ACCELERATION AND THE DIFFERENTIAL GEOMETRY OF SCREWS

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by
Celal Cem SARIOĞLU

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We certify that we have read this thesis and **"ACCELERATION AND THE DIFFERENTIAL GEOMETRY OF SCREWS"** completed by **CELAL CEM SARIOĞLU** under supervision of **PROF.DR. ÖMER KÖSE** and that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Prof. Dr. Ömer KÖSE

Supervisor

Prof. Dr. Şuur NİZAMOĞLU

(Committee Member)

Yrd. Doç. Dr. Bedia AKYAR MØLLER

(Committee Member)

Approved by the

Graduate School of Natural and Applied Sciences

Prof.Dr. Cahit HELVACI

Director

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ABSTRACT

In this dissertation, by using the concept of "differential screw" which characterizes the difference between two consecutive screws, a formula for acceleration at all points on moving dual unit sphere (D.U.S.) is derived with respect to the reference system attached to the fixed axode. Then some special points on moving D.U.S.; acceleration centers, inflection points and Bresse complex are studied.

Keywords : kinematics, dual vectors, dual spherical motion, Euler angles, differential screw, acceleration, acceleration center, inflection points, Bresse complex

ÖZET

Bu çalışmada, birbirine çok yakın iki vida arasındaki uzaklığı karakterize eden diferansiyel vida kavramı kullanılarak hareketli dual birim küre üzerindeki noktaların ivmesi için sabit regle yüzey üzerindeki koordinat sistemine geçilerek bir formül verilmiştir. Hareketli dual birim kürenin bazı özel noktaları; ivme merkezleri, büküm noktaları ve Bresse yüzeyleri çalışılmıştır

Anahtar sözcükler : kinematik, dual vektörler, dual küresel hareket, Euler açıları, diferansiyel vida, ivme, ivme merkezi, büküm noktaları, Bresse yüzeyi

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CHAPTER ONE
INTRODUCTION

Dual numbers were introduced in the 19th century by Clifford, and their application to rigid body kinematics was subsequently generalized by Kotolnikow and Study in their principle of transference, and also dual numbers algebra and principle of transference are used by (Köse, 1974), (Müller, 1963) and (Bottema et al.1979) in their study.

The general spatial displacement of a rigid body (a screw displacement) consists of a rotation through an angle θ about and a translation through a distance θ^* along a finite screw. Therefore, by the principle of transference, a screw displacement corresponds a rotation about a fixed point in three dimensional dual space, in other words dual spherical motion. In addition, a dual spherical motion can be represented by an orthogonal matrix and it can be expressed in terms of a set of Euler angles, see (Bottema, et al., 1979) and (Rooney, 1978).

A simple approach for the investigation of the existence of the acceleration center and the distribution of acceleration field corresponding to the acceleration center are studied by (Moher, 1997).

The locus of spatial inflection points, a twisted cubic curve common to three ruled quadric surfaces is derived in a coordinate system attached to the fixed axode, by using the differential screw and axode geometry by (Bokelberg et al. 1992). In addition acceleration center for general spatial motion with screw axes in the direction of x_2-axes and differential screw in x_2x_3 plane, and Bresse circle in planar motion are presented, in their present paper (Ridley et al. 1992).

In the preliminaries, algebra of dual numbers, dual functions, dual vectors, E.Study theorem, a dual angle between dual vectors, line complex, line congruence and ruled surfaced are studied.

The relative spatial motion (screw displacement) between two line spaces can be visualized as a twist about an axis. By extension, when the motion is divided into infinitesimal steps, each displacement becomes associated with an instantaneous screw axis. During the continuous motions, and, fixing one of the line spaces, the line of instantaneous screw axis (I.S.A) trace out two ruled surfaces, one in the fixed body, and one in the moving body, and also these axodes always meet tangentially in a common line along the I.S.A. So the arrangement of generators of axodes must be such that, as infinitesimal twisting around the current I.S.A is completed, one generator, from each axode becoming perfectly aligned along the line of the next I.S.A., therefore, consecutive screws can be considered to lie on both fixed axode and moving axode.

On the other hand, the relative spatial motion can be represented by a dual spherical motion, and axodes in fixed and moving line spaces correspond dual curves on fixed and moving dual unit spheres (D.U.S.). Therefore, the screw consecutive to current screw (dual Darboux vector), can be considered to be same consecutive screw for fixed D.U.S. Thus, acceleration at all points in moving space can be calculated with aid of the differential screw between the consecutive screws for fixed D.U.S.

After then, using our previous result we will analyze some special points of D.U.S.; acceleration centers, inflection points and Bresse complex.

1.1 Dual Numbers

Dual numbers are numbers expressed by $\hat{x} = x + \varepsilon x^*$ with $x, x^* \in IR$ and ε satisfies $\varepsilon^2 = 0$. The set of all dual numbers is denoted by

$$ID = \{\hat{x} = x + \varepsilon x^* : x, x^* \in IR\}.$$

Here x and x^* are called real and dual parts of dual number $\hat{x}$, respectively. Equality, addition and multiplication of dual numbers $\hat{x}$ and $\hat{y}$ are defined as follows

1. Equality: $\hat{x} = \hat{y}$ if and only if $x = y$ and $x^* = y^*$
2. Addition: $\hat{x} + \hat{y} = (x + \varepsilon x^*) + (y + \varepsilon y^*) = (x + y) + \varepsilon (x^* + y^*)$ (1.1.1)
3. Multiplication: $\hat{x}\hat{y} = (x + \varepsilon x^*)(y + \varepsilon y^*) = xy + \varepsilon (xy^* + x^* y)$.

In addition, the equation $\hat{a} + \hat{y} = \hat{x}$ can be uniquely solved with respect to $\hat{a}$.

The set of all dual numbers is an abelian ring under the operations addition and multiplication, since the division

$$\frac{\hat{x}}{\hat{y}} = \frac{x + \varepsilon x^*}{y + \varepsilon y^*} = \frac{(x + \varepsilon x^*)(y - \varepsilon y^*)}{(y + \varepsilon y^*)(y - \varepsilon y^*)} = \frac{x}{y} + \varepsilon \frac{x^* y - xy^*}{y^2} \qquad (1.1.2)$$

is indefinite for $y = 0$.

1.2 Dual Variable Functions

Definition 1.2.1 Let G be a nonempty subset of ID. A function f defined on G is a rule which assigns to each $\hat{x}$ in G a dual number $\hat{y}$. The dual number $\hat{y}$ is called the value of f at $\hat{x}$ and denoted by $f(\hat{x})$; that is, $\hat{y} = f(\hat{x})$. Let us denote the real and dual parts of $\hat{y}$ by $Re\, \hat{y} = u(x, x^*)$ and $Du\, \hat{y} = u^*(x, x^*)$, respectively. Then $\hat{y} = f(\hat{x})$ can be separated into real and dual parts as

$$\hat{y} = f(\hat{x}) = u(x, x^*) + \varepsilon u^*(x, x^*).$$

In this case, the function f can be written as $f = u + \varepsilon u^*$, where $u : IR \times IR \to IR$ and $u^* : IR \times IR \to IR$ are two variable functions.

The derivative of f at $\hat{x}_0$ is defined by

$$f'(\hat{x}_0) = \frac{df(\hat{x}_0)}{d\hat{x}} = \lim_{\Delta \hat{x} \to 0} \frac{f(\hat{x}_0 + \Delta \hat{x}) - f(\hat{x}_0)}{\Delta \hat{x}},$$

where $\Delta \hat{x} = \Delta x + \varepsilon \Delta x^*$.

Let $f = u + \varepsilon u^*$ and $g = v + \varepsilon v^*$ be two dual variables functions, then derivatives of the functions $f + g$, fg and $\frac{f}{g}$ $(v \neq 0)$ are given by

$$(f+g)' = f' + g',\ (fg)' = f'g + fg' \text{ and } \left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}.$$

Definition 1.2.2 A function differentiable at every point $\hat{x}_0$ in G is called an analytic function in G.

Definition 1.2.3 Let f be a dual variable analytic function in a region G, then the series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(\hat{x}_0)}{n!} (\hat{x} - \hat{x}_0)^n$$

is called Taylor expansion series of the function f at $\hat{x}_0$, where $\hat{x}_0 \in G$.

Theorem 1.2.1 For every $n \in IN$ and $\hat{x} \in ID$,

$$\hat{x}^n = (x + \varepsilon x^*)^n = x^n + \varepsilon n x^* x^{n-1}.$$

Proof We use the induction method.

For $n = 1$, $\quad \hat{x}^1 = x + \varepsilon x^*$;

For $n = 2$, $\quad \hat{x}^2 = (x + \varepsilon x^*)^2 = x^2 + \varepsilon 2x^* x$;

For $n=3$, $\quad \hat{x}^3 = (x+\varepsilon\, x^*)^2 (x+\varepsilon\, x^*) = (x^2+\varepsilon\, 2x^* x)(x+\varepsilon\, x^*) = x^3+\varepsilon\, 3x^* x^2$.

Suppose that $\hat{x}^{n-1} = (x+\varepsilon\, x^*)^{n-1} = x^{n-1}+\varepsilon\,(n-1)x^* x^{n-2}$ is true. Since $\hat{x}^n = \hat{x}^{n-1}\hat{x}$ and by (1.1.1), we get

$$\begin{aligned}\hat{x}^n = \hat{x}^{n-1}\hat{x} &= \left(x^{n-1}+\varepsilon\,(n-1)x^* x^{n-2}\right)(x+\varepsilon\, x^*)\\ &= x^{n-1}x+\varepsilon\left((n-1)x^* x^{n-2}x + x^{n-1}x^*\right)\\ &= x^n+\varepsilon\, n x^* x^{n-1}.\end{aligned}$$

Theorem 1.2.2 A dual variable analytic function f is separated into real and dual parts as

$$f(\hat{x}) = f(x+\varepsilon\, x^*) = f(x)+\varepsilon\, x^* f'(x).$$

Proof Since the function f is analytic, it can be expanded in its Taylor series at $\hat{x}_0 = 0$ as

$$f(\hat{x}) = f(0)+\frac{f'(0)}{1!}\hat{x}+\frac{f''(0)}{2!}\hat{x}^2+\cdots+\frac{f^{(n)}(0)}{n!}\hat{x}^n+\cdots.$$

By Theorem 1.2.1, we have

$$\begin{aligned}f(\hat{x}) &= f(0)+\frac{f'(0)}{1!}(x+\varepsilon\, x^*)+\frac{f''(0)}{2!}(x^2+\varepsilon\, 2x^* x)+\cdots+\\ &\qquad \frac{f^{(n)}(0)}{n!}\left(x^n+\varepsilon\, n x^* x^{n-1}\right)+\cdots\\ &= \left[f(0)+\frac{f'(0)}{1!}+\frac{f''(0)}{2!}x^2+\cdots+\frac{f^{(n)}(0)}{n!}x^n+\cdots\right]+\\ &\qquad \varepsilon\, x^*\left[f'(0)+\frac{f''(0)}{1!}x+\cdots+\frac{f^{(n)}(0)}{(n-1)!}x^{n-1}+\cdots\right].\end{aligned}$$

It is clear that the first part of right hand side of the equality is the Taylor series expansion of $f(x)$ at $x_0 = 0$ and the second part is the Taylor series expansion of $f'(x)$ at $x_0 = 0$.

Hence,

$$f(\hat{x}) = f(x)+\varepsilon\, x^* f'(x).$$

As a result of this theorem, we obtain

$$\sin\hat{x} = \sin\left(x+\varepsilon x^{*}\right) = \sin x + \varepsilon x^{*}\cos x; \qquad (1.2.1)$$

$$\cos\hat{x} = \cos\left(x+\varepsilon x^{*}\right) = \cos x - \varepsilon x^{*}\sin x; \qquad (1.2.2)$$

$$\sqrt{\hat{x}} = \sqrt{x+\varepsilon x^{*}} = \sqrt{x} + \varepsilon\frac{x^{*}}{2\sqrt{x}} \quad (x>0); \qquad (1.2.3)$$

$$\frac{1}{\hat{x}} = \frac{1}{x+\varepsilon x^{*}} = \frac{1}{x} - \varepsilon\frac{x^{*}}{x^{2}} \quad (x\neq 0). \qquad (1.2.4)$$

1.3 Dual Vectors

Definition 1.3.1 If *K* is a commutative ring with identity 1, a module is an algebraic system which behaves like a vector space, with *K* playing the role of the scalar field. To be precise, we say that a commutative group *V* is a module over *K* (or a *K*-module) if the following scalar multiplication conditions are satisfied

M1: $(c_1+c_2)\alpha = c_1\alpha + c_2\alpha,\ c_1,c_2\in K,\ \alpha\in V;$

M2: $c\left(\alpha_1+\alpha_2\right) = c\alpha_1 + c\alpha_2,\ c\in K,\ \alpha_1,\alpha_2\in V;$

M3: $(c_1c_2)\,\alpha = c_1(c_2\alpha),\ c_1,c_2\in K,\ \alpha\in V;$

M4: $1\alpha = \alpha,\ \alpha\in V.$

The set of all ordered triples of dual numbers

$$ID^3 = ID\times ID\times ID = \left\{\hat{x} : \hat{x} = \left(\hat{x}_1,\hat{x}_2,\hat{x}_3\right), \hat{x}_i\in ID, i=1,2,3\right\} \qquad (1.3.1)$$

is a module over *ID* with the operations (addition and scalar multiplication) defined by

$$+ : ID^3\times ID^3 \to ID^3$$

$$\left(\hat{x},\hat{y}\right)\to \hat{x}+\hat{y} = \left(\hat{x}_1,\hat{x}_2,\hat{x}_3\right)+\left(\hat{y}_1,\hat{y}_2,\hat{y}_3\right) = \left(\hat{x}_1+\hat{y}_1,\hat{x}_2+\hat{y}_2,\hat{x}_3+\hat{y}_3\right) \qquad (1.3.2)$$

and

$$\cdot : ID \times ID^3 \to ID^3$$
$$\left(\hat{\lambda},\hat{x}\right) \to \hat{\lambda}\hat{x} = \hat{\lambda}\left(\hat{x}_1,\hat{x}_2,\hat{x}_3\right) = \left(\hat{\lambda}\hat{x}_1,\hat{\lambda}\hat{x}_2,\hat{\lambda}\hat{x}_3\right). \quad (1.3.3)$$

(Köse, 1974).

Definition 1.3.2 The elements of *ID*-module are called dual vectors and from now on they will be denoted by bold characters such as $\hat{\boldsymbol{x}}$. In addition, from (1.3.2) and (1.3.3), every dual vector $\hat{\boldsymbol{x}}$ can be written as

$$\begin{aligned}\hat{\boldsymbol{x}} &= \left(\hat{x}_1,\hat{x}_2,\hat{x}_3\right) = \left(x_1 + \varepsilon\, x_1^*, x_2 + \varepsilon\, x_2^*, x_3 + \varepsilon\, x_3^*\right) = \left(x_1,x_2,x_3\right) + \varepsilon\left(x_1^*,x_2^*,x_3^*\right) \\ &= \boldsymbol{x} + \varepsilon\, \boldsymbol{x}^*,\end{aligned}$$

where $\boldsymbol{x} = \left(x_1,x_2,x_3\right)$ and $\boldsymbol{x}^* = \left(x_1^*,x_2^*,x_3^*\right)$.

Thus, addition and scalar multiplication of dual vectors, respectively, are given by

$$\hat{\boldsymbol{x}} + \hat{\boldsymbol{y}} = \left(\boldsymbol{x} + \boldsymbol{y}\right) + \varepsilon\left(\boldsymbol{x}^* + \boldsymbol{y}^*\right) \quad (1.3.4)$$

and

$$\hat{\lambda}\hat{\boldsymbol{x}} = \left(\lambda + \varepsilon\,\lambda^*\right)\left(\boldsymbol{x} + \varepsilon\,\boldsymbol{x}^*\right) = \lambda\boldsymbol{x} + \varepsilon\left(\lambda^*\boldsymbol{x} + \lambda\boldsymbol{x}^*\right). \quad (1.3.5)$$

Corollary 1.3.1 The three dimensional real vector space IR^3 is a subspace of ID^3 and $ID^3 = IR^3 + \varepsilon\, IR^3$ (direct sum of real vector spaces), where $\varepsilon\, IR^3$ is the set of the vectors $\varepsilon\,\boldsymbol{x}^* = \varepsilon\left(x_1^*,x_2^*,x_3^*\right)$. Therefore, if $\{\boldsymbol{e}_1,\boldsymbol{e}_2,\boldsymbol{e}_3\}$ is a basis of IR^3, then it is also a basis of ID^3 over *ID*.

Thus, the scalar product and vectorial (cross) product of ordinary vectors extend in a natural way from IR^3 to ID^3 and provides *ID*-bilinear operations in *ID*.

Definition 1.3.3 A scalar product on ID^3 is a function which assigns to each ordered pair of vectors $\hat{x}$ and $\hat{y}$ in ID^3 a scalar $\hat{x}\cdot\hat{y}=\sum_{i=1}^{3}\hat{x}_i\hat{y}_i$ in such a way that for all $\hat{x},\hat{y},\hat{z}$ in ID^3 and all scalars $\hat{\lambda}$ in ID.

1. $\hat{x}\cdot\hat{y}=\hat{y}\cdot\hat{x}$;
2. $(\hat{\lambda}\hat{x})\cdot\hat{y}=\hat{\lambda}(\hat{x}\cdot\hat{y})=\hat{x}\cdot(\hat{\lambda}\hat{y})$;
3. $(\hat{x}+\hat{y})\cdot\hat{z}=\hat{x}\cdot\hat{z}+\hat{y}\cdot\hat{z}$;
4. $\hat{x}\cdot\hat{x}=0$ if and only if $\boldsymbol{x}=\boldsymbol{0}$.

Therefore the scalar product of two dual vectors can be written as

$$\hat{x}\cdot\hat{y}=(\boldsymbol{x}+\varepsilon\,\boldsymbol{x}^*)\cdot(\boldsymbol{y}+\varepsilon\,\boldsymbol{y}^*)=\boldsymbol{x}\cdot\boldsymbol{y}+\varepsilon(\boldsymbol{x}\cdot\boldsymbol{y}^*+\boldsymbol{x}^*\cdot\boldsymbol{y}) \qquad (1.3.6)$$

Definition 1.3.4 The vectorial product on ID^3 is an operator defined by

$$\times : ID^3\times ID^3\to ID^3$$

$$\begin{aligned}\hat{x}\times\hat{y}&=(\hat{x}_1,\hat{x}_2,\hat{x}_3)\times(\hat{y}_1,\hat{y}_2,\hat{y}_3)=(\hat{x}_2\hat{y}_3-\hat{x}_3\hat{y}_2,\hat{x}_3\hat{y}_1-\hat{x}_1\hat{y}_3,\hat{x}_1\hat{y}_2-\hat{x}_2\hat{y}_1)\\&=(\boldsymbol{x}+\varepsilon\,\boldsymbol{x}^*)\times(\boldsymbol{y}+\varepsilon\,\boldsymbol{y}^*)\\&=\boldsymbol{x}\times\boldsymbol{y}+\varepsilon(\boldsymbol{x}\times\boldsymbol{y}^*+\boldsymbol{x}^*\times\boldsymbol{y})\end{aligned} \qquad (1.3.7)$$

Definition 1.3.5 The norm of the dual vector $\hat{x}$ is defined by $(\hat{x}\cdot\hat{x})^{\frac{1}{2}}$. From (1.2.3) and (1.3.6), we obtain

$$\|\hat{x}\|=\|\boldsymbol{x}\|\left(1+\varepsilon\frac{\boldsymbol{x}\cdot\boldsymbol{x}^*}{\|\boldsymbol{x}\|^2}\right)=\|\boldsymbol{x}\|+\varepsilon\frac{\boldsymbol{x}\cdot\boldsymbol{x}^*}{\|\boldsymbol{x}\|}=x+\varepsilon\,x^*,\ (\boldsymbol{x}\neq\boldsymbol{0}), \qquad (1.3.8)$$

where $x=\|\boldsymbol{x}\|$ and $x^*=\dfrac{\boldsymbol{x}\cdot\boldsymbol{x}^*}{\|\boldsymbol{x}\|}$.

Definition 1.3.6 A dual vector $\hat{x}$ with norm 1 is called a dual unit vector. It follows from (1.3.8), $\hat{x}$ is a dual unit vector if and only if the relations

$$\boldsymbol{x}^2 = \boldsymbol{x}\cdot\boldsymbol{x} = 1 \quad \text{and} \quad \boldsymbol{x}\cdot\boldsymbol{x}^* = 0 \tag{1.3.9}$$

hold simultaneously.

Theorem 1.3.2 Let $\hat{x} = \boldsymbol{x} + \varepsilon\,\boldsymbol{x}^*$ be a dual vector with $\boldsymbol{x} \neq \boldsymbol{0}$. Then the vector $\hat{x}_0 = \dfrac{\hat{x}}{\|\hat{x}\|}$ is a dual unit vector and it can be written in the form

$$\hat{x}_0 = \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|} + \varepsilon\,\frac{\boldsymbol{x}^* - h\boldsymbol{x}}{\|\boldsymbol{x}\|} = \boldsymbol{x}_0 + \varepsilon\,\boldsymbol{x}_0^*, \tag{1.3.10}$$

where $\boldsymbol{x}_0 = \dfrac{\boldsymbol{x}}{\|\boldsymbol{x}\|}$, $\boldsymbol{x}_0^* = \dfrac{\boldsymbol{x}^* - h\boldsymbol{x}}{\|\boldsymbol{x}\|}$ and $h = \dfrac{\boldsymbol{x}\cdot\boldsymbol{x}^*}{\|\boldsymbol{x}\|^2}$.

Proof Let $\hat{x} = \boldsymbol{x} + \varepsilon\,\boldsymbol{x}^*$ be a dual vector with $\boldsymbol{x} \neq \boldsymbol{0}$. Then by (1.3.8), we have

$$\|\hat{x}\| = \|\boldsymbol{x}\|\left(1 + \varepsilon\,\frac{\boldsymbol{x}\cdot\boldsymbol{x}^*}{\|\boldsymbol{x}\|^2}\right) = \|\boldsymbol{x}\|\,(1 + \varepsilon\,h),$$

where $h = \dfrac{\boldsymbol{x}\cdot\boldsymbol{x}^*}{\|\boldsymbol{x}\|^2}$. Therefore, the dual vector $\hat{x}_0$ is found as

$$\hat{x}_0 = \frac{\hat{x}}{\|\hat{x}\|} = \frac{\boldsymbol{x} + \varepsilon\,\boldsymbol{x}^*}{\|\boldsymbol{x}\|(1+\varepsilon\,h)} = \frac{(\boldsymbol{x} + \varepsilon\,\boldsymbol{x}^*)(1-\varepsilon\,h)}{\|\boldsymbol{x}\|(1+\varepsilon\,h)(1-\varepsilon\,h)} = \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|} + \varepsilon\,\frac{\boldsymbol{x}^* - h\boldsymbol{x}}{\|\boldsymbol{x}\|} = \boldsymbol{x}_0 + \varepsilon\,\boldsymbol{x}_0^*.$$

Now we will show that the dual vector $\hat{x}_0$ is a dual unit vector, i.e., the vectors $\boldsymbol{x}_0$ and $\boldsymbol{x}_0^*$ satisfy the equation (1.3.9).

$$\boldsymbol{x}_0^2 = \boldsymbol{x}_0\cdot\boldsymbol{x}_0 = \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|}\cdot\frac{\boldsymbol{x}}{\|\boldsymbol{x}\|} = \frac{\boldsymbol{x}^2}{\|\boldsymbol{x}\|^2} = \frac{\|\boldsymbol{x}\|^2}{\|\boldsymbol{x}\|^2} = 1$$

and

$$\boldsymbol{x}_0\cdot\boldsymbol{x}_0^* = \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|}\cdot\frac{\boldsymbol{x}^* - h\boldsymbol{x}}{\|\boldsymbol{x}\|} = \frac{\boldsymbol{x}\cdot\boldsymbol{x}^*}{\|\boldsymbol{x}\|^2} - h\frac{\boldsymbol{x}^2}{\|\boldsymbol{x}\|^2} = h - h\frac{\|\boldsymbol{x}\|^2}{\|\boldsymbol{x}\|^2} = h - h = 0.$$

Hence, the vector $\hat{\boldsymbol{x}}_0$ is a dual unit vector with the same sense as $\hat{\boldsymbol{x}}$. In addition it is clear that

$$\hat{\boldsymbol{x}} = \|\hat{\boldsymbol{x}}\|\hat{\boldsymbol{x}}_0 = \|\boldsymbol{x}\|\left(1+\varepsilon\frac{\boldsymbol{x}\cdot\boldsymbol{x}^*}{\|\boldsymbol{x}\|^2}\right)\hat{\boldsymbol{x}}_0 = \|\boldsymbol{x}\|(1+\varepsilon\, h)\hat{\boldsymbol{x}}_0. \qquad (1.3.11)$$

Definition 1.3.7 Let $\hat{\boldsymbol{x}} = \boldsymbol{x}+\varepsilon\,\boldsymbol{x}^* \in ID^3$, then the dual unit vector

$$\hat{\boldsymbol{x}}_0 = \frac{\hat{\boldsymbol{x}}}{\|\hat{\boldsymbol{x}}\|} = \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|} + \varepsilon\frac{\boldsymbol{x}^* - h\boldsymbol{x}}{\|\boldsymbol{x}\|} = \boldsymbol{x}_0 + \varepsilon\,\boldsymbol{x}_0^*$$

and the real number

$$h = \frac{\boldsymbol{x}\cdot\boldsymbol{x}^*}{\|\boldsymbol{x}\|^2}$$

are called axis and pitch of dual vector $\hat{\boldsymbol{x}}$, respectively.

In addition, if

1. h is a finite number, then $\boldsymbol{x} \neq \boldsymbol{0}$ and $\boldsymbol{x}^* \neq \boldsymbol{0}$. In this case the dual vector $\hat{\boldsymbol{x}}$ is called screw;

2. $h = 0$, then by (1.3.9), $\hat{\boldsymbol{x}} = \|\boldsymbol{x}\|\hat{\boldsymbol{x}}_0$ and the dual vector $\hat{\boldsymbol{x}}$ and its axis $\hat{\boldsymbol{x}}_0$ correspond a congruent directed straight line in IR^3 (In theorem 1.4.1, it is proved that there exist a one to one correspondence between the dual unit vectors and directed straight lines in IR^3);

3. $h = \infty$, then $\boldsymbol{x} = \boldsymbol{0}$ and the dual vector $\hat{\boldsymbol{x}} = \varepsilon\,\boldsymbol{x}^*$ is pure dual vector.

1.4 The Study Mapping

Definition 1.4.1 The set $\{\hat{\boldsymbol{x}} = \boldsymbol{x}+\varepsilon\,\boldsymbol{x}^* : \|\hat{\boldsymbol{x}}\| = 1,\ \boldsymbol{x},\boldsymbol{x}^* \in IR^3\}$ is called a dual unit sphere (D.U.S.) in ID^3.

Theorem (E.Study) 1.4.1 There exist a one to one correspondence between the points on D.U.S. and directed straight lines (spears) in IR^3.

Proof A straight line ℓ in IR^3 is completely determined by its direction vector $\boldsymbol{u}$ and a point M on it with respect to the reference system $\{O, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$, and its vectorial equation is

$$(\boldsymbol{x}-\boldsymbol{m}) \times \boldsymbol{u} = \boldsymbol{0}, \tag{1.4.1}$$

where X is an arbitrary point on ℓ and $\boldsymbol{x}$ is location vector of the point X. (see Fig.1.4.1)

By replacing $\boldsymbol{u}$ with $\lambda \boldsymbol{u}$ ($\lambda \in IR$, $\lambda \neq 0$) in equation (1.4.1) and rearranging it, we obtain the same equation. So, we can choose $\boldsymbol{u}$ as a unit vector $\boldsymbol{u}_0$ in the direction of $\boldsymbol{u}$. Then,

$$(\boldsymbol{x}-\boldsymbol{m}) \times \boldsymbol{u}_0 = \boldsymbol{0}$$

and this equation is satisfied if and only if

$$\boldsymbol{u}_0^* = \boldsymbol{x} \times \boldsymbol{u}_0 = \boldsymbol{m} \times \boldsymbol{u}_0. \tag{1.4.2}$$

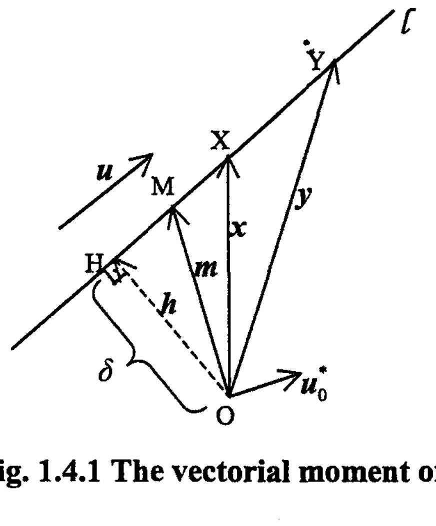

Fig. 1.4.1 The vectorial moment of a directed straight line.

The vector $\boldsymbol{u}_0^*$ is called vectorial moment of the vector $\boldsymbol{u}_0$ with respect to the origin, and it is independent of choice of the point X on ℓ and it is dependent of choice of origin (it is proved in theorem 1.4.2). In fact, for an arbitrary point Y on ℓ (different from X), the equation of ℓ is

$$(\boldsymbol{y}-\boldsymbol{m})\times \boldsymbol{u}_0 = \boldsymbol{0}$$

and implies that

$$\boldsymbol{y}\times \boldsymbol{u}_0 = \boldsymbol{m}\times \boldsymbol{u}_0 .$$

From (1.4.1), we get

$$\boldsymbol{y}\times \boldsymbol{u}_0 = \boldsymbol{m}\times \boldsymbol{u}_0 = \boldsymbol{x}\times \boldsymbol{u}_0 = \boldsymbol{u}_0^* .$$

Let H be perpendicular foot of the origin on ℓ. Hence we also have

$$\boldsymbol{u}_0^* = \boldsymbol{h}\times \boldsymbol{u}_0 . \tag{1.4.3}$$

The length of the vector $\boldsymbol{u}_0^*$ is

$$\|\boldsymbol{u}_0^*\| = \|\boldsymbol{h}\times \boldsymbol{u}_0\| = \|\boldsymbol{h}\|\|\boldsymbol{u}_0\| \sin\frac{\pi}{2} = \|\boldsymbol{h}\| = \delta . \tag{1.4.4}$$

Therefore, the dual unit vector $\hat{\boldsymbol{u}}_0$ defined by

$$\hat{\boldsymbol{u}}_0 = \boldsymbol{u}_0 + \varepsilon\, \boldsymbol{u}_0^*$$

is uniquely determined.

Now we shall show that the dual unit vector $\hat{\boldsymbol{u}}_0 = \boldsymbol{u}_0 + \varepsilon\, \boldsymbol{u}_0^*$ corresponds to an oriented straight line in IR^3. Consider the plane P through the origin with normal vector $\boldsymbol{u}_0^*$, and draw a circle C with radius $\delta = \|\boldsymbol{u}_0^*\|$ and centered at the origin in P. The line in P, perpendicular to the vector $\boldsymbol{u}_0$ and through the origin, intersect the circle C at two points, say M_1 and M_2. Only one of the tangent lines at M_1 and M_2 in the direction $\boldsymbol{u}_0$ corresponds to given dual unit vector $\hat{\boldsymbol{u}}_0 = \boldsymbol{u}_0 + \varepsilon\, \boldsymbol{u}_0^*$. This proves the theorem (see Fig. 1.4.2).

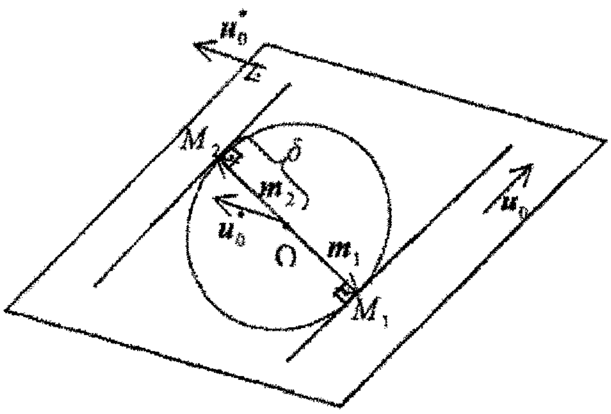

Fig. 1.4.2 The straight line corresponding to dual unit vector $\hat{u}_0 = u_0 + \varepsilon u_0^*$.

Theorem 1.4.2 Let P be a point on IR^3 and $\{O, e_1, e_2, e_3\}$ be standard right handed reference system in IR^3. Then the vectorial moment of the vector u_0 with respect to the point P is

$$u_{0p}^* = PO \times u_0 + u_0^* . \qquad (1.4.5)$$

Proof The vectorial moment of the vector u_0 with respect to the point P is

$$u_{0p}^* = PX \times u_0 = (PO + OX) \times u_0 = PO \times u_0 + x \times u_0 = PO \times u_0 + u_0^* .$$

(see Fig. 1.4.3)

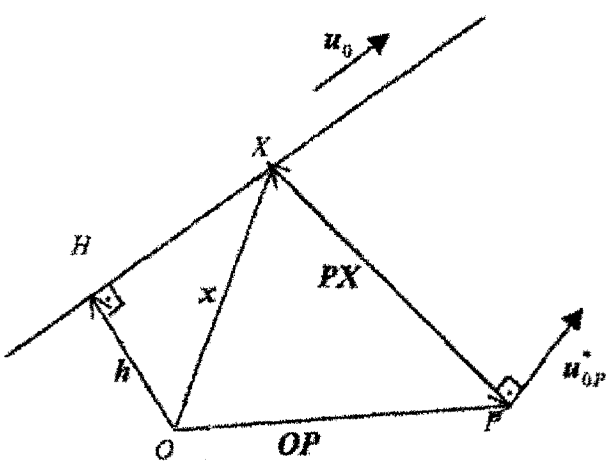

Fig. 1.4.3 The vectorial moment of a directed straight line with respect to a point P.

This theorem shows that the dual unit vector corresponding to directed straight line with respect to the right handed reference system $\{P, e_1, e_2, e_3\}$ is

$$\hat{u}_0 = u_0 + \varepsilon\, u_{0p}^* = u_0 + \varepsilon \left(PO \times u_0 + u_0^* \right) \qquad (1.4.6)$$

Definition 1.4.2 Let $u_0 = (u_{01}, u_{02}, u_{03})$ and $u_0^* = (u_{01}^*, u_{02}^*, u_{03}^*)$ be two vectors in IR^3 satisfying the conditions (1.3.9). The six components of the pair of vectors $(u_0, u_0^*) = (u_{01}, u_{02}, u_{03}; u_{01}^*, u_{02}^*, u_{03}^*)$ are called normed homogenous Plucker coordinates of a line.

1.5 The Dual Angle

Theorem 1.5.1 Let θ and θ^*, respectively, be the angle and the shortest distance between the directed straight lines in IR^3 determined by the dual unit vectors $\hat{x}_0 = x_0 + \varepsilon\, x_0^*$ and $\hat{y}_0 = y_0 + \varepsilon\, y_0^*$. Then

$$\hat{x}_0 \cdot \hat{y}_0 = \cos\hat{\theta}. \qquad (1.5.1)$$

Proof Let θ and θ^*, respectively, be the angle and the shortest distance between two oriented straight lines, say ℓ_1 and ℓ_2, corresponding to dual unit vectors $\hat{x}_0 = x_0 + \varepsilon\, x_0^*$ and $\hat{y}_0 = y_0 + \varepsilon\, y_0^*$. It is clear that

$$x_0 \cdot y_0 = \cos\theta.$$

Since the vectorial moments x_0^* and y_0^* of the vectors x_0 and y_0 are independent of choice of points on ℓ_1 and ℓ_2, then we choose perpendicular foots A and B of common perpendicular on ℓ_1 and ℓ_2. Therefore,

$$x_0^* = a \times x_0 \ , \ y_0^* = b \times y_0$$

and

$$\begin{aligned} \boldsymbol{x}_0 \cdot \boldsymbol{y}_0^* + \boldsymbol{x}_0^* \cdot \boldsymbol{y}_0 &= \boldsymbol{x}_0 \cdot (\boldsymbol{b} \times \boldsymbol{y}_0) + (\boldsymbol{a} \times \boldsymbol{x}_0) \cdot \boldsymbol{y}_0 \\ &= -\boldsymbol{b} \cdot (\boldsymbol{x}_0 \times \boldsymbol{y}_0) + \boldsymbol{a} \cdot (\boldsymbol{x}_0 \times \boldsymbol{y}_0) \\ &= (\boldsymbol{a} - \boldsymbol{b}) \cdot (\boldsymbol{x}_0 \times \boldsymbol{y}_0). \end{aligned}$$

In addition, since the unit vector in the direction of common perpendicular is

$$\boldsymbol{n} = \pm \frac{\boldsymbol{x}_0 \times \boldsymbol{y}_0}{\|\boldsymbol{x}_0 \times \boldsymbol{y}_0\|},$$

then

$$\boldsymbol{a} - \boldsymbol{b} = \pm\theta^* \frac{\boldsymbol{x}_0 \times \boldsymbol{y}_0}{\|\boldsymbol{x}_0 \times \boldsymbol{y}_0\|} = \pm\theta^* \boldsymbol{n}.$$

Therefore,

$$\begin{aligned} \boldsymbol{x}_0 \cdot \boldsymbol{y}_0^* + \boldsymbol{x}_0^* \cdot \boldsymbol{y}_0 &= (\boldsymbol{a} - \boldsymbol{b}) \cdot (\boldsymbol{x}_0 \times \boldsymbol{y}_0) \\ &= \pm\theta^* \frac{1}{\|\boldsymbol{x}_0 \times \boldsymbol{y}_0\|} (\boldsymbol{x}_0 \times \boldsymbol{y}_0)^2 \\ &= \pm\theta^* \|\boldsymbol{x}_0 \times \boldsymbol{y}_0\| \\ &= \pm\theta^* \sin\theta. \end{aligned}$$

Hence, we get

$$\begin{aligned} \hat{\boldsymbol{x}}_0 \cdot \hat{\boldsymbol{y}}_0 &= \boldsymbol{x}_0 \cdot \boldsymbol{y}_0 + \varepsilon (\boldsymbol{x}_0 \cdot \boldsymbol{y}_0^* + \boldsymbol{x}_0^* \cdot \boldsymbol{y}_0) \\ &= \cos\theta \pm \varepsilon\,\theta^* \sin\theta. \end{aligned} \tag{1.5.2}$$

Taking the appropriate sign in (1.5.2) and by using (1.2.2) we obtain (1.5.1). This proves the theorem (see Fig. 1.5.1).

In conclusion, the dual number $\hat{\theta} = \theta + \varepsilon\,\theta^*$ is called the dual angle between the dual unit vectors $\hat{\boldsymbol{x}}_0$ and $\hat{\boldsymbol{y}}_0$, where θ and θ^* are, respectively, the angle and the shortest distance between the two directed straight lines corresponding to dual unit vectors $\hat{\boldsymbol{x}}_0$ and $\hat{\boldsymbol{y}}_0$.

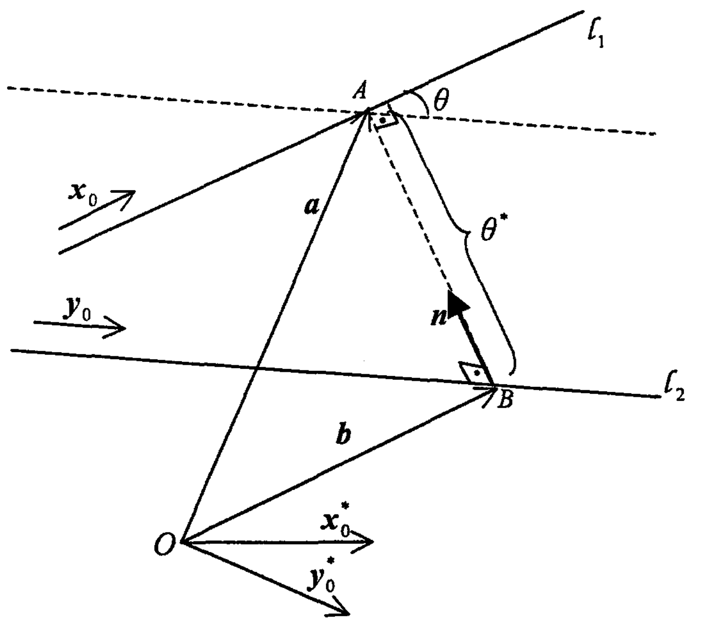

Fig. 1.5.1 The geometric meaning of the dual angle.

Moreover, if the dual number $\hat{\theta} = \theta + \varepsilon\,\theta^*$ is the dual angle between the axes of dual vectors $\hat{x} = x + \varepsilon\,x^*$ and $\hat{y} = y + \varepsilon\,y^*$, then

$$\hat{x} \cdot \hat{y} = \|\hat{x}\|\,\|\hat{y}\| \cos\hat{\theta}\,. \tag{1.5.3}$$

Indeed, since the axes of dual vectors $\hat{x}$ and $\hat{y}$ are dual unit vectors, then the equation (1.5.1) satisfied. Besides, the scalar product of dual vectors satisfies the *ID*-bilinear operations. Then by (1.3.9) and (1.5.1), we get

$$\hat{x} \cdot \hat{y} = (\|\hat{x}\|\hat{x}_0) \cdot (\|\hat{y}\|y_0) = \|\hat{x}\|\,\|\hat{y}\|(\hat{x}_0 \cdot \hat{y}_0) = \|\hat{x}\|\,\|\hat{y}\| \cos\hat{\theta}\,.$$

As a consequence of theorem 1.5.1 we can analyze the positions of directed straight lines, say ℓ_1 and ℓ_2, corresponding to dual unit vectors $\hat{x}_0 = x_0 + \varepsilon\,x_0^*$ and $\hat{y}_0 = y_0 + \varepsilon\,y_0^*$. Let θ and θ^*, respectively, be the angle and the shortest distance between the two directed straight lines ℓ_1 and ℓ_2. Then:

1. $\hat{x}_0 \cdot \hat{y}_0$ has only dual part if and only if $\theta = \frac{\pi}{2}$ and $\theta^* \neq 0$. Therefore, the directed straight lines ℓ_1 and ℓ_2 are perpendicularly skew lines.

2. $\hat{x}_0 \cdot \hat{y}_0 = 0$ if and only if $\theta = \frac{\pi}{2}$ and $\theta^* = 0$. Consequently the directed straight lines ℓ_1 and ℓ_2 are perpendicular lines.

3. If $\hat{x}_0 \cdot \hat{y}_0$ has only real part then $\theta^* = 0$. Hence, the directed straight lines ℓ_1 and ℓ_2 are intersecting lines.

4. If $\hat{x}_0 \cdot \hat{y}_0 = 1$ (or $\hat{x}_0 \cdot \hat{y}_0 = -1$) then $\theta = 0$ (or $\theta = \pi$). Therefore, the directed straight lines ℓ_1 and ℓ_2 are parallel and they are in the same (or opposite) direction. Also if $\theta^* = 0$ then the directed straight lines ℓ_1 and ℓ_2 are congruent and they are in the same (or opposite) direction.

1.6 The Line Complex

We proved in theorem 1.4.1 the dual unit vector $\hat{x}_0 = x_0 + \varepsilon\, x_0^*$ determines a directed straight line in IR^3. Here $x_0 = (x_{01}, x_{02}, x_{03})$ is the direction vector of directed straight line and the vector $x_0^* = (x_{01}^*, x_{02}^*, x_{03}^*)$ is the vectorial moment of the vector x_0 with respect to the origin. Since $\hat{x}_0$ is a dual unit vector then the following conditions are satisfied

$$x_0^2 = x_0 \cdot x_0 = x_{01}^2 + x_{02}^2 + x_{03}^2 = 1, \tag{1.6.1}$$

$$x_0 \cdot x_0^* = x_{01}x_{01}^* + x_{02}x_{02}^* + x_{03}x_{03}^* = 0. \tag{1.6.2}$$

In addition, if there exists another condition including the homogenous Plucker coordinates of directed straight line $\hat{x}_0 = x_0 + \varepsilon\, x_0^*$ such as

$$F\left(x_{01}, x_{02}, x_{03}; x_{01}^{*}, x_{02}^{*}, x_{03}^{*}\right) = 0, \tag{1.6.3}$$

then the directed straight line $\hat{\boldsymbol{x}}_0 = \boldsymbol{x}_0 + \varepsilon\, \boldsymbol{x}_0^*$ has three parameters.

Definition 1.6.1 Any set of straight lines depending on three parameters, i.e., ∞^3 number of straight lines in IR^3, is called a line complex in IR^3. If u, v and w denote the parameters of line complex, then the dual unit vector $\hat{\boldsymbol{x}}_0$ can be written as

$$\hat{\boldsymbol{x}}_0(u,v,w) = \boldsymbol{x}_0(u,v,w) + \varepsilon\, \boldsymbol{x}_0^*(u,v,w).$$

1.7 The Line Congruence

If a line complex with three independent conditions (1.6.1), (1.6.2) and (1.6.3) has another condition including the homogenous Plucker coordinates of directed straight line $\hat{\boldsymbol{x}}_0 = \boldsymbol{x}_0 + \varepsilon\, \boldsymbol{x}_0^*$ such as

$$G\left(x_{01}, x_{02}, x_{03}; x_{01}^{*}, x_{02}^{*}, x_{03}^{*}\right) = 0, \tag{1.7.1}$$

then we have a set of straight lines depending on two parameters.

Definition 1.7.1 Any set of straight lines depending on two parameters, i.e., ∞^2 number of straight lines in IR^3, is called a line congruence in IR^3. If u and v denote the parameters of line congruence, then the dual unit vector $\hat{\boldsymbol{x}}_0$ can be written as

$$\hat{\boldsymbol{x}}_0(u,v) = \boldsymbol{x}_0(u,v) + \varepsilon\, \boldsymbol{x}_0^*(u,v).$$

Moreover, line congruence is the set of common lines of two line complexes.

1.8 The Ruled Surface

If line congruence with four independent conditions (1.6.1), (1.6.2), (1.6.3) and (1.7.1) has another condition including the homogenous Plucker coordinates of directed straight line $\hat{\boldsymbol{x}}_0 = \boldsymbol{x}_0 + \varepsilon\, \boldsymbol{x}_0^*$ such as

$$H\left(x_{01}, x_{02}, x_{03}; x_{01}^*, x_{02}^*, x_{03}^*\right) = 0, \tag{1.8.1}$$

then we have a set of straight lines depending on a parameter.

Definition 1.8.1 Any set of straight lines depending on a parameter, i.e., ∞^1 number of straight lines in IR^3, is called a ruled surface in IR^3. If t denotes the parameter of ruled surface, then the dual unit vector $\hat{\boldsymbol{x}}_0$ is represented by

$$\hat{\boldsymbol{x}}_0(t) = \boldsymbol{x}_0(t) + \varepsilon\, \boldsymbol{x}_0^*(t).$$

In addition, the continuous change of t causes the motion of the dual unit vector $\hat{\boldsymbol{x}}_0(t)$ on the D.U.S. and draws a dual curve called spherical image of the ruled surface $\hat{\boldsymbol{x}}_0(t)$ on the D.U.S. Let us denote the dual arc-element of the dual curve $\hat{\boldsymbol{x}}_0(t)$ on D.U.S. by $d\hat{\sigma} = d\sigma + \varepsilon d\sigma^*$. It is clear that, for small increments

$$(d\hat{\sigma})^2 = (d\hat{\boldsymbol{x}}_0)^2 \tag{1.8.2}$$

or

$$(d\sigma)^2 + 2\varepsilon\, d\sigma\, d\sigma^* = (d\boldsymbol{x}_0)^2 + 2\varepsilon \left(d\boldsymbol{x}_0 \cdot d\boldsymbol{x}_0^*\right). \tag{1.8.3}$$

Hence, we get

$$(d\sigma)^2 = (d\boldsymbol{x}_0)^2 \text{ and } d\sigma\, d\sigma^* = d\boldsymbol{x}_0 \cdot d\boldsymbol{x}_0^*. \tag{1.8.4}$$

It is well known that, $d\hat{\sigma}$ measures the distance between the end points of neighbour vectors $\hat{x}_0(t)$ and $\hat{x}_0(t+dt)$ on D.U.S. Also $d\hat{\sigma}$ is the dual angle between the neighbour vectors $\hat{x}_0(t)$ and $\hat{x}_0(t+dt)$ on D.U.S. Therefore, $d\sigma$ and $d\sigma^*$ represent the angle and the distance between the neighbour rulings of the ruled surface, respectively.

Since the scalar product $d\hat{x}_0 \cdot d\hat{x}_0 = dx_0 \cdot dx_0 + 2\varepsilon\, dx_0 \cdot dx_0^*$ is invariant under the coordinate transformations, then the ratio

$$\varDelta = \frac{dx_0 \cdot dx_0^*}{dx_0 \cdot dx_0} = \frac{d\sigma\, d\sigma^*}{(d\sigma)^2} = \frac{d\sigma^*}{d\sigma} \tag{1.8.5}$$

is invariant under the coordinate transformations.

Definition 1.8.2 The ratio $\varDelta$ given by (1.8.5) is called the distribution parameter of the ruled surface.

Definition 1.8.3 A ruled surface having the intersecting neighbor ruling lines is called developable ruled surface. In this case $\varDelta = 0$.

As it is discussed before, a ruled surface corresponds a dual curve $\hat{x}_0(t) = x_0(t) + \varepsilon\, x_0^*(t)$ on D.U.S. Since by principle of transference, at each time t the dual unit vector $\hat{x}_0(t) = x_0(t) + \varepsilon\, x_0^*(t)$ represents a directed straight line passing through the point $p(t)$ with the direction $x_0(t)$, where $p(t)$ is a point in IR^3 satisfying the equation

$$x_0^*(t) = p(t) \times x_0(t). \tag{1.8.6}$$

Therefore, the equation of the ruled surface is

$$L(t,\xi) = p(t) + \xi\, x_0(t) \tag{1.8.7}$$

(see Fig. 1.8.1).

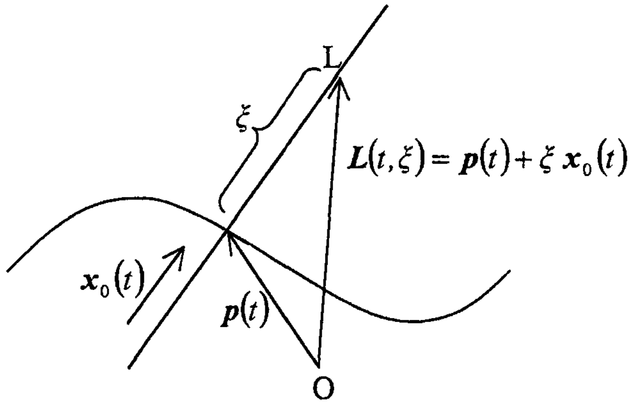

Fig. 1.8.1 The ruled surface corresponding to dual curve $\hat{x}_0(t)=x_0(t)+\varepsilon\, x_0^*(t)$ on D.U.S.

By the rule of vectorial division, $p(t)$ is found from (1.8.6) as

$$p(t)=\frac{(p(t)\cdot x_0(t))x_0(t)-x_0^*(t)\times x_0(t)}{\|x_0(t)\|^2}=x_0(t)\times x_0^*(t)+(p(t)\cdot x_0(t))x_0(t)$$
$$=x_0(t)\times x_0^*(t)+\mu x_0(t),$$

where $\mu=p(t)\cdot x_0(t)$ is a real scalar. Hence, the equation (1.8.7) becomes

$$L(t,\lambda)=x_0(t)\times x_0^*(t)+\lambda x_0(t),\ \ (\lambda=\xi+\mu). \tag{1.8.8}$$

CHAPTER TWO
THE SPATIAL MOTION

2.1 Dual Spherical Motions

We assume that we are given two triples of points on the D.U.S. by means of two orthogonal right handed frame fields $\hat{e}=\{\hat{e}_1,\hat{e}_2,\hat{e}_3\}$ and $\hat{r}=\{\hat{r}_1,\hat{r}_2,\hat{r}_3\}$. Then any point on the D.U.S. can be written unambiguously as a linear combination of $\hat{e}_1,\hat{e}_2$ and $\hat{e}_3$ as well as of $\hat{r}_1,\hat{r}_2$ and $\hat{r}_3$. We have therefore a point $\hat{X}$ on the D.U.S.:

$$\sum_{i=1}^{3}\hat{X}_i\hat{e}_i=\sum_{i=1}^{3}\hat{x}_i\hat{r}_i\,. \tag{2.1.1}$$

The column vectors

$$\hat{\boldsymbol{X}}=\begin{bmatrix}\hat{X}_1\\ \hat{X}_2\\ \hat{X}_3\end{bmatrix} \text{ and } \hat{\boldsymbol{x}}=\begin{bmatrix}\hat{x}_1\\ \hat{x}_2\\ \hat{x}_3\end{bmatrix}$$

are position vectors of the point $\hat{X}$ on D.U.S. with respect to $\hat{e}=\{\hat{e}_1,\hat{e}_2,\hat{e}_3\}$ and $\hat{r}=\{\hat{r}_1,\hat{r}_2,\hat{r}_3\}$, respectively. We derive from (2.1.1)

$$\hat{X}_i=(\hat{e}_i.\hat{r}_1)\hat{x}_1+(\hat{e}_i.\hat{r}_2)\hat{x}_2+(\hat{e}_i.\hat{r}_3)\hat{x}_3\ ,(i=1,2,3). \tag{2.1.2}$$

Also, any element of frame field $\hat{r}$ can be written as a linear combination of $\hat{e}_1,\hat{e}_2$ and $\hat{e}_3$ as

$$\hat{\boldsymbol{r}}_k = \sum_{i=1}^{3} \hat{a}_{ik} \hat{\boldsymbol{e}}_i, \quad (k = 1,2,3). \tag{2.1.2}$$

Putting $\hat{\boldsymbol{e}}_i . \hat{\boldsymbol{r}}_k = \hat{a}_{ik}$ and introducing the dual matrix

$$\hat{A} = (\hat{a}_{ik}) = (\alpha_{ik}) + \varepsilon(\alpha^*_{ik}) = A + \varepsilon A^*, \tag{2.1.3}$$

we may write

$$\hat{R} = \hat{A}'\hat{E} \text{ and } \hat{\boldsymbol{X}} = \hat{X}'\hat{E} = \hat{x}'\hat{R} = \hat{\boldsymbol{x}}. \tag{2.1.4}$$

Here,

$$\hat{R} = \begin{bmatrix} \hat{\boldsymbol{r}}_1 \\ \hat{\boldsymbol{r}}_2 \\ \hat{\boldsymbol{r}}_3 \end{bmatrix} \text{ and } \hat{E} = \begin{bmatrix} \hat{\boldsymbol{e}}_1 \\ \hat{\boldsymbol{e}}_2 \\ \hat{\boldsymbol{e}}_3 \end{bmatrix}$$

are the column matrices. The vectors $\hat{\boldsymbol{X}}$ and $\hat{\boldsymbol{x}}$ are the position vectors of $\hat{X}$ with respect to $\hat{e}$ and $\hat{r}$, i.e., *E* and *R*. We see that (2.1.4) expresses that

$$\hat{X} = \hat{A}\hat{x} = Ax + \varepsilon(Ax^* + A^*x) = X + \varepsilon X^*. \tag{2.1.5}$$

It is clear that, $\hat{A}$ is an orthogonal transformation matrix from the frame field *R* onto *E*. We suppose the frame field *E* is fixed whereas the vectors $\hat{\boldsymbol{r}}_1, \hat{\boldsymbol{r}}_2$ and $\hat{\boldsymbol{r}}_3$ of frame field R are functions of a real parameter *t* (time). Then we say that, R moves with respect to *E*. We may interpret this as follows: the dual unit sphere *K* rigidly connected with *R* moves over the dual unit sphere K' rigidly connected with *E*.

The dual unit spheres K and K' are called moving D.U.S. and fixed D.U.S., respectively. Then the motion is called a dual spherical motion and will be denoted by K/K'. If the dual unit spheres K and K', respectively, correspond to the line spaces H and H', then the dual spherical motion K/K' corresponds to the spatial motion in 3-space denoted by H/H'. Since $\hat{A}$ are an orthogonal matrix we have $\hat{A}\hat{A}' = I$ and therefore the differentiation of $\hat{A}\hat{A}' = I$ with respect to t gives that

$$\frac{d\hat{A}}{dt}\hat{A}' + \hat{A}\frac{d\hat{A}'}{dt} = 0, \tag{2.1.6}$$

where 0 is a 3×3 zero matrix. This shows that $\frac{d\hat{A}}{dt}\hat{A}^t$ is a skew-symmetric matrix.

So, we define $\frac{d\hat{A}}{dt}\hat{A}^t$ as

$$\frac{d\hat{A}}{dt}\hat{A}^t = \begin{bmatrix} 0 & -\hat{w}_3 & \hat{w}_2 \\ \hat{w}_3 & 0 & -\hat{w}_1 \\ -\hat{w}_2 & \hat{w}_1 & 0 \end{bmatrix} = \hat{\Omega}\cdot \tag{2.1.7}$$

The dual velocity of $\hat{x}$ on K is defined as

$$\hat{v} = \frac{d\hat{X}}{dt} = \frac{d\hat{A}}{dt}\hat{x} + \hat{A}\frac{d\hat{x}}{dt}, \tag{2.1.8}$$

where $\frac{d\hat{x}}{dt}$ is the velocity of $\hat{x}$ relative to K. If $\hat{x}$ is fixed on K, then $\frac{d\hat{x}}{dt} = 0$ and the equation (2.1.8) becomes

$$\hat{v} = \frac{d\hat{X}}{dt} = \frac{d\hat{A}}{dt}\hat{x} = \frac{d\hat{A}}{dt}\hat{A}^t\hat{A}\hat{x} = \frac{d\hat{A}}{dt}\hat{A}^t\hat{X} = \hat{\Omega}\hat{X}\,. \tag{2.1.9}$$

Introducing the vector $\hat{w} = (\hat{w}_1, \hat{w}_2, \hat{w}_3)$, we may write

$$\hat{v} = \hat{w}\times\hat{X} = \hat{v}^t = \hat{X}^t\hat{\Omega}^t = -\hat{X}^t\hat{\Omega} \quad . \tag{2.1.10}$$

If $\hat{X}$ is fixed on K', then we have

$$\frac{d\hat{X}}{dt} = \frac{d\hat{x}^t}{dt}\hat{R} + \hat{x}^t\frac{d\hat{R}}{dt} = 0\,. \tag{2.1.11}$$

Since

$$\frac{d\hat{R}}{dt} = \frac{d\hat{A}^t}{dt}\hat{E} = \frac{d\hat{A}^t}{dt}A\hat{R},$$

we get

$$\left(\frac{d\hat{x}'}{dt} - \hat{x}'\hat{A}'\hat{\Omega}\hat{A}\right)\hat{R} = 0$$

or

$$\frac{d\hat{x}}{dt} = -\hat{A}'\hat{\Omega}\hat{A}\hat{x} = -\hat{A}'\hat{\Omega}\hat{X} = -\hat{A}'v = -\hat{A}'\frac{d\hat{A}}{dt}\hat{x}. \tag{2.1.12}$$

In that case, the point $\hat{X}$ is called a fixed point on K' and equation (2.1.11) is defined as a fixed point condition.

The vector $\hat{w}$ is called dual angular velocity vector (the dual Darboux vector) of the motion K/K' and the dual unit vector

$$\hat{w}_0 = \frac{\hat{w}}{\|\hat{w}\|} = \frac{w}{w} + \varepsilon\left(\frac{w^*}{w} - \frac{w^*}{w^2}w\right) \tag{2.1.13}$$

with the same sense as $\hat{w}$ is the instantaneous screw axes (I.S.A.) of the motion, where $\|\hat{w}\| = w + \varepsilon w^*$, $w = \|w\|$ and $w^* = \dfrac{w \cdot w^*}{w}$.

The dual number $w + \varepsilon w^*$ is called dual angular speed of K/K'. Moreover the pitch of the motion K/K' is given by

$$h = \frac{w^*}{w} = \frac{w \cdot w^*}{\|w\|^2}. \tag{2.1.14}$$

The point on K coinciding with $\hat{w}_0$ at instant t has the dual velocity zero; this point also denoted by $\hat{p}$. We shall call $\hat{p}$ the pole point of the motion K/K'.

Therefore, the relative spatial motion between two line spaces H and H' can be visualized as a twist about a finite screw. By extension, when the motion is divided into infinitesimal steps, each displacement becomes associated with an I.S.A. The orientation and location of the I.S.A. constantly changes during continuous motion, and fixing one of the line spaces as a reference, we assumed H', the line of the I.S.A. then traces out two ruled surfaces, one in the fixed line space H' and one in the moving line space H, these ruled surfaces are called fixed axode and moving axode, respectively. The fixed axode and moving axode always meet tangentially in a common line along the I.S.A. and glide upon the each other. In order for generators from each axode to coalesce along the I.S.A. both the fixed axode and moving axode must have the same parameter of distribution along their common axis, and two striction curves on fixed axode and moving axode must always have the central point of the I.S.A. in common (Bokelberg et.al.,1992) (see Fig.2.1.1).

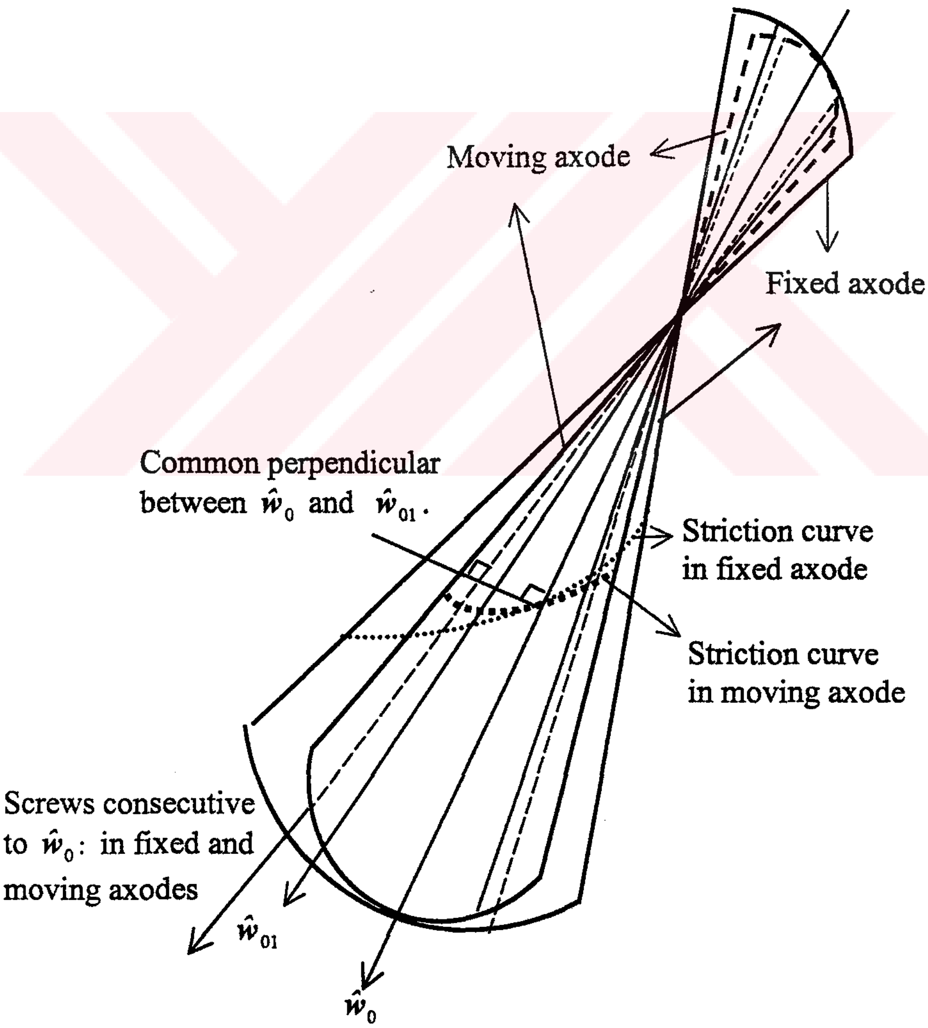

Fig.2.1.1 Portions of typical fixed and moving axodes.

The dual acceleration of the fixed point $\hat{x}$ on K is defined as

$$\begin{aligned}\hat{\boldsymbol{a}} &= \frac{d^2\hat{\boldsymbol{X}}}{dt^2} = \frac{d}{dt}\left(\frac{d\hat{\boldsymbol{X}}}{dt}\right) = \frac{d}{dt}\left(\hat{\boldsymbol{w}}\times\hat{\boldsymbol{X}}\right)\\ &= \frac{d\hat{\boldsymbol{w}}}{dt}\times\hat{\boldsymbol{X}} + \hat{\boldsymbol{w}}\times\frac{d\hat{\boldsymbol{X}}}{dt}\\ &= \frac{d\hat{\boldsymbol{w}}}{dt}\times\hat{\boldsymbol{X}} + \hat{\boldsymbol{w}}\times\left(\hat{\boldsymbol{w}}\times\hat{\boldsymbol{X}}\right)\end{aligned} \tag{2.1.15}$$

or in matrix notation

$$\begin{aligned}\hat{a} &= \frac{d^2\hat{X}}{dt^2} = \frac{d}{dt}\left(\frac{d\hat{X}}{dt}\right) = \frac{d}{dt}\left(\hat{\Omega}\hat{X}\right)\\ &= \frac{d\hat{\Omega}}{dt}\hat{X} + \hat{\Omega}\frac{d\hat{X}}{dt}\\ &= \frac{d\hat{\Omega}}{dt}\hat{X} + \hat{\Omega}\hat{\Omega}\hat{X} = \left(\frac{d\hat{\Omega}}{dt} + \hat{\Omega}^2\right)\hat{X}.\end{aligned} \tag{2.1.16}$$

2.2 Dual Euler Angles

In section 2.1, we discussed that any motion (rotation about an axis $\hat{\boldsymbol{w}}_0$ and translation along $\hat{\boldsymbol{w}}_0$) in line space can be represented by a dual spherical motion associated with a dual orthogonal matrix. For most purposes, however, it is better to express the dual orthogonal rotation matrix in terms of a set of dual Euler angles. First of all we will introduce the dual rotation matrix about the x_1-, x_2- and x_3-axes through dual angles $\hat{\theta}_1 = \theta_1 + \varepsilon\theta_1^*$, $\hat{\theta}_2 = \theta_2 + \varepsilon\theta_2^*$ and $\hat{\theta}_3 = \theta_3 + \varepsilon\theta_3^*$, respectively. Here θ_1, θ_1^*, θ_2, θ_2^*, θ_3 and θ_3^*, therefore $\hat{\theta}_1$, $\hat{\theta}_2$ and $\hat{\theta}_3$, are functions of t and for the simplicity we will denote them without t as long as we do not claim they are constant.

The dual orthogonal 3×3 matrix for a screw displacement through an angle θ_1 and a distance θ_1^* about the x_1-axis is

$$\hat{A}_1 = R_{x_1}\left(\hat{\theta}_1\right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\hat{\theta}_1 & \sin\hat{\theta}_1 \\ 0 & -\sin\hat{\theta}_1 & \cos\hat{\theta}_1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_1 & \sin\theta_1 \\ 0 & -\sin\theta_1 & \cos\theta_1 \end{bmatrix} + \varepsilon\theta_1^* \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\sin\theta_1 & \cos\theta_1 \\ 0 & -\cos\theta_1 & -\sin\theta_1 \end{bmatrix} \qquad (2.2.1)$$

and its Darboux matrix and Darboux vector are

$$\hat{\Omega}_1 = \frac{d\hat{A}_1}{dt}\hat{A}_1^t = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\frac{d\hat{\theta}_1}{dt}\sin\hat{\theta}_1 & \frac{d\hat{\theta}_1}{dt}\cos\hat{\theta}_1 \\ 0 & -\frac{d\hat{\theta}_1}{dt}\cos\hat{\theta}_1 & -\frac{d\hat{\theta}_1}{dt}\sin\hat{\theta}_1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\hat{\theta}_1 & -\sin\hat{\theta}_1 \\ 0 & \sin\hat{\theta}_1 & \cos\hat{\theta}_1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{d\hat{\theta}_1}{dt} \\ 0 & -\frac{d\hat{\theta}_1}{dt} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{d\theta_1}{dt} \\ 0 & -\frac{d\theta_1}{dt} & 0 \end{bmatrix} + \varepsilon \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{d\theta_1^*}{dt} \\ 0 & -\frac{d\theta_1^*}{dt} & 0 \end{bmatrix} = \Omega_1 + \varepsilon\,\Omega_1^* \qquad (2.2.2)$$

and

$$\hat{w}_1 = \left(-\frac{d\hat{\theta}_1}{dt}, 0, 0\right) = \left(-\frac{d\theta_1}{dt}, 0, 0\right) + \varepsilon\left(-\frac{d\theta_1^*}{dt}, 0, 0\right) = w_1 + \varepsilon\, w_1^* . \qquad (2.2.3)$$

The dual orthogonal 3×3 matrix for a screw displacement through an angle θ_2 and a distance θ_2^* about the x_2 – axis is

$$\hat{A}_2 = R_{x_2}\left(\hat{\theta}_2\right) = \begin{bmatrix} \cos\hat{\theta}_2 & 0 & -\sin\hat{\theta}_2 \\ 0 & 1 & 0 \\ \sin\hat{\theta}_2 & 0 & \cos\hat{\theta}_2 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta_2 & 0 & -\sin\theta_2 \\ 0 & 1 & 0 \\ \sin\theta_2 & 0 & \cos\theta_2 \end{bmatrix} + \varepsilon\theta_2^* \begin{bmatrix} -\sin\theta_2 & 0 & -\cos\theta_2 \\ 0 & 0 & 0 \\ \cos\theta_2 & 0 & -\sin\theta_2 \end{bmatrix} \qquad (2.2.4)$$

and its Darboux matrix and Darboux vector are

$$\hat{\Omega}_2 = \frac{d\hat{A}_2}{dt}\hat{A}_2^t = \begin{bmatrix} -\frac{d\hat{\theta}_2}{dt}\sin\hat{\theta}_2 & 0 & -\frac{d\hat{\theta}_2}{dt}\cos\hat{\theta}_2 \\ 0 & 0 & 0 \\ \frac{d\hat{\theta}_2}{dt}\cos\hat{\theta}_2 & 0 & -\frac{d\hat{\theta}_2}{dt}\sin\hat{\theta}_2 \end{bmatrix} \begin{bmatrix} \cos\hat{\theta}_2 & 0 & \sin\hat{\theta}_2 \\ 0 & 1 & 0 \\ -\sin\hat{\theta}_2 & 0 & \cos\hat{\theta}_2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & -\frac{d\hat{\theta}_2}{dt} \\ 0 & 0 & 0 \\ \frac{d\hat{\theta}_2}{dt} & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & -\frac{d\theta_2}{dt} \\ 0 & 0 & 0 \\ \frac{d\theta_2}{dt} & 0 & 0 \end{bmatrix} + \varepsilon \begin{bmatrix} 0 & 0 & -\frac{d\theta_2^*}{dt} \\ 0 & 0 & 0 \\ \frac{d\theta_2^*}{dt} & 0 & 0 \end{bmatrix} = \Omega_2 + \varepsilon\,\Omega_2^* \qquad (2.2.5)$$

and

$$\hat{w}_2 = \left(0, -\frac{d\hat{\theta}_2}{dt}, 0\right) = \left(0, -\frac{d\theta_2}{dt}, 0\right) + \varepsilon\left(0, -\frac{d\theta_2^*}{dt}, 0\right) = w_2 + \varepsilon\, w_2^*. \qquad (2.2.6)$$

The dual orthogonal 3×3 matrix for a screw displacement through an angle θ_3 and a distance θ_3^* about the x_3 −axis is

$$\hat{A}_3 = R_{x_3}\left(\hat{\theta}_3\right) = \begin{bmatrix} \cos\hat{\theta}_3 & \sin\hat{\theta}_3 & 0 \\ -\sin\hat{\theta}_3 & \cos\hat{\theta}_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta_3 & \sin\theta_3 & 0 \\ -\sin\theta_3 & \cos\theta_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \varepsilon\theta_3^* \begin{bmatrix} -\sin\theta_3 & \cos\theta_3 & 0 \\ -\cos\theta_3 & -\sin\theta_3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{2.2.7}$$

and its Darboux matrix and Darboux vector are

$$\hat{\Omega}_3 = \frac{d\hat{A}_3}{dt}\hat{A}_3^t = \begin{bmatrix} -\frac{d\hat{\theta}_3}{dt}\sin\hat{\theta}_3 & \frac{d\hat{\theta}_3}{dt}\cos\hat{\theta}_3 & 0 \\ -\frac{d\hat{\theta}_3}{dt}\cos\hat{\theta}_3 & -\frac{d\hat{\theta}_3}{dt}\sin\hat{\theta}_3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \cos\hat{\theta}_3 & -\sin\hat{\theta}_3 & 0 \\ \sin\hat{\theta}_3 & \cos\hat{\theta}_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \frac{d\hat{\theta}_3}{dt} & 0 \\ -\frac{d\hat{\theta}_3}{dt} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \frac{d\theta_3}{dt} & 0 \\ -\frac{d\theta_3}{dt} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \varepsilon \begin{bmatrix} 0 & \frac{d\theta_3^*}{dt} & 0 \\ -\frac{d\theta_3^*}{dt} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \Omega_3 + \varepsilon\,\Omega_3^* \tag{2.2.8}$$

and

$$\hat{w}_3 = \left(0,0,-\frac{d\hat{\theta}_3}{dt}\right) = \left(0,0,-\frac{d\theta_3}{dt}\right) + \varepsilon\left(0,0,-\frac{d\theta_3^*}{dt}\right) = w_3 + \varepsilon\, w_3^*. \tag{2.2.6}$$

Finally it is useful to express a dual matrix $\hat{A}$ in terms of a dual Euler angles. The matrix $\hat{A}$ is obtained by combining three successive intermediate screw displacements about the x_1-, x_2- and x_3-axes. Thus consider a screw displacement about the x_3-axis through an angle θ_3 and a distance θ_3^*, then about the new position of x_2-axis through an angle θ_2 and a distance θ_2^*, and finally about the new position of x_1 − axis through an angle θ_1 and a distance θ_1^*, which they can relate directly to appropriate definitions of yaw, pitch, and roll (see Fig.2.2.1).

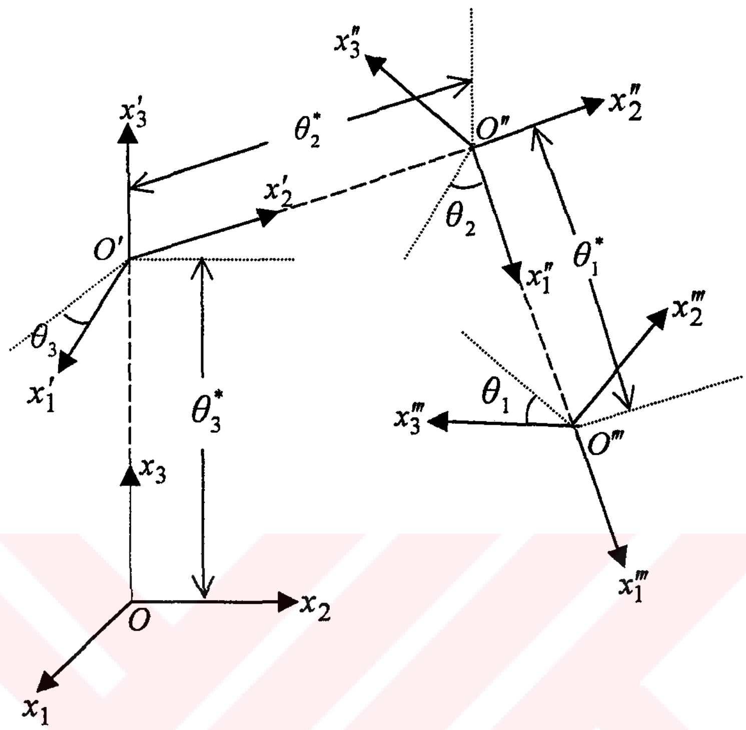

Fig.2.2.1 Screw displacements about the coordinate axes x_3, x_2 and x_1 through an angle θ_3, θ_2 and θ_1 and a distance θ_3^*, θ_2^* and θ_1^*, respectively.

The combinations of equations (2.2.1), (2.2.4) and (2.2.7) successively in that order gives the dual matrix

$$\hat{A} = \hat{A}_1\hat{A}_2\hat{A}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos\,\hat{\theta}_1 & sin\,\hat{\theta}_1 \\ 0 & -sin\,\hat{\theta}_1 & cos\,\hat{\theta}_1 \end{bmatrix} \begin{bmatrix} cos\,\hat{\theta}_2 & 0 & -sin\,\hat{\theta}_2 \\ 0 & 1 & 0 \\ sin\,\hat{\theta}_2 & 0 & cos\,\hat{\theta}_2 \end{bmatrix} \begin{bmatrix} cos\,\hat{\theta}_3 & sin\,\hat{\theta}_3 & 0 \\ -sin\,\hat{\theta}_3 & cos\,\hat{\theta}_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \hat{a}_{11} & \hat{a}_{12} & \hat{a}_{13} \\ \hat{a}_{21} & \hat{a}_{22} & \hat{a}_{23} \\ \hat{a}_{31} & \hat{a}_{32} & \hat{a}_{33} \end{bmatrix}. \qquad (2.2.10)$$

Here

$$\hat{a}_{11} = cos\,\hat{\theta}_2\, cos\,\hat{\theta}_3,$$

$$\hat{a}_{12} = cos\,\hat{\theta}_2\, sin\,\hat{\theta}_3,$$

$$\hat{a}_{13} = -sin\,\hat{\theta}_2,$$

$$\hat{a}_{21} = -\cos\hat{\theta}_1 \sin\hat{\theta}_3 + \sin\hat{\theta}_1 \sin\hat{\theta}_2 \cos\hat{\theta}_3,$$

$$\hat{a}_{22} = \cos\hat{\theta}_1 \cos\hat{\theta}_3 + \sin\hat{\theta}_1 \sin\hat{\theta}_2 \sin\hat{\theta}_3,$$

$$\hat{a}_{23} = \sin\hat{\theta}_1 \cos\hat{\theta}_2,$$

$$\hat{a}_{31} = \sin\hat{\theta}_1 \sin\hat{\theta}_3 + \cos\hat{\theta}_1 \sin\hat{\theta}_2 \cos\hat{\theta}_3,$$

$$\hat{a}_{32} = -\sin\hat{\theta}_1 \cos\hat{\theta}_3 + \cos\hat{\theta}_1 \sin\hat{\theta}_2 \sin\hat{\theta}_3$$

and

$$\hat{a}_{33} = \cos\hat{\theta}_1 \cos\hat{\theta}_2.$$

By separating $\hat{A}$ into real and dual parts, we get

$$\hat{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} + \varepsilon \begin{bmatrix} a^*_{11} & a^*_{12} & a^*_{13} \\ a^*_{21} & a^*_{22} & a^*_{23} \\ a^*_{31} & a^*_{32} & a^*_{33} \end{bmatrix} = A + \varepsilon A^*, \tag{2.2.11}$$

where

$$a_{11} = \cos\theta_2 \cos\theta_3$$

$$a_{12} = \cos\theta_2 \sin\theta_3$$

$$a_{13} = -\sin\theta_2,$$

$$a_{21} = -\cos\theta_1 \sin\theta_3 + \sin\theta_1 \sin\theta_2 \cos\theta_3,$$

$$a_{22} = \cos\theta_1 \cos\theta_3 + \sin\theta_1 \sin\theta_2 \sin\theta_3,$$

$$a_{23} = \sin\theta_1 \cos\theta_2,$$

$$a_{31} = \sin\theta_1 \sin\theta_3 + \cos\theta_1 \sin\theta_2 \cos\theta_3,$$

$$a_{32} = -\sin\theta_1 \cos\theta_3 + \cos\theta_1 \sin\theta_2 \sin\theta_3,$$

$$a_{33} = \cos\theta_1 \cos\theta_2,$$

$$a^*_{11} = -\theta^*_2 \sin\theta_2 \cos\theta_3 - \theta^*_3 \cos\theta_2 \sin\theta_3$$

$$a^*_{12} = -\theta^*_2 \sin\theta_2 \sin\theta_3 + \theta^*_3 \cos\theta_2 \cos\theta_3$$

$$a^*_{13} = -\theta^*_2 \sin\theta_2,$$

$$a^*_{21} = \theta^*_1 (\sin\theta_1 \sin\theta_3 + \cos\theta_1 \sin\theta_2 \cos\theta_3) + \theta^*_2 \sin\theta_1 \cos\theta_2 \cos\theta_3 - \theta^*_3 (\cos\theta_1 \cos\theta_3 + \sin\theta_1 \sin\theta_2 \sin\theta_3),$$

$$a_{22}^{*} = \theta_1^{*}\left(-\sin\theta_1 \cos\theta_3 + \cos\theta_1 \sin\theta_2 \sin\theta_3\right) + \theta_2^{*} \sin\theta_1 \cos\theta_2 \sin\theta_3 + \theta_3^{*}\left(-\cos\theta_1 \sin\theta_3 + \sin\theta_1 \sin\theta_2 \cos\theta_3\right),$$

$$a_{23}^{*} = \theta_1^{*} \cos\theta_1 \cos\theta_2 - \theta_2^{*} \sin\theta_1 \sin\theta_2,$$

$$a_{31}^{*} = \theta_1^{*}\left(\cos\theta_1 \sin\theta_3 - \sin\theta_1 \sin\theta_2 \cos\theta_3\right) + \theta_2^{*} \cos\theta_1 \cos\theta_2 \cos\theta_3 + \theta_3^{*}\left(\sin\theta_1 \cos\theta_3 - \cos\theta_1 \sin\theta_2 \sin\theta_3\right),$$

$$a_{32}^{*} = -\theta_1^{*}\left(\cos\theta_1 \cos\theta_3 + \sin\theta_1 \sin\theta_2 \sin\theta_3\right) + \theta_2^{*} \cos\theta_1 \cos\theta_2 \sin\theta_3 + \theta_3^{*}\left(\sin\theta_1 \sin\theta_3 + \cos\theta_1 \sin\theta_2 \cos\theta_3\right)$$

and

$$a_{33}^{*} = -\theta_1^{*} \sin\theta_1 \cos\theta_2 - \theta_2^{*} \cos\theta_1 \sin\theta_2 .$$

Therefore, the Darboux matrix of the motion given by the matrix (2.2.10) is

$$\begin{aligned}
\hat{\Omega} &= \frac{d\hat{A}}{dt}\hat{A}^t = \frac{d}{dt}\left(\hat{A}_1\hat{A}_2\hat{A}_3\right)\left(\hat{A}_1\hat{A}_2\hat{A}_3\right)^t \\
&= \left(\frac{d\hat{A}_1}{dt}\hat{A}_2\hat{A}_3 + \hat{A}_1\frac{d\hat{A}_2}{dt}\hat{A}_3 + \hat{A}_1\hat{A}_2\frac{d\hat{A}_3}{dt}\right)\left(\hat{A}_3^t\hat{A}_2^t\hat{A}_1^t\right) \\
&= \frac{d\hat{A}_1}{dt}\hat{A}_1^t + \hat{A}_1\frac{d\hat{A}_2}{dt}\hat{A}_2^t\hat{A}_1^t + \hat{A}_1\hat{A}_2\frac{d\hat{A}_3}{dt}\hat{A}_3^t\hat{A}_2^t\hat{A}_1^t \\
&= \hat{\Omega}_1 + \hat{A}_1\hat{\Omega}_2\hat{A}_1^t + \hat{A}_1\hat{A}_2\hat{\Omega}_3\hat{A}_2^t\hat{A}_1^t .
\end{aligned}$$

Using (2.2.1), (2.2.2), (2.2.4), (2.2.5), (2.2.7) and (2.2.8), the Darboux matrix is calculated as

$$\hat{\Omega} = \Omega + \varepsilon\Omega^{*} = \begin{bmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{bmatrix} + \varepsilon \begin{bmatrix} 0 & -w_3^{*} & w_2^{*} \\ w_3^{*} & 0 & -w_1^{*} \\ -w_2^{*} & w_1^{*} & 0 \end{bmatrix}, \qquad (2.2.12)$$

and therefore the Darboux vector is

$$\hat{w} = w + \varepsilon\, w = \left(w_1, w_2, w_3\right) + \varepsilon\left(w_1^{*}, w_2^{*}, w_3^{*}\right), \qquad (2.2.13)$$

where

$$w_1 = -\frac{d\theta_1}{dt} + \frac{d\theta_3}{dt} sin\,\theta_2, \tag{2.2.14}$$

$$w_2 = -\frac{d\theta_2}{dt} cos\,\theta_1 - \frac{d\theta_3}{dt} sin\,\theta_1\, cos\,\theta_2, \tag{2.2.15}$$

$$w_3 = \frac{d\theta_2}{dt} sin\,\theta_1 - \frac{d\theta_3}{dt} cos\,\theta_1\, cos\,\theta_2, \tag{2.2.16}$$

$$w_1^* = -\frac{d\theta_1^*}{dt} + \frac{d\theta_3^*}{dt} sin\,\theta_2 + \frac{d\theta_3}{dt}\theta_2^*\, cos\,\theta_2, \tag{2.2.17}$$

$$w_2^* = -\frac{d\theta_2^*}{dt} cos\,\theta_1 + \frac{d\theta_2}{dt}\theta_1^*\, sin\,\theta_1 - \frac{d\theta_3^*}{dt} sin\,\theta_1\, cos\,\theta_2 - \frac{d\theta_3}{dt}\theta_1^*\, cos\,\theta_1\, cos\,\theta_2 + \frac{d\theta_3}{dt}\theta_2^*\, sin\,\theta_1\, sin\,\theta_2, \tag{2.2.18}$$

$$w_3^* = \frac{d\theta_2^*}{dt} sin\,\theta_1 + \frac{d\theta_2}{dt}\theta_1^*\, cos\,\theta_1 - \frac{d\theta_3^*}{dt} cos\,\theta_1\, cos\,\theta_2 + \frac{d\theta_3}{dt}\theta_1^*\, sin\,\theta_1\, cos\,\theta_2 + \frac{d\theta_3}{dt}\theta_2^*\, cos\,\theta_1\, sin\,\theta_2. \tag{2.2.19}$$

Moreover, the pitch of the motion is

$$h = \frac{w \cdot w^*}{\|w\|^2} \tag{2.2.20}$$

$$= \frac{\frac{d\theta_1}{dt}\frac{d\theta_1^*}{dt} + \frac{d\theta_2}{dt}\frac{d\theta_2^*}{dt} + \frac{d\theta_3}{dt}\frac{d\theta_3^*}{dt} - \left(\frac{d\theta_1}{dt}\frac{d\theta_3^*}{dt} + \frac{d\theta_3}{dt}\frac{d\theta_1^*}{dt}\right) sin\,\theta_2 - \frac{d\theta_1}{dt}\frac{d\theta_3}{dt}\theta_2^*\, cos\,\theta_2}{\left(\frac{d\theta_1}{dt}\right)^2 + \left(\frac{d\theta_2}{dt}\right)^2 + \left(\frac{d\theta_3}{dt}\right)^2 - 2\frac{d\theta_1}{dt}\frac{d\theta_3}{dt} sin\,\theta_2}.$$

When the displacement becomes infinitesimal, $\theta_i \to d\theta_i$ and $\theta_i^* \to d\theta_i^*$, and therefore, $\hat{\theta}_i \to d\hat{\theta}_i$, $(i = 1,2,3)$. In addition, for small displacements, $sin\, d\theta_i \to d\theta_i$ and $cos\, d\theta_i \to 1$, $(i = 1,2,3)$. Thus the screw displacement matrix given by (2.2.10) reduces the following form

$$\hat{\beta}=\hat{R}_{x_1}(d\hat{\theta}_1)\hat{R}_{x_2}(d\hat{\theta}_2)\hat{R}_{x_3}(d\hat{\theta}_3)=\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}+\varepsilon\begin{bmatrix} r_{11}^* & r_{12}^* & r_{13}^* \\ r_{21}^* & r_{22}^* & r_{23}^* \\ r_{31}^* & r_{32}^* & r_{33}^* \end{bmatrix}, \qquad (2.2.21)$$

where

$$r_{11}=1,\ r_{12}=d\theta_3,\ r_{13}=-d\theta_2,$$

$$r_{21}=-d\theta_3+d\theta_1 d\theta_2 d\theta_3,\ r_{22}=1+d\theta_1 d\theta_2 d\theta_3,\ r_{23}=d\theta_1,$$

$$r_{31}=d\theta_2+d\theta_1 d\theta_3,\ r_{32}=-d\theta_1+d\theta_2 d\theta_3,\ r_{33}=1$$

$$r_{11}^*=-d\theta_2^* d\theta_2-d\theta_3^* d\theta_3,\ r_{12}^*=-d\theta_2^* d\theta_2 d\theta_3+d\theta_3^*,\ r_{13}^*=-d\theta_2^* d\theta_2,$$

$$r_{21}^*=d\theta_1^*(d\theta_1 d\theta_3+d\theta_2)+d\theta_2^* d\theta_1-d\theta_3^*(1+d\theta_1 d\theta_2 d\theta_3),$$

$$r_{22}^*=d\theta_1^*(-d\theta_1+d\theta_2 d\theta_3)+d\theta_2^* d\theta_1 d\theta_3+d\theta_3^*(-d\theta_3+d\theta_1 d\theta_2),$$

$$r_{23}^*=-d\theta_1^*-d\theta_2^* d\theta_1 d\theta_2,$$

$$r_{31}^*=d\theta_1^*(d\theta_3-d\theta_1 d\theta_2)+d\theta_2^*+d\theta_3^*(d\theta_1-d\theta_2 d\theta_3),$$

$$r_{32}^*=-d\theta_1^*(1+d\theta_1 d\theta_2 d\theta_3)+d\theta_2^* d\theta_3+d\theta_3^*(d\theta_1 d\theta_3+d\theta_2)$$

and

$$r_{33}^*=-d\theta_1^* d\theta_1-d\theta_2^* d\theta_2.$$

Moreover, it is convenient to use $d\theta_i=\frac{d\theta_i}{dt}dt$ and $d\theta_i^*=\frac{d\theta_i^*}{dt}dt$, $(i=1,2,3)$.

Ignoring small quantities of the second order, the infinitesimal screw displacement matrix can be simplified to

$$\hat{\beta}=\begin{bmatrix} 1 & \frac{d\theta_3}{dt}dt & -\frac{d\theta_2}{dt}dt \\ -\frac{d\theta_3}{dt}dt & 1 & \frac{d\theta_1}{dt}dt \\ \frac{d\theta_2}{dt}dt & -\frac{d\theta_1}{dt}dt & 1 \end{bmatrix}+\varepsilon\begin{bmatrix} 0 & \frac{d\theta_3^*}{dt}dt & -\frac{d\theta_2^*}{dt}dt \\ -\frac{d\theta_3^*}{dt}dt & 0 & \frac{d\theta_1^*}{dt}dt \\ \frac{d\theta_2^*}{dt}dt & -\frac{d\theta_1^*}{dt}dt & 0 \end{bmatrix} \qquad (2.2.22)$$

or

$$\hat{\beta}=I+(\Gamma+\varepsilon\Gamma^*)dt=I+\hat{\Gamma}dt, \qquad (2.2.23)$$

where

$$\hat{\Gamma}=\Gamma+\varepsilon\Gamma^*,\quad \Gamma=\begin{bmatrix}0 & \frac{d\theta_3}{dt} & -\frac{d\theta_2}{dt}\\ -\frac{d\theta_3}{dt} & 0 & \frac{d\theta_1}{dt}\\ \frac{d\theta_2}{dt} & -\frac{d\theta_1}{dt} & 0\end{bmatrix}\text{ and }\Gamma^*=\begin{bmatrix}0 & \frac{d\theta_3^*}{dt} & -\frac{d\theta_2^*}{dt}\\ -\frac{d\theta_3^*}{dt} & 0 & \frac{d\theta_1^*}{dt}\\ \frac{d\theta_2^*}{dt} & -\frac{d\theta_1^*}{dt} & 0\end{bmatrix}.$$

Note that

$$\hat{\Gamma}=\hat{\Omega}_1+\hat{\Omega}_2+\hat{\Omega}_3. \tag{2.2.24}$$

Example 2.2.1 Let us consider the dual spherical motion given by the matrix (2.2.10) with $\theta_1(t)=\theta_2(t)=\theta_3(t)=\theta_1^*(t)=\theta_2^*(t)=\theta_3^*(t)=t$. Then the dual matrix $\hat{A}$ is

$$\hat{A}=A+\varepsilon A^*=\begin{bmatrix}\cos^2 t & \cos t\sin t & -\sin t\\ -\sin t\cos t+\sin^2 t\cos t & \cos^2 t+\sin^3 t & \sin t\cos t\\ \sin^2 t+\sin t\cos^2 t & -\sin t\cos t+\sin^2 t\cos t & \cos^2 t\end{bmatrix}+$$

$$\varepsilon\begin{bmatrix}-t\sin 2t & t\cos 2t & t\sin t\\ -t(\cos 2t+\sin t(\cos 2t+\cos^2 t)) & t(-\sin 2t+3\sin^2 t\cos t) & t\cos 2t\\ t(\sin 2t+\cos t(\cos 2t-\sin^2 t)) & t(-\cos 2t+\sin t(\cos 2t+\cos^2 t)) & -t\sin 2t\end{bmatrix}.$$

Therefore, by (2.2.13)-(2.2.20) the dual Darboux vector and the pitch of the motion are

$$\hat{w}=w+\varepsilon w^*=(-1+\sin t,\ -\cos t-\sin t\cos t,\ \sin t-\cos^2 t)+$$
$$\varepsilon(-1+\sin t+t\cos t,\ -\cos t-\sin t\cos t+t\sin t-t\cos 2t,$$
$$\sin t-\cos^2 t+t\cos t+t\sin 2t),$$

and

$$h=\frac{3-2\sin t-t\cos t}{3-2\sin t}.$$

Besides, from (2.1.13) the instantaneous screw axis of the motion is found as

$$\hat{w}_0 = w_0 + \varepsilon\, w_0^* = \left(\frac{-1+\sin t}{\sqrt{3-2\sin t}},\ -\frac{\cos t+\sin t\cos t}{\sqrt{3-2\sin t}},\ \frac{\sin t-\cos^2 t}{\sqrt{3-2\sin t}} \right) +$$

$$\varepsilon \left(\frac{t\cos t(2-\sin t)}{(3-2\sin t)^{\frac{3}{2}}},\ \frac{t\sin t(2-2\sin t+\sin^2 t)-t\cos 2t}{(3-2\sin t)^{\frac{3}{2}}},\right.$$

$$\left. \frac{t\sin 2t(3-2\sin t)+t\cos t(2-\sin t+\sin^2 t)}{(3-2\sin t)^{\frac{3}{2}}} \right).$$

Hence, the ruled surface (axode) determined by the I.S.A. is

$$L_{w_0}(t,\lambda) = w_0 \times w_0^* + \lambda w_0$$

$$= \left(\frac{t(2\sin 4t\cos t+8\cos t\sin 2t+\cos 4t+10\sin^3 t+6\sin t)}{4(3-2\sin t)^2} + \right.$$

$$\frac{t\cos 2t(2\sin t+2\cos 2t+15)}{4(3-2\sin t)^2} + \frac{\lambda(\sin t-1)}{\sqrt{3-2\sin t}},$$

$$\frac{t\sin 2t(2\sin t-3)(\sin t-1)+t\cos t\sin^2 t(3-2\sin t)}{(3-2\sin t)^2} - \frac{\lambda\cos t(\sin t+1)}{\sqrt{3-2\sin t}},$$

$$\frac{t\cos 2t(2\sin^2 t-5\sin t+4)-t\cos^2 t(2\sin^2 t-3\sin t+4)+t(6-5\sin t)}{(3-2\sin t)^2} +$$

$$\left. \frac{\lambda(\sin t-\cos^2 t)}{\sqrt{3-2\sin t}} \right)$$

(See Fig. 2.2.2).

Let us consider the straight line through $(0,1,0)$ with the direction $x_0 = (0,0,1)$. Then by study mapping this directed straight line corresponds the dual unit vector

$$\hat{x}_0 = x_0 + \varepsilon x_0^* = (0,0,1) + \varepsilon(0,1,0) \times (0,0,1) = (0,0,1) + \varepsilon(1,0,0)$$

and it draws a dual curve on the fixed D.U.S., that is

$$\hat{X}_0 = \hat{A}\hat{x}_0 = Ax_0 + \varepsilon\left(Ax_0^* + A^*x_0\right) = X_0 + \varepsilon X_0^*,$$

where

$$X_0 = Ax_0 = \left(-\sin t, \sin t\cos t, \cos^2 t\right)$$

and

$$X_0^* = Ax_0^* + A^*x_0 = \left(\cos^2 t + t\sin t, \sin^2 t\cos t - \sin t\cos t + t\cos 2t,\right.$$
$$\left.\sin^2 t + \sin t\cos^2 t - t\sin 2t\right).$$

Then, the ruled surface corresponding the dual curve $\hat{X}_0(t)$ is

$$L(t,\lambda) = X_0 \times X_0^* + \lambda X_0 = \left(\sin t\cos t - t\cos^2 t - \lambda\sin t,\ \cos^2 t + \sin^3 t +\right.$$
$$t\sin t\left(\cos^2 t - \sin 2t\right) + \lambda\sin t\cos t,\ \sin^2 t\cos t - \sin t\cos t -$$
$$\left.t\left(\sin t\cos 2t + \sin^2 t\cos t\right) + \lambda\cos^2 t\right)$$

(See Fig. 2.2.3).

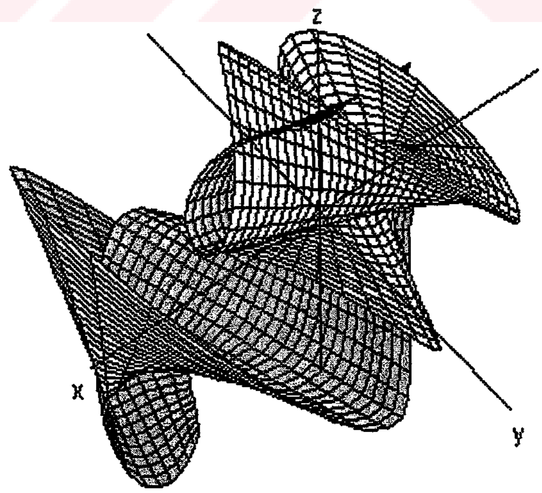

Fig.2.2.2 The ruled surface $L_{w_0}(t,\lambda) = w_0 \times w_0^* + \lambda w_0$ with domain $D = \{(t,\lambda) : t,\lambda \in (-5,5)\}$.

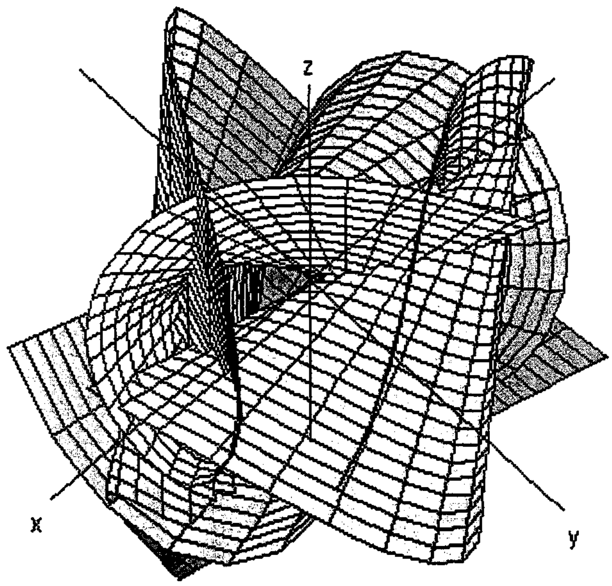

Fig. 2.2.3 The ruled surface $L(t,\lambda)$ with domain $D=\{(t,\lambda):t,\lambda\in(-5,5)\}$.

2.3 The Differential Screw

The difference between two consecutive screws $\hat{w}$ and $\hat{w}_1$, can be represented by a differential screw $\frac{d\hat{w}}{dt}$. For general continuous spatial motion the consecutive screws are $\hat{w}$ and $\hat{w}_1=\hat{w}(t+dt)$. Therefore

$$\hat{w}_1=\hat{w}+\frac{d\hat{w}}{dt}dt. \qquad (2.3.1)$$

First of all, we will obtain two consecutive screws and the differential screw on fixed axode. Since the fixed axode and moving axode always meet tangentially in a common line along the I.S.A., for infinitesimal motion $\hat{w}$ and $\hat{w}_1$ can be considered to lie on both fixed axode and moving axode. Therefore the screw consecutive $\hat{w}$ on fixed axode is

$$\hat{\bar{w}}_1=\left(I+\hat{\Gamma}dt\right)\left(\hat{w}+\frac{d\hat{w}}{dt}dt\right)=\hat{w}+\left(\frac{d\hat{w}}{dt}+\hat{\Gamma}\hat{w}\right)dt. \qquad (2.3.2)$$

Here, we ignored the small quantities of the second order and $\hat{\Gamma}$ is a dual skew symmetric matrix in (2.2.23).

Hence, by using (2.3.1), the differential screw, from now on it will be denoted by $\hat{w}_d$, is found in the form

$$\hat{w}_d = \frac{d\hat{w}}{dt} + \hat{\Gamma}\hat{w} \qquad (2.3.3)$$

or introducing the vector $\hat{\gamma} = \left(-\frac{d\hat{\theta}_1}{dt}, -\frac{d\hat{\theta}_2}{dt}, -\frac{d\hat{\theta}_3}{dt} \right)$, we may write

$$\hat{w}_d = \frac{d\hat{w}}{dt} + \hat{\gamma} \times \hat{w}, \qquad (2.3.4)$$

where $\hat{\gamma} = \gamma + \varepsilon\gamma^*$, $\gamma = \left(-\frac{d\theta_1}{dt}, -\frac{d\theta_2}{dt}, -\frac{d\theta_3}{dt} \right)$, $\gamma^* = \left(-\frac{d\theta_1^*}{dt}, -\frac{d\theta_1^*}{dt}, -\frac{d\theta_1^*}{dt} \right)$, and w and w^* are real and dual parts of the dual Darboux vector $\hat{w}$ and their components are given by (2.2.14)-(2.2.19).

In addition, it can be easily seen from (2.2.3), (2.2.6), (2.2.9) and (2.2.24) that

$$\hat{\gamma} = \hat{w}_1 + \hat{w}_2 + \hat{w}_3. \qquad (2.3.5)$$

Moreover, if the dual vectors $\hat{\gamma}$ and $\hat{w}$ are parallel, then the screws $\hat{w}_1$ and $\hat{\bar{w}}_1$ consecutive to current screw $\hat{w}$ coincide, i.e., the axodes in moving line space $H$and in fixed line space H' coincides. This case is possible if and only if the axode either a plane or a straight line. Therefore, the dual spherical motion K / K' is symmetric motion, i.e., $\hat{A} = \hat{A}'$.

By separating (2.3.4) into real and dual parts, we obtain

$$\hat{w}_d = w_d + \varepsilon\, w_d^* = \frac{dw}{dt} + \gamma \times w + \varepsilon \left(\frac{dw^*}{dt} + \gamma \times w^* + \gamma^* \times w \right). \tag{2.3.6}$$

Hence, from (2.2.14)-(2.2.19) and (2.3.6),

$$\hat{w}_d = w_d + \varepsilon\, w_d^* = \left(\hat{w}_{d1}, \hat{w}_{d2}, \hat{w}_{d3}\right) + \varepsilon\left(\hat{w}_{d1}^*, \hat{w}_{d2}^*, \hat{w}_{d3}^*\right)$$

yields:

$$w_{d1} = -\frac{d^2\theta_1}{dt^2} + \frac{d^2\theta_3}{dt^2} \sin\theta_2 + \frac{d\theta_2}{dt}\frac{d\theta_3}{dt}\left(\cos\theta_1 \cos\theta_2 - \cos\theta_1 + \cos\theta_2\right) - \left(\frac{d\theta_3}{dt}\right)^2 \sin\theta_1 \cos\theta_2 - \left(\frac{d\theta_2}{dt}\right)^2 \sin\theta_1, \tag{2.3.7}$$

$$w_{d2} = -\frac{d^2\theta_2}{dt^2}\cos\theta_1 - \frac{d^2\theta_3}{dt^2}\sin\theta_1 \cos\theta_2 + 2\frac{d\theta_1}{dt}\frac{d\theta_2}{dt}\sin\theta_1 + \frac{d\theta_1}{dt}\frac{d\theta_3}{dt}\left(1 - 2\cos\theta_1 \cos\theta_2\right) + \frac{d\theta_2}{dt}\frac{d\theta_3}{dt}\sin\theta_1 \sin\theta_2 - \left(\frac{d\theta_3}{dt}\right)^2 \sin\theta_2, \tag{2.3.8}$$

$$w_{d3} = \frac{d^2\theta_2}{dt^2}\sin\theta_1 - \frac{d^2\theta_3}{dt^2}\cos\theta_1 \cos\theta_2 + \frac{d\theta_1}{dt}\frac{d\theta_2}{dt}\left(2\cos\theta_1 - 1\right) + 2\frac{d\theta_1}{dt}\frac{d\theta_3}{dt}\sin\theta_1 \cos\theta_2 + \frac{d\theta_2}{dt}\frac{d\theta_3}{dt}\sin\theta_2\left(1 + \cos\theta_1\right), \tag{2.3.9}$$

$$w_{d1}^* = -\frac{d^2\theta_1^*}{dt^2} + \frac{d^2\theta_3^*}{dt^2}\sin\theta_2 + \frac{d^2\theta_3}{dt^2}\theta_2^* \cos\theta_2 - 2\frac{d\theta_2}{dt}\frac{d\theta_2^*}{dt}\sin\theta_1 + \frac{d\theta_2}{dt}\frac{d\theta_3}{dt}\left[\theta_1^* \sin\theta_1\left(1 - \cos\theta_2\right) - \theta_2^*\left(1 + \cos\theta_1\right)\sin\theta_2\right] - \left(\frac{d\theta_2}{dt}\right)^2 \theta_1^* \cos\theta_1 - \left(\frac{d\theta_3}{dt}\right)^2\left(\theta_1^* \cos\theta_1 \cos\theta_2 - \theta_2^* \sin\theta_1 \sin\theta_2\right) - 2\frac{d\theta_3}{dt}\frac{d\theta_3^*}{dt}\sin\theta_1 \cos\theta_2 + \left(\frac{d\theta_2^*}{dt}\frac{d\theta_3}{dt} + \frac{d\theta_2}{dt}\frac{d\theta_3^*}{dt}\right)\left(\cos\theta_1 \cos\theta_2 - \cos\theta_1 + \cos\theta_2\right), \tag{2.3.10}$$

$$w^*_{d2} = -\frac{d^2\theta^*_2}{dt^2}\cos\theta_1 + \frac{d^2\theta_2}{dt^2}\theta^*_1 \sin\theta_1 - \frac{d^2\theta^*_3}{dt^2}\sin\theta_1 \cos\theta_2 -$$

$$\frac{d^2\theta_3}{dt^2}\left(\theta^*_1 \cos\theta_1 \cos\theta_2 + \theta^*_2 \sin\theta_1 \sin\theta_2\right)+$$

$$2\left(\frac{d\theta^*_1}{dt}\frac{d\theta_2}{dt} + \frac{d\theta_1}{dt}\frac{d\theta^*_2}{dt}\right)\sin\theta_1 + 2\frac{d\theta_1}{dt}\frac{d\theta_2}{dt}\theta^*_1 \cos\theta_1 - 2\frac{d\theta_3}{dt}\frac{d\theta^*_3}{dt}\sin\theta_2 -$$

$$\left(\frac{d\theta_3}{dt}\right)^2 \theta^*_2 \cos\theta_2 + \left(\frac{d\theta^*_1}{dt}\frac{d\theta_3}{dt} + \frac{d\theta_1}{dt}\frac{d\theta^*_3}{dt}\right)\left(1 - 2\cos\theta_1 \cos\theta_2\right)+$$

$$2\frac{d\theta_1}{dt}\frac{d\theta_3}{dt}\left(\theta^*_1 \sin\theta_1 \cos\theta_2 + \theta^*_2 \cos\theta_1 \sin\theta_2\right)+$$

$$\left(\frac{d\theta^*_2}{dt}\frac{d\theta_3}{dt} + \frac{d\theta_2}{dt}\frac{d\theta^*_3}{dt}\right)\sin\theta_1 \sin\theta_2 + \frac{d\theta_2}{dt}\frac{d\theta_3}{dt}\left(\theta^*_1 \cos\theta_1 \sin\theta_2 + \theta^*_2 \sin\theta_1 \cos\theta_2\right)$$

(2.3.11)

and

$$w^*_{d3} = \frac{d^2\theta^*_2}{dt^2}\sin\theta_1 + \frac{d^2\theta_2}{dt^2}\theta^*_1 \cos\theta_1 - \frac{d^2\theta^*_3}{dt^2}\cos\theta_1 \cos\theta_2 +$$

$$\frac{d^2\theta_3}{dt^2}\left(\theta^*_1 \sin\theta_1 \cos\theta_2 + \theta^*_2 \cos\theta_1 \sin\theta_2\right)+$$

$$\left(\frac{d\theta^*_1}{dt}\frac{d\theta_2}{dt} + \frac{d\theta_1}{dt}\frac{d\theta^*_2}{dt}\right)\left(2\cos\theta_1 - 1\right) - 2\frac{d\theta_1}{dt}\frac{d\theta_2}{dt}\theta^*_1 \sin\theta_1 -$$

$$2\left(\frac{d\theta^*_1}{dt}\frac{d\theta_3}{dt} + \frac{d\theta_1}{dt}\frac{d\theta^*_3}{dt}\right)\sin\theta_1 \cos\theta_2 +$$

$$2\frac{d\theta_1}{dt}\frac{d\theta_3}{dt}\left(\theta^*_1 \cos\theta_1 \cos\theta_2 - \theta^*_2 \sin\theta_1 \sin\theta_2\right)+$$

$$\frac{d\theta_2}{dt}\frac{d\theta_3}{dt}\left(-\theta^*_1 \sin\theta_1 \sin\theta_2 + \theta^*_2 \cos\theta_2\left(1 + \cos\theta_1\right)\right)+$$

$$\left(\frac{d\theta^*_2}{dt}\frac{d\theta_3}{dt} + \frac{d\theta_2}{dt}\frac{d\theta^*_3}{dt}\right)\sin\theta_2\left(1 + \cos\theta_1\right). \qquad (2.3.12)$$

Although, the determination of velocities at all points of a moving D.U.S. K by reference to $\hat{w}$ with the formula (2.1.10), it is not possible to proceed similarly for accelerations of all points by reference to a single "acceleration screw". In fact both

$\hat{w}$ and $\hat{w}_d$ are needed. Since the differential screw, $\hat{w}_d$, has to lie normal to the common perpendicular between consecutive screws $\hat{w}$ and $\hat{\bar{w}}$. Then total acceleration of a point will therefore be the vector sum of two components: a dual angular acceleration about the axis of $\hat{w}_d$ and a dual centripetal acceleration directed from the point and normal to the axis of $\hat{w}$ (Bokelberg et.al., 1992). Then we get,

$$\hat{a} = \frac{d^2\hat{X}}{dt^2} = \hat{w}_d \times \hat{X} + \hat{w} \times \left(\hat{w} \times \hat{X}\right) \qquad (2.3.15)$$

or in matrix notation

$$\hat{a} = \hat{\Omega}_d\hat{X} + \hat{\Omega}\hat{\Omega}\hat{X} = \left(\hat{\Omega}_d + \hat{\Omega}^2\right)\hat{X}, \qquad (2.3.16)$$

where $\hat{\Omega}_d = \begin{bmatrix} 0 & -\hat{w}_{d3} & \hat{w}_{d2} \\ \hat{w}_{d3} & 0 & -\hat{w}_{d1} \\ -\hat{w}_{d2} & \hat{w}_{d1} & 0 \end{bmatrix}$.

2.4 The Acceleration Center

An acceleration center is a point on moving D.U.S. K that has no acceleration in the dual spherical motion K / K'. In other words, a point $\hat{X}$ is a point of acceleration center if and only if

$$\hat{a} = \frac{d^2\hat{X}}{dt^2} = \hat{w}_d \times \hat{X} + \hat{w} \times \left(\hat{w} \times \hat{X}\right) = 0 \qquad (2.4.1)$$

or in matrix notation

$$\hat{a} = \hat{\Omega}_d\hat{X} + \hat{\Omega}\hat{\Omega}\hat{X} = \left(\hat{\Omega}_d + \hat{\Omega}^2\right)\hat{X} = 0 \qquad (2.4.2)$$

2.5 The Inflection Ruled Surface

An inflection point on D.U.S. is one that instantaneously exhibiting a dual spherical motion in its path, and therefore its acceleration and velocity vectors are parallel. In other words, a point $\hat{X}$ is a point of inflection if and only if

$$\hat{\boldsymbol{a}} \times \hat{\boldsymbol{v}} = \boldsymbol{0} \text{ or } \hat{\boldsymbol{a}} = \hat{\lambda}\hat{\boldsymbol{v}}, \tag{2.5.1}$$

where $\hat{\lambda}$ is a dual parameter.

Let $\hat{x}$ be a fixed point on moving D.U.S. K and $\hat{X}$ its image on fixed D.U.S. K'. Then by (2.1.9), (2.3.16) and (2.5.1), the following equations hold:

$$\left(\hat{\Omega}_d + \hat{\Omega}^2\right)\hat{X} = \hat{\lambda}\hat{\Omega}\hat{X}$$

or

$$\left(\hat{\Omega}_d + \hat{\Omega}^2 - \hat{\lambda}\hat{\Omega}\right)\hat{X} = 0. \tag{2.5.2}$$

The system of linear equations (2.5.2) has non-trivial infinite solutions if and only if

$$det\left(\hat{\Omega}_d + \hat{\Omega}^2 - \hat{\lambda}\hat{\Omega}\right) = 0. \tag{2.5.3}$$

On the other hand

$$\left(\hat{\Omega}_d + \hat{\Omega}^2 - \hat{\lambda}\hat{\Omega}\right) = \begin{bmatrix} 0 & -\hat{w}_{d3} & \hat{w}_{d2} \\ \hat{w}_{d3} & 0 & -\hat{w}_{d1} \\ -\hat{w}_{d2} & \hat{w}_{d1} & 0 \end{bmatrix} + \begin{bmatrix} -\hat{w}_2^2 - \hat{w}_3^2 & \hat{w}_1\hat{w}_2 & \hat{w}_1\hat{w}_3 \\ \hat{w}_1\hat{w}_2 & -\hat{w}_1^2 - \hat{w}_3^2 & \hat{w}_2\hat{w}_3 \\ \hat{w}_1\hat{w}_3 & \hat{w}_2\hat{w}_3 & -\hat{w}_1^2 - \hat{w}_2^2 \end{bmatrix}$$

$$-\hat{\lambda}\begin{bmatrix} 0 & -\hat{w}_{d3} & \hat{w}_{d2} \\ \hat{w}_{d3} & 0 & -\hat{w}_{d1} \\ -\hat{w}_{d2} & \hat{w}_{d1} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -\hat{w}_2^2 - \hat{w}_3^2 & -\hat{w}_{d3} + \hat{w}_1\hat{w}_2 + \hat{\lambda}\hat{w}_3 & \hat{w}_{d2} + \hat{w}_1\hat{w}_3 - \hat{\lambda}\hat{w}_2 \\ \hat{w}_{d3} + \hat{w}_1\hat{w}_2 - \hat{\lambda}\hat{w}_3 & -\hat{w}_1^2 - \hat{w}_3^2 & -\hat{w}_{d1} + \hat{w}_2\hat{w}_3 + \hat{\lambda}\hat{w}_1 \\ -\hat{w}_{d2} + \hat{w}_1\hat{w}_3 + \hat{\lambda}\hat{w}_2 & \hat{w}_{d1} + \hat{w}_2\hat{w}_3 - \hat{\lambda}\hat{w}_1 & -\hat{w}_1^2 - \hat{w}_2^2 \end{bmatrix}$$

(2.5.4)

and therefore, its determinant is

$$det\left(\hat{\Omega}_d + \hat{\Omega}^2 - \hat{\lambda}\hat{\Omega}\right) = \hat{w}_1^2\hat{w}_{d1}^2 + \hat{w}_2^2\hat{w}_{d2}^2 + \hat{w}_3^2\hat{w}_{d3}^2 + 2\left(\hat{w}_1\hat{w}_2\hat{w}_{d1}\hat{w}_{d2} + \hat{w}_1\hat{w}_3\hat{w}_{d1}\hat{w}_{d3} + \hat{w}_2\hat{w}_3\hat{w}_{d2}\hat{w}_{d3}\right) - \|\hat{w}\|^2\|\hat{w}_d\|^2$$

$$= \left(\hat{w}\cdot\hat{w}_d\right)^2 - \|\hat{w}\|^2\|\hat{w}_d\|^2$$

$$= -\|w \times w_d\| \qquad (2.5.5)$$

and it is independent of choice of $\hat{\lambda}$.

Thus, the necessary and sufficient condition for the existence of points of inflection of the motion is that the dual vectors $\hat{w}_d$ and $\hat{w}$ are parallel for some value of t, i.e.,

$$\hat{w}_d = \hat{\mu}\hat{w} \text{ or } \hat{\Omega}_d = \hat{\mu}\hat{\Omega},\ t \in G, \qquad (2.5.6)$$

where G is the set of solutions of the equation

$$\|w \times w_d\| = 0. \qquad (2.5.7)$$

Therefore $\hat{\mu}$ is the function of t and it is determined by the formula

$$\hat{\mu}(t) = \frac{\hat{w}_d \cdot \hat{w}}{\|\hat{w}\|^2},\ t \in G. \qquad (2.5.8)$$

Hence by (2.5.6), the system of equations (2.5.2) is simplified to

$$\left(\hat{\Omega}^2 + \hat{\rho}\hat{\Omega}\right)\hat{X} = 0, \qquad (2.5.9)$$

where $\hat{\rho} = \hat{\mu} - \hat{\lambda}$. By separating (2.5.9) into real and dual parts, we get

$$\left(\Omega^2 + \rho\Omega\right)X = 0 \qquad (2.5.10)$$

and

$$\left(\Omega\Omega^* + \Omega^*\Omega + \rho^*\Omega + \rho\Omega^*\right)X + \left(\Omega^2 + \rho\Omega\right)X^* = 0. \qquad (2.5.11)$$

Here

$$\Omega^2 + \rho\Omega = \begin{bmatrix} -w_2^2 - w_3^2 & w_1w_2 - \rho w_3 & w_1w_3 + \rho w_2 \\ w_1w_2 + \rho w_3 & -w_1^2 - w_3^2 & w_2w_3 - \rho w_1 \\ w_1w_3 - \rho w_2 & w_2w_3 + \rho w_1 & -w_1^2 - w_2^2 \end{bmatrix}$$

and

$$\Omega\Omega^* + \Omega^*\Omega + \rho^*\Omega + \rho\Omega^* =$$

$$\begin{bmatrix} -2(w_2w_2^* + w_3w_3^*) & w_1^*w_2 + w_1w_2^* - \rho^*w_3 - \rho w_3^* & w_1^*w_3 + w_1w_3^* + \rho^*w_2 + \rho w_2^* \\ w_1^*w_2 + w_1w_2^* + \rho^*w_3 + \rho w_3^* & -2(w_1w_1^* + w_3w_3^*) & w_2^*w_3 + w_2w_3^* - \rho^*w_1 - \rho w_1^* \\ w_1^*w_3 + w_1w_3^* - \rho^*w_2 - \rho w_2^* & w_2^*w_3 + w_2w_3^* + \rho^*w_1 + \rho w_1^* & -2(w_1w_1^* + w_2w_2^*) \end{bmatrix}.$$

In addition, since $\hat{X}$ is a point on D.U.S., then the following equations are also satisfied:

$$X_1^2 + X_2^2 + X_3^2 = 1$$

and

$$X_1X_1^* + X_2X_2^* + X_3X_3^* = 0.$$

Hence the inflection points of the motion are the solution of following systems :

$$\begin{gathered} -(w_2^2 + w_3^2)X_1 + (w_1w_2 - \rho w_3)X_2 + (w_1w_3 + \rho w_2)X_3 = 0, \\ (w_1w_2 + \rho w_3)X_1 - (w_1^2 + w_3^2)X_2 + (w_2w_3 - \rho w_1)X_3 = 0, \\ (w_1w_3 - \rho w_2)X_1 + (w_2w_3 + \rho w_1)X_2 - (w_1^2 + w_2^2)X_3 = 0, \\ X_1^2 + X_2^2 + X_3^2 = 1 \end{gathered} \tag{2.5.12}$$

and

$$-2\left(w_2w_2^* + w_3w_3^*\right)X_1 + \left(w_1^*w_2 + w_1w_2^* - \rho^*w_3 - \rho w_3^*\right)X_2 +$$
$$\left(w_1^*w_3 + w_1w_3^* + \rho^*w_2 + \rho w_2^*\right)X_3 - \left(w_2^2 + w_3^2\right)X_1^* + \left(w_1w_2 - \rho w_3\right)X_2^* +$$
$$\left(w_1w_3 + \rho w_2\right)X_3^* = 0,$$

$$\left(w_1^*w_2 + w_1w_2^* + \rho^*w_3 + \rho w_3^*\right)X_1 + -2\left(w_1w_1^* + w_3w_3^*\right)X_2 +$$
$$\left(w_2^*w_3 + w_2w_3^* - \rho^*w_1 - \rho w_1^*\right)X_3 + \left(w_1w_2 + \rho w_3\right)X_1^* - \left(w_1^2 + w_3^2\right)X_2^* +$$
$$\left(w_2w_3 - \rho w_1\right)X_3^* = 0,$$

$$(2.5.13)$$

$$\left(w_1^*w_3 + w_1w_3^* - \rho^*w_2 - \rho w_2^*\right)X_1 + \left(w_2^*w_3 + w_2w_3^* + \rho^*w_1 + \rho w_1^*\right)X_2 -$$
$$2\left(w_1w_1^* + w_2w_2^*\right)X_3 + \left(w_1w_3 - \rho w_2\right)X_1^* + \left(w_2w_3 + \rho w_1\right)X_2^* - \left(w_1^2 + w_2^2\right)X_3^* = 0,$$

$$X_1X_1^* + X_2X_2^* + X_3X_3^* = 0.$$

In addition, we know that $\hat{\mu}$ is the function of t ($t \in G$), $\hat{\lambda}$ is a dual parameter and $\hat{\rho} = \hat{\mu} - \hat{\lambda}$. In the case of $\hat{\rho} = \hat{\mu}$, i.e. $\hat{\lambda} = 0$, the solution of the systems of equations (2.5.12) and (2.5.13) is an acceleration center of the motion.

We showed that the necessary and sufficient condition for the existence points of inflection of the motion is $\|\boldsymbol{w} \times \boldsymbol{w}_d\| = 0$ or $\boldsymbol{w} \times \boldsymbol{w}_d = \boldsymbol{0}$, for some value of t. Therefore, the following equations must be satisfied

$$\begin{aligned} \hat{w}_2\hat{w}_{d3} - \hat{w}_3\hat{w}_{d2} &= 0 \\ \hat{w}_3\hat{w}_{d1} - \hat{w}_1\hat{w}_{d3} &= 0 \\ \hat{w}_1\hat{w}_{d2} - \hat{w}_2\hat{w}_{d3} &= 0 \end{aligned} \qquad (2.5.14)$$

By using (2.2.14)-(2.2.19) and (2.3.9)-(2.3.14), we obtain the following equations:

$$\hat{w}_2\hat{w}_{d3} - \hat{w}_3\hat{w}_{d2} = -\frac{1}{2}\left(\frac{d\hat{\theta}_3}{dt}\right)^3 cos\,\hat{\theta}_1\,sin\,2\hat{\theta}_2 + \frac{d\hat{\theta}_1}{dt}\left(\frac{d\hat{\theta}_2}{dt}\right)^2\left(cos\,\hat{\theta}_1 - 2\right) +$$

$$\frac{d\hat{\theta}_1}{dt}\left(\frac{d\hat{\theta}_3}{dt}\right)^2\left(cos\,\hat{\theta}_1\,cos\,\hat{\theta}_2 - 2\,cos^2\,\hat{\theta}_2\right) + \frac{d\hat{\theta}_2}{dt}\left(\frac{d\hat{\theta}_3}{dt}\right)^2 sin\,\hat{\theta}_1\,sin\,\hat{\theta}_2\left(cos\,\hat{\theta}_2 + 1\right) -$$

$$\frac{d\hat{\theta}_3}{dt}\left(\frac{d\hat{\theta}_2}{dt}\right)^2 sin\,\hat{\theta}_1\left(1 + cos\,\hat{\theta}_2\right) + \frac{d\hat{\theta}_1}{dt}\frac{d\hat{\theta}_2}{dt}\frac{d\hat{\theta}_3}{dt} sin\,\hat{\theta}_1\left(sin\,\hat{\theta}_2\,cos\,\hat{\theta}_2 - 1\right) +$$

$$\left(\frac{d\hat{\theta}_2}{dt}\frac{d^2\hat{\theta}_3}{dt^2} - \frac{d^2\hat{\theta}_2}{dt^2}\frac{d\hat{\theta}_3}{dt}\right) cos\,\hat{\theta}_2 = 0, \qquad (2.5.15)$$

$$\hat{w}_3\hat{w}_{d1} - \hat{w}_1\hat{w}_{d3} = \frac{1}{2}\left(\frac{d\hat{\theta}_3}{dt}\right)^3 sin\,\hat{\theta}_1\,cos^2\,\hat{\theta}_2 - \left(\frac{d\hat{\theta}_2}{dt}\right)^3 sin^2\,\hat{\theta}_1 +$$

$$\frac{d\hat{\theta}_2}{dt}\left(\frac{d\hat{\theta}_3}{dt}\right)^2\left(cos^2\,\hat{\theta}_2\,sin^2\,\hat{\theta}_1 + cos\,2\hat{\theta}_1\,cos\,\hat{\theta}_2 - cos\,\hat{\theta}_1 - 1\right) +$$

$$\frac{d\hat{\theta}_3}{dt}\left(\frac{d\hat{\theta}_2}{dt}\right)^2\left[sin\,2\hat{\theta}_1\,cos\,\hat{\theta}_2 + sin\,\hat{\theta}_1\left(cos\,\hat{\theta}_2 - sin\,\hat{\theta}_2\right)\right] +$$

$$\frac{d\hat{\theta}_2}{dt}\left(\frac{d\hat{\theta}_1}{dt}\right)^2\left(2\,cos\,\hat{\theta}_1 - 1\right) - \frac{d\hat{\theta}_1}{dt}\left(\frac{d\hat{\theta}_2}{dt}\right)^2 sin\,\hat{\theta}_1\,sin\,2\hat{\theta}_2 +$$

$$2\frac{d\hat{\theta}_3}{dt}\left(\frac{d\hat{\theta}_1}{dt}\right)^2 sin\,\hat{\theta}_1\,cos\,\hat{\theta}_2 + \frac{d\hat{\theta}_1}{dt}\frac{d\hat{\theta}_2}{dt}\frac{d\hat{\theta}_3}{dt} sin\,\hat{\theta}_2\left(2 - cos\,\hat{\theta}_1\right) +$$

$$\left(\frac{d\hat{\theta}_1}{dt}\frac{d^2\hat{\theta}_2}{dt^2} - \frac{d^2\hat{\theta}_1}{dt^2}\frac{d\hat{\theta}_2}{dt}\right) sin\,\hat{\theta}_1 + \left(\frac{d^2\hat{\theta}_1}{dt^2}\frac{d\hat{\theta}_3}{dt} - \frac{d\hat{\theta}_1}{dt}\frac{d^2\hat{\theta}_3}{dt^2}\right) cos\,\hat{\theta}_1\,cos\,\hat{\theta}_2 +$$

$$\left(\frac{d\hat{\theta}_2}{dt}\frac{d^2\hat{\theta}_3}{dt^2} - \frac{d^2\hat{\theta}_2}{dt^2}\frac{d\hat{\theta}_3}{dt}\right) sin\,\hat{\theta}_1\,sin\,\hat{\theta}_2 = 0 \qquad (2.5.16)$$

and

$$\hat{w}_1\hat{w}_{d2} - \hat{w}_2\hat{w}_{d1} = -\frac{1}{2}\left(\frac{d\hat{\theta}_2}{dt}\right)^3 sin\,2\hat{\theta}_1 - \left(\frac{d\hat{\theta}_3}{dt}\right)^3\left(sin^2\,\hat{\theta}_1\,cos^2\,\hat{\theta}_2 - sin^2\,\hat{\theta}_2\right) +$$

$$\frac{d\hat{\theta}_1}{dt}\left(\frac{d\hat{\theta}_3}{dt}\right)^2\left(cos\,\hat{\theta}_1\,sin\,2\hat{\theta}_2 + 2\,sin\,\hat{\theta}_2\right) + \frac{d\hat{\theta}_3}{dt}\left(\frac{d\hat{\theta}_1}{dt}\right)^2\left(2\,cos\,\hat{\theta}_1\,cos\,\hat{\theta}_2 - 1\right) +$$

$$\frac{d\hat{\theta}_2}{dt}\left(\frac{d\hat{\theta}_3}{dt}\right)^2\left(\frac{1}{2}cos^2\hat{\theta}_2\, sin\, 2\hat{\theta}_1 + cos\,\hat{\theta}_2\, sin\, 2\hat{\theta}_1 + sin\,\hat{\theta}_1\right) +$$

$$\frac{d\hat{\theta}_3}{dt}\left(\frac{d\hat{\theta}_2}{dt}\right)^2\left(cos\, 2\hat{\theta}_1\, cos\,\hat{\theta}_2 - cos^2\hat{\theta}_1 + cos\,\hat{\theta}_1\, cos\,\hat{\theta}_2\right) - 2\frac{d\hat{\theta}_2}{dt}\left(\frac{d\hat{\theta}_1}{dt}\right)^2 sin\,\hat{\theta}_1 +$$

$$\left(\frac{d\hat{\theta}_1}{dt}\frac{d^2\hat{\theta}_2}{dt^2} - \frac{d^2\hat{\theta}_1}{dt^2}\frac{d\hat{\theta}_2}{dt}\right)cos\,\hat{\theta}_1 + \left(\frac{d^2\hat{\theta}_1}{dt^2}\frac{d\hat{\theta}_3}{dt} - \frac{d\hat{\theta}_1}{dt}\frac{d^2\hat{\theta}_3}{dt^2}\right)sin\,\hat{\theta}_1\, cos\,\hat{\theta}_2 +$$

$$\left(\frac{d\hat{\theta}_2}{dt}\frac{d^2\hat{\theta}_3}{dt^2} - \frac{d^2\hat{\theta}_2}{dt^2}\frac{d\hat{\theta}_3}{dt}\right)cos\,\hat{\theta}_1\, sin\,\hat{\theta}_2 + \frac{d\hat{\theta}_1}{dt}\frac{d\hat{\theta}_2}{dt}\frac{d\hat{\theta}_3}{dt}sin\,\hat{\theta}_1\, sin\,\hat{\theta}_2 = 0. \quad (2.5.17)$$

2.6 Bresse Complex

We now examine those points of the moving D.U.S. K that have zero tangential component of acceleration namely for which

$$\hat{a}\cdot\hat{v} = \frac{d^2\hat{X}}{dt^2}\cdot\frac{d\hat{X}}{dt} = 0. \quad (2.6.1)$$

With aid of the equations (2.1.10) and (2.3.15), the condition for zero tangential acceleration can be expressed as

$$\left[\hat{w}_d \times \hat{X} + \hat{w}\times\left(\hat{w}\times\hat{X}\right)\right]\cdot\left(\hat{w}\times\hat{X}\right) = 0 \quad (2.6.2)$$

or

$$\left(\hat{w}_d \times \hat{X}\right)\cdot\left(\hat{w}\times\hat{X}\right) = 0. \quad (2.6.3)$$

By using the property of mixed product, we obtain

$$\left(\hat{w}\cdot\hat{X}\right)\left(\hat{w}_d\cdot\hat{X}\right) = \hat{w}\cdot\hat{w}_d \quad (2.6.4)$$

or

$$\hat{w}_1\hat{w}_{d1}\hat{X}_1^2+\hat{w}_2\hat{w}_{d2}\hat{X}_2^2+\hat{w}_3\hat{w}_{d3}\hat{X}_3^2+\left(\hat{w}_1\hat{w}_{d2}+\hat{w}_{d1}\hat{w}_2\right)\hat{X}_1\hat{X}_2+\left(\hat{w}_1\hat{w}_{d3}+\hat{w}_{d1}\hat{w}_3\right)\hat{X}_1\hat{X}_3+\left(\hat{w}_2\hat{w}_{d3}+\hat{w}_{d2}\hat{w}_3\right)\hat{X}_2\hat{X}_3-\left(\hat{w}_1\hat{w}_{d1}+\hat{w}_2\hat{w}_{d2}+\hat{w}_3\hat{w}_{d3}\right)=0. \quad (2.6.5)$$

On the other hand, since $\hat{X}$ is a point on D.U.S. then its coordinates satisfy the following natural conditions

$$X_1^2+X_2^2+X_3^2=1 \quad (2.6.6)$$

$$X_1X_1^*+X_2X_2^*+X_3X_3^*=0. \quad (2.6.7)$$

By separating the equations (2.6.5) and (2.6.6) into real and dual parts and using the equations (2.6.6) and (2.6.7), we get the following system of equations

$$\begin{aligned}
&w_1w_{d1}X_1^2+w_2w_{d2}X_2^2+w_3w_{d3}X_3^2+\left(w_1w_{d2}+w_{d1}w_2\right)X_1X_2+\left(w_1w_{d3}+w_{d1}w_3\right)X_1X_3+\left(w_2w_{d3}+w_{d2}w_3\right)X_2X_3-\left(w_1w_{d1}+w_2w_{d2}+w_3w_{d3}\right)=0\\
&X_1^2+X_2^2+X_3^2=1\\
&X_1X_1^*+X_2X_2^*+X_3X_3^*=0.\\
&\left(w_1^*w_{d1}+w_1w_{d1}^*\right)X_1^2+\left(w_2^*w_{d2}+w_2w_{d2}^*\right)X_2^2+\left(w_3^*w_{d3}+w_3w_{d3}^*\right)X_3^2+\\
&\quad\left(w_1^*w_{d2}+w_1w_{d2}^*+w_2^*w_{d1}+w_2w_{d1}^*\right)X_1X_2+\\
&\quad\left(w_1^*w_{d3}+w_1w_{d3}^*+w_3^*w_{d1}+w_3w_{d1}^*\right)X_1X_3+\\
&\quad\left(w_2^*w_{d3}+w_2w_{d3}^*+w_3^*w_{d2}+w_3w_{d2}^*\right)X_2X_3+\\
&\quad\left[2w_1w_{d1}X_1+\left(w_1w_{d2}+w_2w_{d1}\right)X_2+\left(w_1w_{d3}+w_3w_{d1}\right)X_3\right]X_1^*+\\
&\quad\left[\left(w_1w_{d2}+w_2w_{d1}\right)X_1+2w_2w_{d2}X_2+\left(w_2w_{d3}+w_3w_{d2}\right)X_3\right]X_2^*+\\
&\quad\left[\left(w_1w_{d3}+w_3w_{d1}\right)X_1+\left(w_2w_{d3}+w_3w_{d2}\right)X_2+2w_3w_{d3}X_3\right]X_3^*-\\
&\quad\left(w_1^*w_{d1}+w_1w_{d1}^*+w_2^*w_{d2}+w_2w_{d2}^*+w_3^*w_{d3}+w_3w_{d3}^*\right)=0
\end{aligned} \quad (2.6.8)$$

In general, the system of equations (2.6.8) represents three parameter families of lines, linear complex, in fixed line space H. We call this linear complex Bresse complex. At an instant t, it is a two parameter family of lines, linear congruence, in line space H.

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