DOKUZ EYLÜL UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

SYMMETRIC FUNCTIONS AND *Q*-BERNSTEIN POLYNOMIALS

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İZMİR

SYMMETRIC FUNCTIONS
AND
Q-BERNSTEIN POLYNOMIALS

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We have read the thesis entitled **"SYMMETRIC FUNCTIONS AND Q-BERNSTEIN POLYNOMIALS"** completed by **ARZU GÖCÜMOĞLU** under supervision of **DOÇ.DR. HALİL ORUÇ** and we certify that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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Arzu GÖCÜMOĞLU

SYMMETRIC FUNCTIONS
AND
***Q*-BERNSTEIN POLYNOMIALS**

ABSTRACT

In this thesis, the properties of symmetric functions and total positivity of Bernstein basis are investigated. We discuss a special function blossom and see how subdivide a Bezier curve. Also, we give the blossom of Bernstein and q-Bernstein polynomials by using the elementary symmetric polynomials. Finally, the blossom values of q-Bernstein polynomials are obtained.
Keywords: Symmetric functions, total positivity, Bernstein basis, blossom, Bezier curve, *q*-Bernstein polynomials.

SİMETRİK FONKSİYONLAR
VE
Q-BERNSTEIN POLİNOMLARI

ÖZ

Bu tezde simetrik fonksiyon özellikleri ve Bernstein bazlarının tüm positifliği incelendi. Tomurcuk fonksiyonu (blossom) üzerinde çalışıldı ve Bezier eğrisinin nasıl alt bölümlere ayrılabildiği görüldü. Ayrıca, simetrik fonksiyonlar kullanılarak Bernstein ve q-Bernstein polinomlarının tomurcuk fonsiyon değeri verildi. Son olarak q-Bernstein polinomlarının tomurcuk fonsiyon değeri elde edildi.

Anahtar sözcükler: Simetrik fonksiyonlar, tüm positiflik, Bernstein bazları, tomurcuk fonksiyon, Bezier eğrisi, q-Bernstein polinomları.

CONTENTS

CHAPTER ONE
INTRODUCTION

In this chapter, we will outline some basic definitions and theorems about symmetric functions, Bernstein polynomials, Bézier representation and total positivity before introducing the blossom function. We will also show that Bernstein basis is totally positive.

1.1 Symmetric Functions

The study of symmetric functions is based on symmetric polynomials. Symmetric polynomials are fundamental in the theory of rings and there are various kind of symmetric polynomials. We will be concerned with two of them.

Definition 1.1.1 The *rth* elementary symmetric function $\sigma_r\left(x_0, x_1, ..., x_n\right)$ for $r \geq 1$, is the sum of all products of r distinct variables chosen from the $n+1$ variables $x_0, x_1, ..., x_n$. That is,

$$\sigma_r\left(x_0, x_1, ..., x_n\right) = \sum_{0 \leq i_1 < i_2 < ... < i_r \leq n} x_{i_1} ... x_{i_r}.$$

We define $\sigma_0\left(x_0, x_1, ..., x_n\right) = 1$ for $r = 0$ and $\sigma_r\left(x_0, x_1, ..., x_n\right) = 0$ either $r > n+1$ or $r < 0$.

For example,

$$\sigma_1\left(x_0, x_1, x_2\right) = x_0 + x_1 + x_2,$$
$$\sigma_2\left(x_0, x_1\right) = x_0 x_1,$$
$$\sigma_3\left(x_0, x_1, x_2\right) = x_0 x_1 x_2.$$

Definition 1.1.2 The complete symmetric function $\tau_r\left(x_0, x_1, ..., x_n\right)$ of degree r in variables $x_0, x_1, ..., x_n$ is the sum of all monomials of total degree *r*. That is,

$$\tau_r\left(x_0, x_1, ..., x_n\right) = \sum_{i_0+...+i_n=r} x_0^{i_0} x_1^{i_1} ... x_n^{i_n} \ ,$$

where $i_0, ..., i_n \in \{0,1,...,r\}$. Also it is convenient to define $\tau_r\left(x_0, x_1, ..., x_n\right) = 1$ when $r = 0$ and $\tau_r\left(x_0, x_1, ..., x_n\right) = 0$ for $r < 0$.

For example,

$$\begin{aligned}
\tau_1\left(x_0, x_1, x_2\right) &= x_0 + x_1 + x_2, \\
\tau_2\left(x_0, x_1\right) &= x_0^2 + x_1^2 + x_0 x_1, \\
\tau_3\left(x_0, x_1, x_2\right) &= x_0^3 + x_1^3 + x_2^3 + x_0^2 x_2 + x_1^2 x_0 + x_1^2 x_2 + x_2^2 x_0 + x_2^2 x_1 + x_1 x_2 x_3.
\end{aligned}$$

The generating function for the elementary function is

$$\left(1 + x_0 x\right) ... \left(1 + x_n x\right) = \sum_{r=0}^{n+1} \sigma_r\left(x_0, x_1, ..., x_n\right) x^r \qquad (1.1.1)$$

and

$$\frac{1}{\left(1 - x_0 x\right) ... \left(1 - x_n x\right)} = \sum_{r=0}^{\infty} \tau_r\left(x_0, x_1, ..., x_n\right) x^r \qquad (1.1.2)$$

is the generating function for the complete symmetric functions. (See Phillips, 2003).

Note that the sum of the elementary symmetric functions involves $\binom{n+1}{r}$ terms and complete symmetric functions have $\binom{n+r}{r}$ terms. Giving particular values $x_i = 1$, $i = 0, ..., n$ in (1.1.1) and (1.1.2), we obtain

$$\left(1 + x\right)^{n+1} = \sum_{r=0}^{n+1} \binom{n+1}{r} x^r = \sum_{r=0}^{n+1} \sigma_r\left(1,1,...,1\right) x^r,$$

$$\frac{1}{\left(1 - x\right)^{n+1}} = \sum_{r=0}^{\infty} \binom{n+r}{r} x^r = \sum_{r=0}^{\infty} \tau_r\left(1,1,...,1\right) x^r,$$

where $\binom{n+1}{r} = \sigma_r(1,1,\ldots,1)$ and $\binom{n+r}{r} = \tau_r(1,1,\ldots,1)$.

1.2 Bernstein Polynomials

Bernstein polynomials give a constructive proof of Weierstrass's Theorem. Sergei Natanovich Bernstein introduced the following polynomials in 1912, for a function *f* defined on [0, 1]

$$B_n(f;x) = \sum_{r=0}^{n} f\left(\frac{r}{n}\right)\binom{n}{r} x^r (1-x)^{n-r} \qquad (1.2.1)$$

for each positive integer *n*. Bernstein form is a linear combination of Bernstein basis polynomials $\binom{n}{r} x^r (1-x)^{n-r}$, defined on the space of *n*-degree polynomial, where $r = 0,1,\ldots,n$.

It is easily checked from the equation (1.2.1) for all $n \geq 1$,

$$B_n(f;0) = f(0) \quad \text{and} \quad B_n(f;1) = f(1).$$

We say that $B_n f$ interpolates *f* at the points 0 and 1. This property of Bernstein polynomial is called end point interpolation.

The operator $B_n : C[0,1] \to C[0,1]$ is called Bernstein operator. If *f* is a continuous function then, $B_n f$ uniformly converges to *f*. The Bernstein operator is linear,

$$B_n(af + bg) = aB_n f + bB_n g,$$

for any function *f*, *g* defined on [0, 1] and *a*, *b* are real constants.

We remark that the relations $B_n\left(1:x\right)=1$ and $B_n\left(x;x\right)=x$ are significant in CAGD (Computer Aided Geometric Design). While studying on Bernstein polynomials we refer to some properties which are linearity, convexity, variation diminishing. Before saying Bernstein operator is *monotone* operator or, equivalently, a *positive* operator, we will give an explanation about monotonicity of a linear operator. An operator L from $C\left[a,b\right]$ to $C\left[a,b\right]$ is a monotone operator if it maps $f\left(x\right)\geq g\left(x\right)$ into $\left(Lf\right)\left(x\right)\geq\left(Lg\right)\left(x\right)$, $x\in\left[a,b\right]$.

We can see from the definition of Bernstein polynomial that B_n is a monotone operator. In particular, if we choose $f\left(x\right)\geq 0$ in $\left[0,1\right]$, then $B_n\left(f;x\right)\geq 0$ in $\left[0,1\right]$.

Forward difference is important and used for some calculations in the following chapter. For any function *f* we define,

$$\Delta^0 f_i = f_i\,, \tag{1.2.3}$$

recursively

$$\Delta^{n+1} f_i = \Delta^n f_{i+1} - \Delta^n f_i\,, \tag{1.2.4}$$

where $i=0,1,\ldots,k$ and $n=0,1,\ldots,k-i-1$.

Theorem 1.2.1 The Bernstein polynomial may be expressed in the form

$$B_n\left(f;x\right)=\sum_{r=0}^{n}\binom{n}{r}\Delta^r f\left(0\right)x^r\,, \tag{1.2.2}$$

where Δ is the forward difference operator, with step size $h=1/n$.

For the proof see Phillips, 2003.

For example, with $f(x) = x^3$ it follows from (1.2.4) that $f(0) = 0$, $\Delta f(0) = \frac{1}{n^3}$, $\Delta^2 f(0) = \frac{6}{n^3}$, $\Delta^3 f(0) = \frac{6}{n^3}$ then using (1.2.2)

$$B_n\left(x^3; x\right) = \binom{n}{1}\frac{x}{n^3} + \binom{n}{2}\frac{6x^2}{n^3} + \binom{n}{3}\frac{6x^3}{n^3},$$

which can be written as

$$B_n\left(x^3; x\right) = x^3 + \frac{3}{n-2}x^2 + \frac{1}{(n-1)(n-2)}x.$$

Thus the Bernstein polynomials converges uniformly to x^3 like $\frac{1}{n}$ as $n \to \infty$.

Phillips proposed a generalization of Bernstein polynomial based on the *q*-integers. (See Phillips, 2003). For each positive integers *n*, we define

$$B_n(f; x) = \sum_{r=0}^{n} f_r \begin{bmatrix} n \\ r \end{bmatrix} x^r \prod_{t=0}^{n-r-1}\left(1 - q^t x\right) \qquad (1.2.5)$$

where f_r denotes the value of the function *f* at $x = [r]/[n]$, the quotient of the *q*-integers $[r]$ and $[n]$, and $\begin{bmatrix} n \\ r \end{bmatrix}$ denotes *q*-binomial coefficient. We define,

$$[r] = \begin{cases} \left(1 - q^r\right)/\left(1 - q\right), & q \neq 1, \\ r, & q = 1, \end{cases}$$

and *q*-factorial

$$[r]! = [r][r-1]...[1], \qquad r \geq 1,$$

when $r = 0$, $[r]! = 1$. The *q*-binomial coefficient

$$\begin{bmatrix} n \\ r \end{bmatrix} = \frac{[n]!}{[n-r]!\ [r]!}$$

for $n \geq r \geq 0$, and the other cases are zero.

If we put $q=1$ in equation (1.2.5), we obtain the classical Bernstein polynomial, defined by (1.2.1).

We may express the generalized Bernstein polynomial in terms of q-differences in the form

$$B_n(f;x) = \sum_{r=0}^{n} \begin{bmatrix} n \\ r \end{bmatrix} \Delta_q^r f_0 x^r ,$$

where

$$\Delta_q^r f_j = \Delta_q^{r-1} f_{j+1} - q^{r-1} \Delta_q^{r-1} f_j, \quad r \geq 1,$$

with $\Delta_q^0 f_j = f_j = f\left([j]/[n]\right)$.

For evaluating generalized Bernstein polynomials iteratively, the below algorithm is very useful.

for $r=0$ to n
    $f_r^{[0]} := f\left(\frac{[r]}{[n]}\right)$
next r
for $m=1$ to n
    for $r=0$ to $n-m$
        $f_r^{[m]} := \left(q^r - q^{m-1}x\right) f_r^{[m-1]} + x f_{r+1}^{[m-1]}$
    next r
next m

This algorithm is named Generalized de Casteljau algorithm. It starts with some initials values which are the value of q and the values of f at the $n+1$ points. The iterate $f_r^{[m]}$ satisfies the below equality for $0 \le m \le n$

$$f_r^{[m]} = \sum_{t=0}^{m} f_{r+t} \begin{bmatrix} m \\ t \end{bmatrix} x^t \prod_{s=0}^{m-t-1} \left(q^r - q^s x \right),$$

and has the q-difference form

$$f_r^{[m]} = \sum_{s=0}^{m} q^{(m-s)r} \begin{bmatrix} m \\ s \end{bmatrix} \Delta_q^s f_r x^s .$$

Finally for $r = 0$, we have $f_0^{[m]} = B_m\left(f ; x \right)$. We note that the above algorithm generalizes de Casteljau algorithm with the value $q = 1$.

1.3 Bézier Representation

Bernstein polynomials of degree n,

$$B_i^n\left(u\right) = \binom{n}{i} u^i \left(1-u\right)^{n-i}, \qquad i = 0,1,\ldots,n .$$

leads to computing binomial expansion

$$1 = \left(u + \left(1-u\right)\right)^n = \sum_{i=0}^{n} \binom{n}{i} u^i \left(1-u\right)^{n-i} .$$

Bernstein polynomials B_i^n form a basis for all polynomials of degree $\le n$. Hence, every polynomial curve $b\left(u\right)$ of degree $\le n$ has a unique *nth* degree Bézier representation,

$$b(u)=\sum_{i=0}^{n} c_i B_i^n(u), \qquad u\in[0,1].$$

Applying affine parameter transformation

$$u=u(t)=(1-t)a+tb, \qquad a\neq b.$$

where $u(t)$ is the barycentric combination of a and b, then

$$b(u(t))=\sum_{i=0}^{n} b_i B_i^n(t).$$

The coefficients $b_i\in\mathbb{R}^2$ are called Bézier points or control points. They are the vertices of the Bézier polygon of $b(u)$ over the interval $[a,b]$.

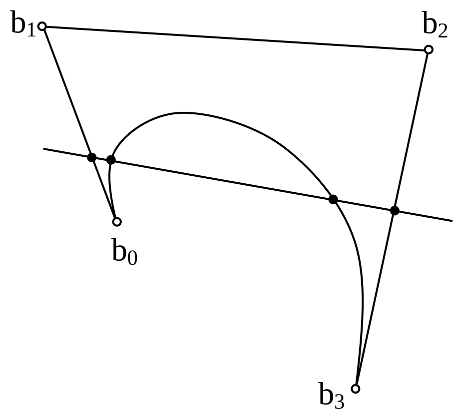
There are some properties of Bézier curves, an important property of Bézier curves is affine invariance. Any point $b(u)$ is an affine combination of the Bézier points. As a consequence, given any affine map ϕ, the image curve $\phi(b)$ has the Bézier points $\phi(b_i)$ over $[a,b]$. It is not necessary to define Bézier curve over the interval $[0,1]$. It can be defined over any arbitrary interval $a\le u\le b$ of the real line.

Using the equality $u=(1-t)a+tb$, we find $t=\frac{u-a}{b-a}$. Then the corresponding expression, is the recursive formula of the de Casteljau algorithm, is of the form

$$b_i^r(u)=\frac{b-u}{b-a}b_i^{r-1}(u)+\frac{u-a}{b-a}b_{i+1}^{r-1}(u).$$

The transition from the interval $[0,1]$ to the interval $[a,b]$ is an affine map. Therefore, we can say that Bézier curves are invariant under affine parameter transformations. The next property is convexity, we mentioned $b(u)$ is an affine combination of the Bézier points and the basis of $b(u)$ can be negative. Since

Bernstein polynomials are positive over the interval $[0,1]$, $b(u)$ is a convex combinations of those points. Hence the curve segment $b[0,1]$ lies in a point set which is formed by Bézier points. And the last property is that a Bézier curve passes through b_0 and b_n we have $b(0)=b_0$, $b(1)=b_n$. This is easily verified by writing $u=0$ and $u=1$. In curve design, preservation of shape properties is very important. Usually a curve is designed with a control polygon which is given by the ordered set of control points. Bézier curves possess the variation diminishing property with respect to their control polygon. It means that any line does not intersect the curve more than it intersects the control polygon. This implies that the shape of the curve can not change sign more than its control polygon. This is the reason that Bézier curve lies in the convex hull of its control polygon. We can illustrate variation diminishing property for Bézier curve,



we see from the figure that the line intersects the Bézier curve and its control polygon in two different points.

1.4 Total Positivity and Totally Positive Basis

We begin this section by defining a totally positive matrix before giving results on basis conversion and the total positivity of the generalized Bernstein basis. The total positivity plays a fundamental role in the theory of approximations and convexity as well as in statistics.

Definition 1.4.1 A real matrix **A** = (**a**$_{ij}$) is said to be totally positive if all its minors are positive, that is

$$\det\begin{bmatrix} a_{i_1,j_1} & \cdots & a_{i_1,j_k} \\ \vdots & & \vdots \\ a_{i_k,j_1} & \cdots & a_{i_k,j_k} \end{bmatrix} > 0, \qquad (1.4)$$

for all $i_1 < i_2 < ... < i_k$ and all $j_1 < j_2 < ... < j_k$.

Total positivity concerns all the minors of the matrix *A*, not just its elements. If we examine this fact in a simple example for the given matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

all minors of the matrix $M_{ij} > 0$, it is clear from $M_{11} = 3$, $M_{12} = 2$, $M_{13} = 1$, $M_{21} = 2$, $M_{22} = 2$, $M_{23} = 1$, $M_{31} = 1$, $M_{32} = 1$, $M_{33} = 1$. So, we say that the matrix *A* is total positive.

An important example of a totally positive matrix is the Vandermonde matrix

$$V = V\left(x_0, ..., x_n\right) = \begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & & x_1^n \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{bmatrix}$$

Let $0 < x_0 < ... < x_n$, since det $V\left(x_0, ..., x_n\right) = \prod_{i>j}\left(x_i - x_j\right)$ we can easily see that $\det V > 0$, and we now show that the minors

$$\det\begin{bmatrix} x_i^j & x_i^{j+1} & \cdots & x_i^{j+k} \\ x_{i+1}^j & x_{i+1}^{j+1} & \cdots & x_{i+1}^{j+k} \\ \vdots & \vdots & & \vdots \\ x_{i+k}^j & x_{i+k}^{j+1} & \cdots & x_{i+k}^{j+k} \end{bmatrix}$$

are positive for all nonnegative *i, j, k* such that $i+k$, $j+k \le n$. The above determinant may be expressed as

$$\begin{vmatrix} x_i^j & x_i^{j+1} & \cdots & x_i^{j+k} \\ x_{i+1}^j & x_{i+1}^{j+1} & \cdots & x_{i+1}^{j+k} \\ \vdots & \vdots & & \vdots \\ x_{i+k}^j & x_{i+k}^{j+1} & \cdots & x_{i+k}^{j+k} \end{vmatrix} = \begin{bmatrix} x_i^j & x_{i+1}^j & \cdots & x_{i+k}^j \end{bmatrix} \cdot \begin{vmatrix} 1 & x_i & x_i^2 & \cdots & x_i^k \\ 1 & x_{i+1} & x_{i+1}^2 & \cdots & x_{i+1}^k \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_{i+k} & x_{i+k}^2 & \cdots & x_{i+k}^k \end{vmatrix}$$

equivalently $\left(x_i \ldots x_{i+k}\right)^j \det V\left(x_i, \ldots, x_{i+k}\right) > 0$. So the Vandermonde matrix V is strictly totally positive. (See Phillips, 2003).

Definition 1.4.2 A matrix **A**, which may be finite or infinite, is said to be m-banded if there exists an integer *l* such that $a_{i,j} \ne 0$ implies that $l \le j - i \le l + m$.

This is equivalent to saying that all matrix elements are zero outside a diagonally bordered band. For example, a tridiagonal matrix has bandwidth 3. Note that every finite matrix is banded.

Theorem 1.4.1 A finite matrix is totally positive if and only if it is a product of 1-banded matrices with non-negative elements.

For proof of this theorem, see de Boor and Pinkus [13].

Definition 1.4.3 We say that a sequence $\left(\phi_0, \ldots, \phi_n\right)$ of real valued functions on an interval I is totally positive if, for any points $0 < x_0 < \ldots < x_n$ in *I*, the collocation matrix $\left(\phi_j\left(x_i\right)\right)_{i,j=0}^{n}$ is totally positive.

If $\phi = (\phi_0, ..., \phi_n)$ is a total positive basis in an interval *I* using the above definition we obtain the following properties:

(i) If *f* is increasing function from an interval *J* into *I* then $(\phi_0 \circ f, ..., \phi_n \circ f)$ is totally positive on *J*, where $\phi_0 \circ f$ denotes the composition of ϕ_0 and *f*.

(ii) If *g* is a positive function on *I*, then $(g\phi_0, ..., g\phi_n)$ is totally positive on *I*.

(iii) If *A* is a constant $(m+1)$ x $(n+1)$ totally positive matrix and

$$\psi_i = \sum_{j=0}^{n} a_{ij}\phi_j, \quad i = 0, ..., m,$$

then $\psi_0, ..., \psi_m$ is totally positive on *I*. (See Oruç & Goodman, 1998).

As further result we will show the Bernstein basis is totally positive using the above properties of total positivity. When we think the monomial functions $\phi_i = t^i$, $i = 0,1,...,n$, then the monomial basis $(1, t, t^2, ..., t^n)$ is a totally positive basis on interval $[0, \infty)$ since the Vandermonde matrix $V(x_0, ..., x_n)$ is totally positive. If we change the variable $t = x/(1-x)$, then it is obviously that *t* is an increasing function for the given all values x_i, $i = 0,1,...,n$, where $0 < x_0 < x_1 < ... < x_n < 1$. Using the first property above, we can say that the sequence of functions

$$\left(1, \frac{x}{1-x}, \frac{x^2}{(1-x)^2}, ..., \frac{x^n}{(1-x)^n}\right)$$

is totally positive on [0,1]. Multiplying this basis by a positive function $(1-x)^n$, $x \in [0,1]$, we have the following sequence of functions,

$$\left((1-x)^n, x(1-x)^{n-1}, x^2(1-x)^{n-2}, ..., x^n\right)$$

which is totally positive on [0, 1] from the second property.

If we multiply this basis by a $(n+1)$ x $(n+1)$ diagonal matrix D which is totally positive

$$\begin{bmatrix} \binom{n}{0} & & & \\ & \binom{n}{1} & & \\ & & \ddots & \\ & & & \binom{n}{n} \end{bmatrix} \cdot \begin{bmatrix} (1-x)^n \\ x(1-x)^{n-1} \\ \vdots \\ x^n \end{bmatrix} = \begin{bmatrix} (1-x)^n \\ \binom{n}{1} x(1-x)^{n-1} \\ \vdots \\ \binom{n}{n} x^n \end{bmatrix}$$

$$\left((1-x)^n, \binom{n}{1} x(1-x)^{n-1}, \binom{n}{2} x^2 (1-x)^{n-2}, \ldots, \binom{n}{n} x^n \right)$$

is a basis for classical Bernstein polynomials of degree at most n and totally positive on [0, 1]. The above argument can be seen in Phillips, 2003.

CHAPTER TWO
BLOSSOMING

We discuss here a special function which is called *blossom*. It can be named also *polarization* instead of blossoming because in geometric modeling polar forms are called blossom, and blossoms are used in the most of the algorithms underlying computer aided geometric design. Blossomming is very important in the geometric modeling. In the following two sections, we underline blossom properties, deal with subdivision and see how to subdivide a Bézier curve. Subdivision is essential in application because the control polygons converge to the Bézier curve under recursive subdivision. Moreover, we compute the blossom of a Bézier curve $p(t)$ using de Casteljau algorithm and explain the relation of these terms.

2.1 Introduction to Blossom

A blossom $b[t_1, t_2, ..., t_n]$ of a polynomial $p(t)$ is a multivariate function from $\mathbb{R}^n$ to $\mathbb{R}^2$ or $\mathbb{R}^3$. We will be concerned with the function $b : \mathbb{R}^n \to \mathbb{R}^2$ in this thesis. The blossom $b[t_1, t_2, ..., t_n]$ of the polynomial $p(t)$ is the unique function with the following properties. (See Ramshaw, 1986).

The first property is called *symmetry*, it states that the order of the variables is irrelevant. Thus, $b[t_1, t_2, ..., t_n] = b[\prod(t_1, t_2, ..., t_n)]$ where $\prod(t_1, t_2, ..., t_n)$ is a permutation of the arguments $t_1, t_2, ..., t_n$.

The blossom $b[t_1, t_2, ..., t_n]$ is linear in each variable t_i, it can be named "*multilinear function*" but the term "*multiaffine*" is more appropriate. Here is an example for $n = 2$ that helps to explain this property. If the first argument t_1 of a blossom $b[t_1, t_2]$ is a barycentric combination of two points a_1, a_2, then the blossom function,

$$b\left[\left(a_1\alpha + a_2\beta\right), t_2\right] = \alpha b\left[a_1, t_2\right] + \beta b\left[a_2, t_2\right]$$

where $\alpha + \beta = 1$.

Thus, we can say that the blossom $b\left[t_1, t_2\right]$ is affine with respect to its first argument. By the symmetry property, the same property holds for the other arguments t_2 in place of t_1. Blossom function is affine in each variable separately. More generally,

let $t_k = \left(1-\alpha\right) n_k + \alpha m_k$ is the affine combination of the points $n_k, m_k \in \mathbb{R}$

$$b\left[t_1, ..., \left(1-\alpha\right) n_k + \alpha m_k, ..., t_n\right] = \left(1-\alpha\right) b\left[t_1, ..., n_k, ..., t_n\right] + \alpha b\left[t_1, ..., m_k, ..., t_n\right].$$

The blossom $b\left[t_1, t_2, ..., t_n\right]$ is also affine for the arguments $\left\{t_1, ..., t_{k-1}, t_{k+1}, ..., t_n\right\}$, not just the one.

In the third one, the diagonal property, the blossom back to the original polynomial, exactly, if all arguments of blossom function are equal $t = t_1, ..., t_n$, we obtain a polynomial curve. For an *n*-degree polynomial $p\left(t\right)$, we have

$$p\left(t\right) = b\left[t, t, ..., t\right].$$

Now taken any cubic polynomial

$$p\left(t\right) = a_3 t^3 + a_2 t^2 + a_1 t + a_0$$

we find that the blossom function of $p\left(t\right)$ is

$$b\left[t_1, t_2, t_3\right] = a_3 t_1 t_2 t_3 + a_2 \frac{\left(t_1 t_2 + t_2 t_3 + t_1 t_3\right)}{3} + a_1 \frac{\left(t_1 + t_2 + t_3\right)}{3} + a_0,$$

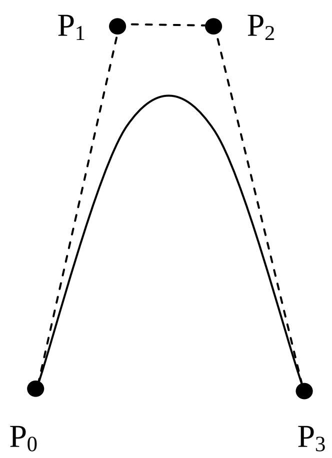
and if we take $t = t_1 = t_2 = t_3$ we obtain original polynomial $p(t)$.

2.2 Blossom of Bézier Curve

We know that Bézier curves are polynomial curves represented in the Bernstein basis. Over the interval [0, 1], *n*-degree Bézier curve *P*(*t*) is given by

$$P(t) = \sum_{i=0}^{n} P_i B_i^n(t) \tag{2.2.1}$$

where P_i are control points. As we take *t* from 0 to 1, we trace out the curve, see figure



The blossom of a polynomial provides us to find the Bézier control points of the corresponding curve. These points control the shape of the curve. In this section, for the control points of Bézier curve we use the notation b_0, b_1, b_2 instead of the P_0, P_1, P_2. Recall the de Casteljau algorithm with $q = 1$. We use this algorithm to subdivide Bézier curves.

For $n = 2$, given $b_0, b_1, b_2 \in \mathbb{R}^2$ and $t \in \mathbb{R}$, we obtain quadratic Bézier curve $p(t)$ with the following construction

$$b_0^1(t) = (1-t)b_0 + tb_1,$$
$$b_1^1(t) = (1-t)b_1 + tb_2,$$

using the points $b_0^1(t)$ and $b_1^1(t)$ we have

$$p(t) = b_0^2(t) = (1-t)b_0^1(t) + tb_1^1(t).$$

Explicitly,

$$p(t) = (1-t)^2 b_0 + 2t(1-t)b_1 + t^2 b_2.$$

This construction leads to repeated linear interpolation for $t \in [0,1]$. The below Bézier curve is affinely invariant, because piecewise linear interpolation is affinely invariant.

$b_0^2(t)$

$(1-t)$ ↗ ↖ t

$b_0^1(t)$ $b_1^1(t)$

$(1-t)$ ↗ ↖ t $(1-t)$ ↗ ↖ t

b_0 b_1 b_2

is the geometric interpretation of the de Casteljau algorithm. Each computed point is found by taking the two points and multiplying them by the respective labels on the edges, and summing the two resulting products.

To subdivide a Bézier curve using blossoms with the same control points of Bézier curve b_0, b_1, b_2, we illustrate

$$b_0^2\left[t_1,t_2\right]$$

$(1 - t_2)$ ↗ ↖ t_2

$b_0^1\left(t_1\right)$ $b_1^1\left(t_1\right)$

$(1 - t_1)$ ↗ ↖ t_1 $(1 - t_1)$ ↗ ↖ t_1

b_0 b_1 b_2

by replacing t with a different parameter t_i where $i = 1, 2, \ldots$ on each level of the triangle, we can compute the blossom of a Bézier curve $p\left(t\right)$. From the below figure, we can express the blossom which is defined over [0, 1].

$$b_0^1\left(t_1\right) = \left(1 - t_1\right)b_0 + t_1 b_1,$$

$$b_1^1\left(t_1\right) = \left(1 - t_1\right)b_1 + t_1 b_2,$$

$$b_0^2\left[t_1,t_2\right] = \left(1 - t_2\right)b_0^1\left(t_1\right) + t_2 b_1^1\left(t_1\right),$$

Explicitly,

$$b_0^2\left[t_1,t_2\right] = b\left[t_1,t_2\right] = \left(1-t_1\right)\left(1-t_2\right)b_0 + \left[\left(1-t_1\right)t_2 + t_1\left(1-t_2\right)\right]b_1 + t_1 t_2 b_2 \qquad (2.2.2)$$

It is straightforward to check that $b\left[t_1,t_2\right]$ is the blossom of the curve $p\left(t\right)$, since $b\left[t_1,t_2\right]$ is clearly symmetric, from the equation (2.2.2)

$$b\left[t_2,t_1\right] = \left(1-t_2\right)\left(1-t_1\right)b_0 + \left[\left(1-t_2\right)t_1 + t_2\left(1-t_1\right)\right]b_1 + t_2 t_1 b_2$$

then

$$b\left[t_1,t_2\right] = b\left[t_2,t_1\right]$$

and we show $b\left[t_1,t_2\right]$ is affine for the first argument t_1,

$$
\begin{aligned}
b\left[(\alpha r+\beta s),t_2\right] &= (1-t_2)\left(1-(\alpha r+\beta s)\right)b_0+\left[(1-t_2)(\alpha r+\beta s)\right.\\
&\quad \left.+t_2\left(1-(\alpha r+\beta s)\right)\right]b_1+t_2(\alpha r+\beta s)b_2\\
&= (1-t_2)\left((\alpha+\beta)-(\alpha r+\beta s)\right)b_0+\left[\alpha r(1-t_2)\right.\\
&\quad + \left.\beta s(1-t_2)+t_2\left(\alpha+\beta-(\alpha r+\beta s)\right)\right]b_1+t_2(\alpha r+\beta s)b_2\\
&= (1-t_2)\left(\alpha(1-r)+\beta(1-s)\right)b_0+\left[\alpha r(1-t_2)+\beta s(1-t_2)\right.\\
&\quad + \left.\alpha(1-r)t_2+\beta(1-s)t_2\right]b_1+(t_2\alpha r+t_2\beta s)b_2\\
&= \alpha(1-r)(1-t_2)b_0+\beta(1-s)(1-t_2)b_0+\alpha\left[r(1-t_2)+(1-r)t_2\right]b_1\\
&\quad +\beta\left[s(1-t_2)+(1-s)t_2\right]b_1+\alpha rt_2b_2+\beta st_2b_2\\
&= \alpha b[r,t_2]+\beta b[s,t_2]
\end{aligned}
$$

and similarly by the symmetric property $b[t_1,t_2]$ is affine for the second one. This is the reason that $b[t_1,t_2]$ is multiaffine.

For a third property, taken $t_1=t_2=t$

$$b[t,t]=(1-t)^2b_0+2(1-t)tb_1+t^2b_2$$

is the Bézier curve with the control points b_0,b_1,b_2. Finally, we proved $b[t_1,t_2]$ is the blossom of Bézier curve.

The blossom values are computed by linear interpolation, we find that the Bézier points b_i are found as the blossom values. For a *n*-degree polynomial, we express the Bézier points,

$$b_i=b\left[0^{<n-i>},1^{<i>}\right]$$

where $0^{<n-i>}$ means that $n-i$ times 0 and $1^{<i>}$ denotes i times 1. Moreover, each multiaffine function $b[t_1,t_2,...,t_n]$ can be written as unique linear combination of this function

$$b[t_1,t_2,...,t_n]=(1-t_1)b[0,t_2,...,t_n]+t_1b[1,t_2,...,t_n] \qquad (2.2.3)$$

where the function defined over the interval [0, 1]. If we take any arbitrary interval $[\alpha,\beta]$ instead of [0,1], then the local coordinate of Bézier curve t can be written

$$t=\frac{\beta-t}{\beta-\alpha}\alpha+\frac{t-\alpha}{\beta-\alpha}\beta,$$

and the equation (2.2.3) can be expressed in interval $[\alpha,\beta]$, (See Farin, 2002),

$$b[t_1,t_2,...,t_n]=\frac{\beta-t_1}{\beta-\alpha}b[\alpha,t_2,...,t_n]+\frac{t_1-\alpha}{\beta-\alpha}b[\beta,t_2,...,t_n].$$

One may express the general form of blossom by introducing a new parameter t_r in each column of the de Casteljau algorithm as follows:

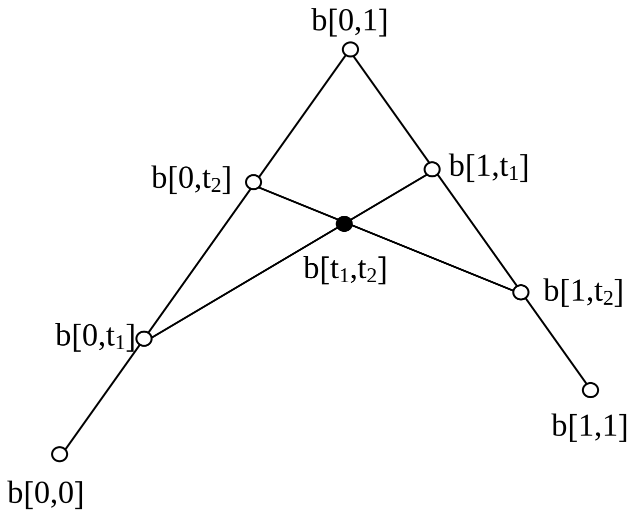
For $r=1,2,...,n$ and $i=0,1,...,n-r$, compute

$$b[t_1,t_2,...,t_r]=\frac{\beta-t_r}{\beta-\alpha}b[t_1,...,t_{r-1},\alpha]+\frac{t_r-\alpha}{\beta-\alpha}b[t_1,...,t_{r-1},\beta].$$

If we rewrite the equation (2.2.2) using the blossom values, we have

$$b[t_1,t_2]=(1-t_1)(1-t_2)b[0,0]+[(1-t_1)t_2+t_1(1-t_2)]b[0,1]+t_1t_2b[1,1].$$

This is shown in the following figure,



Each node in the diagram indicates one of the blossom values computed by de Casteljau algorithm, and $b[t_1, t_2]$ is the resulting values in this computation.

CHAPTER THREE
RESULTS ON BLOSSOM OF BERNSTEIN POLYNOMIALS

In this chapter, we will express the blossom of Bernstein and *q*-Bernstein polynomials in general form using the elementary symmetric polynomials. In addition, we will point out how Bernstein coefficients may be expressed using the blossom values. Finally, we demonstrate that the blossom of Bernstein polynomials interpolates control points of the polynomial curve.

3.1 Blossom of Bernstein Polynomials

By the main theorem, for every polynomial curve $p(t)$ there exists a unique *n*-variate symmetric polynomial in *nth* degree polynomial space. (See Prautzsch & Boehm & Paluszny, 2002). If we consider a polynomial curve $p(t)$ which is written as a linear combination of *nth* degree polynomials $C_i(t)$,

$$p(t)=\sum_{i=0}^{n} c_i C_i(t), \tag{3.1.1}$$

then we write the blossom of $p(t)$

$$b[t_1,t_2,...,t_n]=\sum_{i=0}^{n} c_i C_i[t_1,t_2,...,t_n]$$

where $C_i[t_1,t_2,...,t_n]$ denotes the blossom of $C_i(t)$.

If we take $C_i(t)=\binom{n}{i}t^i$, $i=0,1,...,n$, in equation (3.1.1), we get the elementary symmetric polynomials

$$C_i[t_1,t_2,...,t_n]=\sum_{1\le k_1<...<k_i\le n} t_{k_1}\dots t_{k_i}$$

since the summation involves $\binom{n}{i}$ terms for each value *i*.

For example, taking a polynomial $p(t)$ for $n=2$ in the form

$$p(t)=\sum_{i=0}^{2} c_i \binom{n}{i} t^i$$
$$= c_0 + 2c_1 t + c_2 t^2$$

the blossom of $p(t)$, with using symmetric polynomial, is

$$b[t_1,t_2]=\sum_{i=0}^{2} c_i \sum_{1\le k_1<k_i\le 2} t_{k_1} t_{k_i}$$
$$= c_0 + c_1(t_1+t_2) + c_2(t_1 t_2),$$

the elementary symmetric polynomial $C_i[t_1,t_2]$ clearly satisfies the three properties of symmetric polynomials.

In equation (3.1.1), if we take $C_i(t)$ as Bernstein basis

$$B_i^n(t)=\binom{n}{i} t^i (1-t)^{n-i},$$

then the symmetric polynomial of $B_i^n(t)$ has the following form, (See Prautzsch & Boehm & Paluszny, 2002)

$$B_i^n[t_1,t_2,...,t_n]=\sum_{\substack{k_1<...<k_i \\ l_1<...<l_{n-i}}} t_{k_1}\dots t_{k_i}(1-t_{l_1})\dots(1-t_{l_{n-i}}).$$

It follows from the symmetric polynomials that we can represent the blossom of Bernstein polynomial $B_n(f;t)$ using the symmetric polynomial

$$b\left[t_1,t_2,...,t_n\right]=\sum_{i=0}^{n}c_iB_i^n\left[t_1,t_2,...,t_n\right]$$

$$=\sum_{i=0}^{n}c_i\sum_{\substack{k_1<...<k_i\\ l_1<...<l_{n-i}}}t_{k_1}\dots t_{k_i}\left(1-t_{l_1}\right)\dots\left(1-t_{l_{n-i}}\right) \tag{3.1.2}$$

where $c_i = f\left(\frac{i}{n}\right)$ and the function *f* is defined on $\left[0,1\right]$.

We remark that Bernstein polynomials were given in a difference form in the first chapter

$$B_n\left(f;t\right)=\sum_{i=0}^{n}\binom{n}{i}\Delta^i f\left(0\right)t^i \tag{3.1.3}$$

where Δ is the forward difference operator.

As a consequence of the expression, if we consider Bernstein polynomial in this form, then the representation of blossom will be different from the equation (3.1.2). Taking $\binom{n}{i}t^i=C_i\left(t\right)$ and $\Delta^i f\left(0\right)=c_i$ in equation (3.1.3), the blossom of Bernstein polynomial can also be written

$$b\left[t_1,t_2,...,t_n\right]=\sum_{i=0}^{n}c_i\sum_{1\le k_1<...<k_i\le n}t_{k_1}\dots t_{k_i} \tag{3.1.4}$$

with the elementary symmetric polynomial

$$\sum_{1\le k_1<...<k_i\le n}t_{k_1}\dots t_{k_i}\ .$$

Let us show that $b\left[t_1,t_2,...,t_n\right]$ satisfies the three properties of blossom. Since $C_i\left[t_1,t_2,...,t_n\right]$ is symmetric polynomial, linear combination of the symmetric polynomial is also symmetric. Hence the first property is hold.

If any argument t_{k_j} , $1 < j < i$, of the blossom is a barycentric combination of two points *r* and *s*,

$$t_{k_j} = \alpha r + (1-\alpha) s, \qquad r, s \in \mathbb{R}$$

the blossom is

$$b\left[t_1, ..., (\alpha r + (1-\alpha) s), ..., t_n\right] = \sum_{i=0}^{n} c_i \sum_{1 \le k_1 < ... < k_i \le n} t_{k_1} \ldots (\alpha r + (1-\alpha) s) \ldots t_{k_i}$$

since the symmetric polynomial is affine in each variable,

$$\begin{aligned} b\left[t_1, ..., (\alpha r + (1-\alpha) s), ..., t_n\right] &= \sum_{i=0}^{n} c_i \left(\alpha \sum_{1 \le k_1 < ... < k_i \le n} t_{k_1} \ldots r \ldots t_{k_i} \right. \\ &\qquad \left. + \ (1-\alpha) \sum_{1 \le k_1 < ... < k_i \le n} t_{k_1} \ldots s \ldots t_{k_i} \right) \\ &= \ \alpha \sum_{i=0}^{n} c_i \sum_{1 \le k_1 < ... < k_i \le n} t_{k_1} \ldots r \ldots t_{k_i} \\ &\qquad + \ (1-\alpha) \sum_{i=0}^{n} c_i \sum_{1 \le k_1 < ... < k_i \le n} t_{k_1} \ldots s \ldots t_{k_i}) \end{aligned}$$

then we deduce that

$$b\left[t_1, ..., (\alpha r + (1-\alpha) s), ..., t_n\right] = \alpha b\left[t_1, ..., r, ..., t_n\right] + (1-\alpha) b\left[t_1, ..., s, ..., t_n\right].$$

So, the affine property is satisfied. For the diagonal property of blossom, take $t_1 = t, ..., t_n = t$. Then we obtain,

$$\sum_{1 \le k_1 < ... < k_i \le n} t_{k_1} \ldots t_{k_i} = \binom{n}{i} t^i$$

and $C_i(t)=\binom{n}{i}t^i$, $c_i=\Delta^i f(0)$. It follows that

$$
\begin{aligned}
b[t,...,t] &= \sum_{i=0}^{n} \Delta^i f(0)\binom{n}{i} t^i \\
&= B_n(f;t).
\end{aligned}
$$

Thus, the equation (3.1.4) is the blossom of Bernstein polynomial.

From the previous chapter, we know that the Bézier points are represented as the blossom values. Here, we will analyze the relation between the coefficients of Bernstein polynomial and blossom values.

For n = 3, with the given blossom values, we obtain the blossom of Bézier curve

$$
\begin{aligned}
b[t_1,t_2,t_3] = b[0,0,0]&\{(1-t_1)(1-t_2)(1-t_3)\} \\
&+ b[1,0,0]\{t_1(1-t_2)(1-t_3)+t_2(1-t_1)(1-t_3)+t_3(1-t_1)(1-t_2)\} \\
&+ b[1,1,0]\{t_1t_2(1-t_3)+t_1t_3(1-t_2)+t_3t_2(1-t_1)\}+b[1,1,1]t_1t_2t_3.
\end{aligned}
$$

Substituting n = 3 in equation (3.1.4), we obtain the blossom of Bernstein polynomial in the form

$$
\begin{aligned}
b[t_1,t_2,t_3] &= \sum_{i=0}^{3} \Delta^i f_0 \sum_{1\le k_1<...<k_i\le 3} t_{k_1}\dots t_{k_i} \\
&= \Delta^0 f_0 + \Delta^1 f_0(t_1+t_2+t_3)+\Delta^2 f_0(t_1t_2+t_1t_3+t_2t_3)+\Delta^3 f_0(t_1t_2t_3)
\end{aligned}
$$

Comparing the coefficients 1, $(t_1+t_2+t_3)$, $(t_1t_2+t_1t_3+t_2t_3)$, $(t_1t_2t_3)$ in the above equations, we obtain the blossom values of Bernstein polynomial

$$
\begin{aligned}
\Delta^0 f_0 &= b[0,0,0] \\
\Delta^1 f_0 &= b[1,0,0]-b[0,0,0]
\end{aligned}
$$

$$\Delta^2 f_0 = b[1,1,0] - 2b[1,0,0] + b[0,0,0]$$
$$\Delta^3 f_0 = b[1,1,1] - 3b[1,1,0] + 3b[1,0,0] - b[0,0,0].$$

The blossom values are the Bézier control points of the curve

$$b\left[0^{<n-i>},1^{<i>}\right] = b_i, \qquad i = 0,1,...,n.$$

Thus, the blossom of Bézier curve interpolates to all control points b_i. Using the Bernstein- Bézier equality, we conclude that the blossom of Bernstein polynomial interpolates all points f_i.

3.2 Blossom of *q*-Bernstein Polynomials

We recall from the first chapter the expression of *q*-Bernstein polynomial in the form

$$B_n^q(f;t) = \sum_{i=0}^{n} f_i \begin{bmatrix} n \\ i \end{bmatrix} t^i \prod_{s=0}^{n-i-1} (1-q^s t) \tag{3.2.1}$$

where

$$B_i^{n,q}(t) = \begin{bmatrix} n \\ i \end{bmatrix} t^i \prod_{s=0}^{n-i-1} (1-q^s t), \quad 0 \le i \le n,$$

is the generalized Bernstein basis.

Using the above expression (3.2.1), we could not find a symmetric polynomial of the generalized Bernstein basis. Since the symmetry property of blossom is disqualified, the blossom of *q*-Bernstein polynomial can not be reached from this expression.

It will be reasonable to use the following representation for *q*-Bernstein polynomial,

$$B_n^q(f;t) = \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix} \Delta_q^i f_0 t^i. \qquad (3.2.2)$$

If we multiply and divide the equation (3.2.2) by $\binom{n}{i}$, then it becomes

$$B_n^q(f;t) = \sum_{i=0}^{n} \frac{\begin{bmatrix} n \\ i \end{bmatrix} \Delta_q^i f_0}{\binom{n}{i}} \binom{n}{i} t^i.$$

Then we express the blossom of q-Bernstein polynomials

$$b[t_1, t_2 ..., t_n] = \sum_{i=0}^{n} c_i \sum_{1 \le k_1 < ... < k_i \le n} t_{k_1} \dots t_{k_i} \qquad (3.2.3)$$

where $c_i = \frac{\begin{bmatrix} n \\ i \end{bmatrix} \Delta_q^i f_0}{\binom{n}{i}}$.

Now, one may see that the blossom properties, symmetry-multiaffinity-diagonality, are hold for the equation (3.2.3). Here, we will give an example of blossom for a quadratic q-Bernstein polynomial.

Substituting $n = 2$ in equation (3.2.3), we have

$$b[t_1, t_2] = c_0 + c_1(t_1 + t_2) + c_2 t_1 t_2$$

where

$$c_0 = \frac{\begin{bmatrix} 2 \\ 0 \end{bmatrix} \Delta_q^0 f_0}{\binom{2}{0}} = f_0,$$

$$c_1 = \frac{\begin{bmatrix} 2 \\ 1 \end{bmatrix} \Delta_q^1 f_0}{\binom{2}{1}} = \frac{(1+q)(f_1 - f_0)}{2},$$

$$c_2 = \frac{\begin{bmatrix} 2 \\ 2 \end{bmatrix} \Delta_q^2 f_0}{\binom{2}{2}} = f_2 - f_1 - q(f_1 - f_0).$$

Then the blossom is written in the form

$$b[t_1, t_2] = f_0 + \frac{(1+q)(f_1 - f_0)}{2}(t_1 + t_2) + [f_2 - f_1 - q(f_1 - f_0)]t_1 t_2.$$

For $t_1 = t_2 = 0,$ the blossom interpolates the initial point f_0,

$$b[0, 0] = f_0$$

and for $t_1 = 0,\ t_2 = 1,$ we obtain that

$$b[0,1] = f_0 + \frac{(1+q)(f_1 - f_0)}{2} \neq f_1.$$

To find the value of t_2, it must satisfy

$$b[0, t_2] = f_0 + \frac{(1+q)(f_1 - f_0)}{2} t_2 = f_1.$$

It follows that

$$t_2 = \frac{2}{1+q}.$$

So, we have

$$b[0, \frac{2}{1+q}] = f_1,$$

and taken the arguments $t_1 = t_2 = 1,$ we see that

$$\begin{aligned} b[1,1] &= f_0 + (1+q)(f_1 - f_0) + f_2 - f_1 - q(f_1 - f_0) \\ &= f_2. \end{aligned}$$

Thus, we deduce that the blossom of quadratic q-Bernstein polynomial interpolates the points f_0, f_1 and f_2.

To generalize the above argument, it is easy to see that

$$b[0^{<n>}] = f_0 \text{ and } b[1^{<n>}] = f_n.$$

The latter equation shows that the blossom of q-Bernstein polynomial satisfies the end point interpolation property. In order to find the other $n-1$ points $f_1, ..., f_{n-1}$ from the blossom function, we impose transformation matrix between the two representations of the same curve. Since q-Bernstein basis and classical Bernstein basis span the space of any polynomial curve $p(t)$, we have

$$\begin{aligned} p(t) &= b_0 B_0^n(t) + ... + b_n B_n^n(t) \\ &= c_0 B_0^{n,q}(t) + ... + c_n B_n^{n,q}(t), \end{aligned}$$

where $b_0, ..., b_n$ and $c_0, ..., c_n$ are respective control points of the curve.

Now we can find a nonsingular transformation $(n+1)\times(n+1)$ matrix $\mathrm{T}^{n,q}$ between Bernstein coefficients and q-Bernstein coefficients, such that

$$[c_0, c_1, ..., c_n]\mathrm{T}^{n,q} = [b_0, b_1, ..., b_n]. \qquad (3.2.4)$$

Then

$$b_j = \sum_{i=0}^{n} t_{ij} c_i, \quad 0 \le j \le n,$$

is the linear transformation, where t_{ij} are the entries of the transformation matrix $\mathrm{T}^{n,q}$. The entries of the matrix $\mathrm{T}^{n,q}$ are

$$t_{ij} = \frac{\begin{bmatrix} n \\ i \end{bmatrix}}{\binom{n}{j}} (1-q)^{j-i} S(n-1-i, j-i),$$

where $S(n, j)$ is the Stirling polynomial defined by the sum of $\binom{n}{j}$ products of j distinct factors chosen from the set $\{[1],[2],...,[n]\}$. (See Oruç & Phillips, 2002).

Thus,

$$[c_0, c_1, ..., c_n] = [b_0, b_1, ..., b_n]\left(\mathrm{T}^{n,q}\right)^{-1}.$$

A general formulation to find the blossom values of q-Bernstein polynomial is not as clear as in Bernstein polynomials. However, the above formula expresses the blossom values of q-Bernstein polynomial in terms of control points of its Bernstein representation.

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