x_n)} = \sum_{\alpha\in A_\gamma} i\alpha_j e^{i(\alpha_1 x_1 + \ldots + \alpha_n x_n)}$$

$$\frac{\partial^2 u_\gamma}{\partial x_j^2} = -\sum_{\alpha\in A_\gamma} \alpha_j^2 e^{i\langle\alpha,x\rangle} .$$

By the definition of A_γ, we have $|\alpha_j| = |\gamma_j| \Rightarrow \alpha_j^2 = \gamma_j^2$, which together with the last equation above imply

$$-\Delta u_\gamma = \sum_{j=1}^{n} \sum_{\alpha\in A_\gamma} \gamma_j^2 e^{i\langle\alpha,x\rangle} = |\gamma|^2 \sum_{\alpha\in A_\gamma} e^{i\langle\alpha,x\rangle} = |\gamma|^2 u_\gamma$$

Hence the equation (2.2) is satisfied. Now let us show that the function u_γ satisfies the boundry conditions (1.8): We know that $\frac{\partial u_\gamma}{\partial n}$ is the derivative of the function u_γ in the direction of the vector n which is the normal to the boundary of F. By definition of F, the boundary ∂F lies in the hyperplanes $\Pi_k = \{x \in R^n : (x, e_k) = 0\}$ or on its shifts $a_k e_k + \Pi_k$, where

$k=1,\dots n$; $e_1=(1,0,\dots,0), e_2=(0,1,0,\dots,0)\dots e_n=(0,0,\dots,1)$. So normal to the Π_k and $a_k e_k+\Pi_k$ are e_k or $-e_k$, respectively. Therefore, $\frac{\partial u_\gamma}{\partial n}$ coinside with the first partial derivatives $\frac{\partial u_\gamma}{\partial x_j}$ or $-\frac{\partial u_\gamma}{\partial x_j}$ of u_γ . Thus we need to show that,

$$\left.\frac{\partial u_\gamma}{\partial x_j}\right|_{x\in\Pi_j} = \left.\sum_{\alpha\in A_\gamma}\frac{\partial}{\partial x_j}e^{i(\alpha_1x_1+\dots+\alpha_nx_n)}\right|_{x\in\Pi_j} = \left.\sum_{\alpha\in A_\gamma} i\alpha_j e^{i(\alpha_1x_1+\dots\alpha_jx_j+\alpha_nx_n)}\right|_{x\in\Pi_j} = 0, \qquad 2.3$$

and

$$\left.\frac{\partial u_\gamma}{\partial x_j}\right|_{x\in a_je_j+\Pi_j} = \left.\sum_{\alpha\in A_\gamma}\frac{\partial}{\partial x_j}e^{i(\alpha_1x_1+\dots+\alpha_nx_n)}\right|_{x\in a_je_j+\Pi_j} = \left.\sum_{\alpha\in A_\gamma} i\alpha_j e^{i(\alpha_1x_1+\dots\alpha_jx_j+\alpha_nx_n)}\right|_{x\in a_je_j+\Pi_j} = 0. \qquad 2.4$$

In the set A_γ we have the vectors $(\pm\frac{m_1\pi}{a_1},\pm\frac{m_2\pi}{a_2},\dots,\frac{m_j\pi}{a_j},\dots,\pm\frac{m_n\pi}{a_n})$ and $(\pm\frac{m_1\pi}{a_1},\pm\frac{m_2\pi}{a_2},\dots,-\frac{m_j\pi}{a_j},\dots,\pm\frac{m_n\pi}{a_n})$, $j=1,2,\dots,n$. Therefore the right-hand side of (2.3) and (2.4) consist of the terms $i\frac{m_j\pi}{a_j}e^{i(\pm\frac{m_1\pi}{a_1}x_1\pm\dots+\frac{m_j\pi}{a_j}x_j\pm\dots\pm\frac{m_n\pi}{a_n}x_n)}$ and $-i\frac{m_j\pi}{a_j}e^{i(\pm\frac{m_1\pi}{a_1}x_1\pm\dots-\frac{m_j\pi}{a_j}x_j\pm\dots\pm\frac{m_n\pi}{a_n}x_n)}$ which will be cancelled when $x_j=0,a_j$ under the summation notation . Hence u_γ satisfies the boundry conditions (1.8).

Lemma 2.2: Let $\sum_{\alpha\in A\gamma}e^{i\langle\alpha,x\rangle}$ and $\sum_{\beta\in A_\omega}e^{i\langle\beta,x\rangle}$ be the eigenfunctions of the operator L^0, where $\gamma,\omega\in\frac{\Gamma}{2}$. Then $\sum_{\alpha\in A_\gamma}e^{i\langle\alpha,x\rangle}\sum_{\beta\in A_\omega}e^{i\langle\beta,x\rangle}=\sum_{\beta\in A_\omega}\sum_{\alpha\in A_{\gamma+\beta}}e^{i\langle\alpha,x\rangle}$.

Proof: We can write

$$\sum_{\alpha\in A_\gamma}e^{i\langle\alpha,x\rangle}\sum_{\beta\in A_\omega}e^{i\langle\beta,x\rangle} = \sum_{\alpha\in A_\gamma,\beta\in A_\omega}e^{i\langle\alpha+\beta,x\rangle} = \sum_{\{y:y=\alpha+\beta,\alpha\in A_\gamma,\beta\in A_\omega\}}e^{i\langle y,x\rangle}$$

and

$$\sum_{\beta\in A_\omega}\sum_{\alpha\in A_{\gamma+\beta}} e^{i\langle\alpha,x\rangle} = \sum_{\{y:y\in A_{\gamma+\beta},\beta\in A_\omega\}} e^{i\langle y,x\rangle}$$

to prove the lemma it is enough to show that the sets $C\equiv\{y: y=\alpha+\beta, \alpha\in A_\gamma, \beta\in A_\omega\}$ and $D\equiv\{y: y\in A_{\gamma+\beta}, \beta\in A_\omega\}$ are equal.

Let $y=\alpha+\beta\in C$ then $\alpha\in A_\gamma$ and $\beta\in A_\omega$, that is, $\alpha_i=\mp\gamma_i$ and $\beta_i=\mp\omega_i$,by (2.3). Then $y_i=\gamma_i+\omega_i$ or $y_i=-\gamma_i-\omega_i$ or $y_i=\gamma_i-\omega_i$ or $y_i=-\gamma_i+\omega_i$ $\forall i=1,2,...,n$. Equivalently we can write

$$y_i=\mp(\gamma_i+\omega_i) \quad \text{or} \quad y_i=\mp(\gamma_i-\omega_i) \quad \forall i=1,2,...,n \qquad 2.5$$

Now let $z\in D$ then $z\in A_{\gamma+\beta}$, where $\beta\in A_\omega$. Again by (2.3), $z_i=\mp(\gamma_i+\beta_i)$ and $\beta_i=\mp\omega_i$. Hence we have

$$z_i=\mp(\gamma_i+\omega_i) \quad \text{or} \quad z_i=\mp(\gamma_i-\omega_i) \quad \forall i=1,2,...,n \qquad 2.6$$

From (2.5) and (2.6), it is clear that the sets C and D are equal to each other. So the lemma is proved . □

The eigenfunctions $\{\sum_{\alpha\in A\gamma} e^{i\langle\alpha,x\rangle}\}_{\gamma\in\frac{\Gamma}{2}}$ of the self-adjoint operator L^0 forms an orthonormal basis in $L_2(F)$. Hence the potential $q(x)$ in the operator $L(q)$ can be written in the following form;

$$q(x)=\sum_{\gamma\in\frac{\Gamma}{2}} q_\gamma \sum_{\alpha\in A\gamma} e^{i\langle\alpha,x\rangle} \qquad 2.7$$

where $q_\gamma=\int_F q(x)\overline{\sum_{\alpha\in A\gamma} e^{i\langle\alpha,x\rangle}}dx$ (without loss of generality we can take $q_0=\int_F q(x)dx=0$. Other wise we replace $q(x)$ by $q(x)-\frac{q_0}{\mu(F)}$.) is the fourier coefficient of the potential $q(x)$ with respect to the basis $\{\sum_{\alpha\in A\gamma} e^{i\langle\alpha,x\rangle}\}_{\gamma\in\frac{\Gamma}{2}}$.

Moreover for big parameter ρ we can write $q(x)\in W_2^l(F)$ as

$$q(x)=\sum_{\gamma\in\Gamma(\rho^\alpha)} q_\gamma \sum_{\beta\in A_\gamma} e^{i\langle\beta,x\rangle}+O(\rho^{-l\alpha}) \qquad 2.8$$

where $\Gamma(\rho^\alpha)=\{\gamma\in\frac{\Gamma}{2}:0<|\gamma|<\rho^\alpha\}$, $\alpha>0$ and $O(\rho^{-l\alpha})$ is a function with norm of order $\rho^{-l\alpha}$. Indeed, by (1.20), $q(x)\in W_2^l(F)$ means that

$$\sum_{\gamma\in\Gamma}(q(x),e^{i\langle\gamma,x\rangle})^2(1+|\gamma|^{2l})<\infty .$$

Since we have only 2^n terms in $\sum_{\beta\in A_\gamma} e^{i\langle\beta,x\rangle}$,

$$\sum_{\gamma\in\frac{\Gamma}{2}}(q(x),\sum_{\beta\in A_\gamma} e^{i\langle\beta,x\rangle})^2(1+|\gamma|^{2l})<\infty$$

It implies that

$$|q_\gamma|^2\le\frac{1}{|\gamma|^{2l}}|a_\gamma|^2 \qquad 2.9$$

where $a_\gamma=(D^\alpha q(x),\sum_{\beta\in A_\gamma} e^{i\langle\beta,x\rangle})$, $|\alpha|\le l$ and $\sum_{\gamma\in\frac{\Gamma}{2}}|a_\gamma|<\infty$. Therefore, we have

$$\left\|\sum_{|\gamma|\ge\rho^\alpha} q_\gamma \sum_{\beta\in A_\gamma} e^{i\langle\beta,x\rangle}\right\|\le(\sum_{|\gamma|\ge\rho^\alpha}|q_\gamma|^2)^{\frac{1}{2}}\le(\frac{1}{\rho^{2l\alpha}}\sum_{|\gamma|\ge\rho^\alpha}|a_\gamma|^2)^{\frac{1}{2}}=O(\rho^{-l\alpha}),$$

which is equivalent to (2.8).

Remark: Note that it follows from (2.9) that

$$\sum_\gamma|q_\gamma|\le\sum_\gamma\frac{|a_\gamma|}{|\gamma|^l}\le(\sum_\gamma|a_\gamma|^2)^{\frac{1}{2}}(\sum_\gamma\frac{1}{|\gamma|^{2l}})^{\frac{1}{2}}\le\infty$$

the sum $(\sum_\gamma\frac{1}{|\gamma|^{2l}})^{\frac{1}{2}}$ converges for $l>\frac{n}{2}$.

2.2 Asymptotic formulas for the eigenvalues of the operator L

Let us introduce the following notations:

$$M = \sum_{\gamma \in \frac{\Gamma}{2}} |q_\gamma| \qquad 2.10$$

$$V_b(\rho^\alpha) = \{x \in R^n : \left| |x|^2 - |x+b|^2 \right| < \rho^\alpha\}$$

$$U(\rho^\alpha, m) = R^n \quad \backslash \bigcup_{b \in \Gamma(m\rho^\alpha)} V_b(\rho^\alpha)$$

where $\Gamma(m\rho^\alpha) = \{b \in \frac{\Gamma}{2} : 0 < |b| < m|\rho^\alpha|\}$. The domain $U(\rho^\alpha, m)$ is said to be a non-resonance domain and the eigenvalue $|\gamma|^2$ is called a non-resonance eigenvalue if $\gamma \in U(\rho^\alpha, m)$. The domains $V_\gamma(\rho^\alpha)$ for all $\gamma \in \Gamma(m\rho^\alpha)$ are called resonance domains and the eigenvalue $|\gamma|^2$ is called resonance eigenvalue if $\gamma \in V_b(\rho^\alpha)$. It is not hard to see that $U(\rho^\alpha, m)$ has asymptotically full measure in R^n in the sense that

$$\frac{\mu(U(\rho^\alpha, m) \cap S_\rho)}{\mu(S_\rho)} \longrightarrow 1, \qquad 2.11$$

where $S_\rho = \{x \in R^n : |x| = \rho\}$. To prove (2.11), we calculate $\mu(V_b(\rho^\alpha) \cap S_\rho)$. Let $x \in V_b(\rho^\alpha)$, then, by definition of $V_b(\rho^\alpha)$, we have $\left| |x|^2 - |x+b|^2 \right| < \rho^\alpha$.

$$\left| |x|^2 - |x+b|^2 \right| = \left| \langle x, x \rangle - \langle x+b, x+b \rangle \right| = \left| -2\langle x, b \rangle - |b|^2 \right| = \rho^\alpha \quad \Rightarrow \quad -2\langle x, b \rangle - |b|^2 = \pm \rho^\alpha \Rightarrow$$

$$\langle x, b \rangle + \frac{|b|^2}{2} \mp \frac{\rho^\alpha}{2} = 0 \quad \Rightarrow \quad \left\langle x + \frac{b}{2} \mp \frac{\rho^\alpha b}{2|b|^2}, b \right\rangle = 0.$$

If $\Pi_b = \{x : \langle x, b \rangle = 0\}$, then $V_b(\rho^\alpha)$ contained between the planes

$$\Pi_b + \left(\frac{b^2}{2} + \frac{\rho^\alpha b}{2|b|^2}\right) \text{ and } \Pi_b + \left(\frac{b^2}{2} - \frac{\rho^\alpha b}{2|b|^2}\right) \qquad 2.12$$

The planes (2.12) are parallel and the distance between them is $\frac{\rho^\alpha}{2|b|} = O(\rho^\alpha)$ (between them and the origin are also $O(\rho^\alpha)$). Therefore, it is well known that,

$$\mu(V_b(\rho^{\alpha})\bigcap S_{\rho})=O(\rho^{n-2+\alpha}).$$

The number of $b\in\Gamma(m\rho^{\alpha})$ is $\rho^{n\alpha}$. Hence

$$\mu(\bigcup_b(V_b(\rho^{\alpha})\bigcap S_{\rho}))=O(\rho^{n-2+(n+1)\alpha})=O(\rho^{-\alpha})\mu(S_{\rho}),$$

if $-1+(n+1)\alpha<-\alpha$, i.e., $\alpha<\frac{1}{n+2}$; where $\mu(S_{\rho})=O(\rho^{n-1})$, from which we get (2.11).

In the proofs , we denote by c_i, a_i, $i=1,2,...$, the constants whose exact values are not important.

Lemma 2.3: Let $\Psi_N(x)$ be the eigenfunction of the Schrödinger operator L defined by (2.1)-(2.2) corresponding to the eigenvalue Λ_N . Then the following formula holds

$$(\Lambda_N-|\gamma|^2)(\Psi_N(x),\sum_{\alpha\in A_\gamma}e^{i\langle\alpha,x\rangle})=(\Psi_N(x),q(x)\sum_{\alpha\in A_\gamma}e^{i\langle\alpha,x\rangle}) \qquad 2.13$$

where (.,.) is the inner product in $L_2(F)$.

Proof: The eigenfunction $\Psi_N(x)$ and the corresponding eigenvalue Λ_N satisfy the equation

$$-\Delta\Psi_N(x)+q(x)\Psi_N(x)=\Lambda_N\Psi_N(x) \qquad 2.14$$

If we multiply both sides of the equation (2.14) by $\sum_{\alpha\in A_\gamma}e^{i\langle\alpha,x\rangle}$, we have

$$(-\Delta\Psi_N(x)+q(x)\Psi_N(x),\sum_{\alpha\in A_\gamma}e^{i\langle\alpha,x\rangle})=(\Lambda_N\Psi_N(x),\sum_{\alpha\in A_\gamma}e^{i\langle\alpha,x\rangle})$$

$$(-\Delta\Psi_N(x),\sum_{\alpha\in A_\gamma}e^{i\langle\alpha,x\rangle})+(q(x)\Psi_N(x),\sum_{\alpha\in A_\gamma}e^{i\langle\alpha,x\rangle})=\Lambda_N(\Psi_N(x),\sum_{\alpha\in A_\gamma}e^{i\langle\alpha,x\rangle})$$

$$(\Psi_N(x), -\Delta\sum_{\alpha\in A_\gamma} e^{i\langle\alpha,x\rangle}) + (\Psi_N(x), q(x)\sum_{\alpha\in A_\gamma} e^{i\langle\alpha,x\rangle}) = \Lambda_N(\Psi_N(x), \sum_{\alpha\in A_\gamma} e^{i\langle\alpha,x\rangle})$$

$$(\Psi_N(x), |\gamma|^2\sum_{\alpha\in A_\gamma} e^{i\langle\alpha,x\rangle}) + (\Psi_N(x), q(x)\sum_{\alpha\in A_\gamma} e^{i\langle\alpha,x\rangle}) = \Lambda_N(\Psi_N(x), \sum_{\alpha\in A_\gamma} e^{i\langle\alpha,x\rangle})$$

$$(\Lambda_N - |\gamma|^2)(\Psi_N(x), \sum_{\alpha\in A_\gamma} e^{i\langle\alpha,x\rangle}) = (\Psi_N(x), q(x)\sum_{\alpha\in A_\gamma} e^{i\langle\alpha,x\rangle})$$

□

Lemma 2.4: Let $|\gamma|^2$ be the eigenvalue of the operator L_0 of order ρ^2. Then there is N such that $\left|\Lambda_N - |\gamma|^2\right| \le 2M$ and $\left|(\Psi_N(x), \sum_{\alpha\in A_\gamma} e^{i\langle\alpha,x\rangle})\right| > c_1\rho^{-n/2}$, where M is the number defined in (2.10).

Proof: We know that the set of eigenfunctions $\{\Psi_N(x)\}$ of the self-adjoint operator L forms a complete system in $L_2(F)$ and by Paseval's equality, we have

$$\left.\begin{aligned} &\left\|\sum_{\alpha\in A_\gamma} e^{i\langle\alpha,x\rangle}\right\|^2 = \sum_N \left|(\Psi_N(x), \sum_{\alpha\in A_\gamma} e^{i\langle\alpha,x\rangle})\right|^2 = 1 \\ &\sum_N \left|(\Psi_N, \sum_{\alpha\in A_\gamma} e^{i\langle\alpha,x\rangle})\right|^2 = \sum_{N:\left|\Lambda_N-|\gamma|^2\right|>2M} \left|(\Psi_N, \sum_{\alpha\in A_\gamma} e^{i\langle\alpha,x\rangle})\right|^2 + \sum_{N:\left|\Lambda_N-|\gamma|^2\right|\le 2M} \left|(\Psi_N, \sum_{\alpha\in A_\gamma} e^{i\langle\alpha,x\rangle})\right|^2 \end{aligned}\right\} \quad (2.15)$$

Using the equation (2.13) we have

$$\sum_{N:\left|\Lambda_N-|\gamma|^2\right|>2M} \left|(\Psi_N(x), \sum_{\alpha\in A_\gamma} e^{i\langle\alpha,x\rangle})\right|^2 = \sum_{N:\left|\Lambda_N-|\gamma|^2\right|>2M} \left|(\Psi_N(x), q(x)\sum_{\alpha\in A_\gamma} e^{i\langle\alpha,x\rangle})\right|^2 \frac{1}{\left|\Lambda_N - |\gamma|^2\right|^2} \le$$

$$\le \frac{1}{4M^2} \sum_{N:\left|\Lambda_N-|\gamma|^2\right|>2M} \left|(\Psi_N(x), q(x)\sum_{\alpha\in A_\gamma} e^{i\langle\alpha,x\rangle})\right|^2 \le$$

$$\le \frac{1}{4M^2} \sum_{N:\left|\Lambda_N-|\gamma|^2\right|>2M} \|q(x)\|^2 \|\Psi_N(x)\|^2 \left\|\sum_{\alpha\in A_\gamma} e^{i\langle\alpha,x\rangle}\right\|^2 \le \frac{1}{4}$$

then by (2.15), we have ;

$$\sum_{N:\left|\Lambda_N-|\gamma|^2\right|\le 2M}\left|(\Psi_N(x),\sum_{\alpha\in A_\gamma}e^{i\langle\alpha,x\rangle})\right|^2>\frac{3}{4} \qquad 2.16$$

The number of the eigenvalues Λ_N of the operator L in the interval $I=[|\gamma|^2-2M,|\gamma|^2+2M]$ is less than $c_2\rho^n$. Because the number of elements w of $\frac{\Gamma}{2}$, satisfying $|w|<c\rho^\alpha$ is less than $c\rho^\alpha$. Indeed it comes from the following:

a) The number of eigenvalue $|w|^2$ of L_0, lying in $[|\gamma|^2-3M,|\gamma|^2+3M]$, is less than $c\rho^\alpha$.

b) By general perturbation theory, the N-th eigenvalue of L_t lies in the M-neighbourhood of the N-th eigenvalue of L_t^0.

Using this fact and (2.16), we have; $\exists N\in I$ such that

$$\frac{3}{4}<\sum_{N:\left|\Lambda_N-|\gamma|^2\right|\le 2M}\left|(\Psi_N(x),\sum_{\alpha\in A_\gamma}e^{i\langle\alpha,x\rangle})\right|^2<c_2\rho^n\left|(\Psi_N(x),\sum_{\alpha\in A_\gamma}e^{i\langle\alpha,x\rangle})\right|^2$$

$$\Rightarrow\left|(\Psi_N(x),\sum_{\alpha\in A_\gamma}e^{i\langle\alpha,x\rangle})\right|>c_1\rho^{-\frac{n}{2}}$$

Lemma is proved. □

Lemma 2.5: Let $\gamma\in U(\rho^\alpha,m)$, i.e. $|\gamma|^2$ be the non-resonance eigenvalue of L^0 and Λ_N be the eigenvalue of $L(q)$ lying in the interval $I=[|\gamma|^2-2M,|\gamma|^2+2M]$ then $\left|\Lambda_N-|\gamma+b|^2\right|>\frac{1}{2}\rho^\alpha$ for all $b\in\Gamma(m\rho^\alpha)$.

Proof: If $\gamma\in U(\rho^\alpha,m)$ then $\forall b\in\Gamma(m\rho^\alpha)$ and we have the following inequality

$$\left||\gamma|^2-|\gamma+b|^2\right|\ge\rho^\alpha$$

which implies , together with the fact that $\Lambda_N\in I$;

$$\left|\Lambda_N-|\gamma+b|^2\right|=\left|\Lambda_N-|\gamma+b|^2+|\gamma|^2-|\gamma|^2\right|\ge\left|\left||\gamma|^2-|\gamma+b|^2\right|-\left|\Lambda_N-|\gamma|^2\right|\right|\ge\left|\rho^\alpha-2M\right|$$

where ρ^{α} is sufficiently large so that

$$\left|\Lambda_N - |\gamma + b|^2\right| \geq \frac{1}{2}\rho^{\alpha}$$

□

Theorem 2.6 : Let $\gamma \in U(\rho^{\alpha}, m)$; i.e. $|\gamma|^2$ be non-resonance eigenvalue of the operator L^0. Then there exists an eigenvalue Λ_N of the operator L satisfying the following formula :

$$\Lambda_N = |\gamma|^2 + O(\rho^{-\alpha}) \qquad 2.17$$

Proof: By lemma 2.4, there is an index N such that $\left|\Lambda_N - |\gamma|^2\right| \leq 2M$ and $\left|(\Psi_N(x), \sum_{\alpha \in A_\gamma} e^{i\langle \alpha, x\rangle})\right| > c_1 \rho^{-n/2}$. We prove that this eigenvalue satisfies (2.17). Substituting the decomposition (2.8) of the potential $q(x)$ in (2.13) we have;

$$(\Lambda_N - |\gamma|^2)(\Psi_N(x), \sum_{\alpha \in A_\gamma} e^{i\langle \alpha, x\rangle}) = (\Psi_N(x), \sum_{\gamma_1 \in \Gamma(\rho^{\alpha})} q_{\gamma_1} \sum_{\beta_1 \in A_{\gamma_1}} e^{i\langle \beta_1, x\rangle} \sum_{\alpha \in A_\gamma} e^{i\langle \alpha, x\rangle}) + O(\rho^{-l\alpha})$$

$$= \sum_{\gamma_1 \in \Gamma(\rho^{\alpha})} q_{\gamma_1} (\Psi_N(x), \sum_{\beta_1 \in A_{\gamma_1}} e^{i\langle \beta_1, x\rangle} \sum_{\alpha \in A_\gamma} e^{i\langle \alpha, x\rangle}) + O(\rho^{-l\alpha})$$

Using lemma 2.2 , we get

$$(\Lambda_N - |\gamma|^2)(\Psi_N(x), \sum_{\alpha \in A_\gamma} e^{i\langle \alpha, x\rangle}) = \sum_{\gamma_1 \in \Gamma(\rho^{\alpha})} q_{\gamma_1} (\Psi_N(x), \sum_{\beta_1 \in A_{\gamma_1}} \sum_{\alpha \in A_{\gamma+\beta_1}} e^{i\langle \alpha, x\rangle}) + O(\rho^{-l\alpha})$$

$$= \sum_{\gamma_1 \in \Gamma(\rho^{\alpha})} \sum_{\beta_1 \in A_{\gamma_1}} q_{\gamma_1} (\Psi_N(x), \sum_{\alpha \in A_{\gamma+\beta_1}} e^{i\langle \alpha, x\rangle}) + O(\rho^{-l\alpha}),$$

since $\gamma + \beta_1 \in \frac{\Gamma}{2}$, i.e.; $|\gamma + \beta_1|^2$ is an eigenvalue of the operator L^0 with the corresponding eigenfunction $\sum_{\alpha \in A_{\gamma+\beta_1}} e^{i\langle \alpha, x\rangle}$, we can use (2.13) in the last equation and we get ;

$$(\Lambda_N - |\gamma|^2)(\Psi_N(x), \sum_{\alpha \in A_\gamma} e^{i\langle \alpha, x\rangle}) = \sum_{\gamma_1 \in \Gamma(\rho^{\alpha})} \sum_{\beta_1 \in A_{\gamma_1}} q_{\gamma_1} \frac{(\Psi_N(x), q(x) \sum_{\alpha \in A_{\gamma+\beta_1}} e^{i\langle \alpha, x\rangle})}{\Lambda_N - |\gamma + \beta_1|^2} + O(\rho^{-l\alpha}),$$

Here, by lemma 2.5, for the denominator we have estimation

$$\left|\Lambda_N - |\gamma+\beta_1|^2\right| > \frac{1}{2}\rho^\alpha, \qquad 2.18$$

because $|\beta_1| = |\gamma_1|$ and $\gamma_1 \in \Gamma(\rho^\alpha)$, i.e., $\beta_1 \in \Gamma(\rho^\alpha)$. Again sustituting the decomposition of $q(x)$ in the above equation and using $\beta_1 \in \Gamma(\rho^\alpha)$ we obtain ;

$$(\Lambda_N - |\gamma|^2)(\Psi_N(x), \sum_{\alpha\in A_\gamma} e^{i\langle\alpha,x\rangle}) =$$

$$= \sum_{\gamma_1\in\Gamma(\rho^\alpha)}\sum_{\beta_1\in A_{\gamma_1}} q_{\gamma_1} \frac{(\Psi_N(x), \sum_{\gamma_2\in\Gamma(\rho^\alpha)} q_{\gamma_2} \sum_{\alpha\in A_{\gamma_2}} e^{i\langle\alpha,x\rangle} \sum_{\alpha\in A_{\gamma+\beta_1}} e^{i\langle\alpha,x\rangle}) + O(\rho^{-l\alpha})}{\Lambda_N - |\gamma+\beta_1|^2} + O(\rho^{-l\alpha})$$

$$= \sum_{\gamma_1\in\Gamma(\rho^\alpha)}\sum_{\beta_1\in A_{\gamma_1}} q_{\gamma_1} \frac{(\Psi_N(x), \sum_{\gamma_2\in\Gamma(\rho^\alpha)} q_{\gamma_2} \sum_{\alpha\in A_{\gamma_2}} e^{i\langle\alpha,x\rangle} \sum_{\alpha\in A_{\gamma+\beta_1}} e^{i\langle\alpha,x\rangle})}{\Lambda_N - |\gamma+\beta_1|^2} + O(\rho^{-l\alpha})$$

$$= \sum_{\gamma_1,\gamma_2\in\Gamma(\rho^\alpha)}\sum_{\beta_1\in A_{\gamma_1}} q_{\gamma_1}q_{\gamma_2} \frac{(\Psi_N(x), \sum_{\alpha\in A_{\gamma_2}} e^{i\langle\alpha,x\rangle} \sum_{\alpha\in A_{\gamma+\beta_1}} e^{i\langle\alpha,x\rangle})}{\Lambda_N - |\gamma+\beta_1|^2} + O(\rho^{-l\alpha})$$

$$= \sum_{\gamma_1,\gamma_2\in\Gamma(\rho^\alpha)}\sum_{\substack{\beta_1\in A_{\gamma_1}\\ \beta_2\in A_{\gamma_2}}} q_{\gamma_1}q_{\gamma_2} \frac{(\Psi_N(x), \sum_{\alpha\in A_{\gamma+\beta_1+\beta_2}} e^{i\langle\alpha,x\rangle})}{\Lambda_N - |\gamma+\beta_1|^2} + O(\rho^{-l\alpha}),$$

if the terms with coefficient $(\Psi_N(x), \sum_{\alpha\in A_\gamma} e^{i\langle\alpha,x\rangle})$ are isolated , we obtain

$$\left.\begin{aligned}(\Lambda_N - |\gamma|^2)(\Psi_N(x), \sum_{\alpha\in A_\gamma} e^{i\langle\alpha,x\rangle}) &= \sum_{\gamma_1,\gamma_2\in\Gamma(\rho^\alpha)}\sum_{\substack{\beta_2=-\beta_1\\ \beta_1\in A_{\gamma_1}\\ \beta_2\in A_{\gamma_2}}} q_{\gamma_1}q_{\gamma_2} \frac{(\Psi_N(x), \sum_{\alpha\in A_\gamma} e^{i\langle\alpha,x\rangle})}{\Lambda_N - |\gamma+\beta_1|^2}\\ &+ \sum_{\gamma_1,\gamma_2\in\Gamma(\rho^\alpha)}\sum_{\substack{\beta_2\neq-\beta_1\\ \beta_1\in A_{\gamma_1}\\ \beta_2\in A_{\gamma_2}}} q_{\gamma_1}q_{\gamma_2} \frac{(\Psi_N(x), \sum_{\alpha\in A_{\gamma+\beta_1+\beta_2}} e^{i\langle\alpha,x\rangle})}{\Lambda_N - |\gamma+\beta_1|^2} + O(\rho^{-m\alpha})\end{aligned}\right\} \qquad 2.19$$

by using lemma 2.3 in the last term of the equation (2.19) , we get

$$(\Lambda_N - |\gamma|^2)(\Psi_N(x), \sum_{\alpha\in A_\gamma} e^{i\langle\alpha,x\rangle}) = \sum_{\gamma_1,\gamma_2\in\Gamma(\rho^\alpha)}\sum_{\substack{\beta_2=-\beta_1\\ \beta_1\in A_{\gamma_1}\\ \beta_2\in A_{\gamma_2}}} q_{\gamma_1}q_{\gamma_2} \frac{(\Psi_N(x), \sum_{\alpha\in A_\gamma} e^{i\langle\alpha,x\rangle})}{\Lambda_N - |\gamma+\beta_1|^2}$$

$$+\sum_{\gamma_1,\gamma_2\in\Gamma(\rho^{\alpha})}\sum_{\substack{\beta_2\neq-\beta_1\\ \beta_1\in A_{\gamma_1}\\ \beta_2\in A_{\gamma_2}}} q_{\gamma_1}q_{\gamma_2}\frac{(\Psi_N(x),q(x)\sum_{\alpha\in A_{\gamma+\beta_1+\beta_2}}e^{i\langle\alpha,x\rangle})+O(\rho^{-l\alpha})}{(\Lambda_N-|\gamma+\beta_1|^2)(\Lambda_N-|\gamma+\beta_1+\beta_2|^2)}+O(\rho^{-l\alpha})$$

Since $\beta_1+\beta_2\in\Gamma(2\rho^{\alpha})$, by lemma (2.5), the estimation

$$\left|\Lambda_N-|\gamma+\beta_1+\beta_2|^2\right|>\frac{1}{2}\rho^{\alpha} \qquad 2.20$$

holds.Using (2.18) and (2.20), substituting the expansion of $q(x)$ into the last equation and isolating the coefficient of $\sum_{\alpha\in A_{\gamma}}e^{i\langle\alpha,x\rangle}$, we have;

$$(\Lambda_N-|\gamma|^2)(\Psi_N(x),\sum_{\alpha\in A_{\gamma}}e^{i\langle\alpha,x\rangle})=\sum_{\gamma_1,\gamma_2\in\Gamma(\rho^{\alpha})}\sum_{\substack{\beta_2=-\beta_1\\ \beta_1\in A_{\gamma_1}\\ \beta_2\in A_{\gamma_2}}} q_{\gamma_1}q_{\gamma_2}\frac{(\Psi_N(x),\sum_{\alpha\in A_{\gamma}}e^{i\langle\alpha,x\rangle})}{\Lambda_N-|\gamma+\beta_1|^2}$$

$$+\sum_{\gamma_1,\gamma_2\in\Gamma(\rho^{\alpha})}\sum_{\substack{\beta_2\neq-\beta_1\\ \beta_1\in A_{\gamma_1}\\ \beta_2\in A_{\gamma_2}}} q_{\gamma_1}q_{\gamma_2}\frac{(\Psi_N(x),\sum_{\gamma_3\in\Gamma(\rho^{\alpha})}q_{\gamma_3}\sum_{\alpha\in A_{\gamma_3}}e^{i\langle\alpha,x\rangle}\sum_{\alpha\in A_{\gamma+\beta_1+\beta_2}}e^{i\langle\alpha,x\rangle})+O(\rho^{-l\alpha})}{(\Lambda_N-|\gamma+\beta_1|^2)(\Lambda_N-|\gamma+\beta_1+\beta_2|^2)}+O(\rho^{-l\alpha})$$

$$=\sum_{\gamma_1,\gamma_2\in\Gamma(\rho^{\alpha})}\sum_{\substack{\beta_2=-\beta_1\\ \beta_1\in A_{\gamma_1}\\ \beta_2\in A_{\gamma_2}}} q_{\gamma_1}q_{\gamma_2}\frac{(\Psi_N(x),\sum_{\alpha\in A_{\gamma}}e^{i\langle\alpha,x\rangle})}{\Lambda_N-|\gamma+\beta_1|^2}$$

$$+\sum_{\gamma_1,\gamma_2,\gamma_3\in\Gamma(\rho^{\alpha})}\sum_{\substack{\beta_2\neq-\beta_1\\ \beta_1\in A_{\gamma_1}\\ \beta_2\in A_{\gamma_2}\\ \beta_3\in A_{\gamma_3}}} q_{\gamma_1}q_{\gamma_2}q_{\gamma_3}\frac{(\Psi_N(x),\sum_{\alpha\in A_{\gamma+\beta_1+\beta_2+\beta_3}}e^{i\langle\alpha,x\rangle})}{(\Lambda_N-|\gamma+\beta_1|^2)(\Lambda_N-|\gamma+\beta_1+\beta_2|^2)}+O(\rho^{-l\alpha})$$

$$=\sum_{\gamma_1,\gamma_2\in\Gamma(\rho^{\alpha})}\sum_{\substack{\beta_2=-\beta_1\\ \beta_1\in A_{\gamma_1}\\ \beta_2\in A_{\gamma_2}}} q_{\gamma_1}q_{\gamma_2}\frac{(\Psi_N(x),\sum_{\alpha\in A_{\gamma}}e^{i\langle\alpha,x\rangle})}{\Lambda_N-|\gamma+\beta_1|^2}$$

$$+\sum_{\gamma_1,\gamma_2,\gamma_3\in\Gamma(\rho^{\alpha})}\sum_{\substack{\beta_3=-(\beta_1+\beta_2)\\ \beta_1\in A_{\gamma_1},\beta_2\in A_{\gamma_2}\\ \beta_3\in A_{\gamma_3}}} q_{\gamma_1}q_{\gamma_2}q_{\gamma_3}\frac{(\Psi_N(x),\sum_{\alpha\in A_{\gamma}}e^{i\langle\alpha,x\rangle})}{(\Lambda_N-|\gamma+\beta_1|^2)(\Lambda_N-|\gamma+\beta_1+\beta_2|^2)}$$

$$+\sum_{\substack{\gamma_1,\gamma_2,\gamma_3\in\Gamma(\rho^\alpha)}}\ \sum_{\substack{\beta_3\neq-(\beta_1+\beta_2)\\ \beta_1\in A_{\gamma_1},\beta_2\in A_{\gamma_2}\\ \beta_3\in A_{\gamma_3}}} q_{\gamma_1}q_{\gamma_2}q_{\gamma_3}\frac{(\Psi_N(x),\ \sum_{\alpha\in A_{\gamma+\beta_1+\beta_2+\beta_3}} e^{i\langle\alpha,x\rangle})}{(\Lambda_N-|\gamma+\beta_1|^2)(\Lambda_N-|\gamma+\beta_1+\beta_2|^2)}+O(\rho^{-l\alpha})$$

$$=\left(\sum_{\gamma_1,\gamma_2\in\Gamma(\rho^\alpha)}\ \sum_{\substack{\beta_2=-\beta_1\\ \beta_1\in A_{\gamma_1}\\ \beta_2\in A_{\gamma_2}}}\frac{q_{\gamma_1}q_{\gamma_2}}{\Lambda_N-|\gamma+\beta_1|^2}\right.$$

$$\left.+\sum_{\gamma_1,\gamma_2,\gamma_3\in\Gamma(\rho^\alpha)}\ \sum_{\substack{\beta_3=-(\beta_1+\beta_2)\\ \beta_1\in A_{\gamma_1},\beta_2\in A_{\gamma_2}\\ \beta_3\in A_{\gamma_3}}}\frac{q_{\gamma_1}q_{\gamma_2}q_{\gamma_3}}{(\Lambda_N-|\gamma+\beta_1|^2)(\Lambda_N-|\gamma+\beta_1+\beta_2|^2)}\right)(\Psi_N(x),\sum_{\alpha\in A_\gamma}e^{i\langle\alpha,x\rangle})$$

$$+\sum_{\gamma_1,\gamma_2,\gamma_3\in\Gamma(\rho^\alpha)}\ \sum_{\substack{\beta_3\neq-(\beta_1+\beta_2)\\ \beta_1\in A_{\gamma_1},\beta_2\in A_{\gamma_2}\\ \beta_3\in A_{\gamma_3}}} q_{\gamma_1}q_{\gamma_2}q_{\gamma_3}\frac{(\Psi_N(x),\ \sum_{\alpha\in A_{\gamma+\beta_1+\beta_2+\beta_3}} e^{i\langle\alpha,x\rangle})}{(\Lambda_N-|\gamma+\beta_1|^2)(\Lambda_N-|\gamma+\beta_1+\beta_2|^2)}+O(\rho^{-l\alpha})$$

By the same method as above, iterating m times we obtain the following formula:

$$(\Lambda_N-|\gamma|^2)(\Psi_N(x),\sum_{\alpha\in A_\gamma}e^{i\langle\alpha,x\rangle})=\left(\sum_{i=1}^{l}S_i\right)(\Psi_N(x),\sum_{\alpha\in A_\gamma}e^{i\langle\alpha,x\rangle})+C_l+O(\rho^{-l\alpha})\qquad 2.21$$

where

$$S_i=\sum_{\substack{\gamma_1,\gamma_2,\dots,\gamma_{i+1}\\ \in\Gamma(\rho^\alpha)}}\ \sum_{\substack{\beta_{i+1}=-(\beta_1+\beta_2+\dots+\beta_i)\\ \beta_1\in A_{\gamma_1},\beta_2\in A_{\gamma_2},\dots,\beta_{i+1}\in A_{\gamma_{i+1}}}}\frac{q_{\gamma_1}q_{\gamma_2}\dots q_{\gamma_{k+1}}}{(\Lambda_N-|\gamma+\beta_1|^2)\dots(\Lambda_N-|\gamma+\beta_1+\beta_2+\dots+\beta_i|^2)}\qquad 2.22$$

$$C_l=\sum_{\substack{\gamma_1,\gamma_2,\dots,\gamma_{l+1}\\ \in\Gamma(\rho^\alpha)}}\ \sum_{\substack{\beta_{l+1}\neq-(\beta_1+\beta_2+\dots+\beta_l)\\ \beta_1\in A_{\gamma_1},\beta_2\in A_{\gamma_2},\dots,\beta_{l+1}\in A_{\gamma_{l+1}}}}\frac{q_{\gamma_1}q_{\gamma_2}\dots q_{\gamma_{l+1}}(\Psi_N(x),\ \sum_{\alpha\in A_{\gamma+\beta_1+\dots+\beta_{l+1}}}e^{i\langle\alpha,x\rangle})}{(\Lambda_N-|\gamma+\beta_1|^2)\dots(\Lambda_N-|\gamma+\beta_1+\beta_2+\dots+\beta_l|^2)}\qquad 2.23$$

Now to obtain the formula (2.17), let us calculate the order of S_i and C_m

$$|S_i|\leq\sum_{\substack{\gamma_1,\gamma_2,\dots,\gamma_{i+1}\\ \in\Gamma(\rho^\alpha)}}\ \sum_{\substack{\beta_{i+1}=-(\beta_1+\beta_2+\dots+\beta_i)\\ \beta_1\in A_{\gamma_1},\beta_2\in A_{\gamma_2},\dots,\beta_{i+1}\in A_{\gamma_{i+1}}}}\left|\frac{q_{\gamma_1}q_{\gamma_2}\dots q_{\gamma_{k+1}}}{(\Lambda_N-|\gamma+\beta_1|^2)\dots(\Lambda_N-|\gamma+\beta_1+\beta_2+\dots+\beta_i|^2)}\right|$$

$$\leq \sum_{\substack{\gamma_1,\gamma_2,\ldots,\gamma_{l+1} \\ \in\Gamma(\rho^\alpha)}} \sum_{\substack{\beta_{l+1}=-(\beta_1+\beta_2+\ldots+\beta_l) \\ \beta_1\in A_{\gamma_1},\beta_2\in A_{\gamma_2},\ldots,\beta_{l+1}\in A_{\gamma_{l+1}}}} \frac{|q_{\gamma_1}||q_{\gamma_2}|\ldots|q_{\gamma_{l+1}}|}{\left|\Lambda_N-|\gamma+\beta_1|^2\right|\ldots\left|\Lambda_N-|\gamma+\beta_1+\beta_2+\ldots+\beta_i|^2\right|}$$

$\forall i=1,2,\ldots,m$, $\gamma_i\in\Gamma(\rho^\alpha)$ and $\beta_i\in A_{\gamma_i}$ $\Rightarrow$ $|\gamma_i|=|\beta_i|<\rho^\alpha$ and $|\beta_1+\beta_2+\ldots+\beta_i|<m\rho^\alpha$ $\forall i=1,2,\ldots,m$, so we can use lemma 2.5 and the equation (2.10) which defines the number M, and we have

$$|S_i|\leq\frac{M^{i+1}}{\frac{1}{2^i}\rho^{i\alpha}}=a_i\rho^{-i\alpha} \qquad \forall i=1,2,\ldots,l$$

which implies $S_i=O(\rho^{-i\alpha})$ $\forall i=1,2,\ldots,l$ and

$$\left|\sum_{i=1}^{l}S_i\right|\leq a_1\rho^{-\alpha}+a_2\rho^{-2\alpha}+\ldots+a_l\rho^{-l\alpha}\leq\max_{i=1,2,\ldots,l}\{a_i\}\ l\rho^{-\alpha}$$

$$\sum_{i=1}^{l}S_i=O(\rho^{-\alpha}) \qquad 2.24$$

$$\|C_l\|\leq \sum_{\substack{\gamma_1,\gamma_2,\ldots,\gamma_{l+1} \\ \in\Gamma(\rho^\alpha)}} \sum_{\substack{\beta_{l+1}\neq-(\beta_1+\beta_2+\ldots+\beta_l) \\ \beta_1\in A_{\gamma_1},\beta_2\in A_{\gamma_2},\ldots,\beta_{l+1}\in A_{\gamma_{l+1}}}} \left|\frac{q_{\gamma_1}q_{\gamma_2}\ldots q_{\gamma_{l+1}}(\Psi_N(x),\ \sum_{\alpha\in A_{\gamma+\beta_1+\ldots+\beta_{l+1}}}e^{i\langle\alpha,x\rangle})}{(\Lambda_N-|\gamma+\beta_1|^2)\ldots(\Lambda_N-|\gamma+\beta_1+\beta_2+\ldots+\beta_l|^2)}\right|$$

$$\leq \sum_{\substack{\gamma_1,\gamma_2,\ldots,\gamma_{l+1} \\ \in\Gamma(\rho^\alpha)}} \sum_{\substack{\beta_{l+1}\neq-(\beta_1+\beta_2+\ldots+\beta_l) \\ \beta_1\in A_{\gamma_1},\beta_2\in A_{\gamma_2},\ldots,\beta_{l+1}\in A_{\gamma_{l+1}}}} \frac{|q_{\gamma_1}|\ldots|q_{\gamma_{l+1}}|\left\|(\Psi_N(x),\ \sum_{\alpha\in A_{\gamma+\beta_1+\ldots+\beta_{l+1}}}e^{i\langle\alpha,x\rangle})\right\|}{\left|\Lambda_N-|\gamma+\beta_1|^2\right|\ldots\left|\Lambda_N-|\gamma+\beta_1+\beta_2+\ldots+\beta_l|^2\right|}$$

by using lemma 2.5 and $\left\|(\Psi_N(x),\ \sum_{\alpha\in A_{\gamma+\beta_1+\ldots+\beta_{l+1}}}e^{i\langle\alpha,x\rangle})\right\|=1$, we have; $\|C_l\|\leq\dfrac{M^{l+1}}{\frac{1}{2^l}\rho^{-l\alpha}}$ and

$$C_l=O(\rho^{-l\alpha}) \qquad 2.25$$

If we substitute (2.24) and (2.25) into the equation (2.21), we get

$$(\Lambda_N-|\gamma|^2)(\Psi_N(x),\sum_{\alpha\in A_\gamma}e^{i\langle\alpha,x\rangle})=O(\rho^{-\alpha})(\Psi_N(x),\sum_{\alpha\in A_\gamma}e^{i\langle\alpha,x\rangle})+O(\rho^{-l\alpha}) \qquad 2.26$$

dividing both side of the equation (2.26) by $(\Psi_N(x),\sum_{\alpha\in A_\gamma}e^{i\langle\alpha,x\rangle})$, we get

$$\Lambda_N-|\gamma|^2=O(\rho^{-\alpha})+\frac{O(\rho^{-l\alpha})}{(\Psi_N(x),\sum_{\alpha\in A_\gamma}e^{i\langle\alpha,x\rangle})} \qquad 2.27$$

by lemma 2.4 $\left|(\Psi_N(x), \sum_{\alpha\in A_\gamma} e^{i\langle\alpha,x\rangle})\right| > c_1\rho^{-n/2}$ which implies

$$\left|\frac{O(\rho^{-l\alpha})}{(\Psi_N(x), \sum_{\alpha\in A_\gamma} e^{i\langle\alpha,x\rangle})}\right| \le \frac{\rho^{-l\alpha}}{c_1\rho^{-n/2}} = c_3\rho^{-(l\alpha-n/2)} \qquad 2.28$$

since m is the iteration number , we can choose m to make $(m-n/2)>0$ so that

$$\frac{O(\rho^{-l\alpha})}{(\Psi_N(x), \sum_{\alpha\in A_\gamma} e^{i\langle\alpha,x\rangle})} = O(\rho^{-\alpha}) \qquad 2.29$$

substituting (2.29) into (2.27), we obtain the desired result. □

Let $d=\left[\frac{n}{2\alpha}\right]+1$, where $\left[\frac{n}{2\alpha}\right]$ is the integer part of $\frac{n}{2\alpha}$. Then $\rho^{-(m\alpha-n/2)} \le \rho^{-(m-d)\alpha}$ so instead of (2.28) we can write

$$\left|\frac{O(\rho^{-l\alpha})}{(\Psi_N(x), \sum_{\alpha\in A_\gamma} e^{i\langle\alpha,x\rangle})}\right| \le c_3\rho^{-(l-d)\alpha} \Rightarrow \frac{O(\rho^{-l\alpha})}{(\Psi_N(x), \sum_{\alpha\in A_\gamma} e^{i\langle\alpha,x\rangle})} = O(\rho^{-(l-d)\alpha}) \qquad 2.30$$

Let us denote the term S_i in equation (2.22) as follows:

$$S_i(\Lambda_N) = \sum_{\substack{\gamma_1,\gamma_2,\dots,\gamma_{i+1} \\ \in\Gamma(\rho^\alpha)}} \sum_{\substack{\beta_{i+1}=-(\beta_1+\beta_2+\dots+\beta_i) \\ \beta_1\in A_{\gamma_1},\beta_2\in A_{\gamma_2},\dots,\beta_{i+1}\in A_{\gamma_{i+1}}}} \frac{q_{\gamma_1}q_{\gamma_2}\cdots q_{\gamma_{i+1}}}{(\Lambda_N-|\gamma+\beta_1|^2)\dots(\Lambda_N-|\gamma+\beta_1+\beta_2+\dots+\beta_i|^2)}$$

2.31

Theorem 2.7: Let $\gamma\in U(\rho^\alpha,m)$ then there is an eigenvalue Λ_N of the operator L satisfying the formulas

$$\Lambda_N = |\gamma|^2 + F_{k-1} + O(\rho^{-k\alpha}), \quad \forall k=1,2,\dots,l-d \qquad 2.32$$

where

$$F_0=0,\ F_1=\sum_{\gamma_1\in\Gamma(\rho^\alpha)}\sum_{\beta\in A_{\gamma_1}} \frac{|q_{\gamma_1}|^2}{|\gamma|^2-|\gamma-\beta_1|^2},\ F_s=\sum_{i=1}^{m} S_i(|\gamma|^2+F_{s-1}),\ s=2,3,\dots,l \qquad 2.33$$

Proof: We prove that for the eigenvalue Λ_N satisfying the formula (2.17) the formulas (2.32) hold. Let us prove it by mathematical induction on k:

for $k=1$; by theorem 2.6, Λ_N satisfy the equation $\Lambda_N = |\gamma|^2 + F_0 + O(\rho^{-\alpha})$,

where $F_0 = 0$,

for $k=j$; assume that it is true, i.e.

$$\Lambda_N = |\gamma|^2 + F_{j-1} + O(\rho^{-j\alpha}) \qquad 2.34$$

for $k=j+1$; we must prove that

$$\Lambda_N = |\gamma|^2 + F_j + O(\rho^{-(j+1)\alpha})$$

To prove this we put the expression (2.34) into $S_i(\Lambda_N)$ in the equation (2.31) and divide both side of (2.21) by $(\Psi_N(x), \sum_{\alpha\in A_\gamma} e^{i\langle\alpha,x\rangle})$ then by using (2.30), we get

$$\Lambda_N = |\gamma|^2 + \sum_{i=1}^{l} S_i(|\gamma|^2 + F_{j-1} + O(\rho^{-j\alpha})) + O(\rho^{-(l-d)\alpha}) \qquad 2.35$$

adding and subtracting the term $\sum_{i=1}^{l} S_i(|\gamma|^2 + F_{j-1})$ in (2.35), we have

$$\Lambda_N = |\gamma|^2 + \sum_{i=1}^{l} S_i(|\gamma|^2 + F_{j-1} + O(\rho^{-j\alpha})) \pm \sum_{i=1}^{l} S_i(|\gamma|^2 + F_{j-1}) + O(\rho^{-(l-d)\alpha})$$

$$= |\gamma|^2 + \left[\sum_{i=1}^{l} S_i(|\gamma|^2 + F_{j-1} + O(\rho^{-j\alpha})) - S_i(|\gamma|^2 + F_{j-1})\right] + \sum_{i=1}^{l} S_i(|\gamma|^2 + F_{j-1}) + O(\rho^{-(l-d)\alpha})$$

2.36

by (2.33) $\sum_{i=1}^{l} S_i(|\gamma|^2 + F_{j-1}) = F_j$, so we need only to show that the expression in square bracket is equal to $O(\rho^{-(j+1)})$.

In order to understand the calculations clearly, we first calculate the orders of $S_1(|\gamma|^2 + F_{j-1} + O(\rho^{-j\alpha})) - S_1(|\gamma|^2 + F_{j-1})$, $S_2(|\gamma|^2 + F_{j-1} + O(\rho^{-j\alpha})) - S_2(|\gamma|^2 + F_{j-1})$ and $S_3(|\gamma|^2 + F_{j-1} + O(\rho^{-j\alpha})) - S_3(|\gamma|^2 + F_{j-1})$:

$S_1(|\gamma|^2 + F_{j-1} + O(\rho^{-\alpha})) - S_1(|\gamma|^2 + F_{j-1}) =$

$$= \frac{q_{\gamma_1} q_{\gamma_2}}{|\gamma|^2 + F_{j-1} + O(\rho^{-\alpha}) - |\gamma + \beta_1|^2} - \frac{q_{\gamma_1} q_{\gamma_2}}{|\gamma|^2 + F_{j-1} - |\gamma + \beta_1|^2}$$

equalizing the denominator, we get

$$= q_{\gamma_1} q_{\gamma_2} \left[\frac{(|\gamma|^2 + F_{j-1} - |\gamma + \beta_1|^2) - (|\gamma|^2 + F_{j-1} + O(\rho^{-j\alpha}) - |\gamma + \beta_1|^2)}{(|\gamma|^2 + F_{j-1} - |\gamma + \beta_1|^2)(|\gamma|^2 + F_{j-1} + O(\rho^{-j\alpha}) - |\gamma + \beta_1|^2)} \right]$$

$$= q_{\gamma_1} q_{\gamma_2} \left[\frac{-O(\rho^{-j\alpha})}{(|\gamma|^2 + F_{j-1} - |\gamma + \beta_1|^2)(|\gamma|^2 + F_{j-1} + O(\rho^{-\alpha}) - |\gamma + \beta_1|^2)} \right]$$

Since γ is non-resonance eigenvalue, we have the inequality $\left||\gamma|^2 - |\gamma + \beta_1|^2\right| \geq \rho^{\alpha}$,

hence $\left||\gamma|^2 + F_{j-1} - |\gamma + \beta_1|^2\right| \geq \rho^{\alpha}$ and $\left||\gamma|^2 + F_{j-1} + O(\rho^{-j\alpha}) - |\gamma + \beta_1|^2\right| > \frac{1}{2}\rho^{\alpha}$ by lemma 2.5 and (2.34). Then, using (2.10) we have;

$$\left|S_1(|\gamma|^2 + F_{j-1} + O(\rho^{-j\alpha})) - S_1(|\gamma|^2 + F_{j-1})\right| \leq c_4 \frac{\rho^{-j\alpha}}{\rho^{2\alpha}} < c_4 \rho^{-(j+2)\alpha}$$

$$S_1(|\gamma|^2 + F_{j-1} + O(\rho^{-j\alpha})) - S_1(|\gamma|^2 + F_{j-1}) = O(\rho^{-(j+2)\alpha})$$

$$S_2(|\gamma|^2 + F_{j-1} + O(\rho^{-j\alpha})) - S_2(|\gamma|^2 + F_{j-1}) = q_{\gamma_1} q_{\gamma_2} q_{\gamma_3} \left[\frac{1}{(|\gamma|^2 + F_{j-1} + O(\rho^{-j\alpha}) - |\gamma + \beta_1|^2)} \right.$$

$$\left. \frac{1}{(|\gamma|^2 + F_{j-1} + O(\rho^{-j\alpha}) - |\gamma + \beta_1 + \beta_2|^2)} - \frac{1}{(|\gamma|^2 + F_{j-1} - |\gamma + \beta_1|^2)(|\gamma|^2 + F_{j-1} - |\gamma + \beta_1 + \beta_2|^2)} \right]$$

after equalizing the denominators of the above equation, we obtain

$S_2(|\gamma|^2 + F_{j-1} + O(\rho^{-j\alpha})) - S_2(|\gamma|^2 + F_{j-1}) =$

$$= q_{\gamma_1} q_{\gamma_2} q_{\gamma_3} \left[\frac{-O(\rho^{-j\alpha})(|\gamma|^2 + F_{j-1} + O(\rho^{-j\alpha}) - |\gamma + \beta_1|^2)}{(|\gamma|^2 + F_{j-1} + O(\rho^{-j\alpha}) - |\gamma + \beta_1|^2)(|\gamma|^2 + F_{j-1} + O(\rho^{-j\alpha}) - |\gamma + \beta_1 + \beta_2|^2)} \right.$$

$$\frac{1}{(|\gamma|^2 + F_{j-1} - |\gamma + \beta_1|^2)(|\gamma|^2 + F_{j-1} - |\gamma + \beta_1 + \beta_2|^2)}$$

$$-\frac{O(\rho^{-j\alpha})(|\gamma|^2+F_{j-1}-|\gamma+\beta_1+\beta_2|^2)}{(|\gamma|^2+F_{j-1}-|\gamma+\beta_1|^2)(|\gamma|^2+F_{j-1}-|\gamma+\beta_1+\beta_2|^2)}$$

$$\left.\frac{1}{(|\gamma|^2+F_{j-1}+O(\rho^{-j\alpha})-|\gamma+\beta_1|^2)(|\gamma|^2+F_{j-1}+O(\rho^{-j\alpha})-|\gamma+\beta_1+\beta_2|^2)}\right]$$

$$=q_{\gamma_1}q_{\gamma_2}q_{\gamma_3}.$$

$$\left[\frac{-O(\rho^{-j\alpha})}{(|\gamma|^2+F_{j-1}+O(\rho^{-j\alpha})-|\gamma+\beta_1+\beta_2|^2)(|\gamma|^2+F_{j-1}-|\gamma+\beta_1|^2)(|\gamma|^2+F_{j-1}-|\gamma+\beta_1+\beta_2|^2)}\right.$$

$$\left.-\frac{O(\rho^{-j\alpha})}{(|\gamma|^2+F_{j-1}-|\gamma+\beta_1|^2)(|\gamma|^2+F_{j-1}+O(\rho^{-j\alpha})-|\gamma+\beta_1|^2)(|\gamma|^2+F_{j-1}+O(\rho^{-j\alpha})-|\gamma+\beta_1+\beta_2|^2)}\right]$$

Hence we have

$$S_2(|\gamma|^2+F_{j-1}+O(\rho^{-j\alpha}))-S_2(|\gamma|^2+F_{j-1})=O(\rho^{-(j+3)\alpha})$$

$$S_3(|\gamma|^2+F_{j-1}+O(\rho^{-j\alpha}))-S_3(|\gamma|^2+F_{j-1})=\left[\frac{q_{\gamma_1}q_{\gamma_2}q_{\gamma_3}q_{\gamma_4}}{(|\gamma|^2+F_{j-1}+O(\rho^{-j\alpha})-|\gamma+\beta_1|^2)}\right.$$

$$\frac{1}{(|\gamma|^2+F_{j-1}+O(\rho^{-j\alpha})-|\gamma+\beta_1+\beta_2|^2)(|\gamma|^2+F_{j-1}+O(\rho^{-j\alpha})-|\gamma+\beta_1+\beta_2+\beta_3|^2)}$$

$$\left.-\frac{q_{\gamma_1}q_{\gamma_2}q_{\gamma_3}q_{\gamma_4}}{(|\gamma|^2+F_{j-1}-|\gamma+\beta_1|^2)(|\gamma|^2+F_{j-1}-|\gamma+\beta_1+\beta_2|^2)(|\gamma|^2+F_{j-1}-|\gamma+\beta_1+\beta_2+\beta_3|^2)}\right]$$

$$=-q_{\gamma_1}q_{\gamma_2}q_{\gamma_3}q_{\gamma_4}O(\rho^{-j\alpha})\left[\frac{(|\gamma|^2+F_{j-1}+O(\rho^{-j\alpha})-|\gamma+\beta_1+\beta_2|^2)}{(|\gamma|^2+F_{j-1}+O(\rho^{-j\alpha})-|\gamma+\beta_1+\beta_2|^2)}\right.$$

$$\frac{(|\gamma|^2+F_{j-1}+O(\rho^{-j\alpha})-|\gamma+\beta_1+\beta_2+\beta_3|^2)}{(|\gamma|^2+F_{j-1}+O(\rho^{-j\alpha})-|\gamma+\beta_1|^2)(|\gamma|^2+F_{j-1}+O(\rho^{-j\alpha})-|\gamma+\beta_1+\beta_2+\beta_3|^2)}$$

$$\frac{1}{(|\gamma|^2+F_{j-1}-|\gamma+\beta_1|^2)(|\gamma|^2+F_{j-1}-|\gamma+\beta_1+\beta_2|^2)(|\gamma|^2+F_{j-1}-|\gamma+\beta_1+\beta_2+\beta_3|^2)}$$

$$+\frac{(|\gamma|^2+F_{j-1}-|\gamma+\beta_1|^2)(|\gamma|^2+F_{j-1}-|\gamma+\beta_1+\beta_2+\beta_3|^2)}{(|\gamma|^2+F_{j-1}-|\gamma+\beta_1|^2)(|\gamma|^2+F_{j-1}-|\gamma+\beta_1+\beta_2|^2)(|\gamma|^2+F_{j-1}-|\gamma+\beta_1+\beta_2+\beta_3|^2)}$$

$$\frac{1}{(|\gamma|^2+F_{j-1}+O(\rho^{-j\alpha})-|\gamma+\beta_1|^2)(|\gamma|^2+F_{j-1}+O(\rho^{-j\alpha})-|\gamma+\beta_1+\beta_2|^2)}$$

$$\frac{1}{(|\gamma|^2+F_{j-1}+O(\rho^{-j\alpha})-|\gamma+\beta_1+\beta_2+\beta_3|^2)}+$$

$$+\frac{(|\gamma|^2+F_{j-1}-|\gamma+\beta_1|^2)(|\gamma|^2+F_{j-1}-|\gamma+\beta_1+\beta_2|^2)}{(|\gamma|^2+F_{j-1}-|\gamma+\beta_1|^2)(|\gamma|^2+F_{j-1}-|\gamma+\beta_1+\beta_2|^2)(|\gamma|^2+F_{j-1}-|\gamma+\beta_1+\beta_2+\beta_3|^2)}$$

$$\frac{1}{(|\gamma|^2+F_{j-1}+O(\rho^{-j\alpha})-|\gamma+\beta_1|^2)(|\gamma|^2+F_{j-1}+O(\rho^{-j\alpha})-|\gamma+\beta_1+\beta_2|^2)}$$

$$\left.\frac{1}{(|\gamma|^2+F_{j-1}+O(\rho^{-j\alpha})-|\gamma+\beta_1+\beta_2+\beta_3|^2)}\right]$$

simplifying the above equation and using the inequalities which are proved in this chapter, we get

$$S_3(|\gamma|^2+F_{j-1}+O(\rho^{-j\alpha}))-S_3(|\gamma|^2+F_{j-1})=O(\rho^{-(j+4)\alpha})$$

in the same way we obtain

$$S_i(|\gamma|^2+F_{j-1}+O(\rho^{-j\alpha}))-S_i(|\gamma|^2+F_{j-1})=O(\rho^{-(j+1+i)}) \qquad \forall i=1,2,\dots,l$$

since $\rho^{-(j+1+i)}<\rho^{-(j+1)}$, $\forall i=1,2,\dots,l$, we can write

$$\sum_{i=1}^{l} S_i(|\gamma|^2 + F_{j-1} + O(\rho^{-j\alpha})) - S_i(|\gamma|^2 + F_{j-1}) = O(\rho^{-(j+1)})$$

substituting this to (2.36) we get

$$\Lambda_N = |\gamma|^2 + F_j + O(\rho^{-(j+1)\alpha}) + O(\rho^{-(l-d)\alpha})$$

since $1 \le j+1 \le l-d$, $\rho^{-(l-d)\alpha} \le \rho^{-(j+1)\alpha}$ which implies the the result for $k = j+1$. □

REFERENCES

Eastham, M. S. P. (1973), The spectral theory of periodic differential equations. Edinburgh : Academic Press.

Feldman, J., Knorrer, H. & Trubowits, E. (1990), The perturbatively stable spectrum of a periodic Schrödinger operator. Inventory of Mathematics, 100 259-300.

Feldman, J., Knorrer, H. & Trubowits, E. (1991), The perturbatively unstable eigenvalues of a periodic Schrödinger operator. Comment. Math. Helvetica, 66 557-579.

Hald, O. H. & McLaughlin, J. R. (1996). Inverse nodal problems: Finding the potential from nodal lines. Memoirs of the American Mathematical Society.

Karpeshina, Y. E. (1990) Perturbation theory for the Schrödinger operator with a non-smooth periodic potential. Proc. Steklov Math. Int., 188, 88-116

Karpeshina, Y. E. (1992) Perturbation theory formulae for the Schrödinger operator with a non-smooth periodic potential. Math USSR-Sb. 71, 101-123

Reed, M. & Simon, B. (1978). Methods of modern mathematical physics. IV. Analysis of operators Academic Press, New York.

Veliev, O. (1987) Asymtotic formulae for eigenvalues of Schrödinger operators and Bethe-Sommerfeld Conjecture. J. of Functional Analysis and Appl. USSR 21 no:2, 1-15

Veliev, O. (1988) On the spectral analysis of the multidimentional periodic operators. J. Theory of Functions, Functional Analysis and Appl. USSR no:49, 17-34