# COMPUTATION OF EIGENVALUES OF REGULAR STURM-LIOUVILLE PROBLEMS WITH PERIODIC BOUNDARY CONDITIONS



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**by**
**Volkan ÖĞER**



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## M.Sc THESIS EXAMINATION RESULT FORM

We certify that we have read this thesis and **"COMPUTATION OF EIGENVALUES OF REGULAR STURM-LIOUVILLE PROBLEMS WITH PERIODIC BOUNDARY CONDITIONS"** completed by **VOLKAN ÖĞER** under supervision of **DOÇ.DR. ŞENNUR SOMALI** and that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Doç. Dr. Şennur Somalı

Supervisor

Prof. Dr. [illegible] Gökmen

(Committee Member)

Prof. Dr. Coşkun İşçi

(Committee Member)

Approved by the

Graduate School of Natural and Applied Sciences

Prof.Dr. Cahit HELVACI

Director

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# ABSTRACT

Computation of eigenvalues of regular Sturm-Liouville problems with homogeneous and periodic boundary conditions respectively is considered. For each problem, using the Richardson Extrapolation based on the finite difference, the accuracy of the eigenvalues are improved. Numerical results demonstrate the usefulness of the correction.

# ÖZET

Bu çalışmada, homojen ve periyodik sınır değerli Sturm-Liouville problemlerinin özdeğerlerinin hesaplanması ve iyileştirilmesi ele alınmıştır. Problemlere sonlu farklar yöntemi üzerine Richardson ekstrapolayonu uygulanıp hesaplanan özdeğerlerdeki hata azaltılmıştır. Numerik sonuçlar metodun uygulanabilirliğini desteklemektedir.

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# CHAPTER ONE
# INTRODUCTION

In this thesis, we investigate the computation of eigenvalues of Regular Sturm-Liouville Problem (SLP)

$$-y'' + q(x)y = \lambda y$$

for $q(x) \in C^1[0,\pi]$ with homogeneous boundary conditions,

$$y(0) = y(\pi) = 0$$

and also for $q(x) \in C^1[0,1]$ and $q(x) = q(x+1)$ with t-periodic boundary conditions, $t \in (0,2\pi) - \{\pi\}$, (Veliev & Duman, 2002)

$$y(1) = e^{it} y(0)$$
$$y'(1) = e^{it} y'(0).$$

There have been a number of papers (see Anderssen and de Hoog, 1984; Andrew, 1988, 1988, 1989; Andrew and Paine, 1986) dealing with the same problem with different boundary conditions in different methods. A survey paper related to this problem can be found in (Andrew, 1994). Andrew (Andrew, 1989) used the approach to improve finite difference eigenvalue estimates of periodic Sturm-Liouville problems. It is proved in (Andrew, 1989) that the application of the correction technique to the classical finite difference scheme studied in (Evans, 1971) reduces the error from $O(k^4h^2)$ to $O(kh^2)$. It is well known that when finite difference methods are used to approximate the eigenvalues, $\lambda_1 < \lambda_2 < \lambda_3 < ....$, of Sturm-Liouville Problem, the error in the approximation to $\lambda_k$ is known to increase rapidly with *k*.

In this thesis, we apply the Richardson Extrapolation to the approximate eigenvalues computed by finite difference method for improving the results. In the error analysis of computed eigenvalues we form on equivalent system for the SLP in order to investigate the behavior of convergence and the maximum number of extrapolation depending on the number of eigenvalues and also the asymptotic expansion for the computed eigenvalues. In Chapter 3, the theoretical improved results are established numerically by solving the problems in (Ghelardoni, 2001)

## 1.1. Computation of the Eigenvalues of SLP with Homogeneous Boundary Conditions and Its Asymptotic Expansion

We consider the regular Sturm-Liouville problem (SLP),

$$-y''(x)+q(x)y(x)=\lambda y(x), \quad 0<x<\pi \tag{1.1}$$

where

$$q(x)\in C^1[0,\pi]$$

with homogeneous boundary conditions

$$\begin{aligned} y(0)&=0\\ y(\pi)&=0. \end{aligned} \tag{1.2}$$

Introducing the new depending variable

$$y'(x)=z(x),$$

(1.1) and (1.2) can be written as

$$\begin{bmatrix} y'(x)\\ z'(x)\end{bmatrix}=\begin{bmatrix}0&1\\0&0\end{bmatrix}\begin{bmatrix} y(x)\\ z(x)\end{bmatrix}+(q(x)-\lambda)\begin{bmatrix}0&0\\1&0\end{bmatrix}\begin{bmatrix} y(x)\\ z(x)\end{bmatrix},$$

$$\begin{bmatrix}1&0\\0&0\end{bmatrix}\begin{bmatrix} y(0)\\ z(0)\end{bmatrix}+\begin{bmatrix}0&0\\1&0\end{bmatrix}\begin{bmatrix} y(\pi)\\ z(\pi)\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}.$$

Defining

$$Y(x)=\begin{bmatrix} y(x)\\ z(x)\end{bmatrix},$$

$$A=\begin{bmatrix}0&1\\0&0\end{bmatrix}, B=\begin{bmatrix}0&0\\1&0\end{bmatrix},$$

$$C_1=\begin{bmatrix}1&0\\0&0\end{bmatrix}, C_2=\begin{bmatrix}0&0\\1&0\end{bmatrix},$$

we have

$$Y'(x)=AY(x)+(q(x)-\lambda)BY(x), \quad 0<x<\pi \tag{1.3}$$

$$C_1Y(0)+C_2Y(\pi)=\underline{0}. \tag{1.4}$$

We consider the partition of the interval $[0,\pi]$

$$x_j=jh, \quad j=0,1,\ldots,n, \quad h=\frac{\pi}{n}.$$

Applying the finite difference scheme to the SLP (1.3) gives the following system

$$Y_{j+1} - Y_j = hAY_{j+1} + h(q_j - \lambda_h)BY_j$$

$$C_1Y_0 + C_1Y_n = \underline{0}$$

where

$$-y_{j+1} + (h^2 q_j + 2)y_j - y_{j-1} = h^2\lambda_h y_j,$$

$$\frac{y_{j+1} - y_j}{h} = z_{j+1},\ q_j = q(x_j), \quad j = 0,1,\ldots,n-1$$

and $\lambda_h$ is the computed approximate eigenvalue to $\lambda$, $Y_j$ is the approximation to $Y(x_j)$.

Since

$$(I - hA)^{-1} = (I + hA),$$

we rewrite $Y_{j+1}$ as

$$Y_{j+1} = (I + hA)(I + h(q_j - \lambda_h)B)Y_j = M_jY_j, \tag{1.5}$$

where

$$M(x_j) = M_j,$$

$$M_j = \begin{bmatrix} 1 + h^2(q_j - \lambda_h) & h \\ h(q_j - \lambda_h) & 1 \end{bmatrix}.$$

The eigenvalues and the corresponding eigenvectors of $M_j$ are

$$\mu_{1,2}^j = \frac{1}{2}(2 - h^2\lambda_h + h^2 q_j \mp \sqrt{-4 + (2 - h^2\lambda_h + h^2 q_j)^2}),$$

$$v_{1,2}^j = \begin{bmatrix} 1 \\ -\frac{1}{h}\left[1 + h^2(q_j - \lambda_h) - \mu_{1,2}^j\right] \end{bmatrix}.$$

Let

$$2 - h^2(\lambda_h - q_j) = 2\cos\theta(x_j, \lambda_h) = 2\cos\theta_j, \tag{1.6}$$

then

$$\mu_{1,2}^j = \cos\theta_j \mp i\sin\theta_j = e^{\mp i\theta_j} \text{ and}$$

$$v_1^j = \begin{bmatrix} 1 \\ \frac{1}{h}\left(1 - e^{-i\theta_j}\right) \end{bmatrix}, \; v_2^j = \begin{bmatrix} 1 \\ \frac{1}{h}\left(1 - e^{i\theta_j}\right) \end{bmatrix}.$$

Since $M_j$ is diagonalizable matrix, there exists an invertible matrix $P(x_j) = P_j$,

$$P_j = (v_1^j, v_2^j)$$

such that

$$M_j = P_j D_j P_j^{-1}, \tag{1.7}$$

where

$$D_j = \begin{bmatrix} e^{i\theta_j} & 0 \\ 0 & e^{-i\theta_j} \end{bmatrix}.$$

Substituting (1.7) into (1.5),

$$Y_{j+1} = P_j D_j P_j^{-1} Y_j$$

and premultiplying by $P_{j+1}^{-1}$, we get

$$P_{j+1}^{-1} Y_{j+1} = P_{j+1}^{-1} P_j D_j P_j^{-1} Y_j .$$

Using the transformation

$$Z_j = P_j^{-1} Y_j ,$$

we can rewrite system as,

$$Z_{j+1} = P_{j+1}^{-1} P_j D_j Z_j . \tag{1.8}$$

From Taylor's theorem, we have

$$P_{j+1}^{-1} = P_j^{-1} + h \frac{d}{dx} P^{-1}(x)\Big|_{x=x_j} + \frac{h^2}{2}\frac{d^2}{dx^2} P^{-1}(x)\Big|_{x=\xi_j}, \quad x_j < \xi_j < x_{j+1} \tag{1.9}$$

Substituting (1.9) into (1.8), we get

$$Z_{j+1} = \left( I + h \frac{d}{dx} P^{-1}(x)\Big|_{x=x_j} P_j + \frac{h^2}{2}\frac{d^2}{dx^2} P^{-1}(x)\Big|_{x=\xi_j} P_j \right) D_j Z_j ,$$

where

$$\frac{d}{dx} P^{-1}(x)\Big|_{x=x_j} P_j D_j = i\, f_j\, S ,$$

and $f_j = \dfrac{1}{e^{i\theta_j} - e^{-i\theta_j}} \dfrac{\partial}{\partial x}\theta\Big|_{x=x_j}, S = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}.$

Since

$$\left\| \frac{d^2}{dx^2} P^{-1}(x) \Big|_{x=\xi_j} P_j D_j \right\|_\infty \leq const.h\,,$$

we obtain

$$Z_{j+1} = \left(D_j + ihf_j S + O(h^3)\right) Z_j\,, \qquad j = 0,1,\ldots,n-1 \tag{1.10}$$

where *const* is a constant independent of $h$.

**Consider the case $q(x) = 0$, $\forall x \in [0,\pi]$.**

Then the eigenvalues of the original problem (1.1) and (1.2) are known to be

$$\lambda = k^2\,.$$

Since $\theta(\lambda) = 2 - h^2\lambda_h$ does not depend on $x$, $P$ will be the constant matrix and the approximate problem (1.10) for this case can be written as

$$Z_{j+1} = DZ_j\,, \qquad j = 0,1,\ldots,n-1\,,$$

where

$$D = \begin{bmatrix} e^{i\theta(\lambda_h)} & 0 \\ 0 & e^{-i\theta(\lambda_h)} \end{bmatrix}.$$

Hence the backward substitution gives

$$Z_n = D^n Z_0\,.$$

From boundary conditions (1.4) and

$$Y_j = P_j Z_j\,,$$

we have

$$C_1 P_0 Z_0 + C_2 P_n Z_n = \underline{0}\,,$$

(i.e.)

$$\left(C_1 P + C_2 P D^n\right) Z_0 = \underline{0}\,.$$

Defining

$$\psi(\lambda_h) = \det\left[C_1 P + C_2 P D^n\right],$$

then we have a non-trivial solution $Z_0 \in \Re^{2x1}$, if $\lambda_h$ is an eigenvalue of the problem (1.1) and (1.2), that is the solution of the equation

$$\psi(\lambda_h) = 0\,.$$

Therefore we obtain

$$\psi(\lambda_h)=\begin{vmatrix}1 & 1\\ e^{-ni\theta} & e^{ni\theta}\end{vmatrix}=e^{ni\theta}-e^{-ni\theta}=2\sin n\theta=0$$

and

$$\theta=k\frac{\pi}{n}=kh\,,\qquad k=1,2...,n-1.$$

From (1.6)

$$\theta=\arccos(\frac{1}{2}(2-h^2\lambda_h)),$$

we obtain the approximate eigenvalues,

$$\lambda_{h,k}=\frac{1}{h^2}(2-2\cos kh)=\frac{4}{h^2}\sin^2(\frac{kh}{2}),\qquad k=1,2...,n-1.$$

**For the case** $\boldsymbol{q(x)\neq 0}$, we already have (1.10), by back substitution, we get

$$\begin{aligned}Z_n=\big[&D_{n-1}D_{n-2}D_{n-3}....D_1D_0\\&+ih(f_0D_{n-1}D_{n-2}D_{n-3}....D_1S+f_1D_{n-1}D_{n-2}D_{n-3}....SD_0\\&+f_2D_{n-1}D_{n-2}D_{n-3}....SD_1D_0+f_3D_{n-1}D_{n-2}D_{n-3}....SD_2D_1D_0\\&+...........\\&+f_{n-1}SD_{n-2}D_{n-3}....D_1D_0)+O(h^2)\big]Z_0\quad,\end{aligned}$$

where

$$f_jD_{n-1}D_{n-2}D_{n-3}...D_{j+1}SD_{j-1}...D_1D_0=\frac{\frac{\partial}{\partial x}\theta_j}{e^{i\theta_j}-e^{-i\theta_j}}\begin{bmatrix}-e^{-ia_j} & e^{ib_j}\\ e^{-ib_j} & -e^{ia_j}\end{bmatrix}$$

with

$$a_j=(\theta_0+....+\theta_{j-1})+(\theta_{j+1}+....+\theta_{n-1}),$$

$$b_j=(\theta_0+....+\theta_{j-1})-(\theta_{j+1}+....+\theta_{n-1}).$$

Using the same consideration as in $q(x)=0$, from the boundary condition, it is obtained that;

$$\det\left[C_1P_0+C_2P_n\left\{\prod_{j=0}^{n-1}D_{n-1-j}+ih\sum_{j=0}^{n-1}f_jD_{n-1}D_{n-2}D_{n-3}...D_{j+1}SD_{j-1}...D_1D_0+O(h^2)\right\}\right]=0.$$

As a consequence we have

$$\psi(\lambda) = e^{i(\theta_0+....+\theta_{n-1})} - e^{-i(\theta_0+....+\theta_{n-1})} + 2ih\sum_{j=0}^{n-1}\frac{\frac{\partial}{\partial x}\theta_j}{e^{i\theta_j} - e^{-i\theta_j}}(e^{ib_j} - e^{-ib_j} - e^{-ia_j} - e^{ia_j}) + O(h^2),$$

i.e.,

$$\psi(\lambda) = 2\sin(\theta_0 + .... + \theta_{n-1}) + 2ih\sum_{j=1}^{n-1}\frac{\partial}{\partial x}\theta_j\frac{1}{\sin\theta_j}(\sin b_j - \sin a_j) + O(h^2),$$

where

$$\frac{\partial}{\partial x}\theta(x_j,\lambda) = \frac{hq'(x_j)}{\sqrt{4(\lambda - q_j) - h^2(\lambda - q_j)^2}}.$$

If $\lambda$ is an eigenvalue of the SLP (1.1) and (1.2), then

$$\psi(\lambda) = 0 \tag{1.11}$$

So the approximation $\lambda_h$ to $\lambda$ is obtained solving the equation (1.11).

Performing one iteration of Newton's method for $\psi(\lambda_h)$ the starting with $k^2$ , it is obtained

$$\lambda_h^{(k)} = k^2 - \frac{\psi(k^2)}{\psi_\lambda(k^2)},\ \psi_\lambda(k^2) \neq 0,$$

where

$$\psi_\lambda = \frac{d}{d\lambda}\psi(\lambda).$$

Because the Sturm-Liouville Problem is regular, the algebraic and geometric multiplicities of each eigenvalues are the same. Thus for each simple root $k^2$, there is a corresponding simple root $\lambda_h^{(k)}$.

To find the asymptotic expansion of the approximate eigenvalue, we expand the Taylor series of $\lambda_h^{(k)}$ around $h = 0$, by ignoring the term $iO(h)$ since it does not effect the terms, which contain $h^0, h^2, h^4$

Considering the real part of Taylor series, it is obtained that

$$\lambda_h^{(k)} = k^2 - \frac{\sin(\theta_0 + .... + \theta_{n-1})}{\cos(\theta_0 + .... + \theta_{n-1})\sum_{j=0}^{n-1}\frac{\partial}{\partial\lambda}\theta_j}$$

$$\lambda_h^{(k)} = k^2 - \frac{1}{h}\frac{\tan\left(\sum_{j=0}^{n-1}\arccos(1+\frac{h^2}{2}(q_j-k^2))\right)}{\sum_{j=0}^{n-1}\frac{1}{\sqrt{4(k^2-q_j)-h^2(k^2-q_j)^2}}}$$

and

$$\lambda_h^{(k)} = \left[k^2 - \frac{2\sum_{j=0}^{n-1}\sqrt{k^2-q_j}}{\sum_{j=0}^{n-1}\frac{1}{\sqrt{k^2-q_j}}}\right]$$

$$+h^2\left[\frac{-3\left(\sum_{j=0}^{n-1}\sqrt{k^2-q_j}\right)^2+\left(\sum_{j=0}^{n-1}\frac{1}{\sqrt{k^2-q_j}}\right)\left(\sum_{j=0}^{n-1}(k^2-q_j)^{3/2}+8\left(\sum_{j=0}^{n-1}\sqrt{k^2-q_j}\right)^3\right)}{12\left(\sum_{j=0}^{n-1}\frac{1}{\sqrt{k^2-q_j}}\right)^2}\right]$$

$$+O(k^6h^4).$$

Since $q(x)$ is continuous on $0 \le x \le \pi$,

$$\left|\sum_{j=0}^{n-1}\sqrt{k^2-q_j}\right| \le n\sqrt{k^2-q_{\max}}, \quad \text{for fixed } k.$$

$$\left|\sum_{j=0}^{n-1}\frac{1}{\sqrt{k^2-q_j}}\right| \le n\frac{1}{\sqrt{k^2-q_{\min}}} \text{ are hold.}$$

As a consequence, the coefficient term of $h^2$ is of order $O(k^4)$, i.e.

$$\lambda_h^{(k)} = \left[k^2 - \frac{2\sum_{j=0}^{n-1}\sqrt{k^2-q_j}}{\sum_{j=0}^{n-1}\frac{1}{\sqrt{k^2-q_j}}}\right] + O(h^2k^4)\ . \tag{1.12}$$

## 1.2. Error Analysis and Extrapolation

In this section, we analyze the behavior of convergence due to the asymptotic expansion of the error for the eigenvalues Sturm-Liouville Problem.

We now that, for the regular Strum-Liouville Problem

$$-y''(x) = \lambda y(x), \quad 0 < x < \pi$$

$$y(0) = 0$$

$$y(\pi) = 0,$$

the eigenvalues are

$$\lambda_k = k^2,$$

and the corresponding eigenfunctions are

$$y_k(x) = c_k \sin(kx), \quad k = 1,2....$$

The error of our numerical result is

$$E_k^{(n)} = k^2 - \frac{4}{h^2}\sin^2(\frac{kh}{2}), \quad k = 1,...,n-1 \tag{1.13}$$

which satisfies

$$E_k^{(n)} = O(k^4 h^2).$$

This clearly illustrates the rapid growth of $E_k^{(n)}$ as a function $k$. To get more accuracy we use the Richardson Extrapolation method.

The asymptotic expansion of the error for the eigenvalues is obtained as

$$E_k^{(n)} = \frac{1}{3}\frac{1}{2^2}k^4h^2 - \frac{2}{45}\frac{1}{2^4}k^6h^4 + \frac{1}{315}\frac{1}{2^6}k^8h^6 - \frac{2}{141715}\frac{1}{2^8}k^{10}h^8 + ......$$

or simply

$$E_k^{(n)} = \sum_{j=1}^{\infty} a_{2j}\frac{1}{2^{2j}}k^{2j+2}h^{2j},$$

where $a_{2j}$ is constant, which given later in this section.

If we denote $\lambda_k^{(n)}[m]$ as eigenvalues after $m$ extrapolations

$$\lambda_k^{(n)}[0] = k^2 - E_k^{(n)},$$

$$\lambda_k^{(n)}[0] = k^2 - \sum_{j=1}^{\infty} a_{2j}\frac{1}{2^{2j}}k^{2j+2}h^{2j},$$

where $\lambda_h^{(k)} = \lambda_h^{(k)}[0]$ is the approximate eigenvalue obtained from finite difference form of (1.1) and (1.2).

According to algorithm of Richardson Extrapolation, corrected eigenvalues are computed as follows:

$$\lambda_k^{(n)}[m] = \frac{4^m \lambda_k^{(2n)}[m-1] - \lambda_k^{(n)}[m-1]}{4^m - 1} \tag{1.14}$$

***Theorem 1.1:*** (Kincaid & Cheney, 1996) The quantities $\lambda_k^{(n)}[m]$ defined in (1.14) satisfy the following formula

$$\lambda_k^{(n)}[m] = k^2 + \sum_{j=m+1}^{\infty} A_{j,m+1}\, k^{2j+2}\, h^{2j} \tag{1.15}$$

***Proof:***

When $m = 0$, the equation (1.15) is

$$\lambda_k^{(n)}[0] = k^2 + \sum_{j=1}^{\infty} \mathrm{A}_{j,1}\, k^{2j+2}\, h^{2j}.$$

Thus we can let $A_{j,1} = -a_{2j} \dfrac{1}{2^{2j}}$. Now proceed by induction on *m*. We assume the equation (1.15) is valid for some *m-1*, and on that basis we prove it for *m*. From equations (1.14) and (1.15), we have

$$\lambda_k^{(n)}[m] = \frac{1}{4^m - 1}\left[4^m\left(k^2 + \sum_{j=m}^{\infty} A_{j,m}\, k^{2j+2} \left(\frac{\mathrm{h}}{2}\right)^{2j}\right) - \left(k^2 + \sum_{j=m}^{\infty} A_{j,m}\, k^{2j+2}\, \mathrm{h}^{2j}\right)\right],$$

$$\lambda_k^{(n)}[m] = k^2 + \sum_{j=m}^{\infty} A_{j,m}\, k^{2j+2}\, h^{2j} \left(\frac{1}{4^m - 1}\right)\left(\frac{4^m}{2^{2j}} - 1\right), \text{ and}$$

$$\lambda_k^{(n)}[m] = k^2 + \sum_{j=m}^{\infty} A_{j,m} \left(\frac{4^m - 4^j}{4^m - 1}\right)\frac{1}{2^{2j}} k^{2j+2} h^{2j}. \tag{1.16}$$

Thus $A_{j,m+1}$ should be defined by

$$A_{j,m+1} = A_{j,m}\left(\frac{4^m - 4^j}{4^m - 1}\right)\frac{1}{2^{2j}}. \tag{1.17}$$

Notice that $A_{m,m+1} = 0$, and thus equation (1.15) can be written as

$$\lambda_k^{(n)}[m] = k^2 + \sum_{j=m+1}^{\infty} A_{j,m+1}\, k^{2j+2} h^{2j} .$$

***Theorem 1.2:*** For a certain fixed integer *k*, the error $E_k^{(n)}[m]$ of the corrected difference eigenvalues at *m* extrapolation step of Sturm-Liouville Problem (1.1) and (1.2) with (1.5) satisfying the estimate

$$E_k^{(n)}[m] \le \left| \frac{1}{2^{P(m)}} k^{2(m+2)} h^{2(m+1)} \right| , \tag{1.18}$$

$$\text{where} \quad P(m) = \begin{cases} m^2 + 7m + 2 & m \le 4 \\ m^2 + 9m - 5 & m \ge 5 \end{cases} .$$

***Proof:***

We analyze the coefficients $A_{j,m+1}$ of $k^{2j+2} h^{2j}$, $j = m+1..$ in (1.14) after $m^{th}$ extrapolation. From (1.17), we can write $A_{j,t+1}$ in terms of $A_{j,1}$

$$\begin{aligned} A_{j,m+1} &= A_{j,m} \frac{4^m - 4^j}{4^m - 1} \frac{1}{2^{2j}} \\ &= A_{j1,m-1} \frac{4^{m-1} - 4^j}{4^{m-1} - 1} \frac{1}{2^{2j}} \frac{4^m - 4^j}{4^m - 1} \frac{1}{2^{2j}} \\ &= ....... \\ &= A_{j,1} \frac{4 - 4^j}{4 - 1} .... \frac{4^m - 4^j}{4^m - 1} \left( \frac{1}{2^{2j}} \right)^m \end{aligned}$$

$$A_{j,m+1} = A_{j,1} \left[ \prod_{s=1}^{m} \frac{4^s - 4^j}{4^s - 1} \right] \left( \frac{1}{2^{2j}} \right)^m$$

and we know that

$$A_{j,1} = -a_{2j} \frac{1}{2^{2j}} .$$

Substituting into the last equation, we get

$$A_{j,m+1} = -a_{2j} \frac{1}{2^{2j}} \frac{1}{2^{2(j-1)j}} \left[ \prod_{s=1}^{m} \frac{4^s - 4^j}{4^s - 1} \right]$$

and

$$\prod_{s=1}^{m} \frac{4^s - 4^j}{4^s - 1} = 2^{j(j-1)}(-1)^{j-1}$$

Thus

$$A_{j,m+1} = (-1)^j a_{2j} \frac{1}{2^{2j}} \frac{1}{2^{2(j-1)j}} 2^{j(j-1)} = (-1)^j a_{2j} \frac{1}{2^{j(j+1)}}.$$

After these arrangements, $A_{j,m+1}$ gets the form

$$A_{j,m+1} = (-1)^j a_{2j} \frac{1}{2^{j(j+1)}}.$$

Defining the coefficients $a_{2j}$ by

$$a_{2j} = const. \begin{cases} \dfrac{1}{2^{4(j-1)}}, & \mathrm{j} \le 5 \\ \dfrac{1}{2^{6(j-1)-7}}, & \mathrm{j} \ge 6 \end{cases},$$

where $const \approx 1$ which is independent of *j*, we can write the equation (1.15) as in the form

$$\lambda_k^{(n)}[m] = k^2 + \sum_{j=m+1}^{\infty} (-1)^j \frac{1}{2^{j(j+1)}} k^{2j+2} h^{2j} \begin{cases} \dfrac{1}{2^{4(j-1)}}, & \mathrm{j} \le 5 \\ \dfrac{1}{2^{6(j-1)-7}}, & \mathrm{j} \ge 6 \end{cases}.$$

The error after *m* extrapolation ($E_k^{(n)}[m]$) will be

$$E_k^{(n)}[m] \le \left| \frac{1}{2^{(m+1)(m+2)}} k^{2(m+1)+2} h^{2(m+1)} \begin{cases} \dfrac{1}{2^{4m}}, & \mathrm{m} \le 4 \\ \dfrac{1}{2^{6m-7}}, & \mathrm{m} \ge 5 \end{cases} \right|$$

or

$$E_k^{(n)}[m] \le \left| \frac{1}{2^{P(m)}} k^{2(m+2)} h^{2(m+1)} \right|$$

$$\text{where} \quad P(m) = \begin{cases} m^2 + 7m + 2 & m \le 4 \\ m^2 + 9m - 5 & m \ge 5 \end{cases}.$$

Now we can find eigenvalue number ($k_{max}$), which has minimized error after $m$ extrapolations, or we can construct extrapolation number, which is needed to minimize error for all eigenvalues, for given $n$.

***Remark 1.1:*** For given $n$, for the regular SLP (1.1) and (1.2) with (1.5), If

$$E_k^{(n)}[m] < 1 \qquad \text{for } k=1,\dots, k_{max}$$

then the maximum number of corrected eigenvalue is

$$k_{\max} = \left[\left[ \left( 2^{P(m)} \left( \frac{n}{\pi} \right)^{2(m+1)} \right)^{\frac{1}{2(m+2)}} \right]\right], \tag{1.19}$$

where

$$P(m) = \begin{cases} m^2 + 7m + 2 & m \le 4 \\ m^2 + 9m - 5 & m \ge 5 \end{cases}.$$

***Proof:***

For given $n$, after $m$ extrapolation, the error (1.18) will be

$$E_k^{(n)}[m] < 1$$

then

$$\frac{1}{2^{P(m)}} k^{2(m+2)} h^{2(m+1)} < 1$$

and

$$k < \left( \frac{2^{P(m)}}{h^{2(m+1)}} \right)^{\frac{1}{2(m+2)}}$$

Since $\frac{\pi}{n} = h$, it is obtained that

$$k < \left( 2^{P(t)} \left( \frac{n}{\pi} \right)^{2(t+1)} \right)^{\frac{1}{2(t+2)}}.$$

So, we have the assertion

$$k_{\max} = \left[\left[ \left( 2^{P(m)} \left( \frac{n}{\pi} \right)^{2(m+1)} \right)^{\frac{1}{2(m+2)}} \right]\right],$$

$$\text{where} \quad P(m) = \begin{cases} m^2 + 7m + 2 & m \le 4 \\ m^2 + 9m - 5 & m \ge 5 \end{cases}.$$

***Remark 1.2:*** For given *n*, for the regular SLP (1.1) and (1.2) with (1.5), if

$$E_k^{(n)}[m_{\min}] < 1 \text{ for } k=1..(n-1),$$

then the number of extrapolations satisfy

$$m_{\min} = \frac{1}{\log 4}\left[ -\log(\frac{2^7 n^2}{(n-1)^2 \pi^2}) + \sqrt{-4\log 2 \log(\frac{2^2 n^2}{(n-1)^4 \pi^2}) + \left( \log(\frac{2^7 n^2}{(n-1)^2 \pi^2}) \right)^2} \right] \quad (1.20a)$$

where $P(m) = m^2 + 7m + 2$,

$$m_{\min} = \frac{1}{\log 4}\left[ -\log(\frac{2^9 n^2}{(n-1)^2 \pi^2}) + \sqrt{-4\log 2 \log(\frac{2^{-5} n^2}{(n-1)^4 \pi^2}) + \left( \log(\frac{2^9 n^2}{(n-1)^2 \pi^2}) \right)^2} \right] \quad (1.20b)$$

where $P(m) = m^2 + 9m - 5$.

***Proof:***

For given *n*, we have the equation (1.18), and the worse error occurs at $(n\text{-}1)^{th}$ eigenvalue. So choose $k=(n\text{-}1)$ for equation (1.18), we get

$$\frac{1}{2^{P(m)}}(n-1)^{2(m+2)} h^{2(m+1)} \le 1.$$

Substituting $\frac{\pi}{n} = h$ into the last inequality and after some calculations give,

$$(n-1)^{2(m+2)} \leq 2^{P(m)} \left(\frac{n}{\pi}\right)^{2(m+1)}.$$

Let's take natural logarithms of above inequality, we have

$$2(m+2)\log(n-1) \leq P(m)\log 2 + 2(m+1)\log\left(\frac{n}{\pi}\right). \tag{1.21}$$

Since we constructed *P(m)* as a piecewise function, we investigate the roots of (1.21) separately. If $P(m) = m^2 + 7m + 2$ then the inequality (1.21) gets the form

$$m^2 \log 2 + m \log\left(\frac{2^7}{(n-1)^2}\left(\frac{n}{\pi}\right)^2\right) + \log\left(\frac{2^2}{(n-1)^4}\left(\frac{n}{\pi}\right)^2\right) \geq 0. \tag{1.22}$$

If $P(m) = m^2 + 9m - 5$ then

$$m^2 \log 2 + m \log\left(\frac{2^9}{(n-1)^2}\left(\frac{n}{\pi}\right)^2\right) + \log\left(\frac{2^{-5}}{(n-1)^4}\left(\frac{n}{\pi}\right)^2\right) \geq 0. \tag{1.23}$$

Hence positive root of (1.22)

$$m_{\min} = \frac{1}{\log 4}\left[-\log\left(\frac{2^7 n^2}{(n-1)^2 \pi^2}\right) + \sqrt{-4\log 2 \log\left(\frac{2^2 n^2}{(n-1)^4 \pi^2}\right) + \left(\log\left(\frac{2^7 n^2}{(n-1)^2 \pi^2}\right)\right)^2}\right]$$

where $P(m) = m^2 + 7m + 2$, and positive root of (1.23)

$$m_{\min} = \frac{1}{\log 4}\left[-\log\left(\frac{2^9 n^2}{(n-1)^2 \pi^2}\right) + \sqrt{-4\log 2 \log\left(\frac{2^{-5} n^2}{(n-1)^4 \pi^2}\right) + \left(\log\left(\frac{2^9 n^2}{(n-1)^2 \pi^2}\right)\right)^2}\right]$$

where $P(m) = m^2 + 9m - 5$.

**Conclusion 1.1:** For given *n*, to get the error estimates $E_k^{(n)}\left[m_{\min}\right] \le h^{2r}$ for all *k=1..(n-1)* , for $r \in Z^+$ of the corrected eigenvalues, the number of extrapolation which is necessary and sufficient is

$$m_{\min} = \frac{1}{\log 4}\left[ -\log(\frac{2^7 n^2}{(n-1)^2 \pi^2}) + \sqrt{-4\log 2 \log(\frac{2^2 n^{2(1-r)}}{(n-1)^4 \pi^{2(1-r)}}) + \left( \log(\frac{2^7 n^2}{(n-1)^2 \pi^2}) \right)^2} \right] \quad (1.24a)$$

where $P(m) = m^2 + 7m + 2$,

$$m_{\min} = \frac{1}{\log 4}\left[ -\log(\frac{2^9 n^2}{(n-1)^2 \pi^2}) + \sqrt{-4\log 2 \log(\frac{2^{-5} n^{2(1-r)}}{(n-1)^4 \pi^{2(1-r)}}) + \left( \log(\frac{2^9 n^2}{(n-1)^2 \pi^2}) \right)^2} \right] \quad (1.24b)$$

where $P(m) = m^2 + 9m - 5$.

**Conclusion 1.2:** Since $q(x)$ is continuous on $[0,\pi]$,

$$q_{\min} \le q(x) \le q_{\max},$$

the asymptotic error formula (1.12) of the eigenvalues of SLP (1.1) and (1.2) when $q(x) \ne 0$, satisfy the similar inequality

$$\left| E_k^{(n)} \right| \le \sum_{j=1}^{\infty} \bar{a}_{2j} \frac{1}{2^{2j}} k^{2j+2} h^{2j},$$

where $\bar{a}_{2j}$ is a constant depend on the maximum and minimum value of the function $q(x)$, and to get the error estimates as

$$E_k^{(n)}\left[m_{\min}\right] \le h^{2r}, \quad \text{for } r \in Z^+,$$

the number of extrapolation is (1.24a) and (1.24b).

All of these conclusions are discussed and demonstrated by the numerical results in the last chapter.

# CHAPTER TWO

# COMPUTATION OF THE EIGENVALUES OF STURM-LIOUVILLE PROBLEM WITH *t*-PERIODIC BOUNDARY CONDITIONS AND ASYMPTOTIC EXPANSION

## 2.1. Computation of the Eigenvalues of Sturm-Liouville Problem with *t*-Periodic Boundary Conditions and Asymptotic Expansion

Consider the Sturm-Liouville Problem on $[0,1]$ with the t-periodic boundary conditions

$$-y''(x)+q(x)y(x)=\lambda y(x), \qquad 0<x<1 \tag{2.1}$$

$$\begin{aligned} y(1)&=e^{it}y(0) \\ y'(1)&=e^{it}y'(0), \end{aligned} \tag{2.2}$$

where $t\in(0,2\pi)-\{\pi\}$ and

$$q(x)=q(x+1). \tag{2.3}$$

As in the Chapter 1, the problem is converted to the system

$$Y'(x)=AY(x)+(q(x)-\lambda)BY(x), \qquad 0<x<1, \tag{2.4}$$

$$Y(1)=e^{it}Y(0).$$

Taking $h=\dfrac{1}{n}$, $n\in\mathrm{N}$, define $x_j=jh$, $q_j=q(x_j)$, $j=0,1,..,n$. By using the same method and the similar consideration given in Chapter 1 we obtained

$$Z_n=\left[\prod_{j=0}^{n-1}D_{n-1-j}+ih\sum_{j=0}^{n-1}f_jD_{n-1}D_{n-2}D_{n-3}...D_{j+1}SD_{j-1}...D_1D_0+O(h^2)\right]Z_0. \tag{2.5}$$

From boundary conditions (2.2) we have

$$Y_n=e^{it}Y_0.$$

Since $Y_j = P_j Z_j$,

$$P_n Z_n = e^{it} P_0 Z_0$$

and from (2.3) (i.e. $q_n = q_0$)

$$Z_n = e^{it} Z_0 .$$

Using (2.5) , we get

$$\left[\prod_{j=0}^{n-1} D_{n-1-j} + ih\sum_{j=0}^{n-1} f_j D_{n-1} D_{n-2} D_{n-3} \ldots D_{j+1} S D_{j-1} \ldots D_1 D_0 + O(h^2) - e^{it} I\right] Z_0 = \underline{0} .$$

By the same idea as in Chapter 1, the following must be

$$\det\left[\prod_{j=0}^{n-1} D_{n-1-j} + ih\sum_{j=0}^{n-1} f_j D_{n-1} D_{n-2} D_{n-3} \ldots D_{j+1} S D_{j-1} \ldots D_1 D_0 + O(h^2) - e^{it} I\right] = 0 .$$

**For the case $q(x) = 0$, $\forall x$** , the determinant is

$$\det\left[D^n - e^{it} I\right] = 0 ,$$

$$\text{i.e. } e^{\mp in\theta} - e^{it} = 0 ,$$

where

$$\theta = \arccos\left[\frac{1}{2}\left(2 - h^2 \lambda_h\right)\right].$$

Solving this, the approximate eigenvalue $\lambda_h$ is computes as

$$\lambda_h^{(k)} = \frac{4}{h^2} \sin^2\left[\frac{h}{2}(t + 2k\pi)\right], \quad t \in [0, 2\pi), \quad k = 1, \ldots, n-1$$

**For the case $q(x) \neq 0$**, as in the Chapter 1 we obtained the determinant

$$\psi(\lambda_h) = \left(e^{-i(\theta_0 + \ldots + \theta_{n-1})} - e^{it}\right)\left(e^{i(\theta_0 + \ldots + \theta_{n-1})} - e^{it}\right) + O(ih)$$

$$\psi(\lambda_h) = e^{it}(e^{it} + e^{-it}) - e^{it}(e^{-i(\theta_0 + \ldots + \theta_{n-1})} + e^{i(\theta_0 + \ldots + \theta_{n-1})}) + O(ih)$$

$$\psi(\lambda_h) = e^{it} \cos t - e^{it} \cos(\theta_0 + \ldots + \theta_{n-1}) + O(ih)$$

Ignoring $O(ih)$ term, by Newton's method taking the starting value as $(t + 2k\pi)$ with one step, $\lambda_h^{(k)}$ is computed,

$$\lambda_h^{(k)} = (t + 2k\pi)^2 - \frac{\psi\left[(t + 2k\pi)^2\right]}{\psi_\lambda\left[(t + 2k\pi)^2\right]},$$

where

$$\psi_\lambda = \frac{d}{d\lambda}\psi(\lambda)$$

and

$$\lambda_h^{(k)} = (t+2k\pi)^2 - \frac{\cos t - \cos(\theta_0 + \ldots + \theta_{n-1})}{\sin(\theta_0 + \ldots + \theta_{n-1})\sum_{j=0}^{n-1}\frac{\partial}{\partial\lambda}\theta_j}$$

$$= (t+2k\pi)^2 - \frac{1}{h}\left[\frac{\cos t - \cos(\sum_{j=0}^{n-1}\arccos(1-\frac{h^2}{2}c_J))}{\sin(\sum_{j=0}^{n-1}\arccos(1-\frac{h^2}{2}c_j))\sum_{j=0}^{n-1}\frac{1}{\sqrt{4c_j - h^2c_j^2}}}\right],$$

where

$$c_j = ((t+2k\pi)^2 - q_j).$$

Using Taylor Expansion of $\lambda_h^{(k)}$ at $h=0$ gives the asymptotic expansion of the computed approximate eigenvalues

$$\lambda_h^{(k)} = \frac{1}{h^2}\left[\frac{2(\cos t - 1)}{\left(\sum_{j=0}^{n-1}\frac{1}{\sqrt{c_j}}\right)\left(\sum_{j=0}^{n-1}\sqrt{c_j}\right)}\right] +$$

$$+\left[(t+2k\pi)^2 - \frac{1}{4\left(\sum_{j=0}^{n-1}\frac{1}{\sqrt{c_j}}\right)^2} + \frac{\sum_{j=0}^{n-1}\sqrt{c_j}}{\left(\sum_{j=0}^{n-1}\frac{1}{\sqrt{c_j}}\right)} + \frac{\frac{1}{12}\sum_{j=0}^{n-1}\left(\sqrt{c_j}\right)^3 - \frac{1}{3}\left(\sum_{j=0}^{n-1}\sqrt{c_j}\right)^3}{\left(\sum_{j=0}^{n-1}\frac{1}{\sqrt{c_j}}\right)\left(\sum_{j=0}^{n-1}\sqrt{c_j}\right)^2}\right.$$

$$\left.\left.+\cos t\left[\frac{1}{4\left(\sum_{j=0}^{n-1}\frac{1}{\sqrt{c_j}}\right)^2} - \frac{\frac{1}{12}\sum_{j=0}^{n-1}\left(\sqrt{c_j}\right)^3 - \frac{1}{3}\left(\sum_{j=0}^{n-1}\sqrt{c_j}\right)^3}{\left(\sum_{j=0}^{n-1}\frac{1}{\sqrt{c_j}}\right)\left(\sum_{j=0}^{n-1}\sqrt{c_j}\right)^2}\right]\right]\right] + O(h^2k^4).$$

Since $q(x)$ is continuous on $[0,1]$,

$$q_{\min} \le q(x) \le q_{\max}$$

The first term of the asymptotic expansion of $\lambda_k^{(n)}$ is bounded by

$$\frac{1}{h^2}\left[\frac{2(\cos t-1)}{\left(\sum_{j=0}^{n-1}\frac{1}{\sqrt{c_j}}\right)\left(\sum_{j=0}^{n-1}\sqrt{c_j}\right)}\right] \le \frac{1}{h^2}\left[\frac{2(\cos t-1)}{\left(\frac{n}{\sqrt{(t+2k\pi)^2-q_{\max}}}\right)\left(n\sqrt{(t+2k\pi)^2-q_{\min}}\right)}\right]$$

$$\le n^2h^2\frac{2(\cos t-1)}{\sqrt{\frac{(t+2k\pi)^2-q_{\min}}{(t+2k\pi)^2-q_{\max}}}} \le const$$

where $nh=1$ and $const$ is a constant independent of h and k.

## 2.2. Error Analysis and Extrapolation

We now that, for the regular Strum-Liouville Problem when, $q(x)=0$

$$-y''(x)=\lambda y(x), \quad 0<x<1$$

$$y(1)=e^{it}y(0)$$
$$y'(1)=e^{it}y'(0),$$

the eigenvalues are

$$\lambda_k=(t+2k\pi)^2, \quad k=1,2,....$$

The error of our numerical result is

$$E_k^{(n)}=(t+2k\pi)^2-\frac{4}{h^2}\sin^2\left(\frac{h}{2}(t+2k\pi)\right), \quad k=1,..,n. \tag{2.6}$$

This satisfies

$$E_k^{(n)}=O(k^4h^2).$$

This clearly illustrates the rapid growth of $E_k^{(n)}$ as a function $k$. To improve the approximate eigenvalues we use Richardson Extrapolation method.

The asymptotic expansion of the error for the eigenvalues is

$$E_k^{(n)} = \frac{1}{3}\frac{1}{2^2}(t+2k\pi)^4 h^2 - \frac{2}{45}\frac{1}{2^4}(t+2k\pi)^6 h^4 + \frac{1}{315}\frac{1}{2^6}(t+2k\pi)^8 h^6 + \ldots\ldots$$

or simply

$$E_k^{(n)} = \sum_{j=1}^{\infty} a_{2j} \frac{1}{2^{2j}}(t+2k\pi)^{2j+2} h^{2j},$$

where $a_{2j}$ is constant, which is defined in Chapter 1.

The error after $m$ extrapolation ($E_k^{(n)}[m]$) is constructed in chapter 1 (inequality (1.18)). Of course $k$ in (1.18) is replaced by $(t+2k\pi)$.

Hence

$$E_k^{(n)}[m] \le \left| \frac{1}{2^{P(m)}}(t+2k\pi)^{2(m+2)} h^{2(m+1)} \right|, \tag{2.7}$$

$$\text{where} \quad P(m) = \begin{cases} m^2+7m+2 & m \le 4 \\ m^2+9m-5 & m \ge 5 \end{cases}.$$

***Remark 2.1:*** For given $n$, for the regular SLP (2.1) and (2.2), if

$$E_k^{(n)}[m] < 1 \qquad \text{for } k=1,.., k_{max}$$

then the maximum number of corrected eigenvalue is

$$k_{\max} = \left[\left| \frac{1}{2\pi}\left(2^{P(m)} n^{2(m+1)}\right)^{\frac{1}{2(m+2)}} - t \right|\right],$$

where

$$P(m) = \begin{cases} m^2+7m+2 & m \le 4 \\ m^2+9m-5 & m \ge 5 \end{cases}.$$

***Proof:***

Proof is similar to proof of Remark 1.1 in chapter 1.

Since $t \in (0,2\pi) - \{\pi\}$, we can rearrange the error (2.7) as

$$E_k^{(n)}[m] < \left| \frac{1}{2^{P(m)}} (2\pi + 2k\pi)^{2(m+2)} h^{2(m+1)} \right|,$$

or simply

$$E_k^{(n)}[m] < \left| \frac{1}{2^{P(m)}} (2\pi)^{2(m+2)} (1+k)^{2(m+2)} h^{2(m+1)} \right|,$$

and $2\pi < 2^3$

$$E_k^{(n)}[m] < \left| \frac{1}{2^{P(m)}} 2^{6(m+2)} (1+k)^{2(m+2)} h^{2(m+1)} \right|.$$

Therefore

$$E_k^{(n)}[m] < \left| \frac{1}{2^{G(m)}} (1+k)^{2(m+2)} h^{2(m+1)} \right|, \tag{2.8}$$

where

$$G(m) = \begin{cases} m^2 + m - 10 & m \le 4 \\ m^2 + 3m - 17 & m \ge 5 \end{cases}.$$

***Remark 2.2:*** For given *n*, for the regular SLP (2.1) and (2.2), if

$E_k^{(n)}[m_{\min}] \le 1$ for all *k=1..(n-1),*

The number of extrapolation which is necessary and sufficient is

$$m_{\min} = \frac{1}{2}\left( -1 + \sqrt{1 + \frac{4\log(2^{10} n^2)}{\log 2}} \right),$$

where $G(m) = m^2 + m - 10$,

$$m_{\min} = \frac{1}{2}\left( -3 + \sqrt{9 + \frac{4\log(2^{17} n^2)}{\log 2}} \right),$$

where $G(m) = m^2 + 3m - 17$.

***Proof:***

It is similar to the proof of Remark 1.2.

***Conclusion 2.1:*** For given *n*, to get the error estimates $E_k^{(n)}[m_{\min}] \le h^{2r}$ for all *k=1..(n-1)* , for $r \in Z^+$ of the corrected eigenvalues of SLP (2.1) and (2.2), the number of extrapolation which is necessary and sufficient is

$$m_{\min} = \frac{1}{2}\left( -1 + \sqrt{1 + \frac{4\log\left(2^{10} n^{2+2r}\right)}{\log 2}} \right),$$

where $G(m) = m^2 + m - 10$,

$$m_{\min} = \frac{1}{2}\left( -3 + \sqrt{9 + \frac{4\log\left(2^{17} n^{2+2r}\right)}{\log 2}} \right),$$

where $G(m) = m^2 + 3m - 17$.

All computation results are given in Chapter 3.

# CHAPTER THREE
# NUMERICAL RESULTS

To illustrate the Richardson Extrapolation based on finite difference method for Sturm-Liouville Problems with homogeneous and t-periodic boundary conditions, the following problem used in the papers (Andrew & Paine, 1985) is considered.

***Example 3.1:***

$$-y''(x)+q(x)y(x)=\lambda y(x) \tag{3.1.1}$$

$$y(0)=y(\pi)=0 \tag{3.1.2}$$

where $x\in[0,\pi]$, $q(x)\in C^1[0,\pi]$.

Error estimates with 30 subintervals were calculated and are presented in Table 3.1 and Table 3.2, and the calculated eigenvalues are shown in Figure 3.1 and Figure 3.2.

**Table 3.1** Error estimates for (3.1.1) and (3.1.2) with *n=30* for $q(x)=0$

| $k$ | $\lvert\lambda_k-\lambda_k^n\rvert$ | $\lvert\lambda_k-\lambda_k^n[1]\rvert$ | $\lvert\lambda_k-\lambda_k^n[2]\rvert$ |
|---|---|---|---|
| 1 | 0,00091 | $8{,}34916\text{x}10^{-8}$ | $1{,}85874\text{x}10^{-12}$ |
| 2 | 0,0146 | $5{,}33957\text{x}10^{-6}$ | $2{,}58549\text{x}10^{-10}$ |
| 3 | 0,07377 | $6{,}07467\text{x}10^{-5}$ | $6{,}69691\text{x}10^{-9}$ |
| 25 | 284,678 | 17,5064 | 0,1412 |
| 26 | 327,011 | 21,8785 | 0,19169 |
| 27 | 373,17 | 27,0879 | 0,25709 |
| 28 | 423,229 | 33,2467 | 0,34093 |
| 29 | 477,243 | 40,4753 | 0,44739 |

**Table 3.2** Error estimates for (3.1.1) and (3.1.2) with $n$=*30* for $q(x)=e^x$

| $k$ | $\lambda_k^n$ | $\left\|\lambda_k^n - \lambda_k^n[1]\right\|$ | $\left\|\lambda_k^n[1] - \lambda_k^n[2]\right\|$ | $\left\|\lambda_k^n[2] - \lambda_k^n[3]\right\|$ |
|---|---|---|---|---|
| 1 | 4,89155 | 0,00512 | 9,40362x10$^{-7}$ | 1,53078 x10$^{-10}$ |
| 2 | 10,0145 | 0,03072 | 1,71546x10$^{-5}$ | 3,57413 x10$^{-9}$ |
| 3 | 15,9221 | 0,09724 | 4,78974x10$^{-5}$ | 4,18342 x10$^{-8}$ |
| 25 | 347,032 | 267,625 | 17,2539 | 0,14696 |
| 26 | 355,534 | 305,805 | 21,5181 | 0,20088 |
| 27 | 362,064 | 347,167 | 26,5526 | 0,27296 |
| 28 | 366,69 | 391,486 | 32,5168 | 0,36350 |
| 29 | 372,983 | 433,86 | 40,8161 | 0,39481 |

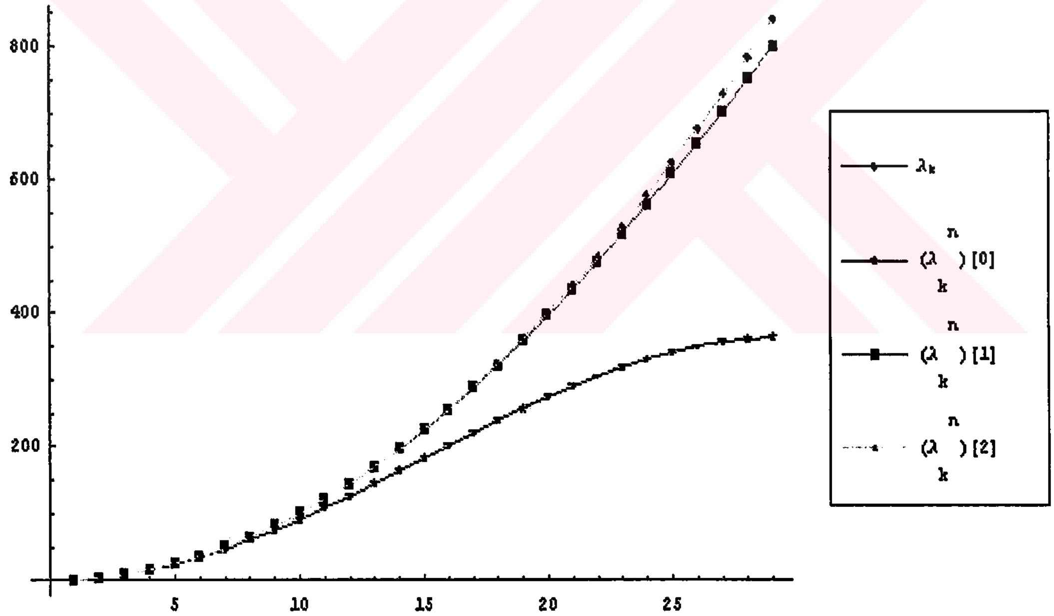


**Figure 3.1** (*n=30*, for *q(x)=0* )

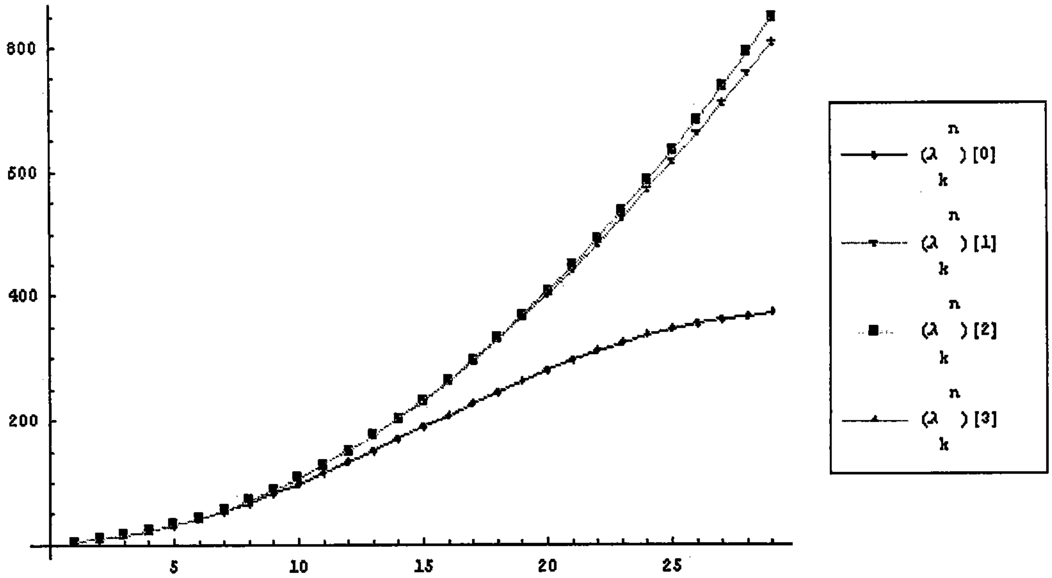


**Figure 3.2** (*n=30*, for $q(x)= e^x$)

**Table 3.3** Error estimates for (3.1.1) and (3.1.2) with $n=100$ for $q(x)=e^x$

| $k$ | $\lambda_k^n$ | $\lvert\lambda_k^n-\lambda_k^n[1]\rvert$ | $\lvert\lambda_k^n[1]-\lambda_k^n[2]\rvert$ | $\lvert\lambda_k^n[2]-\lambda_k^n[3]\rvert$ | $\lvert\lambda_k^n[3]-\lambda_k^n[4]\rvert$ |
|---|---|---|---|---|---|
| 1 | 4,89621 | 4,6051x10$^{-4}$ | 7,65411x10$^{-9}$ | 2,71871x10$^{-12}$ | 4,6203x10$^{-11}$ |
| 2 | 10,0424 | 0,00275 | 1,38391x10$^{-7}$ | 2,71871x10$^{-11}$ | 1,68399x10$^{-12}$ |
| 3 | 16,0105 | 0,00873 | 3,8152x10$^{-7}$ | 2,04032x10$^{-11}$ | 2,03535x10$^{-11}$ |
| 96 | 4043,4 | 4747,8 | 427,75 | 4,71557 | 0,00757 |
| 97 | 4050,08 | 4907,78 | 453,068 | 5,11499 | 0,00829 |
| 98 | 4054,75 | 5071,18 | 479,565 | 5,54255 | 0,00908 |
| 99 | 4061,05 | 5232,5 | 508,572 | 5,91784 | 0,01125 |

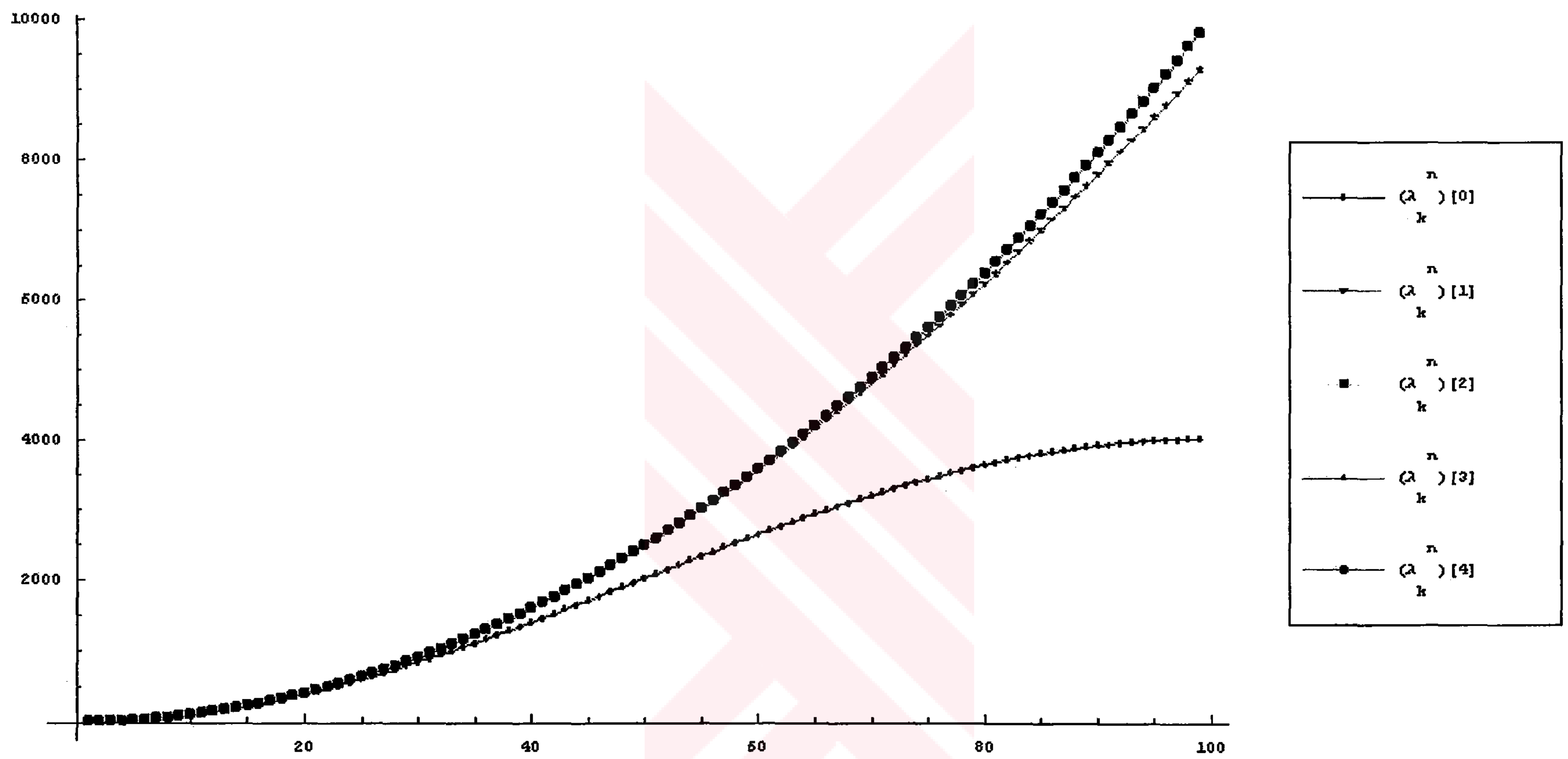


Figure 3.3 ($n=100$ , for $q(x)=e^x$)

When we look at Table 3.1 we can see easily two extrapolations is enough to get error less than 1. In Chapter 1 we conclude that the same extrapolation number, which is needed for $q(x)=0$ is also valid for $q(x)\neq 0$. And in Table 3.2 we see that the difference between eigenvalues after 2nd and 3rd extrapolations is nearly 0. It means the 2nd extrapolation is very close to exact eigenvalues. Also from Table 3.3 we can say 3rd extrapolation is enough for n=100, $q(x)=e^x$. So in Figure 3.3 the eigenvalues after 3rd and 4th extrapolations are so close that we cannot see the 3rd extrapolation eigenvalues because of 4th extrapolation eigenvalues.

***Example 3.2:***

$$-y''(x)+q(x)y(x)=\lambda y(x) \tag{3.2.1}$$

$$y(1)=e^{it}y(0)$$
$$y'(1)=e^{it}y'(0) \tag{3.2.2}$$

where $x \in [0,1]$, $q(x) \in C^1[0,1]$ and $q(x)=q(x+1)$ and $t=2$.

Error estimates with 50 subintervals were calculated and are presented in Table 3.4 and Table 3.5, and the calculated eigenvalues are shown in Figure 3.4 and Figure 3.5.

**Table 3.4** Error estimates for (3.2.1) and (3.2.2) with $n=50$ for $q(x)=0$

| $k$ | $\left\|\lambda_k-\lambda_k^n\right\|$ | $\left\|\lambda_k-\lambda_k^n[1]\right\|$ | $\left\|\lambda_k-\lambda_k^n[2]\right\|$ | $\left\|\lambda_k-\lambda_k^n[3]\right\|$ | $\left\|\lambda_k-\lambda_k^n[4]\right\|$ |
|---|---|---|---|---|---|
| 1 | 0,15677 | $3,58654\text{x}10^{-5}$ | $1,01385\ \text{x}10^{-9}$ | $3,37366\ \text{x}10^{-11}$ | $8,3125\ \text{x}10^{-10}$ |
| 2 | 1,49643 | $1,05936\ \text{x}10^{-3}$ | $1,00182\ \text{x}10^{-7}$ | $5,64171\ \text{x}10^{-11}$ | $8,24059\ \text{x}10^{-10}$ |
| 3 | 6,26252 | $9,09186\ \text{x}10^{-3}$ | $1,76658\ \text{x}10^{-6}$ | $2,24986\ \text{x}10^{-10}$ | $5,88159\ \text{x}10^{-10}$ |
| 23 | 11577,4 | 908,183 | 9,30043 | 0,01441 | $3,74687\ \text{x}10^{-6}$ |
| 24 | 13365,1 | 1149,14 | 12,8735 | 0,02176 | $6,17044\ \text{x}10^{-6}$ |
| 47 | 88111,8 | 35531,1 | 1835,12 | 13,1613 | 0,01512 |
| 48 | 92057,4 | 39021,1 | 2124,89 | 15,9928 | 0,01923 |
| 49 | 96004,9 | 42720,4 | 2451,1 | 19,3454 | 0,02433 |

**Table 3.5** Error estimates for (3.2.1) and (3.2.2) with $n=50$ for $q(x)=\sin(2\pi x)$

| $k$ | $\lambda_k^n$ | $\left\|\lambda_k^n-\lambda_k^n[1]\right\|$ | $\left\|\lambda_k^n[1]-\lambda_k^n[2]\right\|$ | $\left\|\lambda_k^n[2]-\lambda_k^n[3]\right\|$ | $\left\|\lambda_k^n[3]-\lambda_k^n[4]\right\|$ | $\left\|\lambda_k^n[4]-\lambda_k^n[5]\right\|$ |
|---|---|---|---|---|---|---|
| 1 | 68,4565 | 0,15674 | $3,58415\text{x}10^{-5}$ | $7,58504\ \text{x}10^{-10}$ | $1,14314\ \text{x}10^{-9}$ | $1,14872\ \text{x}10^{-8}$ |
| 2 | 210,683 | 1,45538 | 0,00105 | $1,00265\ \text{x}10^{-7}$ | $2,2311\ \text{x}10^{-11}$ | $4,98385\ \text{x}10^{-9}$ |
| 3 | 428,442 | 6,25344 | 0,00908 | $1,76653\ \text{x}10^{-6}$ | $1,23026\ \text{x}10^{-9}$ | $4,77405\ \text{x}10^{-9}$ |
| 23 | 9888,77 | 10699,2 | 898,883 | 9,28605 | 0,01440 | $3,76046\ \text{x}10^{-6}$ |
| 24 | 9981,65 | 12216 | 1136,26 | 12,852 | 0,02175 | $6,19175\ \text{x}10^{-6}$ |
| 47 | 281,231 | 52580,7 | 33696 | 1821,96 | 13,1462 | 0,01511 |
| 48 | 111,234 | 53036,3 | 36896,2 | 2108,5 | 15,9736 | 0,01922 |
| 49 | 18,3452 | 53284,4 | 40265,4 | 2431,74 | 15,3216 | 0,02432 |

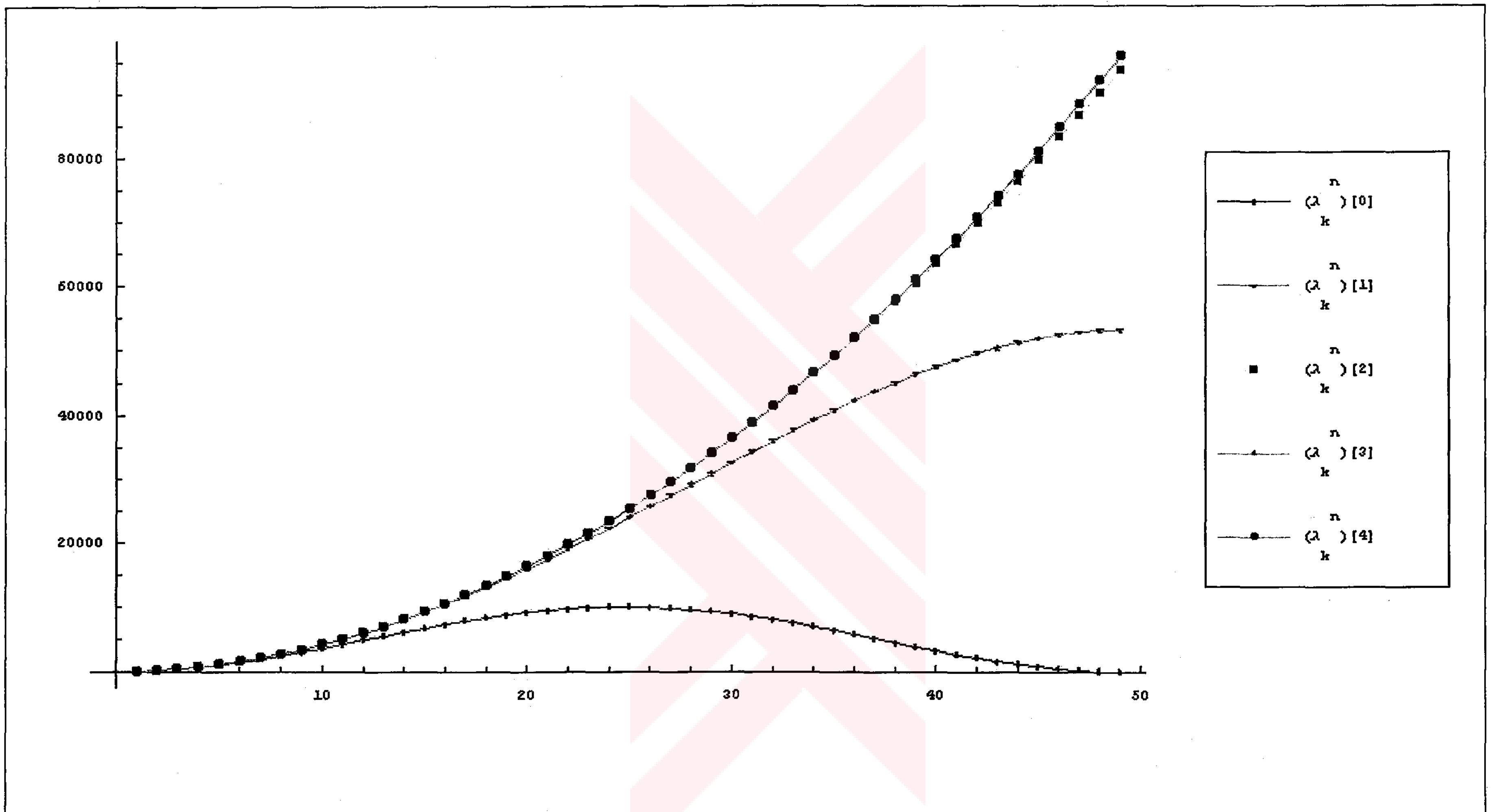


Figure 3.4 ($n=50$, for $q(x)=0$)

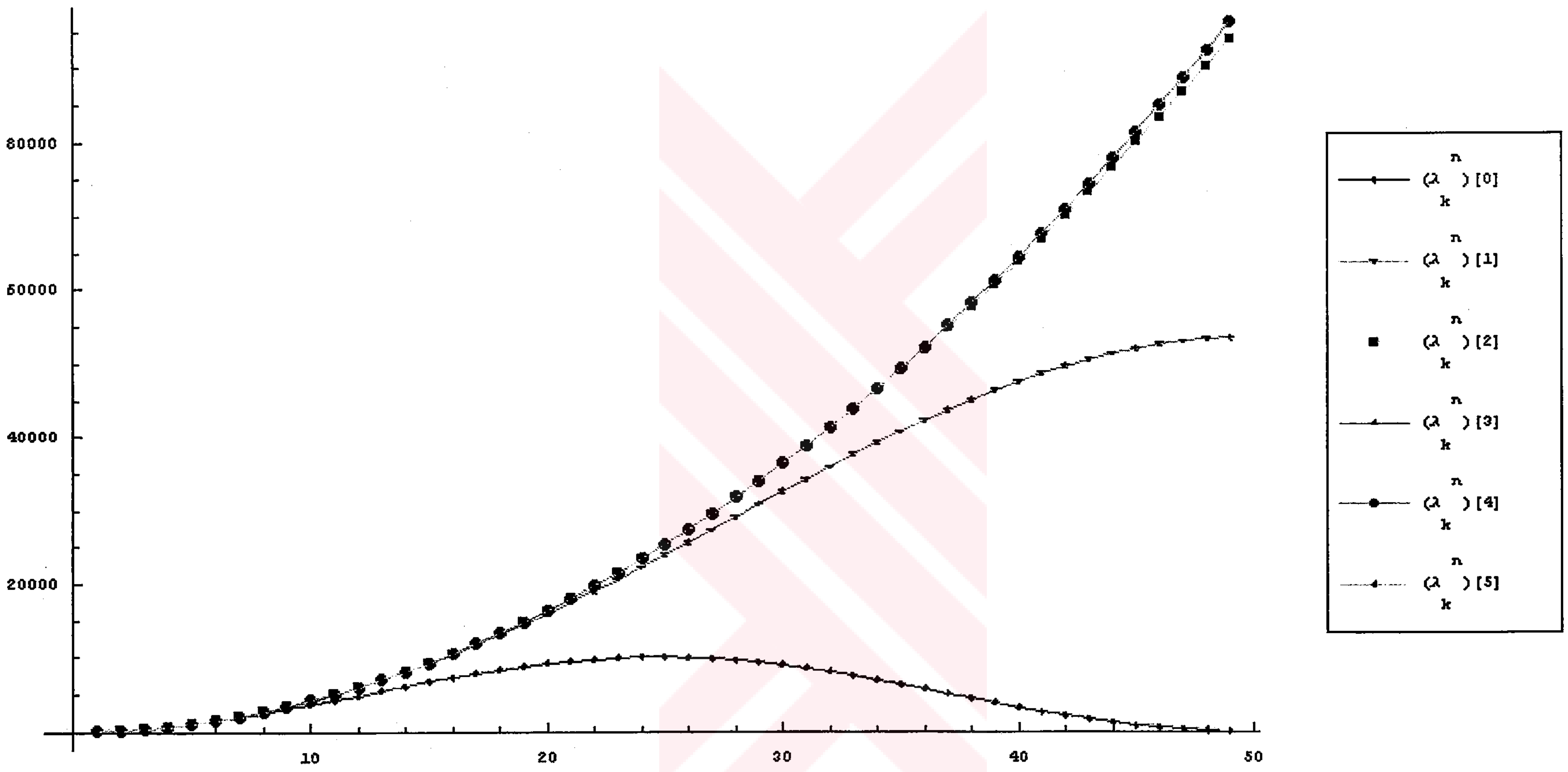


Figure 3.5 (*n=50*, for *q(x)= sin2πx* )

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