DOKUZ EYLÜL UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

ON THE SOLUTIONS OF NONLINEAR DIFFERENTIAL EQUATIONS

by

Berrak ÖZGÜR

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İZMİR

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We have read the thesis entitled **“ON THE SOLUTIONS OF NONLINEAR DIFFERENTIAL EQUATIONS”** completed by **BERRAK ÖZGÜR** under supervision of **PROF. DR. GONCA ONARGAN** and we certify that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

PROF. DR. GONCA ONARGAN

Supervisor

YRD. DOÇ. DR. MELDA DUMAN

(Jury Member)

YRD. DOÇ. DR. HAKAN EPİK

(Jury Member)

Prof.Dr. Cahit HELVACI
Director
Graduate School of Natural and Applied Sciences

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ON THE SOLUTIONS OF NONLINEAR DIFFERENTIAL EQUATIONS

ABSTRACT

An approximate method for the solution of non-linear singularly perturbed problems for second order ordinary differential equations with a boundary layer is studied. The region is divided into two parts as inner and outer regions. The original second order singularly perturbed problem is replaced by an asymptotically equivalent first order problem and solved in the inner and outer regions. Then, the solutions of inner and outer region problems are matched to obtain an approximate uniform solution to the original problem.

Keywords : Singular perturbation, boundary layer, matching technique, approximate solution, asymptotic expansion, uniform solution

DOĞRUSAL OLMAYAN DİFERANSİYEL DENKLEMLERİN ÇÖZÜMÜ ÜZERİNE

ÖZ

Sınır tabakaya sahip, ikinci mertebe doğrusal olmayan tekil pertürbe adi diferansiyel denklemler için bir yaklaşım metodu çalışıldı. Bölge, iç ve dış bölge olarak ikiye bölündü. Orijinal ikinci mertebe tekil pertürbe denklem, asimptotik olarak denk birinci mertebe denkleme dönüştürüldü ve iç ve dış bölgelerde çözüldü. Daha sonra, iç ve dış bölge problemlerinin çözümleri, orijinal probleme yaklaşık düzgün çözüm bulmak için eşlendi.

Anahtar sözcükler : Tekil pertürbasyon, sınır tabaka, eşleme yöntemi, yaklaşık çözüm, asimptotik açılım, düzgün çözüm

CONTENTS

CHAPTER ONE

INTRODUCTION

Lots of differential equations which arise as models of physical systems cannot be solved analytically. So, we can solve them by means of some numerical methods. However if there are some dimensionless parameter in equations, we can solve them by using some asymptotic methods and then we have an approximate solution.

When the methods do not yield an exact closed-form solution of a differential equation or when the exact solution is too complicated to be useful, then we should try to ascertain the approximate nature of the solution. The first step toward an approximate solution is called local analysis (Bender&Orszag, 1978). The methods of local analysis are Taylor series solutions, method of Fuchs and Frobenius, method of dominant balance, asymptotic series expansions of solutions.

The purpose of local analysis is to represent the solutions of equations which cannot be solved in closed-form as simple expressions in terms of elementary functions. The results of a local analysis are valid in a sufficiently small neighborhood of a point. Ultimately, a uniform approximation to the behavior of the solution over an entire interval may be found by piecing together neighborhoods in which the local behavior is known. This piecing-together process uses the techniques of global analysis such as Boundary Layer Theory, WKB Theory (named after Wentzel, Kramers, Brillouin), Multiple Scale Analysis. (Bender&Orszag,1978)

Local analysis methods such as Taylor series solution, methods of Fuchs and Frobenius method of dominant balance, asymptotic series expansions are powerful tools, but they cannot provide global information on the behavior of solutions at two distantly separated points. They cannot predict how a change in initial conditions at $x = 0$ will affect the asymptotic behavior as $x \to \infty$ (Bender&Orszag, 1978). To solve such kind problems, we must use the methods of global analysis such as Boundary Layer Theory, WKB Theory, Multiple-Scale Theory which are perturbative in character.

Perturbation theory is a collection of methods for the systematic analysis of the global behavior of solutions to differential and difference equations.
(Bender&Orszag, 1978)

Boundary layer theory and WKB theory are a collection of perturbative methods for obtaining an asymptotic approximation to the solution $y(x)$ of a differential equation whose highest derivative is multiplied by a small parameter (perturbing parameter) ε. Solutions to such equations usually develop regions of rapid variation as $\varepsilon \to 0$. If the thickness of these regions approaches 0 as $\varepsilon \to 0$, they are called boundary layers, and boundary layer theory may be used to approximate $y(x)$. If the extent of these regions remains finite as $\varepsilon \to 0$, we must use WKB theory. For linear equations boundary layer theory is special case of WKB theory, but boundary layer theory also applies directly to nonlinear equations while WKB theory does not.
(Bender&Orszag, 1978)

Multiple scale theory is used when ordinary perturbative methods fail to give a uniformly accurate approximation to $y(x)$ for both small and large values of *x*. Some (although not certainly all) perturbation problems which yield to boundary layer or WKB analysis can also be solved using multiple-scale analysis.
(Bender&Orszag, 1978)

WKB theory, is a powerful tool for obtaining a global approximation to the solution of a linear differential equation whose highest derivative is multiplied by a small parameter ε ; it contains boundary layer theory as a special case.
(Bender&Orszag, 1978)

Singularly perturbed differential equations arise in modelling of various physical processes. Equations of this type typically exhibit solutions with layers; that is, the domain of the differential equation contains narrow regions where the solution varies very fast whereas away from this region the solution behaves smoothly and varies slowly. To handle this type of problem, the basic idea is, to divide the domain of integration into inner and outer regions.

A detailed discussion on the analytic theory of singular perturbation problem is given by Bender&Orszag (1978), Bush (1992), Hinch (1991), Holmes(1995), Johnson (2005), Nayfeh (1993), Nayfeh (1973), O'Malley (1991), Simmonds & Man (1998). Solving singular perturbation problems by using numerical methods have been suggested ((Kadalbajoo & Reddy, 1988), (Kadalbajoo & Kumar, 2008)). A singular perturbation boundary value problem depending on a parameter λ have studied and numerical results have obtained (Amiraliyev & Duru, 2005). Series solutions of boundary layer flows with nonlinear Navier boundary conditions have obtained by means of the homotopy analysis method (Cheng, Liao, Mohapatra & Vajravelu, 2008).

In this thesis, we investigate mainly finding approximate analytic solutions of nonlinear singularly perturbed initial value problems and boundary value problems for second order ordinary differential equations with a boundary layer.

In Chapter Two, we give some basic concepts in asymptotic analysis, such as gauge functions, order symbols, asymptotic sequences and asymptotic expansions.

In Chapter Three, we give general information about perturbation theory and matching process.

In Chapter Four, we give detailed information about the boundary layer theory which is the main part of the thesis and some illustrative examples.

In Chapter Five, the applications of the boundary layer theory are presented for chosen nonlinear problems.

CHAPTER TWO

BASIC CONCEPTS IN ASYMPTOTIC ANALYSIS

2.1 Gauge Functions

We consider the limit of functions such as $f(\varepsilon)$ as ε tends to zero. This limit might depend on whether ε tends to zero from the right, denoted by $\varepsilon \to 0+$, or from the left, denoted by $\varepsilon \to 0-$.

If the limit of $f(\varepsilon)$ exists (i.e. , it doesn't have an essential singularity at $\varepsilon = 0$ such as $\sin \varepsilon^{-1}$) , then there are three possibilities (Nayfeh, 1993)

$$\left.\begin{array}{l} f(\varepsilon) \to 0 \\ f(\varepsilon) \to A \\ f(\varepsilon) \to \infty \end{array}\right\} \qquad as \quad \varepsilon \to 0 \qquad , \qquad 0 < |A| < \infty$$

Therefore, to narrow down the above classification, we subdivide each class according to the rate at which they tend to zero or infinity. To accomplish this, we compare the rate at which these functions tend to zero or infinity with the rate at which known functions tend to zero and infinity. These comparison functions are called gauge functions. The simplest and most useful of these are powers of ε

$$1, \varepsilon, \varepsilon^2, \varepsilon^3, \ldots$$

and the inverse powers of ε

$$\varepsilon^{-1}, \varepsilon^{-2}, \varepsilon^{-3}, \varepsilon^{-4}, \ldots$$

For small ε, we know that

$$1 > \varepsilon > \varepsilon^2 > \varepsilon^3 > \varepsilon^4 > \ldots$$

and

$$\varepsilon^{-1} < \varepsilon^{-2} < \varepsilon^{-3} < \varepsilon^{-4} < \ldots$$

Let us determine the rate at which the function $\sin\varepsilon$ tends to zero or infinity. Let us use the Taylor series expansion, then we have

$$\sin\varepsilon = \varepsilon - \frac{\varepsilon^3}{3!} + \frac{\varepsilon^5}{5!} - \ldots$$

so that $\sin\varepsilon \to 0$ as $\varepsilon \to 0$ because

$$\lim_{\varepsilon\to 0}\frac{\sin\varepsilon}{\varepsilon} = \lim_{\varepsilon\to 0}\left(1 - \frac{\varepsilon^2}{2!} + \frac{\varepsilon^4}{4!} - \ldots\right) = 1 \quad .$$

Now let's look at $\dfrac{1}{\sin\varepsilon}$.

$$\frac{1}{\sin\varepsilon} = \frac{1}{\varepsilon - \frac{\varepsilon^3}{3!} + \frac{\varepsilon^5}{5!} - \ldots} = \frac{1}{\varepsilon\left(1 - \frac{\varepsilon^2}{3!} + \frac{\varepsilon^4}{4!} - \ldots\right)} \quad ,$$

so that $\dfrac{1}{\sin\varepsilon} \to \infty$ as $\dfrac{1}{\varepsilon}$ because

$$\lim_{\varepsilon\to 0}\frac{\frac{1}{\sin\varepsilon}}{\frac{1}{\varepsilon}} = \lim_{\varepsilon\to 0}\frac{\varepsilon}{\sin\varepsilon} = \lim_{\varepsilon\to 0}\frac{1}{1 - \frac{\varepsilon^2}{3!} + \ldots} = 1$$

2.2 Order Symbols

Instead of saying $\sin\varepsilon$ tends to zero at the same rate that ε tends to zero, we say $\sin\varepsilon$ is order ε as $\varepsilon \to 0$ or $\sin\varepsilon$ is big 'oh' of ε as $\varepsilon \to 0$ and we write it as $\sin\varepsilon = O(\varepsilon)$ as $\varepsilon \to 0$. (Nayfeh, 1993)

Let us define big oh "O" , and little oh "o" symbols.

If

$$\lim_{\varepsilon\to 0}\frac{f(\varepsilon)}{g(\varepsilon)}=k \quad , \quad 0<|k|<\infty$$

then we can write that

$$f(\varepsilon)=O[g(\varepsilon)] \quad \text{as} \quad \varepsilon\to 0$$

Let us give some examples.

$$\cos\varepsilon = O(1)$$
$$\sin\varepsilon = O(\varepsilon)$$
$$\cos\varepsilon - 1 = O(\varepsilon^2)$$

Now let us define little oh “o” .

If

$$\lim_{\varepsilon\to 0}\frac{f(\varepsilon)}{g(\varepsilon)}=0$$

then we can write that

$$f(\varepsilon)=o[g(\varepsilon)] \quad \text{as} \quad \varepsilon\to 0$$

Let us give some examples.

$$\sin\varepsilon = o(1)$$
$$\cos\varepsilon = o(\varepsilon^{-1})$$

In the definitions of big oh “O” and little oh “o” , the functions $g(\varepsilon)$ are gauge functions.

Also, if

$$\lim_{\varepsilon\to 0}\frac{f(\varepsilon)}{g(\varepsilon)}=1$$

then we say that f is asymptotic to g as $\varepsilon\to 0$ and we write

$$f(\varepsilon)\sim g(\varepsilon)$$

Thus $\dfrac{1}{3+2x^2} \sim \dfrac{1}{2x^2}$ as $|x| \to \infty$

2.3 Asymptotic Sequences

The set of functions

$$\{\phi_n(x)\} \quad , \quad n = 0,1,2,...$$

is an asymptotic sequence as $x \to a$, if

$$\phi_{n+1}(x) = o[\phi_n(x)] \quad \text{as} \quad x \to a$$

for every n . (Johnson, 2005)

Some examples for asymptotic series are

$$\varepsilon^n , \varepsilon^{\frac{n}{3}} , (\ln \varepsilon)^{-n} , (\sin \varepsilon)^n$$

2.4. Asymptotic Expansions

The series of terms written as

$$\sum_{n=0}^{N} c_n \phi_n(x) + O(\phi_{n+1}) \quad ,$$

where the c_n are the constants, is an asymptotic expansion of $f(x)$, with respect to the asymptotic sequence $\{\phi_n(x)\}$ if, for every $N \geq 0$,

$$f(x) - \sum_{n=0}^{N} c_n \phi_n(x) = o[\phi_n(x)] \quad \text{as} \quad x \to a \ .$$

If this expansion exists, it is unique in that the coefficients, c_n , are completely determined. (Johnson, 2005)

Clearly , an asymptotic series is a special case of an asymptotic expansion.

$$J_0(x) \sim \sqrt{\frac{2}{\pi x}} \left[u \ \cos\left(x - \frac{\pi}{4} \right) + v \sin\left(x - \frac{\pi}{4} \right) \right] \quad \text{as} \quad x \to \infty$$

where

$$u(x) = 1 - \frac{1^2.3^2}{4^2.2^2.2!\,x^2} + \frac{1^2.3^2.5^2.7^2}{4^4.2^4.4!\,x^4} + \ldots$$

$$v(x) = \frac{1}{4.2x} - \frac{1^2.3^2.5^2}{4^3.2^3.3!\,x^3} + \ldots$$

is an asymptotic expansion of Bessel’s function.

CHAPTER THREE

PERTURBATION THEORY

Perturbation theory is a collection of methods for the systematic analysis of the global behavior of solutions to differential and difference equations. The general procedure of perturbation theory is to identify a small parameter, usually denoted by ε, such that when $\varepsilon = 0$ the problem becomes soluble. The global solution to the given problem can then be studied by a local analysis about $\varepsilon = 0$. For example, the differential equation

$$y'' = \left(1 + \frac{\varepsilon}{1+x^2}\right)y$$

can only be solved in terms of elementary functions when $\varepsilon = 0$. A perturbative solution is constructed by local analysis about $\varepsilon = 0$ as a series of powers of ε;

$$y(x) = y_0(x) + \varepsilon y_1(x) + \varepsilon^2 y_2(x) + ...$$

This series is called a perturbation series. It has a property that $y_n(x)$ can be computed in terms of $y_0(x), y_1(x), ..., y_n(x)$ as long as the problem obtained by setting $\varepsilon = 0$, $y'' = y$ is soluble, which it is in this case. Notice that the perturbation series for $y(x)$ is local in ε but that it is global in x. If ε is very small, we expect that $y(x)$ will be well approximated by only a few terms of the perturbation series. (Bender&Orszag, 1978)

The thematic approach of perturbation theory is to decompose a given problem into an infinite number of relatively easy ones. (Bender&Orszag, 1978)

In perturbation theory, it is convenient to have an asymptotic order relation that express the relative magnitudes of two functions more precisely than << but less precisely than ~ . We define

$$f(x) = O[g(x)] \quad , \quad x \to x_0$$

and say that $f(x)$ is at most of order $g(x)$ as $x \to x_0$ or $f(x)$ is O of $g(x)$ as $x \to x_0$, if $\dfrac{f(x)}{g(x)}$ is bounded for x near x_0, that is $\left|\dfrac{f(x)}{g(x)}\right| < M$ for some constant M if x is sufficiently close to x_0. Observe that if $f(x) \sim g(x)$ or if $f(x) << g(x)$ as $x \to x_0$, then $f(x) = O[g(x)]$ as $x \to x_0$. If $f << g$ as $x \to x_0$, then any $M > 0$ satisfies the definition, while if $f \sim g$ $(x \to x_0)$ only $M > 1$ can work. (Bender&Orszag, 1978)

In perturbation theory we may calculate just a few terms in a perturbation series. Whether or not this series is convergent, the notation '*O*' is very useful for expressing the order of magnitude of the first neglected term when that term has not been calculated explicitly. (Bender&Orszag, 1978)

3.1 What is a perturbation problem?

We called an equation presenting a physical process a perturbation problem if it depends on a small dimensionless parameter ε .

There are two types of perturbation problems:

i)Regular perturbation problem

ii)Singular perturbation problem

We define a regular perturbation problem as one whose perturbation series is a power series in ε having a nonvanishing radius of convergence. A basic feature of all regular perturbation problems is that the exact solution for small but nonzero $|\varepsilon|$ smoothly approaches the unperturbed or zeroth-order solution as $\varepsilon \to 0$. We define a singular perturbation problem as one whose perturbation series either does not take the form of a power series or , if it does the power series has a vanishing radius of convergence. (Bender&Orszag,1978)

3.2 Example

(Approximate solution of an initial value problem) (Bender&Orszag, 1978)

Consider the initial-value problem

$$y'' = f(x)y \quad , \quad y(0) = 1 \quad , \quad y'(0) = 1 \tag{3.1}$$

where $f(x)$ is continuous. This problem has no closed-form solution except for very special choices for $f(x)$. Nevertheless, it can be solved perturbatively.

First, we introduce an ε in a such way that the unperturbed problem is solvable:

$$y'' = \varepsilon f(x)y \quad , \quad y(0) = 1 \quad , \quad y'(0) = 1 \tag{3.2}$$

Second, we assume a perturbation expansion for *y(x)* of the form

$$y(x) = \sum_{n=0}^{\infty} \varepsilon^n y_n(x) \tag{3.3}$$

where $y_0(0) = 1 \quad , \quad y_0'(0) = 1$ and

$$y_n(0) = 0 \quad , \quad y_n'(0) = 0 \quad (n \geq 1).$$

The zeroth order problem $y'' = 0$ is obtained by setting $\varepsilon = 0$ and the solution which satisfies the initial conditions is $y_0 = 1 + x$. The nth order problem $(n \geq 1)$ is obtained by substituting (3.3) into (3.2) and setting the coefficient of ε^n $(n \geq 1)$ equal to 0. The result is

$$y_n'' = y_{n-1} f(x) \quad , \quad y_n(0) = y_n'(0) = 0 \tag{3.4}$$

Observe that perturbation theory has replaced the intractable differential equation (3.1) with a sequence of inhomogeneous equations (3.4). In general, any inhomogeneous equation may be solved routinely by the method of variation of parameters whenever the solution of the associated homogeneous equation is known. Here the homogeneous equation is precisely the unperturbed equation. Thus, it is clear why it is so crucial that the unperturbed equation is soluble.

The solution to (3.4) is

$$y_n = \int_0^x dt \int_0^t ds\, f(s)\, y_{n-1}(s) \quad , \quad n \geq 1 \tag{3.5}$$

Equation (3.5) gives a simple iterative procedure for calculating successive terms in the perturbation series (3.3).

$$y(x) = 1 + x + \varepsilon \int_0^x dt \int_0^t ds (1+s) f(s) + \varepsilon^2 \int_0^x dt \int_0^t ds\, f(s) \int_0^s dv \int_0^v du (1+u) f(u) + \ldots \tag{3.6}$$

Third, we must sum this series. It is easy to show that when N is large, the Nth term in this series is bounded in absolute value by $\varepsilon^N x^{2N} K^N (1+|x|)/(2N)!$, where K is an upper bound for $|f(t)|$ in the interval $0 \leq |t| \leq |x|$. Thus the series (3.6) is convergent for all x. We also conclude that if $x^2 K$ is small, then the perturbation series is rapidly convergent for $\varepsilon = 1$ and an accurate solution to the original problem may be achieved by taking only a few terms.

3.3 Asymptotic Matching

Asymptotic matching is an important perturbative method which is used often in both boundary layer theory and WKB theory to determine analytically the approximate global properties of the solution to a differential equation. Asymptotic matching is usually used to determine a uniform approximation to the solution of a differential equation and to find other global properties of differential equations such as eigenvalues. Asymptotic matching may also be used to develop approximations to integrals. (Bender&Orszag, 1978)

The principle of asymptotic matching is simple. The interval on which a boundary value problem is posed is broken into a sequence of two or more overlapping subintervals. Then, on each subinterval perturbation theory is used to obtain an asymptotic approximation to the solution of the differential equation valid on this interval. Finally, the matching is done by requiring that the asymptotic approximations have the same functional form on the overlap of every pair of intervals. This gives a sequence of asymptotic approximations to the solution of the differential equation; by construction , each approximation satisfies all the boundary conditions given at various points on the interval. Thus, the end result is an approximate solution to a boundary value problem valid over the entire interval. (Bender&Orszag, 1978)

3.3.1 Van Dyke's Matching Principle

According the Van Dyke's matching rule;

The m-term inner expansion of (the n-term outer expansion) equals the n-term outer expansion of (the m-term inner expansion) (3.7)

where m and n may be any two integers that need not to be equal. To determine the m-term inner expansion of the (n-term outer expansion) , we rewrite the first n-terms of the outer expansion in terms of the inner variable, expand it for small ε with the inner variable being kept fixed, and truncate the resulting expansion after m-terms, and conversely for the right hand side of (3.7). (Nayfeh, 1993)

CHAPTER FOUR
BOUNDARY LAYER THEORY

Bender&Orszag (1978) noted that, the general perturbative methods are necessary to perform global analysis. There are three specific analytic techniques of global approximation theory; boundary layer theory, WKB theory and multiple scale analysis. And the most elementary one of the perturbative methods for solving a differential equation whose highest derivative is multiplied by the perturbing parameter, is the boundary layer technique.

Boundary layer theory is first studied by Prandtl.

Prandtl noted that when a fluid of low viscosity such as air or water flows about an obstacle, the ratio of viscous to inertial forces is small everywhere except in a narrow layer near the boundary of the obstacle. Using this observation, he was able to simplify considerably the analysis of the governing Navier-Stokes equations. His idea was that there is a region far from the obstacle where the flow is essentially the like the flow of an inviscid fluid. On the other hand, near the obstacle, where viscosity plays an important role in making the velocity equal to zero on the surface, the velocity changes much more rapidly along a perpendicular to the surface than along the surface itself. This suggests that we introduce the boundary layer as a short interval within which the solution of the differential equation changes very rapidly. (Simmonds and Man,1998)

4.1 What is a boundary layer?
What is a boundary layer problem?

A boundary layer is a narrow region where the solution of a differential equation changes rapidly. By definition,the thickness of a boundary layer must approach zero as $\varepsilon \to 0$. (Bender&Orszag, 1978)

Boundary layer problem is one of the types of singular perturbation problem.

If the highest derivative in the equation is multiplied by a small parameter ε , then this equation is called a boundary layer problem.

We give a simple boundary layer problem whose solution exhibits boundary layer structure.

4.2 Example

We give an exactly solvable boundary layer problem whose solution exhibits boundary layer structure. (Bender&Orszag,1978)

$$\varepsilon\, y'' + (1+\varepsilon) y' + y = 0 \quad , \quad y(0) = 0 \ , \ y(1) = 1 \tag{4.1}$$

The exact solution of this equation is

$$y(x) = \frac{e^{-x} - e^{-\frac{x}{\varepsilon}}}{e^{-1} - e^{-\frac{1}{\varepsilon}}} \tag{4.2}$$

In the limit $\varepsilon \to 0+$, this solution becomes discontinuous at $x = 0$. For very small ε the solution $y(x)$ is slowly varying for $\varepsilon << x \le 1$. However, on the small interval $0 \le x \le O(\varepsilon)$ ($\varepsilon \to 0+$) it has an abrupt and a rapid change. This small interval of rapid change is called a boundary layer. (The notation $0 \le x \le O(\varepsilon)$ means that the thickness of the boundary layer is proportional to ε as $\varepsilon \to 0+$.) The region of slow variation of $y(x)$ is called the outer region and the boundary layer region is called the inner region.

Boundary layer theory is a collection of perturbation methods for solving differential equations whose solution exhibit boundary layer structure. When the solution to a differential equation is slowly varying except in isolated boundary layers, then it may be relatively easy to obtain a leading-order approximation to that solution for small ε without directly solving the differential equation. (Bender&Orszag, 1978)

There are two standart approximations that one makes in boundary layer theory. In the outer region (away from the boundary layer) $y(x)$ slowly varying, so it is valid to neglect any derivatives of $y(x)$ which are multiplied by ε . Inside a boundary layer the derivatives of $y(x)$ are large, but the boundary layer is so narrow that we may approximate the coefficient functions of the differential equations by constants. Thus we can replace a single differential equation by a sequence of much simpler approximate equations in each of several inner and outer regions. In every region the solution of the approximate equation will contain one or more unknown constants of integration. These constants are then determined from the boundary and initial conditions using technique of asymptotic matching. (Bender&Orszag, 1978)

We explain these ideas by the following initial value problem.

4.3 Example

(First order nonlinear boundary value problem) (Bender&Orszag, 1978)

Consider the differential equation,

$$(x-\varepsilon y)y' + xy = e^{-x} \quad , \qquad y(1) = \frac{1}{\varepsilon} \tag{4.3}$$

We wish to determine a leading order perturbative approximation to $y(0)$ as $\varepsilon \to 0+$.

Although this is only a first order differential equation, it is nonlinear and is much too difficult to solve in closed form. However,in regions where y and y' are not large (such regions are called outer regions), it is valid to neglect $\varepsilon yy'$ compared with e^{-x} . Thus, in outer regions we approximate the solution to (4.3) by the solution to the outer equation

$$xy'_{out} + xy_{out} = e^{-x}$$

The equation is easy to solve because it is linear. The solution which satisfies $y_{out}(1) = \frac{1}{\varepsilon}$ is

$$y_{out} = (1 + \ln x)e^{-x} \tag{4.4}$$

Note that it is valid to impose the initial condition $y(1) = \frac{1}{\varepsilon}$ on $y_{out}(x)$ because x=1 lies in an outer region ; x=1 is in an outer region because (4.3) implies that $y'(1) = 0$, so $y(1)$ and $y'(1)$ are of order 1 as $\varepsilon \to 0+$.

As $x \to 0+$, both $y_{out}(x)$ and $y'_{out}(x)$ become larger. Thus, near x=0 the term $\varepsilon yy'$ is no longer negligible compared with e^{-x}. From the outer solution we can estimate that the thickness δ of the region in which $\varepsilon yy'$ is not small is given by

$$\frac{\delta}{\ln \delta} = O(\varepsilon) \quad , \quad \varepsilon \to 0+$$

Thus, $\delta \to 0+$ as $\varepsilon \to 0+$ (in fact $\delta = O(\varepsilon \ln \varepsilon)$ as $\varepsilon \to 0+$) , and there is a boundary layer of thickness δ at $x = 0$.

In the boundary layer (the inner region) , x is small so it is valid to approximate e^{-x} by 1. Furthermore, since y varies rapidly in the narrow boundary layer , we may neglect xy compared with xy' . Hence, in the inner region we approximate the solution to (4.3) by the solution to the inner equation

$$(x - \varepsilon y_{in})y_{in}{}' = 1$$

This is a linear equation if we regard x as the dependent variable. Its solution is

$$x = \varepsilon(y_{in} + 1) + Ce^{y_{in}} \tag{4.5}$$

where C is an unkown constant of integration. Since $x = 0$ is in the inner region, we may use (4.5) to find an approximation to $y(0) = 0$.

C is determined by asymptotically matching the outer and inner solutions (4.4) and (4.5) . Take *x* small but not as small as δ , say $x = O\left(\varepsilon^{\frac{1}{2}}\right)$. Then (4.4) implies that $y_{out} \sim 1 + \ln x$ as $\varepsilon \to 0+$ and (4.5) implies that $x \sim Ce^{y_{in}}$ as $\varepsilon \to 0+$. Thus, $C = \frac{1}{e}$ and a leading order implicit equation for $y_{in}(0)$ is

$$0 = \varepsilon\left[y_{in}(0) + 1\right] + e^{y_{in}(0) - 1} \tag{4.6}$$

When $\varepsilon = 0.1$ and 0.01 , the numerical solution of (4.6) are $y_{in}(0) \sim -1.683$ and $y_{in}(0) \sim -2.942$, respectively. These results compare favorably with the numerical solution to (4.3) which gives $y(0) \sim -1.508$ when $\varepsilon = 0.1$ and $y(0) \sim -2.875$ when $\varepsilon = 0.01$. For both values of ε the relative error between the perturbative and the numerical solution for $y(0)$ is about $\frac{1}{2}\varepsilon \ln \varepsilon$.

4.4 Mathematical structure of boundary layers: inner, outer limits. Thickness of boundary layer and intermediate limits

We take again the boundary value problem (given by (Bender&Orszag, 1978)) in section 4.2:

$$\varepsilon\, y'' + (1 + \varepsilon) y' + y = 0 \quad , \quad y(0) = 0 \, , \; y(1) = 1 \tag{4.1}$$

which has a boundary layer at *x*=0 when $\varepsilon \to 0+$ .The function

$$y(x) = \frac{e^{-x} - e^{-\frac{x}{\varepsilon}}}{e^{-1} - e^{-\frac{1}{\varepsilon}}} \tag{4.2}$$

has two components : e^{-x} , a slowly varying function on the entire interval $[0,1]$, and $e^{\frac{-x}{\varepsilon}}$, a rapidly varying function in the boundary layer $x \le O(\delta)$, where $\delta = O(\varepsilon)$ is the thickness of the boundary layer.

In boundary layer theory we treat the solution *y* of the differential equation as a function of two independent variables, *x* and ε . The goal of the analysis is to find a global approximation to y as a function of x ; this is achieved by performing a local analysis of y as $\varepsilon \to 0+$.

To explain the appearence of the boundary layer we introduce the notion of an inner and outer limit of the solution. The outer limit of the solution (4.2) is obtained by choosing a fixed x outside the boundary layer, that is, $\delta << x \le 1$,and allowing $\varepsilon \to 0+$. Thus, the outer limit is

$$y_{out}(x) \equiv \lim_{\varepsilon \to 0+} y(x) = e^{1-x} \quad . \tag{4.7}$$

The difference between the outer limit of the exact solution and the exact solution itself, $|y(x) - y_{out}(x)|$ is exponentially small in the limit $\varepsilon \to 0$ when $\delta << x$.

Similarly, we can formally take the outer limit of the differential equation (4.1) ; the result of keeping x fixed and letting $\varepsilon \to 0+$ is simply

$$y'_{out} + y_{out} = 0 \tag{4.8}$$

which is satisfied by (4.7). Because the outer limit of (4.2) is a first order differential equation , its solution cannot in general be required to satisfy both boundary conditions $y(0) = 0$ and $y(1) = 1$; the outer limit of (4.2) satisfies $y(1) = 1$ but not $y(0) = 0$.

In other words, the small ε limit of the solution is not everywhere close to the solution of the unperturbed differential equation (4.8) (the differential equation (4.1) with $\varepsilon = 0$). Thus the problem (4.1) is a singular perturbation problem. The singular behavior (the appearance of a discontinuity in $y(x)$ as $\varepsilon \to 0+$) occurs because the highest order derivative in (4.1) disappears when $\varepsilon = 0$.

The exact solution satisfies the boundary condition $y(0) = 0$ by developing a boundary layer in the neighborhood of $x = 0$. To examine the nature of this boundary layer, we consider the inner limit in which $\varepsilon \to 0+$ with $x \le O(\varepsilon)$. In this case x lies inside the boundary layer at $x = 0$. For this limit it is convenient to let $x = \varepsilon X$ with X fixed and finite. X is called an inner variable. X is a better variable than x to describe y in the boundary layer because, as $\varepsilon \to 0+$, y varies rapidly as a function of x but slowly as a function of X. (Bender&Orszag, 1978)

Determination of boundary layer thickness $\delta(\varepsilon)$:

Lots of boundary layers have the thickness $\delta = \varepsilon$. But, the thickness of a boundary layer need not be of order ε as $\varepsilon \to 0+$. To determine δ, we will use distinguished limits. We now explain the determination of the thickness of boundary layer for this problem.

In the inner region we let,

$$X = \frac{x}{\delta(\varepsilon)} \quad , \qquad \varepsilon \to 0+$$
$$y(x) = Y(X)$$

Then

$$\frac{dy}{dx} = \frac{dY}{dX}\frac{dX}{dx} = \frac{1}{\delta(\varepsilon)}\frac{dY}{dX}$$
$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{dY}{dX}\frac{dX}{dx}\right)$$
$$= \frac{1}{\delta(\varepsilon)}\frac{d^2Y}{dX^2}\frac{dX}{dx} = \frac{1}{\delta(\varepsilon)^2}\frac{d^2Y}{dX^2}$$

Substituting them into (4.1) , we get

$$\frac{\varepsilon}{\delta(\varepsilon)^2}\frac{d^2Y}{dX^2} + (1+\varepsilon)\frac{1}{\delta(\varepsilon)}\frac{dY}{dX} + Y = 0 \qquad (4.9)$$

To determine $\delta(\varepsilon)$ we must compare the coefficient of the highest order derivative with the others. That is we consider

$$\frac{\varepsilon}{\delta(\varepsilon)^2} \quad , \quad \frac{1}{\delta(\varepsilon)} \quad , \quad \frac{\varepsilon}{\delta(\varepsilon)} \quad , \quad 1$$

and investigate the following possibilities :

Case1: If $\dfrac{\varepsilon}{\delta(\varepsilon)^2} \sim \dfrac{1}{\delta(\varepsilon)} \quad \Rightarrow \quad \dfrac{\varepsilon}{\delta(\varepsilon)} \sim 1 \quad \Rightarrow \delta(\varepsilon) \sim \varepsilon$

Hence $\dfrac{\varepsilon}{\delta(\varepsilon)^2} \sim \dfrac{1}{\varepsilon}$ and $\dfrac{1}{\varepsilon} \to \infty$.

Taking $\delta(\varepsilon) \sim \varepsilon$ the other coefficients become $\dfrac{\varepsilon}{\delta(\varepsilon)} \sim 1$, 1 .

Since the coefficient of the highest derivative is the largest one, this is the case that we want.

Case2: If $\dfrac{\varepsilon}{\delta(\varepsilon)^2} \sim \dfrac{\varepsilon}{\delta(\varepsilon)} \quad \Rightarrow \quad \delta(\varepsilon) \sim 1$

Hence $\dfrac{\varepsilon}{\delta(\varepsilon)^2} \sim \varepsilon$ and this coefficient is small.

Taking $\delta(\varepsilon) \sim 1$, the other coefficients become $\dfrac{1}{\delta(\varepsilon)} \sim 1$, 1 .

Here the coefficient of the highest derivative is smaller than the others and we don't want this situation to occur.

Case3:If $\dfrac{\varepsilon}{\delta(\varepsilon)^2} \sim 1 \quad \Rightarrow \quad \delta(\varepsilon)^2 \sim \varepsilon \quad \Rightarrow \quad \delta(\varepsilon) \sim \sqrt{\varepsilon}$

Hence $\dfrac{\varepsilon}{\delta(\varepsilon)^2} \sim 1$

Taking $\delta(\varepsilon) = \sqrt{\varepsilon}$ the other coefficients become $\dfrac{1}{\delta(\varepsilon)} \sim \dfrac{1}{\sqrt{\varepsilon}}$, 1 .

Here $\dfrac{1}{\sqrt{\varepsilon}}$ is much larger than the coefficient of the highest derivative and we don't

want also this situation to occur.

Finally, we determined that $\delta(\varepsilon) \sim \varepsilon$ and this choice in (4.9) gives the leading order equation,

$$\frac{d^2Y}{dX^2} + (1+\varepsilon)\frac{dY}{dX} + \varepsilon Y = 0$$

The choice $\delta(\varepsilon) \sim \varepsilon$ is called a distinguished limit because it involves a nontrivial relation (a dominant balance) between two or more terms of the equation (4.9).(Bender&Orszag,1978). Here, two terms are of comparable size and the others are smaller. Case2 and Case3 are undistinguished. Generally, only the distinguished limit gives a nontrivial boundary layer structure which is asymptotically matchable to the outer solution.(Bender&Orszag, 1978)

After the determination of boundary layer thickness we return our problem.

It follows from (4.2) that

$$y_{in}(x) = Y_{in}(X) = \lim_{\varepsilon \to 0+} y(\varepsilon X) = e - e^{1-X} \tag{4.10}$$

Taking the inner limit of (15) , $\varepsilon \to 0+$, X fixed, gives

$$\frac{d^2Y_{in}(X)}{dX^2} + \frac{dY_{in}(X)}{dX} = 0 \tag{4.11}$$

Observe that, the inner limit function (4.10) does satisfy (14.11) together with the boundary condition $Y_{in}(0) = 0$.

Boundary layer analysis is extremely useful because it allows us to construct an approximate solution to a given differential equation, when an exact answer is not attainable. This is because the inner and outer limits of an insoluble differential equation are often soluble. Once y_{in} and y_{out} have been determined, they must be asymptotically matched. This asymptotic match occurs on the overlap region which is defined by the intermediate limit

$$x \to 0,\, X = \frac{x}{\varepsilon} \to \infty,\ \varepsilon \to 0+ \ .$$

A glance at (4.7) and (4.10) shows that the intermediate limits of $y_{out}(x)$ and

$y_{in}(x) = Y_{in}(X)$ agree:

$$\lim_{x\to 0} y_{out}(x) = \lim_{X\to\infty} Y_{in}(X) = e$$

This common limit verifies the asymptotic match between the inner and outer solutions… The above matching condition provides the second boundary condition for the solution of (4.11): $Y_{in}(\infty) = e$. Observe that although the x region is finite, $0 \le x \le 1$, the size of the matching region in terms of the inner variable is infinite. As we emphasized, the extent of the matching region must always be infinite.

A main problem in boundary layer theory is the question of whether or not an overlap region for any given problem actually exists. Since one's ability to construct a matched asymptotic expansion depend on the presence of this overlap region,its existence is crucial to the solution of the problem. How did we know , for example , that the intermediate limits of y_{out} and Y_{in} would agree? That is , how did we know that the inner and outer limits of the differential equation (4.1) have a common region of validity?

To answer these questions we will perform a complete perturbative solution of (4.1) to all orders in powers of ε , and not just to leading order.

Outer solution:

First, we examine the outer solution. We seek a perturbation expansion of the outer solution of the form

$$y_{out}(x) \sim \sum_{n=0}^{\infty} y_n(x)\varepsilon^n \qquad , \qquad \varepsilon \to 0+ \tag{4.12}$$

and restate the boundary condition $y(1) = 1$ as

$$y_0(1) = 1,\ y_1(1) = 0,\ y_2(1) = 0,\ \ldots \tag{4.13}$$

Now $y_{out}(x)$ in (4.12) is not the same as $y_{out}(x)$ in (4.7) , $y_{out}(x)$ in (4.7) is the first term $y_0(x)$ in (4.12).

Substituting (4.12) into (4.1) and collecting powers of ε gives a sequence of differential equations:

$$y_0' + y_0 = 0 \qquad , \qquad y_0(1) = 1$$
$$y_n' + y_n = -y_{n-1}'' - y_{n-1}' \qquad , \qquad y_n(1) = 0 \qquad n \geq 1$$

The solution to these equations is

$$y_0 = e^{1-x} \tag{4.14}$$
$$y_n = 0 \qquad , \qquad n \geq 1$$

Thus, the leading order outer solution , $y_{out} = e^{1-x}$, is correct to all orders in perturbation theory. This is the reason why in the outer region , $x >> \varepsilon$,the difference between $y(x)$ and $y_{out}(x)$ is at most exponentially small (subdominant) : $\left|y - y_{out}\right| = O\left(\varepsilon^n\right)$ for all n as $\varepsilon \to 0+$.

Inner solution:

We perform a similar expansion of the inner solution. We assume a perturbation series of the form

$$Y_{in}(X) \sim \sum_{n=0}^{\infty} \varepsilon^n Y_n(X) \qquad , \qquad \varepsilon \to 0+ \tag{4.15}$$

and restate the boundary condition $Y_{in} = y(0) = 0$ as

$$Y_n(0) = 0 \qquad , \qquad n \geq 0 \tag{4.16}$$

Substituting (4.15) into (4.10) gives the sequence of differential equations:

$$Y_0'' + Y_0' = 0 \qquad , \qquad Y(0) = 0$$
$$Y_n'' + Y_n' = -Y_{n-1}' - Y_{n-1} \qquad , \qquad Y_n(0) = 0 \qquad , \qquad n \geq 1$$

These equations may be solved by means of the integrating factor e^X . The results are

$$Y_0(X) = A_0\left(1 - e^{-X}\right)$$
$$Y_n(X) = \int_0^X \left(A_n e^{-z} - Y_{n-1}(z)\right) dz \quad , \; n \geq 1 \tag{4.17}$$

where the A_n are arbitrary integration constants.

Does this inner solution match asymptotically, order by order in powers of ε, to $y_{out}(x)$? To see if this is so, we subtitute $x = \varepsilon X$ into y_{out} in (4.14) and expand in powers of ε:

$$y_{out}(x) = e^{1-x} = e\left(1 - \varepsilon X + \frac{\varepsilon^2 X^2}{2!} - \frac{\varepsilon^3 X^3}{3!} + \ldots\right) \tag{4.18}$$

Returning to equation (4.17),we take *X* large ($X \to \infty$) and obtain $Y_0(X) \sim A_0$ $(X \to \infty)$.

Thus comparing the first term of (4.18), we have $A_0 = e$,as we already know. Now that Y_0 is known , we may compute Y_1 from (4.17):

$$Y_1(X) = (A_1 + A_0)\left(1 - e^{-X}\right) - eX$$

Asymptotic matching with y_{out} (comparing $Y_1(X)$,when $X \to \infty$, with the second term of (4.18)) gives $A_1 = -e$, so $Y_1(X) = -eX$. Similarly,

$$Y_n(X) = e[(-1^n / n!)]X^n$$

Hence the full expansion is

$$Y_{in}(X) = e\sum_{n=0}^{\infty} \varepsilon^n \frac{(-1)^n X^n}{n!} - e^{1-X} = e^{1-\varepsilon X} - e^{1-X} \tag{4.19}$$

Evidently, the inner expansion is a valid asymptotic expansion not only for values of X inside the boundary layer ($X = O(1)$) but also for large values of X [$X = O\left(\varepsilon^{-\alpha}\right)$, $0 < \alpha < 1$] . At the same time the expansion for $y_{out}(x)$ is valid for $\varepsilon << x \leq 1$ ($\varepsilon \to 0+$).[$y_{out}(x)$ is not valid for $x = O(\varepsilon)$ because it does not satisfy the boundary condition $y(0) = 0$; nor does it have the boundary layer term e^{1-X} which is present in $Y_{in}(X)$.] We conclude that to all orders in powers of ε it is possible to match asymptotically the inner and outer expansions because they have a common region of validity : $\varepsilon << x << 1$ $(\varepsilon \to 0+)$.

We have been able to demonstrate explicitly the existence of the overlap region for this problem because it is soluble to all orders in perturbation theory. In general , however, such a calculation is too difficult. Instead, our approach will always be to assume that an overlap region exists and then to verify the consistency of this assumption by performing an asymptotic match. In the above simple boundary value problem , we found that the size of the overlap region was independent of the order of perturbation theory. In general, however , the extent of the matching region may vary with the order of perturbation theory.

Uniform approximation:

Our final point concerns the construction of the uniform approximation to $y(x)$.

The formula used to construct a uniform approximation is

$$y_{unif}(x) = y_{out}(x) + y_{inner}(x) - y_{match}(x)$$

where $y_{match}(x)$ is the approximation to $y(x)$ in the matching region and $y_{unif}(x)$ is a uniform approximation to $y(x)$… For the boundary layer solution to (4.1) , it is easy to verify that if $y_{out}(x)$, $y_{in}(x)$ and $y_{match}(x)$ are calculated to nth order in perturbation theory, then it can be shown that

$$\left| y(x) - y_{unif}(x) \right| = O(\varepsilon^{n+1}) \qquad (\varepsilon \to 0+ \;\; ; \quad 0 \le x \le 1)$$

In definition of $y_{unif}(x)$, $y_{match}(x)$ is defined as

$$y_{match}(x) = y_{out}(y_{in}(x)) \qquad ,$$

or

$$y_{match}(x) = y_{in}(y_{out}(x))$$

Since $y_{unif}(x)$ reproduces the outer expansion in the outer domain and the inner expansion in the inner domain, we can say that it is valid everywhere.

Let’s find $y_{out}(y_{in}(x))$ for this problem.

$$Y_{in}(X) = e^{1-\varepsilon X} - e^{1-X}$$

$$= e^{1-x} - e^{1-\frac{x}{\varepsilon}} \qquad (\varepsilon \to +0)$$

$$= e^{1-x}$$

Hence

$$y_{out}(y_{in}(x)) = e^{1-x}$$

and $y_{unif}(x) = e^{1-x} - e^{1-X}$ is the finite order uniform approximation to $y(x)$.

It is remarkable, however, that this expression, which is the result of summing up perturbation theory to infinite order , is actually not equal to $y(x)$ in (4.2). Thus, although the perturbation series for $y_{unif}(x)$ is asymptotic to $y(x)$ as $\varepsilon \to 0+$, the asymptotic series does not converge to $y(x)$ as n, the order of perturbation theory, tends to ∞; there is an exponentially small error, of order $e^{\frac{-1}{\varepsilon}}$, which remains undetermined. Boundary layer theory is indeed a singular and, not a regular, perturbation theory.

Why is boundary layer theory a singular perturbation theory? The singular nature of boundary layer theory is intrinsic to both the inner and outer expansions. The outer expansion is singular because there is an abrupt change in the order of the differential equation when $\varepsilon = 0$. By contrast, the inner expansion is a regular perturbation expansion for finite *X*. However, since asymptotic matching takes place in the limit $X \to \infty$, the inner expansion is also singular. Another manifestation of the singular limit $\varepsilon \to 0$ is the location of the boundary layer in (4.2); when the limit $\varepsilon \to 0+$ is replaced by $\varepsilon \to 0-$, the boundary layer abruptly jumps from *x*=0 to *x*=1.

4.5 Higher Order Boundary Layer Theory

In sections 4.1 and 4.4 we formulated the procedure for finding the leading-order boundary layer approximation to the solution of an ordinary differential equation , i.e. , to obtain outer and inner solutions and then asymptotically match them in an overlap region. The self-consistency of boundary layer theory depends on the success of asymptotic matching. Ordinarily, if the inner and outer solutions match to all orders in ε, then boundary layer theory gives an asymptotic approximation to the exact solution of the differential equation. (Bender&Orszag, 1978)

We will solve one problem in Chapter Five, to illustrate how boundary layer theory is used to construct higher order approximations.

CHAPTER FIVE

NONLINEAR BOUNDARY LAYER PROBLEMS

Boundary layer analysis applies to nonlinear as well as linear equations. (Bender&Orszag, 1978)

In this section, first of all, we give an example for nonlinear problems. Then we will give three nonlinear problems that we solved.

5.1 Example

Boundary layer analysis of a nonlinear problem (Bender&Orszag, 1978)

We consider the following nonlinear boundary value problem of a type first proposed by Carrier:

$$\varepsilon y'' + 2\left(1 - x^2\right)y + y^2 = 1, \qquad y(-1) = y(1) = 0 \tag{5.1}$$

If we attemp a leading order boundary layer analysis of (5.1), we are immediately surprised to find that the outer equation obtained by setting $\varepsilon = 0$ is an algebraic equation rather than a differential equation:

$$y^2{}_{out} + 2\left(1 - x^2\right)y_{out} - 1 = 0$$

Because this equation is quadratic, it has two solutions

$$y_{out,\pm}(x) = x^2 - 1 \pm \sqrt{1 + (1 - x^2)} \tag{5.2}$$

In Figure 1 we plot the two outer solutions. Observe that neither one satisfies the boundary conditions at $x = \pm 1$. Therefore, there must be boundary layers at $x = -1$ and $x = +1$ which allow the boundary conditions to be satisfied. The question is, which of the two outer solutions can be joined the inner solutions which satisfy the boundary conditions?

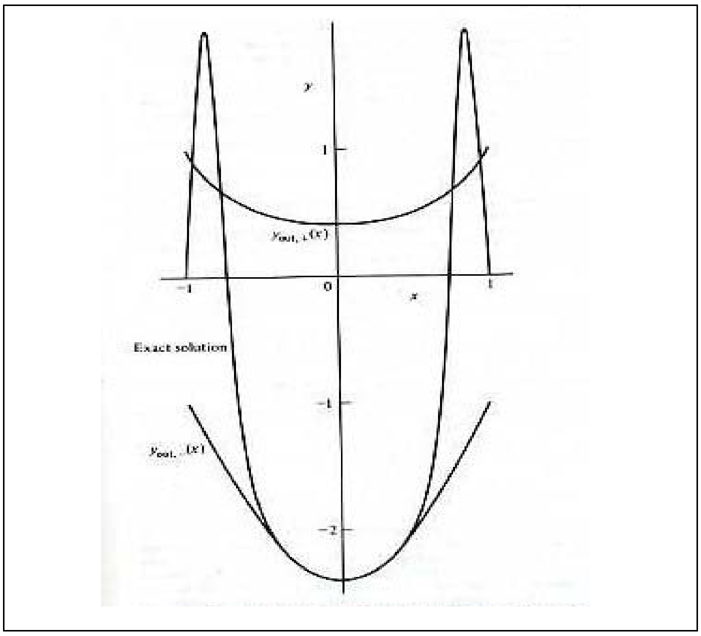

Figure 1 Exact solution for the nonlinear boundary value problem in (5.1) for $\varepsilon = 0.01$. Also shown are the two outer approximations $y_{out,\pm}$ in (5.2)

Let us examine the boundary layer at $x = 1$. If we substitute the inner variables $X = \dfrac{(1-x)}{\delta}$, $Y_{in}(X) = y(x)$ into (5.1) , we obtain in leading order

$$\frac{d^2Y_{in}}{dX^2} + \frac{\delta^2}{\varepsilon}\left(Y^2{}_{in} - 1\right) = 0$$

Thus, the distinguished limit is $\delta = \sqrt{\varepsilon}$. The solution to the leading order inner equation

$$\frac{d^2Y_{in}}{dX^2} + \left(Y^2{}_{in} - 1\right) = 0 \tag{5.3}$$

must satisfy the boundary condition $Y_{in}(0) = 0$ and match asymptotically with one (or both) of the outer solutions. That is, Y_{in} must approach either ± 1 as $X \to +\infty$.

Is it possible for Y_{in} to approach 1 as $X \to +\infty$? Suppose we let $Y_{in} = 1 + W(X)$. If $Y_{in} \to 1$, then $W(X) \to 0$ and we can replace (5.3) with the approximate linear equation $W'' + 2W = 0$. However, solutions to this equations oscillate as $X \to +\infty$

and do not approach to 0. This simple analysis shows that it is not possible for Y_{in} to match to $y_{out,+}$ in (5.2).

Fortunately, the same argument suggests that it is possible for Y_{in} to match to $y_{out,-}$. Let $Y_{in} = -1 + W(X)$. Now if $Y_{in} \to -1$ then $W \to 0$ and we can replace (5.3) by the approximate linear equation $W'' - 2W = 0$. Since this equation has a solution which decays to 0 exponentially , it is at least consistent to assume that Y_{in} matches asymptotically with $y_{out,-}$.

Having established this much, let us solve the inner equation exactly. Substituting $Y_{in} = -1 + W(X)$ into (5.3) gives the autonomous equation

$$W'' + W^2 - 2W = 0 \tag{5.4}$$

subject to the boundary conditions $W(\infty) = 0$, $W(0) = 1$. Also since we expect W to decay exponentially as $X \to +\infty$, we may assume that $W'(\infty) = 0$. To solve (5.4) we multiply by $W'(X)$, integrate the equation once, and determine the integration constant by setting $X = \infty$. We obtain

$$\frac{1}{2}(W')^2 + \frac{1}{3}W^3 - W^2 = 0,$$

which is a separable first order equation:

$$\frac{dW}{W\sqrt{2 - \frac{2W}{3}}} = \pm dX$$

integrating this equation gives

$$-\sqrt{2}\tanh^{-1}\sqrt{1 - \frac{W}{3}} = \pm X + C$$

The integration constant is determined by the requirement that $W = 1$ at $X = 0$. Hence, there are two solutions:

$$Y_{in}(X) = -1 + 3\sec h^2\left(\pm\left(\frac{X}{\sqrt{2}} + \tanh^{-1}\sqrt{\frac{2}{3}}\right)\right) \tag{5.5}$$

There are two inner solutions at $x = -1$ which satisfy the boundary condition $y(-1) = 0$ and match to the lower outer solution $y_{out,-}$ in (5.2).

We can combine the outer with the two inner solutions to form a single uniform approximation valid over the entire interval $-1 \le x \le 1$:

$$y_{unif}(x) = x^2 - 1 - \sqrt{1 + (1 - x^2)} + 3\sec h^2\left(\pm \frac{1-x}{\sqrt{2\varepsilon}} + \tanh^{-1}\sqrt{\frac{2}{3}} \right) + 3\sec h^2\left(\pm \frac{1+x}{\sqrt{2\varepsilon}} + \tanh^{-1}\sqrt{\frac{2}{3}} \right) \tag{5.6}$$

Notice that the solution in (5.6) is not unique. There are actually four different solutions depending on the two choices of plus or minus signs in in the boundary layer. For one of the choice of sign, $y_{unif}(x)$ in the boundary layer rapidly descends from its boundary value $y(\pm 1) = 0$ until it joins on the outer solution $y_{out,-}$. For the other choice of sign , $y_{unif}(x)$ rises rapidly until it reaches a maximum and then descends and joins onto the outer solution. It is easy to see that this maximum value of y_{unif} is 2 because the maximum value of sech is 1. It is a glorious triumph of boundary later theory that all four solutions actually exist and are extremely well approximated by the leading order uniform approximation in (5.6). See Figures 1 to 3.

The analysis does not end here, however. The existence of four solutions to (5.1) may lead one to wonder if there are stil more solutions. One may begin by asking whether there can be any internal boundary layers. We will now show that internal boundary layers are consistent.

Assume there is an internal boundary layer at $x = 0$. The thickness of such a boundary layer is $\delta = \sqrt{\varepsilon}$. The leading-order equation is

$$Y''_{in}(X) + 2Y_{in} + Y^2{}_{in} = 1 \tag{5.7}$$

Since $y_{out,-}(0) = -1 - \sqrt{2}$, the boundary condition on Y_{in} in (5.7) are $\lim_{X \to \pm\infty} Y_{in}(X) = -1 - \sqrt{2}$. The exact solution to (5.7) which satisfies these boundary conditions contains an arbitrary parameter A :

$$Y_{in} = 3\sqrt{2}\sec h^2\left(2^{-1/4} x / \sqrt{\varepsilon} + A\right) - 1 - \sqrt{2}$$

Noted that if $A = \pm\infty$ then there is no internal boundary-layer structure. However, for all finite values of A there is a narrow region in which Y rises abruptly to a sharp peak at which it attains a maximum value of $2\sqrt{2-1} \sim 1.8$ …. In fact, in Figs 4 to 6 we see that for each solution in Figs. 1 to 3 there is another solution which is almost identical except that it exhibits a boundary layer at $x = 0$. What is more, the maximum in the boundary layer is close to 1.8 .

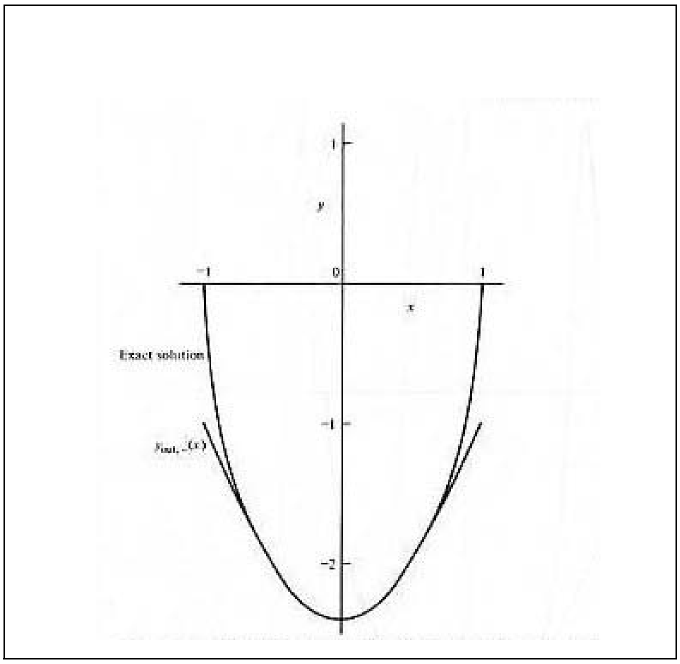

Figure 2 A different solution for the equation as in Figure 1. y_{unif} becomes a good approximation to the plotted solution for the upper choice of sign.

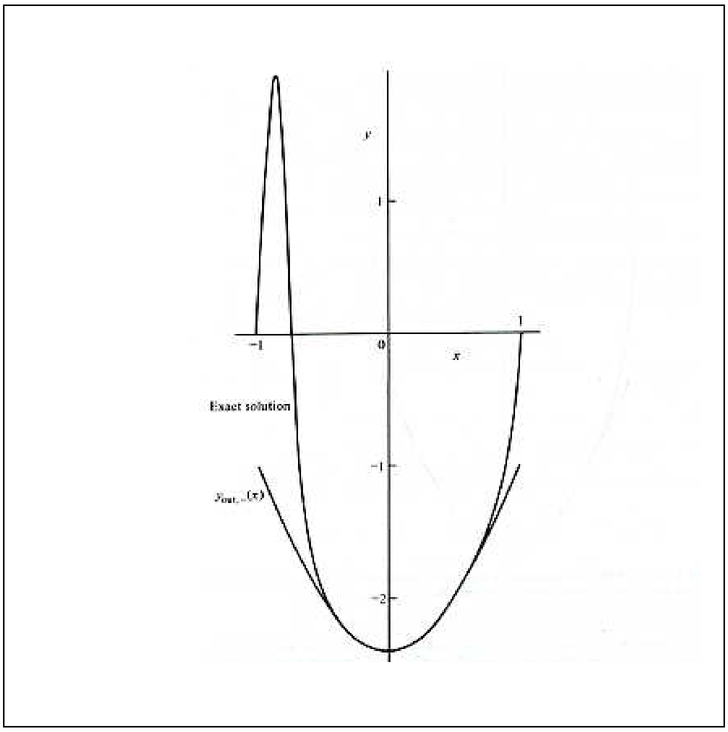

Figure 3 Same differential equation as in Figure 1. y_{unif} in (5.6) is a good approximation to the plotted solution for one upper sign and one lower sign. There is also another solution which is the reflection about the y axis of the one shown here.

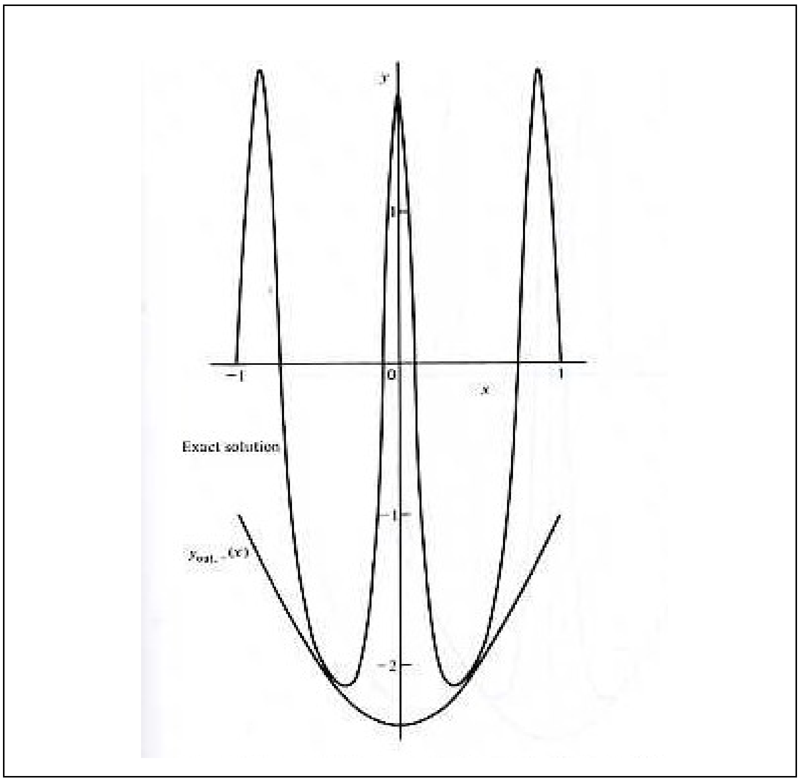

Figure 4 An exact solution to the boundary value problem in (5.1). Apart from the internal boundary layer at $x = 0$, this solution is nearly identical to the solution in Figure 1. The outer approximation $y_{out,-}(x)$ in (5.2) is a good approximation to $y(x)$ between the boundary layers.

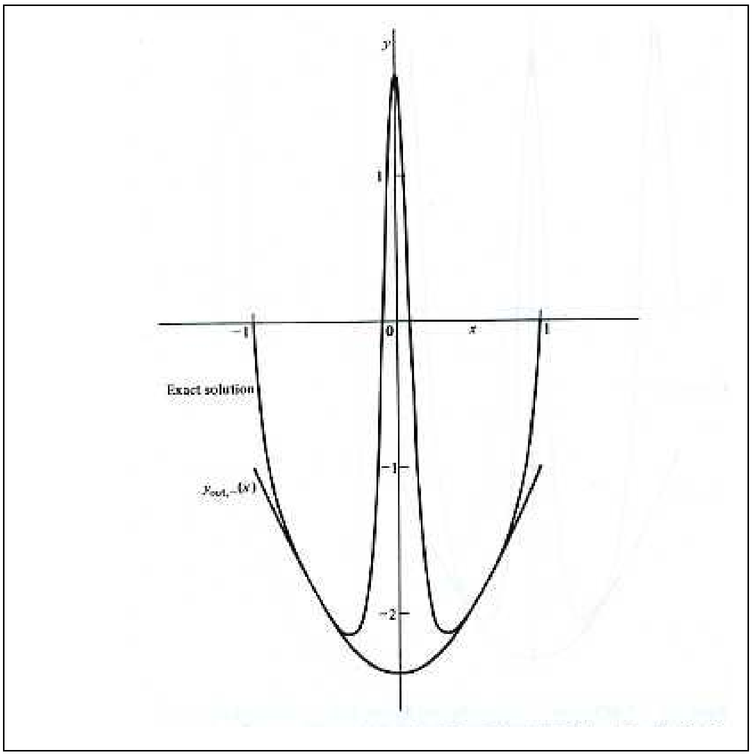

Figure 5 An exact solution to the boundary value problem in (5.1). Apart from the internal boundary layer at $x = 0$, this solution is nearly identical to that in Figure 2.

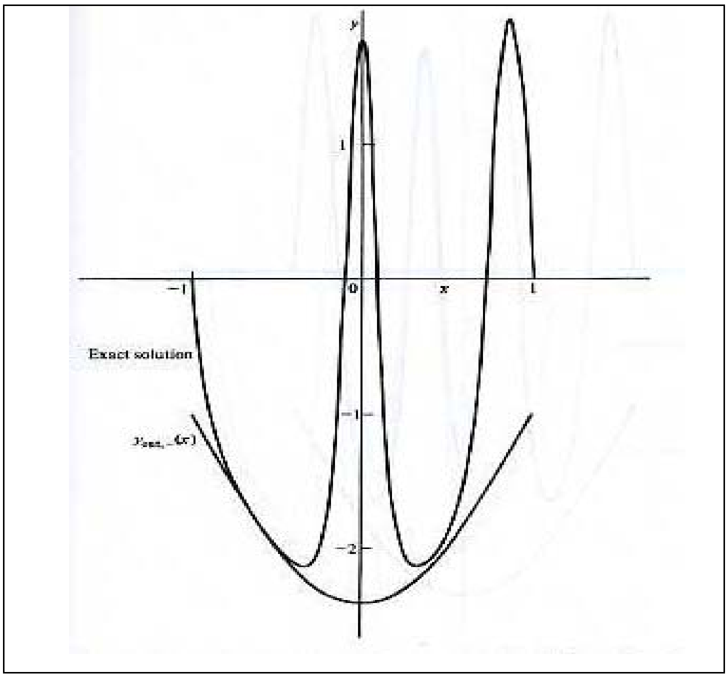

Figure 6 An exact solution to the boundary value problem in (5.1). Apart from the internal boundary layer at $x = 0$, this solution is nearly identical to that in Figure 3 reflected about the y axis.

5.2 Problems and Solutions

Here, we give three nonlinear problems that we solved.

5.2.1 Problem 1

This problem is given in Bender&Orszag (1978) with leading order analysis. Now we will find its two-term approximate solution.

We consider the boundary value problem

$$\varepsilon y'' + 2y' + e^{y} = 0 \tag{5.8}$$

$$y(0) = y(1) = 0 \tag{5.9}$$

where the small parameter ε, $0 < \varepsilon << 1$, multiplies the highest derivative, and hence, a boundary layer is expected.

If e^y were a linear function of y , there would be a boundary layer at *x*=0 (and no boundary layer at *x*=1) because the coefficient of y' is positive. This nonlinear problem also has just one boundary layer at *x*=0. (Bender&Orszag, 1978)

Let us write an expansion for y(x) as

$$y(x) = y_0(x) + \varepsilon y_1(x) + ... \tag{5.10}$$

and then substitute it into (5.8).

$$\varepsilon(y_0'' + \varepsilon y_1'' + ...) + 2\left(y_0' + \varepsilon y_1' + ...\right) + e^{(y_0 + \varepsilon y_1 + ...)} = 0$$

Now, we will collect the terms w.r.t ε power and we will solve the differential equation that we get.

$$\begin{aligned}
&\varepsilon y_0'' + \varepsilon^2 y_1'' + 2y_0' + 2\varepsilon y_1' + e^{(y_0 + \varepsilon y_1 + ...)} + ... = 0 \\
&\varepsilon y_0'' + \varepsilon^2 y_1'' + 2y_0' + 2\varepsilon y_1' + \left(e^{y_0} . e^{\varepsilon y_1} . \; ....\right) + ... = 0 \\
&\varepsilon y_0'' + \varepsilon^2 y_1'' + 2y_0' + 2\varepsilon y_1' + e^{y_0}\left(1 + \varepsilon y_1 + ...\right) + ... = 0 \\
&\left(2y_0' + e^{y_0}\right) + \varepsilon\left(y_0'' + 2y_1' + e^{y_0}\, y_1\right) + O(\varepsilon^2) = 0
\end{aligned}$$

ε^0 order : $$2y_0' + e^{y_0} = 0 \tag{5.11}$$

ε^1 order : $$2y_1' + e^{y_0}\, y_1 = -y_0'' \tag{5.12}$$

After solving these equations, we will have y_0 and y_1. Then we will substitute them into (5.10) and hence we will get an approximate solution for $y(x)$. But there are some important things in our analysis.

First of all, we have seen that our equations (5.11) and (5.12) are of first order while equation (5.8) is of second order. Here, we have two boundary conditions and they can't satisfy two boundary conditions. Because of this, we think that there is a boundary layer and our expansion for y(x) is not valid near the boundary layer. Now, the important question is "Where is the boundary layer?" and "Which condition must be eliminated?".

First of all, let us solve equations (5.11) and (5.12).

$$2y_0' + e^{y_0} = 0 \tag{5.11}$$

The solution of (5.11) is

$$y_0(x) = \ln\left(\frac{2}{x+C}\right) \tag{5.13}$$

And , then

$$2y_1' + e^{y_0}\, y_1 = -y_0'' \tag{5.12}$$

$$y_0(x) = \ln\left(\frac{2}{x+C}\right) \;\Rightarrow\; e^{y_0} = \frac{2}{x+C}$$

$$y_0'(x) = -\frac{1}{x+C}$$

$$y_0''(x) = \frac{1}{(x+C)^2}$$

$$\Rightarrow 2y_1' + \frac{2}{x+C}\, y_1 = -\frac{1}{(x+C)^2}$$

$$y_1' + \frac{1}{x+C}\, y_1 = -\frac{1}{2(x+C)^2} \tag{5.14}$$

Here

$$p(x) = \frac{1}{x+C} \quad , \quad q(x) = -\frac{1}{2(x+C)^2}$$

Let us find the integrating factor

$$\mu(x) = e^{\int \frac{1}{x+C} dx} \qquad (u = x+C \quad , \quad du = dx)$$

$$= e^{\int \frac{du}{u}} = e^{\ln(x+C)+a} = A(x+C)$$

Let us multiply both sides of (5.14) with $\mu(x)$.

$$A(x+C)y_1{}' + Ay_1 = -\frac{A}{2(x+C)}$$

$$\frac{d}{dx}\left(A(x+C)\,y_1\right) = -\frac{A}{2(x+C)}$$

$$A(x+C)y_1 = -\frac{A}{2}\int \frac{dx}{x+C} \qquad , u = x+C \text{ , } du = dx$$

$$= -\frac{A}{2}\int \frac{du}{u} = -\frac{A}{2}\left(\ln(x+C)+D\right)$$

$$y_1(x) = -\frac{1}{2(x+C)}\left(\ln(x+C)+D\right)$$

Now , it's time to determine where the boundary layer is. Let us examine the case that the boundary layer is near $x=1$. Hence the outer solution must satisfy $y(0)=0$ condition.

$$y_0(x) = \ln\left(\frac{2}{x+C}\right)$$

$$y_0(0) = \ln\left(\frac{2}{C}\right) = 0 \;\Rightarrow\; C = 2$$

$$y_1(x) = -\frac{1}{2(x+2)}\left(\ln(x+2)+D\right)$$

$$y_1(x) = -\frac{1}{4}\left(\ln 2 + D\right) = 0 \Rightarrow D = -\ln 2$$

After determining the constants C and D, now we have an expansion for $y_{out}(x)$ as

$$y_{out}(x) = \ln\left(\frac{2}{x+2}\right) + \varepsilon\left(-\frac{1}{2(x+2)}\left[\ln(x+2) - \ln 2\right]\right) + \ldots \qquad (5.15)$$

As we mentioned before, this outer solution is valid on the interval except boundary layer. Since this outer solution isn't valid for boundary layer and the solution of (5.8) varies so fast , we need a new variable for boundary layer. We call this new variable as “stretched”or “magnified” variable.

Our hypothesis for boundary layer location is that there is a boundary layer near $x=1$. Hence let us introduce our new variables for the region. We call the new independent variable as X and the new dependent variable as Y.

Let us determine the new variables. In the equations that we solved, we use Y instead of Y_{in}.

$$X=\frac{1-x}{\delta(\varepsilon)}\ ,\ Y(X)=y(x) \tag{5.16}$$

Since we changed the variables, we must write our equation (5.8) in terms of X and Y. To do this change, we must define our new differentiation term.

$$\frac{dy}{dx}=\frac{dY}{dX}\frac{dX}{dx}=-\frac{1}{\delta(\varepsilon)}\frac{dY}{dX}\ ,\ \frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=-\frac{1}{\delta(\varepsilon)}\frac{d^2Y}{dX^2}\frac{dX}{dx}=\frac{1}{\delta(\varepsilon)^2}\frac{d^2Y}{dX^2} \tag{5.17}$$

Let us substitute (5.17) into (5.8), then we get

$$\frac{\varepsilon}{\delta(\varepsilon)^2}\frac{d^2Y}{dX^2}-2\frac{1}{\delta(\varepsilon)}\frac{dY}{dX}+e^Y=0 \tag{5.18}$$

Here , $\delta(\varepsilon)$ is the thickness of the boundary layer. Now, we must determine $\delta(\varepsilon)$ by using balance between the terms. We have two possibilities for this analysis.

i) $\dfrac{\varepsilon}{\delta(\varepsilon)^2}\sim\dfrac{1}{\delta(\varepsilon)}\ \Rightarrow\ \dfrac{\varepsilon}{\delta(\varepsilon)}\sim 1\ \Rightarrow\ \delta(\varepsilon)\sim\varepsilon$

If $\delta(\varepsilon)\sim\varepsilon$ then equation (5.18) becomes

$$\frac{d^2Y}{dX^2}-2\frac{dY}{dX}+\varepsilon e^Y=0 \tag{5.19}$$

This is the case we want. We call (5.19) as the distinguished limit for the equation.

ii) $\dfrac{\varepsilon}{\delta(\varepsilon)^2}\sim 1\ \Rightarrow\ \varepsilon\sim\delta(\varepsilon)^2\ \Rightarrow\ \delta(\varepsilon)\sim\sqrt{\varepsilon}$

If $\delta(\varepsilon)\sim\sqrt{\varepsilon}$ then equation (5.18) becomes

$$\frac{d^2Y}{dX^2}-2\frac{1}{\sqrt{\varepsilon}}\frac{dY}{dX}+e^{Y}=0 \qquad (5.20)$$

In this case, the coefficient of the second term is $\frac{1}{\sqrt{\varepsilon}}$ and $\frac{1}{\sqrt{\varepsilon}} \to \infty$ as $\varepsilon \to 0+$.

And the coefficient of the derivative term is very small compared with $\frac{1}{\sqrt{\varepsilon}}$. We don't want this situation to occur, hence we omit second case for $\delta(\varepsilon)$.

Now, we have determined the thickness of boundary layer as $\delta(\varepsilon) \sim \varepsilon$. Our new equation for the boundary layer is now given by (5.19).

$$\frac{d^2Y}{dX^2}-2\frac{dY}{dX}+\varepsilon e^{Y}=0 \qquad (5.19)$$

As we have done before, now we will write an expansion for $Y(X)$ as

$$Y(X)=Y_0(X)+\varepsilon Y_1(X)+... \qquad (5.21)$$

and substitute (5.21) into (5.19), then we get

$$\left(Y_0'' +\varepsilon Y_1'' +...\right)-2\left(Y_0' +\varepsilon Y_1' +...\right)+\varepsilon e^{(Y_0+\varepsilon Y_1+...)}=0 \qquad (5.22)$$

Now we have equation series to solve. To determine our expansion for $Y(X)$, we will find $Y_0(X)$ and $Y_1(X)$.

Let us write the series expansion for $e^{(Y_0+\varepsilon Y_1+...)}$ term.

$$\left(Y_0'' +\varepsilon Y_1'' +...\right)-2\left(Y_0' +\varepsilon Y_1' +...\right)+\varepsilon\left(1+Y_0+\varepsilon Y_1+...\right)=0 \qquad (5.23)$$

Let us collect the terms w.r.t. their ε power.

ε^0 order : $\quad Y_0'' -2Y_0' =0 \qquad (5.24)$

ε^1 order : $\quad Y_1'' - 2Y_1' + 1 + Y_0 = 0$ (5.25)

Now let us solve (5.24) and (5.25).

$$Y_0'' - 2Y_0' = 0 \tag{5.24}$$

$$m^2 - 2m = 0$$

$$m(m-2) = 0$$

$$Y_0(X) = A_0 + B_0 e^{2X} \tag{5.26}$$

Since we think that there is a boundary layer near $x = 1$, then we will use the boundary condition $y(1) = 0$. But this condition is valid for x, so we need to rearrange this condition for X .

$$X = \frac{1-x}{\delta(\varepsilon)} = \frac{1-x}{\varepsilon} \quad \Rightarrow \quad X = 0 \quad \text{for} \quad x = 1$$

New condition is $Y(0) = 0$. Let us use it.

$$Y_0(X) = A_0 + B_0 e^{2X} \quad \Rightarrow \quad Y_0(0) = A_0 + B_0 = 0 \quad \Rightarrow \quad B_0 = -A_0$$

Now we have

$$Y_0(X) = A_0 - A_0 e^{2X} = A_0\left(1 - e^{2X}\right) \tag{5.27}$$

After this, let us solve (5.25).

$$Y_1'' - 2Y_1' = -Y_0 - 1 \tag{5.25}$$

$$Y_1'' - 2Y_1' = -A_0\left(1 - e^{2X}\right) - 1 \tag{5.25a}$$

$$Y_{1,h}(X) = C_0 + D_0 e^{2X}$$

$$Y_{1,p}(X) = aX + bXe^{2X}$$

$$Y_{1,p}'(X) = a + be^{2X} + 2bXe^{2X}$$

$$Y_{1,p}''(X) = 2be^{2X} + 2be^{2X} + 4bXe^{2X}$$

Let us substitute them into (5.25a)

$$4be^{2X}+4bXe^{2X}-2a-2be^{2X}-4bXe^{2X}=-A_0\left(1-e^{2X}\right)-1$$

$$2b=A_0 \qquad -2a=-\left(A_0+1\right)$$

$$b=\frac{A_0}{2} \qquad a=\frac{1}{2}\left(A_0+1\right)$$

$$Y_{1,p}(X)=\frac{1}{2}\left(A_0+1\right)X+\frac{1}{2}A_0Xe^{2X}$$

Hence,

$$Y_1(X)=C_0+D_0e^{2X}+\frac{1}{2}\left(A_0+1\right)X+\frac{1}{2}A_0Xe^{2X} \tag{5.28}$$

Using the boundary condition $Y_1(0)=0$, we get

$$Y_1(X)=C_0\left(1-e^{2X}\right)+\frac{1}{2}\left(A_0+1\right)X+\frac{1}{2}A_0Xe^{2X} \tag{5.29}$$

Since our approximate solution for $Y(X)$ in the form

$$Y(X)=Y_0(X)+\varepsilon Y_1(X)+\ldots$$

then we get

$$Y(X)=A_0\left(1-e^{2X}\right)+\varepsilon\left[C_0\left(1-e^{2X}\right)+\frac{1}{2}\left(A_0+1\right)X+\frac{1}{2}A_0Xe^{2X}\right]+\ldots \tag{5.30}$$

Since we think that there is a boundary layer near $x=1$; now we have two solutions valid for different regions. The outer solution $y(x)$ valid for the outer region i.e. $0\le x<\varepsilon$ and the inner solution $Y(X)$ valid for the inner region i.e. $\varepsilon<x\le 1$.

Now our aim is to determine a uniform solution valid in the whole interval.

To determine the location of the boundary layer, we assume that it exists at one of the ends. Then, we carry out one-term expansions. If neighboring expansions can be matched, our assumption is correct; otherwise, the boundary layer exists at the other end. (Nayfeh, 1993)

To match the inner and outer solution, first we replace y and Y with y^o and y^i. Then we write $y^o(x)$ in terms of X and $y^i(X)$ in terms of *x*. After this, we will expand them for small ε and the terms *x* and *X* fixed. Doing this, we will get $\left(y^o\right)^i$ and $\left(y^i\right)^o$ and then we will matched the terms. Hence, we will find the undetermined coefficients in y^o and y^i .

In the end of this matching process, we will get $\left(y^o\right)^i$ and $\left(y^i\right)^o$ and they will be the same. We will call this expansion y_{match} .

Now let us return to our problem (5.8) and (5.9).

We get the outer expansion

$$y^o(x)=\ln\left(\frac{2}{x+2}\right)+\varepsilon\left(-\frac{1}{2(x+2)}\left[\ln(x+2)-\ln 2\right]\right)+\ldots \tag{5.15}$$

Let us write the Taylor expansions of the terms

$$\ln\left(\frac{2}{x+2}\right)=-\frac{1}{2}x+\ldots$$

$$-\frac{1}{2(x+2)}=-\frac{1}{2}\left(\frac{1}{2}-\frac{1}{4}x+\ldots\right)$$

$$\ln(x+2)=\ln 2+\frac{1}{2}x+\ldots$$

Now

$$y_{out}(x)\sim-\frac{1}{2}x+\varepsilon\left(-\frac{1}{2}\left(\frac{1}{2}-\frac{1}{4}x+\ldots\right)\left(\ln 2+\frac{1}{2}x+\ldots-\ln 2\right)\right)$$

And for the inner solution we get

$$Y(X) = A_0(1-e^{2X}) + \varepsilon\left[C_0(1-e^{2X}) + \frac{1}{2}(A_0+1)X + \frac{1}{2}A_0Xe^{2X}\right] + \ldots$$

Writing the Taylor expansion of the terms, we have

$$e^{2X} = 1 + 2X + \ldots$$

Writing it into the inner solution,

$$Y(X) \sim A_0(-2X+\ldots) + \varepsilon\left[C_0(-2X+\ldots) + \frac{1}{2}(A_0+1)X + \frac{1}{2}A_0X(1+2X+\ldots)\right]$$

For the matching at leading order, we keep only the first terms and match them. But, this is impossible. Also, for the first order matching, let us consider the terms with coefficient ε^0, ε^1. Keeping only the terms of order ε, x, and discarding the terms of order $\varepsilon^2, x^2, \varepsilon x$ for the outer expansion and $\varepsilon^2, \varepsilon^2X^2, \varepsilon^2X$ for the inner expansion , then we have

$$y_{out}(x) = -\frac{1}{2}x + \ldots$$

$$Y(X) = -2A_0X - 2\varepsilon C_0X + \frac{1}{2}\varepsilon(A_0+1)X + \frac{1}{2}\varepsilon A_0X + \ldots$$

These terms also cannot be matched.

Hence, we couldn't get y_{match}. Now let us return to our hypothesis on the location of boundary layer. Since the matching process has failed, we think that the boundary layer must be at the other end of the interval.

Now, we will do the same things, thinking that the boundary layer is at $x=0$.
Hence, our outer solution

$$y_{out}(x) = \ln\left(\frac{2}{x+C}\right) + \varepsilon\left(-\frac{1}{2(x+C)}\left[\ln(x+C)+D\right]\right) + \ldots$$

must satisfy $y(1)=0$ condition

$$y_0(x) = \ln\left(\frac{2}{x+C}\right)$$

$$y_0(1) = \ln\left(\frac{2}{1+C}\right) = 0 \quad \Rightarrow \; 1 + C = 2 \;\; \Rightarrow \;\; C = 1$$

$$y_1(x) = -\frac{1}{2(x+C)}(\ln(x+C) + D)$$

$$y_1(1) = -\frac{1}{2.2}(\ln 2 + D) = -\frac{1}{4}(\ln 2 + D) = 0 \;\Rightarrow\; D = -\ln 2$$

Since we have found C and D, we can write $y_{out}(x)$.

$$y_{out}(x) = \ln\left(\frac{2}{x+1}\right) + \varepsilon\left(-\frac{1}{2(x+1)}\left[\ln(x+1) - \ln 2\right]\right) + \ldots \tag{5.31}$$

Now, it's time to construct our new inner variables valid near $x = 0$.

$$X = \frac{x}{\delta(\varepsilon)} \quad , \;\; Y(X) = y(x) \tag{5.32}$$

Let us write (5.8) in terms of X and Y. First of all, we will define the new derivative terms

$$\frac{dy}{dx} = \frac{dY}{dX}\frac{dX}{dx} = \frac{1}{\delta(\varepsilon)}\frac{dY}{dX} \quad , \;\; \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{1}{\delta(\varepsilon)}\frac{d^2Y}{dX^2}\frac{dX}{dx} = \frac{1}{\delta(\varepsilon)^2}\frac{d^2Y}{dX^2} \tag{5.33}$$

Let us substitute (5.33) into (5.8), then we get

$$\frac{\varepsilon}{\delta(\varepsilon)^2}\frac{d^2Y}{dX^2} + 2\frac{1}{\delta(\varepsilon)}\frac{dY}{dX} + e^Y = 0 \tag{5.34}$$

We have done the analysis to determine $\delta(\varepsilon)$. We have found that $\delta(\varepsilon) = \varepsilon$. Hence $X = \dfrac{x}{\varepsilon}$.

Here, the distinguished limit for this problem is

$$\frac{d^2Y}{dX^2} + 2\frac{dY}{dX} + \varepsilon\, e^Y = 0 \tag{5.35}$$

To solve this equation, we give an expansion for $Y(X)$ as

$$Y(X) = Y_0(X) + \varepsilon Y_1(X) + \ldots \tag{5.21}$$

and then substitute (5.21) into (5.35), we get

$$\left(Y_0'' + \varepsilon Y_1'' + \ldots\right) + 2\left(Y_0' + \varepsilon Y_1' + \ldots\right) + \varepsilon\, e^{(Y_0 + \varepsilon Y_1 + \ldots)} = 0 \tag{5.36}$$

$$\left(Y_0'' + \varepsilon Y_1'' + \ldots\right) + 2\left(Y_0' + \varepsilon Y_1' + \ldots\right) + \varepsilon\left(1 + Y_0 + \varepsilon Y_1 + \ldots\right) = 0 \tag{5.37}$$

Now, we will collect the terms w.r.t. their ε power and solve the equation that we get for $Y_0(X)$ and $Y_1(X)$.

$$\varepsilon^0 \quad \text{order}: \; Y_0'' + 2Y_0' = 0 \tag{5.38}$$

$$\varepsilon^1 \quad \text{order}: \; Y_1'' + 2Y_1' + 1 + Y_0 = 0 \tag{5.39}$$

Let us solve (5.38) and (5.39).

$$Y_0'' + 2Y_0' = 0 \tag{5.38}$$

$$m^2 + 2m = 0$$
$$m(m+2) = 0$$
$$m_1 = 0 \quad , \quad m_2 = -2$$

$$\Rightarrow \; Y_0(X) = A_0 + B_0 e^{-2X}$$

and we have the boundary condition $y(0) = 0$. Since $X = \dfrac{x}{\varepsilon}$, $y(0) = 0$ condition converts into $Y(0) = 0$. Using this condition,

$$Y_0(0) = A_0 + B_0 = 0$$

$$\Rightarrow \; Y_0(X) = A_0(1 - e^{-2X}) \tag{5.40}$$

Now we will use $Y_0(X)$ to solve $Y_1(X)$.

$$Y_1'' + 2Y_1' = -Y_0 - 1 \tag{5.39}$$

$$Y_1'' + 2Y_1' = -A_0\left(1 - e^{-2X}\right) - 1 \tag{5.39a}$$

$$Y_{1,h}(X) = C_0 + D_0 e^{-2X}$$
$$Y_{1,p}(X) = aX + bXe^{-2X}$$
$$Y_{1,p}'(X) = a + be^{-2X} - 2bXe^{-2X}$$
$$Y_{1,p}''(X) = -2be^{-2X} - 2be^{-2X} + 4bXe^{-2X}$$

Substituting them into (5.39a) , we get

$$-4be^{-2X} + 4bXe^{-2X} + 2a + 2be^{-2X} - 4bXe^{-2X} = -(A_0 + 1) + A_0 e^{-2X}$$

$$2a - 2be^{-2X} = -(A_0 + 1) + A_0 e^{-2X}$$

$$2a = -(A_0 + 1) \qquad\qquad -2be^{-2X} = A_0 e^{-2X}$$

$$a = -\frac{1}{2}(A_0 + 1) \qquad\qquad b = -\frac{1}{2}A_0$$

$$\Rightarrow\; Y_{1,p}(X) = -\frac{1}{2}(A_0 + 1)X - \frac{1}{2}A_0 Xe^{-2X}$$

$$\Rightarrow\; Y_1(X) = Y_{1,h}(X) + Y_{1,p}(X) = C_0 + D_0 e^{-2X} - \frac{1}{2}(A_0 + 1)X - \frac{1}{2}A_0 Xe^{-2X}$$

We have the boundary condition $Y_1(0) = 0$.

$$Y_1(0) = C_0 + D_0 = 0 \quad \Rightarrow \quad D_0 = -C_0$$

$$\Rightarrow\; Y_1(X) = C_0 - C_0 e^{-2X} - \frac{1}{2}(A_0 + 1)X - \frac{1}{2}A_0 Xe^{-2X}$$

$$Y_1(X) = C_0\left(1 - e^{-2X}\right) - \frac{1}{2}(A_0 + 1)X - \frac{1}{2}A_0 Xe^{-2X} \tag{5.41}$$

Hence the inner expansion is

$$Y(X) = A_0\left(1 - e^{-2X}\right) + \varepsilon\left[C_0\left(1 - e^{-2X}\right) - \frac{1}{2}(A_0 + 1)X - \frac{1}{2}A_0 Xe^{-2X}\right] + \ldots \tag{5.42}$$

Now let us perform the matching process.

The outer expansion

$$y_{out}(x) = \ln\left(\frac{2}{x+1}\right) + \varepsilon\left[-\frac{1}{2(x+1)}\left(\ln(x+1) - \ln 2\right)\right] + \ldots$$

To perform the matching process in the leading order, let us expand the first term and keep only first term of this expansion.

$$\ln\left(\frac{2}{x+1}\right) \sim \ln 2$$

Then

$$y_{out}(x) \sim \ln 2$$

The inner expansion is

$$Y(X) = A_0\left(1 - e^{-2X}\right) + \varepsilon\left[C_0\left(1 - e^{-2X}\right) - \frac{1}{2}(A_0 + 1)X - \frac{1}{2}A_0 X e^{-2X}\right] + \ldots$$

To perform the matching process in the leading order, let us expand the first term and keep only first term of this expansion.

$$A_0\left(1 - e^{-2X}\right) \sim A_0$$

According to matching, we get

$$A_0 = \ln 2$$

Now, for the first order matching, we do the same thing. But, in this case we keep the terms of ε and x order.

$$\ln\left(\frac{2}{x+1}\right) \sim \ln 2 - \varepsilon x$$

$$-\frac{1}{2}\left(\frac{1}{1+x}\right) \sim -\frac{1}{2} + \frac{1}{2}x - \frac{1}{2}x^2$$

$$\ln(1+x) \sim x$$

Hence

$$y_{out}(x) \sim \ln 2 + \frac{1}{2}\varepsilon \ln 2$$

And, for the inner expansion ($e^{-2X} \to 0$, $X \to \infty$)

$$Y(X) \sim A_0 + \varepsilon\left[C_0 - \frac{1}{2}(A_0 + 1)X\right]$$

Keeping only the terms involving ε, X and discarding other terms, we get

$$Y(X) \sim A_0 + \varepsilon C_0$$

According to matching, we have

$$A_0 = \ln 2 \quad , \quad C_0 = \frac{1}{2}\ln 2$$

Finding these constants, we get

$$Y(X) = \ln 2\left(1 - e^{-2X}\right) + \varepsilon\left[\frac{1}{2}\ln 2\left(1 - e^{-2X}\right) - \frac{1}{2}(1 + \ln 2)X - \frac{1}{2}(\ln 2)Xe^{-2X}\right] + \ldots$$

Now, we will find $y_{match}(x)$.

Two-term inner expansion :

$$y^i(X) = \ln 2\left(1 - e^{-2X}\right) + \varepsilon\left[\frac{1}{2}\ln 2\left(1 - e^{-2X}\right) - \frac{1}{2}(1 + \ln 2)X - \frac{1}{2}(\ln 2)Xe^{-2X}\right]$$

Writing $X = \dfrac{x}{\varepsilon}$, we have

$$y^i(x) = \ln 2\left(1 - e^{-2\frac{x}{\varepsilon}}\right) + \varepsilon\left[\frac{1}{2}\ln 2\left(1 - e^{-2\frac{x}{\varepsilon}}\right) - \frac{1}{2}(1 + \ln 2)\frac{x}{\varepsilon} - \frac{1}{2}(\ln 2)\frac{x}{\varepsilon}e^{-2\frac{x}{\varepsilon}}\right]$$

Since $e^{\frac{-2x}{\varepsilon}} \to 0$, $x \to 1$; then we get

$$\left(y^i(x)\right)^o = \ln 2 + \varepsilon\left[\frac{1}{2}\ln 2 - \frac{1}{2}(1 + \ln 2)x\right]$$

Hence,

$$y_{match}(x)=\left(y^i(x)\right)^o=\ln 2+\varepsilon\left[\frac{1}{2}\ln 2-\frac{1}{2}(1+\ln 2)x\right]$$

Now, we will find the uniform expansion.

$$y_{unif}(x)=y_{out}(x)+y_{in}(x)-y_{match}(x)$$

Then,

$$y_{unif}(x)=\ln\left(\frac{2}{1+x}\right)+\ln 2\left(1-e^{\frac{-2x}{\varepsilon}}\right)-\ln 2-\frac{1}{2}(1+\ln 2)x-\frac{1}{2}(\ln 2)x\,e$$
$$\varepsilon\left[-\frac{1}{2(1+x)}\left(\ln(1+x)-\ln 2\right)+\frac{1}{2}\ln 2\left(1-e^{\frac{-2x}{\varepsilon}}\right)-\frac{1}{2}\ln 2+\frac{1}{2}(1+\ln 2)x\right]+O(\varepsilon)^2 \tag{5.43}$$

5.2.2 Problem 2

Let us find one-term outer and inner expansions of the equation given as

$$\varepsilon\ddot{x}+\dot{x}+\sin x=0 \tag{5.44a}$$

$$x(0)=1\ ,\ \ \dot{x}(0)=1 \tag{5.44b}$$

First of all let us give an expansion for x .

$$x=x_0+\varepsilon x_1+\ldots \tag{5.45}$$

Now let us write (5.45) into (5.44a).

$$\varepsilon\left(\ddot{x}_0+\varepsilon\ddot{x}_1+\ldots\right)+\left(\dot{x}_0+\varepsilon\dot{x}_1+\ldots\right)+\sin\left(x_0+\varepsilon x_1+\ldots\right)=0$$

$$\varepsilon\left(\ddot{x}_0+\varepsilon\ddot{x}_1+\ldots\right)+\left(\dot{x}_0+\varepsilon\dot{x}_1+\ldots\right)+\sin x_0.\cos(\varepsilon x_1)+\sin(\varepsilon x_1).\cos x_0+\ldots=0 \tag{5.46}$$

Since $\sin\varepsilon x_1\sim\varepsilon x_1$ and $\cos\varepsilon x_1\sim 1$ near $x=0$, then we can write (5.46) in the form as below,

$$\dot{x}_0+\sin x_0+\varepsilon\left(\ddot{x}_0+\dot{x}_1+x_1\cos x_0\right)+\varepsilon^2\ddot{x}_1+\ldots=0 \tag{5.47}$$

ε^0 order : $\quad \dot{x}_0+\sin x_0=0$ (5.48)

Now let us solve the equation obtained in (5.48) .

$$\dot{x}_0 + \sin x_0 = 0 \tag{5.48}$$

$$\frac{dx_0}{dt} = -\sin x_0$$

$$\int \frac{dx_0}{\sin x_0} = -\int dt$$

Let us do the necessary trigonometric transformations.

$$u = \tan\frac{x_0}{2} \quad , \quad x_0 = 2\arctan u \quad , \quad dx_0 = \frac{2du}{1+u^2} \quad , \quad \sin x_0 = \frac{2u}{1+u^2}$$

Now our integral is

$$\int \frac{dx_0}{\sin x_0} = \int \frac{2}{1+u^2} \cdot \frac{1}{\frac{2u}{1+u^2}} du = \int \frac{du}{u} = \ln|u| = \ln\left|\tan\left(\frac{x_0}{2}\right)\right|$$

$$\Rightarrow \quad \int \frac{dx_0}{\sin x_0} = -\int dt$$

$$\ln\left|\tan\left(\frac{x_0}{2}\right)\right| = -\int dt$$

$$\ln\left|\tan\left(\frac{x_0}{2}\right)\right| = -t + C_1$$

$$\tan\left(\frac{x_0}{2}\right) = Ce^{-t} \quad , \quad C = e^{C_1}$$

$$\frac{x_0}{2} = \arctan\left(Ce^{-t}\right)$$

$$x_0 = 2\tan^{-1}\left(Ce^{-t}\right) \tag{5.49}$$

Now let us find where the boundary layer is. To find it, we will use the conditions given in (5.44b). Since we have found our outer solution

$$x_0 = 2\tan^{-1}\left(Ce^{-t}\right)$$

from a first order differential equation, then our outer solution doesn't satisfy both conditions given in (5.44b). Consequently, we think that there is a boundary layer near $x = 0$.

Now we will find the inner solution valid near $x=0$. To find it we will introduce the new inner variable.

$$T = \frac{t}{\delta(\varepsilon)} \tag{5.50}$$

And we will write the new derivative terms.

$$\dot{x} = \frac{dx}{dt} = \frac{dX}{dT}\frac{dT}{dt} = \frac{1}{\delta(\varepsilon)}\frac{dX}{dT}$$

$$\ddot{x} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{1}{\delta(\varepsilon)}\frac{d^2X}{dT^2}\frac{dT}{dt} = \frac{1}{\delta(\varepsilon)^2}\frac{dX}{dT}$$

Now our equation is converted into the form

$$\frac{\varepsilon}{\delta(\varepsilon)^2}\ddot{X} + \frac{1}{\delta(\varepsilon)}\dot{X} + \sin X = 0 \tag{5.51}$$

Let us determine the thickness of boundary layer. We have two possibilities.

i) $\dfrac{\varepsilon}{\delta(\varepsilon)^2} \sim \dfrac{1}{\delta(\varepsilon)} \Rightarrow \delta(\varepsilon) \sim \varepsilon$

if $\delta(\varepsilon) \sim \varepsilon$, then $\dfrac{\varepsilon}{\delta(\varepsilon)^2} \sim \dfrac{1}{\varepsilon}$ and $\dfrac{1}{\delta(\varepsilon)} \sim \dfrac{1}{\varepsilon}$. And these two terms are bigger than 1, the coefficient of the third term in (5.51). This is the case we want.

ii) $\dfrac{\varepsilon}{\delta(\varepsilon)^2} \sim 1 \Rightarrow \delta(\varepsilon) \sim \sqrt{\varepsilon}$

if $\delta(\varepsilon) \sim \sqrt{\varepsilon}$, then the coefficient of the second term is $\dfrac{1}{\sqrt{\varepsilon}}$ and it is bigger than the coefficient of the second order derivative term. We don't want this situation.

Hence we have found the thickness of the boundary layer $\delta(\varepsilon) \sim \varepsilon$. Writing this in (5.51),

$$\ddot{X} + \dot{X} + \varepsilon \sin X = 0 \tag{5.52}$$

Our expansion for the inner solution is

$$X = X_0 + \varepsilon X_1 + ... \tag{5.53}$$

Let us substitute it into (5.52),

$$\left(\ddot{X}_0 + \varepsilon\ddot{X}_1 + ...\right) + \left(\dot{X}_0 + \varepsilon\dot{X}_1 + ...\right) + \varepsilon \sin\left(X_0 + \varepsilon X_1 + ...\right) = 0$$

$$\sin\left(X_0 + \varepsilon X_1 + ...\right) = \sin X_0 . \cos\left(\varepsilon X_1\right) + \sin\left(\varepsilon X_1\right) . \cos X_0 + ...$$

And also

$$\cos\left(\varepsilon X_1\right) \sim 1 \quad , \quad \sin\left(\varepsilon X_1\right) \sim \varepsilon X_1$$

Then we can write

$$\ddot{X}_0 + \varepsilon\ddot{X}_1 + \dot{X}_0 + \varepsilon\dot{X}_1 + \varepsilon \sin X_0 + O\left(\varepsilon^2\right) = 0 \tag{5.54}$$

And for one-term inner approximation we have

$$\ddot{X}_0 + \dot{X}_0 = 0 \tag{5.55}$$

Let us solve (5.55).

$$m(m+1) = 0$$

$$m_1 = 0 \ , \ m_2 = -1$$

$$\Rightarrow \ X_0(T) = A + Be^{-T} \tag{5.56}$$

Since our conditions in (5.44b) are valid for x , then we must write them for X .

$x(0) = 1$ and $T = \dfrac{t}{\varepsilon}$, then $t=0 \Rightarrow T = 0$. Hence our new condition is $X(0) = 1$.

We have found that $\dot{x} = \dfrac{1}{\varepsilon}\dot{X}$. Now let us use it .

$$\dot{x} = \frac{1}{\varepsilon}\dot{X} \ \Rightarrow \ \dot{X} = \varepsilon\dot{x} \tag{5.57}$$

Our condition was $\dot{x}(0) = 1$.

$$\dot{x}(0) = 1 \ \Rightarrow \ \dot{X}(0) = \varepsilon\dot{x}(0) \ \Rightarrow \ \dot{X}(0) = \varepsilon \tag{5.58}$$

We have an expansion for X as

$$X = X_0 + \varepsilon X_1 + ... \tag{5.53}$$

Using it, we have

$$\dot{X}(0) = \dot{X}_0(0) + \varepsilon\dot{X}_1(0) + ... = \varepsilon \tag{5.59}$$

Then,

$$\dot{X}_0(0)=0 \quad \text{and} \quad \dot{X}_1(0)=1 \tag{5.60}$$

Now we will use them to find A and B .

$$\Rightarrow \; X_0(T)=A+Be^{-T}$$

$$\Rightarrow \; \dot{X}_0(T)=-Be^{-T}$$

Now we have $X_0(0)=1$ and $\dot{X}_0(0)=0$.

$$\Rightarrow \; X_0(0)=A+B=1$$

$$\Rightarrow \; \dot{X}_0(0)=-B=0$$

Hence $A=1$, $B=0$

$$\Rightarrow \; X_0(T)=1 \tag{5.61}$$

Now we will find C with matching.

One-term outer expansion :

$$x^o = 2\tan^{-1}\left(Ce^{-t}\right)$$

Writing it in terms of T :

$$x^o = 2\tan^{-1}\left(Ce^{-\varepsilon T}\right)$$

Expanding it for small ε ,

$$x^o = 2\tan^{-1}\left(C(1-\varepsilon T+...)\right)$$

One-term inner expansion :

$$\left(x^o\right)^i = 2\tan^{-1} C$$

This must be equal to $X_0(T)=1$. Then we have $C=\tan\dfrac{1}{2}$.

Hence our one-term outer expansion is

$$x_0(t)=2\tan^{-1}\left(\tan\left(\frac{1}{2}\right)e^{-t}\right) \tag{5.62}$$

Hence our one-term inner solution is given by (5.61), and one-term outer solution is given by (5.62) .

5.2.3 Problem 3

Let us consider the differential equation

$$\varepsilon y'' + x^{1/5} y' + \frac{5}{6} y^2 = 0 \qquad , \ -1 \le x \le 1$$
$$y(-1) = \frac{1}{2} \quad , \ y(1) = \frac{1}{3} \tag{5.63}$$

$p(x) = x^{1/5}$, and $p(0) = 0$, hence there is a boundary layer near $x = 0$.

Let us give a one-term outer expansion for the outer solution as

$$y_{out}(x) \sim y_0(x) \tag{5.64}$$

And substituting it into the equation, we have

$$\varepsilon y_0'' + x^{1/5} y_0' + \frac{5}{6} y_0{}^2 = 0 \tag{5.65}$$

We have for ε^0 order,

$$x^{1/5} y_0' + \frac{5}{6} y_0{}^2 = 0 \tag{5.66}$$

This is a Bernoulli differential equation. To solve it, let us introduce the transformation

$$z = y_0{}^{-1} = \frac{1}{y_0}$$
$$\Rightarrow \ y_0 = \frac{1}{z} \ \Rightarrow \ y_0' = -\frac{z'}{z^2} \tag{5.67}$$

After these transformations our equation is converted into

$$\frac{dz}{dx} = \frac{5}{6} x^{-\frac{1}{5}}$$
$$dz = \frac{5}{6} x^{-\frac{1}{5}} dx \tag{5.68}$$

Integrating it we have,

$$z = \frac{25}{24} x^{4/5} + A_0 \tag{5.69}$$

Then

$$y_0 = \frac{24}{25x^{4/5} + 24A_0} \tag{5.70}$$

Hence our one-term outer expansion is

$$y_{out}(x) \sim \frac{24}{25x^{4/5} + 24A_0} \tag{5.71}$$

We have two conditions and let us use them.

$$y(-1) = \frac{1}{2} \;\Rightarrow\; y_{out}(-1) = \frac{24}{25 + 24A_0} = \frac{1}{2} \;\Rightarrow\; A_0 = \frac{23}{24} \tag{5.72}$$

$$y(1) = \frac{1}{3} \;\Rightarrow\; y_{out}(1) = \frac{24}{25 + 24A_0} = \frac{1}{3} \;\Rightarrow\; A_0 = \frac{47}{24} \tag{5.73}$$

Since the boundary layer in this equation is an interior boundary layer then, we have two outer solutions. We have

$$y_{out}(x) \sim \frac{24}{25x^{4/5} + 23} \quad , \quad -1 \le x \le -|x_1| \tag{5.74}$$

$$y_{out}(x) \sim \frac{24}{25x^{4/5} + 47} \quad , \quad |x_1| \le x \le 1 \tag{5.75}$$

Here x_1 is a point near $x = 0$. And also $x_1 = o(1)$, $\frac{\varepsilon}{x_1} = o(1)$.

Now, we will find the inner solution. Let us use the new inner variable near $x = 0$, let us write $X = \frac{x}{\delta(\varepsilon)}$.

$$\frac{dy}{dx} = \frac{dY}{dX}\frac{dX}{dx} = \frac{1}{\delta(\varepsilon)}\frac{dY}{dX} \quad , \quad \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{1}{\delta(\varepsilon)}\frac{d^2Y}{dX^2}\frac{dX}{dx} = \frac{1}{\delta(\varepsilon)^2}\frac{d^2Y}{dX^2} \tag{5.76}$$

Substituting them into the first equation, we get

$$\frac{\varepsilon}{\delta(\varepsilon)^2}Y'' + \frac{1}{\delta(\varepsilon)^{4/5}}X^{1/5}Y' + \frac{5}{6}Y^2 = 0 \tag{5.77}$$

Let us determine $\delta(\varepsilon)$.

$$\frac{\varepsilon}{\delta(\varepsilon)^2} \sim \frac{1}{\delta(\varepsilon)^{4/5}} \;\Rightarrow\; \delta(\varepsilon) \sim \varepsilon^{5/6}$$

hence our equation becomes,

$$Y'' + X^{1/5} Y' + \frac{5}{6}\varepsilon^{2/3} Y^2 = 0 \tag{5.78}$$

Since we get the distinguished limit, now, we write our one-term outer expansion

$$Y(X) \sim Y_0(X) \tag{5.79}$$

Substituting (5.79) into (5.78) , we get

$$Y_0'' + X^{1/5} Y_0' + \frac{5}{6}\varepsilon^{2/3} Y_0^{\,2} = 0 \tag{5.80}$$

First of all, we collect the terms of order ε^0 in (5.80) , and to solve this equation , let us write

$$p = Y_0' \;\;,\;\; \frac{dp}{dX} = Y_0'' \tag{5.81}$$

Then we have,

$$\frac{dp}{dX} + X^{1/5} p = 0 \;\Rightarrow\; \frac{dp}{p} = -X^{1/5} dX \;\Rightarrow\; \ln p = -\frac{5}{6} X^{6/5} + B_0$$

Then,

$$p = Be^{-\frac{5}{6} X^{6/5}} \tag{5.82}$$

Hence, we have

$$Y_0 = B\int_0^X e^{-\frac{5}{6}\hat{X}^{6/5}}\, d\hat{X} + C \tag{5.83}$$

Here $X = \dfrac{x}{\varepsilon^{5/6}} \;\Rightarrow\; X \to \infty$.

Now we will evaluate integral $\displaystyle\int_0^\infty e^{-\frac{5}{6}X^{6/5}}\, dX$.

Let us write

$$u = \frac{5}{6} X^{6/5} \quad , \quad du = X^{1/5} dX \tag{5.84}$$

Hence our integral becomes

$$\int_0^\infty e^{-\frac{5}{6}X^{6/5}}\, dX \;=\; \int_0^\infty \left(\frac{5}{6}\right)^{1/6} u^{-1/6} e^{-u}\, du$$

$$= \left(\frac{5}{6}\right)^{1/6} \Gamma\left(\frac{5}{6}\right)$$

Then, we get

$$Y_0(X) = B\left(\frac{5}{6}\right)^{1/6} \Gamma\left(\frac{5}{6}\right) + C \qquad (5.85a)$$

$$= Bk + C \qquad (5.85b)$$

where $k = \left(\frac{5}{6}\right)^{1/6} \Gamma\left(\frac{5}{6}\right)$.

Now, we have $Y(X) \sim Y_0(X) = Bk + C$ (5.86)

To match the inner and outer solutions, let us write,

$$y_{out}(x) \sim \frac{24}{23} \quad \text{as} \quad x \to 0- \quad ; \quad y_{out}(x) \sim \frac{24}{47} \quad \text{as} \quad x \to 0+ \qquad (5.87)$$

Hence

$$Y \sim \frac{24}{23} \quad \text{as} \quad x \to 0- \quad ; \quad Y \sim \frac{24}{47} \quad \text{as} \quad x \to 0+ \qquad (5.88)$$

Also,

$$Y(X) \sim Bk + C \quad \text{as} \quad x \to 0- \quad ; \quad Y(X) \sim -Bk + C \quad \text{as} \quad x \to 0+ \qquad (5.89)$$

Here, we have found that

$$\begin{aligned} C &\sim 1.5541 \\ Bk &\sim -0.2664 \end{aligned} \qquad (5.90)$$

Hence,

$$Y_0(X) \sim 1.2877 \qquad (5.91)$$

And

$$Y(X) \sim 1.2877 \tag{5.92}$$

Now, we will find the uniform expansions valid on the whole interval. We know that

$$y_{unif}(x) = y_{out}(x) + y_{in}(x) - y_{match}(x) \tag{5.93}$$

Hence,

$$y_{unif,left}(x) = \frac{24}{25x^{4/5} + 23} + 1.2877 - \frac{24}{23} + O(\varepsilon) \quad , \quad x \to 0- \tag{5.94}$$

$$y_{unif,right}(x) = \frac{24}{25x^{4/5} + 47} + 1.2877 - \frac{24}{47} + O(\varepsilon) \quad , \quad x \to 0+ \tag{5.95}$$

Consequently, we have

$$y_{unif,left}(x) = \frac{24}{25x^{4/5} + 23} + 0.2442 + O(\varepsilon) \quad , \quad x \to 0- \tag{5.96}$$

$$y_{unif,right}(x) = \frac{24}{25x^{4/5} + 47} + 0.7771 + O(\varepsilon) \quad , \quad x \to 0+ \tag{5.97}$$

CHAPTER SIX
CONCLUSIONS

In this thesis , we studied boundary layer problems which are one of the types of singular perturbation problems. To solve these problems, we divide the region into inner and outer regions. We give asymptotic expansions for these regions, hence given second order problems are converted into first order problems which are easier to solve. Solving them we get outer and inner expansions. Finally, we match them to get a uniform expansion valid on the whole interval. Hence, we get asymptotic solutions for chosen nonlinear boundary layer problems. In the chosen problems, the boundary layer theory can be used to construct higher-order approximations in the perturbing parameter ε to the solution of the nonlinear differential equations.

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