DOKUZ EYLÜL UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

SIMULATION OF ELECTRIC AND MAGNETIC FIELDS IN ANISOTROPIC MEDIA

by
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İZMİR

SIMULATION OF ELECTRIC AND MAGNETIC FIELDS IN ANISOTROPIC MEDIA

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M.Sc. THESIS EXAMINATION RESULT FORM

We have read the thesis entitled **"SIMULATION OF ELECTRIC AND MAGNETIC FIELDS IN ANISOTROPIC MEDIA"** completed by **ŞENGÜL KEÇELLİ** under supervision of **PROF. DR. VALERY YAKHNO** and we certify that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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SIMULATION OF ELECTRIC AND MAGNETIC FIELDS IN ANISOTROPIC MEDIA

ABSTRACT

In the thesis the time-dependent Maxwell's equations with piecewise constant coefficients are considered. These equations describe the electromagnetic waves in layered anisotropic media. The main problem of the thesis is initial value problems for two layered and three layered media. The main results are the following. The explicit formulae for the solutions of the considered problems are constructed. Finding the explicit formula the method of characteristics and matching conditions have been used. For simulation of electric and magnetic waves symbolic transformation in MATLAB is used.

Keywords: Time-dependent Maxwell's system; electromagnetic waves; Anisotropic layered media; Method of characteristics; Matching conditions; Simulation; MATLAB

ANİZOTROPİK ORTAMDA ELEKTRİK VE MANYETİK ALANLARIN SİMÜLASYONU

ÖZ

Tezde zamana bağlı ve parçalı sabit katsayılı Maxwell denklemleri ele alındı. Bu denklemler, katmanlı anizotropik ortamdaki elektrik ve manyetik dalgaları tanımlar. Tezin ana problemi iki ve üç katmanlı ortamlar için başlangıç değer problemleridir. Ana sonuçlar şu şekilde sıralanabilir. Ele alınan problemlerin çözümleri için kesin formüller oluşturuldu. Kesin formüllerin bulunabilmesi için karakteristikler metodu ve eşleme koşulları kullanıldı. Elektik ve manyetik dalgaların simülasyonları için MATLAB'ta sembolik dönüşümler uygulandı.

Anahtar Sözcükler: Zamana bağlı Maxwell denklemi; Elektromanyetik dalgalar; Katmanlı anizotropik ortam; Elektromanyetik dalga yayılımı; Karakteristikler metodu; Eşleme koşulları; Simülasyon; MATLAB

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CHAPTER ONE

INTRODUCTION

Equations of the time-dependent electric and magnetic fields in homogeneous anisotropic media are given by the following relations called the Maxwell's system (Eom (2004)), (Kong (1986)):

$$curl_x\vec{H} = \mathscr{E}\frac{\partial \vec{E}}{\partial t} + \vec{J}, \tag{1.0.1}$$

$$curl_x\vec{E} = -\mathscr{M}\frac{\partial \vec{H}}{\partial t}, \tag{1.0.2}$$

$$div_x(\mathscr{E}\vec{E}) = \rho, \tag{1.0.3}$$

$$div_x(\mathscr{M}\vec{H}) = 0, \tag{1.0.4}$$

where $x = (x_1, x_2, x_3)$ is a space variable from $\mathbb{R}^3$, t is a time variable from $\mathbb{R}$. $\vec{E}(x,t) = (E_1, E_2, E_3)$, $\vec{H}(x_3,t) = (H_1, H_2, H_3)$ are electric and magnetic fields, $E_k = E_k(x,t)$, $H_k = H_k(x,t)$; $\vec{J}(x,t) = (J_1, J_2, J_3)$ is the electric current density, $J_k = J_k(x,t)$, $k = 1,2,3$; $\mathscr{M}$ is the tensor of the magnetic permeability, $\mathscr{E}$ is the tensor of the dielectric permittivity; ρ is the density of electric charges.

In homogeneous non-dispersive electrically and magnetically anisotropic media the relation between the electric and magnetic fields $\vec{E}$ and $\vec{H}$ and the electric and magnetic flux densities **D** and **B** represented as

$$\mathbf{D} = \mathscr{E}\vec{E}, \quad \mathbf{B} = \mathscr{M}\vec{H},$$

where $\mathscr{E} = (\varepsilon_{ij}(x))_{3\times 3}$ dielectric permittivity and $\mathscr{M} = (\mu_{ij}(x))_{3\times 3}$ magnetic permeability are symmetric positive definite matrices. The matrices $\mathscr{E}$ and $\mathscr{M}$ characterize the electric and magnetic properties of the materials.

For an inhomogeneous isotropic medium $\mathscr{M}$ and $\mathscr{E}$ are positive scalar functions, if the medium is homogeneous isotropic then $\mathscr{M}$ and $\mathscr{E}$ are positive constants (that is, $\mathscr{E} = \varepsilon\mathscr{I}$ $\mathscr{M} = \mu\mathscr{I}$, where $\mathscr{I}$ identity matrix). If we take $\mathscr{E} = (\varepsilon_{ij})_{3\times 3}$ and $\mathscr{M} = (\mu_{ij})_{3\times 3}$ as arbitrary matrices (Eom (2004)) then we say our medium is electrically and magnetically anisotropic. If the dielectric permittivity $\mathscr{E} = (\varepsilon_{ij})_{3\times 3}$ is taken as arbitrary matrix and $\mathscr{M}$ as a constant (that is, $\mathscr{M} = \mu\mathscr{I}$) then the medium is electrically anisotropic. Another example of a medium is magnetically anisotropic where we consider the case $\mathscr{M} = (\mu_{ij})_{3\times 3}$ is an arbitrary matrix and $\mathscr{E}$ is a constant (that is, $\mathscr{E} = \varepsilon\mathscr{I}$) (Kong (1986)).

Let x be a space variable from $\mathbb{R}^3$ and t be a time variable from $\mathbb{R}$, then the Maxwell's is given by the relations (1.0.1)-(1.0.4), where $\mathscr{E} = (\varepsilon_{ij})_{3\times 3}$ and $\mathscr{M} = (\mu_{ij})_{3\times 3}$ are symmetric positive definite matrices. From relations (1.0.1)-(1.0.4) we can find the following

$$\frac{\partial \rho}{\partial t} + div_x J = 0,$$

and is called the conservation law of charges.

We suppose that

$$\vec{E} = 0, \ \ \vec{H} = 0, \ \ \rho = 0, \ \ for \ \ t \leq 0,$$

this means that there is no electric charge at the time $t \leq 0$.

This problem is called initial value problem (IVP) for time-dependent Maxwell's system with piecewise constant coefficients. This system describes the electric and magnetic wave propagation in layered anisotropic media.

To deal with electromagnetic wave propagation different problems and methods of their solving have been made (Kong (1986), Monk (2003), Yakhno et al. (2006)). For instance to solve the problem of electric field equation decomposition method has been suggested (Lindell (1990)). Analytic method of Green's functions constructions have been studied for isotropic and anisotropic materials in (Haba (2004), Ortner & Wagner (2004) Yakhno (2005)). Modeling lossy anisotropic dielectric wave-guides with the method of lines has been made for inhomogeneous biaxial anisotropic media.

Most of the electromagnetic wave problems have been solved by numerical methods, in particular finite element method (Monk (2003), Cohen (2002)), boundary elements method, finite difference method, nodal method (Zienkiewicz & Taylor (2000), Cohen et al. (2003)).

The main goal of the thesis is to find explicit formulae for solution of the stated problem and using these formulae to simulate electric and magnetic field.

This thesis is organized as follows. Firstly we solve the time-dependent Maxwell's system in free space and this is done in Chapter Two. In Section 2.1 we give equations of the electric and magnetic fields. Section 2.2 consists of assumptions and problem set-up for the Maxwell's equations. Using these assumptions and the equations from the Section 2.1 we construct our problem. In Section 2.3 we describe the procedure how to find the explicit formulas for the stated problem in Section 2.2. In the following section we reduce the original problem into first order partial differential equations and this is done in Section 2.3.1. Section 2.3.2 consists of the method to get explicit formulae for the reduced problem. Here we use the method of characteristics to get the formulae. Using the results the Section 2.3.2 and by back substitution of these formulae we get the explicit formula for solution of the IVP of the time-dependent Maxwell's system in Section 2.3.3.

In Chapter Three we solve the IVP related with the Maxwell's system for two layered anisotropic media. Section 3.1 consists of the basic information about the two layered media and we state the differences between the free space and two layered media. Equations of the electric and magnetic fields are also given in this section. In Section 3.2 we state the assumptions and set up the problem. For two layered media to solve the constructed problem matching conditions are needed. These matching conditions are given in this section. Section 3.3 describes the procedure to get the formulae for solution of the problem. In Section 3.3.1 the reduction of the IVP for Maxwell's system is given. Explicit formulae for the reduced IVP are obtained in Section 3.3.2 by using the method of characteristics. Firstly we divide each layer into subregions, this division process is based on the IVPs that are considered in each region. For instance for the first and second layer we have two subregions. One of the subregions of these layers only consists of IVP without matching conditions whereas the other one consists of an IVP and IVP with matching conditions. Solutions of the reduced IVPs for each subregion is computed in Sections 3.3.2.1, 3.3.2.2, 3.3.2.3, 3.3.2.4. But there some values in that solutions that are not defined. These values are the matching conditions and deriving process of these conditions is given in Section 3.3.3. Using the results of Section 3.3.2, 3.3.3 and by back substitution of the solution of the reduced IVP we get explicit formulae for the electric and magnetic fields in two layered anisotropic media, these formulae are stated in Section 3.3.4. In last section of Chapter Three applying symbolic transformation in MATLAB to explicit formulae simulation of the electric and magnetic waves is obtained. These images are presented in Section 3.4 and analysis of these images is given.

Solution of the electric and magnetic fields in three layered anisotropic media is considered in Chapter Four. In Section 4.1 we describe the three layered media and give equations of the electric and magnetic fields. Like as we did in Chapter One and Two we state our assumptions and construct the main problem for three layered media.Here like two layered media to solve the considered problem we also need matching conditions and they are given in Section 4.2. In Section 4.3 applying the same procedure as we used in Chapter Three we get explicit formulae of the problem. In Section 4.3.1 we make reduction of the IVP for Maxwell's system. Section 4.3.2 describes how to solve the reduced initial value problem. Firstly each layer of the media is separated into subregions by means of the main characteristics. After that by considering the IVPs related to each subregion we reorganize these regions and this organization constitutes flats. Then flat by flat we solve our problem. In Sections 4.3.2.1, 4.3.2.2, 4.3.2.3 the reduced IVPs of the main problem are solved for the flat one, flat two and flat three and using the method of characteristics explicit formulae are obtained. Undefined values in these formulas, matching conditions, are derived in Section 4.3.3. Since in the first flat we considered only IVPs without matching conditions then we need to derive matching conditions only for the second and third flat. In Section 4.3.2.1 and 4.3.2.2 these values are defined for the flats two and three. Section 4.3.4 is the last section of the Chapter Four, here by using the results of the last two sections we obtain explicit formulae of the electric and magnetic fields in three layered anisotropic media.

CHAPTER TWO

METHOD OF CHARACTERISTICS FOR FINDING ELECTRIC AND MAGNETIC FIELDS IN FREE SPACE

2.1 Equations of Electric and Magnetic Fields in Free Space

The propagation of electromagnetic waves in homogeneous, electrically and magnetically anisotropic materials is described by the time dependent Maxwell's system with matrices of dielectric permittivity and magnetic permeability.

In this chapter, we find explicit formulae for the solution of the Maxwell's system in free space. Let $x = (x_1, x_2, x_3)$ be a space variable from $\mathbb{R}^3$, t be a time variable from $\mathbb{R}$, then the Maxwell's system is given by the following relations:

$$curl_x \vec{H}(x,t) = \frac{\partial(\mathscr{E}\vec{E}(x,t))}{\partial t} + \vec{J}(x,t), \tag{2.1.1}$$

$$curl_x \vec{E}(x,t) = -\frac{\partial(\mathscr{M}\vec{H}(x,t))}{\partial t}, \tag{2.1.2}$$

$$div_x(\mathscr{E}\vec{E}(x,t)) = \rho(x,t), \tag{2.1.3}$$

$$div_x(\mathscr{M}\vec{H}(x,t)) = 0, \tag{2.1.4}$$

where $\mathscr{M} = \begin{pmatrix} \mu_{11} & \mu_{12} & \mu_{13} \\ \mu_{21} & \mu_{22} & \mu_{23} \\ \mu_{31} & \mu_{32} & \mu_{33} \end{pmatrix}$ and $\mathscr{E} = \begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{pmatrix}$ are symmetric positive definite matrices with constant elements.

And the conservation law of charges is given by the following relation:

$$\frac{\partial \rho(x_3,t)}{\partial t} + div_x \vec{J}(x_3,t) = 0. \tag{2.1.5}$$

Definition 2.1.1. Let $\vec{H}(x) = (H_1(x), H_2(x), H_3(x))$, $H_k(x)$ be a function of $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, $k = 1,2,3$ then divergence of $\vec{H}(x)$ is defined by:

$$div_x \vec{H}(x) = \frac{\partial H_1(x)}{\partial x_1} + \frac{\partial H_2(x)}{\partial x_2} + \frac{\partial H_3(x)}{\partial x_3}.$$

Definition 2.1.2. Let $\vec{H}(x) = (H_1(x), H_2(x), H_3(x))$, $H_k(x)$ be a function of $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, $k = 1,2,3$ then curl of $\vec{H}(x)$ is defined by:

$$curl_x \vec{H}(x) = (\frac{\partial H_3(x)}{\partial x_2} - \frac{\partial H_2(x)}{\partial x_3}, \frac{\partial H_1(x)}{\partial x_3} - \frac{\partial H_3(x)}{\partial x_1}, \frac{\partial H_2(x)}{\partial x_1} - \frac{\partial H_1(x)}{\partial x_2}).$$

2.2 Assumptions and Problem Set Up For Maxwell's System In Free Space

We assume that $\mathscr{E} = \begin{pmatrix} \varepsilon_{11} & 0 & 0 \\ 0 & \varepsilon_{22} & 0 \\ 0 & 0 & \varepsilon_{33} \end{pmatrix}, and \ \ \mathscr{M} = \begin{pmatrix} \mu_{11} & 0 & 0 \\ 0 & \mu_{22} & 0 \\ 0 & 0 & \mu_{33} \end{pmatrix}$ are symmetric positive definite matrices with constant elements.

Let the components of vectors $\vec{H}(x,t) = (H_1, H_2, H_3)$, $\vec{E}(x,t) = (E_1, E_2, E_3)$ depend on x_3 and t only, that is, $H_i = H_i(x_3, t)$, $E_i = E_i(x_3, t)$, $i = 1, 2, 3$; $\vec{J} = (J_1, J_2, J_3)$, where $J_i = J_i(x_3, t)$, i=1,2,3.

Moreover we suppose that:

$$\vec{E} = 0, \ \ \vec{H} = 0, \ \ \rho = 0, \ \ \vec{J} = 0 \ \ for \ \ t \leq 0, \tag{2.2.1}$$

this means that there is no electric charges and currents at time $t \leq 0$; electric and magnetic fields vanish for $t \leq 0$.

Let further $\mathscr{E}_{3\times3}$, $\mathscr{M}_{3\times3}$, $\vec{J}(x_3, t)$ be given.

The main problem is to find electric and magnetic fields, $\vec{E}(x_3, t)$, $\vec{H}(x_3, t)$ respectively, satisfying the IVP $(2.1.1) - (2.1.5)$ and $(2.2.1)$.

2.3 Finding Explicit Formula for Solution of the Problem

An explicit formula for solution of the Maxwell's system is obtained in this section. The method of deriving explicit formulae for the electric and magnetic fields consists of the following steps. On the first step we reduce the initial value problem for the Maxwell's system into another initial value problem; this reduced problem consists of first order linear partial differential equations with initial conditions. On the second step we use the method of characteristics to solve the reduced problem. As a result we get solution for it. On the last step using the formulae obtained on the second step, we get explicit formulae for the electric and magnetic fields; that is, the solution of the problem for Maxwell's system.

2.3.1 Reduction of the Problem for Maxwell's System

By considering the assumptions for the Maxwell's system and using the definitions of $curl_x$ and div_x we can rewrite $curl_x\vec{H}$ and $div_x\vec{H}$ in the following form:

$$\begin{aligned} div_x\vec{H}(x) &= \frac{\partial H_1(x)}{\partial x_1}+\frac{\partial H_2(x)}{\partial x_2}+\frac{\partial H_3(x)}{\partial x_3} \\ &= \frac{\partial H_3(x)}{\partial x_3}, \end{aligned}$$

$$\begin{aligned} curl_x\vec{H}(x) &= \left(\frac{\partial H_3(x)}{\partial x_2}-\frac{\partial H_2(x)}{\partial x_3},\frac{\partial H_1(x)}{\partial x_3}-\frac{\partial H_3(x)}{\partial x_1},\frac{\partial H_2(x)}{\partial x_1}-\frac{\partial H_1(x)}{\partial x_2}\right) \\ &= \left(-\frac{\partial H_2(x)}{\partial x_3},\frac{\partial H_1(x)}{\partial x_3},0\right). \end{aligned}$$

Under assumptions from each component of Maxwell's system (2.1.1)-(2.1.4) we get new sub equations:

$$curl_x\vec{H}=\frac{\partial(\mathscr{E}\vec{E})}{\partial t}+\vec{J},\ \ (2.1.1)\left.\right\}\begin{cases} -\frac{\partial H_2}{\partial x_3}=\frac{\partial(\varepsilon_{11}E_1)}{\partial t}+j_1,\ \ (2.1.1.a) \\ \frac{\partial H_1}{\partial x_3}=\frac{\partial(\varepsilon_{22}E_2)}{\partial t}+j_2,\ \ (2.1.1.b) \\ 0=\frac{\partial(\varepsilon_{33}E_3)}{\partial t}+j_3,\ \ (2.1.1.c) \end{cases}$$

$$curl_x\vec{E}=-\frac{\partial(\mathscr{M}\vec{H})}{\partial t},\ \ (2.1.2)\left.\right\}\begin{cases} -\frac{\partial E_2}{\partial x_3}=-\frac{\partial(\mu_{11}H_1)}{\partial t},\ \ (2.1.2.a) \\ \frac{\partial E_1}{\partial x_3}=-\frac{\partial(\mu_{22}H_2)}{\partial t},\ \ (2.1.2.b) \\ 0=-\frac{\partial(\mu_{33}H_3)}{\partial t},\ \ (2.1.2.c) \end{cases}$$

$$div_x(\mathscr{E}\vec{E})=\rho,\ \ (2.1.3)\ \Big\}\ \frac{\partial(\varepsilon_{33}E_3)}{\partial x_3}=\rho,\ \ (2.1.3)$$

$$div_x(\mathscr{M}\vec{H})=0,\ \ (2.1.4)\ \Big\}\ \frac{\partial(\mu_{33}H_3)}{\partial x_3}=0.\ \ (2.1.4)$$

To find E_1 and H_2 we will consider equations $(2.1.1.a)$ and $(2.1.2.b)$. These equations may be written in the form:

$$-\frac{\partial H_2}{\partial x_3}=\frac{\partial(\varepsilon_{11}E_1)}{\partial t}+j_1,\ \ (2.1.1.a)\ \Big\}\ \frac{\partial(\sqrt{\varepsilon_{11}}E_1)}{\partial t}=-\frac{1}{\sqrt{\varepsilon_{11}\mu_{22}}}\frac{\partial(\sqrt{\mu_{22}}H_2)}{\partial x_3}-\frac{J_1}{\sqrt{\varepsilon_{11}}}, \tag{2.3.1}$$

$$\frac{\partial E_1}{\partial x_3}=-\frac{\partial(\mu_{22}H_2)}{\partial t},\ \ (2.1.2.b)\ \Big\}\ \frac{\partial(\sqrt{\mu_{22}}H_2)}{\partial t}=-\frac{1}{\sqrt{\varepsilon_{11}\mu_{22}}}\frac{\partial(\sqrt{\varepsilon_{11}}E_1)}{\partial x_3}. \tag{2.3.2}$$

Summing (2.3.1) and (2.3.2) and subtracting (2.3.1) from (2.3.2) we find:

$$\frac{\partial(\sqrt{\mu_{22}}H_2+\sqrt{\varepsilon_{11}}E_1)}{\partial t}+\frac{1}{\sqrt{\varepsilon_{11}\mu_{22}}}\frac{\partial(\sqrt{\mu_{22}}H_2+\sqrt{\varepsilon_{11}}E_1)}{\partial x_3}=-\frac{J_1}{\sqrt{\varepsilon_{11}}}, \tag{2.3.3}$$

$$\frac{\partial(\sqrt{\mu_{22}}H_2-\sqrt{\varepsilon_{11}}E_1)}{\partial t}-\frac{1}{\sqrt{\varepsilon_{11}\mu_{22}}}\frac{\partial(\sqrt{\mu_{22}}H_2-\sqrt{\varepsilon_{11}}E_1)}{\partial x_3}=\frac{J_1}{\sqrt{\varepsilon_{11}}}. \tag{2.3.4}$$

To find E_2 and H_1 we will consider equations $(2.1.1.b)$ and $(2.1.2.a)$. These equations may be written in the form:

$$\left.\frac{\partial H_1}{\partial x_3}=\frac{\partial(\varepsilon_{22}E_2)}{\partial t}+j_2,\ \ (2.1.1.b)\ \right\}\ \frac{\partial(\sqrt{\varepsilon_{22}}E_2)}{\partial t}=\frac{1}{\sqrt{\varepsilon_{22}\mu_{11}}}\frac{\partial(\sqrt{\mu_{11}}H_1)}{\partial x_3}-\frac{J_2}{\sqrt{\varepsilon_{22}}}, \tag{2.3.5}$$

$$\left.-\frac{\partial E_2}{\partial x_3}=-\frac{\partial(\mu_{11}H_1)}{\partial t},\ \ (2.1.2.a)\ \right\}\ \frac{\partial(\sqrt{\mu_{11}}H_1)}{\partial t}=\frac{1}{\sqrt{\varepsilon_{22}\mu_{11}}}\frac{\partial(\sqrt{\varepsilon_{22}}E_2)}{\partial x_3}. \tag{2.3.6}$$

Subtracting (2.3.5) from (2.3.6)and summing (2.3.5) and (2.3.6) we find

$$\frac{\partial(\sqrt{\mu_{11}}H_1-\sqrt{\varepsilon_{22}}E_2)}{\partial t}+\frac{1}{\sqrt{\varepsilon_{22}\mu_{11}}}\frac{\partial(\sqrt{\mu_{11}}H_1-\sqrt{\varepsilon_{22}}E_2)}{\partial x_3}=\frac{J_2}{\sqrt{\varepsilon_{22}}}, \tag{2.3.7}$$

$$\frac{\partial(\sqrt{\mu_{11}}H_1+\sqrt{\varepsilon_{22}}E_2)}{\partial t}-\frac{1}{\sqrt{\varepsilon_{22}\mu_{11}}}\frac{\partial(\sqrt{\mu_{11}}H_1+\sqrt{\varepsilon_{22}}E_2)}{\partial x_3}=-\frac{J_2}{\sqrt{\varepsilon_{22}}}. \tag{2.3.8}$$

To solve equations (2.3.3), (2.3.4), (2.3.7), (2.3.8) we will denote,

$$\begin{aligned}\sqrt{\mu_{22}}H_2+\sqrt{\varepsilon_{11}}E_1&=u_1,\\ \sqrt{\mu_{22}}H_2-\sqrt{\varepsilon_{11}}E_1&=u_2,\\ \sqrt{\mu_{11}}H_1-\sqrt{\varepsilon_{22}}E_2&=u_3,\\ \sqrt{\mu_{11}}H_1+\sqrt{\varepsilon_{22}}E_2&=u_4,\end{aligned} \tag{2.3.9}$$

$$v_1=v_2=\frac{1}{\sqrt{\varepsilon_{11}\mu_{22}}};\ \ v_3=v_4=\frac{1}{\sqrt{\varepsilon_{22}\mu_{11}}};$$

$$f_1=-\frac{J_1}{\sqrt{\varepsilon_{11}}};\ \ f_2=\frac{J_1}{\sqrt{\varepsilon_{11}}};\ \ f_3=\frac{J_2}{\sqrt{\varepsilon_{22}}};\ \ f_4=-\frac{J_2}{\sqrt{\varepsilon_{22}}},$$

where $u_i=u_i(x_3,t),\ f_i=f_i(x_3,t),\ \ i=1,2,3,4.$

Then equations (2.3.3), (2.3.4), (2.3.7), (2.3.8) may be written as:

$$\frac{\partial u_i(x_3,t)}{\partial t} + (-1)^{i+1} v_i \frac{\partial u_i(x_3,t)}{\partial x_3} = f_i(x_3,t), \quad i = 1,2,3,4. \tag{2.3.10}$$

And, using (2.2.1) we get initial conditions for (2.3.10) as:

$$u_i(x_3,0) = 0, \quad i = 1,2,3,4. \tag{2.3.11}$$

As a result, we reduced the initial value problem (IVP) for Maxwell's system to another initial value problem (IVP). This reduced problem consists of the equations (2.3.10) and (2.3.11).

2.3.2 Solving Initial Value Problem

Let us consider the equation (2.3.10). This equation is a first order linear partial differential equation (PDE) with the independent variables x_3 and t.
In this equation v_i, $i = 1,2,3,4$, are given constants (coefficients of the PDE (2.3.10)); $f_i(x_3,t)$, $i = 1,2,3,4$, are given functions (inhomogeneous term of PDE (2.3.10)); $u_i(x_3,t)$, $i = 1,2,3,4$, is the unknown function.

To find solution of the initial value problem (IVP)(2.3.10)-(2.3.11) we use the method of characteristics.

Firstly, let us write the equation (2.3.10) in terms of ξ and τ:

$$\frac{\partial u_i}{\partial \tau} + (-1)^{i+1} v_i \frac{\partial u_i}{\partial \xi} = f_i, \quad -\infty < \xi < \infty, \quad \tau > 0, \tag{2.3.12}$$

where $u_i = u_i(\xi,\tau)$, $\quad f_i = f_i(\xi,\tau)$, $i = 1,2,3,4$.

Then equations for characteristics are can be found as:

$$\tfrac{d\xi}{ds} = (-1)^{i+1} v_i, \quad i = 1,2,3,4,$$
$$\tfrac{d\tau}{ds} = 1.$$

Then the characteristic, that is, passing through the point (x_3,t) can be found as:

$$\xi = (-1)^{i+1} v_i(\tau - t) + x_3, \quad i = 1,2,3,4.$$

Now, we can write the equation (2.3.12) along these characteristics in the following form:

$$\frac{du_i((-1)^{i+1} v_i(\tau - t) + x_3, \tau)}{d\tau} = f_i((-1)^{i+1} v_i(\tau - t) + x_3, \tau). \tag{2.3.13}$$

Integrating relation (2.3.13) from $\tau = 0$ to $\tau = t$ we have:

$$\int_0^t \frac{d[u_i(v_i(\tau - t) + x_3, \tau)]}{d\tau} d\tau = \int_0^t f_i((-1)^{i+1} v_i(\tau - t) + x_3, \tau) d\tau, \tag{2.3.14}$$

where $i = 1, 2, 3, 4$.

That is,

$$u_i(x_3, t) = u_i(x_3, 0) + \int_0^t f_i((-1)^{i+1} v_i(\tau - t) + x_3, \tau) d\tau,$$

where $i = 1, 2, 3, 4$.

Finally, using the initial condition (2.3.11) we find the solution of the initial value problem (IVP) (2.3.10)-(2.3.11) in the following form:

$$u_i(x_3, t) = \int_0^t f_i((-1)^{i+1} v_i(\tau - t) + x_3, \tau) d\tau, \tag{2.3.15}$$

where $i = 1, 2, 3, 4$.

2.3.3 Finding Explicit Formulae For The Electric And Magnetic Fields

Now we obtain explicit formula for the solution of the original problem (2.1.1)-(2.1.5), (2.2.1). In the subsection 2.3.1 the relations that depend on $H_k(x_3, t)$ and $E_k(x_3, t)$, $k = 1, 2$, denoted by $u_i(x_3, t)$'s $i = 1, 2, 3, 4$. This was given in (2.3.9). Here by making back substitution of $u_i(x_3, t)$'s $i = 1, 2, 3, 4$ into (2.3.9) we get explicit formulae for the electric and magnetic fields.

Then the formulas for $H_k(x_3, t)$ and $E_k(x_3, t)$, $k = 1, 2$ can be found by means of $u_i(x_3, t)$'s $i = 1, 2, 3, 4$ as:

$$\begin{aligned}
H_1(x_3, t) &= \tfrac{1}{2\sqrt{\mu_{11}}}(u_3 + u_4), \\
H_2(x_3, t) &= \tfrac{1}{2\sqrt{\mu_{22}}}(u_1 + u_2), \\
E_1(x_3, t) &= \tfrac{1}{2\sqrt{\varepsilon_{11}}}(u_1 - u_2), \\
E_2(x_3, t) &= \tfrac{1}{2\sqrt{\varepsilon_{22}}}(u_4 - u_3).
\end{aligned}$$

As a result we get the formulas for $H_k(x_3, t)$ and $E_k(x_3, t)$, $k = 1, 2$, but still remain some unknown functions. We have not considered the unknowns $H_3(x_3, t)$ and $E_3(x_3, t)$. Finding solution of these functions is easier than the other ones; since we consider first order ordinary differential equations (ODEs) with initial conditions.

The solution of $H_3(x_3, t)$ follows from the equations (2.1.2.c) and (2.1.4).

$$\frac{\partial(\mu_{33}H_3)}{\partial t} = 0,$$
$$\frac{\partial(\mu_{33}H_3)}{\partial x_3} = 0,$$

We see that $H_3(x_3,t)$ is independent of x_3 in the first relation and independent of t in the second relation. This situation is valid if and only if $H_3(x_3,t)$ is a constant. Hence,

$$H_3(x_3,t) = h_3, \qquad h_3 : arbitrary\ \ constant.$$

Using the initial condition (2.2.1) we find the solution of $H_3(x_3,t)$ as:

$$H_3(x_3,t) = 0.$$

To find solution of $E_3(x_3,t)$ we consider a similar procedure. The equations that are related with $E_3(x_3,t)$ are (2.1.1.c) and (2.1.3).

$$\frac{\partial(\varepsilon_{33}E_3)}{\partial t} = -j_3,$$
$$\frac{\partial(\varepsilon_{33}E_3)}{\partial x_3} = \rho,$$

Using the conservation law of charges we see that these two relations are equivalent to each other. Hence, let us only consider the relation (2.1.1.c). Then by taking integral with respect to τ from 0 to t and using the initial condition for $E_3(x_3,t)$, we get the solution of $E_3(x_3,t)$ as,

$$E_3(x_3,t) = -\frac{1}{\varepsilon_{33}}\int_0^t J_3(x_3,\tau)d\tau.$$

Then the explicit formulae for the electric and magnetic fields can be stated as:

$$\begin{aligned}
H_1(x_3,t) &= \tfrac{1}{2\sqrt{\mu_{11}\varepsilon_{22}}}\int_0^t \big[J_2(\tfrac{1}{\sqrt{\mu_{11}\varepsilon_{22}}}(\tau-t)+x_3,\tau) - J_2(-\tfrac{1}{\sqrt{\mu_{11}\varepsilon_{22}}}(\tau-t)+x_3,\tau)\big]d\tau,\\
H_2(x_3,t) &= \tfrac{1}{2\sqrt{\mu_{22}\varepsilon_{11}}}\int_0^t \big[-J_1(\tfrac{1}{\sqrt{\mu_{22}\varepsilon_{11}}}(\tau-t)+x_3,\tau) + J_1(-\tfrac{1}{\sqrt{\mu_{22}\varepsilon_{11}}}(\tau-t)+x_3,\tau)\big]d\tau,\\
H_3(x_3,t) &= 0,\\
E_1(x_3,t) &= \tfrac{1}{2\varepsilon_{11}}\int_0^t \big[-J_1(\tfrac{1}{\sqrt{\mu_{22}\varepsilon_{11}}}(\tau-t)+x_3,\tau) - J_1(-\tfrac{1}{\sqrt{\mu_{22}\varepsilon_{11}}}(\tau-t)+x_3,\tau)\big]d\tau,\\
E_2(x_3,t) &= \tfrac{1}{2\varepsilon_{22}}\int_0^t \big[J_2(\tfrac{1}{\sqrt{\mu_{11}\varepsilon_{22}}}(\tau-t)+x_3,\tau) + J_2(-\tfrac{1}{\sqrt{\mu_{11}\varepsilon_{22}}}(\tau-t)+x_3,\tau)\big]d\tau,\\
E_3(x_3,t) &= -\tfrac{1}{\varepsilon_{33}}\int_0^t J_3(x_3,\tau)d\tau.
\end{aligned}$$

As a result we find the explicit formulae for the electric and magnetic fields that is the explicit solution of the Maxwell's system in free space.

CHAPTER THREE

METHOD OF CHARACTERISTICS FOR FINDING ELECTRIC AND MAGNETIC FIELDS IN TWO LAYERED MEDIA

3.1 Equations of Electric and Magnetic Fields in Two Layered Media

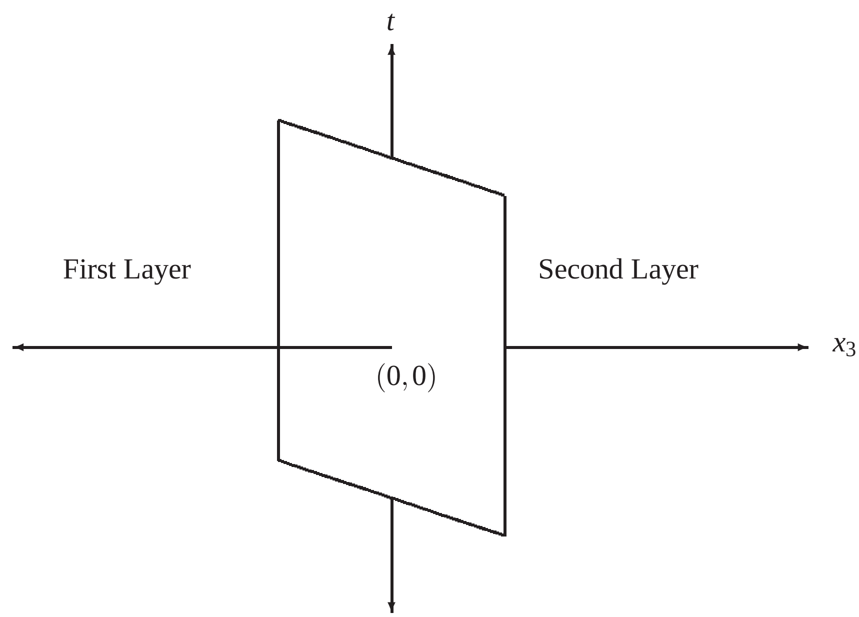


Figure 3.1 Two layered media.

In this chapter, we consider the initial value problem for the time-dependent Maxwell's system in homogeneous, anisotropic materials in two layered media. To solve this problem we follow a similar procedure as we applied in the last chapter for free space, but the main problem, which should be solved, has some differences with the problem for free space.

The first difference is domain, on which we study. In free space we consider the whole space, that is, $-\infty < x_3 < \infty$, $t > 0$; but here we separate the whole space in two layers. Each layer consists of a half space. Then we define the first layer as $-\infty < x_3 < 0$, $t > 0$; and the second one as: $0 < x_3 < \infty$, $t > 0$. We denote each layer by a notation writing the number of the layer, on which we study, in parentheses. This notation is shown like a power, that is, $\square^{(k)}$, where $k = 1, 2$.

The second difference is the conditions, which are used. In free space we consider only initial conditions, but here we also need another ones which are called matching conditions. To find explicit formula for solution of the original problem, they should be derived.

The last difference is the initial value problem, which is reduced from the original problem for Maxwell's system. In free space there was just one initial value problem to be considered, but here we also divide each layer into subregions and in each subregion we consider different initial value problems. In one of the subregion we consider an initial value problem as we did before, but in the other one we solve an initial value problem with matching conditions.

As a result except these differences, we solve the main problem by a similar process as we did before.

Now, let $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ be a space variable and $t \in \mathbb{R}$ be the time variable. Then the Maxwell's system for two the layered media can be written as:

$$curl_x \vec{H}^{(k)} = \mathscr{E}^{(k)} \frac{\partial \vec{E}^{(k)}}{\partial t} + \vec{J}^{(k)}, \tag{3.1.1}$$

$$curl_x \vec{E}^{(k)} = -\mathscr{M}^{(k)} \frac{\partial \vec{H}^{(k)}}{\partial t}, \tag{3.1.2}$$

$$div_x(\mathscr{E}^{(k)} \vec{E}^{(k)}) = \rho^{(k)}, \tag{3.1.3}$$

$$div_x(\mathscr{M}^{(k)} \vec{H}^{(k)}) = 0, \tag{3.1.4}$$

where $k = 1, 2$ and denotes the media.

And the conservation law of charges is given by:

$$\frac{\partial \rho^{(k)}}{\partial t} + div_x \vec{J}^{(k)} = 0, \tag{3.1.5}$$

where $k = 1, 2$.

3.2 Assumptions and Problem Set Up For Maxwell's System

We assume that the electric permittivity matrix $\mathscr{E}^{(k)} = (\varepsilon_{ij})^{(k)}_{3\times 3}$ and the magnetic permeability matrix $\mathscr{M}^{(k)} = (\mu_{ij})^{(k)}_{3\times 3}$, $k = 1, 2$, are symmetric positive definite matrices with constant elements, and they are in the form of:

$$\mathscr{E}^{(k)} = \begin{pmatrix} \varepsilon^{(k)}_{11} & 0 & 0 \\ 0 & \varepsilon^{(k)}_{22} & 0 \\ 0 & 0 & \varepsilon^{(k)}_{33} \end{pmatrix}, and \ \ \mathscr{M} = \begin{pmatrix} \mu^{(k)}_{11} & 0 & 0 \\ 0 & \mu^{(k)}_{22} & 0 \\ 0 & 0 & \mu^{(k)}_{33} \end{pmatrix}.$$

Let the components of vector functions $\vec{H}^{(k)}(x) = (H^{(k)}_1, H^{(k)}_2, H^{(k)}_3)$, $\vec{E}^{(k)}(x) = (E^{(k)}_1, E^{(k)}_2, E^{(k)}_3)$, $k = 1, 2$, depend on x_3 and t only, that is, $H^{(k)}_i = H^{(k)}_i(x_3, t)$, $E^{(k)}_i = E^{(k)}_i(x_3, t)$, $i = 1, 2, 3$; $\vec{J}^{(k)} = (J^{(k)}_1, J^{(k)}_2, J^{(k)}_3)$, where $J^{(k)}_i = J^{(k)}_i(x_3, t)$, $i = 1, 2, 3$; $k = 1, 2$.

Moreover, we suppose that:

$$\vec{E}^{(k)} = 0,\ \ \vec{H}^{(k)} = 0,\ \ \rho^{(k)} = 0,\ \ \vec{J}^{(k)} = 0\ \ for\ \ t \le 0, \tag{3.2.1}$$

this means that there is no electric charges and currents at the time $t \le 0$; electric and magnetic fields vanish for $t \le 0$.

The matching conditions are in the following form:

$$\left.\begin{array}{c} (\vec{E}^{(2)} - \vec{E}^{(1)})|_{x_3=0} \times \vec{n} = 0, \\ (\vec{D}^{(2)} - \vec{D}^{(1)})|_{x_3=0} \cdot \vec{n} = 0, \\ (\vec{H}^{(2)} - \vec{H}^{(1)})|_{x_3=0} \times \vec{n} = 0, \\ (\vec{B}^{(2)} - \vec{B}^{(1)})|_{x_3=0} \cdot \vec{n} = 0, \end{array}\right\} \tag{3.2.2}$$

where $\vec{n} = (0,0,1)$.

Let further that the matrices $\mathscr{E}^{(k)}$ and $\mathscr{M}^{(k)}$ and the current electric density $\vec{J}^{(k)}$ be given, $k = 1,2$.

The main problem is to find $\vec{E}^{(k)}$, $\vec{H}^{(k)}$, $k = 1,2$ satisfying $(3.1.1) - (3.1.4)$ and $(3.2.1)$, $(3.2.2)$.

3.3 Finding Explicit Formula for Solution of the Problem

In this section we find explicit formula for the solution of the initial value problem for Maxwell's system. The procedure of finding solution of the problem consists of the following steps. Firstly we reduce the original problem to another initial value problem. On the second step we divide each layer into subregions and solve the reduced initial value problem related with each subregion. On the third step we derive matching conditions. At the last step using the results of the first, second and third step we find explicit formulae for the electric and magnetic fields; that is, the solution of the Maxwell's system.

3.3.1 Reduction Of The Problem For Maxwell's System

Here under assumptions applying the same procedure, as we used in free space to reduce the Maxwell's system into the first order partial differential equations, we get the reduced problem for the original one. And using the initial conditions $(3.2.1)$ and the matching conditions $(3.2.2)$ we get an initial value problem related with each layer.

After repeating the procedure mentioned above; the first order partial differential equations reduced from Maxwell's system can be found in the following form:

$$\frac{\partial u_i^{(k)}(x_3,t)}{\partial t} + (-1)^{i+1} v_i^{(k)} \frac{\partial u_i^{(k)}(x_3,t)}{\partial x_3} = f_i^{(k)}(x_3,t),\ \ i = 1,2,3,4, \tag{3.3.1}$$

where

$$u_1^{(k)} = \sqrt{\mu_{22}^{(k)}} H_2^{(k)} + \sqrt{\varepsilon_{11}^{(k)}} E_1^{(k)},$$
$$u_2^{(k)} = \sqrt{\mu_{22}^{(k)}} H_2^{(k)} - \sqrt{\varepsilon_{11}^{(k)}} E_1^{(k)},$$
$$u_3^{(k)} = \sqrt{\mu_{11}^{(k)}} H_1^{(k)} - \sqrt{\varepsilon_{22}^{(k)}} E_2^{(k)},$$
$$u_4^{(k)} = \sqrt{\mu_{11}^{(k)}} H_1^{(k)} + \sqrt{\varepsilon_{22}^{(k)}} E_2^{(k)},$$
$$v_1^{(k)} = v_2^{(k)} = \frac{1}{\sqrt{\varepsilon_{11}^{(k)} \mu_{22}^{(k)}}}; \;\; v_3^{(k)} = v_4^{(k)} = \frac{1}{\sqrt{\varepsilon_{22}^{(k)} \mu_{11}^{(k)}}};$$
$$f_1^{(k)} = -\frac{J_1^{(k)}}{\sqrt{\varepsilon_{11}^{(k)}}}; \;\; f_2^{(k)} = \frac{J_1^{(k)}}{\sqrt{\varepsilon_{11}^{(k)}}}; \;\; f_3^{(k)} = \frac{J_2^{(k)}}{\sqrt{\varepsilon_{22}^{(k)}}}; \;\; f_4^{(k)} = -\frac{J_2^{(k)}}{\sqrt{\varepsilon_{22}^{(k)}}},$$

$u_i^{(k)} = u_i^{(k)}(x_3,t)$, $f_i^{(k)} = f_i^{(k)}(x_3,t)$, $i = 1,2,3,4$; and $k = 1,2$ denotes the media.

Initial conditions can be found as:

$$u_i^{(k)}(x_3,0) = 0, \;\; i = 1,2,3,4; \;\; k = 1,2. \tag{3.3.2}$$

Matching conditions are in the form:

$$\left.\begin{aligned} u_i^{(1)}(-0,t) &= \sqrt{\mu_{22}^{(1)}} H_2^{(1)}(-0,t) + (-1)^{(i+1)} \sqrt{\varepsilon_{11}^{(1)}} E_1^{(1)}(-0,t) \quad t > 0, \;\; i = 1,2 \\ u_i^{(1)}(-0,t) &= \sqrt{\mu_{11}^{(1)}} H_1^{(1)}(-0,t) + (-1)^{(i)} \sqrt{\varepsilon_{22}^{(1)}} E_2^{(1)}(-0,t), \quad t > 0, \;\; i = 3,4. \end{aligned}\right\} \tag{3.3.3}$$

$$\left.\begin{aligned} u_i^{(2)}(+0,t) &= \sqrt{\mu_{22}^{(2)}} H_2^{(2)}(+0,t) + (-1)^{(i+1)} \sqrt{\varepsilon_{11}^{(2)}} E_1^{(2)}(+0,t) \quad t > 0, \;\; i = 1,2 \\ u_i^{(2)}(+0,t) &= \sqrt{\mu_{11}^{(2)}} H_1^{(2)}(+0,t) + (-1)^{(i)} \sqrt{\varepsilon_{22}^{(2)}} E_2^{(2)}(+0,t), \quad t > 0, \;\; i = 3,4. \end{aligned}\right\} \tag{3.3.4}$$

The reduced initial value problem related to the layer one consists of (3.3.1), (3.3.2), (3.3.3) for $k = 1$; and for the second layer we consider (3.3.1), (3.3.2), (3.3.4) for $k = 2$.

3.3.2 Solving Reduced Initial Value Problem For Maxwell's System

Now, we solve the reduced initial value problem related with each layer. We divide each layer into subregions, since the problem in each subregion that may differ from the one related with other subregion. This division depends on the characteristics, related with each layer. On the solution steps, we describe this in details. These subregions, without mentioning the characteristics, are shown in the Figure 3.2.

In the Figure 3.2, the subregions related with the first layer is $R1$ and $R2$; $R3$ and $R4$ are related with the second layer.

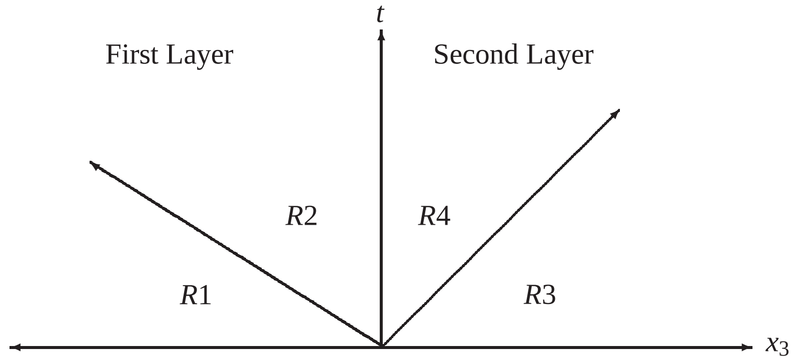


Figure 3.2 Subregions in two layered media.

The layer one, the half space, was defined as; $-\infty < x_3 < 0$, $t > 0$. The reduced initial value problem related with this layer consists of (3.3.1), (3.3.2), (3.3.3) for $k = 1$, that is:

$$\frac{\partial u_i^{(1)}(x_3,t)}{\partial t} + (-1)^{i+1} v_i^{(1)} \frac{\partial u_i^{(1)}(x_3,t)}{\partial x_3} = f_i^{(1)}(x_3,t), \quad i = 1,2,3,4, \tag{3.3.5}$$

$$u_i^{(1)}(x_3,0) = 0, \quad x_3 < 0, \quad i = 1,2,3,4, \tag{3.3.6}$$

$$\left.\begin{aligned} u_i^{(1)}(-0,t) &= \sqrt{\mu_{22}^{(1)}} H_2^{(1)}(-0,t) + (-1)^{(i+1)} \sqrt{\varepsilon_{11}^{(1)}} E_1^{(1)}(-0,t) \quad t > 0, \; i = 1,2 \\ u_i^{(1)}(-0,t) &= \sqrt{\mu_{11}^{(1)}} H_1^{(1)}(-0,t) + (-1)^{(i)} \sqrt{\varepsilon_{22}^{(1)}} E_2^{(1)}(-0,t), \quad t > 0, \; i = 3,4. \end{aligned}\right\} \tag{3.3.7}$$

We use the method of characteristics to find solution of the IVP (3.3.5), (3.3.6), (3.3.7). Equation (3.3.5) can be written in terms of ξ and τ as:

$$\frac{\partial u_i^{(1)}(\xi,\tau)}{\partial \tau} + (-1)^{i+1} v_i^{(1)} \frac{\partial u_i^{(1)}(\xi,\tau)}{\partial \xi} = f_i^{(1)}(\xi,\tau), \quad i = 1,2,3,4. \tag{3.3.8}$$

Equations for characteristics are:

$$\begin{aligned} \tfrac{d\xi}{ds} &= (-1)^{i+1} v_i^{(1)}, \quad i = 1,2,3,4, \\ \tfrac{d\tau}{ds} &= 1. \end{aligned}$$

Then we have:

$$\xi = (-1)^{i+1} v_i^{(1)} + c, \quad i = 1,2,3,4,$$

where c is an arbitrary constant.

Hence, the characteristic, that is, passing through the point (x_3,t) can be found as:

$$\xi = (-1)^{i+1} v_i^{(1)} (\tau - t) + x_3, \quad i = 1,2,3,4. \tag{3.3.9}$$

Equation $(3.3.8)$ along this characteristic may be written in the following form:

$$\frac{du_i^{(1)}((-1)^{i+1}v_i^{(1)}(\tau-t)+x_3,\tau)}{d\tau}=f_i^{(1)}((-1)^{i+1}v_i^{(1)}(\tau-t)+x_3,\tau). \tag{3.3.10}$$

The characteristics for $i=1,2,3,4$ are drawn in Figure 3.3 ($v_1^{(1)}=v_2^{(1)}$ and $v_3^{(1)}=v_4^{(1)}$) .

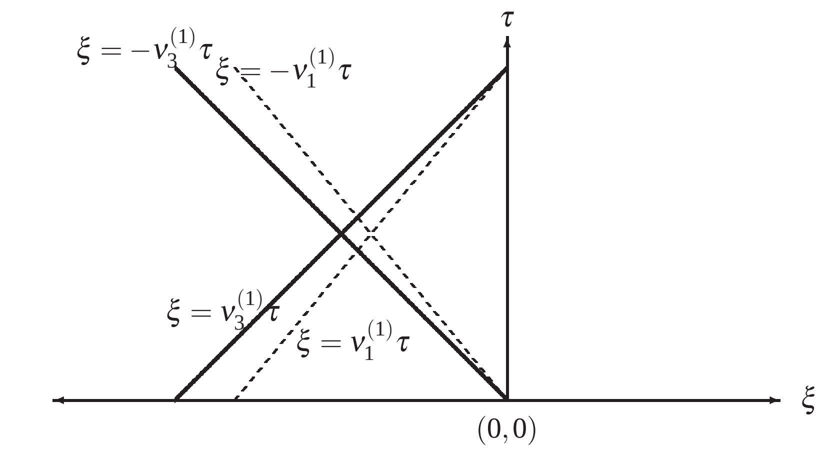


Figure 3.3 Characteristic lines in layer one.

We see from the Figure 3.3 that the characteristics for $i=1$ and $i=3$ are similar with different slopes, and also for $i=2$ and $i=4$ we have similar results. When we divide the layer one into subregions, the characteristics related with $i=1$ and $i=3$ have an significant role.

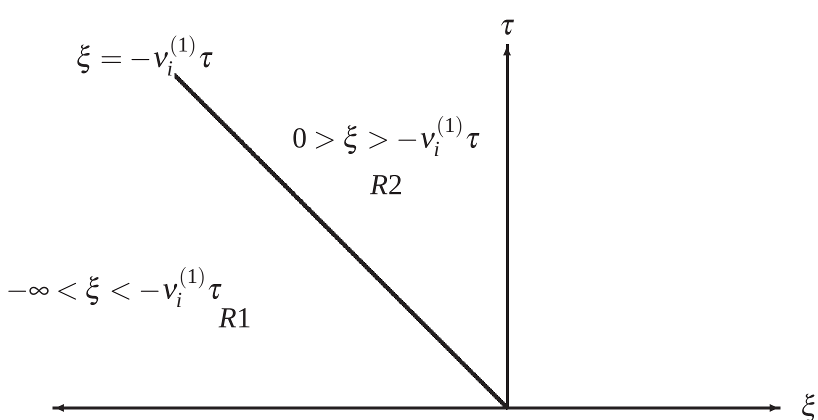


Figure 3.4 Subregions of the first layer.

$R1$ and $R2$ are the subregions of the layer one. We define the region $R1$ and $R2$ as:

$$R1 = \{(x_3,t): \quad -\infty < x_3 < -v_i^{(1)}t, \;\; t > 0, \;\; i = 1,2,3,4\};$$

$$R2 = \{(x_3,t): \quad -v_i^{(1)}t < x_3 < 0, \;\; t > 0, \;\; i = 1,2,3,4\}.$$

In the region $R1$ we consider an initial value problem without any matching conditions and in the region $R2$ we have an initial value problem with matching conditions.

3.3.2.1 *Solving IVP in the Region $R1$*

We have defined the region $R1$ in the following form:

$$R1 = \{-\infty < x_3 < -v_i^{(1)}t, \;\; t > 0 \;\; , i = 1,2,3,4\}.$$

When we take the point (x_3,t) in the region $R1$, we get an IVP. And solution of this problem can be found using the following steps.

Consider the equation $(3.3.10)$. Integrating this equation with respect to τ from $\tau = 0$ to $\tau = t$ we find:

$$u_i^{(1)}(x_3,t) = u_i(x_3 - (-1)^{i+1} \cdot v_i^{(1)}t, 0) + \int_0^t f_i^{(1)}((-1)^{i+1}v_i^{(1)}(\tau - t) + x_3, \tau)d\tau,$$

where $-\infty < x_3 < -v_i^{(1)}t$, $i = 1,2,3,4$; $t > 0$.

Using the initial condition $(3.3.6)$, solution of the IVP can be found as:

$$u_i^{(1)}(x_3,t) = \int_0^t f_i^{(1)}((-1)^{i+1}v_i^{(1)}(\tau - t) + x_3, \tau)d\tau,$$

where $-\infty < x_3 < -v_i^{(1)}t$, $i = 1,2,3,4$; $t > 0$.

3.3.2.2 *Solving IVP in the Region $R2$*

The region $R2$ has been defined as:

$$R2 = \{(x_3,t): \;\; 0 > x_3 > -v_i^{(1)}t, \;\; t > 0, \;\; i = 1,2,3,4\}.$$

If we take the point (x_3,t) in the region $R2$, then there exists two cases to be considered for $(3.3.10)$. First case is that for $i = 1$ and $i = 3$ we have IVP; and for the second one we should consider an IVP with matching conditions for $i = 2$ and $i = 4$.

Case1:

In this case we solve the same IVP for $i = 1$ and $i = 3$ as we have solved in region $R1$. The solution of this IVP has been found as:

$$u_i^{(1)}(x_3,t) = \int_0^t f_i^{(1)}(v_i^{(1)}(\tau - t) + x_3, \tau)d\tau,$$

where $i = 1$ and $i = 3$; $t > 0$.

Case2:

For $i = 2$ and $i = 4$ we are solving an IVP with matching conditions. Integrating the equation (3.3.10) with respect to τ from $t + \frac{x_3}{v_i^{(1)}}$ to t we find:

$$u_i^{(1)}(x_3,t) = u_i^{(1)}(-0, t + \frac{x_3}{v_i^{(1)}}) + \int_{t+\frac{x_3}{v_i^{(1)}}}^t f_i^{(1)}(v_i^{(1)}(t - \tau) + x_3, \tau)d\tau,$$

where $i = 2$ and $i = 4$; $t > 0$.

And, using the matching condition (3.3.7) this solution can be written as:

$$\begin{aligned} u_2^{(1)}(x_3,t) &= \sqrt{\mu_{22}^{(1)}} H_2^{(1)}(-0, t + \frac{x_3}{v_2^{(1)}}) - \sqrt{\varepsilon_{11}^{(1)}} E_1^{(1)}(-0, t + \frac{x_3}{v_2^{(1)}}) \\ &+ \int_{t+\frac{x_3}{v_2^{(1)}}}^t f_2^{(1)}(v_2^{(1)}(t - \tau) + x_3, \tau)d\tau, \end{aligned}$$

$$\begin{aligned} u_4^{(1)}(x_3,t) &= \sqrt{\mu_{11}^{(1)}} H_1^{(1)}(-0, t + \frac{x_3}{v_4^{(1)}}) + \sqrt{\varepsilon_{22}^{(1)}} E_2^{(1)}(-0, t + \frac{x_3}{v_4^{(1)}}) \\ &+ \int_{t+\frac{x_3}{v_4^{(1)}}}^t f_4^{(1)}(v_4^{(1)}(t - \tau) + x_3, \tau)d\tau. \end{aligned}$$

Now, we solve the reduced IVP in the second layer. The second layer is defined as; $\infty > x_3 > 0$, $t > 0$. The reduced initial value problem related with this layer consists of (3.3.1), (3.3.2), (3.3.3) for $k = 2$, that is:

$$\frac{\partial u_i^{(2)}(x_3,t)}{\partial t} + (-1)^{i+1} v_i^{(2)} \frac{\partial u_i^{(2)}(x_3,t)}{\partial x_3} = f_i^{(2)}(x_3,t), \quad i = 1,2,3,4, \tag{3.3.11}$$

$$u_i^{(2)}(x_3, 0) = 0, \quad x_3 > 0, \quad i = 1,2,3,4, \tag{3.3.12}$$

$$\left.\begin{aligned} u_i^{(2)}(+0,t) &= \sqrt{\mu_{22}^{(2)}} H_2^{(2)}(+0,t) + (-1)^{(i+1)} \sqrt{\varepsilon_{11}^{(2)}} E_1^{(2)}(+0,t) \quad t > 0, \ i = 1,2 \\ u_i^{(2)}(+0,t) &= \sqrt{\mu_{11}^{(2)}} H_1^{(2)}(+0,t) + (-1)^{(i)} \sqrt{\varepsilon_{22}^{(2)}} E_2^{(2)}(+0,t), \quad t > 0, \ i = 3,4. \end{aligned}\right\} \tag{3.3.13}$$

Then by following a similar procedure as we used in layer one, the solution of the IVP (3.3.11), (3.3.12), (3.3.13) can be found. After rewriting (3.3.11) in terms of ξ and τ we find the following equation:

$$\frac{du_i^{(2)}((-1)^{i+1}v_i^{(2)}(\tau-t)+x_3,\tau)}{d\tau} = f_i^{(2)}((-1)^{i+1}v_i^{(2)}(\tau-t)+x_3,\tau). \tag{3.3.14}$$

The characteristic lines of this layer are drawn in Figure 3.5. On the figure i used for the values $i=1$ and $i=3$.

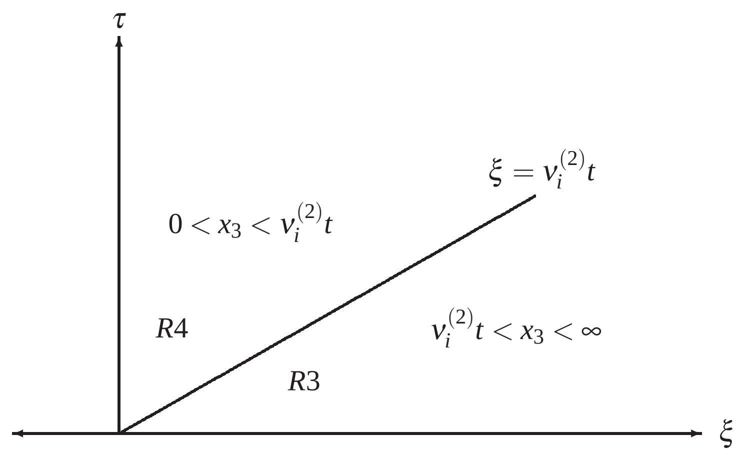


Figure 3.5 Subregions of the second layer.

3.3.2.3 *Solving IVP in the Region $R3$*

The region $R3$ is defined in the following form:

$$R3 = \{(x_3,t): \ \infty > x_3 > v_i^{(2)}t, \ t>0, \ i=1,2,3,4\}.$$

The solution of the IVP in that region is similar with the region $R1$. Hence the solution of the IVP in region $R3$ as can be found as:

$$u_i^{(2)}(x_3,t) = \int_0^t f_i^{(2)}((-1)^{i+1}v_i^{(2)}(\tau-t)+x_3,\tau)d\tau,$$

where $i=1,2,3,4$; $\infty > x_3 > v_i^{(2)}t$, $t>0$.

3.3.2.4 Solving IVP in the Region R4

Firstly let us define the region $R4$.

$$R4 = \{(x_3,t): \ \ 0 < x_3 < v_i^{(2)} t, \ \ t > 0, \ \ i = 1,2,3,4\}.$$

If we take the point (x_3,t) in the region $R4$, then there exists two cases to be considered for $(3.3.14)$. First case is that for $i = 2$ and $i = 4$ we have IVP; and for the second one we should consider a IVP with matching conditions for $i = 1$ and $i = 3$.

Case1:

In this case we solve an IVP for $i = 2$ and $i = 4$. And we find the same result as we have found for the IVP in $R3$ for $i = 2$ and $i = 4$. Then solutions are:

$$u_i^{(2)}(x_3,t) = \int_0^t f_i^{(2)}((-1)^{i+1} v_i^{(2)}(\tau - t) + x_3, \tau) d\tau,$$

where $i = 2$ and $i = 4$; $t > 0$.

Case2:

Now, we consider an IVP with matching conditions for $i = 1$ and $i = 3$. Integrating the equation $(3.3.14)$ with respect to τ from $(t - \frac{x_3}{v_i^{(2)}})$ to t we find:

$$u_i^{(2)}(x_3,t) = u_i^{(2)}(+0, t - \frac{x_3}{v_i^{(2)}}) + \int_{t - \frac{x_3}{v_i^{(2)}}}^{t} f_i^{(2)}((-1)^{i+1} v_i^{(2)}(\tau - t) + x_3, \tau) d\tau$$

where $i = 1$ and $i = 3$; $t > 0$.

And, using the matching condition $(3.3.13)$ this solution can be written as:

$$\begin{aligned} u_1^{(2)}(x_3,t) &= \sqrt{\mu_{22}^{(2)}} H_2^{(2)}(+0, t - \frac{x_3}{v_1^{(2)}}) + \sqrt{\varepsilon_{11}^{(2)}} E_1^{(2)}(+0, t - \frac{x_3}{v_1^{(2)}}) \\ &+ \int_{t - \frac{x_3}{v_1^{(2)}}}^{t} f_1^{(2)}(v_1^{(2)}(\tau - t) + x_3, \tau) d\tau, \end{aligned}$$

$$\begin{aligned} u_3^{(2)}(x_3,t) &= \sqrt{\mu_{11}^{(2)}} H_1^{(2)}(+0, t - \frac{x_3}{v_3^{(2)}}) - \sqrt{\varepsilon_{22}^{(2)}} E_2^{(2)}(+0, t - \frac{x_3}{v_3^{(2)}}) \\ &+ \int_{t - \frac{x_3}{v_3^{(2)}}}^{t} f_3^{(2)}(v_3^{(2)}(\tau - t) + x_3, \tau) d\tau. \end{aligned}$$

As a result solution of the reduced initial value problem for two layered media can be written as:

In $R1$:

$$u_i^{(1)}(x_3,t) = \int_0^t f_i^{(1)}((-1)^{(i+1)}v_i^{(1)}(\tau - t) + x_3, \tau)d\tau, \;\; i = 1,2,3,4,$$

where $-\infty < x_3 < -v_i^{(1)}t; \;\; t > 0, \;\; i = 1,2,3,4.$

In $R2$:

$$\begin{aligned} u_i^{(1)}(x_3,t) &= \int_0^t f_i^{(1)}(v_i^{(1)}(\tau - t) + x_3, \tau)d\tau, \;\; i = 1 \;\; and \;\; i = 3, \\ u_i^{(1)}(x_3,t) &= u_i^{(1)}(-0, t + \frac{x_3}{v_i^{(1)}}) + \int_{t+\frac{x_3}{v_i^{(1)}}}^t f_i^{(1)}(-v_i^{(1)}(t - \tau) + x_3, \tau)d\tau, \;\; i = 2 \;\; and \;\; i = 4, \end{aligned}$$

where $-v_i^{(1)}t < x_3 < 0; t > 0, \;\; i = 1,2,3,4.$

In $R3$:

$$u_i^{(2)}(x_3,t) = \int_0^t f_i^{(2)}((-1)^{(i+1)}v_i^{(2)}(\tau - t) + x_3, \tau)d\tau, \;\; i = 1,2,3,4,$$

where $v_i^{(2)}t < x_3 < \infty; \;\; t > 0, \;\; i = 1,2,3,4.$

In $R4$:

$$\begin{aligned} u_i^{(2)}(x_3,t) &= u_i^{(2)}(+0, t - \frac{x_3}{v_i^{(2)}}) + \int_{t-\frac{x_3}{v_i^{(2)}}}^t f_i^{(2)}(v_i^{(2)}(\tau - t) + x_3, \tau)d\tau, \;\; i = 1 \;\; and \;\; i = 3, \\ u_i^{(2)}(x_3,t) &= \int_0^t f_i^{(2)}(-v_i^{(2)}(\tau - t) + x_3, \tau)d\tau, \;\; i = 2 \;\; and \;\; i = 4, \end{aligned}$$

where $0 < x_3 < v_i^{(2)}; t > 0, \;\; i = 1,2,3,4.$

In the following subsection we derive the matching conditions $u_i^{(1)}(-0, t + \frac{x_3}{v_i^{(1)}})$ for $i = 2$ and $i = 4$; and $u_i^{(2)}(+0, t - \frac{x_3}{v_i^{(2)}})$ for $i = 1$ and $i = 3$.

3.3.3 Deriving Matching Conditions

The main goal of this subsection is to derive the matching conditions.

In last sections we have found the following relations:

$$\begin{pmatrix} u_1^{(1)}(-0,t) \\ u_2^{(1)}(-0,t) \end{pmatrix} = \begin{pmatrix} \sqrt{\mu_{22}^{(1)}} & \sqrt{\varepsilon_{11}^{(1)}} \\ \sqrt{\mu_{22}^{(1)}} & -\sqrt{\varepsilon_{11}^{(1)}} \end{pmatrix} \begin{pmatrix} H_2^{(1)}(-0,t) \\ E_1^{(1)}(-0,t) \end{pmatrix}, \qquad (3.3.15)$$

$$\begin{pmatrix} u_3^{(1)}(-0,t) \\ u_4^{(1)}(-0,t) \end{pmatrix} = \begin{pmatrix} \sqrt{\mu_{11}^{(1)}} & -\sqrt{\varepsilon_{22}^{(1)}} \\ \sqrt{\mu_{11}^{(1)}} & \sqrt{\varepsilon_{22}^{(1)}} \end{pmatrix} \begin{pmatrix} H_1^{(1)}(-0,t) \\ E_2^{(1)}(-0,t) \end{pmatrix}, \qquad (3.3.16)$$

$$\begin{pmatrix} u_1^{(2)}(+0,t) \\ u_2^{(2)}(+0,t) \end{pmatrix} = \begin{pmatrix} \sqrt{\mu_{22}^{(2)}} & \sqrt{\varepsilon_{11}^{(2)}} \\ \sqrt{\mu_{22}^{(2)}} & -\sqrt{\varepsilon_{11}^{(2)}} \end{pmatrix} \begin{pmatrix} H_2^{(2)}(+0,t) \\ E_1^{(2)}(+0,t) \end{pmatrix}, \qquad (3.3.17)$$

$$\begin{pmatrix} u_3^{(2)}(+0,t) \\ u_4^{(2)}(+0,t) \end{pmatrix} = \begin{pmatrix} \sqrt{\mu_{11}^{(2)}} & -\sqrt{\varepsilon_{22}^{(2)}} \\ \sqrt{\mu_{11}^{(2)}} & \sqrt{\varepsilon_{22}^{(2)}} \end{pmatrix} \begin{pmatrix} H_1^{(2)}(+0,t) \\ E_2^{(2)}(+0,t) \end{pmatrix}. \qquad (3.3.18)$$

And using the condition (3.2.2) we get the following relations:

$$\begin{aligned} H_1^{(1)}(-0,t) &= H_1^{(2)}(+0,t), \\ H_2^{(1)}(-0,t) &= H_2^{(2)}(+0,t), \\ E_1^{(1)}(-0,t) &= E_1^{(2)}(+0,t), \\ H_2^{(1)}(-0,t) &= H_2^{(2)}(+0,t). \end{aligned}$$

Firstly let us derive the matching condition $u_2^{(1)}(-0,t)$. To find this value we consider the relations (3.3.15) and (3.3.17) above. Then we have:

$$\begin{aligned} \begin{pmatrix} u_1^{(1)}(-0,t) \\ u_2^{(1)}(-0,t) \end{pmatrix} &= \begin{pmatrix} \sqrt{\mu_{22}^{(1)}} & \sqrt{\varepsilon_{11}^{(1)}} \\ \sqrt{\mu_{22}^{(1)}} & -\sqrt{\varepsilon_{11}^{(1)}} \end{pmatrix} \begin{pmatrix} H_2^{(1)}(-0,t) \\ E_1^{(1)}(-0,t) \end{pmatrix} \\ &= \begin{pmatrix} \sqrt{\mu_{22}^{(1)}} & \sqrt{\varepsilon_{11}^{(1)}} \\ \sqrt{\mu_{22}^{(1)}} & -\sqrt{\varepsilon_{11}^{(1)}} \end{pmatrix} \begin{pmatrix} \sqrt{\mu_{11}^{(2)}} & -\sqrt{\varepsilon_{22}^{(2)}} \\ \sqrt{\mu_{11}^{(2)}} & \sqrt{\varepsilon_{22}^{(2)}} \end{pmatrix}^{-1} \begin{pmatrix} u_1^{(2)}(+0,t) \\ u_2^{(2)}(+0,t) \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2}\left(\frac{\sqrt{\mu_{22}^{(1)}}}{\sqrt{\mu_{22}^{(2)}}} + \frac{\sqrt{\varepsilon_{11}^{(1)}}}{\sqrt{\varepsilon_{11}^{(2)}}}\right) & \frac{1}{2}\left(\frac{\sqrt{\mu_{22}^{(1)}}}{\sqrt{\mu_{22}^{(2)}}} - \frac{\sqrt{\varepsilon_{11}^{(1)}}}{\sqrt{\varepsilon_{11}^{(2)}}}\right) \\ \frac{1}{2}\left(\frac{\sqrt{\mu_{22}^{(1)}}}{\sqrt{\mu_{22}^{(2)}}} - \frac{\sqrt{\varepsilon_{11}^{(1)}}}{\sqrt{\varepsilon_{11}^{(2)}}}\right) & \frac{1}{2}\left(\frac{\sqrt{\mu_{22}^{(1)}}}{\sqrt{\mu_{22}^{(2)}}} + \frac{\sqrt{\varepsilon_{11}^{(1)}}}{\sqrt{\varepsilon_{11}^{(2)}}}\right) \end{pmatrix} \begin{pmatrix} u_1^{(2)}(+0,t) \\ u_2^{(2)}(+0,t) \end{pmatrix}. \end{aligned}$$

Now using the above relation we get the following equalities:

$$\begin{aligned} u_1^{(1)}(-0,t) &= \frac{\sqrt{\mu_{22}^{(1)}} \cdot \sqrt{\varepsilon_{11}^{(2)}} + \sqrt{\mu_{22}^{(2)}} \cdot \sqrt{\varepsilon_{11}^{(1)}}}{2\sqrt{\mu_{22}^{(2)}} \cdot \sqrt{\varepsilon_{11}^{(2)}}} u_1^{(2)}(+0,t) \\ &+ \frac{\sqrt{\mu_{22}^{(1)}} \cdot \sqrt{\varepsilon_{11}^{(2)}} - \sqrt{\mu_{22}^{(2)}} \cdot \sqrt{\varepsilon_{11}^{(1)}}}{2\sqrt{\mu_{22}^{(2)}} \cdot \sqrt{\varepsilon_{11}^{(2)}}} u_2^{(2)}(+0,t), \end{aligned}$$

$$\begin{aligned} u_2^{(1)}(-0,t) &= \frac{\sqrt{\mu_{22}^{(1)}} \cdot \sqrt{\varepsilon_{11}^{(2)}} - \sqrt{\mu_{22}^{(2)}} \cdot \sqrt{\varepsilon_{11}^{(1)}}}{2\sqrt{\mu_{22}^{(2)}} \cdot \sqrt{\varepsilon_{11}^{(2)}}} u_1^{(2)}(+0,t) \\ &+ \frac{\sqrt{\mu_{22}^{(1)}} \cdot \sqrt{\varepsilon_{11}^{(2)}} + \sqrt{\mu_{22}^{(2)}} \cdot \sqrt{\varepsilon_{11}^{(1)}}}{2\sqrt{\mu_{22}^{(2)}} \cdot \sqrt{\varepsilon_{11}^{(2)}}} u_2^{(2)}(+0,t). \end{aligned}$$

To derive $u_2^{(1)}(-0,t)$, firstly we define $u_1^{(2)}(+0,t)$ by means of the first equality above. Then it takes the following form:

$$\begin{aligned} u_1^{(2)}(+0,t) &= \frac{2\sqrt{\mu_{22}^{(2)}}\cdot\sqrt{\varepsilon_{11}^{(2)}}}{\sqrt{\mu_{22}^{(1)}}\cdot\sqrt{\varepsilon_{11}^{(2)}}+\sqrt{\mu_{22}^{(2)}}\cdot\sqrt{\varepsilon_{11}^{(1)}}}u_1^{(1)}(-0,t) \\ &- \frac{\sqrt{\mu_{22}^{(1)}}\cdot\sqrt{\varepsilon_{11}^{(2)}}-\sqrt{\mu_{22}^{(2)}}\cdot\sqrt{\varepsilon_{11}^{(1)}}}{\sqrt{\mu_{22}^{(1)}}\cdot\sqrt{\varepsilon_{11}^{(2)}}+\sqrt{\mu_{22}^{(2)}}\cdot\sqrt{\varepsilon_{11}^{(1)}}}u_2^{(2)}(+0,t). \end{aligned}$$

At this step we substitute $u_1^{(2)}(+0,t)$, $u_2^{(2)}(+0,t)$ and $u_1^{(1)}(-0,t)$ into the equality related with $u_2^{(1)}(-0,t)$. Here $u_1^{(1)}(-0,t)$ is the solution of the initial value problem in $R1$ for $x_3=0$ and $u_2^{(2)}(+0,t)$ is the solution of the initial value problem in $R3$ for $x_3=0$. Then we find $u_2^{(1)}(-0,t)$ in the following form:

$$\begin{aligned} u_2^{(1)}(-0,t) &= \frac{\sqrt{\mu_{22}^{(1)}}\cdot\sqrt{\varepsilon_{11}^{(2)}}-\sqrt{\mu_{22}^{(2)}}\cdot\sqrt{\varepsilon_{11}^{(1)}}}{\sqrt{\mu_{22}^{(1)}}\cdot\sqrt{\varepsilon_{11}^{(2)}}+\sqrt{\mu_{22}^{(2)}}\cdot\sqrt{\varepsilon_{11}^{(1)}}}\cdot\frac{(-1)}{\sqrt{\varepsilon_{11}^{(1)}}}\int_0^t J_1^{(1)}(\nu_1^{(1)}(\tau-t),\tau)d\tau \\ &+ \frac{2\sqrt{\mu_{22}^{(1)}}\cdot\sqrt{\varepsilon_{11}^{(2)}}}{\sqrt{\mu_{22}^{(1)}}\cdot\sqrt{\varepsilon_{11}^{(1)}}+\sqrt{\mu_{22}^{(2)}}\cdot\sqrt{\varepsilon_{11}^{(1)}}}\cdot\frac{1}{\sqrt{\varepsilon_{11}^{(2)}}}\int_0^t J_1^{(2)}(-\nu_2^{(2)}(\tau-t),\tau)d\tau. \end{aligned}$$

Now, we derive the matching condition $u_4^{(1)}(-0,t)$, which is related to the layer one and necessary to solve initial value problem in $R2$.

By a similar procedure as we did before, firstly we write the following equality using (3.3.16) and (3.3.18). Then we get:

$$\begin{pmatrix} u_3^{(1)}(-0,t) \\ u_4^{(1)}(-0,t) \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\left(\frac{\sqrt{\mu_{11}^{(1)}}}{\sqrt{\mu_{11}^{(2)}}}+\frac{\sqrt{\varepsilon_{22}^{(1)}}}{\sqrt{\varepsilon_{22}^{(2)}}}\right) & \frac{1}{2}\left(\frac{\sqrt{\mu_{11}^{(1)}}}{\sqrt{\mu_{11}^{(2)}}}-\frac{\sqrt{\varepsilon_{22}^{(1)}}}{\sqrt{\varepsilon_{22}^{(2)}}}\right) \\ \frac{1}{2}\left(\frac{\sqrt{\mu_{11}^{(1)}}}{\sqrt{\mu_{11}^{(2)}}}-\frac{\sqrt{\varepsilon_{22}^{(1)}}}{\sqrt{\varepsilon_{22}^{(2)}}}\right) & \frac{1}{2}\left(\frac{\sqrt{\mu_{11}^{(1)}}}{\sqrt{\mu_{11}^{(2)}}}+\frac{\sqrt{\varepsilon_{22}^{(1)}}}{\sqrt{\varepsilon_{22}^{(2)}}}\right) \end{pmatrix} \begin{pmatrix} u_3^{(2)}(+0,t) \\ u_4^{(2)}(+0,t) \end{pmatrix}.$$

From the equality above we get following relations:

$$\begin{aligned} u_3^{(1)}(-0,t) &= \frac{\sqrt{\mu_{11}^{(1)}}\cdot\sqrt{\varepsilon_{22}^{(2)}}+\sqrt{\mu_{11}^{(2)}}\cdot\sqrt{\varepsilon_{22}^{(1)}}}{2\sqrt{\mu_{11}^{(2)}}\cdot\sqrt{\varepsilon_{22}^{(2)}}}u_3^{(2)}(+0,t) \\ &+ \frac{\sqrt{\mu_{11}^{(1)}}\cdot\sqrt{\varepsilon_{22}^{(2)}}-\sqrt{\mu_{11}^{(2)}}\cdot\sqrt{\varepsilon_{22}^{(1)}}}{2\sqrt{\mu_{11}^{(2)}}\cdot\sqrt{\varepsilon_{22}^{(2)}}}u_4^{(2)}(+0,t), \end{aligned}$$

$$\begin{aligned} u_4^{(1)}(-0,t) &= \frac{\sqrt{\mu_{11}^{(1)}}\cdot\sqrt{\varepsilon_{22}^{(2)}}-\sqrt{\mu_{11}^{(2)}}\cdot\sqrt{\varepsilon_{22}^{(1)}}}{2\sqrt{\mu_{11}^{(2)}}\cdot\sqrt{\varepsilon_{22}^{(2)}}}u_3^{(2)}(+0,t) \\ &+ \frac{\sqrt{\mu_{11}^{(1)}}\cdot\sqrt{\varepsilon_{22}^{(2)}}+\sqrt{\mu_{11}^{(2)}}\cdot\sqrt{\varepsilon_{22}^{(1)}}}{2\sqrt{\mu_{11}^{(2)}}\cdot\sqrt{\varepsilon_{22}^{(2)}}}u_4^{(2)}(+0,t). \end{aligned}$$

To derive $u_4^{(1)}(-0,t)$, firstly we define $u_3^{(2)}(+0,t)$ by means of the first equality above. Then it takes the following form:

$$u_3^{(2)}(+0,t) = \frac{2\sqrt{\mu_{11}^{(2)}}\cdot\sqrt{\varepsilon_{22}^{(2)}}}{\sqrt{\mu_{11}^{(1)}}\cdot\sqrt{\varepsilon_{22}^{(2)}}+\sqrt{\mu_{11}^{(2)}}\cdot\sqrt{\varepsilon_{22}^{(1)}}}u_3^{(1)}(-0,t) - \frac{\sqrt{\mu_{11}^{(1)}}\cdot\sqrt{\varepsilon_{22}^{(2)}}-\sqrt{\mu_{11}^{(2)}}\cdot\sqrt{\varepsilon_{22}^{(1)}}}{\sqrt{\mu_{11}^{(1)}}\cdot\sqrt{\varepsilon_{22}^{(2)}}+\sqrt{\mu_{11}^{(2)}}\cdot\sqrt{\varepsilon_{22}^{(1)}}}u_4^{(2)}(+0,t).$$

Here $u_3^{(1)}(-0,t)$ is the solution of the initial value problem in $R1$ for $x_3 = 0$ and $u_4^{(2)}(+0,t)$ is the solution of the initial value problem in $R3$ for $x_3 = 0$. Then substituting these values and $u_3^{(2)}(+0,t)$ into the relation related to $u_4^{(1)}(-0,t)$ we find it in the following form:

$$u_4^{(1)}(-0,t) = \frac{\sqrt{\mu_{11}^{(1)}}\cdot\sqrt{\varepsilon_{22}^{(2)}}-\sqrt{\mu_{11}^{(2)}}\cdot\sqrt{\varepsilon_{22}^{(1)}}}{\sqrt{\mu_{11}^{(1)}}\cdot\sqrt{\varepsilon_{22}^{(2)}}+\sqrt{\mu_{11}^{(2)}}\cdot\sqrt{\varepsilon_{22}^{(1)}}}\cdot\frac{1}{\sqrt{\varepsilon_{22}^{(1)}}}\int_0^t J_2^{(1)}(v_3^{(1)}(\tau-t),\tau)d\tau + \frac{2\sqrt{\mu_{11}^{(1)}}\cdot\sqrt{\varepsilon_{22}^{(1)}}}{\sqrt{\mu_{11}^{(1)}}\cdot\sqrt{\varepsilon_{22}^{(2)}}+\sqrt{\mu_{11}^{(2)}}\cdot\sqrt{\varepsilon_{22}^{(1)}}}\cdot\frac{(-1)}{\sqrt{\varepsilon_{22}^{(2)}}}\int_0^t J_2^{(2)}(-v_4^{(2)}(\tau-t),\tau)d\tau.$$

Applying the same procedure as we did for $u_2^{(1)}(-0,t)$ and $u_4^{(1)}(-0,t)$, we can find the matching conditions $u_1^{(2)}(+0,t)$ and $u_3^{(2)}(+0,t)$ in the following form:

$$u_1^{(2)}(+0,t) = \frac{2\sqrt{\mu_{22}^{(2)}}\cdot\sqrt{\varepsilon_{11}^{(2)}}}{\sqrt{\mu_{22}^{(2)}}\cdot\sqrt{\varepsilon_{11}^{(1)}}+\sqrt{\mu_{22}^{(1)}}\cdot\sqrt{\varepsilon_{11}^{(1)}}}\cdot\frac{(-1)}{\sqrt{\varepsilon_{11}^{(1)}}}\int_0^t J_1^{(1)}(v_1^{(1)}(\tau-t),\tau)d\tau + \frac{\sqrt{\mu_{22}^{(2)}}\cdot\sqrt{\varepsilon_{11}^{(1)}}-\sqrt{\mu_{22}^{(1)}}\cdot\sqrt{\varepsilon_{11}^{(2)}}}{\sqrt{\mu_{22}^{(2)}}\cdot\sqrt{\varepsilon_{11}^{(1)}}+\sqrt{\mu_{22}^{(1)}}\cdot\sqrt{\varepsilon_{11}^{(2)}}}\cdot\frac{1}{\sqrt{\varepsilon_{11}^{(2)}}}\int_0^t J_1^{(2)}(-v_2^{(2)}(\tau-t),\tau)d\tau.$$

$$u_3^{(2)}(+0,t) = \frac{2\sqrt{\mu_{11}^{(2)}}\cdot\sqrt{\varepsilon_{22}^{(2)}}}{\sqrt{\mu_{11}^{(2)}}\cdot\sqrt{\varepsilon_{22}^{(1)}}+\sqrt{\mu_{11}^{(1)}}\cdot\sqrt{\varepsilon_{22}^{(2)}}}\cdot\frac{1}{\sqrt{\varepsilon_{22}^{(1)}}}\int_0^t J_2^{(1)}(v_3^{(1)}(\tau-t),\tau)d\tau + \frac{\sqrt{\mu_{11}^{(2)}}\cdot\sqrt{\varepsilon_{22}^{(1)}}-\sqrt{\mu_{11}^{(1)}}\cdot\sqrt{\varepsilon_{22}^{(2)}}}{\sqrt{\mu_{11}^{(2)}}\cdot\sqrt{\varepsilon_{22}^{(1)}}+\sqrt{\mu_{11}^{(1)}}\cdot\sqrt{\varepsilon_{22}^{(2)}}}\cdot\frac{(-1)}{\sqrt{\varepsilon_{22}^{(2)}}}\int_0^t J_2^{(2)}(-v_4^{(2)}(\tau-t),\tau)d\tau.$$

3.3.4 Finding Explicit Formulae For The Electric And Magnetic Fields

Now, we write the explicit formulae for the electric and magnetic fields, that is, the solution of the Maxwell's system. In the last subsections firstly we reduced our original system to an initial value problems; and then we solved these reduced initial value problems; after that we derived matching conditions, which was necessary to solve some of the initial value problems. And now, using all of these we get the explicit formula for the original problem.

Firstly, by making back substitution of $u_i^{(k)}(x_3,t)$, $i=1,2,3,4$, $k=1,2$ we turn back the original problem, Maxwell's system, from the reduced initial value problems. Then $H_i^{(k)}(x_3,t)$, $E_i^{(k)}(x_3,t)$, $i=1,2$, $k=1,2$ can be written by means of $u_i^{(k)}(x_3,t)$ $i=1,2,3,4$, $k=1,2$ in the following form:

$$H_1^{(k)} = \frac{1}{2\sqrt{\mu_{11}^{(k)}}}[u_3^{(k)}(x_3,t)+u_4^{(k)}(x_3,t)],$$
$$H_2^{(k)} = \frac{1}{2\sqrt{\mu_{22}^{(k)}}}[u_1^{(k)}(x_3,t)+u_2^{(k)}(x_3,t)],$$
$$E_1^{(k)} = \frac{1}{2\sqrt{\varepsilon_{11}^{(k)}}}[u_1^{(k)}(x_3,t)-u_2^{(k)}(x_3,t)],$$
$$E_2^{(k)} = \frac{1}{2\sqrt{\varepsilon_{22}^{(k)}}}[-u_3^{(k)}(x_3,t)+u_4^{(k)}(x_3,t)],$$

where $k=1,2$ denotes the media.

After that, substituting $u_i^{(k)}(x_3,t)$, $i=1,2,3,4$, $k=1,2$ and the derived matching conditions explicit formula for solution of the Maxwell's system can be written in the following form:

Explicit formula for $H_1(x_3,t)$:

In $R1: \quad -\infty < x_3 < -v_3^{(1)}t;\ \ t>0$:

$$H_1(x_3,t) = \frac{v_3^{(1)}}{2}\int_0^t [J_2^{(1)}(v_3^{(1)}(\tau-t)+x_3,\tau) - J_2^{(1)}(-v_3^{(1)}(\tau-t)+x_3,\tau)]d\tau,$$

In $R2: \quad -v_3^{(1)}t < x_3 < 0;\ \ t>0$:

$$H_1(x_3,t) = \frac{1}{2\sqrt{\mu_{11}^{(1)}}}\left[\frac{1}{\sqrt{\varepsilon_{22}^{(1)}}}\int_0^t J_2^{(1)}(v_3^{(1)}(\tau-t)+x_3,\tau)d\tau\right.$$
$$+\frac{\sqrt{\mu_{11}^{(1)}\varepsilon_{22}^{(2)}}-\sqrt{\mu_{11}^{(2)}\varepsilon_{22}^{(1)}}}{\sqrt{\varepsilon_{22}^{(1)}}(\sqrt{\mu_{11}^{(1)}\varepsilon_{22}^{(2)}}+\sqrt{\mu_{11}^{(2)}\varepsilon_{22}^{(1)}})}\int_0^{t+\frac{x_3}{v_3^{(1)}}} J_2^{(1)}(v_3^{(1)}(\tau-t)-x_3,\tau)d\tau$$
$$-\frac{2\sqrt{\mu_{11}^{(1)}\varepsilon_{22}^{(1)}}}{\sqrt{\varepsilon_{22}^{(2)}}(\sqrt{\mu_{11}^{(1)}\varepsilon_{22}^{(2)}}+\sqrt{\mu_{11}^{(2)}\varepsilon_{22}^{(1)}})}\int_0^{t+\frac{x_3}{v_3^{(1)}}} J_2^{(2)}(-v_3^{(2)}(\tau-t)+\frac{v_3^{(2)}}{v_3^{(1)}}x_3,\tau)d\tau$$
$$\left.-\frac{1}{\sqrt{\varepsilon_{22}^{(1)}}}\int_{t+\frac{x_3}{v_3^{(1)}}}^t J_2^{(1)}(-v_3^{(1)}(\tau-t)+x_3,\tau)d\tau\right],$$

In $R4: \quad 0 < x_3 < v_3^{(2)}t; \;\; t > 0$:

$$H_1(x_3,t) = \frac{1}{2\sqrt{\mu_{11}^{(2)}}}\left[\frac{1}{\sqrt{\varepsilon_{22}^{(2)}}}\int_{t-\frac{x_3}{v_3^{(2)}}}^{t} J_2^{(2)}(v_3^{(2)}(\tau-t)+x_3,\tau)d\tau\right.$$

$$+\frac{2\sqrt{\mu_{11}^{(2)}\varepsilon_{22}^{(2)}}}{\sqrt{\varepsilon_{22}^{(1)}}(\sqrt{\mu_{11}^{(2)}\varepsilon_{22}^{(1)}}+\sqrt{\mu_{11}^{(1)}\varepsilon_{22}^{(2)}})}\int_0^{t-\frac{x_3}{v_3^{(2)}}} J_2^{(1)}(v_3^{(1)}(\tau-t)+\frac{v_3^{(1)}}{v_3^{(2)}}x_3,\tau)d\tau$$

$$-\frac{\sqrt{\mu_{11}^{(2)}\varepsilon_{22}^{(1)}}-\sqrt{\mu_{11}^{(1)}\varepsilon_{22}^{(2)}}}{\sqrt{\varepsilon_{22}^{(2)}}(\sqrt{\mu_{11}^{(2)}\varepsilon_{22}^{(1)}}+\sqrt{\mu_{11}^{(1)}\varepsilon_{22}^{(2)}})}\int_0^{t-\frac{x_3}{v_3^{(2)}}} J_2^{(2)}(-v_3^{(2)}(\tau-t)-x_3,\tau)d\tau$$

$$\left.-\frac{1}{\sqrt{\varepsilon_{22}^{(2)}}}\int_0^{t} J_2^{(2)}(-v_3^{(2)}(\tau-t)+x_3,\tau)d\tau\right],$$

In $R3: \quad v_3^{(2)}t < x_3 < \infty; \;\; t > 0$:

$$H_1(x_3,t) = \frac{v_3^{(2)}}{2}\int_0^t [J_2^{(2)}(v_3^{(2)}(\tau-t)+x_3,\tau)+J_2^{(2)}(-v_3^{(2)}(\tau-t)+x_3,\tau)]d\tau.$$

Explicit formula for $H_2(x_3,t)$:

In $R1: \quad -\infty < x_3 < -v_1^{(1)}t; \;\; t > 0$:

$$H_2(x_3,t) = \frac{v_1^{(1)}}{2}\int_0^t [-J_1^{(1)}(v_1^{(1)}(\tau-t)+x_3,\tau)+J_1^{(1)}(-v_1^{(1)}(\tau-t)+x_3,\tau)]d\tau,$$

In $R2: \quad -v_1^{(1)}t < x_3 < 0; \;\; t > 0$:

$$H_2(x_3,t) = \frac{1}{2\sqrt{\mu_{22}^{(1)}}}\left[-\frac{1}{\sqrt{\varepsilon_{11}^{(1)}}}\int_0^{t} J_1^{(1)}(v_1^{(1)}(\tau-t)+x_3,\tau)d\tau\right.$$

$$-\frac{\sqrt{\mu_{22}^{(1)}\varepsilon_{11}^{(2)}}-\sqrt{\mu_{22}^{(2)}\varepsilon_{11}^{(1)}}}{\sqrt{\varepsilon_{11}^{(1)}}(\sqrt{\mu_{22}^{(1)}\varepsilon_{11}^{(2)}}+\sqrt{\mu_{22}^{(2)}\varepsilon_{11}^{(1)}})}\int_0^{t+\frac{x_3}{v_1^{(1)}}} J_1^{(1)}(v_1^{(1)}(\tau-t)-x_3,\tau)d\tau$$

$$+\frac{2\sqrt{\mu_{22}^{(1)}\varepsilon_{11}^{(1)}}}{\sqrt{\varepsilon_{11}^{(2)}}(\sqrt{\mu_{22}^{(1)}\varepsilon_{11}^{(2)}}+\sqrt{\mu_{22}^{(2)}\varepsilon_{11}^{(1)}})}\int_0^{t+\frac{x_3}{v_1^{(1)}}} J_1^{(2)}(-v_1^{(2)}(\tau-t)+\frac{v_1^{(2)}}{v_1^{(1)}}x_3,\tau)d\tau$$

$$\left.+\frac{1}{\sqrt{\varepsilon_{11}^{(1)}}}\int_{t+\frac{x_3}{v_1^{(1)}}}^{t} J_1^{(1)}(-v_1^{(1)}(\tau-t)+x_3,\tau)d\tau,\right],$$

In $R4: \quad 0 < x_3 < v_1^{(2)}t; \;\; t > 0$:

$$H_2(x_3,t) = \frac{1}{2\sqrt{\mu_{22}^{(2)}}}\left[-\frac{1}{\sqrt{\varepsilon_{11}^{(2)}}}\int_{t-\frac{x_3}{v_1^{(2)}}}^{t} J_1^{(2)}(v_1^{(2)}(\tau-t)+x_3,\tau)d\tau\right.$$

$$-\frac{2\sqrt{\mu_{22}^{(2)}\varepsilon_{11}^{(2)}}}{\sqrt{\varepsilon_{11}^{(1)}}(\sqrt{\mu_{22}^{(2)}\varepsilon_{11}^{(1)}}+\sqrt{\mu_{22}^{(1)}\varepsilon_{11}^{(1)}})}\int_0^{t-\frac{x_3}{v_1^{(2)}}} J_1^{(1)}(v_1^{(1)}(\tau-t)+\frac{v_1^{(1)}}{v_1^{(2)}}x_3,\tau)d\tau$$

$$+\frac{\sqrt{\mu_{22}^{(2)}\varepsilon_{11}^{(1)}}-\sqrt{\mu_{22}^{(1)}\varepsilon_{11}^{(2)}}}{\sqrt{\varepsilon_{11}^{(2)}}(\sqrt{\mu_{22}^{(2)}\varepsilon_{11}^{(1)}}+\sqrt{\mu_{22}^{(1)}\varepsilon_{11}^{(2)}})}\int_0^{t-\frac{x_3}{v_1^{(2)}}} J_1^{(2)}(-v_1^{(2)}(\tau-t)-x_3,\tau)d\tau$$

$$\left.+\frac{1}{\sqrt{\varepsilon_{11}^{(2)}}}\int_0^t J_1^{(2)}(-v_1^{(2)}(\tau-t)+x_3,\tau)d\tau\right],$$

In $R3: \quad v_1^{(2)}t<x_3<\infty; \;\; t>0$:

$$H_2(x_3,t)=\frac{v_1^{(2)}}{2}\int_0^t[-J_1^{(2)}(v_1^{(2)}(\tau-t)+x_3,\tau)+J_1^{(2)}(-v_1^{(2)}(\tau-t)+x_3,\tau)]d\tau.$$

Explicit formula for $H_3(x_3,t)$:

Explicit formula for $H_3(x_3,t)$ can be find by the same way as we did in free space. Hence the formula is in the following form:

$$H_3(x_3,t)=0, \quad -\infty<x_3<\infty; \quad t>0.$$

Explicit formula for $E_1(x_3,t)$:

In $R1: \quad -\infty<x_3<-v_1^{(1)}t; \;\; t>0$:

$$E_1(x_3,t)=\frac{1}{2\varepsilon_{11}^{(1)}}\int_0^t[-J_1^{(1)}(v_1^{(1)}(\tau-t)+x_3,\tau)-J_1^{(1)}(-v_1^{(1)}(\tau-t)+x_3,\tau)]d\tau,$$

In $R2: \quad -v_1^{(1)}t<x_3<0; \;\; t>0$:

$$E_1(x_3,t)=\frac{1}{2\sqrt{\varepsilon_{11}^{(1)}}}\left[-\frac{1}{\sqrt{\varepsilon_{11}^{(1)}}}\int_0^t J_1^{(1)}(v_1^{(1)}(\tau-t)+x_3,\tau)d\tau\right.$$

$$+\frac{\sqrt{\mu_{22}^{(1)}\varepsilon_{11}^{(2)}}-\sqrt{\mu_{22}^{(2)}\varepsilon_{11}^{(1)}}}{\sqrt{\varepsilon_{11}^{(1)}}(\sqrt{\mu_{22}^{(1)}}\cdot\sqrt{\varepsilon_{11}^{(2)}}+\sqrt{\mu_{22}^{(2)}}\cdot\sqrt{\varepsilon_{11}^{(1)}})}\int_0^{t+\frac{x_3}{v_1^{(1)}}} J_1^{(1)}(v_1^{(1)}(\tau-t)-x_3,\tau)d\tau$$

$$-\frac{2\sqrt{\mu_{22}^{(1)}\varepsilon_{11}^{(1)}}}{\sqrt{\varepsilon_{11}^{(2)}}(\sqrt{\mu_{22}^{(1)}\varepsilon_{11}^{(2)}}+\sqrt{\mu_{22}^{(2)}\varepsilon_{11}^{(1)}})}\int_0^{t+\frac{x_3}{v_1^{(1)}}} J_1^{(2)}(-v_1^{(2)}(\tau-t)+\frac{v_1^{(2)}}{v_1^{(1)}}x_3,\tau)d\tau$$

$$\left.-\frac{1}{\sqrt{\varepsilon_{11}^{(1)}}}\int_{t+\frac{x_3}{v_1^{(1)}}}^t J_1^{(1)}(-v_1^{(1)}(\tau-t)+x_3,\tau)d\tau\right],$$

In $R4:\quad 0<x_3<\nu_1^{(2)}t;\ \ t>0$:

$$E_1(x_3,t)=\frac{1}{2\sqrt{\varepsilon_{11}^{(2)}}}\left[-\frac{1}{\sqrt{\varepsilon_{11}^{(2)}}}\int_{t-\frac{x_3}{\nu_1^{(2)}}}^{t}J_1^{(2)}(\nu_1^{(2)}(\tau-t)+x_3,\tau)d\tau\right.$$
$$-\frac{2\sqrt{\mu_{22}^{(2)}\varepsilon_{11}^{(2)}}}{\sqrt{\varepsilon_{11}^{(1)}}(\sqrt{\mu_{22}^{(2)}\varepsilon_{11}^{(1)}}+\sqrt{\mu_{22}^{(1)}\varepsilon_{11}^{(1)}})}\int_0^{t-\frac{x_3}{\nu_1^{(2)}}}J_1^{(1)}(\nu_1^{(1)}(\tau-t)+\frac{\nu_1^{(1)}}{\nu_1^{(2)}}x_3,\tau)d\tau$$
$$+\frac{\sqrt{\mu_{22}^{(2)}\varepsilon_{11}^{(1)}}-\sqrt{\mu_{22}^{(1)}\varepsilon_{11}^{(2)}}}{\sqrt{\varepsilon_{11}^{(2)}}(\sqrt{\mu_{22}^{(2)}\varepsilon_{11}^{(1)}}+\sqrt{\mu_{22}^{(1)}\varepsilon_{11}^{(2)}})}\int_0^{t-\frac{x_3}{\nu_1^{(2)}}}J_1^{(2)}(-\nu_1^{(2)}(\tau-t)-x_3,\tau)d\tau$$
$$\left.-\frac{1}{\sqrt{\varepsilon_{11}^{(2)}}}\int_0^{t}J_1^{(2)}(-\nu_1^{(2)}(\tau-t)+x_3,\tau)d\tau\right],$$

In $R3:\quad \nu_1^{(2)}t<x_3<\infty;\ \ t>0$:

$$E_1(x_3,t)=\frac{1}{2\varepsilon_{11}^{(1)}}\int_0^t[-J_1^{(2)}(\nu_1^{(2)}(\tau-t)+x_3,\tau)-J_1^{(2)}(-\nu_1^{(2)}(\tau-t)+x_3,\tau)]d\tau.$$

Explicit formula for $E_2(x_3,t)$:

In $R1:\quad -\infty<x_3<-\nu_3^{(1)}t;\ \ t>0$:

$$E_2(x_3,t)=\frac{1}{2\varepsilon_{22}^{(1)}}\int_0^t[-J_2^{(1)}(\nu_3^{(1)}(\tau-t)+x_3,\tau)-J_2^{(1)}(-\nu_3^{(1)}(\tau-t)+x_3,\tau)]d\tau,$$

In $R2:\quad -\nu_3^{(1)}t<x_3<0;\ \ t>0$:

$$E_2(x_3,t)=\frac{1}{2\sqrt{\varepsilon_{22}^{(1)}}}\left[-\frac{1}{\sqrt{\varepsilon_{22}^{(1)}}}\int_0^{t}J_2^{(1)}(\nu_3^{(1)}(\tau-t)+x_3,\tau)d\tau\right.$$
$$+\frac{\sqrt{\mu_{11}^{(1)}\varepsilon_{22}^{(2)}}-\sqrt{\mu_{11}^{(2)}\varepsilon_{22}^{(1)}}}{\sqrt{\varepsilon_{22}^{(1)}}(\sqrt{\mu_{11}^{(1)}\varepsilon_{22}^{(2)}}+\sqrt{\mu_{11}^{(2)}\varepsilon_{22}^{(1)}})}\int_0^{t+\frac{x_3}{\nu_3^{(1)}}}J_2^{(1)}(\nu_3^{(1)}(\tau-t)-x_3,\tau)d\tau$$
$$-\frac{2\sqrt{\mu_{11}^{(1)}\varepsilon_{22}^{(1)}}}{\sqrt{\varepsilon_{22}^{(2)}}(\sqrt{\mu_{11}^{(1)}\varepsilon_{22}^{(2)}}+\sqrt{\mu_{11}^{(2)}\varepsilon_{22}^{(1)}})}\int_0^{t+\frac{x_3}{\nu_3^{(1)}}}J_2^{(2)}(-\nu_3^{(2)}(\tau-t)+\frac{\nu_3^{(2)}}{\nu_3^{(1)}}x_3,\tau)d\tau$$
$$\left.-\frac{1}{\sqrt{\varepsilon_{22}^{(1)}}}\int_{t+\frac{x_3}{\nu_3^{(1)}}}^{t}J_2^{(1)}(-\nu_3^{(1)}(\tau-t)+x_3,\tau)d\tau\right],$$

In $R4:\quad 0<x_3<\nu_3^{(2)}t;\ \ t>0$:

$$E_2(x_3,t)=\frac{1}{2\sqrt{\varepsilon_{22}^{(2)}}}\left[-\frac{1}{\sqrt{\varepsilon_{22}^{(2)}}}\int_{t-\frac{x_3}{\nu_3^{(2)}}}^{t}J_2^{(2)}(\nu_3^{(2)}(\tau-t)+x_3,\tau)d\tau\right.$$

$$-\frac{2\sqrt{\mu_{11}^{(2)}\varepsilon_{22}^{(2)}}}{\sqrt{\varepsilon_{22}^{(1)}}(\sqrt{\mu_{11}^{(2)}\varepsilon_{22}^{(1)}}+\sqrt{\mu_{11}^{(1)}\varepsilon_{22}^{(2)}})}\int_0^{t-\frac{x_3}{v_3^{(2)}}} J_2^{(1)}(v_3^{(1)}(\tau-t)+\frac{v_3^{(1)}}{v_3^{(2)}}x_3,\tau)d\tau$$

$$+\frac{\sqrt{\mu_{11}^{(2)}\varepsilon_{22}^{(1)}}-\sqrt{\mu_{11}^{(1)}\varepsilon_{22}^{(2)}}}{\sqrt{\varepsilon_{22}^{(2)}}(\sqrt{\mu_{11}^{(2)}\varepsilon_{22}^{(1)}}+\sqrt{\mu_{11}^{(1)}\varepsilon_{22}^{(2)}})}\int_0^{t-\frac{x_3}{v_3^{(2)}}} J_2^{(2)}(-v_3^{(2)}(\tau-t)-x_3,\tau)d\tau$$

$$\left.-\frac{1}{\sqrt{\varepsilon_{22}^{(2)}}}\int_0^t J_2^{(2)}(-v_3^{(2)}(\tau-t)+x_3,\tau)d\tau\right],$$

In $R3: \quad v_3^{(2)}t < x_3 < \infty; \;\; t > 0$:

$$E_2(x_3,t) = \frac{1}{2\varepsilon_{22}^{(2)}}\int_0^t [-J_2^{(2)}(v_3^{(2)}(\tau-t)+x_3,\tau)+J_2^{(2)}(-v_3^{(2)}(\tau-t)+x_3,\tau)]d\tau.$$

Explicit formula for $E_3(x_3,t)$:

The formula for $E_3(x_3,t)$ can be found applying the same procedure as we did in free space. Then it can be written in the following form:

$$E_3(x_3,t) = -\frac{1}{\varepsilon_{33}^{(1)}}\int_0^t J_3^{(1)}(x_3,\tau)d\tau, \quad -\infty < x_3 < 0; \quad t > 0,$$

$$E_3(x_3,t) = -\frac{1}{\varepsilon_{33}^{(2)}}\int_0^t J_3^{(2)}(x_3,\tau)d\tau, \quad 0 < x_3 < \infty; \quad t > 0.$$

3.4 Simulation Of Electric And Magnetic Fields

In this section we make simulation of the obtained explicit formulae for the electric and magnetic fields, that is the solution of the Maxwell's system.

For all applications the current density $J(x_3,t)$ is taken in the form

$$J^{(1)}(x_3,t) = e^2\delta(x_3-x^0)\delta(t), \quad J^{(2)}(x_3,t) = 0,$$

where $\delta(x_3-x^0)\delta(t)$ is the Dirac delta function concentrated at the point $x_3 = x^0$ and for the time $t = 0$ in the direction $e^2 = (0,1,0)$.

Using the procedure of Section 3.3 the explicit formulae for the components of the electric and magnetic fields were computed for the given symmetric positive definite matrices $\mathscr{E}$, $\mathscr{M}$. By these formulae the images of $H_1(x_3,t)$, $H_2(x_3,t)$, $H_3(x_3,t)$, $E_1(x_3,t)$, $E_2(x_3,t)$, $E_3(x_3,t)$ were simulated for fixed time t. Some of these images are presented in this section.

Example1: Isotropic media

layer	permittivity tensor - $\mathscr{E}$	permeability tensor - $\mathscr{M}$	source - x^0
1.*Layer*	$\begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$	$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$	-1
2.*Layer*	$\begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	-

Example2: An electrically and magnetically anisotropic medium

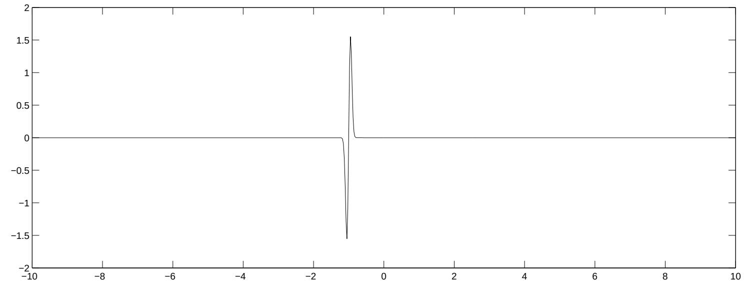
layer	permittivity tensor - $\mathscr{E}$	permeability tensor - $\mathscr{M}$	source - x^0
1.*Layer*	$\begin{pmatrix} 16 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 32 \end{pmatrix}$	$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 7 \end{pmatrix}$	-1
2.*Layer*	$\begin{pmatrix} 11 & 0 & 0 \\ 0 & 13 & 0 \\ 0 & 0 & 44 \end{pmatrix}$	$\begin{pmatrix} 9 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 4 \end{pmatrix}$	-

Analysis of simulation

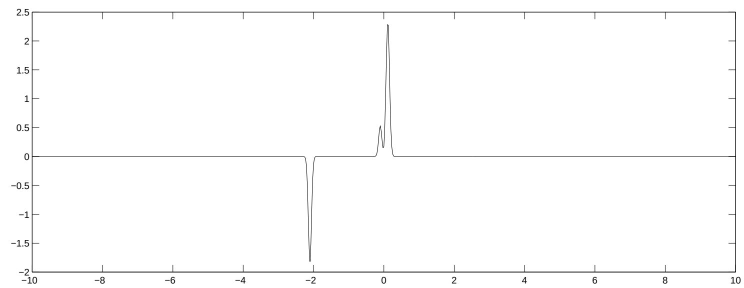
The explicit formula for the component $H_1(x_3,t)$ of the magnetic field was computed for the given symmetric matrices $\mathscr{M}$ and $\mathscr{E}$ in the last section. The result of simulation $H_1(x_3,t)$ is presented in Figure 3.6 and Figure 3.7. In $Example1$ our media is isotropic whereas in other example we have electrically and magnetically anisotropic media. In these two examples we used Dirac delta regularization and for regularization we take $\delta(x_3) \approx \frac{1}{2\sqrt{\pi\varepsilon}} \exp\left(-\frac{x_3^2}{4\varepsilon}\right)$ where $\varepsilon = 0.001$. For the first layer the source is taken as $x_3 = -1$, that is, the Dirac delta is concentrated at $x_3 = -1$ and there is no source at the second layer.

In figures the horizontal axis x_3 is location and the vertical axis is the magnitude of $H_1(x_3,t)$. In

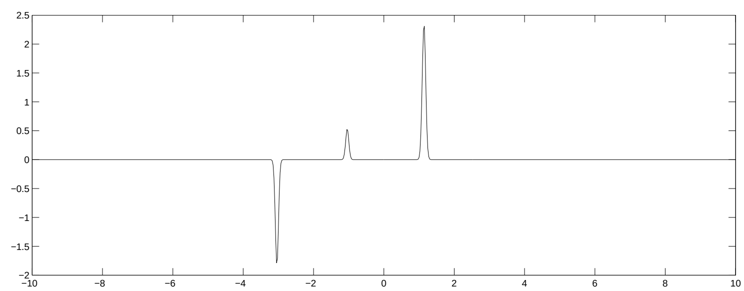
(a) $t = 0.1$



(b) $t = 2.7$



(c) $t = 5$



(d) $t = 18$

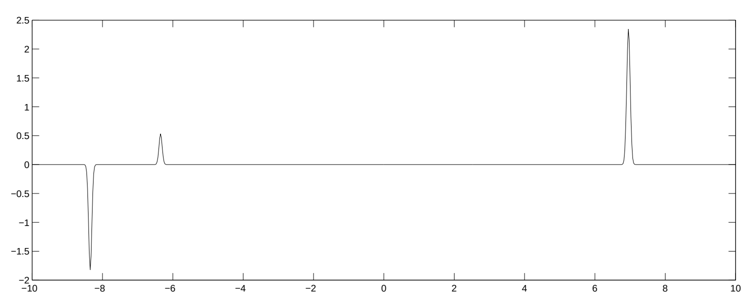
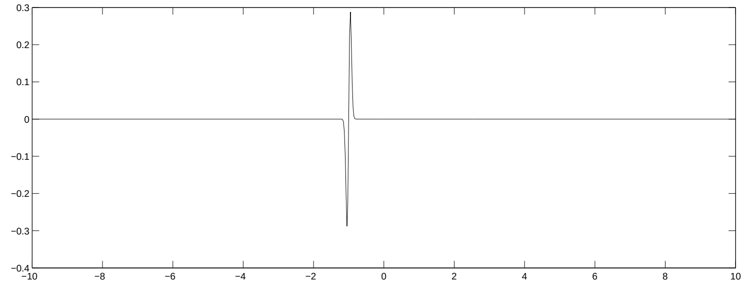
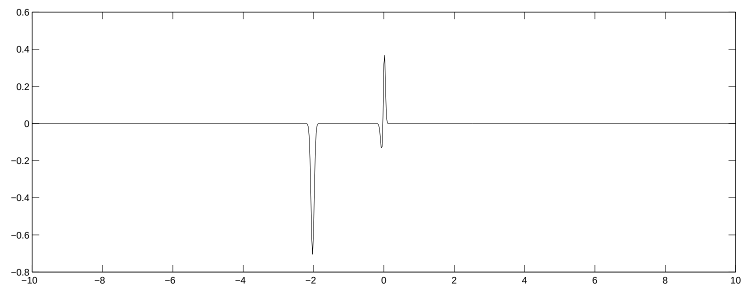


Figure 3.6 The magnetic field $H_1(x_3, t)$.

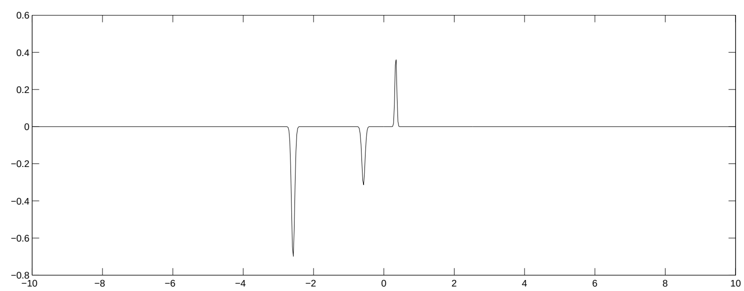
(a) $t = 0.1$



(b) $t = 6.5$



(c) $t = 10$



(d) $t = 25$

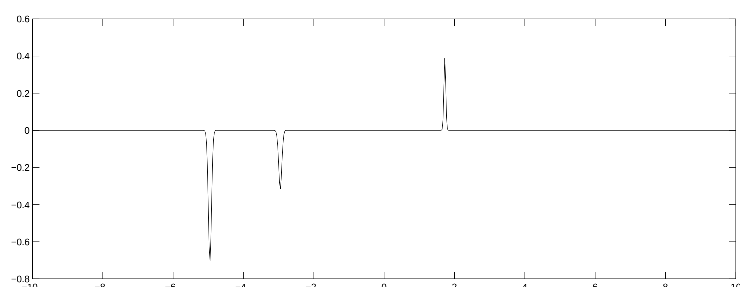


Figure 3.7 The magnetic field $H_1(x_3,t)$.

the Figure 3.6a for $t = 0.1$ at the point $x_3 = -1$ the waves start their propagation. One of the moves to plus infinity whereas the other moves to minus infinity. From the Figure 3.6b for $t = 2.7$ it can be seen that the wave front which moves along the positive direction touches the boundary at $x_3 = 0$ and after that reflected and transmitted waves appear. Magnitude of the reflected wave is smaller than the transmitted wave. The distance between the waves in the first half space, $-\infty < x_3 < 0$, is equivalent and does not change. But the distance between the reflected and transmitted wave become larger time by time and this can be seen from the Figure 3.6c and Figure 3.6d. Analysis of the Figure 3.7 in $Example2$ is similar with the first one. From these two examples the following analysis can be obtained. In $Example1$ we consider isotropic media and in $Example2$ we considered anisotropic media. And we see from the figures that for different media the shapes, magnitudes and the speeds of the waves are different. Reflection and transmission appears at the different times. And we conclude the exactness of our formulae from these examples.

CHAPTER FOUR

METHOD OF CHARACTERISTICS FOR FINDING ELECTRIC AND MAGNETIC FIELDS IN THREE LAYERED MEDIA

4.1 Equations of electric and Magnetic Fields in Three Layered Media

In this section, we find the solution of the initial value problem for the time-dependent Maxwell's system in homogeneous, anisotropic materials in three layered media. We are applying a similar process as we did in free space and two layered media.

Firstly, let us define our domain on which we will study. Like we assumed before, in this section we also assume that the unknown and given functions and vector functions just depend on the third component of the space variable $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ and the time variable $t \in \mathbb{R}$.

Now, we will separate the whole space into three layers where $-\infty < x_3 < 0$, $0 < x_3 < \ell$, $\ell < x_3 < \infty$ denote the first layer, the second layer and the third layer respectively.

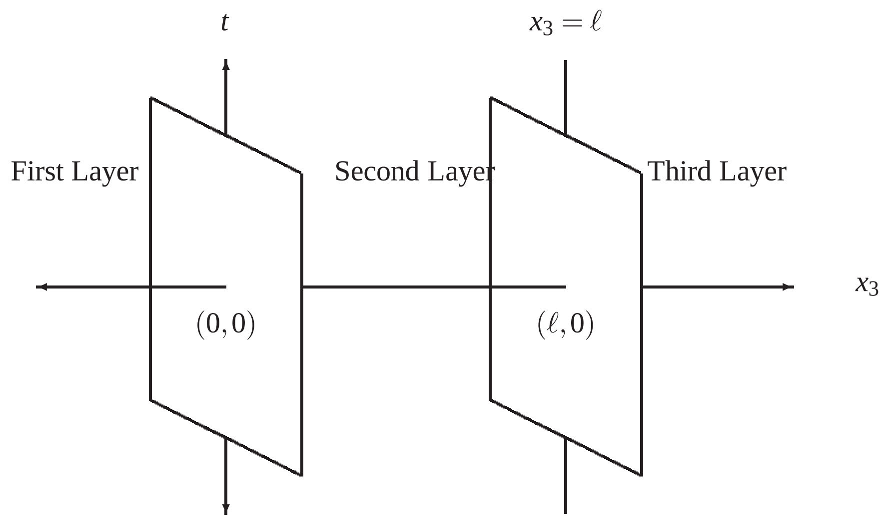


Figure 4.1 Three layered media.

We will denote these layers by a notation writing the number of the layer, in which we study, in parentheses. This notation will be shown like a power, that is, $\square^{(k)}$, where $k = 1, 2, 3$.

Then the Maxwell's system for three layered media can be written as:

$$curl_x \vec{H}^{(k)} = \mathscr{E}^{(k)} \frac{\partial \vec{E}^{(k)}}{\partial t} + \vec{J}^{(k)}, \tag{4.1.1}$$

$$curl_x \vec{E}^{(k)} = -\mathscr{M}^{(k)} \frac{\partial \vec{H}^{(k)}}{\partial t}, \tag{4.1.2}$$

$$div_x (\mathscr{E}^{(k)} \vec{E}^{(k)}) = \rho^{(k)}, \tag{4.1.3}$$

$$div_x (\mathscr{M}^{(k)} \vec{H}^{(k)}) = 0, \tag{4.1.4}$$

where $k = 1,2,3$ and denotes the media.

And the conservation law of charges is given by:

$$\frac{\partial \rho^{(k)}}{\partial t} + div_x \vec{J}^{(k)} = 0, \;\; k = 1,2,3. \tag{4.1.5}$$

4.2 Assumptions and Problem Set-up For Maxwell's System In Three Layered media

We assume that the electric permittivity matrix $\mathscr{E}^{(k)} = (\varepsilon_{ij}^{(k)})_{3\times 3}$ and the magnetic permeability matrix $\mathscr{M}^{(k)} = (\mu_{ij})_{3\times 3}^{(k)}$ are symmetric positive definite matrices with constant elements, and they are in the form of:

$$\mathscr{E}^{(k)} = \begin{pmatrix} \varepsilon_{11}^{(k)} & 0 & 0 \\ 0 & \varepsilon_{22}^{(k)} & 0 \\ 0 & 0 & \varepsilon_{33}^{(k)} \end{pmatrix}, \;\; \mathscr{M} = \begin{pmatrix} \mu_{11}^{(k)} & 0 & 0 \\ 0 & \mu_{22}^{(k)} & 0 \\ 0 & 0 & \mu_{33}^{(k)} \end{pmatrix}.$$

Let the components of vector functions $\vec{H}^{(k)}(x) = (H_1^{(k)}, H_2^{(k)}, H_3^{(k)})$, $\vec{E}^{(k)}(x) = (E_1^{(k)}, E_2^{(k)}, E_3^{(k)})$ depend on x_3 and t only, that is, $H_i^{(k)} = H_i^{(k)}(x_3,t)$, $E_i^{(k)} = E_i^{(k)}(x_3,t)$, $i = 1,2,3$; $\vec{J}^{(k)} = (J_1^{(k)}, J_2^{(k)}, J_3^{(k)})$, where $J_i^{(k)} = J_i^{(k)}(x_3,t)$, $i = 1,2,3$; $k = 1,2,3$.

Moreover, we suppose that:

$$\vec{E}^{(k)} = 0, \;\; \vec{H}^{(k)} = 0, \;\; \rho^{(k)} = 0, \;\; \vec{J}^{(k)} = 0 \;\; for \;\; t \leq 0, \tag{4.2.1}$$

this means that there is no electric charges and currents at the time $t \leq 0$; electric and magnetic fields vanish for $t \leq 0$.

Since we are studying in three layered media, we have matching conditions between these layers. These are given by the followings:

$$\left.\begin{aligned}
(\vec{E}^{(2)}-\vec{E}^{(1)})|_{x_3=0}\times\vec{n}=0,\\
(\vec{D}^{(2)}-\vec{D}^{(1)})|_{x_3=0}\cdot\vec{n}=0,\\
(\vec{H}^{(2)}-\vec{H}^{(1)})|_{x_3=0}\times\vec{n}=0,\\
(\vec{B}^{(2)}-\vec{B}^{(1)})|_{x_3=0}\cdot\vec{n}=0,\\
\\
(\vec{E}^{(3)}-\vec{E}^{(2)})|_{x_3=\ell}\times\vec{n}=0,\\
(\vec{D}^{(3)}-\vec{D}^{(2)})|_{x_3=\ell}\cdot\vec{n}=0,\\
(\vec{H}^{(3)}-\vec{H}^{(2)})|_{x_3=\ell}\times\vec{n}=0,\\
(\vec{B}^{(3)}-\vec{B}^{(2)})|_{x_3=\ell}\cdot\vec{n}=0,
\end{aligned}\right\} \tag{4.2.2}$$

where $\vec{n}=(0,0,1)$.

Let further $\mathscr{E}^{(k)}$, $\mathscr{M}^{(k)}$, $\vec{J}^{(k)}$ be given, $k=1,2,3$.

The main problem is to find $\vec{E}^{(k)}$, $\vec{H}^{(k)}$, $k=1,2,3$ satisfying $(4.1.1)-(4.1.4)$ and $(4.2.1)$, $(4.2.2)$.

4.3 Finding Explicit Formula for Solution of the Problem

Following similar steps as we did for two layered media we find explicit formulae for electric and magnetic fields.

4.3.1 Reduction of the Problem For Maxwell's System

Now using the assumptions and applying the same procedure as we did in free space and two layered media, we reduce the Maxwell's system into the first order partial differential equations. Using the initial conditions $(4.2.1)$ and the matching conditions $(4.2.2)$ we get an initial value problem that is related with each layer.

The first order partial differential equation, that is reduced from Maxwell's system, can be found in the following form:

$$\frac{\partial u_i^{(k)}(x_3,t)}{\partial t}+(-1)^{i+1}v_i^{(k)}\frac{\partial u_i^{(k)}(x_3,t)}{\partial x_3}=f_i^{(k)}(x_3,t),\quad i=1,2,3,4, \tag{4.3.1}$$

where

$$u_1^{(k)}=\sqrt{\mu_{22}^{(k)}}H_2^{(k)}+\sqrt{\varepsilon_{11}^{(k)}}E_1^{(k)},$$

$$u_2^{(k)} = \sqrt{\mu_{22}^{(k)}}H_2^{(k)} - \sqrt{\varepsilon_{11}^{(k)}}E_1^{(k)},$$
$$u_3^{(k)} = \sqrt{\mu_{11}^{(k)}}H_1^{(k)} - \sqrt{\varepsilon_{22}^{(k)}}E_2^{(k)},$$
$$u_4^{(k)} = \sqrt{\mu_{11}^{(k)}}H_1^{(k)} + \sqrt{\varepsilon_{22}^{(k)}}E_2^{(k)},$$
$$v_1^{(k)} = v_2^{(k)} = \frac{1}{\sqrt{\varepsilon_{11}^{(k)}\mu_{22}^{(k)}}}; \;\; v_3^{(k)} = v_4^{(k)} = \frac{1}{\sqrt{\varepsilon_{22}^{(k)}\mu_{11}^{(k)}}};$$
$$f_1^{(k)} = -\frac{J_1^{(k)}}{\sqrt{\varepsilon_{11}^{(k)}}}; \;\; f_2^{(k)} = \frac{J_1^{(k)}}{\sqrt{\varepsilon_{11}^{(k)}}}; \;\; f_3^{(k)} = \frac{J_2^{(k)}}{\sqrt{\varepsilon_{22}^{(k)}}}; \;\; f_4^{(k)} = -\frac{J_2^{(k)}}{\sqrt{\varepsilon_{22}^{(k)}}},$$

$u_i^{(k)} = u_i^{(k)}(x_3,t)$, $f_i^{(k)} = f_i^{(k)}(x_3,t)$, $i = 1,2,3,4$; and $k = 1,2,3$ denotes the media.

Initial conditions can be found as:

$$u_i^{(k)}(x_3,0) = 0, \;\; i = 1,2,3,4; \;\; k = 1,2,3. \tag{4.3.2}$$

Matching conditions can be found as:

$$\left.\begin{aligned} u_i^{(1)}(-0,t) &= \sqrt{\mu_{22}^{(1)}}H_2^{(1)}(-0,t) + (-1)^{(i+1)}\sqrt{\varepsilon_{11}^{(1)}}E_1^{(1)}(-0,t), \quad t>0, \;\; i=1,2; \\ u_i^{(1)}(-0,t) &= \sqrt{\mu_{11}^{(1)}}H_1^{(1)}(-0,t) + (-1)^{(i)}\sqrt{\varepsilon_{22}^{(1)}}E_2^{(1)}(-0,t), \quad t>0, \;\; i=3,4. \end{aligned}\right\} \tag{4.3.3}$$

$$\left.\begin{aligned} u_i^{(2)}(+0,t) &= \sqrt{\mu_{22}^{(2)}}H_2^{(2)}(+0,t) + (-1)^{(i+1)}\sqrt{\varepsilon_{11}^{(2)}}E_1^{(2)}(+0,t), \quad t>0, \;\; i=1,2; \\ u_i^{(2)}(+0,t) &= \sqrt{\mu_{11}^{(2)}}H_1^{(2)}(+0,t) + (-1)^{(i)}\sqrt{\varepsilon_{22}^{(2)}}E_2^{(2)}(+0,t), \quad t>0, \;\; i=3,4. \end{aligned}\right\} \tag{4.3.4}$$

$$\left.\begin{aligned} u_i^{(2)}(\ell^-,t) &= \sqrt{\mu_{22}^{(2)}}H_2^{(2)}(\ell^-,t) + (-1)^{(i+1)}\sqrt{\varepsilon_{11}^{(2)}}E_1^{(2)}(\ell^-,t), \quad t>0, \;\; i=1,2; \\ u_i^{(2)}(\ell^-,t) &= \sqrt{\mu_{11}^{(2)}}H_1^{(2)}(\ell^-,t) + (-1)^{(i)}\sqrt{\varepsilon_{22}^{(2)}}E_2^{(2)}(\ell^-,t), \quad t>0, \;\; i=3,4. \end{aligned}\right\} \tag{4.3.5}$$

$$\left.\begin{aligned} u_i^{(3)}(\ell^+,t) &= \sqrt{\mu_{22}^{(3)}}H_2^{(3)}(\ell^+,t) + (-1)^{(i+1)}\sqrt{\varepsilon_{11}^{(3)}}E_1^{(3)}(\ell^+,t), \quad t>0, \;\; i=1,2; \\ u_i^{(3)}(\ell^+,t) &= \sqrt{\mu_{11}^{(3)}}H_1^{(3)}(\ell^+,t) + (-1)^{(i)}\sqrt{\varepsilon_{22}^{(3)}}E_2^{(3)}(\ell^+,t), \quad t>0, \;\; i=3,4. \end{aligned}\right\} \tag{4.3.6}$$

Since we study in three layered media, we will consider three IVPs that are related with each layer. These IVPs consist of (4.3.1), (4.3.2), (4.3.3) for $k = 1$; (4.3.1), (4.3.2), (4.3.4) and (4.3.5) for $k = 2$; and (4.3.1), (4.3.2), (4.3.6) for $k = 3$, respectively, for the first, second and third layer.

4.3.2 Solving Reduced Initial Value Problem For Maxwell's System

Here we solve the reduced initial value problem (IVP) related with each layer. Firstly, we divide these layers into subregions. After that we decide what kind of initial value problem should be considered in these subregions. That is, we should decide if it is necessary to use matching conditions or not. The following step is that we reorganize these subregions by means of the initial value problems. Then we get flats which consist of some of the subregions. In each flat the initial value problems which should be solved are the same kind. And our last step is solving the initial value problems in each flat.

Now, let us divide our media into subregions by means of the characteristic lines related to each layer. How to find these characteristic lines was shown in the last chapters. Figure 4.2 shows us the subregions and the characteristic lines related to each layer in three layered media.

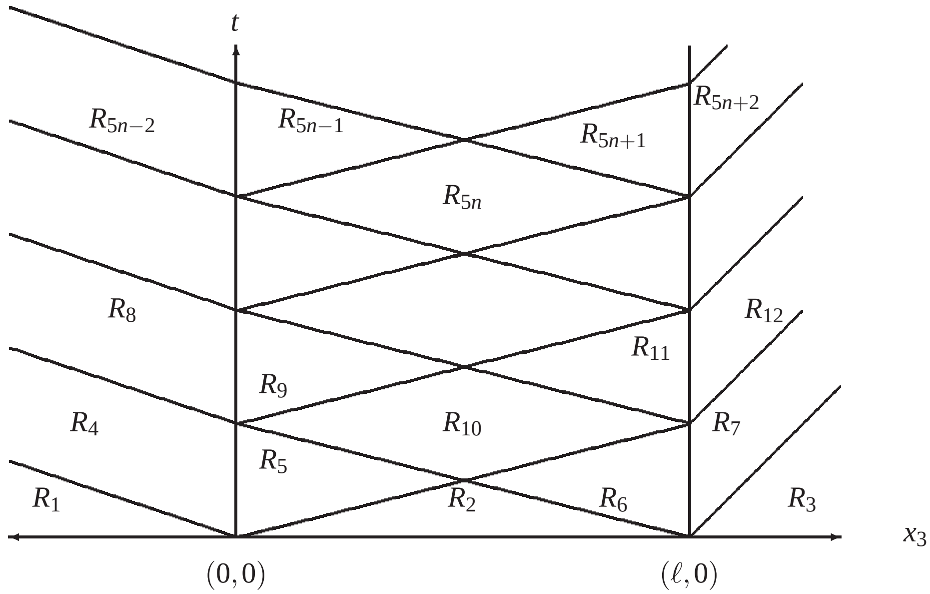


Figure 4.2 Subregions in three layered media.

After naming the subregions let us define our flats by means of them. Our first flat consists of the regions R_1, R_2 and R_3. In this flat we consider an initial value problem without matching conditions in each region. At the second flat we have two initial value problems and one of them is with matching conditions. This flat consists of the regions R_4, R_5, R_6 and R_7. The third flat consists of the regions R_8, R_9, R_{10}, R_{11} and R_{12}. For this flat we do not solve the same kind of initial value problem for all regions. For the regions R_8 and R_{12} we solve two initial value problems and one of them is with matching conditions. For the regions R_9, R_{10} and R_{11} we solve initial value problems all with matching conditions. Now let us define the other flats, that is, the fourth, the fifth and the others. The form of these flats look like the third flat. If we denote the nth flat with $n > 3$ then this flat consists of the regions R_{5n-2}, R_{5n-1}, R_{5n}, R_{5n+1} and R_{5n+2}. And the initial value problems related with each region at the nth flat is in the same form as we did for the third flat.

4.3.2.1 Finding Solution of the IVP at the First Flat

Here we solve an IVP without matching conditions in each region. It was stated that this flat consists of the regions R_1, R_2 and R_3. Firstly we define these regions and then applying a similar process we solve the IVP related with each region.

We define the region R_1 as:

$$R_1 = \{(x_3,t): \quad -\infty < x_3 < -v_i^{(1)}t, \;\; t > 0, \;\; i = 1,2,3,4\}.$$

(Note that $v_1^{(1)} = v_2^{(1)}$ and $v_3^{(1)} = v_4^{(1)}$.) Since the region R_1 is in the first layer we should consider the reduced IVP related with this layer. This IVP consists of the relations (4.3.1) and (4.3.2) for $k = 1$.

Now let us define the region R_2:

$$R_2 = \{(x_3,t): \;\; |x_3 - \frac{\ell}{2}| < -(v_i^{(2)}t - \frac{\ell}{2}), \;\; t > 0, \;\; i = 1,2,3,4\}.$$

Region R_2 lies on the second layer. Hence the IVP related with this region consists of the relations (4.3.1) and (4.3.2) for $k = 2$.

And region R_3 can be defined as:

$$R_3 = \{(x_3,t): \;\; v_i^{(3)}t < x_3 < \infty, \;\; t > 0, \;\; i = 1,2,3,4\}.$$

Since the region R_3 lies on the third layer we consider the relations (4.3.1) and (4.3.2) for $k = 3$, for the IVP in this region.

Then we solve the following IVP for the regions R_1, R_2 and R_3.

$$\frac{\partial u_i^{(k)}(x_3,t)}{\partial t} + (-1)^{i+1} v_i^{(k)} \frac{\partial u_i^{(k)}(x_3,t)}{\partial x_3} = f_i^{(k)}(x_3,t), \;\; i = 1,2,3,4, \;\; k = 1,2,3;$$

$$u_i^{(k)}(x_3,0) = 0, \;\; i = 1,2,3,4, \;\; k = 1,2,3.$$

We use the method of characteristics to solve this IVP. Equation (4.3.1) can be written in terms of ξ and τ as:

$$\frac{\partial u_i^{(k)}(\xi,\tau)}{\partial \tau} + (-1)^{i+1} v_i^{(k)} \frac{\partial u_i^{(k)}(\xi,\tau)}{\partial \xi} = f_i^{(k)}(\xi,\tau), \;\; i = 1,2,3,4, \;\; k = 1,2,3. \tag{4.3.7}$$

Equations for characteristics are:

$$\frac{d\xi}{ds} = (-1)^{i+1} v_i^{(k)},$$
$$\frac{d\tau}{ds} = 1,$$

where $i = 1,2,3,4$ and $k = 1,2,3$.

The equation of the characteristics passing through the point (x_3, t) can be found as:

$$\xi = (-1)^{i+1} v_i^{(k)}(\tau - t) + x_3, \quad i = 1,2,3,4, \quad k = 1,2,3.$$

Equation (4.3.7) along these characteristics can be written in the following form:

$$\frac{du_i^{(k)}((-1)^{i+1} v_i^{(k)}(\tau - t) + x_3, \tau)}{d\tau} = f_i^{(k)}((-1)^{(i+1)} v_i^{(k)}(\tau - t) + x_3, \tau).$$

Integrating last relation with respect to τ from 0 to t we find:

$$u_i^{(k)}(x_3, t) = u_i^{(k)}((-1)^i v_i^{(k)} t + x_3, 0) + \int_0^t f_i^{(k)}((-1)^{(i+1)} v_i^{(k)}(\tau - t) + x_3, \tau) d\tau,$$

where $i = 1,2,3,4$ and $k = 1,2,3$.

Using the initial condition (4.3.2) we find the solution of the reduced IVP at the first flat for the regions R_1, R_2 and R_3 as:

$$u_i^{(k)}(x_3, t) = \int_0^t f_i^{(k)}((-1)^{(i+1)} v_i^{(k)}(\tau - t) + x_3, \tau) d\tau,$$

where $i = 1,2,3,4$ and $k = 1,2,3$.

4.3.2.2 Finding Solution of the IVP at the Second Flat

At this flat, we are solving two IVP's and one of them is with matching conditions. This flat consists of the regions R_4, R_5, R_6, R_7. After some calculations we define the regions in the second flat in the following form:

$$R_4 = \{(x_3, t) : \ -v_i^{(1)} t < x_3 < -v_i^{(1)}(t - \frac{\ell}{v_i^{(2)}}), \ 0 < t < \frac{\ell}{v_i^{(2)}}, \ i = 1,2,3,4\};$$

$$R_5 = \{(x_3, t) : \ 0 < x_3 < \frac{\ell}{2}, \ \frac{x_3}{v_i^{(2)}} < t < \frac{x_3 - \ell}{-v_i^{(2)}}, \ t > 0, \ i = 1,2,3,4\};$$

$$R_6 = \{(x_3, t) : \ \frac{\ell}{2} < x_3 < \ell, \ \frac{x_3 - \ell}{-v_i^{(2)}} < t < \frac{x_3}{v_i^{(2)}}, \ t > 0, \ i = 1,2,3,4\};$$

$$R_7 = \{(x_3, t) : \ \ell < x_3 < \infty, \ \frac{x_3 - \ell}{v_i^{(3)}} < t < \frac{x_3 - \ell}{v_i^{(3)}} + \frac{v_i^{(3)} \ell}{v_i^{(2)}}, \ i = 1,2,3,4\}.$$

Solving IVP in the Region R_4

We have defined this region as:

$$R_4 : -v_i^{(1)} t < x_3 < -v_i^{(1)}(t - \frac{\ell}{v_i^{(2)}}),\ \ 0 < t < \frac{\ell}{v_i^{(2)}},\ \ i = 1,2,3,4;$$

The IVP related to R_4 consists of the relations (4.3.1), (4.3.2) for $k = 1$ and (4.3.3).

Firstly we separate this IVP with matching conditions into two IVP's. For the case $i = 1$ and $i = 3$ we consider an IVP without matching conditions. For the case $i = 2$ and $i = 4$ we have an IVP with matching conditions.

Case1:

Here we consider the case for $i = 1$ and $i = 3$. Applying the same procedure as we did for the region R_1 we can find the solution of the IVP as:

$$u_i^{(1)}(x_3,t) = \int_0^t f_i^{(1)}(v_i^{(1)}(\tau - t) + x_3, \tau)d\tau,\ \ i = 1,3.$$

Case2:

In that case we have an IVP with matching conditions for $i = 2$ and $i = 4$.

Equation of the characteristic lines passing through the point (x_3,t) can be found as:

$$\xi = -v_i^{(1)}(\tau - t) + x_3,\ \ i = 2,4.$$

After that by rewriting equation (4.3.1) for $k = 1$ in terms of ξ and τ and then writing new equation along the characteristic lines we can find the following equation:

$$\frac{du_i^{(1)}(-v_i^{(1)}(\tau - t) + x_3, \tau)}{d\tau} = f_i^{(1)}(-v_i^{(1)}(\tau - t) + x_3, \tau).$$

Then integrating last relation with respect to τ from $(t + \frac{x_3}{v_i^{(1)}})$ to t we find the solution of the IVP for $i = 2,4$ as:

$$u_i^{(1)}(x_3,t) = u_i^{(1)}(-0, t + \frac{x_3}{v_i^{(1)}}) + \int_{t+\frac{x_3}{v_i^{(1)}}}^t f_i^{(1)}(-v_i^{(1)}(\tau - t) + x_3, \tau)d\tau,\ \ i = 2,4.$$

Then the solution of the IVP in the region R_4 can be written as:

$$u_i^{(1)}(x_3,t) = \begin{cases} \int_0^t f_i^{(1)}(v_i^{(1)}(\tau-t)+x_3,\tau)d\tau, & i=1,3; \\ u_i^{(1)}(-0,t+\frac{x_3}{v_i^{(1)}})+\int_{t+\frac{x_3}{v_i^{(1)}}}^t f_i^{(1)}(-v_i^{(1)}(\tau-t)+x_3,\tau)d\tau, & i=2,4. \end{cases}$$

Solving IVP in the Region R_5

We have defined this region as:

$$R_5 : 0 < x_3 < \frac{\ell}{2}, \quad \frac{x_3}{v_i^{(2)}} < t < \frac{x_3-\ell}{-v_i^{(2)}}, \quad t > 0, \quad i = 1,2,3,4;$$

The IVP related to R_5 consists of the relations (4.3.1), (4.3.2) for $k = 2$ and (4.3.4).

By a similar process as we applied in region R_4 we separate the IVP in the region R_5 into two IVPs. But here for the case $i = 1,3$ we have an IVP without matching conditions whereas for the case $i = 2,4$ we have IVP without matching conditions.

Case1:

In this case we solve an IVP for $i = 2,4$. and the solution of this IVP is in the following form:

$$u_i^{(2)}(x_3,t) = \int_0^t f_i^{(2)}(-v_i^{(2)}(\tau-t)+x_3,\tau)d\tau, \quad i = 2,4.$$

Case2:

This case is similar with the Case2 in region R_4. Equation of the characteristic lines passing through the point (x_3,t) can be found in the following form:

$$\xi = v_i^{(2)}(\tau-t)+x_3, \quad i = 1,3.$$

After that by rewriting equation (4.3.1) for $k = 2$ in terms of ξ and τ and then writing new equation along the characteristic lines we can find the following equation:

$$\frac{du_i^{(2)}(v_i^{(2)}(\tau-t)+x_3,\tau)}{d\tau} = f_i^{(2)}(v_i^{(2)}(\tau-t)+x_3,\tau).$$

Then integrating last relation with respect to τ from $(t-\frac{x_3}{v_i^{(2)}})$ to t we find the solution of the IVP for $i=1,3$ as:

$$u_i^{(2)}(x_3,t)=u_i^{(2)}(+0,t-\frac{x_3}{v_i^{(2)}})+\int_{t-\frac{x_3}{v_i^{(2)}}}^{t} f_i^{(2)}(v_i^{(2)}(\tau-t)+x_3,\tau)d\tau,\ \ i=1,3.$$

Then the solution of the IVP in the region R_5 can be written as:

$$u_i^{(2)}(x_3,t)=\begin{cases} u_i^{(2)}(+0,t-\frac{x_3}{v_i^{(2)}})+\int_{t-\frac{x_3}{v_i^{(2)}}}^{t} f_i^{(2)}(v_i^{(2)}(\tau-t)+x_3,\tau)d\tau, & i=1,3; \\ \int_0^t f_i^{(2)}(-v_i^{(2)}(\tau-t)+x_3,\tau)d\tau, & i=2,4. \end{cases}$$

Solving IVP in the Region R_6

We have defined this region as:

$$R_6:\frac{\ell}{2}<x_3<\ell,\ \ \frac{x_3-\ell}{-v_i^{(2)}}<t<\frac{x_3}{v_i^{(2)}},\ \ t>0,\ \ i=1,2,3,4;$$

The IVP related to R_6 consists of the relations (4.3.1), (4.3.2) for $k=2$ and (4.3.5). Firstly we separate this IVP into two cases as we did in the region $R4$.

Case1:

Here we consider an IVP without matching conditions for $i=1$ and $i=3$. And the solution is in the following form:

$$u_i^{(2)}(x_3,t)=\int_0^t f_i^{(2)}(v_i^{(2)}(\tau-t)+x_3,\tau)d\tau,\ \ i=1,3.$$

Case2:

In that case we have an IVP with matching conditions for $i=2$ and $i=4$. Equation of the characteristic lines passing through the point (x_3,t) can be found as:

$$\xi=-v_i^{(2)}(\tau-t)+x_3,\ \ i=2,4.$$

Then integrating the equation (4.3.1) for $k=2$ along these characteristic lines with respect to τ from $(t+\frac{x_3-\ell}{v_i^{(2)}})$ to t we find the solution of the IVP for $i=2,4$ as:

$$u_i^{(2)}(x_3,t)=u_i^{(2)}(\ell^-,t+\frac{x_3-\ell}{v_i^{(2)}})+\int_{t+\frac{x_3-\ell}{v_i^{(2)}}}^{t} f_i^{(2)}(-v_i^{(2)}(\tau-t)+x_3,\tau)d\tau,\ \ i=2,4.$$

Then the solution of the IVP in the region R_6 can be written in the following form:

$$u_i^{(2)}(x_3,t)=\begin{cases}\int_0^t f_i^{(2)}(v_i^{(2)}(\tau-t)+x_3,\tau)d\tau, & i=1,3;\\ u_i^{(2)}(\ell^-,t+\frac{x_3-\ell}{v_i^{(2)}})+\int_{t+\frac{x_3-\ell}{v_i^{(2)}}}^t f_i^{(2)}(-v_i^{(2)}(\tau-t)+x_3,\tau)d\tau, & i=2,4.\end{cases}$$

Solving IVP in the Region R_7

We have defined this region as:

$$R_7:\ell<x_3<\infty,\ \frac{x_3-\ell}{v_i^{(3)}}<t<\frac{x_3-\ell}{v_i^{(3)}}+\frac{v_i^{(3)}\ell}{v_i^{(2)}},\ \ i=1,2,3,4.$$

The IVP related to R_7 consists of the relations (4.3.1), (4.3.2) for $k=3$ and (4.3.6). And solution of this IVP is similar with the IVP in the region R_5.

Case1:

In this case we solve an IVP without matching conditions for $i=2,4$ and the solution of it is in the following form:

$$u_i^{(3)}(x_3,t)=\int_0^t f_i^{(3)}(-v_i^{(3)}(\tau-t)+x_3,\tau)d\tau,\ \ i=2,4.$$

Case2:

Here we have an IVP with matching conditions for $i=1,3$.

Equation of the characteristic lines passing through the point (x_3,t) is in the following form:

$$\xi=v_i^{(3)}(\tau-t)+x_3,\ \ i=1,3.$$

And integrating the equation (4.3.1) for $k=3$ along these characteristic lines with respect to τ from $(t-\frac{x_3-\ell}{v_i^{(3)}})$ to t we find the solution of the IVP for $i=1,3$ as:

$$u_i^{(3)}(x_3,t)=u_i^{(3)}(\ell^+,t-\frac{x_3-\ell}{v_i^{(3)}})+\int_{t-\frac{x_3-\ell}{v_i^{(3)}}}^t f_i^{(3)}(v_i^{(3)}(\tau-t)+x_3,\tau)d\tau,\ \ i=1,3.$$

Then the solution of the IVP in the region R_7 can be written as:

$$u_i^{(3)}(x_3,t)=\begin{cases}u_i^{(3)}(\ell^+,t-\frac{x_3-\ell}{v_i^{(3)}})+\int_{t-\frac{x_3-\ell}{v_i^{(3)}}}^t f_i^{(3)}(v_i^{(3)}(\tau-t)+x_3,\tau)d\tau, & i=1,3;\\ \int_0^t f_i^{(3)}(-v_i^{(3)}(\tau-t)+x_3,\tau)d\tau, & i=2,4.\end{cases}$$

4.3.2.3 Finding Solution of the IVP at the Third Flat

This flat consists of the regions R_8, R_9, R_{10}, R_{11} and R_{12}. For the regions R_8 and R_9 we consider two IVP's and one of them is with matching conditions. For the other regions we have two IVP's with matching conditions. After some calculations we define the regions in the third flat in the following form:

$$R_8 = \left\{ (x_3,t): \quad -\frac{x_3}{v_i^{(1)}} + \frac{\ell}{v_i^{(2)}} < t < -\frac{x_3}{v_i^{(1)}} + \frac{2\ell}{v_i^{(2)}}, \quad -\infty < x_3 < 0, \ \ i = 1,2,3,4 \right\};$$

$$R_9 = \left\{ (x_3,t): \quad \frac{x_3}{v_i^{(2)}} + \frac{\ell}{v_i^{(2)}} < t < -\frac{x_3}{v_i^{(2)}} + \frac{2\ell}{v_i^{(2)}}, \ \ 0 < x_3 < \frac{\ell}{2}, \ \ i = 1,2,3,4 \right\};$$

$$R_{10} = \left\{ (x_3,t): \ \ \frac{x_3-\ell}{-v_i^{(2)}} < t < \frac{x_3}{v_i^{(2)}} + \frac{\ell}{v_i^{(2)}}, \ \ 0 < x_3 < \frac{\ell}{2} \ \ \textit{and} \right.$$

$$\left. \frac{x_3}{v_i^{(2)}} < t < -\frac{x_3}{v_i^{(2)}} + \frac{2\ell}{v_i^{(2)}}, \ \ \frac{\ell}{2} < x_3 < \ell, \ \ i = 1,2,3,4 \right\};$$

$$R_{11} = \left\{ (x_3,t): \quad -\frac{x_3}{v_i^{(2)}} + \frac{2\ell}{v_i^{(2)}} < t < \frac{x_3}{v_i^{(2)}} + \frac{\ell}{v_i^{(2)}}, \ \ \frac{\ell}{2} < x_3 < \ell, \ \ i = 1,2,3,4 \right\};$$

$$R_{12} = \left\{ (x_3,t): \ \ \frac{x_3-\ell}{v_i^{(3)}} + \frac{\ell}{v_i^{(2)}} < t < \frac{x_3-\ell}{v_i^{(3)}} + \frac{2\ell}{v_i^{(2)}}, \ \ \ell < x_3 < +\infty, \ \ i = 1,2,3,4 \right\}.$$

Solving IVP in the Region R_8

The IVP related to R_8 and the form of the solution of this problem is the same with the IVP related to R_4.

Then the solution of the IVP for the region R_8 can be found in the following form:

$$u_i^{(1)}(x_3,t) = \begin{cases} \int_0^t f_i^{(1)}(v_i^{(1)}(\tau - t) + x_3, \tau)d\tau, & i = 1,3; \\ u_i^{(1)}(-0, t + \frac{x_3}{v_i^{(1)}}) + \int_{t+\frac{x_3}{v_i^{(1)}}}^t f_i^{(1)}(-v_i^{(1)}(\tau - t) + x_3, \tau)d\tau, & i = 2,4. \end{cases}$$

The difference between the solution of the IVP's for the regions R_4 and R_8 is that to find the matching condition related to the region R_8 we consider the solution of the IVP in the region R_9 whereas for the matching condition related to R_4 we consider the solution of the IVP for the region R_5.

Solving IVP in the Region R_9

The IVP related to R_9 is the same with the one we considered in the region R_5. And the solution steps are similar. But, here for both cases we have IVP with matching conditions.

Case1:

Here we consider the case for $i=2,4$. The characteristics lines passing through the point (x_3,t) can be found in the following form:

$$\xi=-v_i^{(2)}(\tau-t)+x_3,\ \ i=2,4.$$

And integrating the equation (4.3.1) for $k=2$ along the characteristics with respect to τ from $(t+\frac{x_3-\ell}{v_i^{(2)}})$ to t we find the solution of the IVP in the following form:

$$u_i^{(2)}(x_3,t)=u_i^{(2)}(\ell^-,t+\frac{x_3-\ell}{v_i^{(2)}})+\int_{t+\frac{x_3-\ell}{v_i^{(2)}}}^{t} f_i^{(2)}(-v_i^{(2)}(\tau-t)+x_3,\tau)d\tau,\ \ i=2,4.$$

Case2:

The form of the solution of the IVP in that case is the same with the Case2 in the region R_5.

Then the solution of the IVP in region R_9 can be written in the following form:

$$u_i^{(2)}(x_3,t)=\begin{cases} u_i^{(2)}(+0,t-\frac{x_3}{v_i^{(2)}})+\int_{t-\frac{x_3}{v_i^{(2)}}}^{t} f_i^{(2)}(v_i^{(2)}(\tau-t)+x_3,\tau)d\tau, & i=1,3; \\ u_i^{(2)}(\ell^-,t+\frac{x_3}{v_i^{(2)}})+\int_{t+\frac{x_3-\ell}{v_i^{(2)}}}^{t} f_i^{(2)}(-v_i^{(2)}(\tau-t)+x_3,\tau)d\tau, & i=2,4. \end{cases}$$

The matching condition $u_i^{(2)}(+0,t-\frac{x_3}{v_i^{(2)}})$ for $i=1,3$ can be derived by means of the IVP in the region R_8 and the other matching condition $u_i^{(1)}(\ell^-,t+\frac{x_3}{v_i^{(2)}})$ can be derived by means of the IVP in the region R_7.

Solving IVP in the Region R_{10}

The IVP related to R_{10} and the form of the solution is the same with the IVP in region R_9. And it is in the following form:

$$u_i^{(2)}(x_3,t)=\begin{cases} u_i^{(2)}(+0,t-\frac{x_3}{v_i^{(2)}})+\int_{t-\frac{x_3}{v_i^{(2)}}}^{t} f_i^{(2)}(v_i^{(2)}(\tau-t)+x_3,\tau)d\tau, & i=1,3; \\ u_i^{(2)}(\ell^-,t+\frac{x_3-\ell}{v_i^{(2)}})+\int_{t+\frac{x_3-\ell}{v_i^{(2)}}}^{t} f_i^{(2)}(-v_i^{(2)}(\tau-t)+x_3,\tau)d\tau, & i=2,4. \end{cases}$$

The difference between the solutions of IVPs in this region and the region R_9 is that here, we derive the matching condition $u_i^{(2)}(+0,t-\frac{x_3}{v_i^{(2)}})$ for $i=1,3$ by means of the IVP in the region R_4.

Solving IVP in the Region R_{11}

The IVP and the form of the solution of the IVP related with this region is the same with the region R_9 and R_{10}. Only difference is that when we derive matching condition $u_i^{(2)}(+0, t - \frac{x_3}{v_i^{(2)}})$ for $i = 1,3$ we consider the solution of the IVP in the region R_4 and for the matching condition $u_i^{(2)}(\ell^-, t + \frac{x_3-\ell}{v_i^{(2)}})$ for $i = 2,4$ we consider the solution of the IVP in the region R_{12}.

Solving IVP in the Region R_{12}

The IVP related to R_{12} and the form of the solution of this problem is the same with the IVP in the region R_7. And the form of the solution can be written as:

$$u_i^{(3)}(x_3,t) = \begin{cases} u_i^{(3)}(\ell^+, t - \frac{x_3-\ell}{v_i^{(3)}}) + \int_{t-\frac{x_3-\ell}{v_i^{(3)}}}^{t} f_i^{(3)}(v_i^{(3)}(\tau - t) + x_3, \tau)d\tau, & i = 1,3; \\ \int_0^t f_i^{(3)}(-v_i^{(3)}(\tau - t) + x_3, \tau)d\tau, & i = 2,4. \end{cases}$$

Only difference is that here we derive the matching condition $u_i^{(3)}(\ell^+, t - \frac{x_3-\ell}{v_i^{(3)}})$ for $i = 1,3$ by means of the IVP in the region R_{11}.

4.3.3 Deriving Matching Conditions

Here we are deriving the matching conditions which are necessary to find explicit formulae for the electric and magnetic field vectors. For the first three flat the solutions of the IVPs are found but the values of the matching conditions have not be derived. For the first flat we solved IVPs without matching conditions. But for the second and third flat we also solved IVPs with matching conditions. Hence, now we derive these matching conditions.

4.3.3.1 Deriving Matching Conditions at the Second Flat

Finding Matching Conditions For Region R_4

In order to get explicit formulas of the solution of the problem in this region, we need to find the values $u_i^{(1)}(-0, t + \frac{x_3}{v_i^{(1)}})$ for $i = 2,4$ by means of the solution of the IVP in region R_5. For two layered media these values were defined and they are exactly in the same form what we are looking for.

Now let us write them.

$$u_2^{(1)}(-0,t)=\frac{\sqrt{\mu_{22}^{(1)}\varepsilon_{11}^{(2)}}-\sqrt{\mu_{22}^{(2)}\varepsilon_{11}^{(1)}}}{\sqrt{\mu_{22}^{(1)}\varepsilon_{11}^{(2)}}+\sqrt{\mu_{22}^{(2)}\varepsilon_{11}^{(1)}}}u_1^{(1)}(-0,t)+\frac{2\sqrt{\mu_{22}^{(1)}\varepsilon_{11}^{(1)}}}{\sqrt{\mu_{22}^{(1)}\varepsilon_{11}^{(2)}}+\sqrt{\mu_{22}^{(2)}\varepsilon_{11}^{(1)}}}u_2^{(2)}(+0,t).$$

Then substituting the values $u_1^{(1)}(-0,t)$ and $u_2^{(2)}(+0,t)$ we get the matching condition $u_2^{(1)}(-0,t)$ as:

$$\begin{aligned} u_2^{(1)}(-0,t) &= \frac{\sqrt{\mu_{22}^{(1)}}\cdot\sqrt{\varepsilon_{11}^{(2)}}-\sqrt{\mu_{22}^{(2)}}\cdot\sqrt{\varepsilon_{11}^{(1)}}}{\sqrt{\mu_{22}^{(1)}}\cdot\sqrt{\varepsilon_{11}^{(2)}}+\sqrt{\mu_{22}^{(2)}}\cdot\sqrt{\varepsilon_{11}^{(1)}}}\cdot\frac{(-1)}{\sqrt{\varepsilon_{11}^{(1)}}}\int_0^t J_1^{(1)}(\nu_1^{(1)}(\tau-t),\tau)d\tau \\ &+ \frac{2\sqrt{\mu_{22}^{(1)}}\cdot\sqrt{\varepsilon_{11}^{(1)}}}{\sqrt{\mu_{22}^{(1)}}\cdot\sqrt{\varepsilon_{11}^{(2)}}+\sqrt{\mu_{22}^{(2)}}\cdot\sqrt{\varepsilon_{11}^{(1)}}}\cdot\frac{1}{\sqrt{\varepsilon_{11}^{(2)}}}\int_0^t J_1^{(2)}(-\nu_2^{(2)}(\tau-t),\tau)d\tau. \end{aligned}$$

And the matching condition $u_4^{(1)}(-0,t)$ was in the following form:

$$u_4^{(1)}(-0,t)=\frac{\sqrt{\mu_{11}^{(1)}\varepsilon_{22}^{(2)}}-\sqrt{\mu_{11}^{(2)}\varepsilon_{22}^{(1)}}}{\sqrt{\mu_{11}^{(1)}\varepsilon_{22}^{(2)}}+\sqrt{\mu_{11}^{(2)}\varepsilon_{22}^{(1)}}}u_3^{(1)}(-0,t)+\frac{2\sqrt{\mu_{11}^{(1)}\varepsilon_{22}^{(1)}}}{\sqrt{\mu_{11}^{(1)}\varepsilon_{22}^{(2)}}+\sqrt{\mu_{11}^{(2)}\varepsilon_{22}^{(1)}}}u_4^{(2)}(+0,t).$$

Substituting the values $u_3^{(1)}(-0,t)$ and $u_4^{(2)}(+0,t)$ into the relation above, the matching condition $u_4^{(1)}(-0,t)$ can be found as:

$$\begin{aligned} u_4^{(1)}(-0,t) &= \frac{\sqrt{\mu_{11}^{(1)}\varepsilon_{22}^{(2)}}-\sqrt{\mu_{11}^{(2)}\varepsilon_{22}^{(1)}}}{\sqrt{\mu_{11}^{(1)}\varepsilon_{22}^{(2)}}+\sqrt{\mu_{11}^{(2)}\varepsilon_{22}^{(1)}}}\cdot\frac{1}{\sqrt{\varepsilon_{22}^{(1)}}}\int_0^t J_2^{(1)}(\nu_3^{(1)}(\tau-t),\tau)d\tau \\ &+ \frac{2\sqrt{\mu_{11}^{(1)}\varepsilon_{22}^{(1)}}}{\sqrt{\mu_{11}^{(1)}\varepsilon_{22}^{(2)}}+\sqrt{\mu_{11}^{(2)}\varepsilon_{22}^{(1)}}}\cdot\frac{(-1)}{\sqrt{\varepsilon_{22}^{(2)}}}\int_0^t J_2^{(2)}(-\nu_4^{(2)}(\tau-t),\tau)d\tau. \end{aligned}$$

Finding Matching Conditions For Region R_5

In order to get explicit formulas of the solution of the problem in this region, we need to find the values $u_i^{(2)}(+0,t-\frac{x_3}{\nu_i^{(2)}})$ for $i=1,3$ by means of the solution of the IVP in region R_4. For two layered media these values have been found and they were in the following form:

$$u_1^{(2)}(+0,t)=\frac{2\sqrt{\mu_{22}^{(2)}\varepsilon_{11}^{(2)}}}{\sqrt{\mu_{22}^{(2)}\varepsilon_{11}^{(1)}}+\sqrt{\mu_{22}^{(1)}\varepsilon_{11}^{(2)}}}u_1^{(1)}(-0,t)+\frac{\sqrt{\mu_{22}^{(2)}\varepsilon_{11}^{(1)}}-\sqrt{\mu_{22}^{(1)}\varepsilon_{11}^{(2)}}}{\sqrt{\mu_{22}^{(2)}\varepsilon_{11}^{(1)}}+\sqrt{\mu_{22}^{(1)}\varepsilon_{11}^{(2)}}}u_2^{(2)}(+0,t).$$

Then substituting the values $u_1^{(1)}(-0,t)$ and $u_2^{(2)}(+0,t)$ we find the matching condition $u_1^{(2)}(+0,t)$

as:

$$u_1^{(2)}(+0,t) = -\frac{2\sqrt{\mu_{22}^{(2)}\varepsilon_{11}^{(2)}}}{\sqrt{\varepsilon_{11}^{(1)}}(\sqrt{\mu_{22}^{(2)}\varepsilon_{11}^{(1)}}+\sqrt{\mu_{22}^{(1)}\varepsilon_{11}^{(2)}})}\int_0^t J_1^{(1)}(v_1^{(1)}(\tau-t),\tau)d\tau + \frac{\sqrt{\mu_{22}^{(2)}\varepsilon_{11}^{(1)}}-\sqrt{\mu_{22}^{(1)}\varepsilon_{11}^{(2)}}}{\sqrt{\varepsilon_{11}^{(2)}}(\sqrt{\mu_{22}^{(2)}\varepsilon_{11}^{(1)}}+\sqrt{\mu_{22}^{(1)}\varepsilon_{11}^{(2)}})}\int_0^t J_1^{(2)}(-v_1^{(2)}(\tau-t),\tau)d\tau.$$

And $u_3^{(2)}(+0,t)$ was in the following form:

$$u_3^{(2)}(+0,t) = \frac{2\sqrt{\mu_{11}^{(2)}\varepsilon_{22}^{(2)}}}{\sqrt{\mu_{11}^{(2)}\varepsilon_{22}^{(1)}}+\sqrt{\mu_{11}^{(1)}\varepsilon_{22}^{(2)}}}u_3^{(1)}(-0,t)+\frac{\sqrt{\mu_{11}^{(2)}\varepsilon_{22}^{(1)}}-\sqrt{\mu_{11}^{(1)}\varepsilon_{22}^{(2)}}}{\sqrt{\mu_{11}^{(2)}\varepsilon_{22}^{(1)}}+\sqrt{\mu_{11}^{(1)}\varepsilon_{22}^{(2)}}}u_4^{(2)}(+0,t).$$

Then substituting the values $u_3^{(1)}(-0,t)$ and $u_4^{(2)}(+0,t)$ we find the matching condition $u_3^{(2)}(+0,t)$ as:

$$u_3^{(2)}(+0,t) = \frac{2\sqrt{\mu_{11}^{(2)}\varepsilon_{22}^{(2)}}}{\sqrt{\varepsilon_{22}^{(1)}}(\sqrt{\mu_{11}^{(2)}\varepsilon_{22}^{(1)}}+\sqrt{\mu_{11}^{(1)}\varepsilon_{22}^{(2)}})}\int_0^t J_2^{(1)}(v_3^{(1)}(\tau-t),\tau)d\tau - \frac{\sqrt{\mu_{11}^{(2)}\varepsilon_{22}^{(1)}}-\sqrt{\mu_{11}^{(1)}\varepsilon_{22}^{(2)}}}{\sqrt{\varepsilon_{22}^{(2)}}(\sqrt{\mu_{11}^{(2)}\varepsilon_{22}^{(1)}}+\sqrt{\mu_{11}^{(1)}\varepsilon_{22}^{(2)}})}\int_0^t J_2^{(2)}(-v_3^{(2)}(\tau-t),\tau)d\tau.$$

Finding Matching Conditions For Region R_6

In order to get explicit formulas in this region, the matching conditions $u_i^{(2)}(\ell^-,t+\frac{x_3-\ell}{v_i^{(2)}})$ for $i=2,4$ should be derived by means of the IVP in region R_7. Firstly let us write (4.3.5) and (4.3.6) in the matrix form:

$$\begin{pmatrix} u_1^{(2)}(\ell^-,t) \\ u_2^{(2)}(\ell^-,t)\end{pmatrix} = \begin{pmatrix} \sqrt{\mu_{22}^{(2)}} & \sqrt{\varepsilon_{11}^{(2)}} \\ \sqrt{\mu_{22}^{(2)}} & -\sqrt{\varepsilon_{11}^{(2)}}\end{pmatrix}\begin{pmatrix} H_2^{(2)}(\ell^-,t) \\ E_1^{(2)}(\ell^-,t)\end{pmatrix},$$

$$\begin{pmatrix} u_3^{(2)}(\ell^-,t) \\ u_4^{(2)}(\ell^-,t)\end{pmatrix} = \begin{pmatrix} \sqrt{\mu_{11}^{(2)}} & -\sqrt{\varepsilon_{22}^{(2)}} \\ \sqrt{\mu_{11}^{(2)}} & \sqrt{\varepsilon_{22}^{(2)}}\end{pmatrix}\begin{pmatrix} H_1^{(2)}(\ell^-,t) \\ E_2^{(2)}(\ell^-,t)\end{pmatrix},$$

$$\begin{pmatrix} u_1^{(3)}(\ell^+,t) \\ u_2^{(3)}(\ell^+,t)\end{pmatrix} = \begin{pmatrix} \sqrt{\mu_{22}^{(3)}} & \sqrt{\varepsilon_{11}^{(3)}} \\ \sqrt{\mu_{22}^{(3)}} & -\sqrt{\varepsilon_{11}^{(3)}}\end{pmatrix}\begin{pmatrix} H_2^{(3)}(\ell^+,t) \\ E_1^{(3)}(\ell^+,t)\end{pmatrix},$$

$$\begin{pmatrix} u_3^{(3)}(\ell^+,t) \\ u_4^{(3)}(\ell^+,t)\end{pmatrix} = \begin{pmatrix} \sqrt{\mu_{11}^{(3)}} & -\sqrt{\varepsilon_{22}^{(3)}} \\ \sqrt{\mu_{11}^{(3)}} & \sqrt{\varepsilon_{22}^{(3)}}\end{pmatrix}\begin{pmatrix} H_1^{(3)}(\ell^+,t) \\ E_2^{(3)}(\ell^+,t)\end{pmatrix}.$$

And using the condition (4.2.2) we get the following relations:

$$\begin{aligned} H_1^{(2)}(\ell^-,t) &= H_1^{(3)}(\ell^+,t), \\ H_2^{(2)}(\ell^-,t) &= H_2^{(3)}(\ell^+,t), \\ E_1^{(2)}(\ell^-,t) &= E_1^{(3)}(\ell^+,t), \\ H_2^{(2)}(\ell^-,t) &= H_2^{(3)}(\ell^+,t). \end{aligned}$$

Using these relations above we get the following equalities:

$$\begin{pmatrix} u_1^{(2)}(\ell^-,t) \\ u_2^{(2)}(\ell^-,t) \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{\mu_{22}^{(2)}\varepsilon_{11}^{(3)}}+\sqrt{\mu_{22}^{(3)}\varepsilon_{11}^{(2)}}}{2\sqrt{\mu_{22}^{(3)}\varepsilon_{11}^{(3)}}} & \frac{\sqrt{\mu_{22}^{(2)}\varepsilon_{11}^{(3)}}-\sqrt{\mu_{22}^{(3)}\varepsilon_{11}^{(2)}}}{2\sqrt{\mu_{22}^{(3)}\varepsilon_{11}^{(3)}}} \\ \frac{\sqrt{\mu_{22}^{(2)}\varepsilon_{11}^{(3)}}-\sqrt{\mu_{22}^{(3)}\varepsilon_{11}^{(2)}}}{2\sqrt{\mu_{22}^{(3)}\varepsilon_{11}^{(3)}}} & \frac{\sqrt{\mu_{22}^{(2)}\varepsilon_{11}^{(3)}}+\sqrt{\mu_{22}^{(3)}\varepsilon_{11}^{(2)}}}{2\sqrt{\mu_{22}^{(3)}\varepsilon_{11}^{(3)}}} \end{pmatrix} \begin{pmatrix} u_1^{(3)}(\ell^+,t) \\ u_2^{(3)}(\ell^+,t) \end{pmatrix}.$$

$$\begin{pmatrix} u_3^{(2)}(\ell^-,t) \\ u_4^{(2)}(\ell^-,t) \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{\mu_{11}^{(2)}\varepsilon_{22}^{(3)}}+\sqrt{\mu_{11}^{(3)}\varepsilon_{22}^{(2)}}}{2\sqrt{\mu_{11}^{(3)}\varepsilon_{22}^{(3)}}} & \frac{\sqrt{\mu_{11}^{(2)}\varepsilon_{22}^{(3)}}-\sqrt{\mu_{11}^{(3)}\varepsilon_{22}^{(2)}}}{2\sqrt{\mu_{11}^{(3)}\varepsilon_{22}^{(3)}}} \\ \frac{\sqrt{\mu_{11}^{(2)}\varepsilon_{22}^{(3)}}-\sqrt{\mu_{11}^{(3)}\varepsilon_{22}^{(2)}}}{2\sqrt{\mu_{11}^{(3)}\varepsilon_{22}^{(3)}}} & \frac{\sqrt{\mu_{11}^{(2)}\varepsilon_{22}^{(3)}}+\sqrt{\mu_{11}^{(3)}\varepsilon_{22}^{(2)}}}{2\sqrt{\mu_{11}^{(3)}\varepsilon_{22}^{(3)}}} \end{pmatrix} \begin{pmatrix} u_3^{(3)}(\ell^+,t) \\ u_4^{(3)}(\ell^+,t) \end{pmatrix}.$$

Then by the relations above we get the following equalities:

$$u_1^{(2)}(\ell^-,t) = \frac{\sqrt{\mu_{22}^{(2)}\varepsilon_{11}^{(3)}}+\sqrt{\mu_{22}^{(3)}\varepsilon_{11}^{(2)}}}{2\sqrt{\mu_{22}^{(3)}\varepsilon_{11}^{(3)}}} u_1^{(3)}(\ell^+,t) + \frac{\sqrt{\mu_{22}^{(2)}\varepsilon_{11}^{(3)}}-\sqrt{\mu_{22}^{(3)}\varepsilon_{11}^{(2)}}}{2\sqrt{\mu_{22}^{(3)}\varepsilon_{11}^{(3)}}} u_2^{(3)}(\ell^+,t);$$

$$u_2^{(2)}(\ell^-,t) = \frac{\sqrt{\mu_{22}^{(2)}\varepsilon_{11}^{(3)}}-\sqrt{\mu_{22}^{(3)}\varepsilon_{11}^{(2)}}}{2\sqrt{\mu_{22}^{(3)}\varepsilon_{11}^{(3)}}} u_1^{(3)}(\ell^+,t) + \frac{\sqrt{\mu_{22}^{(2)}\varepsilon_{11}^{(3)}}+\sqrt{\mu_{22}^{(3)}\varepsilon_{11}^{(2)}}}{2\sqrt{\mu_{22}^{(3)}\varepsilon_{11}^{(3)}}} u_2^{(3)}(\ell^+,t);$$

$$u_3^{(2)}(\ell^-,t) = \frac{\sqrt{\mu_{11}^{(2)}\varepsilon_{22}^{(3)}}+\sqrt{\mu_{11}^{(3)}\varepsilon_{22}^{(2)}}}{2\sqrt{\mu_{11}^{(3)}\varepsilon_{22}^{(3)}}} u_3^{(3)}(\ell^+,t) + \frac{\sqrt{\mu_{11}^{(2)}\varepsilon_{22}^{(3)}}-\sqrt{\mu_{11}^{(3)}\varepsilon_{22}^{(2)}}}{2\sqrt{\mu_{11}^{(3)}\varepsilon_{22}^{(3)}}} u_4^{(3)}(\ell^+,t);$$

$$u_4^{(2)}(\ell^-,t) = \frac{\sqrt{\mu_{11}^{(2)}\varepsilon_{22}^{(3)}}-\sqrt{\mu_{11}^{(3)}\varepsilon_{22}^{(2)}}}{2\sqrt{\mu_{11}^{(3)}\varepsilon_{22}^{(3)}}} u_3^{(3)}(\ell^+,t) + \frac{\sqrt{\mu_{11}^{(2)}\varepsilon_{22}^{(3)}}+\sqrt{\mu_{11}^{(3)}\varepsilon_{22}^{(2)}}}{2\sqrt{\mu_{11}^{(3)}\varepsilon_{22}^{(3)}}} u_4^{(3)}(\ell^+,t).$$

Then using the relations above we derive the matching conditions $u_2^{(2)}(\ell^-,t)$ and $u_4^{(2)}(\ell^-,t)$. Firstly let us write $u_2^{(2)}(\ell^-,t)$:

$$u_2^{(2)}(\ell^-,t) = \frac{\sqrt{\mu_{22}^{(2)}\varepsilon_{11}^{(3)}}-\sqrt{\mu_{22}^{(3)}\varepsilon_{11}^{(2)}}}{\sqrt{\mu_{22}^{(2)}\varepsilon_{11}^{(3)}}+\sqrt{\mu_{22}^{(3)}\varepsilon_{11}^{(2)}}} u_1^{(2)}(\ell^-,t) + \frac{2\sqrt{\mu_{22}^{(2)}\varepsilon_{11}^{(2)}}}{\sqrt{\mu_{22}^{(2)}\varepsilon_{11}^{(3)}}+\sqrt{\mu_{22}^{(3)}\varepsilon_{11}^{(2)}}} u_2^{(3)}(\ell^+,t),$$

Here $u_1^{(2)}(\ell^-,t)$ is related with region R_6 and $u_2^{(3)}(\ell^+,t)$ is related with region R_7. Substituting these values we get the matching condition $u_2^{(2)}(\ell^-,t)$ as:

$$\begin{aligned} u_2^{(2)}(\ell^-,t) &= -\frac{\sqrt{\mu_{22}^{(2)}\varepsilon_{11}^{(3)}}-\sqrt{\mu_{22}^{(3)}\varepsilon_{11}^{(2)}}}{\sqrt{\varepsilon_{11}^{(2)}}(\sqrt{\mu_{22}^{(2)}\varepsilon_{11}^{(3)}}+\sqrt{\mu_{22}^{(3)}\varepsilon_{11}^{(2)}})}\int_0^t J_1^{(2)}(v_1^{(2)}(\tau-t)+\ell,\tau)d\tau \\ &+ \frac{2\sqrt{\mu_{22}^{(2)}\varepsilon_{11}^{(2)}}}{\sqrt{\varepsilon_{11}^{(3)}}(\sqrt{\mu_{22}^{(2)}\varepsilon_{11}^{(3)}}+\sqrt{\mu_{22}^{(3)}\varepsilon_{11}^{(2)}})}\int_0^t J_1^{(2)}(v_1^{(2)}(\tau-t)+\ell,\tau)d\tau. \end{aligned}$$

And the matching condition $u_4^{(2)}(\ell^-,t)$ has been found as:

$$u_4^{(2)}(\ell^-,t)=\frac{\sqrt{\mu_{11}^{(2)}\varepsilon_{22}^{(3)}}-\sqrt{\mu_{11}^{(3)}\varepsilon_{22}^{(2)}}}{\sqrt{\mu_{11}^{(2)}\varepsilon_{22}^{(3)}}+\sqrt{\mu_{11}^{(3)}\varepsilon_{22}^{(2)}}}u_3^{(2)}(\ell^-,t)+\frac{2\sqrt{\mu_{11}^{(2)}\varepsilon_{22}^{(2)}}}{\sqrt{\mu_{11}^{(2)}\varepsilon_{22}^{(3)}}+\sqrt{\mu_{11}^{(3)}\varepsilon_{22}^{(2)}}}u_4^{(3)}(\ell^+,t),$$

The values $u_3^{(2)}(\ell^-,t)$ is related with region R_6 and $u_4^{(3)}(\ell^+,t)$ is related with region R_7. Substituting these values we get the matching condition $u_4^{(2)}(\ell^-,t)$ as:

$$\begin{aligned} u_4^{(2)}(\ell^-,t) &= \frac{\sqrt{\mu_{11}^{(2)}\varepsilon_{22}^{(3)}}-\sqrt{\mu_{11}^{(3)}\varepsilon_{22}^{(2)}}}{\sqrt{\varepsilon_{22}^{(2)}}(\sqrt{\mu_{11}^{(2)}\varepsilon_{22}^{(3)}}+\sqrt{\mu_{11}^{(3)}\varepsilon_{22}^{(2)}})}\int_0^t J_2^{(2)}(v_3^{(2)}(\tau-t)+\ell,\tau)d\tau \\ &- \frac{2\sqrt{\mu_{11}^{(2)}\varepsilon_{22}^{(2)}}}{\sqrt{\varepsilon_{22}^{(3)}}(\sqrt{\mu_{11}^{(2)}\varepsilon_{22}^{(3)}}+\sqrt{\mu_{11}^{(3)}\varepsilon_{22}^{(2)}})}\int_0^t J_2^{(3)}(-v_4^{(3)}(\tau-t)+\ell,\tau)d\tau. \end{aligned}$$

Finding Matching Conditions For Region R_7

In order to get explicit formulas in this region we should find the matching conditions $u_i^{(3)}(\ell^+,t-\frac{x_3-\ell}{v_i^{(3)}})$ for $i=1,3$ by means of the IVP in region R_6.
After writing (4.3.5) and (4.3.6) in the matrix form and using some of the relations of (4.2.2) we get the following equalities:

$$u_1^{(3)}(\ell^+,t)=\frac{\sqrt{\mu_{22}^{(3)}\varepsilon_{11}^{(2)}}+\sqrt{\mu_{22}^{(2)}\varepsilon_{11}^{(3)}}}{2\sqrt{\mu_{22}^{(2)}\varepsilon_{11}^{(2)}}}u_1^{(2)}(\ell^-,t)+\frac{\sqrt{\mu_{22}^{(3)}\varepsilon_{11}^{(2)}}-\sqrt{\mu_{22}^{(2)}\varepsilon_{11}^{(3)}}}{2\sqrt{\mu_{22}^{(2)}\varepsilon_{11}^{(2)}}}u_2^{(2)}(\ell^-,t);$$

$$u_2^{(3)}(\ell^+,t)=\frac{\sqrt{\mu_{22}^{(3)}\varepsilon_{11}^{(2)}}-\sqrt{\mu_{22}^{(2)}\varepsilon_{11}^{(3)}}}{2\sqrt{\mu_{22}^{(2)}\varepsilon_{11}^{(2)}}}u_1^{(2)}(\ell^-,t)+\frac{\sqrt{\mu_{22}^{(3)}\varepsilon_{11}^{(2)}}+\sqrt{\mu_{22}^{(2)}\varepsilon_{11}^{(3)}}}{2\sqrt{\mu_{22}^{(2)}\varepsilon_{11}^{(2)}}}u_2^{(2)}(\ell^-,t);$$

$$u_3^{(3)}(\ell^+,t)=\frac{\sqrt{\mu_{11}^{(3)}\varepsilon_{22}^{(2)}}+\sqrt{\mu_{11}^{(2)}\varepsilon_{22}^{(3)}}}{2\sqrt{\mu_{11}^{(2)}\varepsilon_{22}^{(2)}}}u_3^{(2)}(\ell^-,t)+\frac{\sqrt{\mu_{11}^{(3)}\varepsilon_{22}^{(2)}}-\sqrt{\mu_{11}^{(2)}\varepsilon_{22}^{(3)}}}{2\sqrt{\mu_{11}^{(2)}\varepsilon_{22}^{(2)}}}u_4^{(2)}(\ell^-,t);$$

$$u_4^{(3)}(\ell^+,t) = \frac{\sqrt{\mu_{11}^{(3)}\varepsilon_{22}^{(2)}} - \sqrt{\mu_{11}^{(2)}\varepsilon_{22}^{(3)}}}{2\sqrt{\mu_{11}^{(2)}\varepsilon_{22}^{(2)}}} u_3^{(2)}(\ell^-,t) + \frac{\sqrt{\mu_{11}^{(3)}\varepsilon_{22}^{(2)}} + \sqrt{\mu_{11}^{(2)}\varepsilon_{22}^{(3)}}}{2\sqrt{\mu_{11}^{(2)}\varepsilon_{22}^{(2)}}} u_4^{(2)}(\ell^-,t).$$

Then using the relations above we derive the matching conditions $u_1^{(3)}(\ell^+,t)$ and $u_3^{(3)}(\ell^+,t)$. Firstly let us write $u_1^{(3)}(\ell^+,t)$:

$$u_1^{(3)}(\ell^+,t) = \frac{2\sqrt{\mu_{22}^{(3)}\varepsilon_{11}^{(3)}}}{\sqrt{\mu_{22}^{(3)}\varepsilon_{11}^{(2)}} + \sqrt{\mu_{22}^{(2)}\varepsilon_{11}^{(3)}}} u_1^{(2)}(\ell^-,t) + \frac{\sqrt{\mu_{22}^{(3)}\varepsilon_{11}^{(2)}} - \sqrt{\mu_{22}^{(2)}\varepsilon_{11}^{(3)}}}{\sqrt{\mu_{22}^{(3)}\varepsilon_{11}^{(2)}} + \sqrt{\mu_{22}^{(2)}\varepsilon_{11}^{(3)}}} u_2^{(3)}(\ell^+,t),$$

Here $u_1^{(2)}(\ell^-,t)$ is related with region R_6 and $u_2^{(3)}(\ell^+,t)$ is related with region R_7. Substituting these values we get the matching condition $u_1^{(3)}(\ell^+,t)$ as:

$$\begin{aligned} u_1^{(3)}(\ell^+,t) &= -\frac{2\sqrt{\mu_{22}^{(3)}\varepsilon_{11}^{(3)}}}{\sqrt{\varepsilon_{11}^{(2)}}(\sqrt{\mu_{22}^{(3)}\varepsilon_{11}^{(2)}} + \sqrt{\mu_{22}^{(2)}\varepsilon_{11}^{(3)}})} \int_0^t J_1^{(2)}(v_1^{(2)}(\tau-t)+\ell,\tau)d\tau \\ &+ \frac{\sqrt{\mu_{22}^{(3)}\varepsilon_{11}^{(2)}} - \sqrt{\mu_{22}^{(2)}\varepsilon_{11}^{(3)}}}{\sqrt{\varepsilon_{11}^{(3)}}(\sqrt{\mu_{22}^{(3)}\varepsilon_{11}^{(2)}} + \sqrt{\mu_{22}^{(2)}\varepsilon_{11}^{(3)}})} \int_0^t J_1^{(3)}(-v_2^{(2)}(\tau-t)+\ell,\tau)d\tau. \end{aligned}$$

And the matching condition $u_3^{(3)}(\ell^+,t)$ can be found in the following form:

$$u_3^{(3)}(\ell^+,t) = \frac{2\sqrt{\mu_{11}^{(3)}\varepsilon_{22}^{(3)}}}{\sqrt{\mu_{11}^{(3)}\varepsilon_{22}^{(2)}} + \sqrt{\mu_{11}^{(2)}\varepsilon_{22}^{(3)}}} u_3^{(2)}(\ell^-,t) + \frac{\sqrt{\mu_{11}^{(3)}\varepsilon_{22}^{(2)}} - \sqrt{\mu_{11}^{(2)}\varepsilon_{22}^{(3)}}}{\sqrt{\mu_{11}^{(3)}\varepsilon_{22}^{(2)}} + \sqrt{\mu_{11}^{(2)}\varepsilon_{22}^{(3)}}} u_4^{(3)}(\ell^+,t),$$

The values $u_3^{(2)}(\ell^-,t)$ is related with region R_6 and $u_4^{(3)}(\ell^+,t)$ is related with region R_7. Substituting these values we get the matching condition $u_3^{(2)}(\ell^+,t)$ as:

$$\begin{aligned} u_4^{(2)}(\ell^-,t) &= \frac{2\sqrt{\mu_{11}^{(3)}\varepsilon_{22}^{(3)}}}{\sqrt{\varepsilon_{22}^{(2)}}(\sqrt{\mu_{11}^{(3)}\varepsilon_{22}^{(2)}} + \sqrt{\mu_{11}^{(2)}\varepsilon_{22}^{(3)}})} \int_0^t J_2^{(2)}(v_3^{(2)}(\tau-t)+\ell,\tau)d\tau \\ &- \frac{\sqrt{\mu_{11}^{(3)}\varepsilon_{22}^{(2)}} - \sqrt{\mu_{11}^{(2)}\varepsilon_{22}^{(3)}}}{\sqrt{\varepsilon_{22}^{(3)}}(\sqrt{\mu_{11}^{(3)}\varepsilon_{22}^{(2)}} + \sqrt{\mu_{11}^{(2)}\varepsilon_{22}^{(3)}})} \int_0^t J_2^{(3)}(-v_4^{(3)}(\tau-t)+\ell,\tau)d\tau. \end{aligned}$$

4.3.3.2 Deriving Matching Conditions at the Third Flat

Finding Matching Conditions For Region R_8

In order to get explicit formulas in this region we need to derive the matching conditions $u_i^{(1)}(-0,t+\frac{x_3}{v_i^{(1)}})$ for $i=2,4$ by means of the solution of the IVP in region R_9. By following the same way as we did in the second flat at region R_4, we find these values in the following forms:

$$u_2^{(1)}(-0,t)=\frac{\sqrt{\mu_{22}^{(1)}\varepsilon_{11}^{(2)}}-\sqrt{\mu_{22}^{(2)}\varepsilon_{11}^{(1)}}}{\sqrt{\mu_{22}^{(1)}\varepsilon_{11}^{(2)}}+\sqrt{\mu_{22}^{(2)}\varepsilon_{11}^{(1)}}}u_1^{(1)}(-0,t)+\frac{2\sqrt{\mu_{22}^{(1)}\varepsilon_{11}^{(1)}}}{\sqrt{\mu_{22}^{(1)}\varepsilon_{11}^{(2)}}+\sqrt{\mu_{22}^{(2)}\varepsilon_{11}^{(1)}}}u_2^{(2)}(+0,t),$$

Substituting the values $u_1^{(1)}(-0,t)$ related to region R_8 and $u_2^{(2)}(+0,t)$ related to region R_9 we get:

$$u_2^{(1)}(-0,t)=\frac{\sqrt{\mu_{22}^{(1)}\varepsilon_{11}^{(2)}}-\sqrt{\mu_{22}^{(2)}\varepsilon_{11}^{(1)}}}{\sqrt{\mu_{22}^{(1)}\varepsilon_{11}^{(2)}}+\sqrt{\mu_{22}^{(2)}\varepsilon_{11}^{(1)}}}\int_0^t f_1^{(1)}(v_1^{(1)}(\tau-t),\tau)d\tau$$

$$+\frac{2\sqrt{\mu_{22}^{(1)}\varepsilon_{11}^{(1)}}}{\sqrt{\mu_{22}^{(1)}\varepsilon_{11}^{(2)}}+\sqrt{\mu_{22}^{(2)}\varepsilon_{11}^{(1)}}}\left[u_2^{(2)}(\ell^-,t-\frac{\ell}{v_2^{(2)}})+\int_{t-\frac{\ell}{v_2^{(2)}}}^t f_2^{(2)}(-v_2^{(2)}(\tau-t),\tau)d\tau\right],$$

Then substituting the value $u_2^{(2)}(\ell^-,t-\frac{\ell}{v_2^{(2)}})$ which was derived in region R_6, we get:

$$u_2^{(1)}(-0,t)=-\frac{\sqrt{\mu_{22}^{(1)}\varepsilon_{11}^{(2)}}-\sqrt{\mu_{22}^{(2)}\varepsilon_{11}^{(1)}}}{\sqrt{\varepsilon_{11}^{(1)}}(\sqrt{\mu_{22}^{(1)}\varepsilon_{11}^{(2)}}+\sqrt{\mu_{22}^{(2)}\varepsilon_{11}^{(1)}})}\int_0^t J_1^{(1)}(v_1^{(1)}(\tau-t),\tau)d\tau$$

$$+\frac{2\sqrt{\mu_{22}^{(1)}\varepsilon_{11}^{(1)}}}{\sqrt{\mu_{22}^{(1)}\varepsilon_{11}^{(2)}}+\sqrt{\mu_{22}^{(2)}\varepsilon_{11}^{(1)}}}\left[-\frac{\sqrt{\mu_{22}^{(2)}\varepsilon_{11}^{(3)}}-\sqrt{\mu_{22}^{(3)}\varepsilon_{11}^{(2)}}}{\sqrt{\varepsilon_{11}^{(2)}}(\sqrt{\mu_{22}^{(2)}\varepsilon_{11}^{(3)}}+\sqrt{\mu_{22}^{(3)}\varepsilon_{11}^{(2)}})}\right.$$

$$\cdot\int_0^{t-\frac{\ell}{v_2^{(2)}}} J_1^{(2)}(v_1^{(2)}(\tau-t)+2\ell,\tau)d\tau$$

$$+\frac{2\sqrt{\mu_{22}^{(2)}\varepsilon_{11}^{(2)}}}{\sqrt{\varepsilon_{11}^{(3)}}(\sqrt{\mu_{22}^{(2)}\varepsilon_{11}^{(3)}}+\sqrt{\mu_{22}^{(3)}\varepsilon_{11}^{(2)}})}\int_0^{t-\frac{\ell}{v_2^{(2)}}} J_1^{(3)}(-v_2^{(3)}(\tau-t)-\frac{v_2^{(3)}}{v_2^{(2)}}\ell+\ell,\tau)d\tau$$

$$\left.+\frac{1}{\sqrt{\varepsilon_{11}^{(2)}}}\int_{t-\frac{\ell}{v_2^{(2)}}}^t J_1^{(2)}(-v_2^{(2)}(\tau-t),\tau)d\tau\right].$$

The form of the matching condition $u_4^{(1)}(-0,t)$ was defined in region R_4 as:

$$u_4^{(1)}(-0,t)=\frac{\sqrt{\mu_{11}^{(1)}\varepsilon_{22}^{(2)}}-\sqrt{\mu_{11}^{(2)}\varepsilon_{22}^{(1)}}}{\sqrt{\mu_{11}^{(1)}\varepsilon_{22}^{(2)}}+\sqrt{\mu_{11}^{(2)}\varepsilon_{22}^{(1)}}}u_3^{(1)}(-0,t)+\frac{2\sqrt{\mu_{11}^{(1)}\varepsilon_{22}^{(1)}}}{\sqrt{\mu_{11}^{(1)}\varepsilon_{22}^{(2)}}+\sqrt{\mu_{11}^{(2)}\varepsilon_{22}^{(1)}}}u_4^{(2)}(+0,t),$$

Substituting the values $u_3^{(1)}(-0,t)$ related to region R_8 and $u_4^{(2)}(+0,t)$ related to region R_9 into the last relation we get:

$$u_4^{(1)}(-0,t)=\frac{\sqrt{\mu_{11}^{(1)}\varepsilon_{22}^{(2)}}-\sqrt{\mu_{11}^{(2)}\varepsilon_{22}^{(1)}}}{\sqrt{\varepsilon_{22}^{(1)}}(\sqrt{\mu_{11}^{(1)}\varepsilon_{22}^{(2)}}+\sqrt{\mu_{11}^{(2)}\varepsilon_{22}^{(1)}})}\int_0^t J_2^{(1)}(v_3^{(1)}(\tau-t),\tau)d\tau$$

$$+\frac{2\sqrt{\mu_{11}^{(1)}\varepsilon_{22}^{(1)}}}{\sqrt{\mu_{11}^{(1)}\varepsilon_{22}^{(2)}}+\sqrt{\mu_{11}^{(2)}\varepsilon_{22}^{(1)}}}\left[u_4^{(2)}(\ell^-,t-\frac{\ell}{v_4^{(2)}})+\int_{t-\frac{\ell}{v_4^{(2)}}}^t f_4^{(2)}(-v_4^{(2)}(\tau-t),\tau)d\tau\right],$$

And substituting the value $u_4^{(2)}(\ell^-, t-\frac{\ell}{v_4^{(2)}})$, which was defined in region R_6, we get the matching condition $u_4^{(1)}(-0,t)$ as:

$$u_4^{(1)}(-0,t) = \frac{\sqrt{\mu_{11}^{(1)}\varepsilon_{22}^{(2)}}-\sqrt{\mu_{11}^{(2)}\varepsilon_{22}^{(1)}}}{\sqrt{\varepsilon_{22}^{(1)}}(\sqrt{\mu_{11}^{(1)}\varepsilon_{22}^{(2)}}+\sqrt{\mu_{11}^{(2)}\varepsilon_{22}^{(1)}})}\int_0^t J_2^{(1)}(v_3^{(1)}(\tau-t),\tau)d\tau$$

$$+\frac{2\sqrt{\mu_{11}^{(1)}\varepsilon_{22}^{(1)}}}{\sqrt{\mu_{11}^{(1)}\varepsilon_{22}^{(2)}}+\sqrt{\mu_{11}^{(2)}\varepsilon_{22}^{(1)}}}\left[\frac{\sqrt{\mu_{11}^{(2)}\varepsilon_{22}^{(3)}}-\sqrt{\mu_{11}^{(3)}\varepsilon_{22}^{(2)}}}{\sqrt{\varepsilon_{22}^{(2)}}(\sqrt{\mu_{11}^{(2)}\varepsilon_{22}^{(3)}}+\sqrt{\mu_{11}^{(3)}\varepsilon_{22}^{(2)}})}\right.$$

$$\cdot\int_0^{t-\frac{\ell}{v_4^{(2)}}} J_2^{(2)}(v_3^{(2)}(\tau-t)+2\ell,\tau)d\tau$$

$$-\frac{2\sqrt{\mu_{11}^{(2)}\varepsilon_{22}^{(2)}}}{\sqrt{\varepsilon_{22}^{(3)}}(\sqrt{\mu_{11}^{(2)}\varepsilon_{22}^{(3)}}+\sqrt{\mu_{11}^{(3)}\varepsilon_{22}^{(2)}})}\int_0^{t-\frac{\ell}{v_4^{(2)}}} J_2^{(3)}(-v_4^{(3)}(\tau-t)-\frac{v_4^{(3)}}{v_4^{(2)}}\ell+\ell,\tau)d\tau$$

$$-\frac{1}{\sqrt{\varepsilon_{22}^{(2)}}}\int_{t-\frac{\ell}{v_4^{(2)}}}^t J_2^{(2)}(-v_4^{(2)}(\tau-t),\tau)d\tau.$$

Finding Matching Conditions For Region R_9

In order to get explicit formulas in this region we need to derive the matching conditions $u_i^{(2)}(+0,t-\frac{x_3}{v_i^{(2)}})$ for $i=1,3$ by means the IVP related to R_8 and $u_j^{(2)}(\ell^-,t+\frac{x_3-\ell}{v_j^{(2)}})$ for $j=2,4$ by means the IVP related to R_7. But in region R_6 we have derived the matching condition $u_j^{(2)}(\ell^-,t+\frac{x_3-\ell}{v_j^{(2)}})$ for $j=2,4$. Hence we only need to derive the first condition above.

The form of $u_i^{(2)}(+0,t)$ for $i=1,3$ was found at the second flat in region R_5 and it was in the following form:

$$u_1^{(2)}(+0,t) = \frac{2\sqrt{\mu_{22}^{(2)}\varepsilon_{11}^{(2)}}}{\sqrt{\mu_{22}^{(2)}\varepsilon_{11}^{(1)}}+\sqrt{\mu_{22}^{(1)}\varepsilon_{11}^{(2)}}}u_1^{(1)}(-0,t)+\frac{\sqrt{\mu_{22}^{(2)}\varepsilon_{11}^{(1)}}-\sqrt{\mu_{22}^{(1)}\varepsilon_{11}^{(2)}}}{\sqrt{\mu_{22}^{(2)}\varepsilon_{11}^{(1)}}+\sqrt{\mu_{22}^{(1)}\varepsilon_{11}^{(2)}}}u_2^{(2)}(+0,t),$$

Substituting the values $u_1^{(1)}(-0,t)$ related to region R_8 and $u_2^{(2)}(+0,t)$ related to region R_6 we get:

$$u_1^{(2)}(+0,t) = \frac{2\sqrt{\mu_{22}^{(2)}\varepsilon_{11}^{(2)}}}{\sqrt{\mu_{22}^{(2)}\varepsilon_{11}^{(1)}}+\sqrt{\mu_{22}^{(1)}\varepsilon_{11}^{(2)}}}\int_0^t f_1^{(1)}(v_1^{(1)}(\tau-t),\tau)d\tau$$

$$+\frac{\sqrt{\mu_{22}^{(2)}\varepsilon_{11}^{(1)}}-\sqrt{\mu_{22}^{(1)}\varepsilon_{11}^{(2)}}}{\sqrt{\mu_{22}^{(2)}\varepsilon_{11}^{(1)}}+\sqrt{\mu_{22}^{(1)}\varepsilon_{11}^{(2)}}}\left[u_2^{(2)}(\ell^-,t-\frac{\ell}{v_2^{(2)}})+\int_{t-\frac{\ell}{v_2^{(2)}}}^t f_2^{(2)}(-v_2^{(2)}(\tau-t),\tau)d\tau\right],$$

Then substituting the value $u_2^{(2)}(\ell^-,t-\frac{\ell}{v_2^{(2)}})$ which was derived in region R_6, we get:

$$u_1^{(2)}(+0,t) = -\frac{2\sqrt{\mu_{22}^{(2)}\varepsilon_{11}^{(2)}}}{\sqrt{\varepsilon_{11}^{(1)}}(\sqrt{\mu_{22}^{(2)}\varepsilon_{11}^{(1)}}+\sqrt{\mu_{22}^{(1)}\varepsilon_{11}^{(2)}})}\int_0^t J_1^{(1)}(v_1^{(1)}(\tau-t),\tau)d\tau$$

$$+\frac{\sqrt{\mu_{22}^{(2)}\varepsilon_{11}^{(1)}}-\sqrt{\mu_{22}^{(1)}\varepsilon_{11}^{(2)}}}{\sqrt{\mu_{22}^{(2)}\varepsilon_{11}^{(1)}}+\sqrt{\mu_{22}^{(1)}\varepsilon_{11}^{(2)}}}\left[-\frac{\sqrt{\mu_{22}^{(2)}\varepsilon_{11}^{(3)}}-\sqrt{\mu_{22}^{(3)}\varepsilon_{11}^{(2)}}}{\sqrt{\varepsilon_{11}^{(2)}}(\sqrt{\mu_{22}^{(2)}\varepsilon_{11}^{(3)}}+\sqrt{\mu_{22}^{(3)}\varepsilon_{11}^{(2)}})}\right.$$

$$\cdot\int_0^{t-\frac{\ell}{\nu_2^{(2)}}} J_1^{(2)}(\nu_1^{(2)}(\tau-t)+2\ell,\tau)d\tau$$

$$+\frac{2\sqrt{\mu_{22}^{(2)}\varepsilon_{11}^{(2)}}}{\sqrt{\varepsilon_{11}^{(3)}}(\sqrt{\mu_{22}^{(2)}\varepsilon_{11}^{(3)}}+\sqrt{\mu_{22}^{(3)}\varepsilon_{11}^{(2)}})}\int_0^{t-\frac{\ell}{\nu_2^{(2)}}} J_1^{(3)}(-\nu_2^{(3)}(\tau-t)-\frac{\nu_2^{(3)}}{\nu_2^{(2)}}\ell+\ell,\tau)d\tau$$

$$\left.+\frac{1}{\sqrt{\varepsilon_{11}^{(2)}}}\int_{t-\frac{\ell}{\nu_2^{(2)}}}^{t} J_1^{(2)}(-\nu_2^{(2)}(\tau-t),\tau)d\tau\right].$$

The form of the matching condition $u_3^{(2)}(+0,t)$ was found in the region R_5 as:

$$u_3^{(2)}(+0,t)=\frac{2\sqrt{\mu_{11}^{(2)}\varepsilon_{22}^{(2)}}}{\sqrt{\mu_{11}^{(2)}\varepsilon_{22}^{(1)}}+\sqrt{\mu_{11}^{(1)}\varepsilon_{22}^{(2)}}}u_3^{(1)}(-0,t)+\frac{\sqrt{\mu_{11}^{(2)}\varepsilon_{22}^{(1)}}-\sqrt{\mu_{11}^{(1)}\varepsilon_{22}^{(2)}}}{\sqrt{\mu_{11}^{(2)}\varepsilon_{22}^{(1)}}+\sqrt{\mu_{11}^{(1)}\varepsilon_{22}^{(2)}}}u_4^{(2)}(+0,t).$$

Substituting the values $u_3^{(1)}(-0,t)$ related to region R_8 and $u_4^{(2)}(+0,t)$ related to region R_6 we get:

$$u_3^{(2)}(+0,t)=\frac{2\sqrt{\mu_{11}^{(2)}\varepsilon_{22}^{(2)}}}{\sqrt{\varepsilon_{22}^{(1)}}(\sqrt{\mu_{11}^{(2)}\varepsilon_{22}^{(1)}}+\sqrt{\mu_{11}^{(1)}\varepsilon_{22}^{(2)}}}\int_0^t J_2^{(1)}(\nu_3^{(1)}(\tau-t),\tau)d\tau$$

$$+\frac{\sqrt{\mu_{11}^{(2)}\varepsilon_{22}^{(1)}}-\sqrt{\mu_{11}^{(1)}\varepsilon_{22}^{(2)}}}{\sqrt{\mu_{11}^{(2)}\varepsilon_{22}^{(1)}}+\sqrt{\mu_{11}^{(1)}\varepsilon_{22}^{(2)}}}\left[u_4^{(2)}(\ell^-,t-\frac{\ell}{\nu_4^{(2)}})+\int_{t-\frac{\ell}{\nu_4^{(2)}}}^{t} f_4^{(2)}(-\nu_4^{(2)}(\tau-t),\tau)d\tau\right],$$

Then substituting the value $u_4^{(2)}(\ell^-,t-\frac{\ell}{\nu_4^{(2)}})$ which was derived in region R_6, we get:

$$u_3^{(2)}(+0,t)=\frac{2\sqrt{\mu_{11}^{(2)}\varepsilon_{22}^{(2)}}}{\sqrt{\varepsilon_{22}^{(1)}}(\sqrt{\mu_{11}^{(2)}\varepsilon_{22}^{(1)}}+\sqrt{\mu_{11}^{(1)}\varepsilon_{22}^{(2)}})}\int_0^t J_2^{(1)}(\nu_3^{(1)}(\tau-t),\tau)d\tau$$

$$+\frac{\sqrt{\mu_{11}^{(2)}\varepsilon_{22}^{(1)}}-\sqrt{\mu_{11}^{(1)}\varepsilon_{22}^{(2)}}}{\sqrt{\mu_{11}^{(2)}\varepsilon_{22}^{(1)}}+\sqrt{\mu_{11}^{(1)}\varepsilon_{22}^{(2)}}}\left[\frac{\sqrt{\mu_{11}^{(2)}\varepsilon_{22}^{(3)}}-\sqrt{\mu_{11}^{(3)}\varepsilon_{22}^{(2)}}}{\sqrt{\varepsilon_{22}^{(2)}}(\sqrt{\mu_{11}^{(2)}\varepsilon_{22}^{(3)}}+\sqrt{\mu_{11}^{(3)}\varepsilon_{22}^{(2)}})}\right.$$

$$\cdot\int_0^{t-\frac{\ell}{\nu_4^{(2)}}} J_2^{(2)}(\nu_3^{(2)}(\tau-t)+2\ell,\tau)d\tau$$

$$-\frac{2\sqrt{\mu_{11}^{(2)}\varepsilon_{22}^{(2)}}}{\sqrt{\varepsilon_{22}^{(3)}}(\sqrt{\mu_{11}^{(2)}\varepsilon_{22}^{(3)}}+\sqrt{\mu_{11}^{(3)}\varepsilon_{22}^{(2)}})}\int_0^{t-\frac{\ell}{\nu_4^{(2)}}} J_2^{(3)}(-\nu_4^{(3)}(\tau-t)-\frac{\nu_4^{(3)}}{\nu_4^{(2)}}\ell+\ell,\tau)d\tau$$

$$\left.-\frac{1}{\sqrt{\varepsilon_{22}^{(2)}}}\int_{t-\frac{\ell}{\nu_4^{(2)}}}^{t} J_2^{(2)}(-\nu_4^{(2)}(\tau-t),\tau)d\tau\right].$$

Finding Matching Conditions For Region R_{10}

In order to get explicit formulas in this region we need to derive the matching conditions $u_i^{(2)}(+0,t-\frac{x_3}{v_i^{(2)}})$ for $i=1,3$ and $u_j^{(2)}(\ell^-,t+\frac{x_3-\ell}{v_j^{(2)}})$ for $j=2,4$. The value of $u_i^{(2)}(+0,t-\frac{x_3}{v_i^{(2)}})$ for $i=1,3$ is derived at the second flat in region R_5 and $u_j^{(2)}(\ell^-,t+\frac{x_3-\ell}{v_j^{(2)}})$ for $j=2,4$ was derived in the region R_6.

Finding Matching Conditions For Region R_{11}

In order to get explicit formulas in this region we need to derive the matching conditions $u_i^{(2)}(+0,t-\frac{x_3}{v_i^{(2)}})$ for $i=1,3$ and $u_j^{(2)}(\ell^-,t+\frac{x_3-\ell}{v_j^{(2)}})$ for $j=2,4$. The value of $u_i^{(2)}(+0,t-\frac{x_3}{v_i^{(2)}})$ for $i=1,3$ was derived at the second flat in region R_5. Now let us derive $u_j^{(2)}(\ell^-,t+\frac{x_3-\ell}{v_j^{(2)}})$ for $j=2,4$. The form of $u_j^{(2)}(\ell^-,t)$ for $j=2,4$ was found in the region R_6 as:

$$u_2^{(2)}(\ell^-,t)=\frac{\sqrt{\mu_{22}^{(2)}\varepsilon_{11}^{(3)}}-\sqrt{\mu_{22}^{(3)}\varepsilon_{11}^{(2)}}}{\sqrt{\mu_{22}^{(2)}\varepsilon_{11}^{(3)}}+\sqrt{\mu_{22}^{(3)}\varepsilon_{11}^{(2)}}}u_1^{(2)}(\ell^-,t)+\frac{2\sqrt{\mu_{22}^{(2)}\varepsilon_{11}^{(2)}}}{\sqrt{\mu_{22}^{(2)}\varepsilon_{11}^{(3)}}+\sqrt{\mu_{22}^{(3)}\varepsilon_{11}^{(2)}}}u_2^{(3)}(\ell^+,t),$$

Substituting the values of $u_1^{(2)}(\ell^-,t)$ related to R_{11} and $u_2^{(3)}(\ell^+,t)$ related to R_{12} into the last relation above we get:

$$u_2^{(2)}(\ell^-,t)=\frac{\sqrt{\mu_{22}^{(2)}\varepsilon_{11}^{(3)}}-\sqrt{\mu_{22}^{(3)}\varepsilon_{11}^{(2)}}}{\sqrt{\mu_{22}^{(2)}\varepsilon_{11}^{(3)}}+\sqrt{\mu_{22}^{(3)}\varepsilon_{11}^{(2)}}}\Bigg[u_1^{(2)}(+0,t-\frac{\ell}{v_1^{(2)}})$$

$$+\int_{t-\frac{\ell}{v_1^{(2)}}}^{t}f_1^{(2)}(v_1^{(2)}(\tau-t)+\ell,\tau)d\tau\Bigg]$$

$$+\frac{2\sqrt{\mu_{22}^{(2)}\varepsilon_{11}^{(2)}}}{\sqrt{\mu_{22}^{(2)}\varepsilon_{11}^{(3)}}+\sqrt{\mu_{22}^{(3)}\varepsilon_{11}^{(2)}}}\int_0^t f_2^{(3)}(-v_2^{(3)}(\tau-t)+\ell,\tau)d\tau,$$

Substituting $u_1^{(2)}(+0,t-\frac{\ell}{v_1^{(2)}})$ which was derived in region R_5, we get:

$$u_2^{(2)}(\ell^-,t)=\frac{\sqrt{\mu_{22}^{(2)}\varepsilon_{11}^{(3)}}-\sqrt{\mu_{22}^{(3)}\varepsilon_{11}^{(2)}}}{\sqrt{\mu_{22}^{(2)}\varepsilon_{11}^{(3)}}+\sqrt{\mu_{22}^{(3)}\varepsilon_{11}^{(2)}}}\Bigg[-\frac{2\sqrt{\mu_{22}^{(2)}\varepsilon_{11}^{(2)}}}{\sqrt{\varepsilon_{11}^{(1)}}(\sqrt{\mu_{22}^{(2)}\varepsilon_{11}^{(1)}}+\sqrt{\mu_{22}^{(1)}\varepsilon_{11}^{(2)}})}$$

$$\cdot\int_0^{t-\frac{\ell}{v_1^{(2)}}}J_1^{(1)}(v_1^{(1)}(\tau-t)+\frac{v_1^{(1)}}{v_1^{(2)}}\ell,\tau)d\tau$$

$$+\frac{\sqrt{\mu_{22}^{(2)}\varepsilon_{11}^{(1)}}-\sqrt{\mu_{22}^{(1)}\varepsilon_{11}^{(2)}}}{\sqrt{\varepsilon_{11}^{(2)}}(\sqrt{\mu_{22}^{(2)}\varepsilon_{11}^{(1)}}+\sqrt{\mu_{22}^{(1)}\varepsilon_{11}^{(2)}})}\int_0^{t-\frac{\ell}{v_1^{(2)}}}J_1^{(2)}(-v_1^{(2)}(\tau-t)-\ell,\tau)d\tau$$

$$-\frac{1}{\sqrt{\varepsilon_{11}^{(2)}}}\int_{t-\frac{\ell}{v_1^{(2)}}}^{t}J_1^{(2)}(v_1^{(2)}(\tau-t)+\ell,\tau)d\tau\Bigg]$$

$$+\frac{2\sqrt{\mu_{22}^{(2)}\varepsilon_{11}^{(2)}}}{\sqrt{\varepsilon_{11}^{(3)}}(\sqrt{\mu_{22}^{(2)}\varepsilon_{11}^{(3)}}+\sqrt{\mu_{22}^{(3)}\varepsilon_{11}^{(2)}})}\int_0^t J_1^{(3)}(-v_2^{(3)}(\tau-t)+\ell,\tau)d\tau.$$

The form of $u_4^{(2)}(\ell^-,t)$ for $j=2,4$ was found as:

$$u_4^{(2)}(\ell^-,t)=\frac{\sqrt{\mu_{11}^{(2)}\varepsilon_{22}^{(3)}}-\sqrt{\mu_{11}^{(3)}\varepsilon_{22}^{(2)}}}{\sqrt{\mu_{11}^{(2)}\varepsilon_{22}^{(3)}}+\sqrt{\mu_{11}^{(3)}\varepsilon_{22}^{(2)}}}u_3^{(2)}(\ell^-,t)+\frac{2\sqrt{\mu_{11}^{(2)}\varepsilon_{22}^{(2)}}}{\sqrt{\mu_{11}^{(2)}\varepsilon_{22}^{(3)}}+\sqrt{\mu_{11}^{(3)}\varepsilon_{22}^{(2)}}}u_4^{(3)}(\ell^+,t),$$

Substituting the values of $u_3^{(2)}(\ell^-,t)$ related to R_{11} and $u_4^{(3)}(\ell^+,t)$ related to R_{12} into the last relation we get:

$$u_4^{(2)}(\ell^-,t)=\frac{\sqrt{\mu_{11}^{(2)}\varepsilon_{22}^{(3)}}-\sqrt{\mu_{11}^{(3)}\varepsilon_{22}^{(2)}}}{\sqrt{\mu_{11}^{(2)}\varepsilon_{22}^{(3)}}+\sqrt{\mu_{11}^{(3)}\varepsilon_{22}^{(2)}}}\Bigg[u_3^{(2)}(+0,t-\frac{\ell}{v_3^{(2)}})$$
$$+\int_{t-\frac{\ell}{v_3^{(2)}}}^t f_3^{(2)}(v_3^{(2)}(\tau-t)+\ell,\tau)d\tau\Bigg]$$
$$+\frac{2\sqrt{\mu_{11}^{(2)}\varepsilon_{22}^{(2)}}}{\sqrt{\mu_{11}^{(2)}\varepsilon_{22}^{(3)}}+\sqrt{\mu_{11}^{(3)}\varepsilon_{22}^{(2)}}}\int_0^t f_4^{(3)}(-v_4^{(3)}(\tau-t)+\ell,\tau)d\tau,$$

Substituting $u_3^{(2)}(+0,t-\frac{\ell}{v_3^{(2)}})$ which was derived in region R_5, we get:

$$u_4^{(2)}(\ell^-,t)=\frac{\sqrt{\mu_{11}^{(2)}\varepsilon_{22}^{(3)}}-\sqrt{\mu_{11}^{(3)}\varepsilon_{22}^{(2)}}}{\sqrt{\mu_{11}^{(2)}\varepsilon_{22}^{(3)}}+\sqrt{\mu_{11}^{(3)}\varepsilon_{22}^{(2)}}}\Bigg[\frac{2\sqrt{\mu_{11}^{(2)}\varepsilon_{22}^{(2)}}}{\sqrt{\varepsilon_{22}^{(1)}}(\sqrt{\mu_{11}^{(2)}\varepsilon_{22}^{(1)}}+\sqrt{\mu_{11}^{(1)}\varepsilon_{22}^{(2)}})}$$
$$\cdot\int_0^{t-\frac{\ell}{v_3^{(2)}}} J_2^{(1)}(v_3^{(1)}(\tau-t)+\frac{v_3^{(1)}}{v_3^{(2)}}\ell,\tau)d\tau$$
$$-\frac{\sqrt{\mu_{11}^{(2)}\varepsilon_{22}^{(1)}}-\sqrt{\mu_{11}^{(1)}\varepsilon_{22}^{(2)}}}{\sqrt{\varepsilon_{22}^{(2)}}(\sqrt{\mu_{11}^{(2)}\varepsilon_{22}^{(1)}}+\sqrt{\mu_{11}^{(1)}\varepsilon_{22}^{(2)}})}\int_0^{t-\frac{\ell}{v_3^{(2)}}} J_2^{(2)}(-v_3^{(2)}(\tau-t)-\ell,\tau)d\tau$$
$$+\frac{1}{\sqrt{\varepsilon_{22}^{(2)}}}\int_{t-\frac{\ell}{v_3^{(2)}}}^t J_2^{(2)}(v_3^{(2)}(\tau-t)+\ell,\tau)d\tau\Bigg]$$
$$-\frac{2\sqrt{\mu_{11}^{(2)}\varepsilon_{22}^{(2)}}}{\sqrt{\varepsilon_{22}^{(3)}}(\sqrt{\mu_{11}^{(2)}\varepsilon_{22}^{(3)}}+\sqrt{\mu_{11}^{(3)}\varepsilon_{22}^{(2)}})}\int_0^t J_2^{(3)}(-v_4^{(3)}(\tau-t)+\ell,\tau)d\tau.$$

Finding Matching Conditions For Region R_{12}

To get explicit formulas in this region we need to derive the matching conditions $u_i^{(3)}(\ell^+,t-\frac{x_3-\ell}{v_i^{(3)}})$ for $i=1,3$.

The form of $u_1^{(3)}(\ell^+,t)$ was defined before in the following form:

$$u_1^{(3)}(\ell^+,t)=\frac{2\sqrt{\mu_{22}^{(3)}\varepsilon_{11}^{(3)}}}{\sqrt{\mu_{22}^{(3)}\varepsilon_{11}^{(2)}}+\sqrt{\mu_{22}^{(2)}\varepsilon_{11}^{(3)}}}u_1^{(2)}(\ell^-,t)+\frac{\sqrt{\mu_{22}^{(3)}\varepsilon_{11}^{(2)}}-\sqrt{\mu_{22}^{(2)}\varepsilon_{11}^{(3)}}}{\sqrt{\mu_{22}^{(3)}\varepsilon_{11}^{(2)}}+\sqrt{\mu_{22}^{(2)}\varepsilon_{11}^{(3)}}}u_2^{(3)}(\ell^+,t),$$

Substituting the value of $u_1^{(2)}(\ell^-,t)$ related to R_{11} and $u_2^{(3)}(\ell^+,t)$ related to R_{12} into the last relation above we get:

$$u_1^{(3)}(\ell^+,t) = \frac{2\sqrt{\mu_{22}^{(3)}\varepsilon_{11}^{(3)}}}{\sqrt{\mu_{22}^{(3)}\varepsilon_{11}^{(2)}}+\sqrt{\mu_{22}^{(2)}\varepsilon_{11}^{(3)}}}\Bigg[u_1^{(2)}(+0,t-\frac{\ell}{v_1^{(2)}})$$
$$+\int_{t-\frac{\ell}{v_1^{(2)}}}^{t} f_1^{(2)}(v_1^{(2)}(\tau-t)+\ell,\tau)d\tau\Bigg]$$
$$+\frac{\sqrt{\mu_{22}^{(3)}\varepsilon_{11}^{(2)}}-\sqrt{\mu_{22}^{(2)}\varepsilon_{11}^{(3)}}}{\sqrt{\mu_{22}^{(3)}\varepsilon_{11}^{(2)}}+\sqrt{\mu_{22}^{(2)}\varepsilon_{11}^{(3)}}}\int_0^t f_2^{(3)}(-v_2^{(3)}(\tau-t)+\ell,\tau)d\tau,$$

Substituting $u_1^{(2)}(+0,t-\frac{\ell}{v_1^{(2)}})$ which was derived in region R_5, we get:

$$u_1^{(3)}(\ell^+,t) = \frac{2\sqrt{\mu_{22}^{(3)}\varepsilon_{11}^{(3)}}}{\sqrt{\mu_{22}^{(3)}\varepsilon_{11}^{(2)}}+\sqrt{\mu_{22}^{(2)}\varepsilon_{11}^{(3)}}}\Bigg[-\frac{2\sqrt{\mu_{22}^{(2)}\varepsilon_{11}^{(2)}}}{\sqrt{\varepsilon_{11}^{(1)}}(\sqrt{\mu_{22}^{(2)}\varepsilon_{11}^{(1)}}+\sqrt{\mu_{22}^{(1)}\varepsilon_{11}^{(2)}})}$$
$$\cdot\int_0^{t-\frac{\ell}{v_1^{(2)}}} J_1^{(1)}(v_1^{(1)}(\tau-t)+\frac{v_1^{(1)}}{v_1^{(2)}}\ell,\tau)d\tau$$
$$+\frac{\sqrt{\mu_{22}^{(2)}\varepsilon_{11}^{(1)}}-\sqrt{\mu_{22}^{(1)}\varepsilon_{11}^{(2)}}}{\sqrt{\varepsilon_{11}^{(2)}}(\sqrt{\mu_{22}^{(2)}\varepsilon_{11}^{(1)}}+\sqrt{\mu_{22}^{(1)}\varepsilon_{11}^{(2)}})}\int_0^{t-\frac{\ell}{v_1^{(2)}}} J_1^{(2)}(-v_1^{(2)}(\tau-t)-\ell,\tau)d\tau$$
$$-\frac{1}{\sqrt{\varepsilon_{11}^{(2)}}}\int_{t-\frac{\ell}{v_1^{(2)}}}^{t} J_1^{(2)}(v_1^{(2)}(\tau-t)+\ell,\tau)d\tau\Bigg]$$
$$+\frac{\sqrt{\mu_{22}^{(3)}\varepsilon_{11}^{(2)}}-\sqrt{\mu_{22}^{(2)}\varepsilon_{11}^{(3)}}}{\sqrt{\varepsilon_{11}^{(3)}}(\sqrt{\mu_{22}^{(3)}\varepsilon_{11}^{(2)}}+\sqrt{\mu_{22}^{(2)}\varepsilon_{11}^{(3)}})}\int_0^t J_1^{(3)}(-v_2^{(3)}(\tau-t)+\ell,\tau)d\tau.$$

The form of $u_3^{(3)}(\ell^+,t)$ defined before as:

$$u_3^{(3)}(\ell^+,t) = \frac{2\sqrt{\mu_{11}^{(3)}\varepsilon_{22}^{(3)}}}{\sqrt{\mu_{11}^{(3)}\varepsilon_{22}^{(2)}}+\sqrt{\mu_{11}^{(2)}\varepsilon_{22}^{(3)}}}u_3^{(2)}(\ell^-,t)+\frac{\sqrt{\mu_{11}^{(3)}\varepsilon_{22}^{(2)}}-\sqrt{\mu_{11}^{(2)}\varepsilon_{22}^{(3)}}}{\sqrt{\mu_{11}^{(3)}\varepsilon_{22}^{(2)}}+\sqrt{\mu_{11}^{(2)}\varepsilon_{22}^{(3)}}}u_4^{(3)}(\ell^+,t),$$

Substituting the values of $u_3^{(2)}(\ell^-,t)$ related to R_{11} and $u_4^{(3)}(\ell^+,t)$ related to R_{12} into the last relation we get:

$$u_3^{(3)}(\ell^+,t) = \frac{2\sqrt{\mu_{11}^{(3)}\varepsilon_{22}^{(3)}}}{\sqrt{\mu_{11}^{(3)}\varepsilon_{22}^{(2)}}+\sqrt{\mu_{11}^{(2)}\varepsilon_{22}^{(3)}}}\Bigg[u_3^{(2)}(+0,t-\frac{\ell}{v_3^{(2)}})$$
$$+\int_{t-\frac{\ell}{v_3^{(2)}}}^{t} f_3^{(2)}(v_3^{(2)}(\tau-t)+\ell,\tau)d\tau\Bigg]$$
$$+\frac{\sqrt{\mu_{11}^{(3)}\varepsilon_{22}^{(2)}}-\sqrt{\mu_{11}^{(2)}\varepsilon_{22}^{(3)}}}{\sqrt{\mu_{11}^{(3)}\varepsilon_{22}^{(2)}}+\sqrt{\mu_{11}^{(2)}\varepsilon_{22}^{(3)}}}\int_0^t f_4^{(3)}(-v_4^{(3)}(\tau-t)+\ell,\tau)d\tau,$$

Substituting $u_3^{(2)}(+0, t-\frac{\ell}{v_3^{(2)}})$ which was derived in region R_5, we get:

$$u_3^{(3)}(\ell^+,t) = \frac{2\sqrt{\mu_{11}^{(3)}\varepsilon_{22}^{(3)}}}{\sqrt{\mu_{11}^{(3)}\varepsilon_{22}^{(2)}}+\sqrt{\mu_{11}^{(2)}\varepsilon_{22}^{(3)}}}\left[\frac{2\sqrt{\mu_{11}^{(2)}\varepsilon_{22}^{(2)}}}{\sqrt{\varepsilon_{22}^{(1)}}(\sqrt{\mu_{11}^{(2)}\varepsilon_{22}^{(1)}}+\sqrt{\mu_{11}^{(1)}\varepsilon_{22}^{(2)}})}\right.$$

$$\cdot\int_0^{t-\frac{\ell}{v_3^{(2)}}} J_2^{(1)}(v_3^{(1)}(\tau-t)+\frac{v_3^{(1)}}{v_3^{(2)}}\ell,\tau)d\tau$$

$$-\frac{\sqrt{\mu_{11}^{(2)}\varepsilon_{22}^{(1)}}-\sqrt{\mu_{11}^{(1)}\varepsilon_{22}^{(2)}}}{\sqrt{\varepsilon_{22}^{(2)}}(\sqrt{\mu_{11}^{(2)}\varepsilon_{22}^{(1)}}+\sqrt{\mu_{11}^{(1)}\varepsilon_{22}^{(2)}})}\int_0^{t-\frac{\ell}{v_3^{(2)}}} J_2^{(2)}(-v_3^{(2)}(\tau-t)-\ell,\tau)d\tau$$

$$\left.+\frac{1}{\sqrt{\varepsilon_{22}^{(2)}}}\int_{t-\frac{\ell}{v_3^{(2)}}}^{t} J_2^{(2)}(v_3^{(2)}(\tau-t)+\ell,\tau)d\tau\right]$$

$$-\frac{\sqrt{\mu_{11}^{(3)}\varepsilon_{22}^{(2)}}-\sqrt{\mu_{11}^{(2)}\varepsilon_{22}^{(3)}}}{\sqrt{\varepsilon_{22}^{(3)}}(\sqrt{\mu_{11}^{(3)}\varepsilon_{22}^{(2)}}+\sqrt{\mu_{11}^{(2)}\varepsilon_{22}^{(3)}})}\int_0^{t} J_2^{(3)}(-v_4^{(3)}(\tau-t)+\ell,\tau)d\tau.$$

4.3.4 Finding Explicit Formula for Solution of the Problem

By using the relations in Section 4.3.1 the form of the explicit formulae of the components of the electric and magnetic fields can be written as following

$$H_1^{(k)} = \frac{1}{2\sqrt{\mu_{11}^{(k)}}}[u_3^{(k)}(x_3,t)+u_4^{(k)}(x_3,t)],$$
$$H_2^{(k)} = \frac{1}{2\sqrt{\mu_{22}^{(k)}}}[u_1^{(k)}(x_3,t)+u_2^{(k)}(x_3,t)],$$
$$E_1^{(k)} = \frac{1}{2\sqrt{\varepsilon_{11}^{(k)}}}[u_1^{(k)}(x_3,t)-u_2^{(k)}(x_3,t)],$$
$$E_2^{(k)} = \frac{1}{2\sqrt{\varepsilon_{22}^{(k)}}}[-u_3^{(k)}(x_3,t)+u_4^{(k)}(x_3,t)],$$

where $k = 1,2,3$.

Using the results that we get in sections 4.3.2 and 4.3.3 and substituting these results into the relations above obtain the explicit formulae for the electric and magnetic fields for three layered anisotropic media.

Explicit formulae of the electric and magnetic fields at the first flat

$$H_1^{(k)} = \frac{1}{2\sqrt{\mu_{11}^{(k)}}}\int_0^t [f_3^{(k)}(v_3^{(k)}(\tau-t)+x_3,\tau)+f_4^{(k)}(-v_3^{(k)}(\tau-t)+x_3,\tau)]d\tau;$$

$$H_1^{(k)} = \frac{1}{2\sqrt{\mu_{22}^{(k)}}}\int_0^t [f_1^{(k)}(v_1^{(k)}(\tau-t)+x_3,\tau)+f_2^{(k)}(-v_1^{(k)}(\tau-t)+x_3,\tau)]d\tau;$$

$$E_1^{(k)} = \frac{1}{2\sqrt{\varepsilon_{11}^{(k)}}}\int_0^t [f_1^{(k)}(v_1^{(k)}(\tau-t)+x_3,\tau)-f_2^{(k)}(-v_1^{(k)}(\tau-t)+x_3,\tau)]d\tau;$$

$$E_2^{(k)} = \frac{1}{2\sqrt{\varepsilon_{22}^{(k)}}}\int_0^t [-f_3^{(k)}(v_3^{(k)}(\tau-t)+x_3,\tau)+f_4^{(k)}(-v_3^{k}(\tau-t)+x_3,\tau)]d\tau,$$

where $k = 1,2,3$.

Explicit formulae of the electric and magnetic fields at the second flat

Explicit formulae for the region R_4:

$$\begin{aligned} H_1^{(1)}(x_3,t) &= \frac{1}{2\sqrt{\mu_{11}^{(1)}}}\left[\frac{1}{\sqrt{\varepsilon_{22}^{(1)}}}\int_0^t J_2^{(1)}(v_3^{(1)}(\tau-t)+x_3,\tau)+u_4^{(1)}(-0,t+\frac{x_3}{v_3^{(1)}})\right. \\ &- \left.\frac{1}{\sqrt{\varepsilon_{22}^{(1)}}}\int_{t+\frac{x_3}{v_3^{(1)}}}^t J_2^{(1)}(-v_3^{(1)}(\tau-t)+x_3,\tau)\right]d\tau; \end{aligned}$$

$$\begin{aligned} H_2^{(1)}(x_3,t) &= \frac{1}{2\sqrt{\mu_{22}^{(1)}}}\left[-\frac{1}{\sqrt{\varepsilon_{11}^{(1)}}}\int_0^t J_1^{(1)}(v_1^{(1)}(\tau-t)+x_3,\tau)+u_2^{(1)}(-0,t+\frac{x_3}{v_1^{(1)}})\right. \\ &+ \left.\frac{1}{\sqrt{\varepsilon_{11}^{(1)}}}\int_{t+\frac{x_3}{v_1^{(1)}}}^t J_1^{(1)}(-v_1^{(1)}(\tau-t)+x_3,\tau)\right]d\tau; \end{aligned}$$

$$H_3^{(1)}(x_3,t) = 0;$$

$$\begin{aligned} E_1^{(1)}(x_3,t) &= \frac{1}{2\sqrt{\varepsilon_{11}^{(1)}}}\left[-\frac{1}{\sqrt{\varepsilon_{11}^{(1)}}}\int_0^t J_1^{(1)}(v_1^{(1)}(\tau-t)+x_3,\tau)-u_2^{(1)}(-0,t+\frac{x_3}{v_1^{(1)}})\right. \\ &- \left.\frac{1}{\sqrt{\varepsilon_{11}^{(1)}}}\int_{t+\frac{x_3}{v_1^{(1)}}}^t J_1^{(1)}(-v_1^{(1)}(\tau-t)+x_3,\tau)\right]d\tau; \end{aligned}$$

$$\begin{aligned} E_2^{(1)}(x_3,t) &= \frac{1}{2\sqrt{\varepsilon_{22}^{(1)}}}\left[-\frac{1}{\sqrt{\varepsilon_{22}^{(1)}}}\int_0^t J_2^{(1)}(v_3^{(1)}(\tau-t)+x_3,\tau)+u_4^{(1)}(-0,t+\frac{x_3}{v_3^{(1)}})\right. \\ &- \left.\frac{1}{\sqrt{\varepsilon_{22}^{(1)}}}\int_{t+\frac{x_3}{v_3^{(1)}}}^t J_2^{(1)}(-v_3^{(1)}(\tau-t)+x_3,\tau)\right]d\tau; \end{aligned}$$

$$E_3^{(1)}(x_3,t) = -\frac{1}{\varepsilon_{33}^{(1)}}\int_0^t J_3^{(1)}(x_3,\tau)d\tau.$$

The values, matching conditions, $u_2^{(1)}(-0,t+\frac{x_3}{v_1^{(1)}})$ and $u_4^{(1)}(-0,t+\frac{x_3}{v_3^{(1)}})$ derived before in the Section 4.3.3.1 for region R_4.

Explicit formulae for the region R_5:

$$\begin{aligned} H_1^{(2)}(x_3,t) &= \frac{1}{2\sqrt{\mu_{11}^{(2)}}}\left[u_3^{(2)}(+0,t-\frac{x_3}{v_3^{(2)}})+\frac{1}{\sqrt{\varepsilon_{22}^{(2)}}}\int_{t-\frac{x_3}{v_3^{(2)}}}^{t} J_2^{(2)}(v_3^{(2)}(\tau-t)+x_3,\tau)\right. \\ &- \left.\frac{1}{\sqrt{\varepsilon_{22}^{(2)}}}\int_0^t J_2^{(2)}(-v_3^{(2)}(\tau-t)+x_3,\tau)\right]d\tau; \end{aligned}$$

$$\begin{aligned} H_2^{(2)}(x_3,t) &= \frac{1}{2\sqrt{\mu_{22}^{(2)}}}\left[u_1^{(2)}(+0,t-\frac{x_3}{v_1^{(2)}})-\frac{1}{\sqrt{\varepsilon_{11}^{(2)}}}\int_{t-\frac{x_3}{v_1^{(2)}}}^{t} J_1^{(2)}(v_1^{(2)}(\tau-t)+x_3,\tau)\right. \\ &+ \left.\frac{1}{\sqrt{\varepsilon_{11}^{(2)}}}\int_0^t J_1^{(2)}(-v_1^{(2)}(\tau-t)+x_3,\tau)\right]d\tau; \end{aligned}$$

$$H_3^{(2)}(x_3,t)=0;$$

$$\begin{aligned} E_1^{(2)}(x_3,t) &= \frac{1}{2\sqrt{\varepsilon_{11}^{(2)}}}\left[u_1^{(2)}(+0,t-\frac{x_3}{v_1^{(2)}})-\frac{1}{\sqrt{\varepsilon_{11}^{(2)}}}\int_{t-\frac{x_3}{v_1^{(2)}}}^{t} J_1^{(2)}(v_1^{(2)}(\tau-t)+x_3,\tau)\right. \\ &- \left.\frac{1}{\sqrt{\varepsilon_{11}^{(2)}}}\int_0^t J_1^{(2)}(-v_1^{(2)}(\tau-t)+x_3,\tau)\right]d\tau; \end{aligned}$$

$$\begin{aligned} E_2^{(2)}(x_3,t) &= \frac{1}{2\sqrt{\varepsilon_{22}^{(2)}}}\left[-u_3^{(2)}(+0,t-\frac{x_3}{v_3^{(2)}})-\frac{1}{\sqrt{\varepsilon_{22}^{(2)}}}\int_{t-\frac{x_3}{v_3^{(2)}}}^{t} J_2^{(2)}(v_3^{(2)}(\tau-t)+x_3,\tau)\right. \\ &- \left.\frac{1}{\sqrt{\varepsilon_{22}^{(2)}}}\int_0^t J_2^{(2)}(-v_3^{(2)}(\tau-t)+x_3,\tau)\right]d\tau; \end{aligned}$$

$$E_3^{(2)}(x_3,t)=-\frac{1}{\varepsilon_{33}^{(2)}}\int_0^t J_3^{(2)}(x_3,\tau)d\tau.$$

The values, matching conditions, $u_1^{(2)}(+0,t-\frac{x_3}{v_1^{(2)}})$ and $u_3^{(2)}(+0,t-\frac{x_3}{v_3^{(2)}})$ derived before in the Section 4.3.3.1 for region R_5.

Explicit formulae for the region R_6:

$$\begin{aligned} H_1^{(2)}(x_3,t) &= \frac{1}{2\sqrt{\mu_{11}^{(2)}}}\left[\frac{1}{\sqrt{\varepsilon_{22}^{(2)}}}\int_0^t J_2^{(2)}(v_3^{(2)}(\tau-t)+x_3,\tau)+u_4^{(2)}(\ell^-,t+\frac{x_3-\ell}{v_3^{(2)}})\right. \\ &- \left.\frac{1}{\sqrt{\varepsilon_{22}^{(2)}}}\int_{t+\frac{x_3-\ell}{v_3^{(2)}}}^{t} J_2^{(2)}(-v_3^{(2)}(\tau-t)+x_3,\tau)\right]d\tau; \end{aligned}$$

$$H_2^{(2)}(x_3,t) = \frac{1}{2\sqrt{\mu_{22}^{(2)}}}\left[-\frac{1}{\sqrt{\varepsilon_{11}^{(2)}}}\int_0^t J_1^{(2)}(v_1^{(2)}(\tau-t)+x_3,\tau)+u_2^{(2)}(\ell^-,t+\frac{x_3-\ell}{v_1^{(2)}})\right.$$
$$\left. + \frac{1}{\sqrt{\varepsilon_{11}^{(2)}}}\int_{t+\frac{x_3-\ell}{v_1^{(2)}}}^t J_1^{(2)}(-v_1^{(2)}(\tau-t)+x_3,\tau)\right] d\tau;$$

$$H_3^{(1)}(x_3,t) = 0;$$

$$E_1^{(2)}(x_3,t) = \frac{1}{2\sqrt{\varepsilon_{11}^{(2)}}}\left[-\frac{1}{\sqrt{\varepsilon_{11}^{(2)}}}\int_0^t J_1^{(2)}(v_1^{(2)}(\tau-t)+x_3,\tau)+u_2^{(2)}(\ell^-,t+\frac{x_3-\ell}{v_1^{(2)}})\right.$$
$$\left. - \frac{1}{\sqrt{\varepsilon_{11}^{(2)}}}\int_{t+\frac{x_3-\ell}{v_1^{(2)}}}^t J_1^{(2)}(-v_1^{(2)}(\tau-t)+x_3,\tau)\right] d\tau;;$$

$$E_2^{(2)}(x_3,t) = \frac{1}{2\sqrt{\varepsilon_{22}^{(2)}}}\left[-\frac{1}{\sqrt{\varepsilon_{22}^{(2)}}}\int_0^t J_2^{(2)}(v_3^{(2)}(\tau-t)+x_3,\tau)+u_4^{(2)}(\ell^-,t+\frac{x_3-\ell}{v_3^{(2)}})\right.$$
$$\left. - \frac{1}{\sqrt{\varepsilon_{22}^{(2)}}}\int_{t+\frac{x_3-\ell}{v_3^{(2)}}}^t J_2^{(2)}(-v_3^{(2)}(\tau-t)+x_3,\tau)\right] d\tau;$$

$$E_3^{(2)}(x_3,t) = -\frac{1}{\varepsilon_{33}^{(2)}}\int_0^t J_3^{(2)}(x_3,\tau)d\tau.$$

The values, matching conditions, $u_2^{(2)}(\ell^-,t+\frac{x_3-\ell}{v_1^{(2)}})$ and $u_4^{(2)}(\ell^-,t+\frac{x_3-\ell}{v_3^{(2)}})$ derived before in the Section 4.3.3.1 for region R_6.

Explicit formulae for the region R_7:

$$H_1^{(3)}(x_3,t) = \frac{1}{2\sqrt{\mu_{11}^{(3)}}}\left[u_3^{(3)}(\ell^+,t-\frac{x_3-\ell}{v_3^{(3)}})+\frac{1}{\sqrt{\varepsilon_{22}^{(3)}}}\int_{t-\frac{x_3-\ell}{v_3^{(3)}}}^t J_2^{(3)}(v_3^{(3)}(\tau-t)+x_3,\tau)\right.$$
$$\left. - \frac{1}{\sqrt{\varepsilon_{22}^{(3)}}}\int_0^t J_2^{(3)}(-v_3^{(3)}(\tau-t)+x_3,\tau)\right] d\tau;$$

$$H_2^{(3)}(x_3,t) = \frac{1}{2\sqrt{\mu_{22}^{(3)}}}\left[u_1^{(3)}(\ell^+,t-\frac{x_3-\ell}{v_1^{(3)}})-\frac{1}{\sqrt{\varepsilon_{11}^{(3)}}}\int_{t-\frac{x_3-\ell}{v_1^{(3)}}}^t J_1^{(3)}(v_1^{(3)}(\tau-t)+x_3,\tau)\right.$$
$$\left. + \frac{1}{\sqrt{\varepsilon_{11}^{(3)}}}\int_0^t J_1^{(3)}(-v_1^{(3)}(\tau-t)+x_3,\tau)\right] d\tau;$$

$$H_3^{(3)}(x_3,t) = 0;$$

$$E_1^{(3)}(x_3,t) = \frac{1}{2\sqrt{\varepsilon_{11}^{(3)}}}\left[u_1^{(3)}(\ell^+,t-\frac{x_3-\ell}{v_1^{(3)}}) - \frac{1}{\sqrt{\varepsilon_{11}^{(3)}}}\int_{t-\frac{x_3-\ell}{v_1^{(3)}}}^{t} J_1^{(3)}(v_1^{(3)}(\tau-t)+x_3,\tau) - \frac{1}{\sqrt{\varepsilon_{11}^{(3)}}}\int_0^t J_1^{(3)}(-v_1^{(3)}(\tau-t)+x_3,\tau)\right]d\tau;$$

$$E_2^{(3)}(x_3,t) = \frac{1}{2\sqrt{\varepsilon_{22}^{(3)}}}\left[-u_3^{(3)}(\ell^+,t-\frac{x_3-\ell}{v_3^{(3)}}) - \frac{1}{\sqrt{\varepsilon_{22}^{(3)}}}\int_{t-\frac{x_3-\ell}{v_3^{(3)}}}^{t} J_2^{(3)}(v_3^{(3)}(\tau-t)+x_3,\tau) - \frac{1}{\sqrt{\varepsilon_{22}^{(3)}}}\int_0^t J_2^{(3)}(-v_3^{(3)}(\tau-t)+x_3,\tau)\right]d\tau;$$

$$E_3^{(3)}(x_3,t) = -\frac{1}{\varepsilon_{33}^{(3)}}\int_0^t J_3^{(3)}(x_3,\tau)d\tau.$$

The values, matching conditions, $u_1^{(3)}(\ell^+,t-\frac{x_3-\ell}{v_1^{(3)}})$ and $u_3^{(3)}(\ell^+,t-\frac{x_3-\ell}{v_3^{(3)}})$ derived before in the Section 4.3.3.1 for region R_7.

Explicit formulae of the electric and magnetic fields at the third flat

Explicit formulae for the region R_8:

$$H_1^{(1)}(x_3,t) = \frac{1}{2\sqrt{\mu_{11}^{(1)}}}\left[\frac{1}{\sqrt{\varepsilon_{22}^{(1)}}}\int_0^t J_2^{(1)}(v_3^{(1)}(\tau-t)+x_3,\tau) + u_4^{(1)}(-0,t+\frac{x_3}{v_3^{(1)}}) - \frac{1}{\sqrt{\varepsilon_{22}^{(1)}}}\int_{t+\frac{x_3}{v_3^{(1)}}}^{t} J_2^{(1)}(-v_3^{(1)}(\tau-t)+x_3,\tau)\right]d\tau;$$

$$H_2^{(1)}(x_3,t) = \frac{1}{2\sqrt{\mu_{22}^{(1)}}}\left[-\frac{1}{\sqrt{\varepsilon_{11}^{(1)}}}\int_0^t J_1^{(1)}(v_1^{(1)}(\tau-t)+x_3,\tau) + u_2^{(1)}(-0,t+\frac{x_3}{v_1^{(1)}}) + \frac{1}{\sqrt{\varepsilon_{11}^{(1)}}}\int_{t+\frac{x_3}{v_1^{(1)}}}^{t} J_1^{(1)}(-v_1^{(1)}(\tau-t)+x_3,\tau)\right]d\tau;$$

$$H_3^{(1)}(x_3,t) = 0;$$

$$E_1^{(1)}(x_3,t) = \frac{1}{2\sqrt{\varepsilon_{11}^{(1)}}}\left[-\frac{1}{\sqrt{\varepsilon_{11}^{(1)}}}\int_0^t J_1^{(1)}(v_1^{(1)}(\tau-t)+x_3,\tau) - u_2^{(1)}(-0,t+\frac{x_3}{v_1^{(1)}}) - \frac{1}{\sqrt{\varepsilon_{11}^{(1)}}}\int_{t+\frac{x_3}{v_1^{(1)}}}^{t} J_1^{(1)}(-v_1^{(1)}(\tau-t)+x_3,\tau)\right]d\tau;$$

$$
\begin{aligned}
E_2^{(1)}(x_3,t) &= \frac{1}{2\sqrt{\varepsilon_{22}^{(1)}}}\left[-\frac{1}{\sqrt{\varepsilon_{22}^{(1)}}}\int_0^t J_2^{(1)}(v_3^{(1)}(\tau-t)+x_3,\tau)+u_4^{(1)}(-0,t+\frac{x_3}{v_3^{(1)}})\right.\\
&- \left.\frac{1}{\sqrt{\varepsilon_{22}^{(1)}}}\int_{t+\frac{x_3}{v_3^{(1)}}}^t J_2^{(1)}(-v_3^{(1)}(\tau-t)+x_3,\tau)\right]d\tau;
\end{aligned}
$$

$$
E_3^{(1)}(x_3,t) = -\frac{1}{\varepsilon_{33}^{(1)}}\int_0^t J_3^{(1)}(x_3,\tau)d\tau.
$$

The values, matching conditions, $u_2^{(1)}(-0,t+\frac{x_3}{v_1^{(1)}})$ and $u_4^{(1)}(-0,t+\frac{x_3}{v_3^{(1)}})$ derived before in the Section 4.3.3.2 for region R_8.

Explicit formulae for the region R_9:

$$
\begin{aligned}
H_1^{(2)}(x_3,t) &= \frac{1}{2\sqrt{\mu_{11}^{(2)}}}\left[u_3^{(2)}(+0,t-\frac{x_3}{v_1^{(2)}})+\frac{1}{\sqrt{\varepsilon_{22}^{(2)}}}\int_{t-\frac{x_3}{v_3^{(2)}}}^t J_2^{(2)}(v_3^{(2)}(\tau-t)+x_3,\tau)\right.\\
&+ \left.u_4^{(2)}(\ell^-,t+\frac{x_3-\ell}{v_3^{(2)}})-\frac{1}{\sqrt{\varepsilon_{22}^{(2)}}}\int_{t+\frac{x_3-\ell}{v_3^{(2)}}}^t J_2^{(2)}(-v_3^{(2)}(\tau-t)+x_3,\tau)\right]d\tau;
\end{aligned}
$$

$$
\begin{aligned}
H_2^{(2)}(x_3,t) &= \frac{1}{2\sqrt{\mu_{22}^{(2)}}}\left[u_1^{(2)}(+0,t-\frac{x_3}{v_1^{(2)}})-\frac{1}{\sqrt{\varepsilon_{11}^{(2)}}}\int_{t-\frac{x_3}{v_1^{(2)}}}^t J_1^{(2)}(v_1^{(2)}(\tau-t)+x_3,\tau)\right.\\
&+ \left.u_2^{(2)}(\ell^-,t+\frac{x_3-\ell}{v_1^{(2)}})+\frac{1}{\sqrt{\varepsilon_{11}^{(2)}}}\int_{t+\frac{x_3-\ell}{v_1^{(2)}}}^t J_1^{(2)}(-v_1^{(2)}(\tau-t)+x_3,\tau)\right]d\tau;
\end{aligned}
$$

$$
H_3^{(2)}(x_3,t) = 0;
$$

$$
\begin{aligned}
E_1^{(2)}(x_3,t) &= \frac{1}{2\sqrt{\varepsilon_{11}^{(2)}}}\left[u_1^{(2)}(+0,t-\frac{x_3}{v_1^{(2)}})-\frac{1}{\sqrt{\varepsilon_{11}^{(2)}}}\int_{t-\frac{x_3}{v_1^{(2)}}}^t J_1^{(2)}(v_1^{(2)}(\tau-t)+x_3,\tau)\right.\\
&- \left.u_2^{(2)}(\ell^-,t+\frac{x_3-\ell}{v_1^{(2)}})+\frac{1}{\sqrt{\varepsilon_{11}^{(2)}}}\int_{t+\frac{x_3-\ell}{v_1^{(2)}}}^t J_1^{(2)}(-v_1^{(2)}(\tau-t)+x_3,\tau)\right]d\tau;
\end{aligned}
$$

$$
\begin{aligned}
E_2^{(2)}(x_3,t) &= \frac{1}{2\sqrt{\varepsilon_{22}^{(2)}}}\left[-u_3^{(2)}(+0,t-\frac{x_3}{v_1^{(2)}})-\frac{1}{\sqrt{\varepsilon_{22}^{(2)}}}\int_{t-\frac{x_3}{v_3^{(2)}}}^t J_2^{(2)}(v_3^{(2)}(\tau-t)+x_3,\tau)\right.\\
&+ \left.u_4^{(2)}(\ell^-,t+\frac{x_3-\ell}{v_3^{(2)}})-\frac{1}{\sqrt{\varepsilon_{22}^{(2)}}}\int_{t+\frac{x_3-\ell}{v_3^{(2)}}}^t J_2^{(2)}(-v_3^{(2)}(\tau-t)+x_3,\tau)\right]d\tau;
\end{aligned}
$$

$$
E_3^{(2)}(x_3,t) = -\frac{1}{\varepsilon_{33}^{(2)}}\int_0^t J_3^{(2)}(x_3,\tau)d\tau.
$$

The values, matching conditions, $u_i^{(2)}(+0,t-\frac{x_3}{v_i^{(2)}})$, $i=1,3$ and $u_j^{(2)}(\ell^-,t+\frac{x_3-\ell}{v_j^{(2)}})$, $j=2,4$ derived before in the Section 4.3.3.2 for region R_9.

Explicit formulae for the region R_{10}:

$$\begin{aligned}
H_1^{(2)}(x_3,t) &= \frac{1}{2\sqrt{\mu_{11}^{(2)}}}\Bigg[u_3^{(2)}(+0,t-\frac{x_3}{v_1^{(2)}})+\frac{1}{\sqrt{\varepsilon_{22}^{(2)}}}\int_{t-\frac{x_3}{v_3^{(2)}}}^{t} J_2^{(2)}(v_3^{(2)}(\tau-t)+x_3,\tau)\\
&+ u_4^{(2)}(\ell^-,t+\frac{x_3-\ell}{v_3^{(2)}})-\frac{1}{\sqrt{\varepsilon_{22}^{(2)}}}\int_{t+\frac{x_3-\ell}{v_3^{(2)}}}^{t} J_2^{(2)}(-v_3^{(2)}(\tau-t)+x_3,\tau)\Bigg]d\tau;
\end{aligned}$$

$$\begin{aligned}
H_2^{(2)}(x_3,t) &= \frac{1}{2\sqrt{\mu_{22}^{(2)}}}\Bigg[u_1^{(2)}(+0,t-\frac{x_3}{v_1^{(2)}})-\frac{1}{\sqrt{\varepsilon_{11}^{(2)}}}\int_{t-\frac{x_3}{v_1^{(2)}}}^{t} J_1^{(2)}(v_1^{(2)}(\tau-t)+x_3,\tau)\\
&+ u_2^{(2)}(\ell^-,t+\frac{x_3-\ell}{v_1^{(2)}})+\frac{1}{\sqrt{\varepsilon_{11}^{(2)}}}\int_{t+\frac{x_3-\ell}{v_1^{(2)}}}^{t} J_1^{(2)}(-v_1^{(2)}(\tau-t)+x_3,\tau)\Bigg]d\tau;
\end{aligned}$$

$$H_3^{(2)}(x_3,t)=0;$$

$$\begin{aligned}
E_1^{(2)}(x_3,t) &= \frac{1}{2\sqrt{\varepsilon_{11}^{(2)}}}\Bigg[u_1^{(2)}(+0,t-\frac{x_3}{v_1^{(2)}})-\frac{1}{\sqrt{\varepsilon_{11}^{(2)}}}\int_{t-\frac{x_3}{v_1^{(2)}}}^{t} J_1^{(2)}(v_1^{(2)}(\tau-t)+x_3,\tau)\\
&- u_2^{(2)}(\ell^-,t+\frac{x_3-\ell}{v_1^{(2)}})+\frac{1}{\sqrt{\varepsilon_{11}^{(2)}}}\int_{t+\frac{x_3-\ell}{v_1^{(2)}}}^{t} J_1^{(2)}(-v_1^{(2)}(\tau-t)+x_3,\tau)\Bigg]d\tau;
\end{aligned}$$

$$\begin{aligned}
E_2^{(2)}(x_3,t) &= \frac{1}{2\sqrt{\varepsilon_{22}^{(2)}}}\Bigg[-u_3^{(2)}(+0,t-\frac{x_3}{v_1^{(2)}})-\frac{1}{\sqrt{\varepsilon_{22}^{(2)}}}\int_{t-\frac{x_3}{v_3^{(2)}}}^{t} J_2^{(2)}(v_3^{(2)}(\tau-t)+x_3,\tau)\\
&+ u_4^{(2)}(\ell^-,t+\frac{x_3-\ell}{v_3^{(2)}})-\frac{1}{\sqrt{\varepsilon_{22}^{(2)}}}\int_{t+\frac{x_3-\ell}{v_3^{(2)}}}^{t} J_2^{(2)}(-v_3^{(2)}(\tau-t)+x_3,\tau)\Bigg]d\tau;
\end{aligned}$$

$$E_3^{(2)}(x_3,t)=-\frac{1}{\varepsilon_{33}^{(2)}}\int_0^t J_3^{(2)}(x_3,\tau)d\tau.$$

The values, matching conditions, $u_i^{(2)}(+0,t-\frac{x_3}{v_i^{(2)}})$, $i=1,3$ and $u_j^{(2)}(\ell^-,t+\frac{x_3-\ell}{v_j^{(2)}})$, $j=2,4$ derived before in the Section 4.3.3.2 for region R_{10}.

Explicit formulae for the region R_{11}:

$$
\begin{aligned}
H_1^{(2)}(x_3,t) &= \frac{1}{2\sqrt{\mu_{11}^{(2)}}}\Bigg[u_3^{(2)}(+0,t-\frac{x_3}{v_1^{(2)}})+\frac{1}{\sqrt{\varepsilon_{22}^{(2)}}}\int_{t-\frac{x_3}{v_3^{(2)}}}^{t} J_2^{(2)}(v_3^{(2)}(\tau-t)+x_3,\tau)\\
&+ u_4^{(2)}(\ell^-,t+\frac{x_3-\ell}{v_3^{(2)}})-\frac{1}{\sqrt{\varepsilon_{22}^{(2)}}}\int_{t+\frac{x_3-\ell}{v_3^{(2)}}}^{t} J_2^{(2)}(-v_3^{(2)}(\tau-t)+x_3,\tau)\Bigg]d\tau;
\end{aligned}
$$

$$
\begin{aligned}
H_2^{(2)}(x_3,t) &= \frac{1}{2\sqrt{\mu_{22}^{(2)}}}\Bigg[u_1^{(2)}(+0,t-\frac{x_3}{v_1^{(2)}})-\frac{1}{\sqrt{\varepsilon_{11}^{(2)}}}\int_{t-\frac{x_3}{v_1^{(2)}}}^{t} J_1^{(2)}(v_1^{(2)}(\tau-t)+x_3,\tau)\\
&+ u_2^{(2)}(\ell^-,t+\frac{x_3-\ell}{v_1^{(2)}})+\frac{1}{\sqrt{\varepsilon_{11}^{(2)}}}\int_{t+\frac{x_3-\ell}{v_1^{(2)}}}^{t} J_1^{(2)}(-v_1^{(2)}(\tau-t)+x_3,\tau)\Bigg]d\tau;
\end{aligned}
$$

$$
H_3^{(2)}(x_3,t)=0;
$$

$$
\begin{aligned}
E_1^{(2)}(x_3,t) &= \frac{1}{2\sqrt{\varepsilon_{11}^{(2)}}}\Bigg[u_1^{(2)}(+0,t-\frac{x_3}{v_1^{(2)}})-\frac{1}{\sqrt{\varepsilon_{11}^{(2)}}}\int_{t-\frac{x_3}{v_1^{(2)}}}^{t} J_1^{(2)}(v_1^{(2)}(\tau-t)+x_3,\tau)\\
&- u_2^{(2)}(\ell^-,t+\frac{x_3-\ell}{v_1^{(2)}})+\frac{1}{\sqrt{\varepsilon_{11}^{(2)}}}\int_{t+\frac{x_3-\ell}{v_1^{(2)}}}^{t} J_1^{(2)}(-v_1^{(2)}(\tau-t)+x_3,\tau)\Bigg]d\tau;
\end{aligned}
$$

$$
\begin{aligned}
E_2^{(2)}(x_3,t) &= \frac{1}{2\sqrt{\varepsilon_{22}^{(2)}}}\Bigg[-u_3^{(2)}(+0,t-\frac{x_3}{v_1^{(2)}})-\frac{1}{\sqrt{\varepsilon_{22}^{(2)}}}\int_{t-\frac{x_3}{v_3^{(2)}}}^{t} J_2^{(2)}(v_3^{(2)}(\tau-t)+x_3,\tau)\\
&+ u_4^{(2)}(\ell^-,t+\frac{x_3-\ell}{v_3^{(2)}})-\frac{1}{\sqrt{\varepsilon_{22}^{(2)}}}\int_{t+\frac{x_3-\ell}{v_3^{(2)}}}^{t} J_2^{(2)}(-v_3^{(2)}(\tau-t)+x_3,\tau)\Bigg]d\tau;
\end{aligned}
$$

$$
E_3^{(2)}(x_3,t)=-\frac{1}{\varepsilon_{33}^{(2)}}\int_0^t J_3^{(2)}(x_3,\tau)d\tau.
$$

The values, matching conditions, $u_i^{(2)}(+0,t-\frac{x_3}{v_i^{(2)}})$, $i=1,3$ and $u_j^{(2)}(\ell^-,t+\frac{x_3-\ell}{v_j^{(2)}})$, $j=2,4$ derived before in the Section 4.3.3.2 for region R_{11}.

Explicit formulae for the region R_{12}:

$$
\begin{aligned}
H_1^{(3)}(x_3,t) &= \frac{1}{2\sqrt{\mu_{11}^{(3)}}}\Bigg[u_3^{(3)}(\ell^+,t-\frac{x_3-\ell}{v_3^{(3)}})+\frac{1}{\sqrt{\varepsilon_{22}^{(3)}}}\int_{t-\frac{x_3-\ell}{v_3^{(3)}}}^{t} J_2^{(3)}(v_3^{(3)}(\tau-t)+x_3,\tau)\\
&- \frac{1}{\sqrt{\varepsilon_{22}^{(3)}}}\int_0^t J_2^{(3)}(-v_3^{(3)}(\tau-t)+x_3,\tau)\Bigg]d\tau;
\end{aligned}
$$

$$H_2^{(3)}(x_3,t) = \frac{1}{2\sqrt{\mu_{22}^{(3)}}}\Bigg[u_1^{(3)}(\ell^+,t-\frac{x_3-\ell}{v_1^{(3)}}) - \frac{1}{\sqrt{\varepsilon_{11}^{(3)}}}\int_{t-\frac{x_3-\ell}{v_1^{(3)}}}^{t} J_1^{(3)}(v_1^{(3)}(\tau-t)+x_3,\tau) + \frac{1}{\sqrt{\varepsilon_{11}^{(3)}}}\int_0^t J_1^{(3)}(-v_1^{(3)}(\tau-t)+x_3,\tau)\Bigg]d\tau;$$

$$H_3^{(3)}(x_3,t) = 0;$$

$$E_1^{(3)}(x_3,t) = \frac{1}{2\sqrt{\varepsilon_{11}^{(3)}}}\Bigg[u_1^{(3)}(\ell^+,t-\frac{x_3-\ell}{v_1^{(3)}}) - \frac{1}{\sqrt{\varepsilon_{11}^{(3)}}}\int_{t-\frac{x_3-\ell}{v_1^{(3)}}}^{t} J_1^{(3)}(v_1^{(3)}(\tau-t)+x_3,\tau) - \frac{1}{\sqrt{\varepsilon_{11}^{(3)}}}\int_0^t J_1^{(3)}(-v_1^{(3)}(\tau-t)+x_3,\tau)\Bigg]d\tau;$$

$$E_2^{(3)}(x_3,t) = \frac{1}{2\sqrt{\varepsilon_{22}^{(3)}}}\Bigg[-u_3^{(3)}(\ell^+,t-\frac{x_3-\ell}{v_3^{(3)}}) - \frac{1}{\sqrt{\varepsilon_{22}^{(3)}}}\int_{t-\frac{x_3-\ell}{v_3^{(3)}}}^{t} J_2^{(3)}(v_3^{(3)}(\tau-t)+x_3,\tau) - \frac{1}{\sqrt{\varepsilon_{22}^{(3)}}}\int_0^t J_2^{(3)}(-v_3^{(3)}(\tau-t)+x_3,\tau)\Bigg]d\tau;$$

$$E_3^{(3)}(x_3,t) = -\frac{1}{\varepsilon_{33}^{(3)}}\int_0^t J_3^{(3)}(x_3,\tau)d\tau.$$

The values, matching conditions, $u_1^{(3)}(\ell^+,t-\frac{x_3-\ell}{v_1^{(3)}})$ and $u_3^{(3)}(\ell^+,t-\frac{x_3-\ell}{v_3^{(3)}})$ derived before in the Section 4.3.3.2 for region R_{12}.

CHAPTER FIVE

CONCLUSION

In the thesis mathematical model for the electromagnetic wave propagation in homogeneous electrically and magnetically anisotropic media is described by the time-dependent Maxwell's system. Using the method of characteristics explicit formulae for solutions of the time-dependent Maxwell's system is obtained. These solutions are constructed for free space and for layered media. Using these formulae and symbolic transformation in MATLAB the electric and magnetic waves are simulated and images are presented. Images of electric and magnetic fields for different isotropic and anisotropic media have been obtained and their analysis has been given. Our further plan is to get a generalization of this work on layered media where we have more layers than three.

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