DOKUZ EYLÜL UNIVERSITY

GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

THE DUAL EULER PARAMETERS

by

Ayşın ERKAN

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İZMİR

THE DUAL EULER PARAMETERS

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M.Sc. THESIS EXAMINATION RESULT FORM

We have read the thesis entitled **"THE DUAL EULER PARAMETERS"** completed by **AYŞIN ERKAN** under supervision of **ASSIST. PROF. DR. İLHAN KARAKILIÇ** and we certify that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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Ayşın ERKAN

THE DUAL EULER PARAMETERS

ABSTRACT

E.Study mapping states the one to one correspondence between lines of the real three space and the points of the Dual Unit Sphere. In this study, using the Study mapping we obtain the relation between the Euler parameters of the Dual Unit Sphere and the screws of the corresponding motion in the real three space. In the last chapter, the exponential mapping is used to obtain the relation between the dual orthogonal matrices and the dual skew-symmetric matrices for the rotations of the Dual Unit Sphere.

Keywords: Study mapping, Dual Unit Sphere, Euler parameters, dual orthogonal matrix, skew-symmetric matrix, exponential mapping.

DUAL EULER PARAMETRELERİ

ÖZ

Study dönüşümü, reel üç boyutlu doğrular uzayıyla dual birim kürenin noktaları arasında birebir eşleme belirtir. Bu çalışmada, Study dönüşümünü kullanarak, dual kürenin Euler parametreleriyle üç boyutlu reel uzaydaki vida hareketi arasındaki ilişkiyi elde ettik. Son bölümde ise dual birim kürenin dönmelerine ait dual ortogonal matrislerle dual anti-simetrik matrisler arasındaki ilişkiyi üstel dönüşümü kullanarak elde ettik.

Anahtar Sözcükler: Study dönüşümü, dual birim küre, Euler parametreleri, dual ortogonal matris, anti-simetrik matris, üstel dönüşüm.

CONTENTS

Page

CHAPTER ONE
INTRODUCTION

In Chapter One, we discuss the basic properties of dual numbers and dual quantities (dual vectors, dual functions, dual matrices, etc.) using the fundamental definitions of algebra.

The representation of a line is simply done by the normalized Plücker vector. This vector is a point on a unit sphere in $\mathbb{D}^3$. There exists a one to one correspondence between the points on the dual unit sphere and oriented straight lines in $\mathbb{R}^3$ which is given by E. Study. This discussion is given in Chapter Two. Furthermore we study dual angle between dual vectors on the dual unit sphere which express the spatial relationship between skew lines in space.

In Chapter Three, using the E. Study mapping we study dual rotations on the dual unit sphere instead of the transformations in real space. Then using Cayley mapping we obtain dual Rodrigues parameters and the dual Euler parameters. We rewrite the dual Euler parameters by the components of dual Rodrigues vector $\hat{\boldsymbol{b}}$ and the rotation angle $\hat{\phi}$. The dual quaternion $\hat{Z}$ with the dual Euler parameters and the screw $\hat{\boldsymbol{w}}$ for the motion in space are defined. Using the dual quaternion $\hat{Z}$ and the screw $\hat{\boldsymbol{w}}$ we find the transformed screw $\boldsymbol{w}'$. In addition, we compute the coordinates of transformed screw depending on the components of dual Rodrigues vector and the dual angle of the corresponding dual spherical motion.

Exponential mapping is an alternative method of Cayley mapping for finding the relation between the rotation matrices and the skew symmetric matrices. This is the main idea of Chapter Four.

1.1 Dual Numbers

Dual numbers were originally conceived by an English mathematician W.K. Clifford more than a century ago (Clifford (1873)). In late 1940's and early 1950's, these numbers began to be used in the area of screw calculus by a few scientist. Though 1960's and 1970's, dual numbers were extensively applied in the analysis of spatial mechanisms by several investigators. In 1980's, as researches in robotics area have progressed rapidly, these numbers are brought into attentions of some robotics researchers and have been used in the formulation of homogeneous transformation matrices and kinematic equations (Karger & Novak (1985)).

Their first applications to kinematics being attributed to both Kotel'nikov (1895) and Study (1903). A comprehensive analysis of dual numbers and their applications to the kinematic analysis of spatial linkages was conducted by Yang (1963) and Yang & Freudenstein (1964). Veldkamp (1976) and Bottema & Roth (1978) include treatment of theoretical kinematics using dual numbers.

Dual numbers have the form $\hat{a} = a + \varepsilon a^*$ where $\varepsilon^2 = 0$. In this chapter we will discuss the properties of dual numbers and dual quantities and define the dual number algebra. All formal operations involving dual numbers are identical to those of ordinary algebra, while taking into account that $\varepsilon^2 = \varepsilon^3 = ... = 0$. Dual numbers are performed using the laws of conventional algebra in a way similar to complex numbers. On the other hand there is a fundamental difference from complex numbers. As purely dual numbers do not have an inverse, every non-zero complex number has an inverse.

The algebra of dual vectors is analogical with that of the 3-dimensional usual vectors but with components existing of dual numbers. One of the most important properties dual vectors have is that all of the vector identities of real 3×1 vectors carry over to dual vectors. This property is called the principle of transference (Dimentberg (1965), Bottema & Roth (1978), Martinez & Duffy. (1994)). From the principle of transference, dual vectors satisfy all the identities of real vectors.

Functions of dual numbers can be expanded into functions of real numbers by Taylor's series expansion with $\varepsilon^2 = \varepsilon^3 = ... = 0$.

Definition 1.1.1. A dual number A can be defined as an ordered pair

$$A = (a, a^*) \tag{1.1.1}$$

of real numbers a and a^*, with operations of addition and multiplication defined as follows.

Dual numbers of the form $(0, a^*)$ are called pure dual numbers. The real numbers a and a^* in expression (1.1.1) are called the real part and the dual part of A, respectively. We can write simply,

$$ReA = a, \quad DuA = a^*.$$

Let us define the set of all dual numbers by

$$\mathbb{D} = \{(a,a^*) : a,a^* \in \mathbb{R}\}.$$

Two dual numbers (a,a^*) and (b,b^*) are equal whenever they have the same real part and the same dual part. That is,

$$(a,a^*) = (b,b^*) \quad iff \quad a = b \quad and \quad a^* = b^* \tag{1.1.2}$$

The addition operation, $\oplus$, is defined for the dual numbers $A = (a,a^*)$ and $B = (b,b^*)$ as follows;

$$(a,a^*) \oplus (b,b^*) = (a+b,\ a^* + b^*) \tag{1.1.3}$$

and the multiplication operation, $\otimes$, is defined by the equation

$$(a,a^*) \otimes (b,b^*) = (ab,\ ab^* + a^*b) \tag{1.1.4}$$

In particular $(a,0) \oplus (0,a^*)$ and $(0,1) \otimes (a^*,0) = (0,a^*)$. Hence

$$(a,a^*) = (a,0) \oplus (0,1) \otimes (a^*,0) \tag{1.1.5}$$

Any ordered pair $(a,0)$ is to be identified as the real number a, and so the set of dual numbers includes real numbers as a subset. Moreover, the operations defined by equations (1.1.3) and (1.1.4) become the usual operations of addition and multiplication when restricted to the real numbers :

$$(a,0) \oplus (b,0) = (a+b,\ 0+0) = (a+b,0)$$

$$(a,0) \otimes (b,0) = (ab,\ a.0+0.b) = (ab,0).$$

The dual number system is thus a natural extension of the real number system.
Thinking of a real number as either a or $(a,0)$ and letting ε denote the pure dual number $(0,1)$,

we can rewrite equation (1.1.5) as

$$(a,a^*) = (a,0) \oplus \varepsilon \otimes (a^*,0).$$

That is,

$$(a,a^*) = a + \varepsilon a^*. \tag{1.1.6}$$

Also we can note that,

$$\varepsilon^2 = (0,1) \otimes (0,1) = (0.0,\ 0.1 + 1.0) = (0,0).$$

That is, $\varepsilon^2 = 0$ and it is clear that $\varepsilon^2 = \varepsilon^3 = ... = \varepsilon^n = 0$.
In view of identity (1.1.6), equations (1.1.3) and (1.1.4) become

$$(a+\varepsilon a^*) + (b+\varepsilon b^*) = (a+b) + \varepsilon(a^* + b^*), \tag{1.1.7}$$

$$\begin{aligned}(a+\varepsilon a^*)(b+\varepsilon b^*) &= (ab+\varepsilon^2 a^* b^*) + \varepsilon(ab^* + a^* b) \\ &= ab + \varepsilon(ab^* + a^* b),\end{aligned}$$

and also the reciprocal of a dual number $(a+\varepsilon a^*)$ is

$$\frac{1}{a+\varepsilon a^*} = \frac{1}{a+\varepsilon a^*}\frac{a-\varepsilon a^*}{a-\varepsilon a^*} = \frac{a-\varepsilon a^*}{a^2}$$

where $a \neq 0$, i.e., $a+\varepsilon a^*$ is not a pure dual number.

Observe that the right-hand sides of these equations can be obtained by formally manipulating the terms on the left as if they involved only real numbers and by replacing ε^2 by 0 when it occurs.

1.2 Algebraic Properties

Theorem 1.2.1. *The set of dual numbers with respect to addition,* $(\mathbb{D},\oplus)$ *is an abelian group.*

Proof. 1) It is clear that addition is closed on $\mathbb{D}$. For all $A,B \in \mathbb{D}$ we have $A \oplus B \in \mathbb{D}$. 2) For all $A = (a,a^*)$, $B = (b,b^*)$, $C = (c,c^*) \in \mathbb{D}$ addition is associative,

$$\begin{aligned}(A \oplus B) \oplus C &= \Big((a,a^*) \oplus (b,b^*)\Big) \oplus (c,c^*) \\ &= (a+b, a^*+b^*) \oplus (c,c^*) \\ &= \Big((a+b)+c, (a^*+b^*)+c^*) \\ &= \Big(a+(b+c), a^*+(b^*+c^*)) \\ &= (a,a^*) \oplus (b+c, b^*+c^*) \\ &= A \oplus (B \oplus C).\end{aligned}$$

3) $0 = (0,0) \in \mathbb{D}$ is the additive identity in $\mathbb{D}$. $\forall (a,a^*) \in \mathbb{D}$ we have the requirement

$$(a,a^*) \oplus (0,0) = (a+0, a^*+0) = (a,a^*).$$

4) For each $(a,a^*) \in \mathbb{D}$, $(-a,-a^*)$ is the additive inverse. That is,

$$(a,a^*) \oplus (-a,-a^*) = \Big(a+(-a), a^*+(-a^*)\Big) = (0,0).$$

If $A = (a,a^*) \in \mathbb{D}$ then we denote $(-a,-a^*) \in \mathbb{D}$ by $-A$. Thus, $(\mathbb{D},\oplus)$ is a group.
Furthermore,
5) For all A,B we have $A \oplus B = B \oplus A$. That is,

$$(a,a^*) \oplus (b,b^*) = (a+b,\ a^*+b^*) = (b+a,\ b^*+a^*) = (b,b^*) \oplus (a,a^*).$$

Therefore, we can say that $(\mathbb{D},\oplus)$ is an abelian group. □

Theorem 1.2.2. *The set of dual numbers with respect to addition and multiplication,* $(\mathbb{D},\oplus,\otimes)$*, is a commutative ring with identity.*

Proof. We can follow two steps;
i) $(\mathbb{D},\oplus)$ is an abelian group.

ii) Multiplication is associative and it has distributive property over addition and $(1,0)$ is the multiplicative identity.

R1) $(\mathbb{D},\oplus)$ is an abelian group. R2) It is clear that multiplication is closed on $\mathbb{D}$. For all $A,B\in\mathbb{D}$, we have

$$A\otimes B\in\mathbb{D}.$$

R3) Multiplication is associative. That is, for all $A,B,C\in\mathbb{D}$

$$\begin{aligned}(A\otimes B)\otimes C &= \Big((a,a^*)\otimes(b,b^*)\Big)\otimes(c,c^*)\\ &= (ab,\ ab^*+a^*b)\otimes(c,c^*)\\ &= (abc,\ abc^*+ab^*c+a^*bc)\\ &= (a,a^*)\otimes(bc,\ bc^*+b^*c)\\ &= A\otimes(B\otimes C).\end{aligned}$$

R4) Multiplication is distributive over addition. That is,

$$\begin{aligned}(A\oplus B)\otimes C &= \Big((a,a^*)\oplus(b,b^*)\Big)\otimes(c,c^*)\\ &= (a+b,\ a^*+b^*)\otimes(c,c^*)\\ &= \Big((a+b)c,\ (a^*+b^*)c+(a+b)c^*\Big)\\ &= (ac+bc,\ a^*c+ac^*+b^*c+bc^*)\\ &= (ac,\ a^*c+ac^*)\oplus(bc,\ b^*c+bc^*)\\ &= A\otimes C\oplus B\otimes C\ \ for\ all\ A,B,C\in\mathbb{D}.\end{aligned}$$

Hence the right distributive property holds. Similarly $A\otimes(B\oplus C)=A\otimes B\oplus A\otimes C$ for all $A,B,C\in\mathbb{D}$, the left distributive property holds.

Thus $(\mathbb{D},\oplus,\otimes)$ is a ring. Moreover;

R5) For all $A,B\in\mathbb{D}$, we have

$$A\otimes B=(a,a^*)\otimes(b,b^*)=(ab,\ ab^*+a^*b)=(ba,\ ba^*+b^*a)=B\otimes A.$$

Multiplication is commutative. Also, R6) $(1,0) \in \mathbb{D}$ is the identity element with respect to multiplication;

$$(a,a^*) \otimes (1,0) = (1,0) \otimes (a,a^*) = (a,a^*), \ \forall A = (a,a^*) \in \mathbb{D}.$$

Thus, $(\mathbb{D},\oplus,\otimes)$ is a commutative ring with identity. □

1.3 Dual Vectors

Definition 1.3.1. If $\boldsymbol{v},\boldsymbol{v}^* \in \mathbb{R}^3$ then we can define a dual vector $\hat{\boldsymbol{v}}$ in three dimensional dual space $\mathbb{D}^3$, by $\hat{\boldsymbol{v}} = \boldsymbol{v} + \varepsilon \boldsymbol{v}^*$ (Here in after hat over a quantity, such as number, angle, vector, etc., will denote the dual version of that quantity).
The set $\mathbb{D}^3$ is defined by $\mathbb{D}^3 = \{\boldsymbol{a} + \varepsilon \boldsymbol{a}^* : \ \boldsymbol{a},\boldsymbol{a}^* \in \mathbb{R}^3, \ \varepsilon^2 = 0\}$.

The standard algebraic properties for vectors in $\mathbb{R}^3$ can also be defined in $\mathbb{D}^3$. Given $\hat{\boldsymbol{v}},\hat{\boldsymbol{w}} \in \mathbb{D}^3$ and $\hat{d} \in \mathbb{D}$, where $\hat{\boldsymbol{v}} = \boldsymbol{v} + \varepsilon \boldsymbol{v}^*$, $\hat{\boldsymbol{w}} = \boldsymbol{w} + \varepsilon \boldsymbol{w}^*$ and $\hat{\lambda} = \lambda + \varepsilon \lambda^*$ with $\boldsymbol{v},\boldsymbol{v}^*,\boldsymbol{w},\boldsymbol{w}^* \in \mathbb{R}^3$, $\lambda,\lambda^* \in \mathbb{R}$

(i) Equality

$$\hat{\boldsymbol{v}} = \hat{\boldsymbol{w}} \ iff \ \boldsymbol{v} = \boldsymbol{w} \ and \ \boldsymbol{v}^* = \boldsymbol{w}^*$$

(ii) Addition of Dual Vectors

$$\hat{\boldsymbol{v}} + \hat{\boldsymbol{w}} = (\boldsymbol{v} + \varepsilon \boldsymbol{v}^*) + (\boldsymbol{w} + \varepsilon \boldsymbol{w}^*) = (\boldsymbol{v} + \boldsymbol{w}) + \varepsilon(\boldsymbol{v}^* + \varepsilon \boldsymbol{w}^*)$$

(iii) Multiplication of a Dual Vector by a Dual Number

$$\begin{aligned} \hat{\lambda}.\hat{\boldsymbol{v}} &= (\lambda + \varepsilon\lambda^*)(\boldsymbol{v} + \varepsilon \boldsymbol{v}^*) \\ &= \lambda \boldsymbol{v} + \varepsilon \lambda \boldsymbol{v}^* + \varepsilon \lambda^* \boldsymbol{v} + \varepsilon^2 \lambda^* \boldsymbol{v}^* \\ &= \lambda \boldsymbol{v} + \varepsilon(\lambda \boldsymbol{v}^* + \lambda^* \boldsymbol{v}) \end{aligned}$$

(iv) Inner Product of Dual Vectors

$$\begin{aligned}\hat{\boldsymbol{v}}.\hat{\boldsymbol{w}} &= (\boldsymbol{v}+\varepsilon\boldsymbol{v}^*)(\boldsymbol{w}+\varepsilon\boldsymbol{w}^*)\\ &= \boldsymbol{v}\boldsymbol{w}+\varepsilon\boldsymbol{v}\boldsymbol{w}^*+\varepsilon\boldsymbol{v}^*\boldsymbol{w}+\varepsilon^2\boldsymbol{v}^*\boldsymbol{w}^*\\ &= \boldsymbol{v}\boldsymbol{w}+\varepsilon(\boldsymbol{v}\boldsymbol{w}^*+\boldsymbol{v}^*\boldsymbol{w})\\ &= \boldsymbol{w}\boldsymbol{v}+\varepsilon(\boldsymbol{w}^*\boldsymbol{v}+\boldsymbol{w}\boldsymbol{v}^*)\\ &= \hat{\boldsymbol{w}}.\hat{\boldsymbol{v}} \quad (\textit{inner product is commutative})\end{aligned}$$

(v) Cross Product of Dual Vectors

$$\begin{aligned}\hat{\boldsymbol{v}}\times\hat{\boldsymbol{w}} &= (\boldsymbol{v}+\varepsilon\boldsymbol{v}^*)\times(\boldsymbol{w}+\varepsilon\boldsymbol{w}^*)\\ &= \boldsymbol{v}\times\boldsymbol{w}+(\boldsymbol{v}\times\boldsymbol{w}^*+\varepsilon\boldsymbol{v}^*\times\boldsymbol{w})\\ &\neq \hat{\boldsymbol{w}}\times\hat{\boldsymbol{v}} \quad (\textit{cross product is not commutative})\end{aligned}$$

Since, for some nonzero $\hat{u},\hat{v}\in\mathbb{D}$ we have $\hat{u}\hat{v}=0$ (e.g., $2\varepsilon.3\varepsilon=6\varepsilon^2=0$). $\mathbb{D}$ is not a field ($\hat{u}$ and $\hat{v}$ are zero divisors).

The set $\mathbb{D}^3$ satisfies all the axioms of vectors spaces, but its domain $\mathbb{D}$ is only a ring and not a field this is why $\mathbb{D}^3$ is a $\mathbb{D}$-module. However the elements of $\mathbb{D}^3$ are also called dual vectors.

Theorem 1.3.2. $(\mathbb{D}^3,\oplus)$ *is an abelian group.*

Proof. 1) It is clear that addition is closed on $\mathbb{D}^3$. For all $\hat{\boldsymbol{a}}$, $\hat{\boldsymbol{b}}\in\mathbb{D}^3$ we have

$$\begin{aligned}\hat{\boldsymbol{a}}+\hat{\boldsymbol{b}} &= (a_1+\varepsilon a_1^*,\ a_2+\varepsilon a_2^*,\ a_3+\varepsilon a_3^*)+(b_1+\varepsilon b_1^*,\ b_2+\varepsilon b_2^*,\ b_3+\varepsilon b_3^*)\\ &= \Big(a_1+b_1+\varepsilon(a_1^*+b_1^*),\ a_2+b_2+\varepsilon(a_2^*+b_2^*),\ a_3+b_3+\varepsilon(a_3^*+b_3^*)\Big)\in\mathbb{D}^3\end{aligned}$$

2) For all $\hat{\boldsymbol{a}}=(a_1+\varepsilon a_1^*,\ a_2+\varepsilon a_2^*,\ a_3+\varepsilon a_3^*)$, $\hat{\boldsymbol{b}}=(b_1+\varepsilon b_1^*,\ b_2+\varepsilon b_2^*,\ b_3+\varepsilon b_3^*)$, $\hat{\boldsymbol{c}}=(c_1+\varepsilon c_1^*,\ c_2+\varepsilon c_2^*,\ c_3+\varepsilon c_3^*)$, addition is associative;

$$\begin{aligned}(\hat{\boldsymbol{a}}\oplus\hat{\boldsymbol{b}})\oplus\hat{\boldsymbol{c}} &= \Big(a_1+b_1+\varepsilon(a_1^*+b_1^*),a_2+b_2+\varepsilon(a_2^*+b_2^*),a_3+b_3+\varepsilon(a_3^*+b_3^*)\Big)\\ &\quad\oplus(c_1+\varepsilon c_1^*,c_2+\varepsilon c_2^*,c_3+\varepsilon c_3^*)\end{aligned}$$

$$\begin{aligned}
&= \Big((a_1+b_1)+c_1+\varepsilon((a_1^*+b_1^*)+c_1^*),(a_2+b_2)+c_2+\varepsilon((a_2^*+b_2^*)+c_2^*),\\
&\quad (a_3+b_3)+c_3+\varepsilon((a_3^*+b_3^*)+c_3^*)\Big)\\
&= \Big(a_1+(b_1+c_1)+\varepsilon(a_1^*+(b_1^*+c_1^*)),a_2+(b_2+c_2)+\varepsilon(a_2^*+(b_2^*+c_2^*)),\\
&\quad a_3+(b_3+c_3)+\varepsilon(a_3^*+(b_3^*+c_3^*))\Big)\\
&= (a_1+\varepsilon a_1^*,a_2+\varepsilon a_2^*,a_3+\varepsilon a_3^*)\oplus\Big((b_1+c_1)+\varepsilon(b_1^*+c_1^*),(b_2+c_2)\\
&\quad +\varepsilon(b_2^*+c_2^*),(b_3+c_3)+\varepsilon(b_3^*+c_3^*)\Big)\\
&= \hat{\boldsymbol{a}}\oplus(\hat{\boldsymbol{b}}\oplus\hat{\boldsymbol{c}}).
\end{aligned}$$

3) $0=(0+\varepsilon.0,\ 0+\varepsilon.0,\ 0+\varepsilon.0)\in\mathbb{D}^3$ is the additive identity in $\mathbb{D}^3$.

4) For each $\hat{\boldsymbol{a}}\in\mathbb{D}^3$, $-\hat{\boldsymbol{a}}$ is the additive inverse. That is,

$$\begin{aligned}
\hat{\boldsymbol{a}}+(-\hat{\boldsymbol{a}}) &= (a_1+\varepsilon a_1^*,\ a_2+\varepsilon a_2^*,\ a_3+\varepsilon a_3^*)+(-a_1-\varepsilon a_1^*,\ -a_2-\varepsilon a_2^*,\ -a_3-\varepsilon a_3^*)\\
&= \Big(a_1-a_1+\varepsilon(a_1-a_1^*),\ a_2-a_2+\varepsilon(a_2-a_2^*),\ a_3-a_3+\varepsilon(a_3-a_3^*)\Big)\\
&= (0,0,0).
\end{aligned}$$

5) For all $\hat{\boldsymbol{a}}$,$\hat{\boldsymbol{b}}$ we have $\hat{\boldsymbol{a}}\oplus\hat{\boldsymbol{b}}=\hat{\boldsymbol{b}}\oplus\hat{\boldsymbol{a}}$. In other words,

$$\begin{aligned}
\hat{\boldsymbol{a}}\oplus\hat{\boldsymbol{b}} &= (a_1+\varepsilon a_1^*,\ a_2+\varepsilon a_2^*,\ a_3+\varepsilon a_3^*)\oplus(b_1+\varepsilon b_1^*,\ b_2+\varepsilon b_2^*,b_3+\varepsilon b_3^*)\\
&= \Big(a_1+b_1+\varepsilon(a_1^*+b_1^*),\ a_2+b_2+\varepsilon(a_2^*+b_2^*),\ a_3+b_3+\varepsilon(a_3^*+b_3^*)\Big)\\
&= \Big(b_1+a_1+\varepsilon(b_1^*+a_1^*),\ b_2+a_2+\varepsilon(b_2^*+a_2^*),\ b_3+a_3+\varepsilon(b_3^*+a_3^*)\Big)\\
&= (b_1+\varepsilon b_1^*+a_1+\varepsilon a_1^*,\ b_2+\varepsilon b_2^*+a_2+\varepsilon a_2^*,\ b_3+\varepsilon b_3^*+a_3+\varepsilon a_3^*)\\
&= (b_1+\varepsilon b_1^*,\ b_2+\varepsilon b_2^*,\ b_3+\varepsilon b_3^*)\oplus(a_1+\varepsilon a_1^*,\ a_2+\varepsilon a_2^*,\ a_3+\varepsilon a_3^*)\\
&= \hat{\boldsymbol{b}}\oplus\hat{\boldsymbol{a}}.
\end{aligned}$$

Therefore, we can say that $(\mathbb{D}^3,\oplus)$ is an abelian group. □

Definition 1.3.3. Since $\mathbb{D}$ is a ring the additive abelian group $\mathbb{D}$ is a (left) $\mathbb{D}$ - module together with a function $\mathbb{D}\times\mathbb{D}\to\mathbb{D}$ such that for all $\hat{d}$, $\hat{e}\in\mathbb{D}$ and $\hat{a}$, $\hat{b}\in\mathbb{D}$.

(i) $\hat{d}(\hat{a}+\hat{b})=\hat{d}\hat{a}+\hat{d}\hat{b}$

(ii) $(\hat{d}+\hat{e})\hat{a}=\hat{d}\hat{a}+\hat{e}\hat{a}$

(iii) $\hat{d}(\hat{e}\hat{a}) = (\hat{d}\hat{e})\hat{a}$

If $\mathbb{D}$ has an identity element $1_{\mathbb{D}}$ and

(iv) $1_{\mathbb{D}}\,\hat{a} = \hat{a}$ for all $\hat{a} \in \mathbb{D}$

then $\mathbb{D}$ is said to be a unitary $\mathbb{D}$ - module

Theorem 1.3.4. *Since $\mathbb{D}$ is a ring the additive abelian group $\mathbb{D}^3$ is a (left) $\mathbb{D}$ - module together with a function $\mathbb{D} \times \mathbb{D}^3 \to \mathbb{D}^3$ such that for all $\hat{d},\, \hat{e} \in \mathbb{D}$ and $\hat{\boldsymbol{a}},\, \hat{\boldsymbol{b}} \in \mathbb{D}^3$.*

(i) $\hat{d}(\hat{\boldsymbol{a}} + \hat{\boldsymbol{b}}) = \hat{d}\hat{\boldsymbol{a}} + \hat{d}\hat{\boldsymbol{b}}$

(ii) $(\hat{d} + \hat{e})\hat{\boldsymbol{a}} = \hat{d}\hat{\boldsymbol{a}} + \hat{e}\hat{\boldsymbol{a}}$

(iii) $\hat{d}(\hat{e}\hat{\boldsymbol{a}}) = (\hat{d}\hat{e})\hat{\boldsymbol{a}}$

If $\mathbb{D}$ has an identity element $1_{\mathbb{D}}$ and

(iv) $1_{\mathbb{D}}\hat{\boldsymbol{a}} = \hat{\boldsymbol{a}}$ for all $\hat{a} \in \mathbb{D}^3$ then $\mathbb{D}^3$ is said to be a unitary $\mathbb{D}$ - module.

$$\mathbb{D}^3 = \{\hat{\boldsymbol{a}} = a + \varepsilon a^* \,|\, a, a^* \in \mathbb{R}^3,\ \varepsilon^2 = 0\}$$

Proof.

$$\begin{aligned}
(i)\ \ \hat{d}(\hat{\boldsymbol{a}} + \hat{\boldsymbol{b}}) &= \hat{d}\Big((a_1 + \varepsilon a_1^*,\, a_2 + \varepsilon a_2^*,\, a_3 + \varepsilon a_3^*) + (b_1 + \varepsilon b_1^*,\, b_2 + \varepsilon b_2^*,\, b_3 + \varepsilon b_3^*)\Big) \\
&= \hat{d}\Big(a_1 + b_1 + \varepsilon(a_1^* + b_1^*),\, a_2 + b_2 + \varepsilon(a_2^* + b_2^*),\, a_3 + b_3 + \varepsilon(a_3^* + b_3^*)\Big) \\
&= (d_1 + \varepsilon d_1^*,\, d_2 + \varepsilon d_2^*,\, d_3 + \varepsilon d_3^*)\Big(a_1 + b_1 + \varepsilon(a_1^* + b_1^*),\, a_2 + b_2 \\
&\quad + \varepsilon(a_2^* + b_2^*),\, a_3 + b_3 + \varepsilon(a_3^* + b_3^*)\Big) \\
&= (d_1 + \varepsilon d_1^*)\Big(a_1 + b_1 + \varepsilon(a_1^* + b_1^*)\Big) + (d_2 + \varepsilon d_2^*)\Big(a_2 + b_2 + \varepsilon(a_2^* + b_2^*)\Big) \\
&\quad + (d_3 + \varepsilon d_3^*)\Big(a_3 + b_3 + \varepsilon(a_3^* + b_3^*)\Big) \\
&= d_1(a_1 + b_1) + \varepsilon\Big(d_1(a_1^* + b_1^*) + d_1^*(a_1 + b_1)\Big) + d_2(a_2 + b_2) \\
&\quad + \varepsilon\Big(d_2(a_2^* + b_2^*) + d_2^*(a_2 + b_2)\Big) + d_3(a_3 + b_3) \\
&\quad + \varepsilon\Big(d_3(a_3^* + b_3^*) + d_3^*(a_3 + b_3)\Big) \\
&= d_1a_1 + d_1b_1 + \varepsilon d_1a_1^* + \varepsilon d_1b_1^* + \varepsilon d_1^*a_1 + \varepsilon d_1^*b_1 + d_2a_2 + d_2b_2 + \varepsilon d_2a_2^* \\
&\quad + \varepsilon d_2b_2^* + \varepsilon d_2^*a_2 + \varepsilon d_2^*b_2 + d_3a_3 + d_3b_3 + \varepsilon d_3a_3^* + \varepsilon d_3b_3^* + \varepsilon d_3^*a_3 + \varepsilon d_3^*b_3 \\
&= d_1a_1 + \varepsilon(d_1a_1^* + d_1^*a_1) + d_1b_1 + \varepsilon(d_1b_1^* + d_1^*b_1) + d_2a_2 + \varepsilon(d_2a_2^* + d_2^*a_2) \\
&\quad + d_2b_2 + \varepsilon(d_2b_2^* + d_2^*b_2) + d_3a_3 + \varepsilon(d_3a_3^* + d_3^*a_3) + d_3b_3 + \varepsilon(d_3b_3^* + d_3^*b_3) \\
&= \hat{d}\hat{\boldsymbol{a}} + \hat{d}\hat{\boldsymbol{b}}.
\end{aligned}$$

$$
\begin{aligned}
(ii)\ \ (\hat{d}+\hat{e})\hat{\boldsymbol{a}} &= \Big((d_1+\varepsilon d_1^*,\ d_2+\varepsilon d_2^*,\ d_3+\varepsilon d_3^*)+(e_1+\varepsilon e_1^*,\ e_2+\varepsilon e_2^*,\ e_3+\varepsilon e_3^*)\Big)\\
&\quad (a_1+\varepsilon a_1^*,\ a_2+\varepsilon a_2^*,\ a_3+\varepsilon a_3^*)\\
&= \Big(d_1+e_1+\varepsilon(d_1^*+e_1^*),\ d_2+e_2+\varepsilon(d_2^*+e_2^*),\ d_3+e_3+\varepsilon(d_3^*+e_3^*)\Big)\\
&\quad (a_1+\varepsilon a_1^*,\ a_2+\varepsilon a_2^*,\ a_3+\varepsilon a_3^*)\\
&= (d_1+e_1)a_1+\varepsilon\Big((d_1^*+e_1^*)a_1+(d_1+e_1)a_1^*\Big)+(d_2+e_2)a_2\\
&\quad +\varepsilon\Big((d_2^*+e_2^*)a_2+(d_2+e_2)a_2^*\Big)+(d_3+e_3)a_3\\
&\quad +\varepsilon\Big((d_3^*+e_3^*)a_3+(d_3+e_3)a_3^*\Big)\\
&= d_1a_1+\varepsilon(d_1^*a_1+d_1a_1^*)+e_1a_1+\varepsilon(e_1^*a_1+e_1a_1^*)+d_2a_2\\
&\quad +\varepsilon(d_2^*a_2+d_2a_2^*)+e_2a_2+\varepsilon(e_2^*a_2+e_2a_2^*)+d_3a_3+\varepsilon(d_3^*a_3+d_3a_3^*)\\
&\quad +e_3a_3+\varepsilon(e_3^*a_3+e_3a_3^*)\\
&= \hat{d}\hat{\boldsymbol{a}}+\hat{e}\hat{\boldsymbol{a}}.
\end{aligned}
$$

$$
\begin{aligned}
(iii)\ \ \hat{d}(\hat{e}\hat{\boldsymbol{a}}) &= (d_1+\varepsilon d_1^*,\ d_2+\varepsilon d_2^*,\ d_3+\varepsilon d_3^*)\Big((e_1+\varepsilon e_1^*,\ e_2+\varepsilon e_2^*,\ e_3+\varepsilon e_3^*)\\
&\quad (a_1+\varepsilon a_1^*,\ a_2+\varepsilon a_2^*, a_3+\varepsilon a_3^*)\Big)\\
&= (d_1+\varepsilon d_1^*,\ d_2+\varepsilon d_2^*,\ d_3+\varepsilon d_3^*)\Big(e_1a_1+\varepsilon(e_1^*a_1+e_1a_1^*),\ e_2a_2\\
&\quad +\varepsilon(e_2^*a_2+e_2a_2^*),\ e_3a_3+\varepsilon(e_3^*a_3+e_3a_3^*)\Big)\\
&= d_1(e_1a_1)+\varepsilon\Big(d_1(e_1^*a_1+e_1a_1^*)+d_1^*(e_1a_1)\Big)+d_2(e_2a_2)\\
&\quad +\varepsilon\Big(d_2(e_2^*a_2+e_2a_2^*)+d_2^*(e_2a_2)\Big)+d_3(e_3a_3)\\
&\quad +\varepsilon\Big(d_3(e_3^*a_3+e_3a_3^*)+d_3^*(e_3a_3)\Big)\\
&= (d_1e_1)a_1+\varepsilon\Big((d_1e_1^*)a_1+(d_1e_1)a_1^*+(d_1^*e_1)a_1\Big)+(d_2e_2)a_2\\
&\quad +\varepsilon\Big((d_2e_2^*)a_2+(d_2e_2)a_2^*+(d_2^*e_2)a_2\Big)+(d_3e_3)a_3\\
&\quad +\varepsilon\Big((d_3e_3^*)a_3+(d_3e_3)a_3^*+(d_3^*e_3)a_3\Big)\\
&= (d_1e_1)a_1+\varepsilon\Big((d_1e_1^*+d_1^*e_1)a_1+(d_1e_1)a_1^*\Big)+(d_2e_2)a_2\\
&\quad +\varepsilon\Big((d_2e_2^*+d_2^*e_2)a_2+(d_2e_2)a_2^*\Big)+(d_3e_3)a_3\\
&\quad +\varepsilon\Big((d_3e_3^*+d_3^*e_3)a_3+(d_3e_3)a_3^*\Big)\\
&= (\hat{d}\hat{e})\hat{\boldsymbol{a}}.
\end{aligned}
$$

If $\mathbb{D}$ has an identity element $1_{\mathbb{D}}$ and

$$\begin{aligned}
(iv)\ 1_{\mathbb{D}}\hat{\boldsymbol{a}} &= 1_{\mathbb{D}}(a_1+\varepsilon a_1^*,\ a_2+\varepsilon a_2^*,\ a_3+\varepsilon a_3^*) \\
&= \Big(1_{\mathbb{D}}(a_1+\varepsilon a_1^*),\ 1_{\mathbb{D}}(a_2+\varepsilon a_2^*),\ 1_{\mathbb{D}}(a_3+\varepsilon a_3^*)\Big) \\
&= (1_{\mathbb{D}}a_1+1_{\mathbb{D}}\varepsilon a_1^*,\ 1_{\mathbb{D}}a_2+1_{\mathbb{D}}\varepsilon a_2^*,\ 1_{\mathbb{D}}a_3+1_{\mathbb{D}}\varepsilon a_3^*) \\
&= (a_1+\varepsilon a_1^*,\ a_2+\varepsilon a_2^*,\ a_3+\varepsilon a_3^*) \\
&= \hat{\boldsymbol{a}} \quad for\ all\ \hat{\boldsymbol{a}} \in \mathbb{D}^3.
\end{aligned}$$

□

1.4 The Norm of a Dual Vector

Definition 1.4.1. $\mathbb{D}^3$ is a linear space over the real numbers with dimension 6. This bilinear form defines a kind of degenerate scalar product. It induces a "norm" which will be denoted by $\|.\|$.

$$\begin{aligned}
\|\hat{\boldsymbol{v}}\| &= (\boldsymbol{v}\boldsymbol{v})^{1/2} = \Big[(\boldsymbol{v}+\varepsilon\boldsymbol{v}^*)(\boldsymbol{v}+\varepsilon\boldsymbol{v}^*)\Big]^{\frac{1}{2}} \\
&= [\boldsymbol{v}\boldsymbol{v}+2\varepsilon\boldsymbol{v}\boldsymbol{v}^*+\varepsilon^2\boldsymbol{v}^*\boldsymbol{v}^*]^{\frac{1}{2}} = [\boldsymbol{v}\boldsymbol{v}+2\varepsilon\boldsymbol{v}\boldsymbol{v}^*]^{\frac{1}{2}} \\
&= (\|\boldsymbol{v}\|^2+2\varepsilon\boldsymbol{v}\boldsymbol{v}^*)^{\frac{1}{2}} = \|\boldsymbol{v}\|\Big(1+2\varepsilon\frac{\boldsymbol{v}\boldsymbol{v}^*}{\|\boldsymbol{v}\|^2}\Big)^{\frac{1}{2}} \\
&= \|\boldsymbol{v}\|\Big[\Big(1+\varepsilon\frac{\boldsymbol{v}\boldsymbol{v}^*}{\|\boldsymbol{v}\|^2}\Big)^2\Big]^{\frac{1}{2}} = \|\boldsymbol{v}\|\Big(1+\varepsilon\frac{\boldsymbol{v}\boldsymbol{v}^*}{\|\boldsymbol{v}\|^2}\Big) \\
&= \|\boldsymbol{v}\|+\varepsilon\frac{\boldsymbol{v}\boldsymbol{v}^*}{\|\boldsymbol{v}\|} = \Big(\|\boldsymbol{v}\|,\ \frac{\boldsymbol{v}\boldsymbol{v}^*}{\|\boldsymbol{v}\|}\Big).
\end{aligned}$$

1.5 Dual Unit Vectors

Definition 1.5.1. If the norm of a dual vector is $(1,0)$ then this dual vector is called the dual unit vector. If $\hat{\boldsymbol{v}} = \boldsymbol{v} + \varepsilon \boldsymbol{v}^*$ is a dual unit vector then

$$\|\hat{\boldsymbol{v}}\| = \boldsymbol{v} + \varepsilon \frac{\boldsymbol{v}\boldsymbol{v}^*}{\|\boldsymbol{v}\|} = \left(\|\boldsymbol{v}\|, \frac{\boldsymbol{v}\boldsymbol{v}^*}{\|\boldsymbol{v}\|} \right) = (1,0)$$

and which implies $\|\boldsymbol{v}\| = 1$ and $\boldsymbol{v}\boldsymbol{v}^* = 0$.

1.6 Dual Functions

Definition 1.6.1. Let $\mathbb{D}$ and $\mathbb{Y}$ be the sets of dual numbers. A dual function f from a dual set $\mathbb{Y}$ is a rule that assigns a unique element $f(\hat{x}) \in \mathbb{Y}$ to each element $\hat{x} \in \mathbb{D}$.

A symbolic way to say '$\hat{y}$ is a dual function of $\hat{x}$' is writing

$$\hat{y} = f(\hat{x}) \quad (\hat{y}\ equals\ f\ of\ \hat{x})$$

In this notation, the symbol f represents the dual function. The letter $\hat{x}$, called the independent variable, represents the input value of f, and $\hat{y}$, the dependent variable, represents the corresponding output value of f at $\hat{x}$.

The set $\mathbb{D}$ of all possible input values is called the domain of the dual function. The set of all values of $f(\hat{x})$ as $\hat{x}$ varies throughout $\mathbb{D}$ is called the range of the dual function. The range may not include every element in the dual set $\mathbb{Y}$.

Assume that $\hat{y} = y + \varepsilon y^*$ is the value of the function f at $\hat{x} = x + \varepsilon x^*$. In other words;

$$y + \varepsilon y^* = f(x + \varepsilon x^*).$$

Here real part, y, and a dual part, y^*, of $\hat{y}$ depend on the real variables x and x^*. (y and y^* depend on two variables x and x^*). For example, if we take $f(\hat{x}) = (\hat{x})^2$, then

$f(\hat{x}) = f(x+\varepsilon x^*) = (x+\varepsilon x^*)^2 = x^2 + \varepsilon 2xx^*$ since $\varepsilon^2 = 0$. Thus, $y = x^2$ and $y^* = 2xx^*$.
This simple example shows that a function of a dual variable can be expressed in terms of a pair of real valued functions of real variables x and x^*, now let us examine $f(\hat{x})$ as ;

$$f(\hat{x}) = f(x,x^*) + \varepsilon f^*(x,x^*)$$

where $\hat{x} = x + \varepsilon x^*$ is a dual variable, f and f^* are two, generally different, functions of two variables, x and x^*.

Hence similar to the real case we can think of the Taylor series expansion of a dual function with $\varepsilon^2 = \varepsilon^3 = ... = 0$. Let $f(\hat{x})$ be a differentiable function. A function of a single dual number is given by

$$f(\hat{x}) = f(x+\varepsilon x^*) = f(x) + \varepsilon x^* f'(x)$$

provided that the function $f(x)$ has the derivative $f'(x)$.
We can obtain this result similar to the real case by the Taylor series expansion of
$f(\hat{x})$:

$$f(\hat{x}) = f(x_0) + \frac{(\hat{x}-x_0)}{1!} f'(x_0) + ... + \frac{(\hat{x}-x_0)^n}{n!} f^{(n)}(x_0) + ...$$

If we write $\hat{x}$ as $\hat{x} = x + \varepsilon x^*$ and apply Taylor series expansion at $x_0 = 0$ then (Maclaurin series of $f(\hat{x})$ is given by)

$$\begin{aligned} f(x+\varepsilon x^*) &= f(0) + \frac{(x+\varepsilon x^*)}{1!} f'(0) + ... + \frac{(x+\varepsilon x^*)^n}{n!} f^{(n)}(0) + ... \\ &= f(0) + \frac{(x+\varepsilon x^*)}{1!} f'(0) + ... + \frac{x^n + n\varepsilon x^{n-1} x^*}{n!} f^{(n)}(0) + ... \\ &= \left(f(0) + \frac{x}{1!} f'(0) + ... + \frac{x^n}{n!} f^{(n)}(0) + ... \right) \\ &\quad + \varepsilon x^* \left(f'(0) + ... + \frac{x^{n-1}}{(n-1)!} f^{(n)}(0) + ... \right) \end{aligned}$$

where the first part of the expression is the Taylor series expansion of $f(x)$ and the second part is the Taylor series expansion of $f'(x)$. Thus we have

$$f(\hat{x}) = f(x+\varepsilon x^*) = f(x) + \varepsilon x^* f'(x). \tag{1.6.1}$$

This result is also useful in computing basic functions of dual numbers such as following examples.

Example 1.6.2.

$$\begin{aligned}\sin\hat{x} = \hat{x} - \frac{\hat{x}^3}{3!} + \frac{\hat{x}^5}{5!} - \ldots &= x + \varepsilon x^* - \frac{(x^3+3x^2\varepsilon x^*)}{3!} + \frac{(x^5+5x^4\varepsilon x^*)}{5!} - \ldots \\ &= \underbrace{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\right)}_{\sin x} + \varepsilon x^* \underbrace{\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\right)}_{\cos x}\end{aligned}$$

Hence,

$$\sin\hat{x} = \sin(x+\varepsilon x^*) = \sin x + \varepsilon x^* \cos x$$

Similarly,

$$\cos(x+\varepsilon x^*) = \cos x - \varepsilon x^* \sin x,$$

$$\tan(x+\varepsilon x^*) = \tan x + \varepsilon x^*(1+\tan^2 x),$$

$$\cot(x+\varepsilon x^*) = \cot x - \varepsilon x^* \csc^2 x = \cot x - \varepsilon x^*(1+\cot^2 x).$$

Also using the Taylor series expansion of a dual function, a dual number raised to a power is given by

$$(\hat{x})^n = (x+\varepsilon x^*)^n = x^n + \varepsilon n x^* x^{n-1}$$

where n can be any real number.

In particular, when $n=2$;

$$(\hat{x})^2 = (x+\varepsilon x^*)^2 = x^2 + 2\varepsilon x x^*$$

and when $n=\frac{1}{2}$;

$$(\hat{x})^{\frac{1}{2}} = \sqrt{\hat{x}} = \sqrt{x+\varepsilon x^*} = \sqrt{x} + \varepsilon\frac{x^*}{2\sqrt{x}}.$$

The Taylor series expansion also allows us to write the dual form of exponential and logarithmic functions, for example;

$$e^{\hat{x}} = e^{(x+\varepsilon x^*)} = e^x\, e^{\varepsilon x^*} = e^x(1+\varepsilon x^*) = e^x + \varepsilon x^* e^x,$$

$$\ln \hat{x} = \ln(x+\varepsilon x^*) = \ln x + \varepsilon\, \frac{x^*}{x}.$$

In conclusion, all formal operations of dual numbers are the same as those of ordinary algebra followed by setting $\varepsilon^2 = \varepsilon^3 = ... = 0$.

1.7 Dual Matrix

Definition 1.7.1. Dual matrices can be defined likewise, i.e., if A and A^* are two real $n \times n$ matrices, $\hat{A}$ is defined as

$$\hat{A} = A + \varepsilon A^*.$$

3×3 homogeneous dual transformation matrices play an important role in the kinematics and dynamics of robot manipulators.

Definition 1.7.2. Transpose of a dual matrix is defined as follows:

$$\hat{A}^T = (A+\varepsilon A^*)^T = A^T + \varepsilon (A^*)^T$$

where the superscripts ^{T} denote transposes of dual and real matrices.

Definition 1.7.3. An identity dual matrix, denoted by I, is defined, as follows:

$$I = I + \varepsilon 0$$

(where I is a real identity matrix and 0 is a null matrix).

Definition 1.7.4. The inverse of a dual matrix $\hat{A}$ is defined by a dual matrix $\hat{A}^{-1}$ such that $\hat{A}^{-1}\hat{A} = \hat{A}\hat{A}^{-1} = I$ where I is an identity dual matrix.

CHAPTER TWO
THE STUDY MAPPING

The dual representations of a line is simply the Plücker vector written as a dual unit vector. This vector is a point on a unit sphere in $\mathbb{D}^3$ which is also the image of the Plücker quadric in $\mathbb{D}^3$. This representation has all the geometric structure offered by the Plücker coordinates with a simplified computational structure. The computational problem of computing points on a quadric in P^5 is reduced to a problem in a dual form spherical geometry. This is the result of the transfer principle first proposed by Kotel'nikov (1895) and discovered independently by Study (1903). The transfer principle simply states that for any operation defined for a real vector space, there is a dual version with similar interpretation (see Dimentberg (1965) for a discussion of the transfer principle).

In the kinematics and dynamics of robot manipulators, a straight line is one of the fundamental geometrical concepts. The dual number algebra provides us with a particularly simple way of representing a straight line.

The dual unit vector is required for the dual representation of a line. The Plücker vector representation of a line is given by a vector directed along the line and a moment vector.

The points on the dual unit sphere represent lines in $\mathbb{R}^3$. There exist a one to one correspondence between the points on D.U.S. and oriented straight lines in $\mathbb{R}^3$ (Study mapping).

Dual numbers are particularly useful for expression of dual angles, which are, in turn, useful for expressing the spatial relationship between skew lines in space. Skew straight lines in space are separated by a perpendicular distance, d, and the projection of one line onto the other along that perpendicular forms an angle, θ. The dual angle describing the relationship is $\hat{\theta} = \theta + \varepsilon d$.

2.1 Dual Unit Sphere (D.U.S)

Definition 2.1.1. Dual unit vectors define points on a sphere in $\mathbb{D}^3$. This sphere is referred to as the dual unit sphere.

In other words, the set of all dual vectors

$$\{\hat{\boldsymbol{v}} = \boldsymbol{v} + \varepsilon \boldsymbol{v}^* \mid \|\hat{\boldsymbol{v}}\| = (1,0);\ \boldsymbol{v},\ \boldsymbol{v}^* \in \mathbb{R}^3\}$$

is called the dual unit sphere (D.U.S) in $\mathbb{D}^3$.

2.2 Oriented Lines

Definition 2.2.1. An oriented line ℓ which is also called a spear can be defined by a point $p \in \ell$ and a unit direction vector $\boldsymbol{g}$. On the other hand, a unit force on ℓ with respect to the origin O defines the moment vector $\boldsymbol{g}^*$. The norm of the moment vector is the smallest distance from line to the origin where physically the moment vector $\boldsymbol{g}^*$ is defined by $\boldsymbol{g}^* = p \times \boldsymbol{g}$.

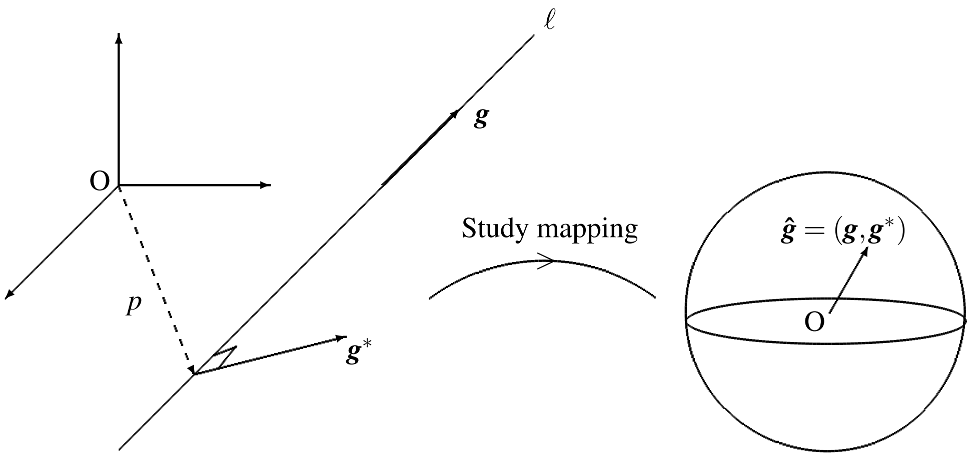

Figure 2.1 Plücker coordinates.

The coordinates $(\boldsymbol{g},\boldsymbol{g}^*) = (\boldsymbol{g}, p \times \boldsymbol{g})$ of the line ℓ with the six components $(g_1, g_2, g_3, g_1{}^*, g_2{}^*, g_3{}^*)$ are called the Plücker coordinates of line ℓ. Since $\boldsymbol{g}$ is a unit vector and $\boldsymbol{v}^*$ (the moment vector), is orthogonal to $\boldsymbol{g}$, we have

$$\boldsymbol{g}.\boldsymbol{g} = 1 \quad \textit{and} \quad \boldsymbol{g}.\boldsymbol{g}^* = 0.$$

E. Study first combined the two parts of the Plücker coordinate vector of a line into a dual vector by letting

$$\hat{\boldsymbol{g}} = \boldsymbol{g} + \varepsilon \boldsymbol{g}^*. \tag{2.2.1}$$

If we compute the norm of (2.2.1), we get

$$\|\hat{\boldsymbol{g}}\|^2 = \hat{\boldsymbol{g}}.\hat{\boldsymbol{g}} = \boldsymbol{g}.\boldsymbol{g} + 2\varepsilon \boldsymbol{g}\boldsymbol{g}^* = 1$$

where $\varepsilon^2 = 0$, $\boldsymbol{g}$ has unit length and $\boldsymbol{g}$ and $\boldsymbol{g}^*$ are orthogonal.

If we substitute the unit dual vector $\hat{\boldsymbol{g}}$ at the center of the D.U.S, it is clear that the unit dual vector $\hat{\boldsymbol{g}}$ corresponds to a point $(\boldsymbol{g}, \boldsymbol{g}^*)$ on the D.U.S. Since the coordinates of the dual point $(\boldsymbol{g}, \boldsymbol{g}^*)$ are the Plücker coordinates of the oriented line ℓ, the oriented line corresponds to a dual point on the D.U.S. (Pottmann & Wallner (2001)).

2.3 The Study Mapping

The mapping which assigns to an oriented line of Euclidean space the dual vector $\hat{\boldsymbol{g}} = \boldsymbol{g} + \varepsilon \boldsymbol{g}^*$, where $(\boldsymbol{g}, \boldsymbol{g}^*)$ are its Plücker coordinates, is called the Study mapping. Therefore the Study mapping constitutes a one to one correspondence between the oriented lines of $\mathbb{R}^3$ space and the dual points of the D.U.S. (Its image is called the Study model of oriented lines of $\mathbb{R}^3$). Moreover the D.U.S is also called the Study sphere.

The angle between the dual vectors is called a dual angle. (The dual angle is useful for expressing the spatial relationship between skew lines in space). Let us denote the angle between the dual unit vectors $\hat{\boldsymbol{g}} = \boldsymbol{g} + \varepsilon \boldsymbol{g}^*$ and $\hat{\boldsymbol{h}} = \boldsymbol{h} + \varepsilon \boldsymbol{h}^*$ by $\hat{\varphi} = \varphi + \varepsilon \varphi^*$.

The scalar product of two dual unit vectors $\hat{\boldsymbol{g}}, \hat{\boldsymbol{h}}$ has a simple geometric meaning in terms of the spears ($\boldsymbol{G}$, $\boldsymbol{H}$ respectively) they represent:

We define the distance $d(\boldsymbol{G},\boldsymbol{H})$ between two lines $\boldsymbol{G},\boldsymbol{H}$ in $\mathbb{R}^3$ as the smallest distance between points $g \in \boldsymbol{G}$ and $h \in \boldsymbol{H}$. The minimum value is attained if g,h are the points where the common perpendicular of $\boldsymbol{G},\boldsymbol{H}$ meets $\boldsymbol{G}$ and $\boldsymbol{H}$, respectively.(cf. figure 2.2)

The common perpendicular of an ordered pair $(\boldsymbol{G},\boldsymbol{H})$ of spears can be given an orientation: If $(\boldsymbol{g},\boldsymbol{g}^*)$ and $(\boldsymbol{h},\boldsymbol{h}^*)$ are the Plücker coordinates of $\boldsymbol{G}$ and $\boldsymbol{H}$, respectively, the common perpendicular N is given an orientation by the vector $\boldsymbol{g} \times \boldsymbol{h}$.

Definition 2.3.1. The dual angle of two spears $\boldsymbol{G},\boldsymbol{H}$ is defined by

$$\hat{\sphericalangle}(\boldsymbol{G},\boldsymbol{H}) = \sphericalangle(\boldsymbol{G},\boldsymbol{H}) + \varepsilon \boldsymbol{d}(\boldsymbol{G},\boldsymbol{H}).$$

Lemma 2.3.2. *The scalar product in $\mathbb{D}^3$ is a Euclidean invariant. If $\boldsymbol{G},\boldsymbol{H}$ are two lines whose Study images are $\hat{\boldsymbol{g}},\hat{\boldsymbol{h}}$, then there is the equation*

$$\hat{\boldsymbol{g}}.\hat{\boldsymbol{h}} = \cos\hat{\varphi} = \cos\varphi - \varepsilon d \sin\varphi$$

where $\varphi = \sphericalangle(\boldsymbol{G},\boldsymbol{H})$, $\hat{\varphi} = \hat{\sphericalangle}(\boldsymbol{G},\boldsymbol{H})$, and $d = d(\boldsymbol{G},\boldsymbol{H})$.

Lemma 2.3.3. *The dual angle is defined as*

$$\hat{\varphi} = \varphi + \varepsilon\varphi^*$$

where φ (real component of $\hat{\varphi}$) is projected angle between lines **G** *and* **H** *and φ^* (dual component of $\hat{\varphi}$) is the shortest distance between the lines* **G** *and* **H** *(length of common perpendicular) Muller (1963).*

Proof. Assume that g,h are the points where their common perpendicular meets the lines $\boldsymbol{G},\boldsymbol{H}$. The scalar product of $\hat{\boldsymbol{g}}$ and $\hat{\boldsymbol{h}}$ is computed by the following:

$$\hat{\boldsymbol{g}}.\hat{\boldsymbol{h}} = \boldsymbol{g}.\boldsymbol{h} + \varepsilon(\boldsymbol{g}.\boldsymbol{h}^* + \boldsymbol{g}^*.\boldsymbol{h})$$

and

$$\hat{\boldsymbol{g}}.\hat{\boldsymbol{h}} = \|\hat{\boldsymbol{g}}\|.\|\hat{\boldsymbol{h}}\|.\cos\hat{\varphi} = \cos\hat{\varphi} = \cos\varphi - \varepsilon\varphi^*\sin\varphi$$

Therefore

$$\hat{\boldsymbol{g}}.\hat{\boldsymbol{h}} = \boldsymbol{g}.\boldsymbol{h} + \varepsilon(\boldsymbol{g}.\boldsymbol{h}^* + \boldsymbol{g}^*.\boldsymbol{h}) = \cos\varphi - \varepsilon\varphi^*\sin\varphi \tag{2.3.1}$$

Using the equality of dual numbers (1.1.2), we have

$$\boldsymbol{g}.\boldsymbol{h} = \cos\varphi.$$

Now we will investigate the dual part φ^* of the dual angle $\hat{\varphi}$.

We know that the dual unit vectors $\hat{\boldsymbol{g}}$ and $\hat{\boldsymbol{h}}$ represent two oriented lines $\boldsymbol{G}$ and $\boldsymbol{H}$, respectively. If we take a unit vector which is perpendicular to both $\boldsymbol{G}$ and $\boldsymbol{H}$ then we can denote it by

$$\boldsymbol{n} = \mp\frac{\boldsymbol{g}\times\boldsymbol{h}}{\|\boldsymbol{g}\times\boldsymbol{h}\|}.$$

A straight line passing through the shortest distance between the oriented lines $\boldsymbol{G}$ and $\boldsymbol{H}$ intersects these lines at two points, say $\boldsymbol{x}$ and $\boldsymbol{y}$, respectively. Also the vectorial moments of the lines $\boldsymbol{G}$ and $\boldsymbol{H}$ with respect to the origin are $\boldsymbol{g}^* = \boldsymbol{x}\times\boldsymbol{g}$ and $\boldsymbol{h}^* = \boldsymbol{y}\times\boldsymbol{h}$, respectively. Hence we can compute the scalar product of $\boldsymbol{h}$ and $\boldsymbol{g}^*$, and the scalar product of $\boldsymbol{g}$ and $\boldsymbol{h}^*$;

$$\boldsymbol{g}^*.\boldsymbol{h} = (\boldsymbol{x}\times\boldsymbol{g}).\boldsymbol{h} = (\boldsymbol{x},\boldsymbol{g},\boldsymbol{h}) = \boldsymbol{x}(\boldsymbol{g}\times\boldsymbol{h}) \tag{2.3.2}$$

$$\boldsymbol{g}.\boldsymbol{h}^* = \boldsymbol{g}.(\boldsymbol{y}\times\boldsymbol{h}) = -(\boldsymbol{y},\boldsymbol{g},\boldsymbol{h}) = -\boldsymbol{y}(\boldsymbol{g}\times\boldsymbol{h}) \tag{2.3.3}$$

The sum of (2.3.2) and (2.3.3) we have,

$$\boldsymbol{g}^*.\boldsymbol{h} + \boldsymbol{g}.\boldsymbol{h}^* = (\boldsymbol{x}-\boldsymbol{y})(\boldsymbol{g}\times\boldsymbol{h}) \tag{2.3.4}$$

If the shortest distance between the lines $\boldsymbol{G}$ and $\boldsymbol{H}$ is denoted by ψ, then the oriented distance is defined by

$$\boldsymbol{x}-\boldsymbol{y} = \psi.\boldsymbol{n} = \mp\psi\frac{\boldsymbol{g}\times\boldsymbol{h}}{\|\boldsymbol{g}\times\boldsymbol{h}\|} \tag{2.3.5}$$

Substituting (2.3.5) into (2.3.4) we get,

$$\boldsymbol{g}.\boldsymbol{h}^* + \boldsymbol{g}^*.\boldsymbol{h} = \mp\psi\frac{(\boldsymbol{g}\times\boldsymbol{h})^2}{\|\boldsymbol{g}\times\boldsymbol{h}\|} = \mp\psi\|\boldsymbol{g}\times\boldsymbol{h}\| = \mp\psi\sin\varphi \tag{2.3.6}$$

From (2.3.1) and (2.3.6) we take

$$-\varphi^*\sin\varphi = \mp\psi\sin\varphi \tag{2.3.7}$$

Depending on the orientation of $\boldsymbol{n}$ we can take the suitable sign and obtain the shortest distance ψ is equal to φ^*. □

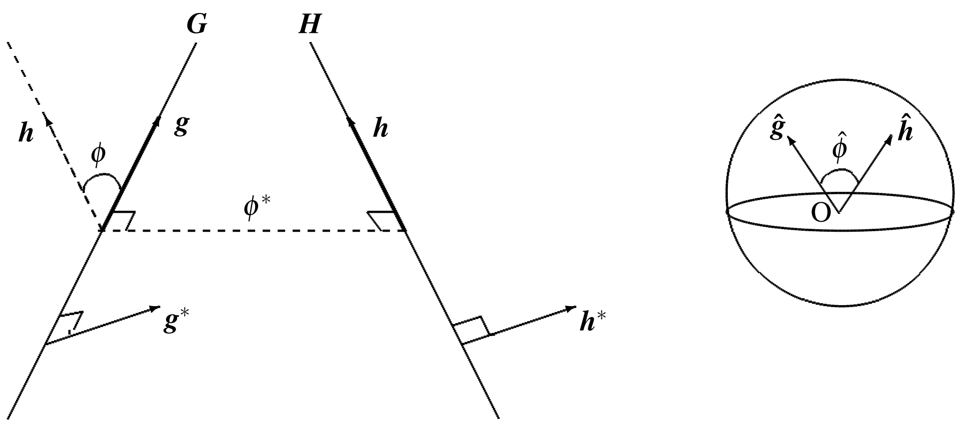

Figure 2.2 The geometric meaning of the dual angle.

2.4 The Dual Angle of Spears

As a summary we can investigate the positions of oriented straight lines, $\boldsymbol{G}$ and $\boldsymbol{H}$ corresponding to dual unit vectors $\hat{\boldsymbol{g}}$ and $\hat{\boldsymbol{h}}$, respectively. If we denote the dual angle between the dual unit vectors $\hat{\boldsymbol{g}}$ and $\hat{\boldsymbol{h}}$ of D.U.S by $\hat{\varphi} = \varphi + \varepsilon\varphi^*$ then φ is the angle between the corresponding oriented straight lines (if these lines are skew, φ is the projected angle) and φ^*

is the shortest distance between these lines.

As a consequence, when we consider the formula (2.3.1) we have the following cases:

1) If $\hat{\boldsymbol{g}}.\hat{\boldsymbol{h}} = 0$ i.e $\varphi = \frac{\pi}{2}$ and $\varphi^* = 0$ then $\boldsymbol{G}$ and $\boldsymbol{H}$ represent perpendicular intersecting straight lines in $\mathbb{R}^3$.

2) If $\hat{\boldsymbol{g}}.\hat{\boldsymbol{h}}$ is equal to a pure dual number or $\boldsymbol{g}.\boldsymbol{h} = 0$ i.e $\varphi = \frac{\pi}{2}$ and $\varphi^* \neq 0$ then oriented straight lines $\boldsymbol{G}$ and $\boldsymbol{H}$ represent skew lines which have perpendicular projections in $\mathbb{R}^3$.

3) If the dual part of $\hat{\boldsymbol{g}}.\hat{\boldsymbol{h}}$ is equal to zero, i.e $\boldsymbol{g}^*.\boldsymbol{h} + \boldsymbol{g}.\boldsymbol{h}^* = 0$ or $\varphi^* = 0$ then $\boldsymbol{G}$ and $\boldsymbol{H}$ represent intersecting straight lines.

4) If $\hat{\boldsymbol{g}}.\hat{\boldsymbol{h}}$ has a real part equal to $+1$ or -1 and dual part different from zero then $\boldsymbol{g}$ and $\boldsymbol{h}$ represent parallel lines in $\mathbb{R}^3$.

5) If $\hat{\boldsymbol{g}}.\hat{\boldsymbol{h}}$ has only real part equal to $+1$ or -1 then $\boldsymbol{G}$ and $\boldsymbol{H}$ represent coincident two lines in $\mathbb{R}^3$.

CHAPTER THREE
THE DUAL EULER PARAMETERS

3.1 Cayley Formula

As we have mentioned before, we examine the rotations of dual unit sphere instead of the rigid body motion in $\mathbb{R}^3$ space (The Study mapping). The trajectory of the rigid body motion is represented by a dual curve on the dual unit sphere. We can obtain this curve by the rotations of a moving dual unit sphere on the fixed dual unit sphere with the same center. This is why we are dealing with rotations of the dual unit sphere. This is also a rigid transformation. Hence any point $\hat{x}$ on the moving dual unit sphere determines the point $\hat{X}$ on the fixed dual unit sphere by a dual rotation matrix $\hat{A}$ such that

$$\hat{X} = \hat{A}\hat{x}.$$

Because of the rigidity of this transformation we have $\|\hat{X}\| = \|\hat{x}\|$ that is,

$$\|\hat{X}\|^2 = \hat{X}^T\hat{X} = (\hat{A}\hat{x})^T\hat{A}\hat{x} = \hat{x}^T\hat{A}^T\hat{A}\hat{x} = \hat{x}^T\hat{x} = \|\hat{x}\|^2$$

which yields $\hat{A}^T\hat{A} = I$, thus $\hat{A}$ is an orthogonal dual matrix.

On the other hand, equality of norms $\|\hat{X}\| = \sqrt{\hat{X}^T\hat{X}} = \sqrt{\hat{x}^T\hat{A}^T\hat{A}\hat{x}} = \sqrt{\hat{x}^T\hat{x}} = \|\hat{x}\|$ implies $\hat{X}^T\hat{X} = \hat{x}^T\hat{x}$ and then we have

$$(\hat{X}-\hat{x})^T(\hat{X}+\hat{x}) = \hat{X}^T\hat{X} + \hat{X}^T\hat{x} - \hat{x}^T\hat{X} - \hat{x}^T\hat{x} = \hat{X}^T\hat{x} - \hat{x}^T\hat{X} = 0$$

where $\hat{X}^T\hat{x} = \hat{a} \in \mathbb{D}$ ($\hat{a}$ denotes any dual number) and $\hat{x}^T\hat{X} = \hat{a}^T = \hat{a} \in \mathbb{D}$. This expresses the orthogonality of $(\hat{X}-\hat{x})$ and $(\hat{X}+\hat{x})$.

Since $\hat{X} = \hat{A}\hat{x}$,

$$\hat{X} + \hat{x} = (\hat{A}+I)\hat{x} \quad or \quad \hat{x} = (\hat{A}+I)^{-1}(\hat{X}+\hat{x}) \quad and \quad (\hat{X}-\hat{x}) = (\hat{A}-I)\hat{x}.$$

Hence we can compute

$$\hat{X}-\hat{x}=(\hat{A}-I)(\hat{A}+I)^{-1}(\hat{X}+\hat{x}).$$

Let us denote $(\hat{A}-I)(\hat{A}+I)^{-1}$ by $\hat{B}$. Since $\hat{X}-\hat{x}$ is orthogonal to $\hat{X}+\hat{x}$. $\hat{B}(\hat{X}+\hat{x})$ is orthogonal to $\hat{X}+\hat{x}$. For a general dual vector $\hat{\mathbf{v}}$, $\hat{B}\hat{\mathbf{v}}$ is orthogonal to $\hat{\mathbf{v}}$. Then we have

$$\hat{\mathbf{v}}^T\hat{B}\hat{\mathbf{v}}=\sum(\hat{b}_{ij}+\hat{b}_{ji})\hat{\mathbf{v}}_\mathbf{i}\hat{\mathbf{v}}_\mathbf{j}=0.$$

This relation holds for every $\hat{\mathbf{v}}$ hence $\hat{b}_{ii}=0$ and $\hat{b}_{ij}=-\hat{b}_{ji}$. Which implies the property $\hat{B}=-\hat{B}^T$, that is, $\hat{B}$ is skew symmetric.

On the other hand skew symmetry of $\hat{B}$ provides $(I-\hat{B})$ not to be singular. A simple computation yields

$$\hat{B}=(\hat{A}-I)(\hat{A}+I)^{-1}\Rightarrow\hat{B}(\hat{A}+I)=(\hat{A}-I)$$
$$\Rightarrow\hat{B}+I=\hat{A}-\hat{B}\hat{A}\Rightarrow(I+\hat{B})(I-\hat{B})^{-1}=\hat{A}$$

Hence we get the Cayley Formula for the dual case:

$$\hat{A}=(I+\hat{B})(I-\hat{B})^{-1}.$$

Let us compute $\hat{A}^T$;

$$\begin{aligned}\hat{A}^T &= (I+\hat{B})^T((I-\hat{B})^{-1})^T\\ &= (I+\hat{B}^T)(I-\hat{B}^T)^{-1}.\end{aligned}$$

Thus we have

$$\hat{A}^T=(I-\hat{B})(I+\hat{B})^{-1}$$

since $\hat{B}$ is skew symmetric.

In fact

$$\hat{A}\hat{A}^T = \hat{A}^T\hat{A} = I.$$

Hence every skew symmetric dual matrix $\hat{B}$ determines an orthogonal dual matrix $\hat{A}$.

If we define the skew symmetric dual matrix $\hat{B}$ by

$$\hat{B} = \begin{pmatrix} 0 & -\hat{b}_3 & \hat{b}_2 \\ \hat{b}_3 & 0 & -\hat{b}_1 \\ -\hat{b}_2 & \hat{b}_1 & 0 \end{pmatrix}$$

then instead of $\hat{B}\hat{\boldsymbol{v}}$ ($\hat{\boldsymbol{v}}$ is a dual vector on the D.U.S.) one can use $\hat{\boldsymbol{b}} \times \hat{\boldsymbol{v}}$ where $\hat{\boldsymbol{b}} = (\hat{b}_1, \hat{b}_2, \hat{b}_3)$. Hence

$$\hat{B}\hat{\boldsymbol{v}} = \hat{\boldsymbol{b}} \times \hat{\boldsymbol{v}}.$$

3.2 Rodrigues' Equations

Given an orthogonal dual matrix $\hat{A}$ we can obtain a skew symmetric dual matrix $\hat{B}$ by the Cayley's formula. It is clear that the relation

$$\hat{X} - \hat{x} = \hat{B}(\hat{X} + \hat{x})$$

between the fixed and the moving frame coordinates can be written in the form

$$\hat{X} - \hat{x} = \hat{\boldsymbol{b}} \times (\hat{X} + \hat{x})$$

This is analogous to the Rodrigues equations in the real case. Let us call $\hat{\boldsymbol{b}}$ the dual Rodrigues vector. Now we define a dual hyperplane perpendicular to $\hat{\boldsymbol{b}}$ and denote the projections of $\hat{X}$ and $\hat{x}$ on this dual plane by $\hat{X}'$ and $\hat{x}'$. Let $\hat{\phi}$ be the angle between $\hat{X}'$ and $\hat{x}'$ ($\hat{\phi}$ is the vertex angle of the rhombus formed by $\hat{X}'$ and $\hat{x}'$ so $\hat{\phi}$ is the rotation angle).

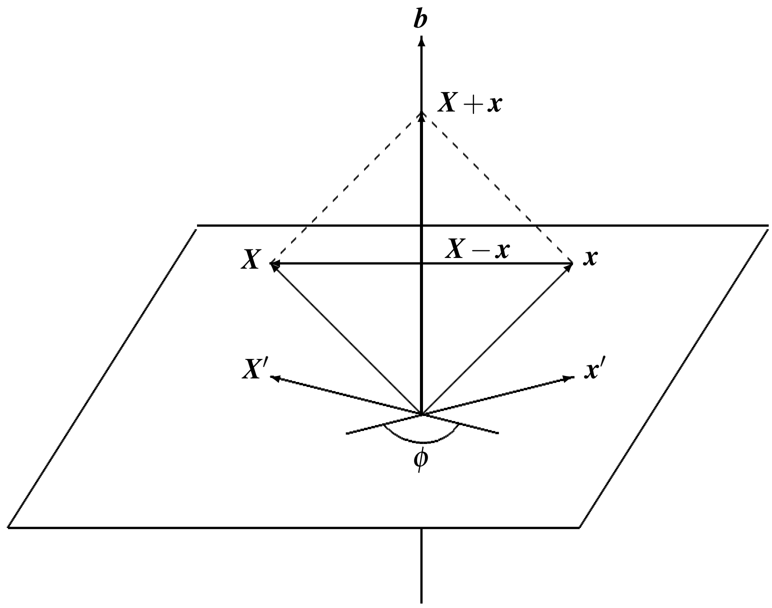

Figure 3.1 The rhombus formed by $\boldsymbol{x}$ and $\boldsymbol{X}$.

It is clear that $\hat{X}-\hat{x}=\hat{\boldsymbol{b}}\times(\hat{X}+\hat{x})$ implies $\hat{X}'-\hat{x}'=\hat{\boldsymbol{b}}\times(\hat{X}'+\hat{x}')$. The norm of $\hat{X}-\hat{x}=\hat{\boldsymbol{b}}\times(\hat{X}+\hat{x})$ is $\|\hat{X}'-\hat{x}'\|=\|\hat{\boldsymbol{b}}\|\|\hat{X}'+\hat{x}'\|$. Hence

$$\|\hat{\boldsymbol{b}}\|=\frac{\|\hat{X}'-\hat{x}'\|}{\|\hat{X}'+\hat{x}'\|}.$$

It is easy to verify from the figure 3.1 that

$$\frac{\|\hat{X}'-\hat{x}'\|}{\|\hat{X}'+\hat{x}'\|}=\tan\frac{\hat{\phi}}{2}.$$

Therefore

$$\|\hat{\boldsymbol{b}}\|=\tan\frac{\hat{\phi}}{2}. \tag{3.2.1}$$

Using the algebra of dual numbers properties we obtain from (3.2.1);

$$\|\hat{\boldsymbol{b}}\|=\|\boldsymbol{b}\|+\varepsilon\frac{\boldsymbol{b}.\boldsymbol{b}^{*}}{\|\boldsymbol{b}\|} \tag{3.2.2}$$

and using (1.6.1) we have

$$\tan\frac{\hat{\phi}}{2} = \tan\frac{\phi}{2} + \varepsilon\frac{\phi^*}{2}\left(1+\tan^2\frac{\phi}{2}\right) \tag{3.2.3}$$

The equality of (3.2.2) and (3.2.3) implies

$$\|\boldsymbol{b}\| + \varepsilon\frac{\boldsymbol{b}\boldsymbol{b}^*}{\|\boldsymbol{b}\|} = \tan\frac{\phi}{2} + \varepsilon\frac{\phi^*}{2}\left(1+\tan^2\frac{\phi}{2}\right) \tag{3.2.4}$$

Thus we have from (3.2.4) the norm of the real Rodrigues vector

$$\|\boldsymbol{b}\| = \tan\frac{\phi}{2} \tag{3.2.5}$$

and

$$\frac{\boldsymbol{b}.\boldsymbol{b}^*}{\|\boldsymbol{b}\|} = \frac{\phi^*}{2}\left(1+\tan^2\frac{\phi}{2}\right). \tag{3.2.6}$$

Let us denote the unit vector parallel to $\boldsymbol{b}$ by $\boldsymbol{s}$ then $\boldsymbol{s} = \frac{\boldsymbol{b}}{\|\boldsymbol{b}\|}$ where $\boldsymbol{s} = (s_1, s_2, s_3)$ the unit Rodrigues vector. So (3.2.6) yields;

$$\boldsymbol{s}.\boldsymbol{b}^* = \frac{\phi^*}{2}\left(1+\tan^2\frac{\phi}{2}\right) \tag{3.2.7}$$

On the other hand let us define the dual Rodrigues vector by $\hat{\boldsymbol{s}}$ where $\hat{\boldsymbol{s}} = \frac{\hat{\boldsymbol{b}}}{\|\hat{\boldsymbol{b}}\|}$. Using the properties of the dual numbers we have

$$\hat{\boldsymbol{s}} = \boldsymbol{s} + \varepsilon\boldsymbol{s}^* = \frac{\hat{\boldsymbol{b}}}{\|\hat{\boldsymbol{b}}\|} = \frac{\boldsymbol{b}}{\|\boldsymbol{b}\|} + \varepsilon\left(\frac{\boldsymbol{b}^*}{\|\boldsymbol{b}\|} - \frac{\boldsymbol{b}(\boldsymbol{b}.\boldsymbol{b}^*)}{\|\boldsymbol{b}\|^3}\right) \tag{3.2.8}$$

where $\boldsymbol{s} = (s_1, s_2, s_3)$ and $\boldsymbol{s}^* = (s_1^*, s_2^*, s_3^*)$. Hence

$$\hat{c}_0 = \cos\frac{\hat{\phi}}{2},\ \ \hat{c}_1 = \sin\frac{\hat{\phi}}{2}\hat{s}_1,\ \ \hat{c}_2 = \sin\frac{\hat{\phi}}{2}\hat{s}_2,\ \ \hat{c}_3 = \sin\frac{\hat{\phi}}{2}\hat{s}_2$$

are known as the dual Euler parameters.

3.3 Quaternions

A quaternion is sometimes referred to as a "hyper complex number". Quaternions and dual numbers were combined and generalized to form what is referred to as "Clifford Algebra" as first discussed by Clifford in 1882. A modern text on quaternions is given by Kuipers (1999). Applications to kinematic analysis is discussed by Yang and Blaschke (1960). A comprehensive introduction to dual quaternions is to be found in (McCarthy (1990)), while an abstract treatment is found in (Chevallier (1991)).

Definition 3.3.1. A quaternion Q is defined as a complex number depending on four units $1, i, j, k$:

$$Q = c_0 + c_1 i + c_2 j + c_3 k, \tag{3.3.1}$$

c_i $(i = 0, 1, 2, 3)$ are real numbers called the components of Q. The addition of quaternions is defined by

$$Q + Q' = (c_0 + c_1 i + c_2 j + c_3 k) + (c'_0 + c'_1 i + c'_2 j + c'_3 k)$$

$$= (c_0 + c'_0) + (c_1 + c'_1)i + (c_2 + c'_2)j + (c_3 + c'_3)k. \tag{3.3.2}$$

The multiplication of two quaternions is distributive with respect to summation and is defined by the following rules for the multiplication of the units:

$$1i = i1 = 1, \quad 1j = j1 = j, \quad 1k = k1 = k,$$

$$i^2 = j^2 = k^2 = -1, \tag{3.3.3}$$

$$jk = -kj = i, \quad ki = -ik = j, \quad ij = -ji = k.$$

Hence

$$QQ' = (c_0 + c_1 i + c_2 j + c_3 k)(c'_0 + c'_1 i + c'_2 j + c'_3 k)$$

$$= \quad (c_0c'_0 - c_1c'_1 - c_2c'_2 - c_3c'_3) + (c_0c'_1 + c_1c'_0 + c_2c'_3 - c_3c'_2)i$$

$$+(c_0c'_2 + c_2c'_0 + c_3c'_1 - c_1c'_3)j + (c_0c'_3 + c_3c'_0 + c_1c'_2 - c_2c'_1)k. \tag{3.3.4}$$

From (3.3.3) it follows that the multiplication is not commutative.

If (c_0, c_1, c_2, c_3) is a quaternion Q the conjugate quaternion $\bar{Q}$ is defined by $(c_0, -c_1, -c_2, -c_3)$. From (3.3.4) it follows that $Q\bar{Q} = \bar{Q}Q = c_0^2 + c_1^2 + c_2^2 + c_3^2$, a non-negative number called the norm of Q. If the norm is equal to 1 then Q is called a unit quaternion.

For a quaternion with $c_0 = 0$ the components (c_1, c_2, c_3) may be considered as those of a Euclidean vector; such a quaternion is called a vector quaternion.

Definition 3.3.2. A dual quaternion $\hat{Q}$ can be written as $\hat{Q} = \hat{c}_0 + i\hat{c}_1 + j\hat{c}_2 + k\hat{c}_3$, where $\hat{c}_0$ is the scalar part (dual number), $(\hat{c}_1, \hat{c}_2, \hat{c}_3)$ is the vector part (dual vector), and i, j, k are the usual quaternion units. The dual unit ε commutes with quaternion units, for example $i\varepsilon = \varepsilon i$. A dual quaternion can be also considered as the sum of two ordinary quaternions, $\hat{Q} = Q + \varepsilon Q^*$. Conjugation of a dual quaternion is defined using classical quaternion conjugation: $\bar{\hat{Q}} = \bar{Q} + \varepsilon \bar{Q}^*$.
$\hat{Q}$ is a unit quaternion if $\sum \hat{c}_i^2 = 1$, which implies $\sum {c_i}^2 = 1$, $\sum c_i c_i^* = 0$; $\hat{Q}$ is a vector quaternion if $\hat{c}_0 = 0$, hence c_0=${c_0}^* = 0$.

Just like ordinary quaternions, dual quaternions are also associative, distributive, but not commutative.

3.4 Euler Parameters

Rotations in real space can be identified by assembling the Euler parameters c_0, c_1, c_2, c_3 of a rotation into the quaternion

$$Z = c_0 + c_1 i + c_2 j + c_3 k$$

or explicitly

$$Z = \cos\frac{\phi}{2} + s_1 \sin\frac{\phi}{2} i + s_2 \sin\frac{\phi}{2} j + s_3 \sin\frac{\phi}{2} k.$$

On the other hand a vector $\boldsymbol{x} = (x, y, z) \in \mathbb{R}^3$ is defined as the vector quaternion

$$\boldsymbol{x} = xi + yj + zk.$$

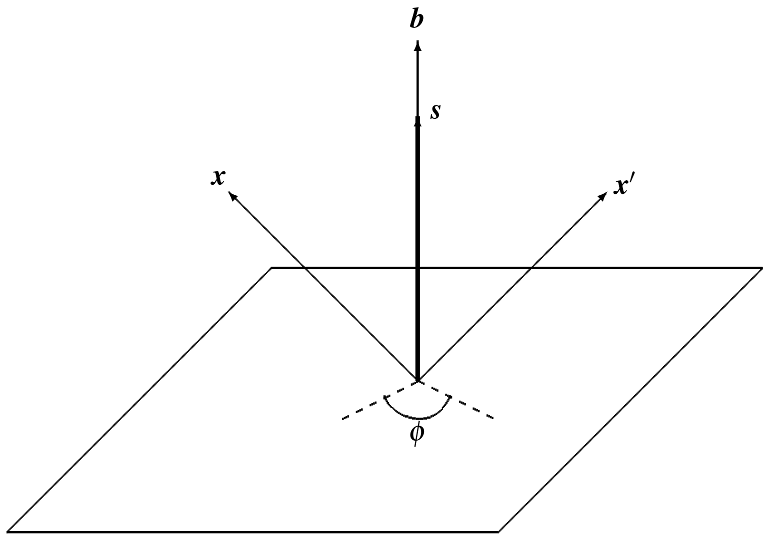

Figure 3.2 The rotation of $\boldsymbol{x}$.

The rotation is now given by the quaternion equation

$$\boldsymbol{x}' = Z\boldsymbol{x}\bar{Z}$$

where

$$\bar{Z} = c_0 - c_1 i - c_2 j - c_3 k$$

is the conjugate of Z.

A spatial displacement can be identified by a coordinate transformation $[T]$ in terms of a rotation matrix $[A]$ and a distance d, $[T] = [A, d]$. This coordinate transformation can be represented by a dual quaternion

$$\hat{Z} = \cos\frac{\hat{\phi}}{2} + \hat{s}_1 \sin\frac{\hat{\phi}}{2} i + \hat{s}_2 \sin\frac{\hat{\phi}}{2} j + \hat{s}_3 \sin\frac{\hat{\phi}}{2} k.$$

The dual quaternion $\hat{Z}$ is sum of the real Z and Z^* components where Z is the quaternion obtained from rotation A and Z^* is the quaternion obtained from

$$Z^* = \frac{1}{2} DZ$$

where D is the quaternion, $D = d_1 i + d_2 j + d_3 k$, formed from the translation vector $\boldsymbol{d} = (d_1, d_2, d_3)$.

The components of the dual quaternion $\hat{Z}$ are known as the dual Euler parameters of the spatial displacement. Using the dual Euler parameters, we can represent the dual orthogonal matrix $\hat{A}$ by

$$[\hat{A}] = I + 2\sin\frac{\hat{\phi}}{2}\cos\frac{\hat{\phi}}{2}[\hat{S}] + 2\sin^2\frac{\hat{\phi}}{2}[\hat{S}^2]$$

If we identify a screw $\boldsymbol{w} = (\boldsymbol{w}, \boldsymbol{v})$ where $\boldsymbol{w}$ is the angular velocity and $\boldsymbol{v}$ is the linear velocity by a dual vector $\hat{\boldsymbol{w}} = \boldsymbol{w} + \varepsilon\boldsymbol{v}$ as the dual quaternion $\hat{\boldsymbol{w}} = (w_1 + \varepsilon v_1)i + (w_2 + \varepsilon v_2)j + (w_3 + \varepsilon v_3)k$ in above transformation $[T]$ then we get the final screw $\hat{\boldsymbol{w}}' = (\boldsymbol{w}', \boldsymbol{v}')$ (where $\boldsymbol{w}'$ is the transformed angular velocity and $\boldsymbol{v}'$ is the transformed linear velocity) which is defined as by $\hat{\boldsymbol{w}}' = \boldsymbol{w}' + \varepsilon\boldsymbol{v}'$

$$\hat{\boldsymbol{w}}' = \hat{Z}\hat{\boldsymbol{w}}\hat{\hat{Z}}$$

where $\hat{\hat{Z}}$ is the conjugate of $\hat{Z}$.

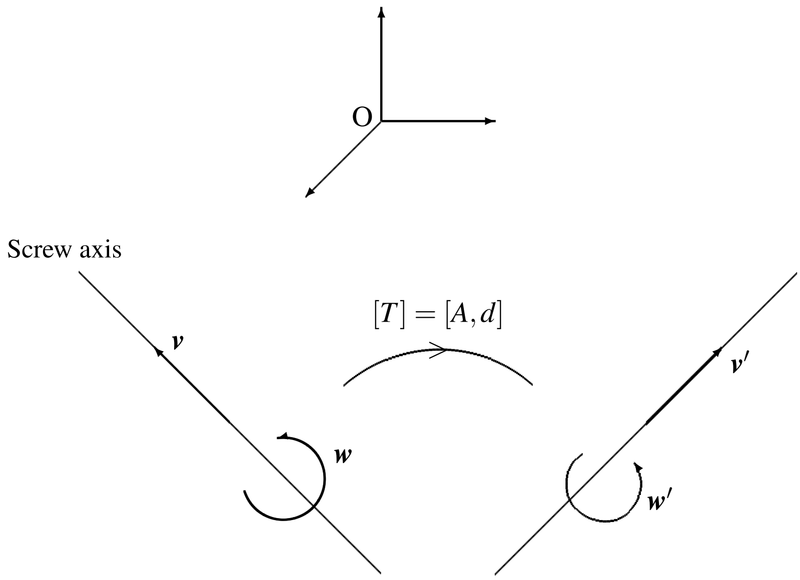

Figure 3.3 Screw transformation

In this chapter, using the E. Study mapping we transfer the motion in $\mathbb{R}^3$ space on the Dual Unit Sphere. Instead of the transformation matrix $[T]$ in real space we use the corresponding dual rotation matrix $\hat{A}$ on the D.U.S. and using the Cayley mapping we find dual Rodrigues parameters and the dual Euler parameters. The dual Euler parameters here are obtained by using the dual Rodrigues vector $\hat{b}$ and the rotation angle $\hat{\phi}$. Then we investigate the results of the dual Euler parameters in the real space.

Defining $\hat{Z} = \hat{c}_0 + \hat{c}_1 i + \hat{c}_2 j + \hat{c}_3 k$ with the dual Euler parameters and the screw in the corresponding spatial displacement by $\hat{\boldsymbol{w}} = \boldsymbol{w} + \varepsilon \boldsymbol{v}$, the final screw (or the transformed screw) $\hat{\boldsymbol{w}}' = \boldsymbol{w}' + \varepsilon \boldsymbol{v}'$ is obtained again by $\hat{\boldsymbol{w}}' = \hat{Z}\hat{\boldsymbol{w}}\hat{\bar{Z}}$ but transformed screw has the coordinates depending on the dual Rodrigues vector $\hat{\boldsymbol{b}}$ and the dual angle $\hat{\phi}$ of the corresponding dual spherical motion.

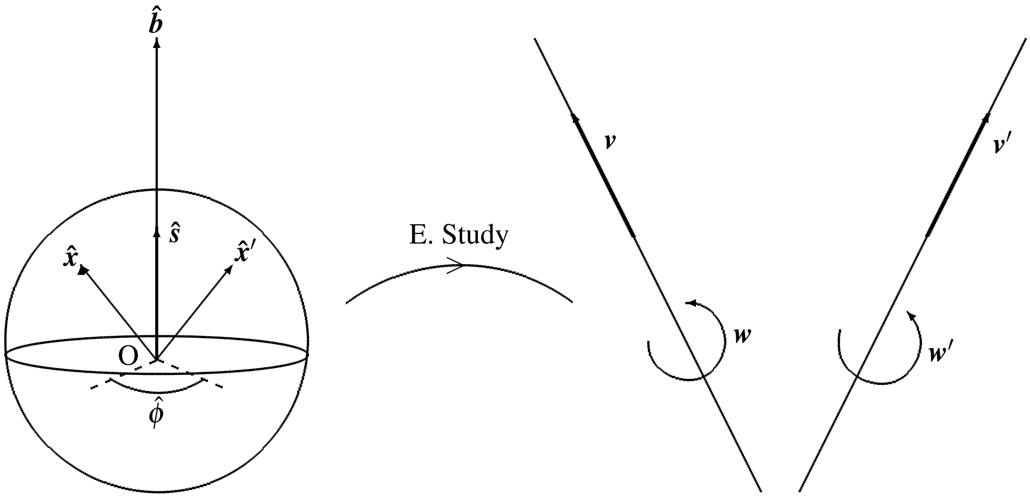

Figure 3.4 The relation between the rotation of D.U.S. and the screw transformation.

$$
\begin{aligned}
\hat{c}_i &= c_i + \varepsilon c_i{}^* = \sin\frac{\hat{\phi}}{2}\,\hat{s}_i = \sin\frac{\hat{\phi}}{2}\,\frac{\hat{b}_i}{\|\boldsymbol{b}\|} \\
&= \left(\sin\frac{\phi}{2} + \varepsilon\,\frac{\phi^*}{2}\cos\frac{\phi}{2}\right)\left(\frac{b_i}{\|\boldsymbol{b}\|} + \varepsilon\left(\frac{b_i{}^*}{\|\boldsymbol{b}\|} - \frac{b_i\,(\boldsymbol{b}\boldsymbol{b}^*)}{\|\boldsymbol{b}\|^3}\right)\right) \\
&= \frac{b_i}{\|\boldsymbol{b}\|}\sin\frac{\phi}{2} + \varepsilon\left(\frac{\phi^*}{2}\cos\frac{\phi}{2}\,\frac{b_i}{\|\boldsymbol{b}\|} + \sin\frac{\phi}{2}\left(\frac{b_i{}^*}{\|\boldsymbol{b}\|} - \frac{b_i(\boldsymbol{b}\boldsymbol{b}^*)}{\|\boldsymbol{b}\|^3}\right)\right) \\
&= \frac{b_i}{\tan\frac{\phi}{2}}\sin\frac{\phi}{2} + \varepsilon\left(\frac{\phi^*}{2}\cos\frac{\phi}{2}\,\frac{b_i}{\tan\frac{\phi}{2}} + \sin\frac{\phi}{2}\left(\frac{b_i{}^*}{\tan\frac{\phi}{2}} - \frac{b_i(\boldsymbol{b}\boldsymbol{b}^*)}{\tan^3\frac{\phi}{2}}\right)\right)
\end{aligned}
$$

$$= \quad b_i \cos\frac{\phi}{2} + \varepsilon\left(\frac{\phi^*}{2}\cos\frac{\phi}{2}\cot\frac{\phi}{2}\, b_i + \cos\frac{\phi}{2}\, b_i{}^* - \frac{(\cos^3\frac{\phi}{2})\, b_i(\boldsymbol{bb}^*)}{\sin^2\frac{\phi}{2}}\right)$$

$$= \quad b_i \cos\frac{\phi}{2} + \varepsilon\left(\frac{\phi^*}{2}\cos\frac{\phi}{2}\cot\frac{\phi}{2}\, b_i + \cos\frac{\phi}{2}\, b_i{}^* - \cos\frac{\phi}{2}\cot^2\frac{\phi}{2}\, b_i(\boldsymbol{bb}^*)\right)$$

$$= \quad b_i \cos\frac{\phi}{2} + \varepsilon\cos\frac{\phi}{2}\left(b_i\ \cot\frac{\phi}{2}\left(\frac{\phi^*}{2} - b_i(\boldsymbol{bb}^*)\cot\frac{\phi}{2}\right) + b_i{}^*\right), \quad i = 1,2,3.$$

Using (3.2.5) and (3.2.6) we have

$$\boldsymbol{bb}^* = \frac{\phi^*}{2}\tan\frac{\phi}{2}\left(1+\tan^2\frac{\phi}{2}\right) \tag{3.4.1}$$

Hence we have from (3.4.1)

$$\begin{aligned}
\hat{c}_i \quad &= \quad c_i + \varepsilon c_i{}^* \\
&= \quad b_i \cos\frac{\phi}{2} + \varepsilon\cos\frac{\phi}{2}\left(b_i\ \cot\frac{\phi}{2}\left(\frac{\phi^*}{2} - \left(\frac{\phi^*}{2}\tan\frac{\phi}{2}\left(1+\tan^2\frac{\phi}{2}\right)\right)\cot\frac{\phi}{2}\right) + b_i{}^*\right) \\
&= \quad b_i \cos\frac{\phi}{2} + \varepsilon\cos\frac{\phi}{2}\left(b_i\ \cot\frac{\phi}{2}\left(\frac{\phi^*}{2} - \frac{\phi^*}{2}\left(1+\tan^2\frac{\phi}{2}\right)\right) + b_i{}^*\right) \\
&= \quad b_i \cos\frac{\phi}{2} + \varepsilon\cos\frac{\phi}{2}\left(b_i\ \cot\frac{\phi}{2}\frac{\phi^*}{2}\left(1-1-\tan^2\frac{\phi}{2}\right) + b_i{}^*\right) \\
&= \quad b_i \cos\frac{\phi}{2} + \varepsilon\cos\frac{\phi}{2}\left(b_i{}^* - b_i\frac{\phi^*}{2}\tan\frac{\phi}{2}\right) \\
&= \quad b_i \cos\frac{\phi}{2} + \varepsilon\left(b_i{}^*\cos\frac{\phi}{2} - b_i\frac{\phi^*}{2}\sin\frac{\phi}{2}\right), \quad i = 1,2,3.
\end{aligned}$$

$$\hat{Z}\hat{\boldsymbol{w}}\bar{\hat{Z}} = \hat{\boldsymbol{w}}' \tag{3.4.2}$$

$$\begin{aligned}
\hat{Z}\hat{\boldsymbol{w}}\bar{Z} \quad = \quad & (\hat{c}_0 + \hat{c}_1 i + \hat{c}_2 j + \hat{c}_3 k)\Big((w_1+\varepsilon v_1)i + (w_2+\varepsilon v_2)j + (w_3+\varepsilon v_3)k\Big) \\
& (\hat{c}_0 - \hat{c}_1 i - \hat{c}_2 j - \hat{c}_3 k)
\end{aligned}$$

$$
\begin{aligned}
= \;& (\hat{c}_0+\hat{c}_1 i+\hat{c}_2 j+\hat{c}_3 k)\Big\{\Big((c_1+\varepsilon {c_1}^*)(w_1+\varepsilon v_1)+(c_2+\varepsilon {c_2}^*)(w_2+\varepsilon v_2)\\
& +(c_3+\varepsilon {c_3}^*)(w_3+\varepsilon v_3)\Big)+\Big((c_0+\varepsilon {c_0}^*)(w_1+\varepsilon v_1)-(c_3+\varepsilon {c_3}^*)(w_2+\varepsilon v_2)\\
& +(c_2+\varepsilon {c_2}^*)(w_3+\varepsilon v_3)\Big)i+\Big((c_0+\varepsilon {c_0}^*)(w_2+\varepsilon v_2)-(c_1+\varepsilon {c_1}^*)(w_3+\varepsilon v_3)\\
& +(c_3+\varepsilon {c_3}^*)(w_1+\varepsilon v_1)\Big)j+\Big((c_0+\varepsilon {c_0}^*)(w_3+\varepsilon v_3)-(c_2+\varepsilon {c_2}^*)(w_1+\varepsilon v_1)\\
& +(c_1+\varepsilon {c_1}^*)(w_2+\varepsilon v_2)\Big)k\Big\}\\
= \;& \Big\{\hat{c}_0\Big(\hat{c}_1\hat{w}_1+\hat{c}_2\hat{w}_2+\hat{c}_3\hat{w}_3\Big)-\hat{c}_1\Big(\hat{c}_0\hat{w}_1-\hat{c}_3\hat{w}_2+\hat{c}_2\hat{w}_3\Big)\\
& -\hat{c}_2\Big(\hat{c}_0\hat{w}_2-\hat{c}_1\hat{w}_3+\hat{c}_3\hat{w}_1\Big)-\hat{c}_3\Big(\hat{c}_0\hat{w}_3-\hat{c}_2\hat{w}_1+\hat{c}_1\hat{w}_2\Big)\Big\}\\
& +\Big\{\hat{c}_1\Big(\hat{c}_1\hat{w}_1+\hat{c}_2\hat{w}_2+\hat{c}_3\hat{w}_3\Big)+\hat{c}_0\Big(\hat{c}_0\hat{w}_1-\hat{c}_3\hat{w}_2+\hat{c}_2\hat{w}_3\Big)\\
& +\hat{c}_2\Big(\hat{c}_0\hat{w}_3-\hat{c}_2\hat{w}_1+\hat{c}_1\hat{w}_2\Big)-\hat{c}_3\Big(\hat{c}_0\hat{w}_2+\hat{c}_1+\hat{w}_3+\hat{c}_3\hat{w}_1\Big)\Big\}i\\
& +\Big\{\hat{c}_2\Big(\hat{c}_1\hat{w}_1-\hat{c}_2\hat{w}_2+\hat{c}_3\hat{w}_3\Big)+\hat{c}_0\Big(\hat{c}_0\hat{w}_2-\hat{c}_1\hat{w}_3+\hat{c}_3\hat{w}_1\Big)\\
& +\hat{c}_3\Big(\hat{c}_0\hat{w}_1-\hat{c}_3\hat{w}_2+\hat{c}_2\hat{w}_3\Big)-\hat{c}_1\Big(\hat{c}_0\hat{w}_3-\hat{c}_2\hat{w}_1+\hat{c}_1\hat{w}_2\Big)\Big\}j\\
& +\Big\{\hat{c}_3\Big(\hat{c}_1\hat{w}_1+\hat{c}_2\hat{w}_2+\hat{c}_3\hat{w}_3\Big)+\hat{c}_0\Big(\hat{c}_0\hat{w}_3-\hat{c}_2\hat{w}_1+\hat{c}_1\hat{w}_2\Big)\\
& +\hat{c}_1\Big(\hat{c}_0\hat{w}_2-\hat{c}_1\hat{w}_3+\hat{c}_3\hat{w}_1\Big)-\hat{c}_2\Big(\hat{c}_0\hat{w}_1-\hat{c}_3\hat{w}_2+\hat{c}_2\hat{w}_3\Big)\Big\}k\\
\\
= \;& \Big\{\Big(\hat{c}_0^2+\hat{c}_1^2+\hat{c}_2^2+\hat{c}_3^2\Big)\hat{w}_1+\Big(2\hat{c}_1\hat{c}_2-2\hat{c}_0\hat{c}_3\Big)\hat{w}_2+\Big(2\hat{c}_1\hat{c}_3-2\hat{c}_0\hat{c}_2\Big)\hat{w}_3\Big\}i\\
& +\Big\{\Big(2\hat{c}_1\hat{c}_2+2\hat{c}_0\hat{c}_3\Big)\hat{w}_1+\Big(\hat{c}_0^2-\hat{c}_1^2+\hat{c}_2^2-\hat{c}_3^2\Big)\hat{w}_2+\Big(2\hat{c}_2\hat{c}_3-2\hat{c}_0\hat{c}_1\Big)\hat{w}_3\Big\}j\\
& +\Big\{\Big(2\hat{c}_1\hat{c}_3-2\hat{c}_0\hat{c}_2\Big)\hat{w}_1+\Big(2\hat{c}_2\hat{c}_3+2\hat{c}_0\hat{c}_1\Big)\hat{w}_2+\Big(\hat{c}_0^2-\hat{c}_1^2-\hat{c}_2^2+\hat{c}_3^2\Big)\hat{w}_3\Big\}k\\
\\
= \;& \hat{w}'_1 i+\hat{w}'_2 j+\hat{w}'_3 k \;\; where \;\; \hat{w}'_i=w'_i+\varepsilon v'_i\,, \quad i=1,2,3.
\end{aligned}
$$

Transformed angular velocity : $\boldsymbol{w}'=(w'_1,w'_2,w'_3)$ and Transformed linear velocity : $\boldsymbol{v}'=(v'_1,v'_2,v'_3)$ where $\hat{\boldsymbol{w}}'=\boldsymbol{w}'+\varepsilon\boldsymbol{v}'$. Hence using

$$\hat{c}_0=c_0+\varepsilon {c_0}^*=\cos\frac{\hat{\phi}}{2}=\cos\frac{\phi}{2}-\varepsilon\frac{\phi^*}{2}\sin\frac{\phi}{2},$$

$$\hat{c}_1=c_1+\varepsilon {c_1}^*=b_1\cos\frac{\phi}{2}+\varepsilon\left({b_1}^*\cos\frac{\phi}{2}-b_1\frac{\phi^*}{2}\sin\frac{\phi}{2}\right),$$

$$\hat{c}_2 = c_2 + \varepsilon c_2{}^* = b_2 \cos\frac{\phi}{2} + \varepsilon\left(b_2{}^* \cos\frac{\phi}{2} - b_2\frac{\phi^*}{2}\sin\frac{\phi}{2}\right),$$

$$\hat{c}_3 = c_3 + \varepsilon c_3{}^* = b_3 \cos\frac{\phi}{2} + \varepsilon\left(b_3{}^* \cos\frac{\phi}{2} - b_3\frac{\phi^*}{2}\sin\frac{\phi}{2}\right),$$

we get the angular velocity $\boldsymbol{w}'$ as,

$$w_1' = \cos^2\frac{\phi}{2}\Big(w_1(1+b_1^2-b_2^2-b_3^2)+(2b_2+2b_1b_3)w_3+(-2b_3+2b_1b_2)w_2\Big),$$

$$w_2' = \cos^2\frac{\phi}{2}\Big(w_2(1+b_2^2-b_1^2-b_3^2)+(2b_3+2b_1b_2)w_1+(-2b_1+2b_1b_3)w_3\Big),$$

$$w_3' = \cos^2\frac{\phi}{2}\Big(w_3(1+b_3^2-b_1^2-b_2^2)+(2b_1+2b_2b_3)w_3+(-2b_2+2b_1b_3)w_1\Big).$$

Similarly we obtain the linear velocity $\boldsymbol{v}'$ as,

$$\begin{aligned}
v_1' \;=\;& w_1\left(2\cos\frac{\phi}{2}\left(-\frac{\phi^*}{2}\sin\frac{\phi}{2}+b_1c_1^*-b_2c_2^*-b_3c_3^*\right)\right)+v_1\left(\cos^2\frac{\phi}{2}\left(1+b_1^2-b_2^2-b_3^2\right)\right)\\
&+w_2\left(2\cos\frac{\phi}{2}\left(b_1c_2^*-b_2c_1^*-c_3^*+\frac{\phi^*}{2}b_3\sin\frac{\phi}{2}\right)\right)+v_2\left(2\cos^2\frac{\phi}{2}\left(b_1b_2-b_3\right)\right)\\
&+w_3\left(2\cos\frac{\phi}{2}\left(b_1c_3^*+b_3c_1^*+c_2^*-\frac{\phi^*}{2}b_2\sin\frac{\phi}{2}\right)\right)+v_3\left(2\cos^2\frac{\phi}{2}\left(b_1b_3+b_2\right)\right),
\end{aligned}$$

$$\begin{aligned}
v_2' \;=\;& w_1\left(2\cos\frac{\phi}{2}\left(b_1c_2^*+b_2c_1^*+c_3^*-\frac{\phi^*}{2}b_3\sin\frac{\phi}{2}\right)\right)+v_1\left(2\cos^2\frac{\phi}{2}\left(b_1b_2+b_3\right)\right)\\
&+w_2\left(2\cos\frac{\phi}{2}\left(-\frac{\phi^*}{2}\sin\frac{\phi}{2}-b_1c_1^*+b_2c_2^*-b_3c_3^*\right)\right)+v_2\left(\cos^2\frac{\phi}{2}\left(1-b_1^2+b_2^2-b_3^2\right)\right)\\
&+w_3\left(2\cos\frac{\phi}{2}\left(b_2c_3^*+b_3c_2^*-c_1^*+\frac{\phi^*}{2}b_1\sin\frac{\phi}{2}\right)\right)+v_3\left(2\cos^2\frac{\phi}{2}\left(b_2b_3-b_1\right)\right),
\end{aligned}$$

$$\begin{aligned}
v_3' \;=\;& w_1\left(2\cos\frac{\phi}{2}\left(b_1c_3^*+b_3c_1^*-c_2^*+\frac{\phi^*}{2}b_2\sin\frac{\phi}{2}\right)\right)+v_1\left(2\cos^2\frac{\phi}{2}\left(b_1b_3-b_2\right)\right)\\
&+w_2\left(2\cos\frac{\phi}{2}\left(b_2c_3^*+b_3c_2^*+c_1^*-\frac{\phi^*}{2}b_1\sin\frac{\phi}{2}\right)\right)+v_2\left(2\cos^2\frac{\phi}{2}\left(b_2b_3+b_1\right)\right)\\
&+w_3\left(2\cos\frac{\phi}{2}\left(-\frac{\phi^*}{2}\sin\frac{\phi}{2}-b_1c_1^*-b_2c_2^*+b_3c_3^*\right)\right)+v_3\left(\cos^2\frac{\phi}{2}\left(1-b_1^2-b_2^2+b_3^2\right)\right)
\end{aligned}$$

where

$$c_i* = b_i^* \cos\frac{\phi}{2} - b_i\frac{\phi^*}{2}\sin\frac{\phi}{2}.$$

3.5 Conclusion

The rotation of the D.U.S. is given by a dual orthogonal matrix $\hat{A}$. Using the Cayley Mapping we obtain the skew symmetric matrix $\hat{B}$ from $\hat{A}$. The components of $\hat{B}$ determines the dual Rodrigues vector $\hat{\boldsymbol{b}}$ and the components of $\hat{\boldsymbol{b}}$, that is, $\hat{b}_1, \hat{b}_2, \hat{b}_3$ are called the dual Rodrigues parameters. The dual Euler parameters $\hat{c}_0, \hat{c}_1, \hat{c}_2, \hat{c}_3$ are obtained from the dual Rodrigues parameters. The dual quaternion $\hat{Z}$ is obtained from the dual Euler parameters. The transformation in $\mathbb{R}^3$ space, which corresponds to the rotation of the D.U.S., provides the transformation of corresponding screws by the formula $\hat{\boldsymbol{w}}' = \hat{\boldsymbol{Z}}\hat{\boldsymbol{w}}\hat{\bar{\boldsymbol{Z}}}$.

As a result we obtain the coordinates of the transformed screw $\hat{\boldsymbol{w}}'$ in terms of the dual Rodrigues parameters of the corresponding dual rotation and the initial screw $\hat{\boldsymbol{w}}$.

CHAPTER FOUR
THE EXPONENTIAL MAPPING

4.1 The Dual Matrix Exponential

The exponential mapping is an alternative method for finding a relation between the rotation matrices and the skew symmetric matrices (Mampetta (Spring 2006)). In this chapter we examine the exponential mapping from $so(3)\times\mathbb{D}$ (the set of dual 3×3 skew symmetric matrices) to $SO(3)\times\mathbb{D}$ (the set of dual 3×3 orthogonal or rotation matrices) (Park (1994)) and (Selig (2004)). Using logarithm function we obtain the skew symmetric matrix $\hat{B}$ from the orthogonal matrix $\hat{A}$ as in the case of Cayley mapping.

The direct calculation shows that a 3×3 skew symmetric dual matrix (Gallier & Xu (2002))

$$\hat{B}=\begin{pmatrix} 0 & -\hat{b}_3 & \hat{b}_2 \\ \hat{b}_3 & 0 & -\hat{b}_1 \\ -\hat{b}_2 & \hat{b}_1 & 0 \end{pmatrix}$$

satisfies a cubic equation

$$\hat{B}^3+\hat{\theta}^2\hat{B}=0,$$

where $\hat{\theta}^2=\hat{b}_1^2+\hat{b}_2^2+\hat{b}_3^2,\ \hat{B}=B+\varepsilon B^*,\ \hat{\theta}=\theta+\varepsilon\theta^*,\ \hat{b}_i=b_i+\varepsilon b_i{}^*\ \ i=1,2,3.$

$$det(\hat{B}-\hat{\lambda}I)=0\Rightarrow\begin{vmatrix} -\hat{\lambda} & -\hat{b}_3 & \hat{b}_2 \\ \hat{b}_3 & -\hat{\lambda} & -\hat{b}_1 \\ -\hat{b}_2 & \hat{b}_1 & -\hat{\lambda} \end{vmatrix}=0$$

$$-\hat{\lambda}(\hat{\lambda}+\hat{b}_1^2)+\hat{b}_3(-\hat{\lambda}\hat{b}_3-\hat{b}_1\hat{b}_2)+\hat{b}_2(\hat{b}_3\hat{b}_1-\hat{\lambda}\hat{b}_2)=0$$

$$-\hat{\lambda}^3-\hat{\lambda}\hat{b}_1^2-\hat{\lambda}\hat{b}_3^2-\hat{b}_1\hat{b}_2\hat{b}_3+\hat{b}_2\hat{b}_3\hat{b}_1-\hat{\lambda}\hat{b}_2^2=0$$

$$-\hat{\lambda}^3-\hat{\lambda}(\hat{b}_1^2+\hat{b}_2^2+\hat{b}_3^2)=0$$

$$\hat{\lambda}^3+\hat{\lambda}(\hat{b}_1^2+\hat{b}_2^2+\hat{b}_3^2)=0$$

$$\Rightarrow\ \hat{B}^3+\hat{B}(\hat{b}_1^2+\hat{b}_2^2+\hat{b}_3^2)=0$$

$$\Rightarrow \ \hat{B}^3 + \hat{B}\hat{\theta}^2 = 0 \quad where \ \ \hat{\theta}^2 = \hat{b}_1^2 + \hat{b}_2^2 + \hat{b}_3^2.$$

A systematic approach will be developed to find the exponential in $so(3) \times \mathbb{D}$. This involves writing the skew symmetric matrix as a sum of mutually annihilating idempotents (Selig (2005)).

Consider the three matrices

$$\begin{aligned} \hat{P}_0 &= \frac{1}{\hat{\theta}^2}(\hat{B} - i\hat{\theta}I_3)(\hat{B} + i\hat{\theta}I_3) \\ &= \frac{1}{\hat{\theta}^2}(\hat{B}^2 + \hat{\theta}^2 I_3) \ = \ \frac{1}{\hat{\theta}^2}\hat{B}^2 + I_3 \end{aligned}$$

$$\hat{P}_+ \ = \ \frac{-1}{2\hat{\theta}^2}\hat{B}(\hat{B} - i\hat{\theta}I_3) \ = \ \frac{-1}{2\hat{\theta}^2}\hat{B}^2 + \frac{i}{2\hat{\theta}}\hat{B}$$

$$\hat{P}_- \ = \ \frac{-1}{2\hat{\theta}^2}\hat{B}(\hat{B} + i\hat{\theta}I_3) \ = \ \frac{-1}{2\hat{\theta}^2}\hat{B}^2 - \frac{i}{2\hat{\theta}}\hat{B}$$

$$\hat{P}_0 \ = \ \frac{1}{\hat{\theta}^2}(\hat{B} - i\hat{\theta}I_3)(\hat{B} + i\hat{\theta}I_3)$$

$$\hat{B} + i\hat{\theta}I_3 \ = \ \begin{pmatrix} 0 & -\hat{b}_3 & \hat{b}_2 \\ \hat{b}_3 & 0 & -\hat{b}_1 \\ -\hat{b}_2 & \hat{b}_1 & 0 \end{pmatrix} + i \begin{pmatrix} \hat{\theta} & 0 & 0 \\ 0 & \hat{\theta} & 0 \\ 0 & 0 & \hat{\theta} \end{pmatrix} = \begin{pmatrix} i\hat{\theta} & -\hat{b}_3 & \hat{b}_2 \\ \hat{b}_3 & i\hat{\theta} & -\hat{b}_1 \\ -\hat{b}_2 & \hat{b}_1 & i\hat{\theta} \end{pmatrix}$$

Similarly,

$$\hat{B} - i\hat{\theta}I_3 \ = \ \begin{pmatrix} -i\hat{\theta} & -\hat{b}_3 & \hat{b}_2 \\ \hat{b}_3 & -i\hat{\theta} & -\hat{b}_1 \\ -\hat{b}_2 & \hat{b}_1 & -i\hat{\theta} \end{pmatrix}.$$

$$\begin{aligned} \hat{P}_0 &= \frac{1}{\hat{\theta}^2}(\hat{B} + i\hat{\theta}I_3)(\hat{B} - i\hat{\theta}I_3) \\ &= \frac{1}{\hat{\theta}^2} \begin{pmatrix} i\hat{\theta} & -\hat{b}_3 & \hat{b}_2 \\ \hat{b}_3 & i\hat{\theta} & -\hat{b}_1 \\ -\hat{b}_2 & \hat{b}_1 & i\hat{\theta} \end{pmatrix} \begin{pmatrix} -i\hat{\theta} & -\hat{b}_3 & \hat{b}_2 \\ \hat{b}_3 & -i\hat{\theta} & -\hat{b}_1 \\ -\hat{b}_2 & \hat{b}_1 & -i\hat{\theta} \end{pmatrix} \end{aligned}$$

$$= \frac{1}{\hat{\theta}^2}\begin{pmatrix} -i^2\hat{\theta}^2-\hat{b}_3^2-\hat{b}_2^2 & -i\hat{b}_3\hat{\theta}+i\hat{\theta}\hat{b}_3+\hat{b}_1\hat{b}_2 & i\hat{b}_2\hat{\theta}+\hat{b_1b_3}-i\hat{\theta}\hat{b}_2 \\ -i\hat{\theta}\hat{b}_3+i\hat{\theta}\hat{b}_3+\hat{b}_1\hat{b}_2 & -\hat{b}_3^2-i^2\hat{\theta}^2-\hat{b}_1^2 & \hat{b}_2\hat{b}_3-i\hat{\theta}\hat{b}_1+\hat{b}_1 i\hat{\theta} \\ i\hat{\theta}\hat{b}_2+\hat{b}_3\hat{b}_1-\hat{b}_2 i\hat{\theta} & \hat{b}_2\hat{b}_3-i\hat{\theta}\hat{b}_1+\hat{b}_1 i\hat{\theta} & -\hat{b}_2^2-\hat{b}_1^2-i^2\hat{\theta}^2 \end{pmatrix}$$

$$= \frac{1}{\hat{\theta}^2}\begin{pmatrix} \hat{\theta}^2-\hat{b}_3^2-\hat{b}_2^2 & \hat{b}_1\hat{b_2} & \hat{b}_1\hat{b}_3 \\ \hat{b}_1\hat{b}_2 & \hat{\theta}^2-\hat{b}_1^2-\hat{b}_3^2 & \hat{b}_2\hat{b}_3 \\ \hat{b}_3\hat{b}_1 & \hat{b}_2\hat{b}_3 & -\hat{b}_2^2-\hat{b}_1^2+\hat{\theta}^2 \end{pmatrix}$$

$$= \begin{pmatrix} 1+\varepsilon.0 & 0 & 0 \\ 0 & 1+\varepsilon.0 & 0 \\ 0 & 0 & 1+\varepsilon.0 \end{pmatrix} + \frac{1}{\hat{\theta}^2}\begin{pmatrix} -\hat{b}_3^2-\hat{b}_2^2 & \hat{b}_1\hat{b_2} & \hat{b}_1\hat{b}_3 \\ \hat{b}_1\hat{b}_2 & -\hat{b}_1^2-\hat{b}_3^2 & \hat{b}_2\hat{b}_3 \\ \hat{b}_3\hat{b}_1 & \hat{b}_2\hat{b}_3 & -\hat{b}_2^2-\hat{b}_1^2 \end{pmatrix}$$

$$= \hat{I}_3+\frac{1}{\hat{\theta}^2}\hat{B}^2$$

since

$$\hat{B}^2 = \begin{pmatrix} 0 & -\hat{b}_3 & \hat{b}_2 \\ \hat{b}_3 & 0 & -\hat{b}_1 \\ -\hat{b}_2 & \hat{b}_1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -\hat{b}_3 & \hat{b}_2 \\ \hat{b}_3 & 0 & -\hat{b}_1 \\ -\hat{b}_2 & \hat{b}_1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -\hat{b}_3^2-\hat{b}_2^2 & \hat{b}_1\hat{b}_2 & \hat{b}_1\hat{b}_3 \\ \hat{b}_1\hat{b}_2 & -\hat{b}_1^2-\hat{b}_3^2 & \hat{b}_2\hat{b}_3 \\ \hat{b}_3\hat{b}_1 & \hat{b}_2\hat{b}_3 & -\hat{b}_2^2-\hat{b}_1^2 \end{pmatrix}$$

Similarly, $\hat{P}_+$ and $\hat{P}_-$ are hold.

It is easy to see that these matrices annihilate each other since, for example,

$$\begin{aligned} \hat{P}_0\hat{P}_+ &= \frac{-1}{2\hat{\theta}^4}\hat{B}(\hat{B}+i\hat{\theta}I_3)(\hat{B}-i\hat{\theta}I_3)^2 \\ &= \frac{-1}{2\hat{\theta}^4}(\hat{B}^3+\hat{\theta}^2\hat{B})(\hat{B}-i\hat{\theta}I_3) \\ &= 0 \end{aligned}$$

using the cubic equation satisfied by $\hat{B}$.

In general we have that, $\hat{P}_0\hat{P}_+ = 0$, $\hat{P}_0\hat{P}_- = 0$ and $\hat{P}_+\hat{P}_- = 0$.

These dual annihilating matrices can be found by expanding the reciprocal of the cubic into partial fractions (Sobczyk (1997)). One consequence of this is that the sum of the dual matrices is the identity matrix

$$\hat{P}_0 + \hat{P}_+ + \hat{P}_- = I_3.$$

This can also be checked by direct computation.That is;

$$\begin{aligned}\hat{P}_0 + \hat{P}_+ + \hat{P}_- &= \left(\frac{1}{\hat{\theta}^2}\hat{B}^2 + I_3\right) + \left(\frac{-1}{2\hat{\theta}^2}\hat{B}^2 + \frac{i}{2\hat{\theta}}\hat{B}\right) + \left(\frac{-1}{2\hat{\theta}^2}\hat{B}^2 - \frac{-i}{2\hat{\theta}}\hat{B}\right)\\ &= I_3.\end{aligned}$$

The fact that these dual annihilating matrices are idempotents is now easily proved, for instance,

$$\hat{P}_0 = I_3\hat{P}_0 = (\hat{P}_0 + \hat{P}_+ + \hat{P}_-)\hat{P}_0 = \hat{P}_0^2 + \hat{P}_+\hat{P}_0 + \hat{P}_-\hat{P}_0 = \hat{P}_0^2$$

and in general, $\hat{P}_0^2 = \hat{P}_0$, $\hat{P}_+^2 = \hat{P}_+$ and $\hat{P}_-^2 = \hat{P}_-$.

The final property we need is that a linear combination of the idempotents gives us back $\hat{B}$,

$$\hat{B} = i\hat{\theta}\hat{P}_- - i\hat{\theta}\hat{P}_+.$$

$$\begin{aligned}i\hat{\theta}\hat{P}_- - i\hat{\theta}\hat{P}_+ &= i\hat{\theta}\left(\frac{-1}{2\hat{\theta}^2}\hat{B}^2 + \frac{i}{2\hat{\theta}}\hat{B}\right) - i\hat{\theta}\left(\frac{-1}{2\hat{\theta}^2}\hat{B}^2 + \frac{i}{2\hat{\theta}}\hat{B}\right)\\ &= \frac{-i}{2\hat{\theta}}\hat{B} + \frac{1}{2}\hat{B} + \frac{i}{2\hat{\theta}}\hat{B}^2 + \frac{1}{2}\hat{B}\\ &= \hat{B}.\end{aligned}$$

The point of these manipulations is that if we raise $\hat{B}$ to some power then because the

'$\hat{P}$' matrices are mutually annihilating there are no cross terms. Moreover since the '$\hat{P}$'s are idempotent only their coefficients are effected by the power

$$\hat{B}^n = (i\hat{\theta}\hat{P}_- - i\hat{\theta}\hat{P}_+)^n = (-i\hat{\theta})^n\hat{P}_+ + (i\hat{\theta})^n\hat{P}_-.$$

Hence the exponential of the dual matrix $\hat{B}$ can be found as

$$e^{\hat{B}} = \hat{P}_0 + e^{-i\hat{\theta}}\hat{P}_+ + e^{i\hat{\theta}}\hat{P}_-.$$

That is,

$$\begin{aligned}
e^{\hat{B}} &= I_3 + \hat{B} + \frac{\hat{B}^2}{2!} + \frac{\hat{B}^3}{3!} + \ldots + \frac{\hat{B}^n}{n!} + \ldots \\
&= I_3 + (i\hat{\theta}\hat{P}_- - i\hat{\theta}\hat{P}_+) + \frac{(i\hat{\theta}\hat{P}_- - i\hat{\theta}\hat{P}_+)^2}{2!} + \ldots + \frac{(i\hat{\theta}\hat{P}_- - i\hat{\theta}\hat{P}_+)^n}{n!} + \ldots \\
&= I_3 + i\hat{\theta}\hat{P}_- - i\hat{\theta}\hat{P}_+ + \frac{(i\hat{\theta}\hat{P}_-)^2 - 2i\hat{P}_- i\hat{\theta}\hat{P}_+ + (-i\hat{\theta}\hat{P}_+)^2}{2!} + \ldots \\
&\quad + \frac{(i\hat{\theta})^n\hat{P}_- + (-i\hat{\theta})^n\hat{P}_+}{n!} + \ldots \\
&= I_3 + i\hat{\theta}\hat{P}_- - i\hat{\theta}\hat{P}_+ + \frac{((i\hat{\theta})^2\hat{P}_- + (-i\hat{\theta})^2\hat{P}_+)}{2!} + \ldots + \frac{((i\hat{\theta})^n\hat{P}_- + (-i\hat{\theta})^n\hat{P}_+)}{n!} \\
&= (\hat{P}_0 + \hat{P}_+ + \hat{P}_-) + i\hat{\theta}\hat{P}_- - i\hat{\theta}\hat{P}_+ + \frac{(i\hat{\theta})^2}{2!}\hat{P}_- + \frac{(-i\hat{\theta})^2}{2!}\hat{P}_+ + \ldots \\
&\quad + \frac{(i\hat{\theta})^n}{n!}\hat{P}_- + \frac{(-i\hat{\theta})^n}{n!}\hat{P}_+ + \ldots \\
&= \hat{P}_0 + \hat{P}_+\left(1 - i\hat{\theta} + \frac{(-i\hat{\theta})^2}{2!} + \ldots + \frac{(-i\hat{\theta})^n}{n!} + \ldots\right) \\
&\quad + \hat{P}_-\left(1 + i\hat{\theta} + \frac{(i\hat{\theta})^2}{2!} + \ldots + \frac{(i\hat{\theta})^n}{n!} + \ldots\right) \\
&= \hat{P}_0 + e^{-i\hat{\theta}}\hat{P}_+ + e^{i\hat{\theta}}\hat{P}_-.
\end{aligned}$$

Now we can replace the idempotents by their definitions in terms of $\hat{B}$ to get

$$\begin{aligned} e^{\hat{B}} &= \frac{1}{\hat{\theta}^2}\hat{B}^2 + I_3 + e^{-i\hat{\theta}}\left(-\frac{1}{2\hat{\theta}^2}\hat{B}^2 + \frac{i}{2\hat{\theta}}\hat{B}\right) + e^{i\hat{\theta}}\left(-\frac{1}{2\hat{\theta}^2}\hat{B}^2 - \frac{i}{2\hat{\theta}}\hat{B}\right) \\ &= I_3 + \frac{i}{2\hat{\theta}}(e^{-i\hat{\theta}} - e^{i\hat{\theta}})\hat{B} - \frac{1}{2\hat{\theta}^2}(e^{i\hat{\theta}} + e^{-i\hat{\theta}} - 2)\hat{B}^2. \end{aligned}$$

Finally, replacing the complex exponential by trigonometric functions we have

$$e^{\hat{B}} = I_3 + \frac{1}{\hat{\theta}}\sin\hat{\theta}\hat{B} + \frac{1}{\hat{\theta}^2}(1-\cos\hat{\theta})\hat{B}^2.$$

That is,

$$\begin{aligned} e^{\hat{B}} &= I_3 + \frac{i}{2\hat{\theta}}(e^{-i\hat{\theta}} - e^{i\hat{\theta}})\hat{B} - \frac{1}{2\hat{\theta}^2}(e^{-i\hat{\theta}} + e^{i\hat{\theta}} - 2)\hat{B}^2 \\ &= I_3 + \frac{i}{2\hat{\theta}}(\cos\hat{\theta} - i\sin\hat{\theta} - \cos\hat{\theta} - i\sin\hat{\theta})\hat{B} \\ &\quad - \frac{1}{2\hat{\theta}^2}(\cos\hat{\theta} - i\sin\hat{\theta} + \cos\hat{\theta} + i\sin\hat{\theta} - 2)\hat{B}^2 \\ &= I_3 + \frac{i}{2\hat{\theta}}(-2i\sin\hat{\theta})\hat{B} - \frac{1}{2\hat{\theta}^2}(2\cos\hat{\theta} - 2)\hat{B}^2 \\ &= I_3 + \frac{1}{\hat{\theta}}\sin\hat{\theta}\hat{B} + \frac{1}{\hat{\theta}^2}(1-\cos\hat{\theta})\hat{B}^2 \end{aligned}$$

since $e^{i\hat{\theta}} = \cos\hat{\theta} + i\sin\hat{\theta}$ and $e^{-i\hat{\theta}} = \cos\hat{\theta} - i\sin\hat{\theta}$.

The inverse function, the logarithm, is not hard to find. Suppose that we are given an arbitrary 3×3 special orthogonal dual matrix, that is, an element of $SO(3)$. $\hat{A}$ say. We can find the angle $\hat{\theta}$ and the anti-symmetric matrix $\hat{B}$ as follows. Notice that $Tr(\hat{I}_3) = 3$, $Tr(\hat{B}) = 0$ and $Tr(\hat{B}^2) = -2\hat{\theta}^2$. Comparing $\hat{A}$ with the exponential of a Lie algebra element, we have

$$\hat{A} = e^{\hat{B}} = I_3 + \frac{1}{\hat{\theta}}\sin\hat{\theta}\hat{B} + \frac{1}{\hat{\theta}^2}(1-\cos\hat{\theta})\hat{B}^2,$$

so the trace of $\hat{A}$ gives

$$\begin{aligned} Tr(\hat{A}) &= Tr(I_3) + \frac{1}{\hat{\theta}} \sin\hat{\theta}\ Tr(\hat{B}) + \frac{1}{\hat{\theta}^2}(1-\cos\hat{\theta})\ Tr(\hat{B}^2) \\ &= 3 + \frac{1}{\hat{\theta}}.0 + \frac{1}{\hat{\theta}^2}(1-\cos\hat{\theta})(-2\hat{\theta}^2) \\ &= 3 - 2 + 2\cos\hat{\theta} \\ &= 1 + 2\cos\hat{\theta}. \end{aligned}$$

Then we have

$$\hat{\theta} = \arccos\Big(\frac{Tr(\hat{A}) - 1}{2}\Big). \tag{4.1.1}$$

To find the anti-symmetric matrix $\hat{B}$ observe that since the matrix $\hat{B}$ is anti-symmetric, its square $\hat{B}^2$ must be symmetric like I_3. Hence, if we compute $\hat{A} - \hat{A}^T$ we will obtain

$$\begin{aligned} \hat{A} - \hat{A}^T &= \Big(I_3 + \frac{1}{\hat{\theta}} \sin\hat{\theta}\hat{B} + \frac{1}{\hat{\theta}^2}(1-\cos\hat{\theta})\hat{B}^2\Big) - \Big(\hat{I}_3 + \frac{1}{\hat{\theta}} \sin\hat{\theta}\hat{B}^T + \frac{1}{\hat{\theta}^2}(1-\cos\hat{\theta})(\hat{B}^T)^2\Big) \\ &= \frac{1}{\hat{\theta}} \sin\hat{\theta}(\hat{B} - \hat{B}^T) + \frac{1}{\hat{\theta}^2}(1-\cos\hat{\theta})\Big(\hat{B}^2 - (\hat{B}^T)^2\Big) \\ &= \frac{1}{\hat{\theta}} \sin\hat{\theta}\ 2\hat{B} \\ &= \frac{2}{\hat{\theta}} \sin\hat{\theta}\ \hat{B} \end{aligned}$$

since

$$\begin{aligned} \hat{B} - \hat{B}^T &= \begin{pmatrix} 0 & -\hat{b}_3 & \hat{b}_2 \\ \hat{b}_3 & 0 & -\hat{b}_1 \\ -\hat{b}_2 & \hat{b}_1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & \hat{b}_3 & -\hat{b}_2 \\ -\hat{b}_3 & 0 & \hat{b}_1 \\ \hat{b}_2 & -\hat{b}_1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -2\hat{b}_3 & 2\hat{b}_2 \\ 2\hat{b}_3 & 0 & 2\hat{b}_1 \\ -2\hat{b}_2 & 2\hat{b}_1 & 0 \end{pmatrix} \end{aligned}$$

$$= 2\begin{pmatrix} 0 & -\hat{b}_3 & \hat{b}_2 \\ \hat{b}_3 & 0 & -\hat{b}_1 \\ -\hat{b}_2 & \hat{b}_1 & 0 \end{pmatrix}$$

$$= 2\hat{B}$$

and

$$\hat{B}^2 - (\hat{B}^T)^2 = \begin{pmatrix} -\hat{b}_3^2 - \hat{b}_2^2 & \hat{b}_1\hat{b}_2 & \hat{b}_1\hat{b}_3 \\ \hat{b}_1\hat{b}_2 & -\hat{b}_1^2 - \hat{b}_3^2 & \hat{b}_2\hat{b}_3 \\ \hat{b}_3\hat{b}_1 & \hat{b}_2\hat{b}_3 & -\hat{b}_2^2 - \hat{b}_1^2 \end{pmatrix} - \begin{pmatrix} -\hat{b}_3^2 - \hat{b}_2^2 & \hat{b}_1\hat{b}_2 & \hat{b}_1\hat{b}_3 \\ \hat{b}_1\hat{b}_2 & -\hat{b}_1^2 - \hat{b}_3^2 & \hat{b}_2\hat{b}_3 \\ \hat{b}_3\hat{b}_1 & \hat{b}_2\hat{b}_3 & -\hat{b}_2^2 - \hat{b}_1^2 \end{pmatrix}$$

$$= 0.$$

Thus we have

$$\hat{A} - \hat{A}^T = \frac{2}{\hat{\theta}} \sin\hat{\theta}\hat{B}$$

and then

$$\hat{B} = \frac{\hat{\theta}}{2\sin\hat{\theta}}(\hat{A} - \hat{A}^T). \quad (4.1.2)$$

Substituting (4.1.1) into (4.1.2) we obtain

$$\hat{B} = \frac{\arccos\left(\frac{Tr(\hat{A})-1}{2}\right)}{2\sin\arccos\left(\frac{Tr(\hat{A})-1}{2}\right)}(\hat{A} - \hat{A}^T).$$

In other words, the logarithm is given by

$$\hat{B} = log(\hat{A}) = \frac{\hat{\theta}}{2\sin\hat{\theta}}(\hat{A} - \hat{A}^T).$$

The method fails when $\hat{\theta} = \pm\pi$, since $\sin\pi = 0$.

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