DOKUZ EYLÜL UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

N-CATEGORIES

by
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İZMİR

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We have read the thesis entitled **"N-CATEGORIES"** completed by **SABRİ KAAN GÜRBÜZER** under supervision of **ASSIST. PROF. DR. BEDİA AKYAR MOLLER** and we certify that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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N-CATEGORIES

ABSTRACT

In this thesis, we examine some different types of categories and try to find a place for some geometrical subjects in category theory. By using functors and natural transformations we approach $n-$categories and higher categories inductively in some different aspects. On the other hand we use some algebraic topological concepts such as simplicial complexes and simplicial sets and give the definition in categorical sense. We also explain the relation $n-$category and homotopy theory.

Keywords: Homotopy, n-category, bicategory, simplex, functor.

N-KATEGORİLER

ÖZ

Bu çalışmada, farklı kategori tipleri incelendi ve bazı geometrik cisimlerin kategori teorisindeki yeri araştırıldı. Funktorlar ve doğal dönüşümler kullanılarak n-kategorilere ve yüksek mertebeden kategorilere tümevarımsal farklı bakış açıları ile yaklaşıldı. Cebirsel topolojideki bazı kavramlar kullanıldı ve kategori teorisindeki tanımları verildi. Ayrıca n-kategoriler ile homotopi teorisi arasındaki ilişkiler açıklandı.

Anahtar Sözcükler: Homotopi, n-kategori, bikategori, simplex, funktor.

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CHAPTER ONE

INTRODUCTION

We will give the motivating ideas of the thesis by saying that category theory is the mathematical study of abstract algebra of functions. Category theory arises from the idea of a system of functions among some objects. One thinks of the composition $g \circ f$ as a sort of product of the functions f and g and consider abstract algebra of algebras of the sort arising from collections of functions. A category is just an algebra, consisting of objects $X, Y, Z, \ldots$ and morphisms $f : X \to Y, g : Y \to Z, \ldots$ that are closed under composition and satisfy certain conditions.

First, let us explain the historical development of category theory. In 1945 the theory was first formulated in Eilenberg and Mac Lane's original paper named *General theory of natural equivalences*. Late in 1940s the main applications were originally in the fields of algebraic topology, particularly homology theory and abstract algebra. In 1950s A. Grothendick et al. began using category theory with a great success in algebraic geometry. In 1960s F.W. Lawvere and others began applying categories to logic, revealing some deep and surprising connections. Also between 1963 and 1966 Lawvere began by characterizing the category of categories. In 1970s applications were already appearing in computer science, philosophy and many other areas. Lawvere's approach, under active development by various mathematicians, logicians and mathematical physicists, lead to what are now called *higher dimensional categories*.

In Chapter Two, we start with the definition of category and describe large and small categories. We continue with some examples and relation between categories and homotopy theory. We show that functors which can be considered as functions connecting with one object and another object, constitute the connection of two categories. After that we give some properties of functors and we investigate the fundamental group of a topological space. We see the relation between topological spaces and group structures by using the fundamental group. Before searching the representable functors, we mention natural transformations among functors and also functor category which consists of natural transformations as morphisms.

In Chapter Three we see the constructions in categories by using the cone structures which are called limits and colimits of a functor in general. Then we give some important examples of limits and colimits in categories and applications in homotopy theory. After giving the equivalence among categories which is also called adjunction of two functors, we finish this chapter with the definition of monads.

In the last Chapter we start by giving the definition of monoidal categories and some related examples. Furthermore, we explain the geometric meaning of simplicial sets which leads us simplicial complex. We also study subdivisions of simplicial complexes. After all, we see bicategories and the definition of $n-$categories. We explain the relation between $n-$categories and Homotopy theory. Finally we compare the definition of Zouhair Tamsamani $n-$categories with the definition of $n-$bicategories.

CHAPTER TWO

CATEGORIES, FUNCTORS AND NATURAL TRANSFORMATIONS

2.1 Categories

Here we start with giving the definition of categories. In order to be prepare the next sections, we define small and locally small categories. We shall list some general categories with their objects and morphisms in a table implicitly. After explaining the homotopy category **Toph**, we shall give the definitions of some special elements of categories with examples.

Definition 2.1.1. A category $\mathscr{C}$ consists of:

- A collection of objects denoted by ob$(\mathscr{C})$
- For every pair $X,Y \in ob(\mathscr{C})$, a collection of morphisms (also referred to as maps or arrows) with domain X and codomain Y, $f : X \to Y$, denoted by $\mathscr{C}(X,Y)$ or $Hom_{\mathscr{C}}(X,Y)$ equipped with
  - for each object $X \in ob(\mathscr{C})$, an identity map $id_X = 1_X \in \mathscr{C}(X,X)$
  - for each $X,Y,Z \in ob(\mathscr{C})$, a composition map

$$\circ_{XYZ} : \mathscr{C}(Y,Z) \times \mathscr{C}(X,Y) \to \mathscr{C}(X,Z)$$
$$(g, f) \mapsto g \circ f = gf$$

These conditions satisfy the following properties:

a. *Unit law*: For all morphism $f : X \to Y$ and $g : Y \to Z$ composition with identity map 1_Y gives $1_Y \circ f = f$ and $g \circ 1_Y = g$.

b. *Associativity*: For given objects and morphisms in the configuration

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$$

have always the equality $h \circ (g \circ f) = (h \circ g) \circ f$.

As 2.1.1 if we have collections of objects and morphisms in a category we can think about domain and codomain as morphisms. Let $\mathscr{C}_0$ and $\mathscr{C}_1$ denote the collection of objects and morphisms in $\mathscr{C}$ respectively, then we have a diagram

$$\mathscr{C}_1 \overset{domain}{\underset{codomain}{\rightrightarrows}} \mathscr{C}_0$$

where the domain function assigns a morphism with its domain (or source) and codomain function assigns a morphism with its codomain (or target). This motivates the definition.

Definition 2.1.2. Given a category $\mathscr{C}$, the *dual* or *opposite* category $\mathscr{C}^{op}$ is defined by:

- $ob(\mathscr{C}) = ob(\mathscr{C}^{op})$,
- $\mathscr{C}(X,Y) = \mathscr{C}^{op}(Y,X)$,
- identities inherited,
- $f^{op} \circ g^{op} = (f \circ g)^{op}$.

It is pointed out here that all of the objects are preserved but the morphisms are reversed. In category theory for any given property, feature or theorem in terms of morphisms, we can immediately obtain its dual by reversing all the arrows and this is often indicated by prefix "co-". One can say that this is the principle of the duality. We will see many examples of the duality later on.

In order to define small categories we give the definition of a universe.

Definition 2.1.3. A *universe* U is a non-empty set which satisfies the followings :

- If $x \in U$ and then $y \in x$, $y \in U$.
- If $x, y \in U$, then $\{x,y\} \in U$.
- If $x \in U$, then $\mathscr{P}(x) \in U$.
- $\{x_i |\ i \in I \in U\} \Rightarrow \bigcup_{i \in I} x_i \in U$.

Definition 2.1.4. A set S is said to be U-*small* if it is isomorphic to an element of U. Let the universe U be fixed and call $u \in U$ small set. Then the universe U is the set of all small sets. Similarly, a function $f : u \to v$ is small when u and v are small sets.

Definition 2.1.5. A category $\mathscr{C}$ is *small* if $ob(\mathscr{C})$ and all of the $\mathscr{C}(X,Y)$ are small sets and *locally small* if each $\mathscr{C}(X,Y)$ is a small set.

Remark 2.1.6. the category of all sets **Set** is not small because the set of its objects is not small set, otherwise we get a contradiction with the universality of fixed U s.t. $U \in U$ and this is contrary to hierarchy, which asserts that there are no infinite chains $...U_n \in U_{n-1} \in U_{n-2} \in ... \in U_0$.

Definition 2.1.7. A category $\mathscr{C}$ is called *discrete* if the only morphisms are identities, that is;

$$\mathscr{C}(X,Y) = \begin{cases} \{1_X\} & \text{if X = Y ;} \\ \emptyset & \text{otherwise .} \end{cases}$$

With aid of this definition any set can be considered as a discrete category with the identity morphisms.

Definition 2.1.8. A *subcategory* $\mathscr{D}$ of $\mathscr{C}$ consists of subcollections

- $ob(\mathscr{D}) \subseteq ob(\mathscr{C})$
- $Hom_{\mathscr{D}} \subseteq Hom_{\mathscr{C}}$

together with composition and identities inherited from $\mathscr{C}$. We say that $\mathscr{D}$ is a *full subcategory* of $\mathscr{C}$ if $\forall X,Y \in \mathscr{D}$, $\mathscr{D}(X,Y) = \mathscr{C}(X,Y)$, and a *luff subcategory* of $\mathscr{C}$ if $ob(\mathscr{D}) = ob(\mathscr{C})$.

In Table 2.1, we give some general categories implicitly where the composition of the maps is ordinary composition.

Table 2.1 General Categories in Mathematics

	objects	**arrows (or morphisms)**
Set	all sets	all functions between sets
Set$_*$	all sets each with a selected base point	base-point-preserving functions
Mon	all monoids	all homomorphisms of monoids
Grp	all groups	all morphisms of groups
Ab	all (additive) abelian groups	all morphisms of abelian groups
Rng	all rings	ring morphisms preserving units
CRng	all commutative rings	ring morphisms preserving units
$\mathscr{R}$**-Mod**	all left modules over the ring $\mathscr{R}$	all linear maps between them
Mod-$\mathscr{R}$	all right $\mathscr{R}$ modules	all linear maps between them
$\mathscr{K}$ **-Mod**	all modules over the commutative ring $\mathscr{K}$	all linear maps between them
Top	all topological spaces	continuous functions
Top$_*$	all topological spaces with selected base point	base-point preserving continuous functions

In table 2.1, one can see that **Set**$_*$ is a subcategory of **Set**. **Set**$_*$ is not full, because the hom-set of **Set**$_*$ includes just base-point preserving functions, but it is a luff subcategory of **Set** since ob(**Set**$_*$)=ob(**Set**). Now we explain the homotopy category **Toph** (also denoted by **[Top]**) explicitly after giving the definition of homotopy.

Definition 2.1.9. Let X,Y be topological spaces and f,g continuous maps from X to Y. A homotopy H is a continuous function from $X \times I$ to Y , where I denotes the unit interval $[0,1]$, satisfying

$H(x,0) = f(x)$ and $H(x,1) = g(x)$ for all $x \in X$. If there exists such a function H then f and g are said to be homotopic. Moreover, homotopy is an equivalence relation with respect to the followings:

- (*reflexive*) Let $H : X \times I \to Y$ be defined by $H(x,t) = f(x)$ for all $t \in I$ where $f : X \to Y$ is continuous. H is continuous because it is the composition of the continuous function f and projection onto the first factor. This means that any continuous function is homotopic to itself.

- (*symmetry*) $H : X \times I \to Y$ be any given homotopy such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$ where f, g are continuous functions from X to Y. Let us define a homotopy $G : X \times I \to Y$ such that $G(x,t) = H(x,1-t)$ for all $(x,t) \in X \times I$. Since H is continuous, G is clearly continuous and homotopy from g to f. This shows that homotopy is symmetric.

- (*transitivity*) For given homotopies $H, G : X \times I \to Y$ between f, g, h such that $H(x,0) = f(x)$, $H(x,1) = G(x,0) = g(x)$ and $G(,1) = h(x)$ let us define a homotopy $F : X \times I \to Y$ by using the Glueing Lemma, that is,

$$F(x,t) = \begin{cases} H(x,2t), & t \in [0, \frac{1}{2}]; \\ G(x,2t-1), & t \in [\frac{1}{2}, 1]. \end{cases}$$

  So we have $F(x,0) = f(x)$, $F(x,1) = h(x)$ and this means that homotopy is transitive.

We denote the homotopy class of continuous functions by $[f]$. According to these, before we construct a subcategory **Toph** of **Top** whose objects are topological spaces and whose morphisms are the homotopy equivalence classes of the continuous functions between topological spaces, we should check whether the composition of the equivalence classes is well-defined or not.

Theorem 2.1.10. *Let X, Y, Z be topological spaces. Suppose that f_0 and f_1 are homotopic maps $X \to Y$ and that g_0 and g_1 are homotopic maps $Y \to Z$. Then $g_0 \circ f_0$ and $g_1 \circ f_1$ are homotopic maps $X \to Z$.*

Proof. Let $H : X \times I \to Y$ be a homotopy from f_0 to f_1. Let $G = g_0 \circ H : X \times I \to Z$ then G is continuous and homotopy from $g_0 \circ f_0$ to $g_0 \circ f_1$. Let $\tilde{f}_1 : X \times I \to Y \times I$ be defined by $\tilde{f}_1(x,t) = (f_1(x),t)$, it is seen that $\tilde{f}_1$ is continuous and suppose that $F : Y \times I \to Z$ is a homotopy from g_0 to g_1. Now we construct a homotopy $K = F \circ \tilde{f}_1 : X \times I \to Z$. So K is continuous and homotopy from $g_0 \circ f_1$ to $g_1 \circ f_1$. We have that $g_0 \circ f_0$ is homotopic to $g_0 \circ f_1$ and $g_0 \circ f_1$ is homotopic to $g_1 \circ f_1$. Since homotopy is transitive $g_0 \circ f_0$ is homotopic to $g_1 \circ f_1$ as desired. □

We continue with an example of one object category, whose morphisms are not just identities.

Example 2.1.11. A *monoid* is a set M with a binary operation $\star : M \times M \longrightarrow M$, obeying the following axioms;

- Associativity : $\forall a,b,c \in M$; $(a \star b) \star c = a \star (b \star c)$.
- Identity element : There exist an element $e \in M$, such that $\forall a \in M$; $a \star e = e \star a = a$. One often sees the additional axiom :
- Closure : $\forall a,b \in M$, $a \star b \in M$ through , strictly speaking , this axiom is implied by the notion of the operation .

A monoid is exactly a semigroup with identity element and according to the definition of monoids, we can construct a category with one object M. Let us take the elements of M as arrows this means that if $a \in M$ then $a : M \longrightarrow M$ such that $a(m) = a \star m$. The associativity and unit laws are satisfied clearly according to definition of the binary operation "$\star$" . For any category $\mathscr{C}$ and any object $X \in \mathscr{C}$, the set of $Hom_{\mathscr{C}}(X,X)$ of all arrows $X \to X$ is a monoid with respect to the composition of arrows.

In the last part of this section, we define some special kinds of objects and morphisms with examples in general categories.

Definition 2.1.12. An element T of $ob(\mathscr{C})$ is called *terminal* if $\forall X \in ob(\mathscr{C})$, there exists a unique morphism $k : X \longrightarrow T$ and dually an element I of $ob(\mathscr{C})$ is called *initial* if $\forall X \in ob(\mathscr{C})$ there exists a unique morphism $k : I \longrightarrow X$. If an object Z is both initial and terminal in a category then it is called *null object* of this category. For example, **Set** all one element sets are terminal and the unique morphism is clearly constant map and similar in **Top** all one point space are terminal. The empty set $\emptyset$ in the category **Set** is accepted as initial object.

Definition 2.1.13. A morphism $m : X \longrightarrow Y$ is *monic* in $\mathscr{C}$, when for any two morphisms $f_1, f_2 : U \longrightarrow X$ the equality $mf_1 = mf_2$ implies $f_1 = f_2$, in otherwords m is monic if it is left cancelable. A morphism $e : X \longrightarrow Y$ is called *epi* in $\mathscr{C}$ if for any two morphisms $g_1, g_2 : Y \longrightarrow U$ the equality $g_1 e = g_2 e$ implies $g_1 = g_2$, or e is epi if it is right cancelable. In **Set** it is clear that monics are injections and since g_1, g_2 are functions then epis must be surjections.

Example 2.1.14. Let us consider the following diagram in **Mon**.

$$\mathbb{N} \xrightarrow{e} \mathbb{Z} \overset{f}{\underset{g}{\rightrightarrows}} (M, e, \star)$$

Now suppose e is an embedding and f,g are two monoid homomorphisms which agree on the non-negative integers. Then

$$f(-1) = f(-1) \star g(1) \star g(-1) = f(-1) \star f(1) \star g(-1) = g(-1)$$

so f and g agree on the whole of $\mathbb{Z}$. This means that e is an epi.

Definition 2.1.15. A morphism $f \in \mathscr{C}(X,Y)$ is an *isomorphism* if $\exists g \in \mathscr{C}(Y,X)$ such that $gf = 1_X$ and $fg = 1_Y$. Moreover f is called *invertible* and g is an *inverse* of f. For instance, in **Toph** given two topological spaces X,Y the morphism $f : X \to Y$ is called *homotopy equivalence* if there exists a continuous morphism $g : Y \to X$ satisfying that $f \circ g$ is homotopic to 1_Y and $g \circ f$ is homotopic to 1_Y. If there exists such a homotopy equivalence f between X,Y then it is said that X and Y are *homotopy equivalent* or *of the same homotopy type*. Another example of the isomorphisms is the bijections in the category **Set**.

2.2 Functors and Natural Transformations

In this section we try to give the relation between two categories with using the functors. Functors are really important because they are like bridge between any two of the mathematical part such that topology and algebra. For example, we use the functors to construct fundamental group of a topological space and this helps us to solve some problems which can not be solved easier. Then we meet with natural transformations as we see in the next sections this gives us an idea to approach to the $n-categories$ and we discribe the functor category. After these we will define Yoneda embedding which is our second aim in this section.

2.2.1 Functors

Definition 2.2.1. Now we can think about the category **Cat** in which the objects are categories and the morphisms are the mappings between categories. The morphisms in such a category are known as *functors*.

We know that a category $\mathscr{C}$ consists of objects and morphisms. So any functor $F : \mathscr{C} \to \mathscr{D}$ must carry objects of $\mathscr{C}$ to objects of $\mathscr{D}$ and morphisms of $\mathscr{C}$ to morphisms of $\mathscr{D}$, such that the following diagram are commutative.

$$\begin{array}{lcccc}
F & : & \mathscr{C} & \longrightarrow & \mathscr{D} \\
ob. & & X & \overset{F}{\longmapsto} & FX \\
morp. & & f \big\downarrow & \longmapsto & \big\downarrow Ff \\
 & & Y & & FY
\end{array}$$

We can combine objects and morphisms in one diagram so we get

$$\begin{array}{ccc} X & \xrightarrow{F} & FX \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle Ff} \\ Y & \xrightarrow[F]{} & FY \end{array}$$

Before giving an example of functor let us look some of its properties. (Baez & Shulman (2006))

- Let $F : \mathscr{C} \to \mathscr{D}$ be a functor, $\forall f, g \in \mathscr{C}(X, X')$ where $X, X' \in ob(\mathscr{C})$, if we have that $Ff = Fg$ implies $f = g$, then F is called *faithful*. This means that F is an injection on morphisms.

- For $F : \mathscr{C} \to \mathscr{D}$ if $\forall h \in \mathscr{D}(FX, FX')$ there exists a morphism $f \in \mathscr{C}(X, X')$ for every pair of $ob(\mathscr{C})$, then F is called *full* and this means that F is surjection on morphisms.

- A functor $F : \mathscr{C} \to \mathscr{D}$ is *essentially surjective* on objects if and only if $\forall Y \in \mathscr{D}$, $\exists X \in \mathscr{C}$ such that $FX \cong Y$.

- In mathematics we are often interested in equipping things with extra structure, staff, or properties. So we can also consider the functors with four different parts :

- F *forgets nothing* if it is an equivalence of categories that is F is faithfull, full and essentially surjective. For example identity functor.

- F *forgets at most properties* if it is faithfull and full. For example, **Ab** $\to$ **Grp** which forgets the property of being abelian, but a homomorphism of abelian groups is just a homomorphism between groups that happen to be abelian.

- F *forgets at most structure* if it is faithfull. For example, the functor from **Top** to **Set**, it forgets the structure of being topological space, but it is still faithfull.

- F *forgets at most staff* if it is arbitrary. For example, **Set** $\times$ **Set** $\to$ **Set**, where we just throw out the second set, is not even faithfull.

Definition 2.2.2. A *contravariant* functor $F : \mathscr{C} \to \mathscr{D}$ is a functor $\mathscr{C}^{op} \to \mathscr{D}$, that is,

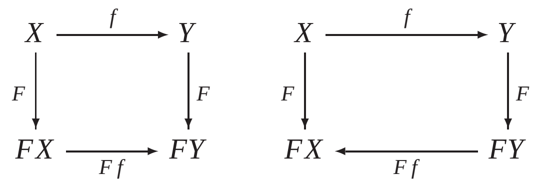
- on objects, $X \to FX$

- on morphisms, $(f : X \to Y) \mapsto (Ff : FY \to FX)$

- identities are preserved

- $F(g \circ f) = Ff \circ Fg$

A *non-contravariant* functor is sometimes reffered to as a *covariant* functor and the following diagrams are commutative.



covariant *contravariant*

Example 2.2.3. Let X be a topological space and I be the interval $[0,1]$, a continuous map α from I to X starting at x and ending at y, that is, $\alpha : I \to X$ such that $\alpha(0) = x$ and $\alpha(1) = y$ for $x, y \in X$, is called *path*. If a path α has the same starting and ending points; such that $\alpha : I \to X$, $\alpha(0) = \alpha(1) = x \in X$, then α is called *a loop* with base point x_0. A homotopy between two paths α and β is a continuous function such that $H : I \times I \to X$ for $s,t \in I$ satisfies the followings:

$$H(s,0) = \alpha(s) \ , H(s,1) = \beta(s)$$
$$H(0,t) = x_0 \ , H(1,t) = x_1$$

Here x_0 is the starting point and x_1 is the ending point of the two curves. Given any two path with same starting and ending point if there exists such a continuous function then the curves are said to be homotopic. By the same procedure 2.1.9 homotopy is also an equivalence relation on paths. The homotopy class of a path α denoted by $[\alpha]$. Let x_0 be the base point of X, the set of all homotopy classes of loops with base point x_0 forms the fundamental group of X at a base point x_0 and it is denoted by $\Pi_1(X,x_0)$ or simply $\Pi_1(X)$ where the binary operation is defined by the composition of the paths, that is,

$$[\alpha] * [\beta] = [\beta \circ \alpha]$$

The composition of paths is given with respect to the parameter $t \in I$, since the ending point of the first path is the starting point of the second one, they can be glued at the common point and we can formulate it by dividing the interval I into the two parts

$$\beta \circ \alpha = \begin{cases} \alpha(2t), & \text{t} \in [0,\frac{1}{2}]; \\ \beta(2t-1), & \text{t} \in [\frac{1}{2},1]. \end{cases}$$

The identity element of the fundamental group is the constant map at the base point x_0 and the inverse homotopy class of a path α is $[\alpha]^{-1} = [\alpha^{-1}] = [\alpha(1-t)]$ the homotopy class of the inverse of α for $t \in I$, that is, α^{-1} follows α backwards.

If $f:(X,x_0)\longrightarrow(Y,y_0)$ is a continuous base point preserving function, such that $f(x_0)=y_0$ for the base points $x_0\in X$ and $y_0\in Y$ respectively, then every loop in X with base point x_0 can be composed with f to yield a loop in Y with the base point y_0. Let α is a loop in X at x_0, since f is continuous $f\circ\alpha$ is a loop in Y at y_0. This composition is compatible with the homotopy equivalence relation and with the composition of loops. Hence we can define a group homomorphism which is called *induced homomorphism*;

$$\begin{aligned} f_*:\Pi_1(X,x_0)&\rightarrow\Pi_1(Y,y_0)\\ [\alpha]&\mapsto[f\circ\alpha]\end{aligned}$$

This operation is compatible with the composition of functions, that is, let $f:(X,x_0)\longrightarrow(Y,y_0)$ and $g:(Y,y_0)\longrightarrow(Z,z_0)$ be continuous base preserving functions then the composition of the induced maps f_* and g_* is defined by the composition of the maps f and g such that

$$\begin{aligned} g_*\circ f_*:\Pi_1(X,x_0)&\rightarrow\Pi_1(Z,z_0)\\ g_*\circ f_*[\alpha]&=[g\circ f\circ\alpha]\end{aligned}$$

According to these construction of induced map, the operation Π_1 can be consider as a covariant functor between **TOP**$_*$ and **Grp**.

$$\begin{array}{lcl} \Pi_1 & : & \mathbf{TOP}_*\longrightarrow\mathbf{Grp}\\ ob. & & (X,x_0)\overset{\Pi_1}{\longmapsto}\Pi_1(X,x_0)\\ & & (X,x_0)\qquad\Pi_1(X,x_0)\\ morp. & & f\downarrow\quad\longmapsto\quad\downarrow f_*\\ & & (Y,y_0)\qquad\Pi_1(Y,y_0)\end{array}
\qquad
\begin{array}{ccc} (X,x_0) & \xrightarrow{\Pi_1} & \Pi_1(X,x_0)\\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f_*}\\ (Y,y_0) & \xrightarrow[\Pi_1]{} & \Pi_1(Y,y_0)\end{array}$$

For any induced homomorphism $f_*=g_*$, we have that f and g are homotopic relative to $\{x_0\}$ and this means that the functor Π_1 is not faithfull. Moreover, one can abandon the group structure of $\Pi_1(X,x_0)$ then Π_1 can be thought as a forgetfull functor between **TOP**$_*$ and **SET**$_*$, which forgets the structure.

2.2.2 *Natural Transformation*

Definition 2.2.4. Given two functors F,G : $\mathscr{C}\rightarrow\mathscr{D}$, a *natural transformation* $\alpha:F\rightarrow G$ is a function which assigns each object X of $\mathscr{C}$ a morphism $\alpha_X=\alpha X:FX\rightarrow GX$ of $\mathscr{D}$ in such that every morphism

$f : X \to X'$ in $\mathscr{C}$ yields the diagram

$$\begin{array}{ccc} X & & FX \xrightarrow{\alpha_X} GX \\ {\scriptstyle f}\downarrow & & {\scriptstyle Ff}\downarrow \qquad\quad \downarrow{\scriptstyle Gf} \\ X' & & FX' \xrightarrow[\alpha_{X'}]{} GX' \end{array}$$

which is commutative.

There are two different types of composition of the natural transformations.

i *Horizontal* : Suppose that $\mathscr{A}, \mathscr{B}, \mathscr{C}$ are categories and $F, G, F^{'}, G^{'}$ are functors, where $\alpha : F \to G$ and $\beta : F' \to G'$ are natural transformations as in the diagram;

$$\mathscr{A} \underset{G}{\overset{F}{\rightrightarrows}} {\Downarrow\alpha}\ \mathscr{B} \underset{G^{'}}{\overset{F^{'}}{\rightrightarrows}} {\Downarrow\beta}\ \mathscr{C}$$

Since F,F' are functors and α, β are natural transformations, the following diagram must be commutative and each of the squares commutes.

$$\begin{array}{ccccccccc} X & \xrightarrow{F} & FX & \xrightarrow{\alpha_X} & GX & \xrightarrow{F} & {}'F'(GX) & \xrightarrow{\beta_{GX}} & G'(GX) \\ {\scriptstyle f}\downarrow & & {\scriptstyle Ff}\downarrow & & {\scriptstyle Gf}\downarrow & & {\scriptstyle F'(Gf)}\downarrow & & \downarrow{\scriptstyle G'(Gf)} \\ Y & \xrightarrow[F]{} & FY & \xrightarrow[\alpha_Y]{} & GY & \xrightarrow[F']{} & F'(GY) & \xrightarrow[\beta_{GY}]{} & G'(GY) \end{array}$$

Hence $\beta \circ \alpha : F' \circ F \to G' \circ G$ is natural.

ii *Vertical* : Let $\mathscr{A}, \mathscr{B}$ be given categories and F, G, H functors Let us construct the composition of two 2-cell such that $\mathscr{A} \overset{F}{\underset{H}{\Downarrow\alpha\ \xrightarrow{G}\ \Downarrow\beta}} \mathscr{B}$; since α and β are natural, the following diagram commutes for $X, Y \in \mathscr{A}$,

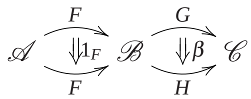
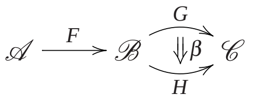
$$\begin{array}{ccccc} Fa & \xrightarrow{\alpha_X} & GX & \xrightarrow{\beta_X} & HX \\ {\scriptstyle Ff}\downarrow & & {\scriptstyle Gf}\downarrow & & \downarrow{\scriptstyle Hf} \\ FY & \xrightarrow[\alpha_Y]{} & GY & \xrightarrow[\beta_Y]{} & HY \end{array} \quad .$$

Hence the composition of the vertical two 2-cells is $\beta \cdot \alpha : F \to H$.

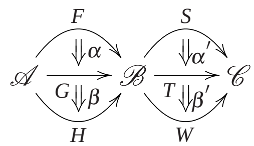
One can consider the particular cases of the horizontal composition :

- $1_H \circ \alpha : HF \to HG$ such that $\mathscr{A} \underset{G}{\overset{F}{\rightrightarrows}} {\Downarrow\alpha}\ \mathscr{B} \underset{H}{\overset{H}{\rightrightarrows}} {\Downarrow 1_H}\ \mathscr{C}$ which we will write as $H\alpha : HF \to HG$

  s.t. $\mathscr{A} \underset{G}{\overset{F}{\rightrightarrows}} {\Downarrow\alpha}\ \mathscr{B} \xrightarrow{H} \mathscr{C}$.

- $\beta \circ 1_F : GF \to HF$ s.t.  which we will write as $\beta F : GF \to HF$ s.t.  .

Proposition 2.2.5. *Given categories, functors and natural transformations in the following figure,*



we have the equality

$$(\beta' \circ \beta) \cdot (\alpha' \circ \alpha) = (\beta' \cdot \alpha') \circ (\beta \cdot \alpha)$$

which is called the middle four interchange law.

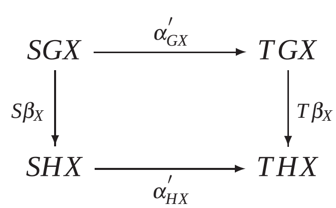
Proof. We give the proof by using the components of the natural transformations. On the right side we have

$$[(\beta' \cdot \alpha') \circ (\beta \cdot \alpha)]_X = (\beta' \cdot \alpha')_{HX} \circ S(\beta \cdot \alpha)_X = \beta'_{HX} \circ \alpha'_{HX} \circ S\beta_X \circ S\alpha_X$$

and on the left side

$$[(\beta' \circ \beta) \cdot (\alpha' \circ \alpha)]_X = \beta'_{HX} \circ T\beta_X \circ \alpha'_{GX} \circ S\alpha_X$$

So we should show that $\alpha'_{HX} \circ S_{\beta_X} = T\beta_X \circ \alpha'_{GX}$. By the naturality of α' we have that



commutes. □

Example 2.2.6. One can construct two different group structure for given any commutative ring K. First, let $GL_n(K)$ be the set of $n \times n$ matrix with entries in the commutative ring K, while $\forall M \in GL_n(K)$ determinant of M is a unit in K, this means that the elements of $GL_n(K)$ are non-singular. Hence the elements of $GL_n(K)$ are compatible with the associativity condition of being group and $GL_n(K)$ has a unit element with respect to matrix multiplication, such that the diagonals are units element of K and the other entries are zero. So $GL_n(K)$ is a group of matrix which is called the general linear group.

Second, let $(K)^*$ denote the set of units of K. $(K)^*$ has clearly a group structure with respect to multiplication of K. One can easily see that GL_n and $(-)^*$ can be thought as functors between **CRng** and **Grp**. Because the determinant is defined by the same formula for all commutative rings K, each morphism $f : K \to K'$ of commutative rings leads us to a commutative diagram

$$\begin{array}{ccc} GL_n(K) & \xrightarrow{det_K} & (K)^* \\ {\scriptstyle GL_n(f)}\downarrow & & \downarrow{\scriptstyle (f)^*} \\ GL_n(K') & \xrightarrow[det_{K'}]{} & (K')^* \end{array}$$

This states that the transformation $det : GL_n \to (-)^*$ is natural between two functors **CRng** $\to$ **Grp**.

Definition 2.2.7. A category $\mathscr{C}$ is called a *groupoid* if every arrow of $\mathscr{C}$ is an isomorphism.

Example 2.2.8. Let $\mathscr{C}$ be a groupoid and suppose that for each object X of $\mathscr{C}$ an arrow μ_X in $\mathscr{C}$ with domain X is given. Then we have a collection $\mu = \{\mu_X | X \in ob(\mathscr{C})\}$. Let us define a functor $F : \mathscr{C} \to \mathscr{C}$ which acts on objects by $F(X) = cod(\mu_X)$. We can consider the following for $\mu_X : X \to Y$;

$$\begin{array}{ccc} X & \xrightarrow{\mu_X} & cod(\mu_X) = F(X) \\ {\scriptstyle \mu_X}\downarrow & & \downarrow{\scriptstyle \mu_{F(X)} = F(\mu_X)} \\ Y & \xrightarrow{\mu_Y} & cod(\mu_Y) = F(Y), \end{array}$$

where the diagram commutes because the horizontal arrows μ_X and μ_Y behave as the functor F. And now we replace X, Y, μ_X by $id(X), id(Y)$ and $id(\mu_X)$ in the left vertical arrow, respectively. Since the diagram commutes for all $X \in ob(\mathscr{C})$ the collection μ becomes a natural transformation between identity functor and F.

2.2.3 Functor Category

Definition 2.2.9. Given categories $\mathscr{C}$ and $\mathscr{D}$ *the functor category* $[\mathscr{C}, \mathscr{D}]$ or $\mathscr{D}^{\mathscr{C}}$ consists of :

- objects are functors $F : \mathscr{C} \to \mathscr{D}$
- morphisms are natural transformations $\alpha : F \to G$, such that :
- identities are natural transformations $1_F : F \to F$, this means that for any $F : \mathscr{C} \to \mathscr{D}$ 1_F has the components $1_{FX} : FX \to FX$; $\forall X \in \mathscr{C}$;
- the composition of the natural transformations is the vertical one.

For given any two functors the set of the morphisms of the functor category is denoted by $[\mathscr{C},\mathscr{D}](F,G)$.

Definition 2.2.10. A *natural isomorphism* $\alpha : F \to G$ is an isomorphism in the functor category; that is, there exists $\beta : G \to F$ such that $\alpha \cdot \beta = 1_G$ and $\beta \cdot \alpha = 1_F$. Moreover two natural transformations are *equal* if and only if all their components are equal.

Proposition 2.2.11. *$\alpha : F \to G$ is a natural isomorphism if and only if each component $\alpha_X : FX \to GX$ is an isomorphism in $\mathscr{D}$*

Proof. Suppose α is a natural isomorphism, and let β be its inverse. Then we have

$$\alpha \cdot \beta = 1_G \implies (\alpha \cdot \beta)_X = 1_{GX} \implies \alpha_X \cdot \beta_X = 1_{GX}$$

and

$$\beta \cdot \alpha = 1_F \implies (\beta \cdot \alpha)_X = 1_{FX} \implies \beta_X \cdot \alpha_X = 1_{FX} \ .$$

So β_X is an inverse for α_X for each $X \in \mathscr{C}$. Thus each component is an isomorphism. Conversely, if each component α_X is an isomorphism, then let β_X be the corresponding inverses for each $X \in \mathscr{C}$. Now given $f \in \mathscr{C}(X,X^{'})$, since α is natural we have that

$$\begin{array}{ccc} FX & \xrightarrow{\alpha_X} & GX \\ {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle Gf} \\ FX^{'} & \xrightarrow[\alpha_{X'}]{} & GX^{'} \end{array}$$

commutes, that is $(Gf) \circ \alpha_X {=} \alpha_{X'} \circ (Ff)$. Let us compose both side with β_X and $\beta_{X'}$ respectively, the we get

$$\beta_{X'} \circ (Gf) \circ \alpha_X \circ \beta_X = \beta_{X'} \circ \alpha_{X'} \circ (Ff) \circ \beta_X .$$

Since β_X and $\beta_{X'}$ are the inverses of α_X and $\alpha_{X'}$ respectively, it follows

$$\beta_{X'} \circ (Gf) \circ 1_{GX} = 1_{FX'} \circ (Ff) \circ \beta_X$$

$$\text{so } \beta_{X'} \circ (Gf) = (Ff) \circ \beta_X$$

Hence the following diagram is commutative

$$\begin{array}{ccc} GX & \xrightarrow{\beta_X} & FX \\ {\scriptstyle Gf}\downarrow & & \downarrow{\scriptstyle Ff} \\ GX^{'} & \xrightarrow[\beta_{X'}]{} & FX^{'} \end{array} \ .$$

So we can define the natural transformation β with components β_X and clearly β is an inverse for α , so α is a natural isomorphism. $\square$

Definition 2.2.12. Given any two categories $\mathscr{C}$ and $\mathscr{D}$ the *equivalence* of these categories consists of two functors F, G and two natural isomorphisms such that $F : \mathscr{C} \to \mathscr{D}$, $G : \mathscr{D} \to \mathscr{C}$ and $\alpha : 1_{\mathscr{C}} \to GF$, $\beta : FG \to 1_{\mathscr{D}}$. Here we mean that FG, GF are clearly the composition of functors and $1_{\mathscr{C}}, 1_{\mathscr{D}}$ are the identities. There is also similar construction in the section of adjunction. If there exists such an equivalence then we say that two categories $\mathscr{C}$ and $\mathscr{D}$ are *equivalent*. It can be shown that if a functor F is full, faithfull and essentially surjective then F is an equivalence of categories.

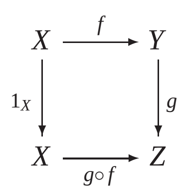
2.2.4 Representables

Let $\mathscr{C}$ be a category and X $\in \mathscr{C}$, using the hom-set, we can define a functor $H^X = \mathscr{C}(X, -) : \mathscr{C} \to$ **Set** with following data;

(i) $H^X(Y) = \mathscr{C}(X,Y)$

(ii) $g \in \mathscr{C}(Y,Z)$, $H^X(g) = \mathscr{C}(g,1) : \mathscr{C}(X,Y) \to \mathscr{C}(X,Z)$ is defined by the composition, such that $H^X(g)(f) = \mathscr{C}(g,1)(f) = g \circ f$.

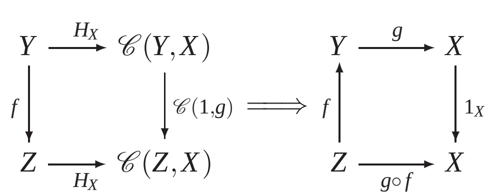
So it is easily seen that this functor is covariant and we get the following commutative diagram,



Now if we put the second parameter as constant value we get another functor $H_X = \mathscr{C}(-, X) : \mathscr{C}^{op} \to$ **Set**, and data;

(i) $H_X(Y) = \mathscr{C}(Y,X)$

(ii) $f \in \mathscr{C}^{op}(Y,Z)$, $H_X(g)(f) = \mathscr{C}(1,g) : \mathscr{C}(Y,X) \to \mathscr{C}(Z,X)$ is defined by the composition, such that $H_X(g)(f) = \mathscr{C}(1,g) = g \circ f$ where the following diagrams commute ;



and this functor is contravariant.

Definition 2.2.13. The functors H^X and H_X are known as *representables* and for each $X \in \mathscr{C}$ one can get the functor H_X, so we have a assignation $X \mapsto H_X$ and we can extend this assignation to a functor known as the *Yoneda embedding*.

$$\begin{array}{rccc} H_\bullet : & \mathscr{C} & \longrightarrow & [\mathscr{C}^{op}, \mathbf{Set}] \\ & X & \longmapsto & H_X \\ & (f : X \to Y) & \longmapsto & (H_f : H_X \to H_Y) \end{array}$$

where H_f is the natural transformation with components

$$\begin{array}{rccc} (H_f)_U : & H_X U & \longrightarrow & H_Y U \\ i.e & \mathscr{C}(U,X) & \longrightarrow & \mathscr{C}(U,Y) \\ & h & \longmapsto & f \circ h \end{array}$$

We need to check that this is a well-defined natural transformation, that is

$$\begin{array}{ccc} \mathscr{C}(U,X) & \xrightarrow{(H_f)_U = f\circ -} & \mathscr{C}(U,Y) \\ {\scriptstyle H_X g = -\circ g}\downarrow & & \downarrow{\scriptstyle H_Y g = -\circ g} \\ \mathscr{C}(U',X) & \xrightarrow[(H_f)_{U'} = f\circ -]{} & \mathscr{C}(U',Y) \end{array}$$

commutes.But along the two legs we just have :

$$\begin{array}{ccc} h & \longmapsto & f \circ h \\ & & \downarrow \\ & & (f \circ h) \circ g \end{array} \qquad and \qquad \begin{array}{ccc} h & & \\ \downarrow & & \\ h \circ g & \longmapsto & f \circ (h \circ g) \end{array}$$

so the naturality condition just says that composition is associative .

Definition 2.2.14. A functor $F : \mathscr{C}^{op} \to \mathbf{Set}$ is *representable* if it is a natural isomorphic to H_X for some $X \in \mathscr{C}$, and a *representation* for F is an object $X \in \mathscr{C}$ together with a natural isomorphism $\alpha : H_X \to F$. Dually, a functor $F : \mathscr{C} \to \mathbf{Set}$ is representable if $F \cong H^X$ for some $X \in \mathscr{C}$, and a representation for F is an object X with a natural isomorphism $\alpha : H^X \to F$.

For naturality of α we have a square : $\forall f : V \to \mathrm{W} \in \mathscr{C}$;

$$\begin{array}{ccc} \mathscr{C}(W,X) & \xrightarrow{\alpha_W} & FW \\ {\scriptstyle H_X f = -\circ f}\downarrow & & \downarrow{\scriptstyle Ff} \\ \mathscr{C}(V,X) & \xrightarrow[\alpha_V]{} & FV \end{array}$$

which must be commutative. Before we end this section, we give an important lemma which is called Yoneda lemma .

Lemma 2.2.15. *Let $\mathscr{C}$ be a locally small category, $F : \mathscr{C}^{op} \to \textbf{\textit{Set}}$. Then there is an isomorphism $FX \cong [\mathscr{C}^{op}, \textbf{\textit{Set}}](H_X, F)$, which is natural in X and F , that is*

$$\begin{array}{ccc} FY & \longrightarrow & [\mathscr{C}^{op}, \textbf{\textit{Set}}](H_Y, F) \\ {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle -\circ H_f} \\ FX & \longrightarrow & [\mathscr{C}^{op}, \textbf{\textit{Set}}](H_X, F) \end{array} \quad \textit{and} \quad \begin{array}{ccc} FX & \longrightarrow & [\mathscr{C}^{op}, \textbf{\textit{Set}}](H_X, F) \\ {\scriptstyle \theta_X}\downarrow & & \downarrow{\scriptstyle \theta\circ -} \\ GX & \longrightarrow & [\mathscr{C}^{op}, \textbf{\textit{Set}}](H_X, G) \end{array}$$

commute for all $f : X \to Y$ and for all $\theta : F \to G$ respectively .

Proof. Given $x \in FX$ let $\hat{x} \in [\mathscr{C}^{op}, \textbf{Set}](H_X, F)$ be defined by components; for $V \in \mathscr{C}^{op}$ $\hat{x}_V : \mathscr{C}(V,X) \to FV$ such that $\hat{x}_V(f) = Ff(x)$. Since F is a contravariant functor, Ff is a map from FX to FV. So given $g : W \to V$ in $\mathscr{C}^{op}$ and $\hat{x}_v$,$\hat{x}^W$ the diagram

$$\begin{array}{ccc} \mathscr{C}(V,X) & \xrightarrow{\hat{x}_V} & FV \\ {\scriptstyle -\circ g}\downarrow & & \downarrow{\scriptstyle Fg} \\ \mathscr{C}(W,X) & \xrightarrow[\hat{x}_W]{} & FW \end{array}$$

So if $f \in \mathscr{C}(V,X)$ then $Fg(\hat{x}_V(f)) = Fg(Ff(x)) = F(f \circ g)(x)$. Given any $\alpha \in [\mathscr{C}^{op}, \textbf{Set}](H_X, F)$, let $\hat{\alpha} \in FX$ be defined by $\hat{\alpha} = \alpha_X(1_X)$. Remember that $1_X \in \mathscr{C}(X,X)$ and $\alpha_X : \mathscr{C}(X,X) \to FX$. Now for $x \in FX$ and $\alpha \in [\mathscr{C}^{op}, Set](H_X, F)$ we have a natural transformation $\hat{x}$ and an element $\hat{\alpha} \in FX$. But we should check that whether $\hat{\hat{()}} = ()$ or not.

$$\hat{\hat{x}} = \hat{x}_X(1_X) = F1_X(x) = 1_{FX}(x) = x \text{ and}$$

$\hat{\alpha} = \alpha_X(1_X) \implies \hat{\hat{\alpha}}_V : \mathscr{C}(V,X) \to FV$ that is for $f \in \mathscr{C}(V,X)$ $\hat{\hat{\alpha}} = Ff(\hat{\alpha}) = Ff(\alpha_X(1_X))$. Moreover, because of the commutativity of the following diagram we have $\alpha_V(1_X \circ f) = Ff(\alpha_X(1_X))$ as required.

$$\begin{array}{ccc} \mathscr{C}(X,X) & \xrightarrow{\alpha_X} & FX \\ {\scriptstyle -\circ f}\downarrow & & \downarrow{\scriptstyle Ff} \\ \mathscr{C}(V,X) & \xrightarrow[\alpha_V]{} & FV \end{array}$$

Here we check that the operation "$\hat{}$" is natural. Let $f : Y \to X$ be a map in $\mathscr{C}^{op}$. We will test the following diagram.

$$\begin{array}{ccc} FX & \longrightarrow & [\mathscr{C}^{op}, \textbf{Set}](H_X, F) \\ {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle -\circ H_f} \\ FY & \longrightarrow & [\mathscr{C}^{op}, \textbf{Set}](H_Y, F) \end{array}$$

In two way we have $x \mapsto \hat{x} \mapsto \hat{x} \circ H_f$ and $x \mapsto Ff(x) \mapsto F\hat{f}(x)$. Explicitly;

$$\mathscr{C}(V,Y) \xrightarrow{(H_f)_V} \mathscr{C}(V,X) \xrightarrow{\hat{x}_V} FV$$

$$g \mapsto f \circ g \mapsto F(f \circ g)(x)$$

and

$$F\hat{f}(x) : \mathscr{C}(V,Y) \to FV \text{ such that } g \mapsto Fg(Ff(x)).$$

We know that $Fg \circ Ff = F(f \circ g)$. So the first diagram in the theorem commutes. Given any $\theta : F \to G$ we should check the second diagram. Let $x \in FX$ we have $x \mapsto \hat{x} \mapsto \theta \circ \hat{x}$ and $x \mapsto \theta_X(x) \mapsto \widehat{\theta_X}(x)$. According to these, $\theta \circ \hat{x}_V(f) = \theta_V \circ Ff(x)$ and $\widehat{\theta_X}(x)(f) = Gf \circ \theta_X(x)$ for any $f \in \mathscr{C}(V,X)$. We can associate this result with the naturality of θ such that

$$\begin{array}{ccc} FX & \xrightarrow{\theta_X} & GX \\ {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle Gf} \\ FV & \xrightarrow[\theta_V]{} & GV \end{array}$$

Hence the second diagram commutes. ☐

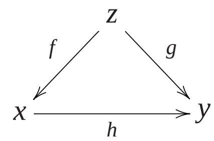
Definition 2.2.16. Given a category $\mathscr{C}$ and an object $X \in ob(\mathscr{C})$, let $M(X)$ be the class of pairs (Y, f) , where $f : Y \to X$ is a monomorphism. Two element (Y, f) and (Z,g) of $M(X)$ are deemed equivalent if there exists an isomorphism $\phi : Y \to Z$ such that $f = g \circ \phi$. A representative class of monomorphisms in $M(X)$ is a subclass of $M(X)$ that is a system of representatives for this equivalence relation. $\mathscr{C}$ is said to be wellpowered provided that each of its objects has a representative class of monomorphisms which is a set. Similarly $E(X)$ denotes the class of pair (f,Y) such that $f : X \to Y$ is an epimorphism. Two elements (f,Y) and (g,Z) of $E(X)$ are deemed equivalent if there exists an epimorphism $\phi : Y \to Z$ such that $g = \phi \circ f$. A representative class of epimorphisms in $E(X)$ is a subclass of $E(X)$ that is a system of representatives for the equivalence relation. $\mathscr{C}$ is said to be cowellpowered provided that each of its objects has a representative class of epimorphisms which is a set. **Set,Gp,Ab,Top** are wellpowered and cowellpowered. The category of ordinal numbers are wellpowered but not cowellpowered.

Before we give a definition of another category constructed by using the functors, we need some extra definitions and a motivation.

Definition 2.2.17. Let $\mathscr{C}$ be a category and $z \in ob(\mathscr{C})$, the category $(z,\mathscr{C})$ is called the category of objects under z with objects all pairs $< f,x >$ and monomorphisms $h :< f,x > \to < g,y >$ those morphisms $h : x \to y$ of $\mathscr{C}$ for which $h \circ f = g$. Thus an object of $(z \downarrow \mathscr{C})$ is just a morphism in $\mathscr{C}$ with domain z and a morphism of $(z,\mathscr{C})$ is a commutative triangle with top vertex z, that is

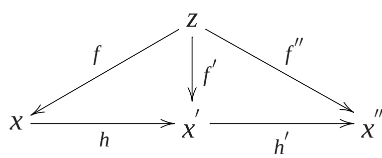
- objects of $(z,\mathscr{C}) : \{< f,x > | f : z \to x;$ for $x \in ob(\mathscr{C})\}$

- morphisms of $(z \downarrow \mathscr{C})$:



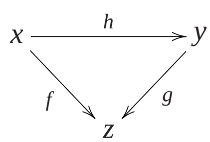
  where $h \in \mathscr{C}(x,y)$ and diagram commutes.

- Since the composition of two commutative diagrams must be commutative in the category $\mathscr{C}$, the composition of the morphisms in $(z \downarrow \mathscr{C})$ is clearly defined, that is, for any maps $h :< f,x > \to < f',x' >$ and $h' :< f',x' > \to < f'',x'' >$; for $< f,x >, < f',x' >, < f'',x'' > \in ob(z \downarrow \mathscr{C})$ the following diagram commutes.



- One can verify that the associativity and unit law hold in this category because the composition is the same as the composition in the category $\mathscr{C}$

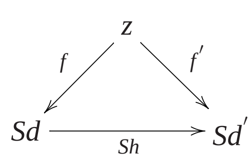
Using the similar idea one can construct the category $(\mathscr{C} \uparrow z)$ which is called the category of objects over z with objects all pairs $< x, f >$ and morphisms $h :< x, f > \to < y,g >$. Here objects are just morphisms in $\mathscr{C}$ with codomain z and morphisms are all commutative diagram for $f : x \to z$ and $g : y \to z$



Now let $S : \mathscr{D} \to \mathscr{C}$ be a functor from the category $\mathscr{D}$ to $\mathscr{C}$, we can define a category $(z \downarrow S)$ of objects S-under z, such that

- objects of $(z \downarrow S)$: all pairs $< f,d >$ for $d \in ob(\mathscr{D})$ and $f \in \mathscr{C}(z,Sd)$

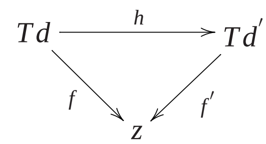
- morphisms of $(z \downarrow S)$: for any morphisms $h : d \to d'$ and the pairs $< f,d >,< f,d' > \in ob(z \downarrow S)$ the following commutative diagram,



Also with the dual notation one can construct a category $(T \downarrow z)$ which is called the category T-over z, where $z \in ob(\mathscr{C})$ and T is a functor from $\mathscr{D}$ to $\mathscr{C}$ such that

- objects of $(T \downarrow z)$: all pairs $< d, f >$ for $d \in ob(\mathscr{D})$ and $f \in \mathscr{C}(Td, z)$

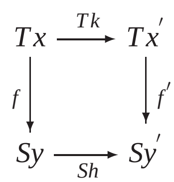
- morphisms of $(T \downarrow z)$: for any morphisms $h : d \to d^{'}$ and the pairs $< d, f >, < d^{'}, f^{'} > \in ob(T \downarrow z)$ the following commutative diagram,



Definition 2.2.18. By combining the four types of categories given above, let $T, S : \mathscr{D} \to \mathscr{C}$ be functors, the category $(T \downarrow S)$ is called *the comma category* and consists of :

- $ob(T \downarrow S)$: the triple $< x, y, f >$ where $x, y \in ob(\mathscr{D})$ and $f \in \mathscr{C}(Tx, Sy)$

- $Hom(T \downarrow S)$: the pair $< k, h >$, such that the diagram commutes



  where $k \in \mathscr{D}(x, x^{'})$, $h \in \mathscr{D}(y, y^{'})$.

- The composite $< k, h > \circ < k^{'}, h^{'} >$ is $< k \circ k^{'}, h \circ h^{'} >$, when the compositions are defined in $\mathscr{D}$.

Let $S = T = \mathbf{1}_{\mathscr{C}}$ where $\mathbf{1}_{\mathscr{C}}$ is the identity functor of $\mathscr{C}$, then $(\mathbf{1}_{\mathscr{C}} \downarrow \mathbf{1}_{\mathscr{C}})$ is exactly the category $\mathscr{C}^2$ of all morphisms of $\mathscr{C}$. Moreover, taking $T, S : \mathscr{C} \to \mathscr{C}$ as a constant functor with the range x and $y \in ob(\mathscr{C})$ respectively; note that constant functors carries morphisms to the identity morphism of the object in the range; then $(T \downarrow S)$ is the category with objects all morphisms $f : x \to y$ and morphisms only the identity morphisms, in otherwords $(T \downarrow S)$ is the set $Hom_{\mathscr{C}}(x, y)$.

Example 2.2.19. Let K is a commutative ring and **CRng** denotes the category of all commutative rings. *A $K-$algebra* is the ring R with identity and a ring homomorphism $f : K \to R$ mapping 1_K to 1_R (identity of K to identity of R) such that the subring $f(K)$ of R is contained in the center of R, that is, $f(K) = \{a \in R | ra = ar \ \forall r \in R\}$. Let R and $R^{'}$ be two commutative rings. *A K-algebra homomorphism* between R and $R^{'}$ is a ring homomorphism $\varphi : R \to R^{'}$ mapping 1_R to $1_{R'}$ such that $\varphi(k \cdot r) = k \cdot \varphi(r)$ for all $k \in K$ and $r \in R$. According to these definitions, the category $(K \downarrow \mathbf{CRng})$ is the category of all $K-$algebras, with the composition of the ring homomorphisms in **CRng**.

CHAPTER THREE

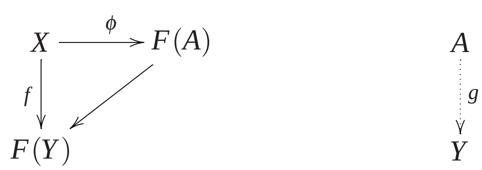
CONSTRUCTIONS IN CATEGORIES

3.1 Limit and Colimit

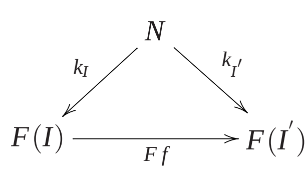
A lot of important properties of categories can be formulated by requiring that limits or colimits of certain kind do exist meaning that certain functor are representable. Here we will define limits and colimits. Later we try to explain the relation between the cone structure and functor. Then we will give the definition some special kinds of limit and colimits such that pullback or equalisers with giving examples in homotopy theory. After we investigate parametrised limits, we will deal with dinatural transformations which are a different kinds of natural transformations.

Definition 3.1.1. Let $F : \mathscr{D} \to \mathscr{C}$ be a functor from a category $\mathscr{D}$ to a category $\mathscr{C}$ and let X be an object of $\mathscr{C}$. A *universal arrow* from X to F consists of a pair (A, ϕ) where A is an object of $\mathscr{D}$ and $\phi : X \to F(A)$ is a morphism in $\mathscr{C}$ such that the following universal mapping property is satisfied:

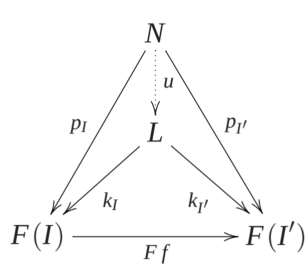
Whenever Y is an object of $\mathscr{D}$ and $f : X \to F(Y)$ is a morphism in $\mathscr{C}$, then there exists a unique morphism $g : A \to Y$ such that the following diagram commutes .



Definition 3.1.2. Let $\mathbb{I}$ and $\mathscr{C}$ be two categories and $F : \mathbb{I} \to \mathscr{C}$ a functor. Here we use the small category $\mathbb{I}$ for indexing. A *cone* of F is an object N of $\mathscr{C}$, together with a family of morphisms $k_I : N \to F(I)$, one for each object I in $\mathbb{I}$ such that for every morphism $f : I \to I^{'}$ in $\mathbb{I}$, we have $F(f) \circ k_I = k_{I'}$ as in the diagram



Definition 3.1.3. A *limit* of a functor is just a universal cone. In detail, a cone (L, k_I) of a functor $F : \mathbb{I} \to \mathscr{C}$ is a limit of that functor if and only if for any cone (N, p_I) of F, there exists precisely one morphism $u : N \to L$ such that $k_I \circ u = p_I$ for all I.



We may say that in the diagram the morphisms p_I factor through L with unique factorization u which is called the mediating morphism. It is possible that a functor F does not have a limit at all. However, if it has two limits then there exists a *unique* isomorphism between the respective limit objects which commutes with the respective cone maps. This isomorphism is given by the unique factorization from one limit to the other. Thus limits are unique up to isomorphism and can be denoted by $\underleftarrow{lim}F$.

Definition 3.1.4. Given any $Y \in \mathscr{C}$, one can define the *constant functor* $\triangle Y$ from $\mathbb{I}$ to $\mathscr{C}$ such that $\forall I \in \mathbb{I}$, $\triangle Y(I) = Y$ and $\forall f \in \mathbb{I}$, $\triangle Y(f) = 1_Y$.

$$\begin{array}{rcl} \triangle - : \ \mathscr{C} & \longrightarrow & [\mathbb{I},\mathscr{C}] \\ Y & \longmapsto & \triangle Y \\ X & & \triangle X \\ f \big\downarrow & \longmapsto & \big\downarrow \triangle f \\ Y & & \triangle Y \end{array}$$

A limit L for F can be thought as a representation for the functor $[\mathbb{I},\mathscr{C}](\triangle -, F) : \mathscr{C}^{op} \to \mathbf{Set}$, that is, there is a natural isomorphism α with $H_L \cong [\mathbb{I},\mathscr{C}](\triangle -, F)$ and we can also denote the limit object $L = \int_I FI$. So we have an isomorphism $\mathscr{C}(-, \int_I FI) \cong [\mathbb{I},\mathscr{C}](\triangle -, F)$. Let us make it explicitly what the functor on the right hand side, call it G and how we can get a universal cone :

$$\begin{array}{rcl} G : \ \mathscr{C}^{op} & \longrightarrow & \mathbf{Set} \\ Y & \longmapsto & [\mathbb{I},\mathscr{C}](\triangle Y, F) \\ Y & & [\mathbb{I},\mathscr{C}](\triangle X, F) \\ f \big\downarrow & \longmapsto & \big\downarrow Gf \\ X & & [\mathbb{I},\mathscr{C}](\triangle Y, F) \end{array}$$

Now we try to explain what does a natural transformation $\triangle Y \to F$ look like. We have :

- for each I$\in \mathbb{I}$, a morphism

$$\begin{array}{rcl} k_I : (\triangle Y)I & \longrightarrow & FI \\ Y & \longrightarrow & FI \end{array}$$

- for all u:I $\to I'$ in $\mathbb{I}$;

$$\begin{array}{ccc} (\triangle Y)I & \longrightarrow & FI \\ {\scriptstyle (\triangle Y)u} \big\downarrow & & \big\downarrow {\scriptstyle Fu} \\ (\triangle Y)I' & \longrightarrow & FI' \end{array}$$

  commutes by naturality , that is

$$\begin{array}{ccccc} & & Y & & \\ & {\scriptstyle k_I} \swarrow & & \searrow {\scriptstyle k_{I'}} & \\ FI & & \xrightarrow[Fu]{} & & FI' \end{array}$$

commutes.

So such a natural transformation is precisely a cone over F with Y as a vertex. Now, consider a representation as above, and let α be its natural isomorphism. Then we have

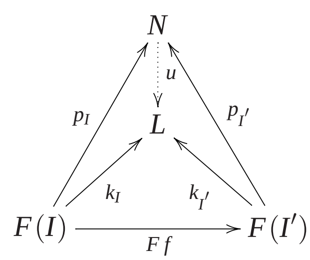
$$\alpha_Y : \mathscr{C}(Y,L) \longrightarrow [\mathbb{I},\mathscr{C}](\triangle Y, F)$$

$$f \longrightarrow Ff(\alpha_L 1_L)$$

that is, the natural transformation is completely determined by $\alpha_L 1_L$. Now, we have a cone given by $\alpha_L 1_L = (k_I)_{I\in\mathbb{I}}$. So given another Y and $f : Y \to L$ on the left hand side, we have $Ff(\alpha_L 1_L)$ with the components $k_I \circ f$, hence we have a bijective correspondence morphisms and cones over F , that is, starting on the right hand side , given any cone $(p_I)_{I\in\mathbb{I}}$ there exists a unique morphism $f : Y \to L$ such that $p_I = k_I \circ f$ for all I; thus $(k_I)_{I\in\mathbb{I}}$ is a universal cone over F.

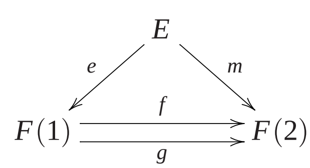
Definition 3.1.5. A category $\mathscr{C}$ is called *complete* if and only if every functor $F : \mathbb{I} \to \mathscr{C}$, where $\mathbb{I}$ is any small category, has a limit, that is "all small limits in $\mathscr{C}$ exist". Similarly, if every such functor with $\mathbb{I}$ finite has a limit, then $\mathscr{C}$ is said to have *finite limits*.

Definition 3.1.6. Also with using the dual notation of limit we can get colimit of a functor F where the morphisms k_I are reversed. The notation of colimit is $\underrightarrow{Lim}F$ or $\int^I FI$ and the diagram shape is the following.



One says that $\mathscr{C}$ is *cocomplete* if and only if every functor $F : \mathbb{I} \to \mathscr{C}$ has a colimit that is all small colimits in $\mathscr{C}$ exist.

Definition 3.1.7. Let $\mathbb{I}$ be a category such that it has just two objects 1 and 2 and two parallel arrows and let F be a functor from $\mathbb{I}$ to $\mathscr{C}$. Then we have a diagram in $\mathscr{C}$ such that $\bullet \rightrightarrows \bullet$ and a cone over this diagram is



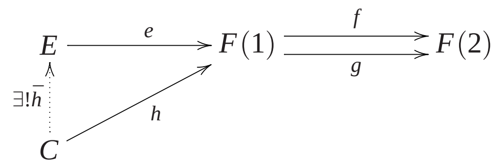
Note that $m = fe = ge$ as all triangles commute; so in fact we can rewrite this more simply as

$$E \xrightarrow{e} F(1) \underset{g}{\overset{f}{\rightrightarrows}} F(2) \qquad \Rightarrow fe = ge.$$

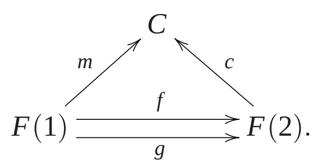
The limit object over F in this diagram is called an *equaliser* and it is a universal cone. Given any cone

$$C \xrightarrow{h} F(1) \underset{g}{\overset{f}{\rightrightarrows}} F(2)$$

there exists a unique factorization $\exists!\overline{h}$ where $h = e\overline{h}$ as in the diagram;



In the category of sets; the equaliser is given by the set $E = \{x \in F(1) | f(x) = g(x)\}$ and by the inclusion map e of the subset E in F(1). With the similar idea we can define a functor $G : \mathbb{I} \to \mathscr{C}$ and a co-cone over the diagram is

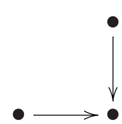


and the colimit object over G in this diagram is called a *coequaliser* and it is a universal cone.

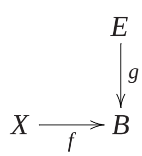
$$F(1) \overset{f}{\underset{g}{\rightrightarrows}} F(2) \xrightarrow{c} C \qquad \Rightarrow cf = cg.$$

In the category of sets, the coequalizer is given by the quotient set $C = F(2)/\sim$ and by the canonical map $c : F(2) \to C$, where $\sim$ is the minimal equivalence relation on $F(2)$ that identifies $f(x)$ and $g(x)$ for all $x \in F(1)$.

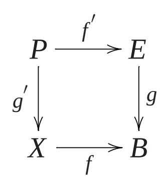
Definition 3.1.8. A *pullback* is a limit of shape



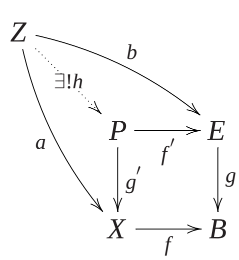
A diagram of this shape in $\mathscr{C}$ is



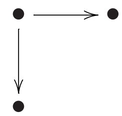
A cone over this diagram is



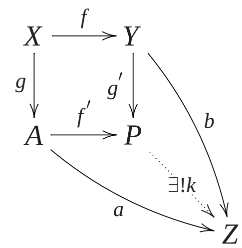
commuting. A pullback is the universal such; so given any commutative square as above we have



a unique h such that $g^{'}h = a$, and $f^{'}h = b$. We say that $g^{'}$ is a pullback for g over f, and that $f^{'}$ is a pullback for f over g. Dually *pushout* is a colimit of shape

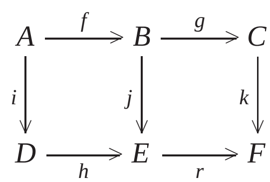


and pushout is the universal such that in the following commutative diagram.



In **Set** the pushout of f and g always exists; it is the disjoint union $A\bigsqcup Y$ with the elements $f(x)$ and $g(x)$ identified for each $x \in X$.

Example 3.1.9. Suppose that two squares in the following rectangle are pullback. We can show that the rectangle is also a pullback.



$k \circ g = r \circ j$, since right square is pullback

$k \circ g \circ f = r \circ j \circ f$,taking the composition of both side with f

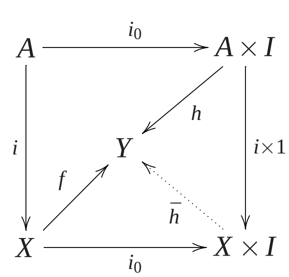
$h \circ i = j \circ f$, since the left square is pullback

$k \circ g \circ f = r \circ h \circ i$, by using the last equality

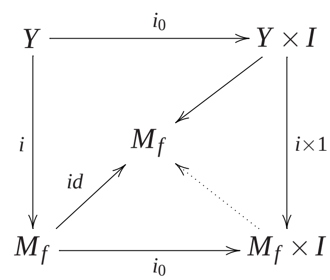
$(r \circ h) \circ i = k \circ (g \circ f)$, this shows the rectangle is pullback.

As an application of pullbacks and pushouts we give some definitions in **Top** using in the Homotopy theory.

Definition 3.1.10. (May (1999)) The morphism $i : A \to X$ is a *cofibration* if and only if it satisfies the homotopy extension property , that is, if the square is commutative for the homotopy h then there exists a homotopy $\overline{h} : X \times I \to Y$.

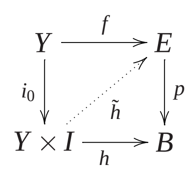


Here $i_0(x) = (x,0)$. The triangle in the upsite is a pushout. In general, we denote the pushout $B \sqcup_g X$ where $i : A \to X$ and $g : A \to B$. One can get the isomorphism $(B \sqcup_g X) \times I \cong (B \times I) \sqcup_{g\times} (X \times I)$. This isomorphism shows that if $i : A \to X$ is a cofibration and $g : A \to B$ is a morphism then the inclusion $B \to B \sqcup_g X$ is also a cofibration. This means that a pushout of a cofibration is also a cofibration. If $A \subset X$ and $i : A \to X$ is a cofibration then the structure $M_i \equiv X \sqcup_i (A \times I)$ is called the *mapping cylinder*. Since the pushouts are universal there exists a unique map between Y and M_i. Now let $f : X \to Y$ be a morphism then we can define a new structure $M_f \equiv Y \sqcup_f (X \times I)$ such that two space X and Y are pasted along the image set of f. So we have the composition $X \xrightarrow{j} M_f \xrightarrow{r} Y$ where $j(x) = (x,1)$, $r(y) = y$ and $r(x,s) = f(x)$ on $X \times I$. If $i : Y \to M_f$ is an inclusion then $r \circ i = id$ and $id \simeq i \circ r$, that is, we can define the homotopy $h : M_f \times I \to M_f$ such that it is surjective from M_f to $i(Y)$ where $h(y,t) = y$ and $h((x,s),t) = (x,(1-t)s)$. This gives a deformation of M_f onto Y with the following diagram.

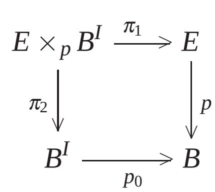


One can define a deformation of M_f onto X with using the inclusion j.

Definition 3.1.11. (May (1999)) The map $p : E \to B$ is a *fibration* if and only if it satisfy the covering homotopy property, that is, with given map p the homotopy $h : Y \times I \to B$ can be lifted a homotopy $\tilde{h} : Y \times I \to E$ as in the following diagram.



Here $\tilde{h}$ must make the diagram commutative. Such a fibration is called Hurewicz fibration. If we take Y in diagram as the cube I^n then this special case is called the Serre fibrations. It is clear that the diagram is a pullback. Usually for a given $p : E \to B$ and $g : A \to B$ we use the notation $A \times_g E$ for the pullback. So if p is a fibration and $g : A \to B$ is a map then the map $A \times_g E \to A$ is also a fibration. Now let us define a space $N_p \equiv E \times_p B^I = \{(e,\beta) | \beta(0) = p(e)\} \subset E \times B^I$ where $B^I = \{\beta | \beta : I \to B$ is a path $\}$. This space is called the mapping path space. Now we have a diagram



here the maps π_1 and π_2 are the projections with respect to first and second factor respectively. So N_p is the pullback of the maps p and p_0 as in the diagram. The map $s : N_p \to E^I$ is called the path lifting function which satisfies for a map $k : E^I \longrightarrow N_p$ $k \circ s = id$ such that $s(e,\beta)(0) = e$ and $p \circ s(e,\beta) = \beta$. For a given any morphism $g : Y \longrightarrow N_p$ is determined with the maps $f : Y \longrightarrow E$ and $h : Y \longrightarrow B^I$. So the lifting of h can be considered as $\tilde{h} = s \circ g$. Hence one can show that if $p : E \longrightarrow B$ is a covering then p is a fibration with a unique path lifting function s because the lifts of paths are determined with the initial point and the function s.

Now we turn back to the category theory and continue giving example of special limits.

Example 3.1.12. Let $\mathbb{I}$ be discrete as in 3.1.3. Then the limit of shape $\mathbb{I}$ is called a *product* denoted by Π and the colimit is called *coproduct* denoted by $\sqcup$. Let $\mathscr{P}$ denote the category of partial ordered sets, that is

- Objects are sets $X, Y, Z, ...$
- Let X and Y are sets then we have :

$$\mathscr{P}(X,Y) = \begin{cases} \emptyset, & \text{if X} \not\subseteq \text{Y;} \\ f_{XY}, & \text{if X} \subseteq \text{Y.} \end{cases}$$

Consider the discrete category $\mathbb{I}$ and the functor $F : \mathbb{I} \rightarrow \mathscr{P}$. The limit object of F is the greatest lower bound of the sets $F(I_n)$, the intersection of the set $F(I_n)$ and we can consider this object as a product of these sets in $\mathscr{P}$. Also the coproduct is the union of the sets $F(I_n)$.

$$\begin{array}{ccc} & \prod_{k=1}^{n} F(I_k) = \bigcap_{k=1}^{n} F(I_k) & \\ \swarrow & & \searrow \\ F(I_{k_1}) \longrightarrow & \cdots & \longrightarrow F(I_{k_n}) \end{array} \qquad \begin{array}{ccc} & \bigsqcup_{k=1}^{n} F(I_k) = \bigcup_{k=1}^{n} F(I_k) & \\ \swarrow & & \searrow \\ F(I_{k_1}) \longrightarrow & \cdots & \longrightarrow F(I_{k_n}) \end{array}$$

Definition 3.1.13. A category $\mathscr{C}$ is called *cartesian closed* if it has a terminal object, any two objects have a product in $\mathscr{C}$ and any two objects have an exponential (a morphism) in $\mathscr{C}$.

Proposition 3.1.14. *Given a functor $F : \mathscr{C}^{op} \times \mathscr{A} \rightarrow$ **Set** such that each $F(-,A) : \mathscr{C}^{op} \rightarrow$ **Set** has a representation $\alpha_A : \mathscr{C}(-,U_A) \rightarrow F(-,A)$, then there is a unique way to extend $A \mapsto U_A$ to a functor $U : \mathscr{A} \rightarrow \mathscr{C}$ such that the α_A are components of a natural transformation $H_{\bullet} \circ U \rightarrow F$.*

Proof. Let us construct U on morphisms that is given any $f : A \rightarrow B$ we seek $Uf : U_A \rightarrow U_B$. In order to satisfy the naturality condition on α, we need

$$\begin{array}{ccc} \mathscr{C}(-,U_A) & \xrightarrow{\alpha_A} & F(-,A) \\ \downarrow & & \downarrow {\scriptstyle F(-,f)} \\ \mathscr{C}(-,U_B) & \xrightarrow{\alpha_B} & F(-,B) \end{array}$$

to commute.Since the horizontal morphisms are isomorphisms, we get a unique morphism on left $H_{U_A} \to H_{U_B}$ making the diagram commute. The Yoneda embedding is full and faithful. So there exists a unique morphism $Uf : U_A \to U_B$ inducing it. It only remains to check that U is functorial, that is, it will make α a natural transformation.

- First we check that $U(1_A) = 1_{U_A}$. We know that $U(1_A)$ is the unique morphism making the naturality square commute. So it suffices to show that 1_{U_A} makes the square commute. We have the diagram

$$\begin{array}{ccc} \mathscr{C}(-,U_A) & \xrightarrow{\alpha_A} & F(-,A) \\ {\scriptstyle 1_{U_A}\circ -}\downarrow & & \downarrow{\scriptstyle F(-,1_A)} \\ \mathscr{C}(-,U_A) & \xrightarrow[\alpha_A]{} & F(-,A) \end{array}$$

  which commutes as required.

- Now we check that $U(g \circ f) = Ug \circ Uf$ for given $A \xrightarrow{f} B \xrightarrow{g} C$. We consider the following diagram

$$\begin{array}{ccc} \mathscr{C}(-,U_A) & \xrightarrow{\alpha_A} & F(-,A) \\ {\scriptstyle H_{U_f}}\downarrow & & \downarrow{\scriptstyle F(-,f)} \\ \mathscr{C}(-,U_B) & \xrightarrow{\alpha_B} & F(-,B) \\ {\scriptstyle H_{U_g}}\downarrow & & \downarrow{\scriptstyle F(-,g)} \\ \mathscr{C}(-,U_C) & \xrightarrow{\alpha_C} & F(-,C) \end{array}$$

  Since each square commutes, the rectangle commutes. The composite on the right hand side is $F(-,g \circ f)$ and by the definition it induces a unique map $H_{U_{g\circ f}}$ on the left hand side. So we have $H_{U_{g\circ f}} = H_{U_g} \circ H_{U_f} = H_{U_g \circ U_f}$ by functoriality, but the Yoneda embedding is full and faithful. Then we have $U(g \circ f) = U_g \circ U_f$ as required.

□

Here we construct a functor which assigns $A \mapsto U_A$ and a representation which is called a parametrised representation.

Proposition 3.1.15. *Define $F : \mathbb{I} \times \mathscr{A} \to \mathscr{C}$ such that each $F(-,A) : \mathbb{I} \to \mathscr{C}$ has a specified limit in $\mathscr{C}$, that is, $\mathscr{C}(-, \int_I F(I,A)) \cong [\mathbb{I},\mathscr{C}](\triangle -, F(-,A))$. Then there is unique way to extend $A \mapsto \int_I F(I,A)$ to a functor $\mathscr{A} \to \mathscr{C}$ such that*

$$\mathscr{C}(Y, \textstyle\int_I F(I,A)) \cong [\mathbb{I},\mathscr{C}](\triangle Y, F(-,A))$$

natural in Y and A.

Now we can restate the definition of a limit to get

$$\mathscr{C}(Y, \int_I FI) \cong \int_I \mathscr{C}(Y, FI) .$$

Let us explain what it means. First, the right hand side is the limit of the functor $\mathscr{C}(Y, F-) : \mathbb{I} \to \mathbf{Set}$. Since **Set** is complete, this functor certainly has a limit. $\int_I \mathscr{C}(Y, FI)$ looks like all tuples $(\alpha_I)_{I \in \mathbb{I}}$ such that $\forall I, \alpha_I \in \mathscr{C}(Y, FI)$ and $\forall u : I \to I^{'}$, $Fu \circ \alpha_I = \alpha_{I'}$. So, this is, precisely a cone over F that is $\int_I \mathscr{C}(Y, FI) = [\mathbb{I}, \mathscr{C}](\triangle Y, F)$. By parametrised limits we have a functor $Y \mapsto \int_I \mathscr{C}(Y, FI)$. So

$$\int_I \mathscr{C}(Y, FI) = [\mathbb{I}, \mathscr{C}](\triangle Y, F) \cong \mathscr{C}(Y, \int_I FI)$$

is natural in Y and F.

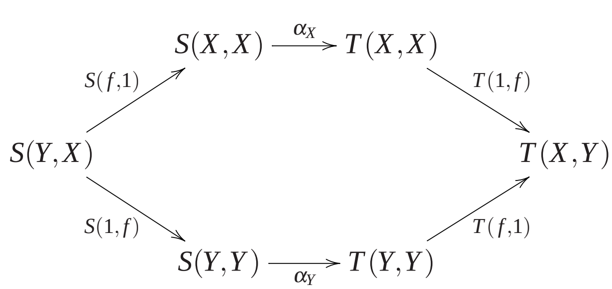
Definition 3.1.16. Let $\mathbb{I} \xrightarrow{G} \mathscr{C} \xrightarrow{F} \mathscr{D}$ be given. We can consider limits over G and limits over FG. Suppose we have a limit cone for G $(\int_I GI \xrightarrow{k_I} GI)_{I \in \mathbb{I}}$. We say F *preserves* this limit if $(F \int_I GI \xrightarrow{Fk_I} FGI)_{I \in \mathbb{I}}$ is a limit cone for FG in $\mathscr{D}$. Note that it must preserve projections.

Definition 3.1.17. Suppose $FG : \mathbb{I} \to \mathscr{D}$ has a limit cone. We say F *reflects* this limit if any cone that goes to a limit cone was already a limit cone itself. That is, given a cone $(Z \xrightarrow{f_I} GI)_{I \in \mathbb{I}}$ such that $(FZ \xrightarrow{Ff_I} FGI)_{I \in \mathbb{I}}$ is a limit cone for FG, then $(Z \xrightarrow{f_I} GI)_{I \in \mathbb{I}}$ is also a limit cone.

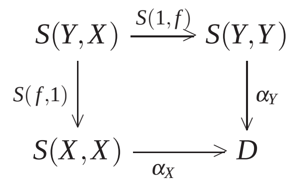
Definition 3.1.18. Suppose $FG : \mathbb{I} \to \mathscr{D}$ has a limit cone. We say F *creates* this limit if there exists a cone $(Z \xrightarrow{f_I} GI)_{I \in \mathbb{I}}$ such that $(FZ \xrightarrow{Ff_I} FGI)_{I \in \mathbb{I}}$ is a limit cone for FG and additionally F reflects limits. That is, given a limit for FG, there is a unique lift to a limit for G up to isomorphism.

Remark 3.1.19. Representable functors and all full and faitfull functor preserve limits.

Definition 3.1.20. Given any two category $\mathscr{C}, \mathscr{D}$ and bifunctors $S, T : \mathscr{C}^{op} \times \mathscr{C} \to \mathscr{D}$ a *dinatural transformation* $\alpha : S \to T$ is a collection of morphisms such that $\forall X \in ob(\mathscr{C})$ a morphism $\alpha_X : S(X,X) \to T(X,X)$ and for $f : X \to Y$ in $\mathscr{C}$ the following diagram

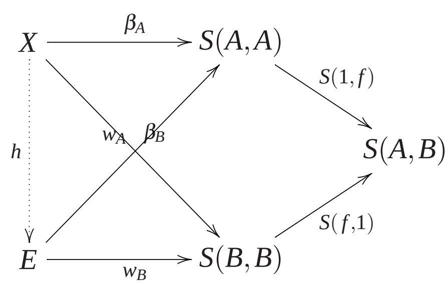


is commutative. If S is dummy in the second variable and T is dummy in the first variable then the dinatural transformation $\alpha : S \to T$ is a natural transformation between functors such that $S_0 : \mathscr{C} \to \mathscr{D}$ and $T_0 : \mathscr{C}^{op} \to \mathscr{D}$. In addition, let S is not dummy and T is dummy in both variable, that is, $\forall X \in ob(\mathscr{C})\ T(X,X) = D \in ob(\mathscr{D})$. Then α looks like a dinatural transformation between S and $D \in ob(\mathscr{D})$. Such a functor is called *extranatural* or *supernatural* transformation. It satisfies the following diagram.



This diagram looks like that the right hand side of the hexagon is collapsed. In the dual notion one can consider the dinatural transformation $\beta : D \to T$ and then the test diagram is obtained from collapsing the left hand side of the hexagon.

Definition 3.1.21. Let $S : \mathscr{C}^{op} \times \mathscr{C} \to \mathscr{D}$ be a functor. The *end* of this functor is a dinatural transformation w such that $E \in ob(\mathscr{D})$ and $w : E \to S$. This natural transformation is sometimes called wedge. Ends are special kinds of limits and they are universal. We mean that $\forall \beta : X \to S$, there exists unique $h : X \to E$ where the components of two dinatural transformations satisfy $\beta_A = w_A h$ for all $A \in ob(\mathscr{C})$, that is, for each $f : A \to B$ in $\mathscr{C}$ all the quadrilaterals in the following diagram commute.



In general, to show the end of the functor S we use just the object E and the notation $\int_A S(A,A)$. It can be considered the dual notion of ends which is called *coend* such that an object D and a dinatural transformation $\alpha : S \to D$ with $S(A,A) \to \int^A S(A,A) = D$.

3.2 Adjunctions and Monads

We know that every group structure is mapped to the set structure by functors. But the main problem is that whether there exist group structures for every sets or not. In this section we will define adjunctions and we will try to find an answer to this problem. After all we will give examples in topology.

3.2.1 Adjunction

Definition 3.2.1. An *adjunction* between two categories $\mathscr{C}$ and $\mathscr{D}$ consists of two functors $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{C}$ and a natural isomorphism $\psi : Hom_{\mathscr{D}}(F-,-) \to Hom_{\mathscr{C}}(-,G-)$ consisting of bijections : $\psi_{X,Y} : Hom_{\mathscr{D}}(FX,Y) \to Hom_{\mathscr{C}}(X,GY)$ for all objects X in $\mathscr{C}$ and Y in $\mathscr{D}$. In order to interpret ψ a *natural isomorphism*, one must recognize $Hom_{\mathscr{D}}(F-,-)$ and $Hom_{\mathscr{C}}(-,G-)$ as functors. In fact, they are both bifunctors from $\mathscr{C}^{op} \times \mathscr{D}$ to **Set** as we have seen. Explicitly, the naturality of ψ means that for all morphisms $f : X \to X^{'}$ in $\mathscr{C}$ and all morphisms $g : Y \to Y^{'}$ in $\mathscr{D}$ the following diagram commutes :

$$\begin{array}{ccc} Hom_{\mathscr{C}}(X,GY) & \xrightarrow{Hom(f,Gg)} & Hom_{\mathscr{C}}(X',GY') \\ {\scriptstyle \psi_{X,Y}}\downarrow & & \downarrow{\scriptstyle \psi_{X',Y'}} \\ Hom_{\mathscr{D}}(FX,Y) & \xrightarrow[Hom(Ff,g)]{} & Hom_{\mathscr{D}}(FX',Y') \end{array}$$

One can give another way to define adjunctions with using units and counit which we will explain now. An adjunction between two categories $\mathscr{C}$ and $\mathscr{D}$ consists of two functors $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{C}$ and two natural transformations $\eta : 1_{\mathscr{C}} \to GF$, $\varepsilon : FG \to 1_{\mathscr{D}}$ called the *unit* and the *co-unit* of the adjunction, respectively. These must satisfy

$$1_F = \varepsilon F \circ F\eta : F \longrightarrow FGF \longrightarrow F$$

$$1_G = G\varepsilon \circ \eta G : G \longrightarrow GFG \longrightarrow G$$

where 1_F and 1_G are the identity transformations on F and G respectively. These equations are sometimes called the zig-zag equations because of the appearance of the corresponding string diagrams. In component form these equations are ;

$$id_{FX} = \varepsilon_{FX} \circ F(\eta_X)$$

$$id_{GY} = G(\varepsilon_Y) \circ \eta_{GY}$$

for each X in $\mathscr{C}$ and each Y in $\mathscr{D}$.

Let us explain this with some example.

Example 3.2.2. Let S be any set and S^{-1} any set disjoint from S such that there is a bijection from S to S^{-1}. Using this bijection for each $s \in S$ let us denote the corresponding element $t \in S^{-1}$ by s^{-1} and similarly for each $t \in S^{-1}$ the corresponding element $s \in S$ by t^{-1} so that the inverse of s^{-1} in S^{-1} will be s in S, that is, $(s^{-1})^{-1} = s$. Now we take a singleton set not contained in $S \cup S^{-1}$ and we call this set $\{1\}$ and we assume that $1^{-1} = 1$. For any $x \in S \cup S^{-1} \cup \{1\}$ let $x^1 = x$. A word on S is by definition a sequence $(s_1, s_2, s_3, \ldots)$ where $s_i \in S \cup S^{-1} \cup \{1\}$ and $s_i = 1$ for all i sufficiently large. We mean that for each sequence there is an n such that $s_i = 1$ for $i \geq n$.Thus we can think of a word as a finite product of elements of S and their inverses. Here we allow the repetitions. To be sure of the uniqueness of this expression we consider the words which have no obvious cancellations between adjacent terms. The word is said to be reduced if $s_{i+1} \neq s_i$ for all i with $s_i \neq 1$ and if $s_k = 1$ for some k then for all $i \geq k$ $s_i = 1$. The reduced word (1,1,1,1,...) is called the empty word and is denoted by 1. By simplifying the notation by writing the reduced word $(s_1^{\varepsilon_1}, s_2^{\varepsilon_2}, s_3^{\varepsilon_3}, ..., s_n^{\varepsilon_n}, 1, 1, 1, ...)$ we write for $s_i \in S$ and $\varepsilon_i = \pm 1$ that $s_1^{\varepsilon_1} s_2^{\varepsilon_2} s_3^{\varepsilon_3} ... s_n^{\varepsilon_n}$. Now by definition, reduced words $r_1^{\delta_1} r_2^{\delta_2} r_3^{\delta_3} ... r_m^{\delta_m}$ and $s_1^{\varepsilon_1} s_2^{\varepsilon_2} s_3^{\varepsilon_3} ... s_n^{\varepsilon_n}$ are equal if and only if $n = m$ and $\delta_i = \varepsilon i$, $1 \leq i \leq n$. Let $F(S)$ be a set of reduced words on S and embed S into $F(S)$ by $s \mapsto (s, 1, 1, 1, ...)$. Under this injection we identify S with its image and henceforth consider S as a subset of $F(S)$. Note that if $S = \emptyset$ the $F(S) = \{1\}$. So we can construct the binary operation on $F(S)$, but we should be sure that the binary product of two reduced word is again a reduced word. Although the definition appears to be complicated it is simply the formal rule for successive cancellation of juxtaposed terms which are inverses of each other. Let $r_1^{\delta_1} r_2^{\delta_2} r_3^{\delta_3} ... r_m^{\delta_m}$ and $s_1^{\varepsilon_1} s_2^{\varepsilon_2} s_3^{\varepsilon_3} ... s_n^{\varepsilon_n}$ be reduced words and assume first that $m \leq n$. Let k be the smallest integer in the range $1 \leq k \leq m+1$ such that $s_k^{\varepsilon_k} \neq r_{m-k+1}^{-\delta_{m-k+1}}$. Then the product of these reduced words is defined to be:

$$(r_1^{\delta_1} r_2^{\delta_2} r_3^{\delta_3} ... r_m^{\delta_m})(s_1^{\varepsilon_1} s_2^{\varepsilon_2} s_3^{\varepsilon_3} ... s_n^{\varepsilon_n}) = \begin{cases} r_1^{\delta_1} ... r_{m-k+1}^{\delta_{m-k+1}} s_k^{\varepsilon_k} ... s_n^{\varepsilon_n}, & \text{if k} \leq \text{m;} \\ s_{m+1}^{\varepsilon_{m+1}} ... s_n^{\varepsilon_n}, & \text{if k=m+1} \leq \text{n;} \\ 1, & \text{if k=m+1 and m=n.} \end{cases}$$

The product is defined similarly when $m \geq n$, so in either case it results in a reduced word. We can easily see that 1 is the identity and the inverse of the reduced word $s_1^{\varepsilon_1} s_2^{\varepsilon_2} s_3^{\varepsilon_3} ... s_n^{\varepsilon_n}$ is the reduced word $s_1^{-\varepsilon_1} s_2^{-\varepsilon_2} s_3^{-\varepsilon_3} ... s_n^{-\varepsilon_n}$. Now let us define for each $s \in S \cup S^1 \cup \{1\}$ $\sigma_s : F(S) \to F(S)$ by

$$\sigma_s(s_1^{\varepsilon_1} s_2^{\varepsilon_2} s_3^{\varepsilon_3} ... s_n^{\varepsilon_n}) = \begin{cases} s \cdot s_1^{\varepsilon_1} s_2^{\varepsilon_2} s_3^{\varepsilon_3} ... s_n^{\varepsilon_n}, & \text{if } s_1^{\varepsilon_1} \neq s^{-1}; \\ s_2^{\varepsilon_2} s_3^{\varepsilon_3} ... s_n^{\varepsilon_n}, & \text{if } s_1^{\varepsilon_1} = s^{-1}. \end{cases}$$

Since $\sigma_{s^1} \circ \sigma_s$ is identity map of $F(S) \to F(S)$, σ_s is a permutation of $F(S)$. Let $A(F)$ be the subgroup of the symmetric group on the set $F(S)$ which is generated by $\{\sigma_s | s \in S\}$. Then the map $s_1^{\varepsilon_1} s_2^{\varepsilon_2} s_3^{\varepsilon_3} ... s_n^{\varepsilon_n} \mapsto \sigma_{s_1}^{\varepsilon_1} \circ \sigma_{s_1}^{\varepsilon_1} \circ ... \circ \sigma_{s_n}^{\varepsilon_1}$ is a set bijection between $F(S)$ and $A(S)$ which respects their binary operations. Since $A(S)$ is a group, hence associative, so is $F(S)$. So $F(S)$ is a group under the binary operation we defined and F is a functor from **Set** to **Grp** according to our construction. Let us define U : **Grp** $\to$ **Set** as a forgetfull functor which forgets the group structure.Let us consider $UF(S)$. Applying F first and then U does not yield the original set S, but we get a fundamental relationship between S and $UF(S)$ which we define above units; $\eta : S \to UF(S)$ that simply sends each element of S to itself in $UF(S)$ and this function satisfies the universal property ; given any function $g : S \to U(G)$, for $G \in ob(\textbf{Grp})$, there is a unique group homomorphism $h : F(S) \to G$ such that $U(h) \circ \eta = g$. In other words, $UF(S)$ is the best possible solution to the problem of inserting elements of S into a group. Composing U and F in the opposite order, we get a counit $\varepsilon : FU(G) \to G$ satisfying the universal property; for any group homomorphism $g : F(S) \to G$, there is a unique function $f : S \to U(G)$ such that $\varepsilon \circ F(h) = g \circ FU(G)$ constitutes the best possible solution to the problem of finding a representation of G as a quotient of a free group and we can express these with the commutative diagrams as follow :

$$\begin{array}{ccc} U(G) & \xrightarrow{\eta_{U(G)}} & UFU(G) \\ & \searrow & \downarrow U\varepsilon_G \\ & & U(G) \end{array} \qquad\qquad \begin{array}{ccc} F(S) & \xrightarrow{F\eta_S} & FUF(S) \\ & \searrow & \downarrow \varepsilon_S \\ & & F(S) \end{array}$$

Example 3.2.3. (Munkres (1975)) Our another example is construction an adjunction between the category of completely regular spaces and the category of the compact Hausdorff spaces **CHaus**. First let us denote the category of completely regular spaces by $\mathfrak{CR}$. Let $X \in ob(\mathscr{CR})$ and C_X be the set of all continuous functions from X to the interval $[0,1]$, simply we denote this set by C. Our first problem is that either there is an embedding $F : X \to Y$ for $Y \in$ **CHaus** or not. We take for every $f \in C$ a copy $[0,1]_f$ of $[0,1]$. This gives us a family of maps $f : X \to [0,1]_f$, with its diagonal map $F = \triangle_{f \in C} f : X \to [0,1]^C$. Because F is diagonal map of continuous functions, it is continuous. The space $[0,1]^C$ is compact according to Tychonoff's theorem and Hausdorff. We should show that $F : X \to FX$ is a homeomorphism. For if $x \neq y$ in X then there is a continuous function $f : X \to [0,1]$ with $f(x) = 0$ and $f(y) = 1$. Then the $f-th$ coordinate of $F(x)$ is 0 and the $f-th$ coordinate of $F(y)$ is 1, so $F(x) \neq F(y)$. This means that F is injective. Let U be open in X. In order to show that FU is open in FX we take $x \in U$ and seek an open set O in $[0,1]^C$ such that $F(x) \in O \cap FX \subseteq FU$. We take a continuous function $f : X \to [0,1]$ such that $f(x) = 0$ and $f(y) = 1$ for $y \in X\ U$ and let $O = \pi_f^{-1}([0,1))$. The $f-th$ coordinate of FX, which is $f(x)$, is 0, so $F(x) \in O$. If y is such that $F(y) \in O$ then we must have $f(y) < 1$, hence $y \in U$; we find $O \cap FX \subseteq FU$. So we show that F is a homeomorphism. In general this embedding denoted by β and the closure of βX in $[0,1]^C$ is a compactification of X, we call it the Cech-Stone compactification of X and we denote it as βX. Of course there exist other ways to construct the compactification of X. For example the unit sphere S^2 is a compactification of $\mathbb{R}^2$ by adding just one point for infinite ∞. This compactification is called one point compactification or *Alexsandorrf compactification*. Let Y and Z be two compactification of the completely regular space X. We call Y a larger compactification of X than Z if there is a continuous map g from Y to Z such that $g(x) = x$ for all $x \in X$. We write $Y \trianglerighteq Z$ or $Z \trianglelefteq Y$. This relation $\trianglerighteq$ is almost partial order but not unique. If Y and Z are compactifications of X with $Y \trianglerighteq Z$ and $Z \trianglerighteq Y$ then Y and Z are homeomorphic and the homeomorphism h can be chosen in such a way that $h(x) = x$ for all $x \in X$. We call two compactifications equivalent if there exists such a homeomorphism. The compactification βX is the largest compactification, that is, if Y is a compactification of the completely regular space X, then $\beta X \trianglerighteq Y$ and this gives us the first characterization of βX. The second characterization is that let f be a continuous function from X to [0,1] then there exist a continuous function $\beta f : \beta X \to [0,1]$ such that $\beta f \restriction_X = f$ and every compactification with this property is equivalent to βX. Let U be a forgetfull functor from **CHaus** to $\mathscr{CR}$ then the extension property makes β a functor from $\mathscr{CR}$ to **CHaus** such that β is left adjoint of U where the bijection $\psi : Hom(\beta X, Y) \to Hom(X, UY)$ is clear with our explanation and natural in X and Y.

Example 3.2.4. Let F : **Grp** $\to$ **Grp** $\times$ **Grp** be the diagonal functor which assigns to every group X the pair (X,X) in the product category **Grp** $\times$ **Grp** and G : **Grp** $\times$ **Grp** $\to$ **Grp** the functor which assigns to each pair (Y_1, Y_2) the product group $Y_1 \times Y_2$. The universal property of the product group shows that G is the right-adjoint to F. the co-unit gives the natural projections from the product of the factors.

Example 3.2.5. For a space X the *suspension* SX is the quotient of $X \times I$ obtaining by collapsing $X \times \{0\}$ to a point and $X \times \{1\}$ to another point. If we think that these points are the chosen base point and collapse the line segment $\{x_0\} \times I$ to the point $\{x_0\} \times \{0\}$ then the new space is homotopy equivalent to SX. We call this new space the *reduced suspension* ΣX. If we identify the points $\{0\}$ and $\{1\}$ in I then we have an identification space $S^1 = I/\{0,1\}$. Let $\{0\} = \{+1\}$ is the base point of S^1 then the reduced suspension space ΣX is actually the same as the smash product $X \wedge S^1$. Let us explain the smash product of two space X and Y. Inside a product $X \times Y$ there are copies of X and Y namely with chosen base points $X \times \{y_0\}$ and $\{0\} \times Y$ for points $x_0 \in X$ and $y_0 \in Y$. The two copies of X and Y in $X \times Y$ intersect only at the point (x_0, y_0). So their union can be identified with the wedge sum of X and Y, that is, take the quotient of the disjoint union $X \sqcup Y$ obtained by identifying x_0 and y_0 to a single point. Then the smash product $X \wedge Y$ is defined to be the quotient $X \times Y / X \vee Y$. Since ΣX and $X \wedge S^1$ are both the quotient of $X \times I$ with $X \times \partial I \cup \{x_0\} \times I$ collapsed to a point we can think that $\Sigma X = X \times S^1$. Now let ΩX denote the loop space of X, that is, $\Omega X = \{\alpha | \alpha : I \to X; \alpha(0) = \alpha(1) = x_0\}$ where $x_0 \in X$ is the chosen base point. Since we stick together the starting and end points of a path in ΩX one may think the points of ΩX as a continuous function from S^1 to X and $\Omega X = (X, x_0)^{S^1}$. From these explanation one can think that Σ and Ω are adjoint functors from the category of pointed compactly generated Haussdorf spaces $\mathbf{CHaus}_*$ to itself such that

$$\Sigma X = X \wedge S^1 \text{ and } \Omega X = (X, x_0)^{S^1}.$$

The unit $X \to \Omega\Sigma X$ of the adjunction send $x \in X$ to the function $< x, - >: I \to \Sigma X$, it has a vivid geometric picture ; it sends each point $x \in X$ to that generator of the cone which passes through x, this generator is a loop from the north pole to the south pole but they are same. Hence a point of $\Omega\Sigma X$. By iteration Σ^n is a left adjoint of Ω^n :$\mathbf{CHaus}_* \to$ $\mathbf{CHaus}_*$. Adjunction has a unit $X \to \Omega^n\Sigma^n X$ which can be written as a composite

$$X \longrightarrow \Omega\Sigma X \xrightarrow{\Omega\eta\Sigma X} \Omega\Omega\Sigma\Sigma X \longrightarrow \cdots .$$

3.2.2 Monads and Algebras

Definition 3.2.6. If $\mathscr{C}$ is a category, a *monad* on $\mathscr{C}$ consists of a functor $T : \mathscr{C} \to \mathscr{C}$ together with two natural transformations $\eta : 1_{\mathscr{C}} \to T$ (where $1_{\mathscr{C}}$ denotes the identity functor on $\mathscr{C}$) and $\mu : T^2 \to T$ here T^2 is the composition of T with itself, that is $T^2 = T \circ T$. these are required the following coherence conditions :

- $\mu \circ T\mu = \mu \circ \mu T$ as natural transformation $T^3 \to T$
- $\mu \circ T\eta = \mu \circ \eta T = 1_T$ as natural transformations $T \to T$ and here 1_T denotes the identity transformation from T to T .

$$\begin{array}{ccc} T^3 & \xrightarrow{T\mu} & T^2 \\ {\scriptstyle \mu T}\downarrow & & \downarrow{\scriptstyle \mu} \\ T^2 & \xrightarrow[\mu]{} & T \end{array} \qquad \begin{array}{ccc} T & \xrightarrow{\eta T} & T^2 \\ {\scriptstyle T\eta}\downarrow & & \downarrow{\scriptstyle \mu} \\ T^2 & \xrightarrow[\mu]{} & T \end{array}$$

Example 3.2.7. We can construct a monad on the category of **Set**. Let X be an object of **Set** and $T(X)$ is the power set of X where for any function f from X to Y in **Set** T(f) be the function between power sets induced by taking direct images under f. For every set X we have a map $\eta_X : X \to T(X)$, which assigns to every element x of X the singleton $\{x\}$. A function $\mu_X : T(T(X)) \to T(X)$ can be given as follows : if Y is a set whose elements are subsets of X , then taking the union of these subsets gives a subset $\eta_X(Y)$ of X . So these data describe a monad.

Note that given an adjunction $(F,G,\eta,\varepsilon) : \mathscr{C} \leftrightarrows \mathscr{D}$ we can always define a monad with using the units and counits of this adjunction such that take $T = GF : \mathscr{D} \to \mathscr{D}$. So we have a natural transformation $\eta : id_{\mathscr{D}} \Rightarrow T$ and a natural transformation $\mu : T^2 \Rightarrow T$ where the components μ_D for $D \in ob\mathscr{D}$ are

$$T^2(D) = GFGF(D) \xrightarrow{G_{(\varepsilon F(D))}} GF(D) = T(D) \ .$$

Conversly every monad arises from and adjunction, but in more than one way. Essentially, there are a maximal and a minimal solution to the problem of finding an adjunction from which a given monad.

Definition 3.2.8. Suppose that (T,η,μ) is given monad on a category $\mathscr{C}$. A *T-algebra* (X,k) is an object X of $\mathscr{C}$ together with an arrow $k : T(X) \to X$ of $\mathscr{C}$ called structure map of the algebra such that the diagrams

$$\begin{array}{ccc} T^2X & \xrightarrow{Tk} & Tx \\ {\scriptstyle \mu_X}\downarrow & & \downarrow{\scriptstyle k} \\ TX & \xrightarrow[k]{} & X \end{array}$$

and

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & TX \\ & \searrow^{1_X} & \downarrow k \\ & & X \end{array}$$

commutes. A morphism $f:(X,k)\to(Y,h)$ of $T-$algebras is an arrow $f:X\to Y$ of $\mathscr{C}$ such that the diagram

$$\begin{array}{ccc} TX & \xrightarrow{Tf} & TY \\ k\downarrow & & \downarrow h \\ X & \xrightarrow[f]{} & Y \end{array}$$

commutes.

The category $\mathscr{C}^T$ of $T-$algebras and their morphisms is called the *Eilenberg-Moore* category of the monad T. Given the monad, we can also define another category which is called the *Kleisli* category of monad T $\mathscr{C}_T$. Its objects are the objects of $\mathscr{C}$ and its arrows from X to Y are the arrows $f:X\to T(Y)$ in $\mathscr{C}$. The identity on an object X is the unit η_X and the composite $g\circ f:X\to T(Z)$ of two arrows $f:X\to T(Y)$ and $g:Y\to T(Z)$ is given by $g\circ f=\mu_Z\circ Tg\circ f$.

Theorem 3.2.9. *There is an adjunction between $T-Alg$ and $\mathscr{C}$ which brings about the given monad T.*

Proof. (van Oosten (2002)) There is an obvious forgetfull functor $U^T:T-Alg\to\mathscr{C}$ which takes (X,h) to X. We claim that U^T has a left adjoint F^T. F^T assigns to an object X the $T-$algebra $T^2(X)\xrightarrow{\mu_X}T(X)$; to $X\xrightarrow{f}Y$ the map $T(f)$; this is an algebra map because of the maturality of μ. That $T^2(X)\xrightarrow{\mu_X}T(X)$ is an algebra follows from the defining axioms for a monad T. Now given any arrow $g:X\to U^T(Y,h)$ we let $\tilde{g}:(T(X),\mu_X)\to(Y,h)$ be the arrow $T(X)\xrightarrow{T(g)}T(Y)\xrightarrow{h}Y$. This is a map of algebras since

$$\begin{array}{ccccc} T^2(X) & \xrightarrow{T^2(g)} & T^2(Y) & \xrightarrow{T(h)} & T(Y) \\ \mu_X\downarrow & & \mu_Y\downarrow & & \downarrow h \\ T(X) & \xrightarrow[g]{} & T(Y) & \xrightarrow[h]{} & Y \end{array}$$

commutes. The left hand square is the naturality of μ; the right hand square is because (Y,h) is a $T-$algebra. Conversely, given a map of algebras $f:(T(X),\mu_X)\to(Y,h)$ we have an arrow $\tilde{f}:X\to Y$ by taking the composite $X\xrightarrow{\eta_X}T(X)\xrightarrow{f}Y$. Now $\tilde{f}:T(X)\to Y$ is the composite $T(X)\xrightarrow{T\eta_X}T^2(X)\xrightarrow{T(f)}T(Y)\xrightarrow{h}Y$. Since f is a $T-$algebra map, that is $T(X)\xrightarrow{T\eta_X}T^2(X)\xrightarrow{\mu_X}T(X)\xrightarrow{f}Y$ which is f by the monad laws. Conversely; $\tilde{g}:X\to Y$ is the composite $X\xrightarrow{\eta_X}T(X)\xrightarrow{T(g)}T(Y)\xrightarrow{h}Y$. By the naturality of η and the fact that (Y,h) is a

$T-$algebra, we conclude that $\tilde{\tilde{g}} = g$. So we have a naturel 1-1 correspondence $\mathscr{C}(X, U^T(Y,h)) \simeq T - Alg(F^T(X), (Y,h))$ and our adjunction. The composite $U^T F^T$ is the functor T, and that unit η of the adjunction is the unit of T, for the counit ε of $F^T \rightarrow U^T$ we have that $T^2 = U^T F^T U^T F^T \xrightarrow{U^T \varepsilon F^T} U^T F^T = T$ is our multiplication. $\square$

CHAPTER FOUR

SIMPLICIAL CATEGORIES AND N-CATEGORIES

4.1 Monoidal Categories

In this section we will give some definitions which we will use in simplicial category.

Definition 4.1.1. A *monoidal category* is a category $\mathscr{M}$ equipped with;

- A binary functor $\otimes : \mathscr{M} \times \mathscr{M} \to \mathscr{M}$ called the tensor product or the monoidal product.
- An object I called the unit object.
- Three natural isomorphism subject to certain coherence condition expressing the fact that the tensor operation;
  - is associative : there is a natural isomorphism α, called *associativity* with components $\alpha_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$.
  - has I left and right identity : there are two natural isomorphism λ and ρ, respectively called *left* and *right* identity, with components $\lambda_A : I \otimes A \to A$ and $\rho_A : A \otimes I \to A$.

The coherence conditions for these natural transformations follow:

- for all A, B, C and D in $\mathscr{M}$, in the diagram :

$$\begin{array}{ccccc} ((A \otimes B) \otimes C) \otimes D & \xrightarrow{\alpha_{A,B,C \otimes D}} & (A \otimes (B \otimes C)) \otimes D & \xrightarrow{\alpha_{A,B \otimes C,D}} & A \otimes ((B \otimes C) \otimes D) \\ \downarrow{\scriptstyle \alpha_{A \otimes B,C,D}} & & & & \downarrow{\scriptstyle A \otimes \alpha_{B,C,D}} \\ (A \otimes B) \otimes (C \otimes D) & & \xrightarrow[\alpha_{A,B,C \otimes D}]{} & & A \otimes (B \otimes (C \otimes D)) \end{array}$$

- for all A and B in $\mathscr{M}$, in diagram

$$\begin{array}{ccc} (A \otimes I) \otimes B & \xrightarrow{\alpha_{A,I,B}} & A \otimes (I \otimes B) \\ & {\scriptstyle \rho_A \otimes B} \searrow \quad \swarrow {\scriptstyle A \otimes \lambda_B} & \\ & a \otimes B & \end{array}$$

  commutes.

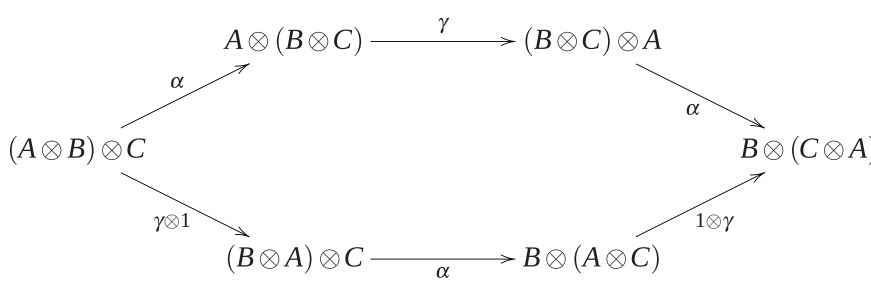
It follows from these three conditions that any such diagram commutes; this is Mac Lane's coherence theorem. This is related to the fact that every monoidal category is monodially equivalent to a strict monoidal category. Let's consider the strict monoidal category. This is a monoidal category where the natural isomorphisms are identities.

Definition 4.1.2. One can construct for any strict monoidal category $\mathscr{M}$ *the free strict monoidal category* $\Sigma(\mathscr{M})$ as follows :

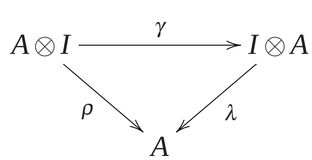
- Its objects are lists (finite sequences) $A_1,......,A_m$ of objects of $\mathscr{M}$;
- there are arrows between two objects $A_1,....,A_m$ and $B_1,....,B_n$ if and only if $m = n$, and then the arrows are lists of arrows $f_i : A_i \to B_i$ of $\mathscr{M}$ where $i = 1,2,....,m$;
- the tensor product of two objects $A_1,...,A_m$ and $B_1,.....,B_n$ is the concatenation $A_1,.....,A_m,B_1,....,B_n$ of the two lists, and, similarly, the tensor product of two morphisms is given by the concatenation of lists. The operation Σ can be considered as functor from $\mathscr{M}$ to $\Sigma(\mathscr{M})$.

Example 4.1.3. Any category with standard categorical products and terminal object is a monoidal category, with the categorical product as tensor and the terminal object as identity. However, in many monoidal categories, the tensor product is neither a categorical product nor coproduct.

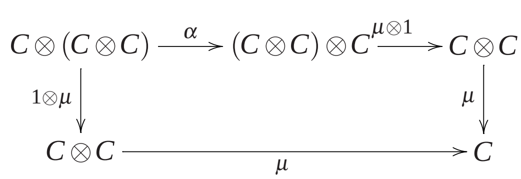
Definition 4.1.4. A *braided monoidal category* is a monoidal category $\mathscr{M}$ equipped with a *braiding*; that is, there is a natural isomorphism $\gamma_{A,B} : A\otimes B \to B\otimes A$ for which the following hexagonal diagrams commute:

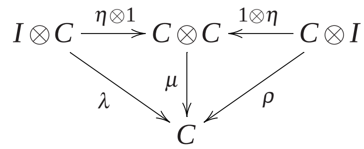


Definition 4.1.5. A *symmetric monoidal category* is a braided monoidal category whose braiding satisfies $\gamma_{A,B}\gamma_{B,A} = 1_{A\otimes B}$ for all objects A and B. In a braided monoidal category, the braidings always commutes with the units as in the diagram :



Definition 4.1.6. A monoid C in a monoidal category $< \mathscr{M}, \otimes, I$ is an object $C \in ob(\mathscr{M})$ together with two morphisms $\mu : C\otimes C \to C$ and $\eta : I \to C$ such that the diagrams





are commute. A morphism $f :< C, \mu, \eta > \to < C', \mu^{'}, \eta^{'} >$ of monoids is a morphism such that

$$f\mu = \mu^{'}(f \otimes f) : C \otimes C \to C' \text{ and } f\eta = \eta^{'} : I \to C' .$$

With these morphisms the monoids in $\mathscr{M}$ constitute a category $\mathbf{Mon}_{\mathscr{M}}$ whose objects are the monoids in $\mathscr{M}$. The operation $< C, \mu, \eta > \mapsto C$ defines a forgetfull functor $U : \mathbf{Mon}_{\mathscr{M}} \to \mathscr{M}$.

4.2 Simplicial Category

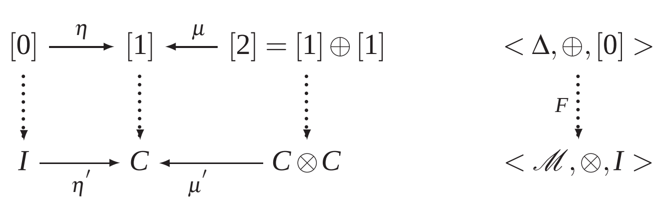
First of all, we will give definition of simplicial category. Secondly, we try to explain the morphisms in this category. Finally, we will give the geometric interpretation of simplicial sets and we will define the subdivision of simplicial complexes.

Definition 4.2.1. The *simplicial category* Δ is defined as the small category whose objects are all finite ordinal numbers $[n] = \{0, 1, 2, ..., n-1\}$ and whose maps are all monotone functions, that is, all functions $f : [n] \longrightarrow [m]$ such that $0 \leq i \leq j < n$ imply $f(i) \leq f(j)$. The empty set $\emptyset = [0]$ is the initial object of the simplicial category and [1] is the terminal object, that is, for any $[n]$ there exist unique maps k and u satisfying that $k : [0] \longrightarrow [n]$ and $u : [n] \longrightarrow [1]$. *Ordinal addition* is a bifunctor, which we denote it by $\oplus : \Delta \times \Delta \longrightarrow \Delta$, defined on ordinals $[n], [m]$ as the usual sum $[n+m]$ and on arrows $f : [n] \longrightarrow [n']$ and $g : [m] \longrightarrow [m']$ as

$$(f \oplus g)(i) = \begin{cases} f(i), & 0 \leq \text{i} < \text{n}; \\ n' \oplus g(i-n), & \text{n} \leq \text{i} < \text{n} \oplus \text{m}. \end{cases} \quad (4.2.1)$$

Moreover; since [1] is terminal and [0] is initial in Δ there are unique arrows $\mu : [2] \longrightarrow [1]$ and $\eta : [0] \longrightarrow [1]$, with these arrows the triple $< \Delta, \oplus, [0] >$ is a strict monoidal category and for the same reason these arrows form a monoid $< [1], \mu, \eta >$ in Δ.

Proposition 4.2.2. *(Mac Lane (1998)) Given a monoid $< C, \mu^{'}, \eta^{'} >$ in a strict monoidal category $< \mathscr{M}, \otimes, I >$ there is a unique morphism $F :< \Delta, \oplus, [0] > \to < \mathscr{M}, \otimes, I >$ such that $F([1]) = c$, $F\mu = \mu^{'}$ and $F\eta = \eta^{'}$ as in the figure*



Proof. First we should show that the arrows of Δ are exactly the iterated formal product. Write $\mu^{(k)}$ for the unique arrow $\mu^{(k)} : [k] \longrightarrow [1]$; thus $\mu^{(0)} = \eta$, $\mu^{(1)}$ is the identity (here we use the notation **1** for identity to not mixed with the ordinal number 1) and $\mu^{(2)} = \mu : [2] \longrightarrow [1]$ let us consider $\mu^{(3)}$;

$$\mu^{(3)} = \mu(\mu \oplus \mathbf{1}) = \mu(\mathbf{1} \oplus \mu) : [3] \longrightarrow [1]$$

This equality holds because it means just the general associative law, that is,

$$\mu \oplus \mathbf{1} = \begin{cases} \mu(i), & \text{i=0,1;} \\ 1 \oplus \mathbf{1}(i-2), & \text{i=2.} \end{cases}$$

$$\mathbf{1} \oplus \mu = \begin{cases} \mathbf{1}(i), & \text{i=0;} \\ 1 \oplus \mu(i-1), & \text{i=1,2.} \end{cases}$$

Since $[1]$ is terminal object in Δ ; one can get the equation by using iteration

$$\mu^n(\mu^{(k_1)} \oplus .... \oplus \mu^{(k_n)}) = \mu^{(k_1 + ... + k_n)} .$$

Moreover, if $f : [m] \longrightarrow [n]$ is any monotone function, let m_i be the number of elements in the subset $f^{-1}(i)$ of $[m]$ then we obtain the equality,

$$f = \mu^{(m_0)} \oplus \mu^{(m_1)} \oplus .... \oplus \mu^{m_{n-1}} \text{ ; where } \textstyle\sum_{i=0}^{n-1} m_i = m$$

This shows that any map f in Δ is a sum of iterated products constructed from μ and η. Now consider the functor required in the proposition. Since $F([1]) = C$ and F is to be a morphism of monoidal categories, F must have $F([n]) = C^{(n)}$; this determines the object function of F. Next, $F\mu = \mu^{'}$ and $F\eta = \eta^{'}$ imply that $F\mu^{(n)} = \mu^{'(n)}$ and the representation of any arrow f in Δ determines the arrow function Ff of F. Thus F is unique and since in Δ composition is given by the equation " $\mu^n(\mu^{(k_1)} \oplus .... \oplus \mu^{(k_n)}) = \mu^{(k_1 + ... + k_n)}$ " which correspond exactly to the general associative law valid in $\mathscr{M}$, this show F is a functor. □

There is another description of the arrows of Δ, which starts by observing that a monotone function $f : [n] \longrightarrow [n^{'}]$ be factored as $f = g \circ h$ where $h : [n] \longrightarrow [n^{''}]$ is surjective and monotone; $g : [n^{''}] \longrightarrow [n^{'}]$ is monotone and injective. This injective function g will be determined just by giving the image set of g,which is a subset of $[n^{''}]$ ordinals in the set $[n^{'}]$.

In particular, from $[n]$ to $[n+1]$ there exactly $n+1$ map which are monotone injective denoted by δ_i^n whose image omits i, thus

$$\delta_i^n : [n] \to [n+1] \ ; \ \delta_i^n\{0,1,....,n-1\} = \{0,1,..,\hat{i},...,n\}$$

where the hat means that i is to be omitted.We display all these arrows as

$$[0] \xrightarrow{\delta_0^0} [1] \underset{\delta_1^1}{\overset{\delta_0^1}{\rightrightarrows}} [2] \ \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow} \ [3] \ , ..., \qquad \delta_0, ..., \delta_n : [n] \longrightarrow [n+1].$$

On the other hand; a monotone $h : [n] \to [n^{''}]$ which is surjective is determined by the subset $\{j \mid h(j) = h(j+1); 0 \leq j \leq n-2\}$ of those $n - n^{''}$ argument j at which h does not increase. In particular there are n such arrows $[n+1] \to [n]$; for $i = 0, ..., n-1$ they are

$$\sigma_i^n : [n+1] \to [n] \text{ where } \sigma_i^n(i) = \sigma_i^n(i+1)$$

We display them as

$$[0] \longleftarrow [1] \xleftarrow{\sigma_0^1} [2] \underset{\sigma_1^2}{\overset{\sigma_0^2}{\leftleftarrows}} [3] \ \substack{\longleftarrow \\ \longleftarrow \\ \longleftarrow} \ [4] \ , ..., \qquad \sigma_0, ..., \sigma_{n-1} : [n+1] \longrightarrow [n].$$

These maps may also be expressed in terms of μ and η. Indeed $\delta_0^0 : [0] \to [1]$ is η and $\sigma_0^2 : [2] \to [1]$ is μ and the descriptions of the morphisms show that

$$\delta_i^n = \quad \mathbf{1}_i \oplus \eta \oplus \mathbf{1}_{n-i} : [n] \longrightarrow [n+1] \qquad i = 0,1,...,n$$
$$\sigma_i^n = \mathbf{1}_i \oplus \mu \oplus \mathbf{1}_{n-i-1} : [n+1] \longrightarrow [n] \qquad i = 0,1,...,n-1$$

Lemma 4.2.3. *In Δ, any arrow $f : [n] \to [n^{'}]$ has a unique representation*

$$f = \delta_{i_1} \circ \delta_{i_2} \circ .... \circ \delta_{i_k} \circ \sigma_{j_1} \circ \sigma_{j_2} .... \circ \sigma_{j_h}$$

where the ordinal numbers h and k satisfy $[n-h+k] = [n^{'}]$ while the string of subscripts i and j satisfy

$$n^{'} > i_1 > ... > i_k \geq 0 \ ; \ n-1 > j_h > ... > j_1 \geq 0$$

Proof. (Mac Lane (1998)) By induction on $i \in [n]$, any monotone f is determined by its image, a subset of $[n^{'}]$, an by the set of those $j \in [n]$ at which does increase $[f(i) = f(j+1)]$. Putting $i_1, ..., i_k$, in reverse order, for those elements of $[n^{'}]$ not in the image and $j_1, ..., j_h$, in order, for the elements j of $[n]$ where f does not increase, it follows that the functions on both sides of the equation are equal. □

In simplical category the morphisms δ and σ satisfy the following axioms :

$$\delta_i\delta_j = \delta_{j+1}\delta_i \quad for \quad i \leq j \tag{4.2.2}$$

$$\sigma_j\sigma_i = \sigma_i\sigma_{j+1} \quad for \quad i \leq j \tag{4.2.3}$$

$$\sigma_j\delta_i = \begin{cases} \delta_i\sigma_{j-1}, & \text{if i < j;} \\ \mathbf{1}_n, & \text{if i=j or i=j+1;} \\ \delta_{i-1}\sigma_j, & \text{if i > j+1.} \end{cases} \tag{4.2.4}$$

These identities may be verified directly by checking the image of both sides with the given condition, For instance, $\delta_i\delta_j : [n] \longrightarrow [n+2]$ for any $i \leq j$ the image;

$$\begin{aligned} \delta_i\delta_j(\{0,1,..,i,...,j,j+1,j+2,...,n-1\}) &= \delta_i(\{0,1,...,i,...,\hat{j},j+1,...,n\}) \\ &= \{0,1,...,\hat{i},...,j,j\hat{+}1,...,n+1\}, \\ \delta_{j+1}\delta_i(\{0,1,..,i,...,j,j+1,j+2,...,n-1\}) &= \delta_{j+1}(\{0,1,...,\hat{i},...,j,j+1,...,n\}) \\ &= \{0,1,...,\hat{i},...,j,j\hat{+}1,...,n+1\}, \end{aligned}$$

where the hat means that the number is omitted. Since the image are equal and the functions are injective, the equality holds. One can verify the others with similar idea.

Definition 4.2.4. In $\mathbb{R}^{n+1}$ the standard n-simplex is the subset given by

$$\Delta_{n+1} = \{(t_0,t_1,...,t_n) \in \mathbb{R}^{n+1} | \textstyle\sum_{i=0}^{n} t_i = 1 \; ; \forall i \; t_i \geq 0\}$$

the vertices of the standard n-simplex are the points

$$\begin{aligned} e_0 &= (1,0,0,...,0), \\ e_1 &= (0,1,0,...,0), \\ e_2 &= (0,0,1,...,0), \\ &\vdots \\ e_n &= (0,0,0,...,1). \end{aligned}$$

For example, the standard 2-simplex $\triangle_3$ in $\mathbb{R}^3$ in figure 4.1, One can construct with arbitrary n+1 points $\{v_0, v_1..., v_n\}$ in $\mathbb{R}^{n+1}$ n-simplex by using the canonical map

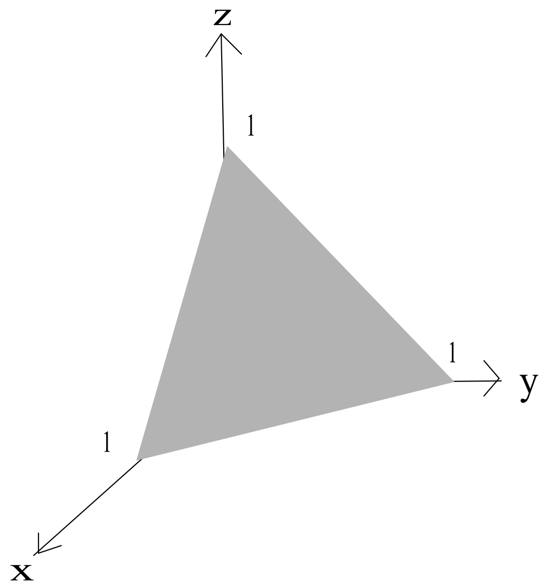


Figure 4.1 Standart 2-simplex.

$$(t_0, t_1, ..., t_n) \longmapsto \sum_{i=0}^{n} t_i v_i$$

The coefficients t_i are called the barycentric coordinates of a point in the n-simplex, this general simplex is often called *affine n-simplex* and the canonical map is called *an affine transformation.*

According the definition of standard n-simplex, we have a functor $\Delta : \Delta \rightarrow$ **TOP** ; on objects $[n+1] \mapsto \triangle_{n+1}$; on arrows $(f : [n+1] \rightarrow [m+1]) \mapsto (\Delta_f : \Delta_{n+1} \rightarrow \Delta_{m+1})$ where the map defined by

$$\Delta_f(t_0, t_1, ..., t_n) = (s_0, s_1, ..., s_m) \; ; \; s_j = \sum_{f(i)=j} t_i \; .$$

Here we should be carefull about the notation, Δ_{n+1} has dimension n and n+1 vertices, while Δ_f is the affine map which sends the vertex i of Δ_{n+1} to the vertex $f(i)$ of Δ_{m+1} and Δ is a subcategory of **TOP**, but the geometric dimension is one less than the arithmetic one used in Δ . By Δ^* we denote the full subcategory of Δ whit objects all the positive ordinals $\{1,2,3,...\}$ omitting only 0 .After here we use the notation Δ instead of Δ^* so that Δ has objects such that $[n] = \{0, 1, 2, ..., n\}$ for all $n \geq 0$ and for the standard n-simplex we will use Δ^n .

Definition 4.2.5. *Simplicial sets* are contravariant functors $K : \Delta^{op} \rightarrow$ **Set** and the natural transformations between simplicial sets are called simplicial maps. If we write this functor as

$$[n+1] \mapsto K_n \; ; \; \delta_i \mapsto d_i \text{ and } \sigma_i \mapsto s_j$$

so that K_n is in geometric dimension n, then the simplicial sets may be described in the traditional way as a list of $K_0, K_1, ..., K_n, ...$ objects of **Set** with arrows $d_i : K_n \rightarrow K_{n-1}$ for $i = 0, 1, ..., n$ and $n > 0$

and $s_i : K_n \to K_{n+1}$ for $i = 0, 1, ..., n$ and $n \geq 0$ called face and degeneracy operators respectively. These arrows satisfy the properties dual to properties 4.2.2, 4.2.3, 4.2.4 of δ and σ;

$$d_i d_{j+1} = d_j d_i \quad for \quad i \leq j \tag{4.2.5}$$

$$s_{j+1} s_i = s_i s_j \quad for \quad i \leq j \tag{4.2.6}$$

$$d_i s_j = \begin{cases} s_{j-1} d_i, & \text{if i < j;} \\ \mathbf{1}, & \text{if i=j or i=j+1;} \\ s_j d_{i-1}, & \text{if i > j+1.} \end{cases} \tag{4.2.7}$$

So with this information we have that for any $\alpha : [m] \to [n]$ there exists $K\alpha : K_n \to K_m$.

Let us use any small category $\mathscr{C}$ instead of the category **Set** in the image of the functor such that $K : \Delta^{op} \to \mathscr{C}$ then this functor K is called the simplicial object in the category $\mathscr{C}$ and it satisfies the conditions listed above for simplicial sets.

Definition 4.2.6. (May (1992))A *simplicial map* $f : K \to K'$ is a natural transformation which commutes with the face and degeneracy operators; that is, f consists of $f_n : K_n \to K'_n$ and

$$f_n d_i = d_i f_{n+1},$$
$$f_n s_i = s_i f_{n-1}.$$

We denote by **sSet** the category of simplicial sets with the natural transformations as simplicial maps.

The *simplicial standard n-simplex* is the simplicial set $\Delta[n] = Hom(-, [n])$, that is, $\Delta[n]$ is the result of applying Δ to $[n]$, so for $\alpha : [m] \to [n]$, $\Delta[\alpha] : \Delta[m] \to \Delta[n]$. Owing to the Yoneda lemma, if K a simplicial set and if $x \in K_n$, then there exists one and only one simplicial map $\Delta_x : \Delta[n] \to K$ that takes $id_{[n]}$ to x. Hence the category **sSet** is complete and cocomplete(3.1.5), wellpowered and cowellpowered (2.2.16).

Definition 4.2.7. Let K be a simplicial set then one writes $x \in K$ when one means $x \in \bigcup_n K_n$. With this understanding, an $x \in K$ is said to be *degenerate* if there exists an epimorphism $\alpha \neq id$ and a $y \in K$ such that $x = (K\alpha)y$, otherwise $x \in K$ is said to be *nondegenerate*. The elements of K_0 which are represents the vertexes of K are nondegenerate. Every $x \in K$ admits a unique representation $x = (K\alpha)y$, where α is an epimorphism and y is nondegenerate. The nondegenerate elements in $\Delta[n]$ are the mononorphisms $\alpha : [m] \to [n]$ $(m \leq n)$.

Definition 4.2.8. A *simplicial subset* of a simplicial set K is a simplicial set L such that L is a *subfunctor* of K that is $L_n \subset K_n$ for all n and the inclusion $L \to K$ is a simplicial map. We use the notation $L \subset K$.

The *n-skeleton* of a simplicial set K is the simplicial subset $K^{(n)}$ $(n \geq 0)$ of K defined by stipulating that $K_p^{(n)}$ is the set of all $x \in K_p$ for which there exists an epimorphism $\alpha : [p] \to [q]$ $(q \leq n)$ and a $y \in K_p$ such that $x = (K\alpha)y$. Therefore $K_p^{(n)} = K_p$ $(p \leq n)$; furthermore, $K^{(0)} \subset K^{(1)} \subset \ldots$ and $K = colimK^{(n)}$ (3.1.6). A proper simplicial subset of $\Delta[n]$ is contained in $\Delta[n]^{(n-1)}$, the *frontier* $\dot{\Delta}[n]$ of $\Delta[n]$. Of course, $\dot{\Delta}[0] = \emptyset$. $K^{(0)}$ is isomorphic to $K_0 \cdot \Delta[0]$. In general, let $K_n^{\#}$ be the set of nondegenerate elements of K_n. Fix a collection $\{\Delta[n]_x : x \in K_n^{\#}\}$ of simplicial standard n-simplexes indexed by $K_n^{\#}$ then the simplicial maps $\Delta_x : \Delta[n] \to K$ $(x \in K_n^{\#})$ determine an arrow $K_n^{\#} \cdot \Delta[n] \to K^{(n)}$ and the commutative diagram

$$\begin{array}{ccc} K_n^{\#} \cdot \dot{\Delta}[n] & \longrightarrow & K^{(n-1)} \\ \downarrow & & \downarrow \\ K_n^{\#} \cdot \Delta[n] & \longrightarrow & K^{(n)} \end{array}$$

is a pushout square. Note that $\dot{\Delta}[n]$ is a coequalizer. For proof consider the diagram

$$\coprod_{0 \leq i < j \leq n} \Delta[n-2]_{i,j} \underset{v}{\overset{u}{\rightrightarrows}} \coprod_{0 \leq i \leq n} \Delta[n-1]_i$$

where u is defined by the $\Delta[\delta_i^{n-1}]$ then the $\Delta[\delta_i^n]$ define a simplicial map $f : \coprod_{0 \leq i \leq n} \Delta[n-1]_i \to \Delta[n]$ that induces an isomorphism $coeq(u,v) \to \dot{\Delta}[n]$.

Definition 4.2.9. The *realization* functor Γ_{Δ^-} is a functor from **sSet** to **Top** such that $\Gamma_{\Delta^-} \circ \Delta = \Delta^-$ this means that it assigns to a simplicial set K a topological space $|K| = \int^n K_n \cdot \Delta^n$, the *geometric realization* of K, and to a simplicial map $f : K \to L$ a continuous function $|f| : |K| \to |L|$, the *geometric realization* of f. In particular, $|\Delta[n]| = \Delta^n$ and $|\Delta[\alpha]| = \Delta^\alpha$. There is an explicit description of $|K|$: Equip K_n with discrete topology and $K_n \times \Delta^n$ with the product topology then $|K|$ be identified with the quotient $\coprod_n K_n \times \Delta^n / \sim$, the equivalence relation being generated by writing $((K\alpha)x, t) \sim (x, \Delta^\alpha t)$. These relations are respected by every simplicial map $f : K \to L$. Denote by $[x,t]$ the equivalence class corresponding to (x,t). The projection $(x,t) \to [x,t]$ of $\coprod_n K_n \times \Delta^n$ onto $|K|$ restricts to a map $\coprod_n K_n^{\#} \times \Delta^{\circ n} \to |K|$ that is a set theoretic bijection. Consequently, if we attach $x \in K_n^{\#}$ the subset e_x of $|K|$ consisting of all $[x,t]$ $(t \in \Delta^{\circ n})$, then the collection $\{e_x : x \in K_n^{\#}(n \geq 0)\}$ partitions $|K|$. It follows from this that a simplicial map $f : K \to L$ is injective if and only if its geometric realization $|f| : |K| \to |L|$ is injective and one can say this condition also for surjective maps. Being a left adjoint, the functor $|-| :$ **sSet** $\to$ **Top** preserves colimits. So, by taking the geometric realization of the diagram $\coprod_{0 \leq i < j \leq n} \Delta[n-2]_{i,j} \underset{v}{\overset{u}{\rightrightarrows}} \coprod_{0 \leq i \leq n} \Delta[n-1]_i$, and unraveling the definitions, one find that $|\dot{\Delta}[n]|$ can be identified with $\dot{\Delta}^n$.

Definition 4.2.10. Given n, let $\overline{\Delta}[n]$ be the simplicial set defined by the following conditions:

a. $\overline{\Delta}[n]$ assigns to an object $[p]$ the set $\overline{\Delta}[n]_p$ of all finite sequence $\mu = (\mu_0, \ldots, \mu_p)$ of monomorphisms in Δ having codomain $[n]$ such that $\forall i, j (0 \leq i \leq j \leq p)$ there is a monomorphism μ_{ij} with $\mu_i = \mu_j \circ \mu_{ij}$.

b. $\overline{\Delta}[n]$ assigns to a morphism $\alpha : [q] \to [p]$ the map $\overline{\Delta}[n]_p \to \overline{\Delta}[n]_q$ taking μ to $\mu \circ \alpha$, that is , $(\mu_0, ...\mu_p) \to (\mu_{\alpha(0)}, ..., \mu_{\alpha(q)})$.

We call $\overline{\Delta}$ the functor $\Delta \to$ **sSet** that sends $[n]$ to $\overline{\Delta}[n]$ and $\alpha : [m] \to [n]$ to $\overline{\Delta}[\alpha] : \overline{\Delta}[m] \to \overline{\Delta}[n]$. The associated realization functor $\Gamma_{\overline{\Delta}}$ is a functor **sSet** $\to$ **sSet** such that $\Gamma_{\overline{\Delta}} \circ \Delta = \overline{\Delta}$. It assigns to a simplicial set K a simplicial set $SdK = \int^{[n]} K_n \cdot \overline{\Delta}[n]$, the *subdivision* of K, and to a simplicial map $f : X \to Y$ a simplicial map $Sdf : SdX \to SdY$, the subdivision of f. In particular, $Sd\Delta[n] = \overline{\Delta}[n]$ and $Sd\Delta[\alpha] = \overline{\Delta}[\alpha]$. On the other hand, the realization functor Γ_Δ associated with the Yoneda embedding Δ is naturally isomorphic to the identity functor id on **sSet**. If $d_n : \overline{\Delta}[n] \to \Delta[n]$ is the simplicial map that sends $\mu = (\mu_0, ..., \mu_p) \in \overline{\Delta}[n]_p$ to $d_n\mu \in \Delta[n]_p : d_n\mu(i) = \mu_i(m_i)$, here $\mu_i : [m_i] \to [n]$,then the d_n determine a natural transformation $d : \overline{\Delta} \to \Delta$, which by functoriality, leads to a natural transformation $d : \Gamma_{\overline{\Delta}} \to \Gamma_\Delta$. Thus, $\forall K, L$ and $\forall f : K \to L$, there is a commutative diagram

$$\begin{array}{ccc} SdK & \xrightarrow{d_K} & K \\ {\scriptstyle Sdf}\downarrow & & \downarrow{\scriptstyle f} \\ SdL & \xrightarrow[d_L]{} & L \end{array}$$

Now given n, we write $\overline{\Delta}^n$ for the geometric realization of $\overline{\Delta}[n]$ $(|\overline{\Delta}[n]|)$ and $\overline{\Delta}^\alpha$ for $|\overline{\Delta}[\alpha]|$. The elements of $\overline{\Delta}^n$ are equivalence classes $[\mu, t]$. Any two representative of μ, t are related by a finite chain of elementary equivalences involving omission of μ_i and t_i if $t_i = 0$ and replacement of t_i and t_{i+1} by $t_i + t_{i+1}$ if $\mu_{i+1} = \mu_i$. Every $[\mu, t]$ has a canonical representative, this means that $[\mu, t]$ can be represented by a pair $(\mu, t) : \mu = (\mu_0, ..., \mu_n) \in \overline{\Delta}[n]_n$ with $\mu_i : [i] \to [n]$ for $0 \le i \le n$ and $t = (t_0, ..., t_n) \in \Delta^n$. So $\mu_n = id_{[n]}$ and there exists a permutation π of $\{01, 2, ..., n\}$ such that $\forall i$, $\mu_i([i]) = \{\pi(0), ..., \pi(i)\}$.

Let M_Δ denote the set of monomorphisms in the simplicial category Δ. Given $\alpha \in M_\Delta$ say $\alpha : [m] \to [n]$, put $b(\alpha) = \frac{1}{m+1} \sum_{i=0}^{m} e_{\alpha(i)} \in \mathbb{R}^{n+1}$.

Lemma 4.2.11. *For each* $n \ge 0$, *the assignment* $[\mu, t] \to \sum_{i=0}^{p} t_i b(\mu_i)$ *is a welldefined homeomorphism* $h_n : \overline{\Delta}^n \to \Delta^n$.

Proof. (see Warner (2000)) □

Remark 4.2.12. Geometrically, $\overline{\Delta}^n$ is the barycentric subdivision of Δ^n.

Before we finish the section, we give a theorem, which is called *subdivision theorem*.

Theorem 4.2.13. *Let* K *be a simplicial set then there is a homeomorphism* $h_K : |SdK| \to |K|$.

Proof. (see Warner (2000)) □

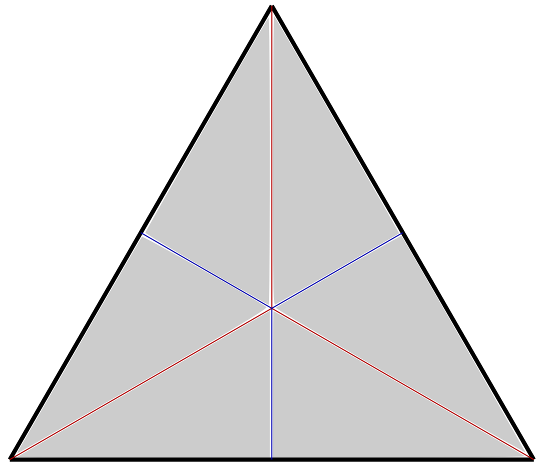


Figure 4.1 Barycentric subdivision of the 2-simplex.

4.3 Bicategories and n-categories

Definition 4.3.1. A *bicategory* $\mathscr{B}$ consists of the following data :

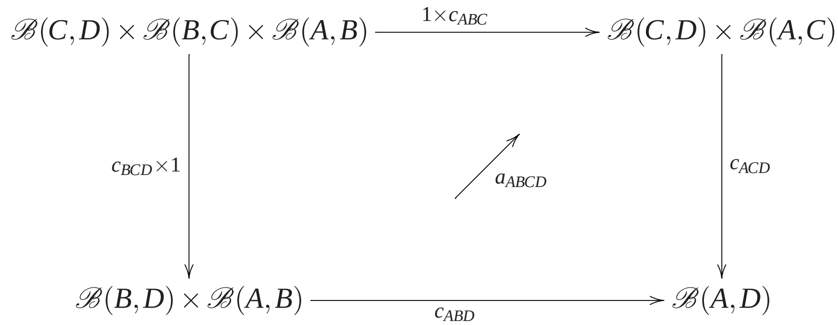
- Collection $ob(\mathscr{B})$, the elements of this collection are called the *0-cells* $(A, B, \ldots)$
- Categories $\mathscr{B}(A,B)$, whose objects are morphisms between the 0-cells and called *1-cells*. The morphisms of $\mathscr{B}(A,B)$ are called *2-cells*, such that $A \underset{g}{\overset{f}{\Downarrow\alpha}} B$ where $A, B \in ob(\mathscr{B})$, $f, g \in ob(\mathscr{B}(A,B))$ and $\alpha \in Hom_{\mathscr{B}(A,B)}$.
- Functors

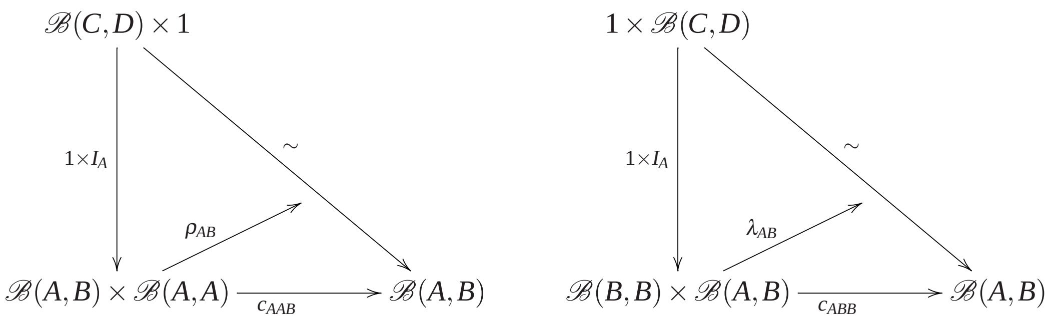
$$c_{ABC} : \mathscr{B}(B,C) \times \mathscr{B}(A,B) \longrightarrow \mathscr{B}(A,C)$$
$$(g, f) \mapsto g \circ f = gf$$
$$(\beta, \alpha) \mapsto \beta * \alpha$$

  Here "$*$" means that the horizontal composition in 2.2.2. Let $\mathbf{1}$ denote the category with one object and identity morphism, then the functor $I_A : \mathbf{1} \longrightarrow \mathscr{B}(A,A)$ is a 1-cell $A \longrightarrow A$. I_A looks like the identity map of the 0-cell A but it is not quite real identity.

- Although we write the compositions of the 1 and 2-cells in usual order, in $\mathscr{B}$ the horizontal composition is not strictly associative, but associative only "up to" a natural isomorphism between iterated composite functors and also the purported identity maps I_A are required to act as identities for the horizontal composition only up to natural isomorphisms. Now we try to explain these isomorphisms with their components. Let $A, B, C, D \in ob(\mathscr{B})$ and f, g, h are the

1-cells in $\mathscr{B}(A,B)$,$\mathscr{B}(B,C)$ and $\mathscr{B}(C,D)$ respectively, then we have the following diagrams



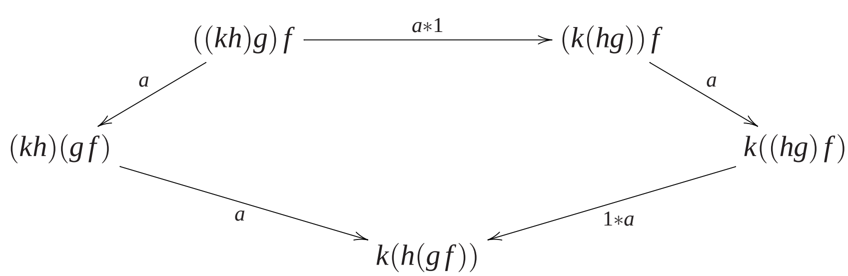


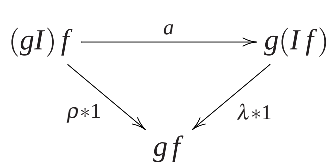
$$a_{hgf} : (hg)f \xrightarrow{\sim} h(gf)$$

$$\rho_f : f\circ I_A \xrightarrow{\sim} f$$

$$\lambda_f : I_B\circ f \xrightarrow{\sim} f$$

So there exist some axioms for this transformations such that the following diagrams must be commute.





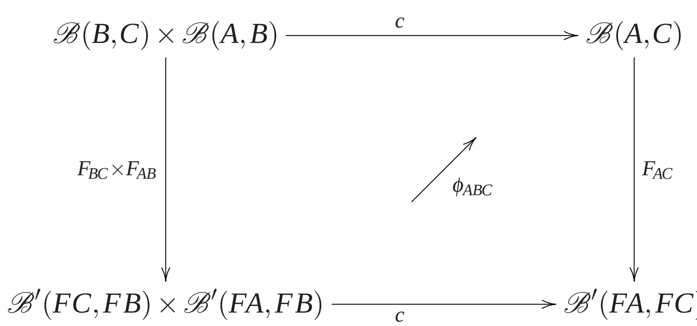
Remark 4.3.2. If we have $(hg)f = h(gf)$ and $If = f = fI$ and similarly for composition of 2-cell. Then the category $\mathscr{B}$ is as expected called 2-category. In this case the axioms hold automatically.

Example 4.3.3. Remember the functor category for given two categories $\mathscr{C}$ and $\mathscr{D}$. If we recreate the data of this category such that the objects are the given categories, 1-cells are the functors and 2-cells are the natural transformation then obviously this category is strict bicategory or as we said just 2-category. Also there is another example **Cat** in the definition of functors which consists of all 1-categories as objects and clearly 1-cells as functor, 2-cells natural transformations. In the section of functors we refer the equivalences in the category **Cat**. Also ın any bicategory $\mathscr{B}$ an equivalence (usually called internal) consists of a pair of 1-cells $f \in \mathscr{B}(A,B)$,$g \in \mathscr{B}(B,A)$ together with two isomorphism α and β in $\mathscr{B}(A,A)$ and $\mathscr{B}(B,B)$ respectively such that $\alpha : 1_A \to g \circ f$ and $\beta : f \circ g \to 1_B$ are natural.

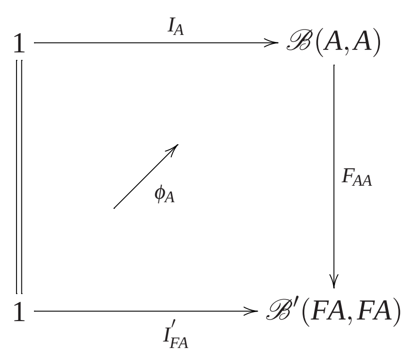
One can define the opposite bicategory of a bicategory $\mathscr{B}$ such that all 1-cells are reversed but not 2-cells. So now the question is what are the morphisms between two bicategory which we call also functors.

Definition 4.3.4. Since in any bicategory composition is associative up to a natural isomorphism, any functor F from $\mathscr{B}$ to $\mathscr{B}'$ consists of the following data:

- Function $F : ob(\mathscr{B}) \to ob(\mathscr{B}')$
- Functors $F_{AB} : \mathscr{B}(A,B) \to \mathscr{B}'(FA,FB)$
- Natural transformations



and

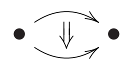


Thus 2-cells $\phi_{gf} : Fg \circ Ff \to F(g \circ f)$ and $\phi_A : I'_{FA} \to FI_A$. It is clear that the morphisms carried by F must satiesfy the axioms written in the definition of bicategories. The exists a familiar variants of this operation. If ϕ_{ABC} and ϕ_A are natural isomorphisms so that $Fg \circ Ff \cong F(g \circ f)$ and $FI \cong I'$

then F is called a *homomorphism*. If the natural transformations are identities then we call F a strict homomorphism.

The functors between bicategories gives us an idea for inductive definition of n-categories. To define 2-categories, where the natural transformations are identities, we think the hom-sets as categories. We can sure that for every 1-category there exists a two category. The definition of discrete categories supports this because of letting the 2-cells and transformations be identity. Now we are allowed to give the definition of n-categories.

Definition 4.3.5. A strict $n-$category consists of the following data:

- 0-cells as object $X, Y, Z, \ldots$
- 1-cells are the morphisms between any two 0-cells, $\bullet \longrightarrow \bullet$
- 2- cells are the morphisms between two 1-cells between two 0-cells , 
- ...
- for each step there exist composition functors similar with functor in the definition of bicategories but the axioms are more complicated.

Although this definition seems elementary, there is a remarkable relation between Homotopy theory and $n-$categories. (Baez (1997)) Let X be a topological space and PX denote the path category of this space. PX consists of the following data:

- objects: for any $x, y \in X$ the paths $\alpha : I \to X$ such that $\alpha(0) = x$ and $\alpha(1) = y$.
- morphisms: Homotopies between two paths with same starting and end points as in the section of functors.

By iterating this procedure, 2-cells are homotopies between homotopies and 3-cells are homotopies between homotopies between homotopies and so on. In homotopy theory, we are interested in the property of map which are preserved by homotopies. So if we rearrange the example such that

- objects are topological spaces $X, Y, Z\ldots$
- 1-cells are the continuous functions between topological spaces.
- 2- cells are homotopies between two functions $f, g : X \to Y$ such that $H : X \times I \to Y$ satisfies

$$H(-,0) = f \text{ and } H(-,1) = g.$$

Now we have a 2-category and we are able to start the procedure with topological spaces. By iteration, we can see that the category **Top** has a $n-$category structure. In fact, we are able to think that for any large n this procedure works, so **Top** has a $w-$category structure. Here the most important fact that we use the unit interval $[0,1]$. If we think the unit interval $[0,1]$ as an arrow from 0 to 1, then this abstract arrow can be seen such an arrow in the graph of a category. Moreover, an attractive property of this arrow is that it can be reversed easily. By this advantage; one thinks that the homotopy theory is a part of the word of $n-$categories. Namely for every topological space there exists a $w-$category $\Pi(X)$. The objects of this category are the points x of X and 1-cells are the paths, 2-cells are paths of paths and so on. But according the property of the unit interval all the $j-$morphisms are equivalences in this category. Hence we call this category as $w-$groupoid. Also one thinks the converse of this idea such that for every $w-$groupoid we are able to obtain a topological space $N(G)$. Maybe this topological space $N(G)$ can be pictured geometrically as intervals, squares,triangle, cubes and so on. In the section of functors we saw the property of the fundamental group Π_1. In general Π can be thought as a weak $w-$fuctor from **Top** to $w-$**Gpd** and by the way we should able to show that in **Cat** the categories **Top** and $w-$**Gpd** are equivalent. In short, working in homotopy theory is the same as working about $w-$groupoids.

After all, we understand that $n-$category structure occurs by using the effects of functors and natural transformations on a given $(n-1)$categories. This means that we use the induction method. But this procedure is commented in different ways. For example, Peter May have used operads to define $n-$categories and Baez and Dolan have used opetopes which are invented by them in 1997. Although it is hard to understand these strategies, one of the main ideas is to paste diagrams according to the route of the maps. This means that maps which are glued must have same source and target. Here we try to explain Tamsamani $n-$categories.

Definition 4.3.6. (Dupont (1978)) It is convenient to interprete each $[n] \in ob(\Delta)$ as a category by defining that the morphisms are the inequalities "$\leq$" .Then Δ becomes a full subcategory of **Cat** formed by the categories $[0],[1],[2],....$ The *nerve* of a (small) category $\mathscr{C}$ is, by definition the simplicial sets :

$$NC : (\Delta)^{op} \to \textbf{Set}$$
$$[n] \to Hom_{\textbf{Cat}}([n],\mathscr{C}) \ .$$

There are natural identifications:

$$NC_0 = ob(\mathscr{C})$$

$$NC_1 = \coprod_{X,Y \in NC_0} Hom_{\mathscr{C}}(X,Y)$$

and more generally for composable maps $f_i : X_{i+1} \to X_i$, where $i = 0,1,...,k-1$ for $k \geq 1$, NC_k is the set of the strings such that

$$X_0 \xleftarrow{f_0} X_1 \xleftarrow{f_1} X_2 \xleftarrow{f_2} \ldots\ldots \xleftarrow{f_{k-2}} X_{k-1} \xleftarrow{f_{k-1}} X_k$$

Furthermore ; let $(f_0, f_1, ..., f_{k-1})$ denote the string defined above. One can define the morphisms ε and η such that $\varepsilon = NC(\delta)$ and $\eta = NC(\sigma)$ where $\varepsilon_i : NC([n]) \to NC([n-1])$ and $\eta : NC([n]) \to NC([n+1])$ are given by

$$\varepsilon_i(f_0, f_1, ..., f_{k-1}) = \begin{cases} (f_1, f_2, ..., f_{k-1}), & \text{i = 0 ;} \\ (f_0, f_1, ..., f_i \circ f_{i+1}, ..., f_{k-1}), & \text{0 < i < k-1 ;} \\ (f_0, f_1, ..., f_{k-2}), & \text{i = k-1} \end{cases}$$

and

$$\eta_i(f_0, f_1, ..., f_{k-1}) = (f_0, f_1, ... f_i, id, f_{i+1}, ..., f_{k-1}) \text{ for } i = 0,1,...,k-1 \text{ .}$$

Functors between categories turn into simplicial maps between their nerves, and the whole construction defines a functor from categories to simplicial sets. This functor is fully faithful. The simplicial sets that arise as nerves of categories are characterised by strict Segal condition: the natural maps

$$NC_{p+q} \to NC_p \times_{NC_0} NC_q \ \forall p,q$$

are isomorphisms. This means that the target of the last arrow is the source of the first arrow.

Definition 4.3.7. (Kock (2006)) Define a Tamsamani 0-category to be a set. A weak $n-$category in the sense of Tamsamani is defined inductively as a functor $NC : \Delta^{op} \to (n-1)$**wCat** such that NC_0 is discrete and satisfying the (nonstrict) Segal condition, namely, that the morphisms $NC_{p+q} \to NC_p \times_{NC_0} NC_q$ should be equimorphisms in (n-1)**wCat**, the category of weak (n - 1)-categories. Equimorphism means fully faithful and essentially surjective, notions which are also defined inductively. Note that an equimorphism is not in general invertible, so there is no longer any well-defined composition like $NC_1 \leftarrow NC_1 \times_{NC_0} NC_1$. This map is now defined only up to homotopy. It exists only in as much as we regard the equimorphism

$$NC_2 \longrightarrow NC_1 \times_{NC_0} NC_1$$

as invertible. The new structure is rather this:

$$\begin{array}{ccc} NC_1 & & NC_1 \times_{NC_0} NC_1 \\ & \nwarrow & \uparrow \cong \\ & & NC_2 \end{array}$$

Before we end this section, we want to attract attention to the relation between the definitions of bicategories and Tamsamani weak $n-$categories. As we see in 4.3.7, Tamsamani weak 2-categories are similar with the definition of bicategories in 4.3.1, (Leinster (2002)). Now we try to explain this similarity.

First take a weak 2-category $NC : \Delta^{op} \to \mathbf{Cat}$ and let us construct a bicategory $\mathscr{B}$. The object set of $\mathscr{B}$ is the set NC_0. Let us define two functors $s, t : NC_1 \to NC_0$ as source and target respectively. These functors express the category NC_1 as a disjoint union $\coprod_{A,B \in NC_0} \mathscr{B}(A, B)$ of categories; the 1-cells from A to B are objects of $\mathscr{B}(A, B)$ and the 2-cells are the morphisms.

Vertical composition of 2-cells in $\mathscr{B}$ is composition in each $\mathscr{B}(A, B)$. To define horizontal composition of 1- and 2-cells, first choose for each k a pseudo-inverse

$$NC_1 \times_{NC_0} \ldots \times_{NC_0} NC_1 \xrightarrow{\psi_k} NC_k$$

to the Segal map ϕ_k and choose natural isomorphisms $\eta_k : \mathbf{1} \to \psi_k \circ \phi_k$ and $\varepsilon_k : \phi_k \circ \psi_k \to \mathbf{1}$. Horizontal composition is then given as

$$NC_1 \times_{NC_0} NC_1 \xrightarrow{\psi_2} NC_2 \xrightarrow{NC(\delta)} NC_1$$

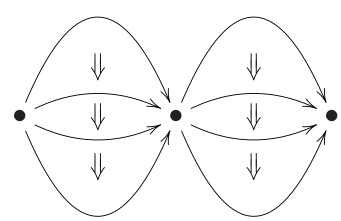
where $\delta : [1] \to [2]$ is the injection whose image omits $1 \in [2]$. The associativity isomorphisms are built up from η_k's and ε_k's and the pentagon in 4.3.1 commutes just as long as the equivalence $(\phi_k, \psi_k, \eta_k, \varepsilon_k)$ was chosen to be an adjunction too. Identities work similarly. So we construct a bicategory.

Conversely, let us take a bicategory $\mathscr{B}$ and construct a weak 2-category $NC : (\Delta^2)^{op} \to \mathbf{Set}$ (its "2-nerve") as follows. An element of $NC_{j,k}$ is a quadruple

$$((a_u)_{0 \le u \le j}, (f^z_{uv})_{0 \le u < v \le j, 0 \le z \le k}, (\alpha^z_{uv})_{0 \le u < v \le j, 1 \le z \le k}, (\iota^z_{uvw})_{0 \le u < v < w \le j, 0 \le z \le k})$$

where

- a_u is an object of $\mathscr{B}$.
- $f^z_{uv} : a_u \to a_v$ is a 1-cell of $\mathscr{B}$.
- $\alpha^z_{uv} : f^{z-1}_{uv} \to f^z_{uv}$ is a 2-cell of $\mathscr{B}$
- $\iota^z_{uvw} : f^z_{vw} \circ f^z_{uv} \to f^z_{uw}$ is an invertible 2-cell of $\mathscr{B}$ such that $\iota^z_{uvw} \circ (\alpha^z_{vw} * \alpha^z_{uv}) = \alpha^z_{uw} \circ \iota^{z-1}_{uvw}$ whenever $0 \le u < v < w \le j$, $1 \le z \le k$ and $\iota^z_{uwx} \circ (\mathbf{1}_{f^z_{wx}} * \iota^z_{uvw})\circ$(associativity isomorphism in $\mathscr{B}$)=$\iota^z_{uvx} \circ (\iota^z_{uwx} * \mathbf{1}_{f^z_{uv}})$ whenever $0 \le u < v < w < x \le j$, $0 \le z \le k$.



$j = 2, k = 3$

This defines the functor NC on objects Δ^2; it is defined on maps by a combination of inserting identities and forgetting data. To get a rough picture of NC let us consider the analogous construction for strict 2-categories, in which we insists that the isomorphisms ι^z_{uvw} are actually equalities. Then an element of $NC_{j,k}$ is a grid of jk 2-cell of width j and eight k. So passing from a bicategory to a weak 2-category and back again gives a bicategory which is isomorphic to the original one and passing from a weak 2-category to a bicategory and back again gives a weak 2-category equivalent to the original one.

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