RIKS THEORY AND AN APPLICATION

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by
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İZMİR

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We certify that we have read the thesis, entitled **"RISK THEORY AND AN APPLICATION"** completed by Banu ÖZAKÇAN ÖZGÜREL under supervision of Prof. Dr. Nilgün MORALI and that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

Insurance risk models, number of claims, claim sizes, claim frequency rate, and calculation of the risk premium are discussed in this thesis. An application depending on data provided by an insurance company involving with motor vehicle insurance supported our study.

In conclusion, we see that the realized values of the five years totals number of claims and number of policies are very close to the values obtained under the planning assumptions.

ÖZET

Bu çalışmada, sigortacılıkta risk modelleri, hasar sıklık oranı, hasar sayıları ve büyüklükleri incelendi. Çalışmamız bir kasko şirketinden alınan beş yıllık veriler ile yapılan bir uygulama ile desteklendi.

Sonuçta, planladığımız varsayımlar altında hesaplanan beş yıllık toplam hasar sayıları ve poliçe sayıları gerçekleşen değerlere oldukça yakın çıkmıştır.

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CHAPTER ONE
INTRODUCTION AND OVERVIEW

1. 1. The International Importance of Risk

People have always sought to reduce uncertainty. This natural risk reduction drive motivated the earliest formations of clans, tribes, and other groups. Group mechanisms ensured a less volatile source of the necessities of life than atomised humans and families could provide. The provided greater physical security and helped its less fortunate members in times of crises.

People today continue their quest to achieve security and reduce uncertainty. We still engage in activities and rely on groups to help reduce the variability of income required to obtain life's necessities and to protect acquired wealth. The group may be our employer, the government, or an insurance company, but the concept is the same. In some ways, however, we are more vulnerable than our ancestors. The physical and economic security formerly provided by the tribe or extended family diminishes with industrialisation. More formalised means are required for mitigating the adverse consequences of unemployment, sickness, old age, death, lawsuits, and destruction of property.

As extensions of human activity, businesses are similarly vulnerable. Risk lies at the hearth of all business operations and processes. With increasing concentrations of property and people, and increasingly complex servicing, manufacturing, and industrial processes, business have come to appreciate that they are exposed to greater risk and that they must manage risks more efficiently and effectively to be competitive.

Risk therefore is universal. Every individual, business, and government must cope with it. All manage risk to a greater or lesser degree, by design or default. As businesses and whole societies continue to grow economically, they face possibilities of unprecedented harm. True, the frequency of loss for many technologically sophisticated consequences of losses that do occur has increased – thus leading to greater risk. Figure 1.1 suggests a stylised version of this phenomenon. (Skipper, 1994)

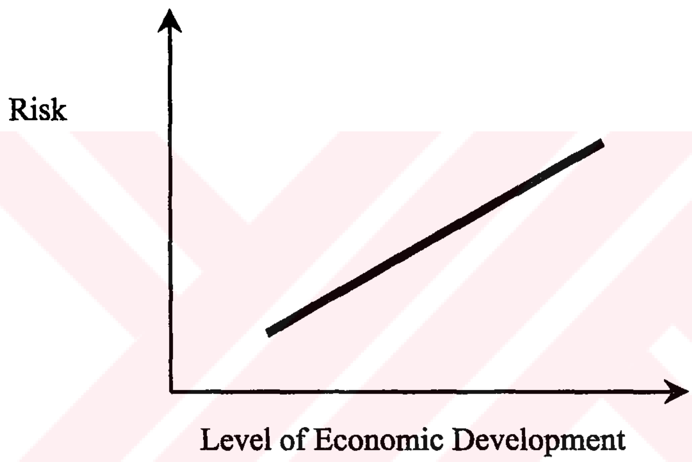

Figure 1.1 Risk and Economic Development

Decisions concerning how to deal with certain risky activities and processes have become incredibly complex. The consequences of wrong decisions or just plain bad luck can have profoundly adverse effect, not only on the decision-makers, but also on employees, customers, suppliers, and citizens. Increasingly, adverse effects can spill across industries and national borders, causing grave harm to the uninvolved. For these reasons, the study of the management of risk arguably is more important today than over.

1. 2. Essential Concepts: Certainty, Uncertainty, and Risk

Certainty is lack of doubt. In *Webster's New Collegiate Dictionary*, one meaning of the term "certainty" is "a state of being free from doubt", a definition well suited to the study of risk management. The antonym of certainty is *uncertainty,* which is "doubt about our ability to predict the future outcome of current actions". Clearly, the term "uncertainty" describes a state of mind. Uncertainty arises when an individual perceives that outcomes cannot be known with certainty.

Risk is potential variation in outcomes. Risk is present in almost everything humans do. When risk is present, the outcome cannot be forecasted precisely. The presence of risk gives rise to uncertainty. *Exposure to risk* is created whenever an act gives rise to possible gain or loss that cannot be predicted. The terms be used in the sense of an object, a situation, or a category of activities giving to outcomes that cannot be predicted. For example, one may hear of dwellings located near a river being exposed to a flood risk. One also may hear of the risk of passing a difficult collage course, or the risk of falling in love.

Risk is an objective concept, so it can be measured. For the purpose of introducing the concept of risk informally, an example is shown in Table 1.1, which depicts the outcomes of two bets: one in which the amount at risk is $1, the other in which the amount at risk is $100. The decision to participate in either bet creates exposure to risk. Most individuals, when asked about their willingness to participate in either bet, indicate that the "$100 coin toss" is less desirable, and many individuals prefer not be at all. Their reluctance to bet is explained mainly as a consequence of *risk aversion*. A risk averse individual prefers predictable outcomes to unpredictably, and tends to be more averse to "high stakes" risks than "low stakes" risks.

Table 1.1. Outcomes of $1 and $100 Coin Tosses

	Outcome × Probability		Expected
	Head × 0.5	Tail × 0.5	Value
$1 Game	$+1\times0.5$	-1×0.5	0
$100 Game	$+100\times0.5$	-100×0.5	0

In Table 1.1, the notion of risk is captured by observing that the dispersion –or range- of possible outcomes is larger for the $100 coin toss than it is for the $1 coin toss. The bets are identical in all other respects. For both bets, *probability of loss* (the proportion of outcomes resulting in loss) is 50 percent, or 0.5. Also, the *expected outcome* or *expected value* (the sum of the outcomes weighted by their respective probabilities) for both bets is zero. The expected value is the average gain or loss that would result from participating in the bet a large number of times.

In the example illustrated in Table 1.1, neither expected value nor probability of loss provides a rationale for an individual believing that the $100 coin toss is less desirable. The probability of loss and the expected loss for both bets are identical. However, if we think about what makes the $100 coin toss less desirable (or at least different from the $1 coin toss), we can see that the deviation of the outcomes from what we expect to occur (zero) is greater in the $100 coin toss. Because the variation is greater with the $100 coin toss, risk-averse individuals find it less desirable than the $1 coin toss.

Although a risk-averse individual finds the "$100 coin toss" less attractive, he or she may be willing to accept this risk if provided compensation. For example, some (but not all) risk-averse individuals are willing to participate in the $100 coin toss if paid $10 for participating the level of necessary compensation is called the *risk premium*, which is the minimum amount over and above the expected value required to induce the individual to participate in the bet. We noted in Table 1.1 that the expected value of the "$100 coin toss" is zero in the absence of compensation. With a $10 risk premium, participating in the bet causes the individual's expected wealth to increase by $10.

A *pure risk* exists when there is a chance of loss but no chance of gain. For example, the owner of an automobile faces the risk associated with a potential collision loss. If a collision occurs, the owner does not gain, so the owner's financial position remains unchanged. A *speculative risk* exists when there is a chance of gain as well as a chance of loss. For instance, investment in a capital project might be profitable or it might prove to be a failure. Pure risks are always distasteful, but speculative risks possess some attractive features.

To a large extent, the distinct between pure and speculative risks is semantic. Typically, a given risk has both pure and speculative elements. For example, the owner of a dwelling faces the risk that the value of the dwelling at the end of a year may be greater or smaller than its current value. The potential variation in the value of the dwelling arises from a number of sources, including possible fire damage and possible changes in the market value of dwellings generally. Customarily the fire risk is considered a pure risk, while possible loss in market value is not. However, the fire risk and the market value risk are both elements of the total risk faced by the dwelling owner

Risk and uncertainty have an important impact on organisations in that they exact a cost, commonly referred to as *the cost of risk.* The cost of risk is a widely discussed concept in professional and academic circles, and though there are discrepancies in definition, the general thrust of the concept is that risk imposes costs on an organisation that would not be incurred in a world of certainty. The most obvious cost is the *cost of losses*.

Risk management is defined as "a general management function that seeks to identify, asses, and address the causes and effects of a uncertainty and risk on an organisation." The purpose of risk management is to enable an organisation to progress toward its goals and objectives (its missions) in the most direct, efficient and effective path.

Risk may be retained by the person or organisation concerned or transferred in part or in whole. The cost of any retained risk has to be met either from current cash flows or has to be funded over a number of years. Insurance is the most important vehicle for risk transfer. It reduces risk by transferring it to professional insurers, thereby pooling the uncertainty of financial loss.

1. 3. Insurance

Insurance is an important part of risk management programs for organisations and individuals. Insurance is a risk financing transfer under which an insurer agrees to accept financial burdens arising from loss. More formally, insurance can be defined as a contractual agreement between two parties: the insurer and the insured. Under the agreement, the insurer agrees to reimburse loss (as defined in the insurance contract) in return for the insured's premium payment.

This definition of insurance focuses on the economic substance of the transaction, not necessarily legal implications. Four elements are required for an insurance transaction under this definition. (Williams at al., 1998)

- a contractual agreement
- a premium payment (or a promise to pay) by the insured,
- a benefit payment conditioned on circumstances (usually of misfortune) defined in the insurance contract,
- the presence of a pool of resources held by the insurer to reimburse claims.

The *pool of resources* is a key element in this definition. Without the pool resources, the transaction has no effect on economic performance, which is the substance of an insurance transaction. An obvious example would be an agreement to reimburse loss written by an entity that is known to be bankrupt. The agreement has no economic substance because the financial burden falling on the holder of the agreement in the event of loss is the same as it would be in the absence of the agreement. Also, an entity cannot insure itself; coverage provided by captive insurers do not fit this definition of insurance unless entities other than the insured provide resources that are subject to claims of the owner-insured. Thus transfer requires the risk, or the responsibility for reimbursing loss, to be assumed by a party outside an organisation's corporate "family".

Not all risks are insurable. The following are the basic criteria for insurability;

1. The loss must be fortuitous – it is necessary to give special consideration to the moral hazards that may contribute to the risk.
2. The frequency and magnitude of the expected loss must be assessable – it is necessary to give special consideration to the availability of relevant information, (both descriptive and numeric) and the expertise available for assessing that information.
3. The circumstances of a loss must be capable of definition.
4. There must not be excessive exposure to loss – it is necessary to give special consideration to accumulations of risk (exposure to catastrophe losses) and cover limits.
5. The premium must be affordable – it is necessary to give special consideration to the importance of the risk relative to the cost of administering it, and to whether insurance is the appropriate means of dealing with the risk (for example, buildings and their contents should not normally be situated on flood plains).
6. The insurance must not threaten the public interest (for example, it should not encourage criminal activity).

We have already given definition of insurance as the tool for the transfer of risk. The *Macquarie Dictionary* defines insurance as the act, system or business of insuring property, life, the person, etc against loss or harm arising in specified contingencies as fire, accident, death, disablement or the like, in consideration of a payment proportionate to the risk involved. It goes on to define the verb insure as to guarantee against loss or harm, to secure indemnity to or on in case of loss, damage or death. In our view, these definitions are rather messy. Insurance is the business of indemnifying a person or organisation for loss or damage, or the liability to compensate for loss or damage arising from specified contingencies such as fire, theft, injury, death, negligence, etc. in consideration for a payment appropriate to the risk involved.

Insurance may be classified into three groups – insurance of person, insurance of property and pecuniary interest, and insurance of liability. General insurance, or Non-Life insurance or Property/Casualty insurance as it is sometimes called, normally relates to the insurance of property and liability.

We have grouped class of insurance under the headings of Stationary and Pecuniary Interest, Moving Property, Liability and Other. Policies are generally classified into their principal class.(Hart at al., 1996) For example; house insurance is predominantly property insurance, but may include personal accident and liability covers; motor insurance is mainly concerned with damage to or loss of the vehicle, but usually includes liability for damage to third party property. Motor bodily injury insurance usually covers injury to third parties but is sometimes extended to include the driver at fault.

Insurance of stationary and pecuniary interest;

1. Houseowners (buildings) and householders (contents)
2. Fire
3. Industrial special risks
4. Glass
5. Engineering
6. Money
7. Fidelity
8. Consequential loss
9. Miscellaneous policies

Insurance of moving property;

1. Motor vehicle
2. Ships and boats
3. Aircraft
4. Cargo
5. Other transportable property

Insurance of liability;

1. Public or products liability
2. Worker's compensations
3. Professional indemnity

Other classes of insurance;

1. Sickness and accident insurance
2. Extended warranty insurance
3. Mortgage insurance
4. Consumer credit insurance
5. Trade credit insurance

We are interested in motor vehicle insurance in this study. Motor vehicle insurance contracts are schedule contracts that the permit the insured to purchase both property and liability insurance under one policy. The contract can be divided, however, in two separate contracts – one providing insurance against physical damage to automobiles, and the other protecting against potential liability arising out of the ownership, maintenance, or use of an automobile. Some automobile insurance contracts, notably those issued by insurers associated with automobile finance companies, cover only physical damage insurance.

Claim is a very important in motor vehicle insurance. A *claim* is an assertion of right to payment, as when a customer notifies manufacturer of an injury from a defective product and expresses a belief that the injury justifies compensation.

Usually, the payment is to occur at some future point in time. A *reported claim* is a claim for which for potentially responsible pary as received notification; otherwise the claim is *unreported.* A *closed claim* is a claim for which the liability for payment has been the resolved and full payment has been made. When liability for payment has not been resolved, the claim is an *open claim* and the estimated amount of payment is called a "*reserve*".

The principles and concepts that flow from the study of insurance have a general impact on risk management, because the concept and theory of risks is at the core of the study of insurance. Virtually every temporal aspects of insurance suggest a strong relationship with risk theory.

The terms "insurance rate" and "insurance premium" often are used interchangeably in ordinary conversation, although they have distinct meanings in the insurance business. An *insurance rate* is the price per unit of coverage or unit of exposure. An *insurance premium* is the total price of coverage. For example, a $50.000 premium for $10 million of fire insurance coverage implies a rate of $0.50 per $100 of fire insurance. Fire insurance often is priced using a rate per $100 of coverage, with the insured choosing the total amount of coverage.

1.4. Premiums

The definition of premiums includes levies and charges (e.g. fire brigade charges) but excludes stamp duty. An estimate of premium from unclosed business should be included. Direct premium revenue is brought to account only when earned. Premium is earned from the date of the attachment of risk and then in accordance with the pattern of risk. Where the result would not be materially different, earned premium may be speared evenly over the period.

Premium rates are set, usually as a percentage of wages, on the basis of industry or occupation. In same states there are varying degrees of cross-subsidisation between classifications to reduce the cost in high-risk industries. The premium formulae often include a measure of self-experience to encourage employers to improve their claims experience.

Premiums must be sufficient to;

- Pay all acquisition and policy administration expenses,
- Provide a sum of money to meet the best estimate of the cost of claims (including claim management expenses)
- Provide a profit margin representing an adequate return on capital invested and compensation for the risk
- Provide for other matters a prudent insurer would make provision for.

The objective of this study is to calculate the risk premiums for a motor vehicle insurance company and compare the results with the risk premium tables, which are used by the company. The data of the study (the total numbers of claims, total claim amounts and the number of issued policies during 1993-1997) are obtained from an insurance company. We see that the realized values of the five years totals of number of claims and number of policies are very close to the values obtained under the planning assumptions. Besides, the total amount of risk premiums which would be collected under the proposed plan would cover the realized total amount of claim sizes.

A relatively extensive review of existing insurance risk models are in Chapter 2. Then, in Chapter 3, we explained risk premium, claim size distributions, and mean claim size, and also discuss the problem of estimating the claim size distribution and mean claim size. The last chapter includes application depending on data we have obtained from the insurance company.

CHAPTER TWO

INSURANCE RISK MODELS

In this chapter some of the more important parametric discrete and continuous probability distributions used in insurance modelling are discussed. An insurer may use many of the continuous random variables to model the costs of claims payable. The discrete distributions may serve either to describe the number of claims or the claims costs payable by an insurer over a specified period of time.

The distributions that will be presented here include both discrete and continuous distributions. The discrete univariate distributions are the Binomial, Poisson, Geometric, Negative Binomial, and Logarithmic distributions. The continuous univariate distributions are the Normal, Lognormal, Gamma, Pareto and Exponential distributions.

2.1. Concept of Probability

To most people, 'probability' is a loosely defined term employed in everyday conversation to indicate the measure of one's belief in the occurrence of a future event. We accept this as a meaningful and practical interpretation of the term but seek a clearer understanding of the context in which it is used, how it is measurement, and how probability assists in solving real problems.

The concept of probability is necessary when dealing with physical, biological, or social mechanisms that generate observations which cannot be predicted with certainty. For example, the blood pressure of a human at a given point in time cannot be predicted with certainty, and we never know the exact load that a bridge will endure before collapsing into a river. Such random events cannot be predicted with certainty, but the relative frequency with which they occur in a long series of trials is remarkably stable.

A gambler tosses a coin four times. This process may be referred to as an ***experiment*** or ***random experiment*** (to emphasise the fact that the outcome is uncertain). At each toss, the gambler will obtain either a 'head' (H) or a 'tail' (T). The particular sequence of 'heads' and 'tails' he obtains is called the ***outcome*** of the experiment, and it is clear that the experiment we have described has $2^4=16$ possible outcomes. The sequence HHTH is one possible outcome. The performance of an experiment is called a ***trial***.

The gambler may count the number of 'heads' he obtains in a trial. This number is a ***random variable***, which may take any one of the non-negative integer values 0, 1, 2, 3 or 4.

Let us imagine that the gambler obtains a sequence containing three 'heads' and one 'tail'. Then we can say that the event 'three heads and one tail' has occurred, or , for example, the event 'an odd number of heads' has occurred, or the event 'more heads then tails' has occurred.

If a random experiment is performed *n* times under identical conditions, and a particular event is observed to occur in *m* of the *n* trials, then an ***estimate*** of the probability that the particular event will occur at any one trial is given by the proportion *m/n*, which must lie in the range 0 to 1. Clearly, the reliability of this estimate of probability depends on the number of times the experiment is performed. A perfectly symmetrical coin tossed three times, for example, might turn up 'heads' twice giving as an estimate of the probability of a 'head' the fraction *2/3*, when intuitively one might regard ½ as a better measure. If, however, the coin were tossed

1000 times and it had no physical bias, a fraction much closer to ½ would emerge. To allow for this, we shall adopt the following definition of probability.

If a random experiment is performed *n* times under identical conditions, and a particular event *E* is observed to occur in *m* of the *n* trials, then the probability of the event occurring is *P(E)* where

$$P(E) = \lim_{n \to \infty} \frac{m}{n} \qquad (2.1)$$

The limit notation on the right-hand side indicates that (theoretically) an infinite number of identical random experiments were performed. In practice, of course, and as a result *m/n* (n finite) provides on estimate only of the underlying probability.

It is clear from (2.1) that

$$0 \le P(E) \le 1 \qquad (2.2)$$

We described a gambler tossing a coin four times and counting the number of 'heads' he obtains. The number is a ***random variable,*** which may take any one of the non-negative integer values 0, 1, 2, 3 or 4. The random variable is ***discrete*** because it is restricted to particular point values; it can not assume all values in any interval (for example, all values between 3 and 4). The number of claims incurred by an insurance company is always a discrete random variable.

Some random variables are, by their very nature, continuous and may assume any value over a continuous interval. Examples include the height of an adult male chosen at random, the amount of rainfall in a certain location during a given period, and the size of an insurance claim (ignoring that fractions of cents cannot be paid!).

For a continuous random variable it is meaningless to seek the probability that the variable takes a particular value because there is an infinite number of values that the variable may take within its range, and the probability that it takes a particular value is, therefore, zero. It is meaningful, however, to seek the probability that the random variable lies in a particular interval.

Let X denote a **random variable** taking on values on a subset of the real line. For any subset A of the real line, we denote the probability that X takes on a value contained in A by $\Pr\{X \in A\}$.

The **distribution function** (d.f.) of the random variable X is defined for any real number x by

$$F_X(x) = \Pr\{X \leq x\} = \Pr\{X \in (-\infty, x]\}. \tag{2.3}$$

The subscript X indicates the random variable under consideration. When there is no ambiguity created function $F_X(x)$ has the following properties:

(a) $F_X(-\infty) = \lim_{x \to -\infty} F_X(x) = 0.$ (2.4)

(b) $F_X(\infty) = \lim_{x \to +\infty} F_X(x) = 1.$ (2.5)

(c) $F_X(x)$ is non-decreasing, i.e. for $x<y$,

$$F_X(x) \leq F_X(y). \tag{2.6}$$

For a discrete random variable X, the function

$$f_X(x) = \Pr\{X = x\} = F(x) - F(x-0) \tag{2.7}$$

is called the ***probability function*** (p.f.) of X.

For a continuous random variable X, the function $f_X(x)$ defined by

$$f_X(x) = \frac{d}{dx} F_X(x) \tag{2.8}$$

at the point where the derivative exists is called the ***probability density function*** (p.d.f.) of X.

2.2. Poisson Distribution

The Poisson distribution will often serve as an appropriate probability distribution for random variables such as the number of telephone calls received at a switchboard during a fixed period of time, the number of atomic particals emitted from a radioactive source which strike a certain target during a fixed period of time, or the number of defects on a specified length of magnetic recording tape. Each of these random variables represents the total number X of occurences of some phenomenon during a fixed period of time or within a fixed region of space. It can be shown that if the physical process which generates these occurences satisfies three specific mathematical conditions, then the distribution of X must be a Poisson distribution.

Assume that we are observing the number of occurances of some phenomenon during a fixed period of time. The first condition is for Poisson distribution that the numbers of occurances in any two *disjoint* intervals of time must be independent of each other. The second condition is that the probability of an occurrence during any particular very hort interval of time must be approximately proportional to the length of that interval. The final condition is that the probability of two or more occurance in any particular very short interval of time must be of a smaller order of magnitude than the probability of just one occurances. Hence, when we consider a very short interval of time, the probability of two or more occurances must be negligible in comparison with the probability of one occurance. Of course, it follows from the second condition that the probability of one occurance in that same interval will itself be negligible in comparison with the probability of no occurances.

Let X be a random variable with a discrete distribution, and suppose that the value of X must be a nonnegative integer. It is said that X has a Poisson distribution with mean μ if the p.d.f. of X is as follows;

$$f_X(x) = \begin{cases} \dfrac{\mu^x e^{-\mu}}{x!} & , \quad x = 0,1,2,3,\ldots \\ 0 & , \quad elsewhere \end{cases} \tag{2.9}$$

where

Mean : $E(X) = \mu$

Variance : $Var(X) = \mu$

2. 3. Binomial Distribution

A Bernoulli experiment is a simplest experiment which is one that may result in either of two possible outcomes. We can think of many examples of such experiments: a flip of a coin (head or tail), performance of a student in a course (pass or fail), the sex of a yet-to-be-born child (male or female), placing a satellite in orbit around the earth (succes or not).

A sequence of Bernoulli trials occurs when a Bernoulli experiment is performed several independent times so that the probability of success, say p, remains the same from trial to trial.

Some experiments consist of the observation of a sequence of identical and independent trials, each of which can result in one of two outcomes. Each item leaving a manufacturing production line is either defective or nondefective. Each shot in a sequence of firings at a target can result in a hit or a miss, and each of n persons questioned prior to a local election will either favour candidate Jones or not.

A binomial experiment is one that has the following properties;

1. The experiment consists of n identical trials.

2. Each trial results in one of two outcomes. For lack of a better nomenclature, the one outcome is called a success, and the other a failure.
3. The probability of success on a single trial is equal to p and remains the same from trial to trial. The probability of failure is equal to $(1-p)=q$.
4. The trials are independent.
5. We are interested in X, the number of successes observed during the n trials.

The binomial probability distribution and its mean, variance, and standard deviation are as follows;

Binomial probability distribution,

$$f_X(x) = \begin{cases} \binom{n}{x} p^x q^{n-x} = \dfrac{n!}{x!(n-x)!} p^x q^{n-x} & , \quad x = 0,1,2,\ldots,n \\ 0 & , \quad elsewhere \end{cases} \tag{2.10}$$

where

Mean : $E(X) = np$

Variance : $Var(X) = npq$

2. 4. The Geometric Distribution

The geometric distribution is defined on sequences of independent Bernoulli trials. Suppose independent Bernoulli trials are performed, each with probability p of success, and we let X be the number of trials necessary to get the first success. Then

$$P(X=1)=p$$

Since the probability of a success is p for each trial. We will observe $X=2$ if and only if we have a failure on the first trial and then a success on the second, so $P(X=2)=pq$. Similarly, for any integer $k \geq 3$, we will observe $X=k$ if and only if we have failures on the first $k-1$ trials, followed by a success on the k^{th} trial, so $P(X=x)= pq^{k-1}$. This random number of trials needed to get the first success is called the geometric random variable with parameter *p*, because the values of its probability function from a geometric progression.

It is said that a random variable *X* has a geometric distribution for which the probability function,

$$f_X(x) = \begin{cases} pq^{x-1} & , \quad x = 1,2,\ldots \\ 0 & , \quad elsewhere \end{cases} \tag{2.11}$$

where q = 1-p

Mean : $E(X) = \dfrac{1}{p}$

Variance : $Var(X) = \dfrac{q}{p^2}$

2. 5. Negative Binomial Distribution

The negative binomial distribution has been used in many disciplines involving count data, such as accident statistics, biological sciences, ecology, epidomiology of noncommunicable events, market research, medical research, and psychology.

Consider a sequence of independent repetitions of a random experiment with constant probabilty *p* of succes. Let the random variable *Y* denote the total number of failures in this sequences before the *r*th success; that is *Y+r* is equal to the number of trials necessary to procedure exactly *r* successes. Here *r* is a fixed positive integer. To determine the p.d.f. of *Y*, let *y* be an element of $\{y : y = 0,1,2,\ldots\}$. Then, by the multiplication rule of probabilities, *Pr(Y=y)=g(y)* is equal to the product of the probability

$$\binom{y+r-1}{r-1} p^{r-1}(1-p)^y$$

Of obtaining exactly *r-1* successes in the first *y+r-1* trials and the probability *p* of a success on the *(y+r)*th trial. A distribution with a p.d.f. of the form *g(y)* is called a *negative binomial distribution*; and any such *g(y)* is called a negative binomial p.d.f.

$$g_Y(y) = \begin{cases} \binom{y+r-1}{r-1} p^r (1-p)^y & , \ y = 0,1,2,\ldots \\ 0 & , \ elsewhere \end{cases} \tag{2.12}$$

2.6. Logarithmic Distribution

The logarithmic distribution is closely related to the negative binomial and Poisson distributions. It is a limiting form of a truncated negative binomial distribution.

The random variable *X* has a ***logarithmic*** or ***logarithmic series*** distribution if its p.d.f. is

$$f_X(x) = \frac{q^x}{-x\log(1-q)} = \frac{1}{x\log(1+\beta)}\left(\frac{\beta}{1+\beta}\right)^x \quad , \ x = 1,2,3,\ldots \tag{2.13}$$

where $0 < q = \frac{\beta}{1+\beta} < 1$, and *log(.)* denotes the natural logarithm.

The mean and the variance are

Mean : $E(X) = \frac{\beta}{\log(1+\beta)}$

Variance : $Var(X) = \frac{\beta(1+\beta)\log(1+\beta) - \beta^2}{[\log(1+\beta)]^2}$

Unlike all the distributions discussed previously, the logarithmic distributions has no probability mass at $x=0$. Another disadvantage of this distribution from the point of view of modelling is that $f(x)$ is strictly decreasing in x.

2.7. Exponential Distribution

An exponential distribution is often used in a practical problem to represent the distribution of the time that elapses before the occurrence of some event. For example, this distribution has been used to represent such periods of time as the period for which a machine or an electronic component will operate without breaking down, the period required to take care of a customer at some service facility, and the period between the arrivals of two successive customers at a facility.

Consider the random variable with p.d.f.

$$f_X(x) = \begin{cases} \lambda e^{-x} & , \quad x > 0 \\ 0 & , \quad elsewhere \end{cases} \tag{2.14}$$

where $\lambda > 0$. The d.f. is

$$F(x) = 1 - e^{-\lambda x}, x > 0 \tag{2.15}$$

the mean and the variance are thus

Mean : $E(X) = \frac{1}{\lambda}$

Variance : $Var(X) = \frac{1}{\lambda^2}$

If the events being considered occur in accordance with a Poisson process, then both the waiting time until an event occurs and the period of time between any two successive events will have exponential distributions. This fact provides theoretical support for the use of the exponential distribution in many types of problem.

2.8. Normal Distribution

The normal distribution is by far the single most important probability distribution in statistics. There are three main reasons for this pre-eminent position of the normal distribution.

The first reason directly related to the mathematical properties of the normal distribution. If a random sample is taken from a normal distribution, then the distribution of various important functions of the observations in the sample can be derived explicitly and will themselves have simple forms. Therefore, it is a mathematical convenience to be able to assume that the distribution from which a random sample was drawn is a normal distribution.

The second reason is that many scientists have observed that the random variables studied in various physical experiments often have distributions that are approximately normal. For example, a normal distribution will usually be a close approximation to the distribution of the heights or weights of individuals in a homogeneous population of people, of corn stalks, or of mice, or to the distribution of the tensile strength of pieces of steel produced by a certain process.

The third reason for the pre-eminence of the normal distribution is the central limit theorem. If a large random sample is taken from some distribution, then even though this distribution is not itself approximately normal, a consequence of the central limit theorem is that many important functions of the observations in the sample will have distributions that are approximately normal. In particular, for a large random sample mean will be approximately normal.

The probabilistic model for the frequency distribution of a continuous random variable involves the selection of a curve, usually smooth, called the probability distribution. Although these distributions may assume a variety of shapes, a large number of a random variables observed in nature possess a frequency distribution that is approximately bell-shaped or, as the statistician would say, is approximately a normal probability distribution.

A random variable of the continuous type that has a p.d.f. of the form of *f(x)* is said to have a *normal distribution*, and any *f(x)* of this form is called a normal p.d.f.

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \quad , \quad -\infty < x < \infty \tag{2.16}$$

Mean : $E(X) = \mu$

Variance : $Var(X) = \sigma^2$

In practice, we seldom encounter variables that range from infinitely large positive values. Nevertheless, many positive random variables (such as height, weight, and times) generate a frequency histogram that is well approximated by a normal distribution. The approximation applies because almost all of the values of a normal random variable lie within three standard deviations of the mean, and in these cases $(\mu \mp 3\sigma)$ almost always encompasses positive.

The normal distribution with mean 0 and variance 1 is called the standard normal distribution. The p.d.f. of the standard normal distribution is usually denoted by the symbol ϕ, and the distribution function is denoted by the symbol ϕ. Thus

$$\phi(x) = f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}, -\infty < x < \infty \tag{2.17}$$

and

$$\Phi(x) = \int_{-\infty}^{x} \phi(u)du \quad \text{for } -\infty < x < \infty \tag{2.18}$$

The d.f. $\Phi(x)$ can not be expressed in closed form in terms of elementary functions. Therefore, probabilities for the standard normal distribution or any other normal distribution can be found only by numerical approximations or by using a table of values of $\Phi(x)$.

2.9. Lognormal Distribution

Consider a random variable X which has the normal distribution with p.d.f. (2.16)

Let $Y=e^x$ so that $X=\log Y$. The p.d.f. of Y is

$$\begin{aligned} f_Y(y) &= f_X(\log y)\frac{1}{y} \\ &= \frac{1}{y\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\log y-\mu}{\sigma}\right)^2}, y>0 \end{aligned} \tag{2.19}$$

where $\sigma > 0, -\infty < \mu < \infty$. The distribution of Y is termed a lognormal distribution. The d.f. if given by

$$F(y) = \phi\left(\frac{\log y - \mu}{\sigma}\right), y > 0 \tag{2.20}$$

where $\phi(.)$ is the standard normal (with mean 0 and variance 1) d.f..

2.10. Gamma Distribution

It is said that a random variable X has a *Gamma distribution* with parameters α>0, β>0, and if X has a continuous distribution for which the p.d.f.

$$f_X(x) = \begin{cases} \dfrac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta} & , \ 0 < x < \infty \\ 0 & , \ elsewhere \end{cases} \tag{2.21}$$

where $\Gamma(\alpha)$ denotes the gamma function which is defined by

$$\Gamma(\alpha) = \int_0^\infty y^{\alpha-1} e^{-1} dy \tag{2.22}$$

If X has a gamma distribution with parameters α and β, then

Mean : $E(X) = \dfrac{\alpha}{\beta}$

Variance : $Var(X) = \dfrac{\alpha(\alpha+1)}{\beta^2} - \left(\dfrac{\alpha}{\beta}\right)^2 = \dfrac{\alpha}{\beta^2}$

2.11. Pareto Distribution

Pareto distribution serves as a useful modelling and predicting tools in a wide variety of socio-economic context. However, there is a definite advantage in focusing discussion on one specific field of application; the size distribution of income.

It was given a warning about the danger of fitting a distribution to claim size data and then using the extrapolated 'tail' to estimate the remote probability of an extremely large claim and the cost of reinsuring such a claim. Clearly, a reinsurer needs to err on the safe side by fitting a 'tail' which does not fade away to zero too quickly, and for this purpose the tail of the Pareto distribution is often more satisfactory than that of the log-normal distribution.

Suppose that a variety X has (conditional on θ) an exponential distribution with mean θ^{-1}. Further, suppose that θ itself has a gamma probability density function. The unconditional distribution of X is a mixture and is called the Pareto Distribution. The distribution function is obtained the Laplace Transform of the Gamma variate.

The Pareto distribution, which is positively skewed has probability density function

$$f_X(x) = \begin{cases} \dfrac{\alpha}{\beta}\left(\dfrac{\beta}{x}\right)^{\alpha+1} & , \ x > \beta \\ 0 & , \ elsewhere \end{cases} \tag{2.23}$$

and distribution function;

$$F(x) = 1 - \left(\frac{\beta}{x}\right)^{\alpha}, x > \beta \tag{2.24}$$

Its mean and variance are given by

Mean : $E(X) = \dfrac{\alpha\beta}{\alpha - 1}$

Variance : $Var(X) = \dfrac{\alpha\beta^2}{\alpha - 2} - \left(\dfrac{\alpha\beta}{\alpha - 1}\right)^2$

it should be noted that, for the mean to exist, α must be greater than 2. These restrictions mean that, in practise, the distribution is somewhat more difficult to use than the log-normal.

2.12. The Central Limit Theorem

If a random sample of size n is taken from a normal distribution with mean μ and variance σ^2, then the sample mean $\overline{X}_n$ has a normal distribution with mean μ and variance σ^2 / n. Whenever a random sample of size n is taken from any distribution with mean μ and variance σ^2. The sample mean $\overline{X}_n$ will have a distribution that is a approximately normal with mean μ and variance σ^2 / n.

If $X_1, X_2, X_3, \ldots, X_n$ is a random sample from a normal distribution with mean μ and variance σ^2, the random variable

$$\frac{\sum_{1}^{n} X_i - n\mu}{\sigma\sqrt{n}} = \frac{\sqrt{n}(\overline{X}_n - \mu)}{\sigma} \tag{2.25}$$

is, for every positive integer n, normally distributed with zero mean and unit variance. In probability theory there is a very elegant theorem called the *Central Limit Theorem*. A special case of this theorem asserts the remarkable and important fact that if $X_1, X_2, X_3, \ldots, X_n$ denote the observations of a random sample of size n from any distribution having positive variance σ^2 (and hence finite mean μ), then the random variable $\sqrt{n}(\overline{X}_x - \mu)/\sigma$ has a limiting standard normal distribution. If this fact can be established it will imply, whenever the conditions of the theorem are satisfied that (for large n) the random variable. $\sqrt{n}(\overline{X}_x - \mu)/\sigma$ has an approximate normal distribution with mean zero and variance 1. It will then be possible to use this approximate normal distribution to compute approximate probabilities concerning $\overline{X}$.

The more general form of the theorem is stated, but it is proved only in the modified case. However, this is exactly the proof of the theorem that would be given if we could use the characteristic function in place of the moment generating function.

CHAPTER THREE
THE RISK PREMIUM

In this chapter, exposure to risk and its measurement are discussed, and examples of the use of the census method are given. We also discuss the problem of estimating the claim size distribution and mean claim size, and the calculation of premiums for policies subject to excesses and excess of loss reinsurance.

The Poisson Distribution is usually satisfactory for studying claim frequency, although, in certain special circumstances, the negative binomial may be more appropriate.(Panjer & Willmot, 1992) Unfortunately, there is no generally applicable distribution for claim size. The Lognormal sometimes applicable and certain authors have also employed the gamma and other distributions for this purpose.(Carter & Doherty, 1974) Reinsurers sometimes prefer to use the Pareto distribution to estimate the risk premium for excess of loss reinsurance, because of its long tail.(Williams at. al., 1998)

3.1. The Risk Premium

The pure premium or risk premium is the premium that would exactly meet the expected cost of the risk covered ignoring management expenses, commissions, contingency loading, etc. To compute it, we need to estimate the claim frequency rate and mean claim size. Multiplying claim frequency rate by mean claim size yields the risk premium.

A claim is an assertion of a right to payment, as when a customer notifies a manufacturer of an injury from a defective product and expenses a belief that the injury justifies compensation. Usually, the payment is to occur at some future point in time. It should not be assumed that all accidents become claims, nor that the amount of the claim will always be the same as the amount of the damage.

The claim size is the sum, which the insurer has to pay on the occurrence of a fire, an accident, death or some other insured event. The sum of the individual claims constitutes the aggregate claim amount, which is one of the key concerns, both in the practical management of an insurance company and in theoretical consideration.

The claim frequency rate is a rate which can be estimated as the number of claims divided by the number of units of exposure.

It is important that the two components, claim frequency rate and claim size be considered separately. Only in exceptional circumstances can the use of 'rate of payment of claims' be considered satisfactory for the assessment of a risk premium.

3.2. The Number of Claims

The claim in general insurance can be divided into two stages;

1. An accident occurs, which result in damage,
2. A claim is lodged with an insurer, which is assessed and then paid.

Both the number of claims and the size of each claim are generally stochastic in practical applications. The behavior of the claim number variable K can be described in terms of its probability distribution, which is determined by the probabilities

$$p = \Pr(K=k), \quad k=0,1,2,\ldots$$

that exactly k claims occur in the given time period.

The risk collective consists of individual risk units, such as houses, buildings, and factories in fire insurance, or insured persons in life insurance. The primary events are the accidents which impinge randomly on the units, giving rise to claims. In individual risk theory the modeling of the risk process is based on consideration of these units as separate entities. The probability the one or more claims will occur in a time period is determined for each unit, as well as the distribution of the claim sizes. The claim number variable K of the whole collective is then obtained as the sum of the claim number variables of the individual risk units and, correspondingly, the aggregate claim amount is the sum of the aggregate claim amounts of the risk units.

Historically, the individual theory came as a first phase in the development of risk theory. In practice, however, it is inconvenient for handling large risk collectives and is therefore usually replaced by a collective approach. A model is developed directly for the claim number variable K and the aggregate claim amount for the whole collective, without any regard to the individual risk units. This method is nowadays commonly recognized as more satisfactory in practice and is adopted here, unless it is explicitly stated otherwise.

Under certain ideal and restricted conditions, the claim number variable K is Poisson distributed.

Insurance claims occur as a sequence of events in such a random way that is not possible to forecast the exact time of occurrence of any one of them, nor the exact total number.

If can be assumed that claims occur independently of each other, then the number of claims in a given time period is Poisson distributed.

These rather loosely described characteristics need to be expressed in a rigorous form which is suitable for mathematical development. For these purpose let us consider the accumulated number *K(t)* of claims occurring during a time period for 0 to *t* as a function of time *t*. It is postulated that this claim number processes satisfies the following three conditions;

1. Claims occur independently.
2. Only one claim can occur at any instant.
3. The probability of a claim in any sub-period (for example, a month) is proportional to the length of that sub-period.

Then the number of claims occurring in any fixed time interval is Poisson distributed.

In considering the first two conditions, it is necessary to emphasize the difference between events and claims. One event may generate a number of claims; for example, when two vehicles collide. For the purpose of fitting a Poisson distribution, multiple claims arising from one event should be treated as a single claim.

Events are in most cases independent. Exceptions tend to occur close together and may be treated as a single event; for example, when fires follow an earthquake. If events are counted in this way, the first two conditions hold reasonably true.

The third condition appears to be reasonable in a stationary environment or when the experience is subject to seasonal fluctuations, provided it is reasonably stable in the short term. Indeed, if the year can be divided into short periods, each of which satisfies the third condition, then the number of claims in each period is a Poisson variable and their sum – the number of claims over the whole year- is also therefore a Poisson variable. Sometimes, however, it is found that the Poisson distribution does not fit very well. This raises the question of whether the expected number of claims is really stationary.

The Poisson distribution has two major advantages:

1.There is only one parameter, the mean number of claims – to be estimated to completely characterize the distribution. The distribution of claim numbers can thus be projected quickly and easily.

2.It is additive: If the distribution for each risk in a pool is Poisson, then the number of claims for the pool is also a Poisson variable.

An alternative derivation of the negative binomial distribution is as the sum of Poisson variable where the Poisson parameter varies from risk to risk in accordance with a gamma distribution rather than over time. This is consistent with the notion of a portfolio of risks, where the underwriters endeavor to select risks with the same characteristics from risk with widely differing characteristics.

If the expected number of claims is large enough, then the Central Limit theorem applies and both of these distributions can be satisfactorily approximated by the normal distribution. The usual criterion is that the normal approximation is satisfactory when there are 50 claims or more.

3.3. The Claim Frequency Rate

So far, we have discussed the distribution of the number of claims. If we know the probability of death and the number of people who lived, we can estimate the number of death. Similarly, if we know the claim frequency rate and the number of policies, we can estimate the number of claims. Therefore, the claim frequency rate can be estimated as the number of claims divided by the number of policies. Both the Poisson and the negative binomial distributions can be expressed in forms that represent the claim frequency.

If we fit a negative binomial distribution to the numbers of claims per risk and assume that this is based on a gamma distribution of individual claim frequencies. It is possible to invert the derivation of the negative of the negative binomial and thereby deduce the parameters of the gamma distribution.

It is often very difficult to choose the most appropriate claim frequency rate for a class of insurance. Consider aviation insurance as an example; should one use claims per aircraft per year, or claims per aircraft, or claims per ton consumed, or what? In each case the numerator of the rate is the number of claims and causes no difficulty. It is the denominator (the exposure) which causes the difficulty. In motor insurance, for example the exposure usually adopted is vehicle year than kilometers driven, which might be theoretically more attractive, but it is not practical.

Once the unit of exposure has been chosen, there is still the problem of calculating the exposure. The insurer with adequate records and a sophisticated computer system may be able to compute exposure exactly.

Claim frequency rate is obtained from the ratio of number of claims to the number of exposure to risk. There are three methods to estimate it;

1. Eighths method
2. Twenty-four rule
3. Census method

3.4. The Amount of the Claims

It should not be assumed that all accident become claims, nor that the amount of the claim will always be the same as the amount of the damage.

One might expect the damage distribution to look something like Figure 3.1 with the accident frequency declining steadily as the amount of damage increases.(Hart at. al., 1996)

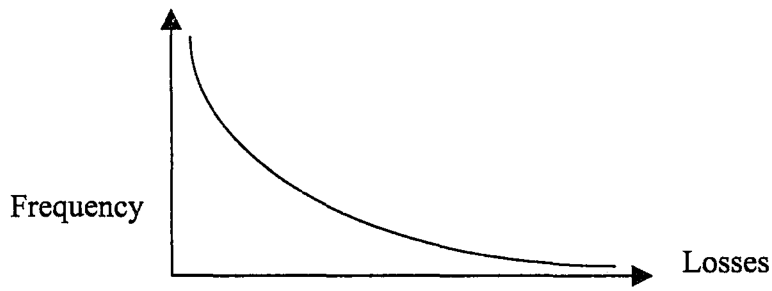

Figure 3.1. Damage Distribution

It is instructive to consider what the claim size distribution might look like and how this compare with the damage distribution. For most classes of insurance, the claim size distribution for a particular risk will look something like Figure 3.2.(Hossack at. al.,1993) There are very few claims for trivial amounts. There is a peak of small or moderate amounts, trailing off slowly for larger amounts, with a much smaller peak around the sum insured for the insured property.

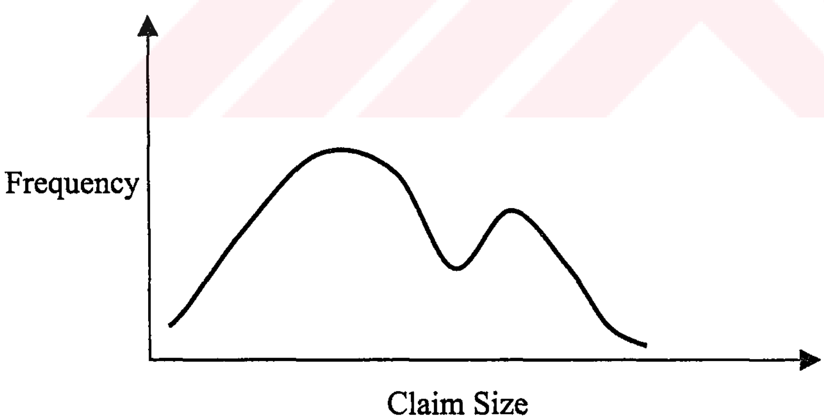

Figure 3.2. Claim Size Distribution

A claim will generally not be made unless there is enough damage to make it worthwhile. The effort of making a claim may be more than the effort of fixing the damage oneself. Some people are prepared to live with minor damage rather than claiming. Some insurance policies encourage this by applying policy excess or similar provision, or experience rating, or both.

If there is policy excess, the policyholder must pay the first part of any damage and the insurer pays only the amount over the excess. This increases the amount of damage required before a claim is worthwhile and reduces the amount of every claim by that excess.

There is no strong theoretical support for any particular claim size distribution. A number of distributions have, however, been used on empirical grounds. None of them provides a very good fit, possibly because of the distortions introduce by bunching and the policy conditions such as excesses and maximum sums insured. Nevertheless, they give a basis from which formal analysis can commence.

The most notable features of claim size distributions are as follows;

1. They are one-sided-negative claims should not arise.
2. They are highly skew.
3. Claims tend to bunch around popular values, often those ending in one or more zeros.

Unfortunately, there is no generally applicable distribution for claim size. The Lognormal is sometimes applied, and certain authors have also employed the gamma and other distribution for this purpose. Reinsurance sometimes prefers to use the Pareto distribution to estimate the risk premium for excess of loss reinsurance, because of its long tail.(Hossack at. al., 1993)

The lognormal distribution is a logarithmic transformation of the normal distribution. It is usually best fitted by fitting a normal distribution to the logarithms of the claim sizes. It covers the main range of claim sizes quite well but tends to run down too quickly at the higher values.

At the end of the distribution – for example, when estimating excess of loss reinsurance premiums – many actuaries prefer to fit the Pareto distribution. It should not be noted, however, that this distribution does not give a satisfactory fit over the whole range of claim sizes.

Another claim size distribution that may be appropriate in some circumstance is the gamma distribution. It gas good flexibility over the principal range of claim size but, like the lognormal, tends to be too light in detail.

In practice, most claims are limited by a maximum sum insured. If this is large, the unbounded distribution may be a reasonable approximation. For smaller sum insured, it may be necessary to modify the distribution by placing all larger losses in a mass at the sum insured. This is also necessary when analyzing retained claims under excess of loss reinsurance. Likewise policy conditions may require modification to loss distribution at the lower end.

Whether a theoretical or an empirical claim size distribution is used, considerable thought must be given to the treatment of large claims. Experience indicates that unless the sample size is very big these claims are likely to be under–represented in the data.(Hart at.al., 1996)

CHAPTER FOUR
APPLICATION

In the application of this study, our aim is to make an analysis depending on data provided by an insurance company which is specialized in motor vehicle insurance. With respect to the confidentiality of the insurance company, the risk premium figures used in practice would not be convenient for our consideration. Hence, our analysis depends on data which are not confidential for the company. These data are presented in the first section of this chapter, following with the method of analysis and the results of further research.

4.1. Data

The insurance company is established in 1992 and the available data comprises of

- the number of policies booked each year
- total amount of the premiums collected for each year
- yearly number of claims
- total claim size for each year

starting from 1993 to 1997. These data are given in Table 4.1.

Table 4.1. Yearly Income and Outgo

Years	Number of Policies	Total Amount of Premium ($)	Number of Claims	Total Claim Size ($)
1993	614	52 190	325	28 340
1994	785	78 500	402	42 150
1995	1 044	96 048	328	51 140
1996	1 472	114 816	499	62 539
1997	2 178	141 570	592	79 856
Total	**6 093**	**483 124**	**2 146**	**264 025**

Besides, the numbers of policies booked by the company October 1, 1997 to October 1, 1998 are also available. This data is presented in Table 4.2.

Table 4. 2. Available number of policies during October 1, 1997 to October 1, 1998

Dates	Number of Policies
October 1, 1997	456
January 1, 1998	812
April 1, 1998	254
July 1, 1998	498
October 1, 1998	260

Number of claims during the time between October 1, 1997 to October 1, 1998 is 183.

4.2. Method

We start the analysis with a search on simple implications of available data.

(1) Depending on yearly total claim sizes and number of claims, mean claim sizes for each year can be calculated by the formula

$$MCS_i = \frac{TCS_i}{c_i} \qquad 4.1.$$

where the notation is

MCS_i : mean claim size for year i

TCS_i : total claim size for year i

C_i : number of claims in year i

(2) Depending on yearly total amount of premiums and number of policies, average office premium per policy for each year can be calculated by the formula

$$AOP_i = \frac{TOP_i}{N_i} \qquad (4.2)$$

where the notation is

AOP_i : average office premium per policies for year i.

TOP_i : total amount of premiums for year i

N_i : number of policies in year i.

(3) Depending on yearly total claim sizes and number of policies, risk premium for each year can be calculated by the formula

$$RP_i = \frac{TCS_i}{N_i} \qquad (4.3)$$

where RP_i denotes risk premium for year i.

(4) Depending on yearly office premium and risk premium, the ratio can be calculated by the formula

$$k_i = \frac{AOP_i}{RP_i} \qquad (4.4)$$

where k_i denotes the ratio of office premium to risk premium for year i.

(5) A fixed claim frequency rate can be calculated depending on the total number of policies and the total number of claims.

In fact, claim frequency rate is the ratio of the number of claims in a period to the exposure to risk for this period. The main problem in calculating claim frequency rate is to find the exposure to risk. As it is explained in Chapter 3 section 3 exposure to risk can be calculated by various methods. One of them is the Census Method.

Let $N(t)$ denote the number of policies booked by the insurance company at time t. During the short interval of time δt immediately following time t, the change in the number of policyholders a premium will be negligible. Each of the $N(t)$ premium policyholders currently on the books will contribute δt years of exposure during that small time interval so that the total number of years' exposure during the interval of time δt following time t will be $N(t)\delta t$. The total exposure during 1997-1998 will, therefore, be approximately

$$\sum_{\substack{t=0 \text{ to } 1 \\ \text{in steps of} \\ \delta t}} N(t)\delta t$$

the calculation becomes more and more accurate the smaller δt becomes, and exact when δt becomes too small. Then in mathematical terms dt is written instead of δt, and becomes

$$\int_0^1 N(t)dt$$

i.e. the area under the $N(t)$ curve between $t=0$ and $t=1$.

For example, when data given in Table 4.2 is considered Figure 4.1 is the approximate graph of *N(t)*.

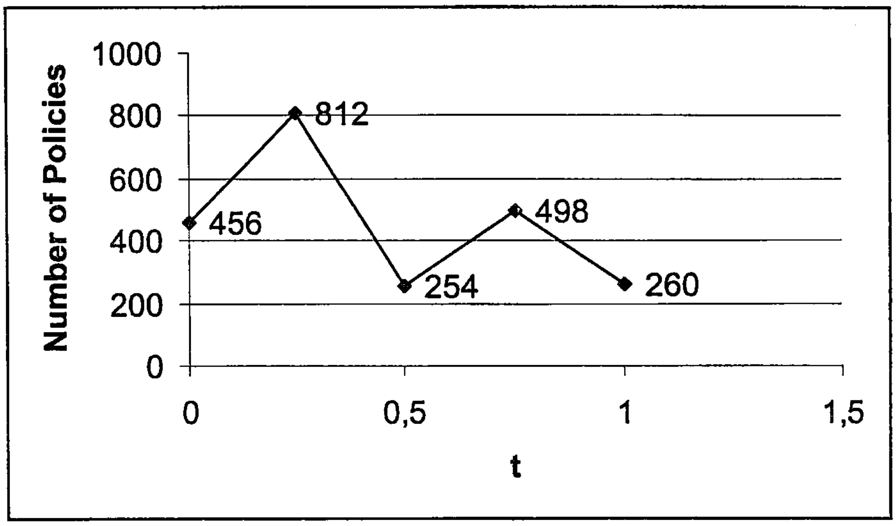

Figure 4.1. Available number of policies during October 1,1997 to October 1,1998

The area under *N(t)* curve is

$$\sum_{t=0}^{3}\frac{1}{2}\left[N\left(\frac{t}{4}\right)+N\left(\frac{t+1}{4}\right)\right]\frac{1}{4}$$

and is calculated to be 481, since the number of claims during this period is reported to be 183, the claim frequency rate is said to be 183/481 and equals 0.38.

When the period is extended to k years and total number of claims booked each year are assumed to be exposure to risk for that year, then total exposure to risk becomes $N_1+N_2+\ldots+N_k$.. The total number of claims given by $C_1+C_2+\ldots+C_k$. therefore yields the claim frequency rate to be calculated by the formula

$$m=\frac{\sum_{i=1}^{k}C_i}{\sum_{i=1}^{k}N_i} \qquad 4.5$$

where m is the claim frequency rate which is assumed to be constant during k years.

(6) On the implications of the mean claim sizes and total number of claims booked by the company the growth rate can be calculated. In the growth rates geometric progression can be assumed or both geometric and arithmetic progressions can be assumed.

4.3. Application

In this section the results obtained by applying each step explained in the method section are presented.

(1) Mean claim sizes for each year are presented in Table 4.3.

Table 4.3. Mean Claim Sizes

Years	Number of Claims	Total Claim Sizes ($)	Average Claim Sizes ($)
1993	325	28 340	88
1994	402	42 150	105
1995	328	51 140	156
1996	499	62 539	125
1997	592	79 856	135

(2) Average office premium for each year are presented in Table 4.4.

Table 4.4. Average Office Premiums

Years	Total Office Premiums	Number of Policies	Average Office Premiums
1993	52 190	614	85
1994	78 500	785	100
1995	96 048	1 044	92
1996	114 816	1 472	78
1997	141 570	2 178	65

(3) Risk premium for each year are presented in Table 4.5.

Table 4.5. Risk Premiums

Years	Total Claim Sizes ($)	Number of Policies	Risk Premiums
1993	28 340	614	47
1994	42 150	785	54
1995	51 140	1 044	50
1996	62 539	1 472	42
1997	79 856	2 178	36

(4) *k* for each year are presented in Table 4.6.

Table 4.6. Yearly ratios of office premiums to risk premiums

Years	Average Office Premiums	Risk Premiums	k
1993	85	47	1.81
1994	100	54	1.85
1995	92	50	1.84
1996	78	42	1.86
1997	65	36	1.81

(5) Claim frequency rate depending on data given Table 4.1 and calculated by formula (4.5) is

$$m = \frac{2146}{6093} = 0{,}35$$

and this can be assumed to be a crude estimate.

(6) The most reasonable yearly mean claim size figures constitute a geometric progression. The first mean claim size in Table 4.3 is $88 and four year after it becomes $135.

Let r denote the growth rate in mean claim size and with the assumption of geometric growth

$$MCS_{i+1} = (1+r)MCS_i$$

Therefore, r can be calculated from

$$135 = (1+r)^4 88$$

and is found to be 11.29%.

The number of policies booked by the insurance company which are given in Table 4.1 on yearly basis implies not only a geometric growth but also an arithmetic growth.

The relation between the successive number of total policies is found to be as follows by trial and error.

$n_2 = 1.2785 n_1$

$n_3 = (1.2785 + 0.063) n_2$

$n_4 = (1.2785 + 2 \times 0.063) n_3$

$n_5 = (1.2785 + 3 \times 0.063) n_4$

or expressed by a general formula as follows

$$n_{i+1} = [1.2785 + (i-1) \times 0.063] n_i$$

4.4. Further research

We start the work with the following assumptions:

(1) In 1993, 614 policyholders are to be booked, and mean claim size is to be \$88.

(2) Claim frequency rate during the following 5 years is to be constant and equal to 35%.

(3) Mean claim size increases by a constant rate which is equal to 11.29%.

(4) Number of policyholders increases by the relation

$$n_{i+1} = [1.2785 + (i-1) \times 0.063] n_i$$

(5) Only half of the claims reported in a year are paid in that year, and the other half of the claims are paid in the succeeding year.

With respect to the above assumptions mean claim sizes, number of policies, number of claims and total claim size paid each year can be calculated. The calculated values are presented in Table 4.7.

Table 4.7. Number of policies and claim amount under the assumptions

i	Years	Average Claim Sizes ($)	Number of Policies	Number of Claims	Total Claim Sizes($)
1	1993	**88**	**614**	215	$\frac{215}{2}(88+98) = 19\ 995$
2	1994	98	785	275	$\frac{275}{2}(98+109) = 28\ 462$
3	1995	109	1 053	369	$\frac{369}{2}(109+121) = 42\ 435$
4	1996	121	1 479	518	$\frac{518}{2}(121+135) = 66\ 304$
5	1997	135	2 170	760	$\frac{760}{2}(135+150) = 108\ 300$

Calculation of claim frequency rate as a ratio of number of claims to the exposure to risk yields a relation which is very useful to estimate the number of claims for each year. The number of claims which are denoted by c_i in Table 4.7 are calculated by

$$c_i = m_i \times n_i \tag{4.6}$$

Total claim size for each year is to be calculated by considering the number of claims produced that year and half of them to be paid mean claim size of that year, and the other half to be paid mean claim size of the succeeding year with respect to assumption (5) stated above:

$$TCS_i = \frac{c_i}{2}(MCS_i + MCS_{i+1})$$

The numerical figures presented in Table 4.7 can be used in order to calculate the expected mean claim sizes by applying formula (4.1), and risk premiums by formula (4.3).

The risk premiums obtained by these calculations are presented in Table 4.8.

Table 4.8. Risk premiums under the assumptions

i	Years	Number of Claims	Number of Policies	Total Claim Sizes	Risk Premium
1	1993	215	614	19 995	32.55
2	1994	275	785	28 462	36.22
3	1995	369	1 053	42 435	40.25
4	1996	518	1 479	66 304	44.80
5	1997	760	2 170	108 300	49.87
Total		**2 137**	**6 101**	**265 496**	

The total row in Table 4.8 contains the most important numerical figures in our analysis. As a result of the analysis total number of policies booked in 5 years is to be 6101, total number of claims in this period is to be 2137, and the total amount of claim sizes to be paid in this period is $ 265 496.

If the calculated risk premiums in Table 4.8 are to be applied during the years 1993-1997, the number of policies realized (data given in Table 4.1) would give rise to collection of total amount of net premiums as presented in Table 4.9.

Table 4.9. Total amount of net premiums

I	Years	Realized Number of Policies	Planned Risk Premium	Total amount of premium
1	1993	614	32.55	19 985.70
2	1994	785	36.22	28 436.62
3	1995	1 044	40.25	42 021.00
4	1996	1 472	44.80	65 945.60
5	1997	2 178	49.87	108 627.75
Total		**6 093**		**265 016.67**

The total rows both in Table 4.1 and 4.9 show that 6093 policy are booked in five years, total number of claims in this period in 2146 (Table 4.1) and the total amount of risk premiums to be collected by the proposed plan is $265 016.67 (Table 4.9).

In conclusion, We see that the realized values of the five years totals of number of claims and number of policies are very close to the values obtained under the planning assumptions. Besides, the total amount of risk premiums which would be collected under the proposed plan would cover the realized total amount of claim sizes.

CHAPTER FIVE
CONCLUSIONS

In this thesis, we have studied the insurance risk models and calculation of risk premiums. We have supported our study with an application depending on data provided by an insurance company involving with motor vehicle insurance.

After making an analysis on the available data, we carried out the research to a further step with some assumptions on the growth of the number of policies and claim sizes. Some assumptions about the payments are also made. Under these assumptions, a risk premium payment plan for five years is proposed. In conclusion, the proposed plan is found to give rise to very close values the realized ones, and be capable to cover the realized expenses.

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