DOKUZ EYLÜL UNIVERSITY

GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

APPROXIMATE ANALYTIC SOLUTIONS OF NONLINEAR INTEGRAL EQUATIONS

by

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September, 2010

İZMİR

APPROXIMATE ANALYTIC SOLUTIONS OF NONLINEAR INTEGRAL EQUATIONS

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by
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M. Sc THESIS EXAMINATION RESULT FORM

We have read the thesis entitled **"APPROXIMATE ANALYTIC SOLUTIONS OF NONLINEAR INTEGRAL EQUATIONS"** completed by **ÖZLEM ERGÜN** under supervision of **PROF. DR. GONCA ONARGAN** and we certify that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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APPROXIMATE ANALYTIC SOLUTIONS OF NONLINEAR INTEGRAL EQUATIONS

ABSTRACT

This thesis is related with nonlinear integral equations, nonlinear systems of integral equations and integro-differential equations. The existence and uniqueness of these equations for Lipschitz continuous kernels are investigated. An analytic method based on He's Homotopy Perturbation Method (HPM) for the solution of nonlinear integral equations and systems are studied and applied. This method is extended for nonlinear integro-differential equations. Moreover, some examples of the mathematics program, solutions are given by MATHEMATICA 7. The approximate solutions of these equations are compared with the analytic approximation methods such as Adomian Decomposition Method (ADM) and Taylor–Series Expansion Method. The comparison shows that the (HPM) is quite conform and efficient for solving nonlinear problems.

Keywords: Linear and nonlinear integral equations, nonlinear systems of integral equations, Homotopy Perturbation Method.

DOĞRUSAL OLMAYAN İNTEGRAL DENKLEMLERİN YAKLAŞIK ANALİTİK ÇÖZÜMLERİ

ÖZ

Bu tezde doğrusal olmayan integral denklemler, doğrusal olmayan integral denklem sistemleri, integro-diferansiyel denklemler incelenmiş, bu denklemlerin çözümlerinin varlık ve tekliği Lipschitz sürekli çekirdekler için araştırılmıştır. Doğrusal olmayan integral denklemlerin ve denklem sistemlerinin çözümü için analitik bir yöntem olan He'nin Homotopi Perturbasyon Yönteminin uygulanması incelenmiştir. Yöntem doğrusal olmayan integro-diferansiyel denklemler için genişletilmiştir. Ayrıca matematik programı MATHEMATICA 7 ile de bazı örneklerin çözümleri verilmiştir. Bu denklemlerin yaklaşık çözümleri analitik yaklaşım yöntemleri olan Adomian Ayrışım Yöntemi (ADM) ve Taylor Serisi Açılım Yöntemi ile karşılaştırılmış ve Homotopi Perturbasyon Yönteminin doğrusal olmayan problemlerin çözümünde uyumlu ve elverişli sonuçlar verdiği gözlenmiştir.

Anahtar kelimeler: Doğrusal ve doğrusal olmayan integral denklemler, doğrusal olmayan integral denklem sistemleri, Homotopi Perturbasyon Yöntemi

CONTENTS

Page

M. Sc THESIS EXAMINATION RESULT FORM ii
ACKNOWLEDGEMENT iii
ABSTRACT iv
ÖZ v

CHAPTER ONE - INTRODUCTION 1

CHAPTER TWO - INTEGRAL EQUATIONS 3

2.1 Introduction 3
2.2 Classification of Integral Equations 4
2.2.1 Fredholm Integral Equations 4
2.2.2 Volterra Integral Equations 5
2.2.3 Singular Integral Equations 6
2.2.4 Linear and Nonlinear Integral Equations 6
2.2.5 Regularity Conditions 8
2.2.6 Special Types of Kernels 9

CHAPTER THREE - NONLINEAR INTEGRAL EQUATIONS 10

3.1 Introduction 10
3.2 Classification of Nonlinear Integral Equations 12
3.2.1 Nonlinear Fredholm Integral Equations of the Second Kind 12
3.2.2 Nonlinear Volterra Integral Equations of the Second Kind 13
3.3 Some examples of Physical Problems Leading to Integral Equations . 15
3.3.1 Duffing's Vibration Problem 15
3.3.2 Bending of a Rod by a Longitudinal Force 16

CHAPTER FOUR - THEORETICAL BACKGROUND 18

4.1 Definitions and Theorems on Nonlinear Integral Equations 18
4.2 An Existence Theorem for Nonlinear Integral Equations of
Volterra Type 21
4.3 An Existence Theorem for Nonlinear Integral Equations of
Fredholm Type 24

CHAPTER FIVE - HOMOTOPY PERTURBATION METHOD 27

5.1 Introduction 27
5.2 What is homotopy?... 29
5.2.1 Example 29
5.3 He's Homotopy Perturbation Method 31
5.4 Homotopy Perturbation Method for Nonlinear Fredholm Integral
Equations of the Second Kind 34
5.5 Homotopy Perturbation Method for Nonlinear Volterra Integral
Equations of the Second Kind 39
5.6 Application of Homotopy Perturbation Method to
Integro-Differential Equations 43

**CHAPTER SIX - SYSTEMS OF NONLINEAR INTEGRAL
EQUATIONS 46**

6.1 Introduction 46
6.2 Systems of Volterra Integral Equations of the First Kind 47
6.2.1 Analysis of Homotopy Perturbation method for Systems of
Volterra Integral Equations of the First Kind 48
6.2 Systems of Fredholm Integral Equations of the Second Type 50
6.2.2 Analysis of Homotopy Perturbation method for Systems of
Fredholm Integral Equations of the Second Kind 51

CHAPTER SEVEN - PROBLEMS ON NONLINEAR INTEGRAL EQUATIONS AND ON NONLINEAR SYSTEMS OF INTEGRAL EQUATIONS.. **54**

CHAPTER EIGHT – CONCLUSIONS **82**

REFERENCES **83**

APPENDICES **86**

CHAPTER ONE

INTRODUCTION

Nonlinear phenomena, that appear in many applications in scientific fields, such as fluid dynamics, solid state physics, plasma physics, mathematical biology and chemical kinetics, can be modelled by partial differential equations and by integral equations as well.

There are many new analytical approximate methods to solve two–point boundary value problems and initial value problems in the literature. Among these, Adomian decomposition method (ADM) (Adomian, 1994) for stochastic and deterministic problems, the Modified Decomposition Method (MDM) (Wazwaz, 1997) and He's Homotopy Perturbation Method (HPM) (He, 1999; 2000; 2003; 2004; 2005) have been receiving much attention in recent years in applied mathematics in general, in the area of series solutions in particular. These methods have been applied to a wide class of functional equations of linear and nonlinear problems. In this study, we investigate He's Homotopy Perturbation Method (HPM) for certain class of nonlinear integral equations and compare these methods for solving the chosen model integral equations.

The application of the Homotopy Perturbation Method in nonlinear problems have been devoted by scientist and engineers, because this method is continuously deform a simple problem which is easy to solve into the under study problem which is difficult to solve.

In Chapter 2, we introduce types of integral equations and some examples for these equations.

Chapter 3 deals with nonlinear Fredholm, Volterra integral equations, their classifications and some examples of physical problems leading to nonlinear integral equations.

The existence and uniqueness theorems for nonlinear Fredholm and Volterra integral equations are given in Chapter 4.

In Chapter 5, we illustrate the basic idea of He's Homotopy Perturbation Method (HPM) which has became a powerful mathematical tool, when it successfully coupled with the perturbation theory. In this chapter we investigate He's Homotopy Perturbation Method (HPM) in details for nonlinear Fredholm and Volterra integral equations. The convergence of the method is also given in this chapter.

In Chapter 6, we give the analysis of He's Homotopy Perturbation Method (HPM) for solving systems of nonlinear Fredholm and Volterra integral equations.

In the last Chapter, we show the efficiency of the Homotopy Perturbation Method (HPM) for chosen problems in the literature. Moreover, some problems of the mathematics program, solutions are given by Mathematica 7.

CHAPTER TWO

INTEGRAL EQUATIONS

2.1 Introduction

An integral equation is an equation in which an unknown function appears under one or more integral signs. Naturally, in such an equation, there can occur other terms as well.

For example, for $a \le x \le b$, $a \le t \le b$, the equations

$$f(x) = \int_a^b \kappa(x,t) g(t) dt \tag{2.1}$$

$$g(x) = f(x) + \int_a^b \kappa(x,t) g(t) dt \tag{2.2}$$

$$g(x) = \int_a^b \kappa(x,t) [g(t)]^2 dt \tag{2.3}$$

where the function $g(x)$ is the unknown function while the other functions are known, are integral equations. The function $\kappa(x,t)$ is called the kernel and the function $f(x)$ is called the free term, in general, the kernel and free term will be complex value functions of the real variables x and t. A condition such as $a \le x \le b$ means that the equation holds for all values of x in the given integral. Thus for the integral equations (2.1), (2.2) and (2.3) we seek a solution $g(x)$ satisfying the equation for all x in [a, b].

In more general case in integral equations the unknown function is dependent not only one variable but on several variables. Such, for example, is the equation

$$g(x) = f(x) + \int_\Omega \kappa(x,t) g(t) dt \tag{2.4}$$

where x and t are n-dimensional vectors and Ω is the region of n-dimensional space. Similarly, we can also consider systems of integral equations with several unknown functions.

2.2 Classification of Integral Equations

The classification of integral equations centers on three basic characteristics which together describe their overall structure:

(1) The kind of an integral equation refers the location of the unknown function. First kind equations have the unknown function present under the integral sign only second and third kind equations also have the unknown function outside the integral.

(2) The historical descriptions Fredholm and Volterra equations are concerned with the integration interval. In a Fredholm integral equation the integral is over a finite interval with fixed end points. In a Volterra integral equation the integral is indefinite.

(3) The term singular is sometimes used when the integration is improper, either because the interval is indefinite, or because the interval is unbounded within the given interval or the kernel becomes infinite at one or more points within the range of integration. Clearly an integral equation can be singular on both counts.

The most general type of linear integral equations is of the form

$$h(x)g(x) = f(x) + \lambda \int_a^x \kappa(x,t)g(t)dt \tag{2.5}$$

where the upper limit may be either variable or fixed. The functions f , h and κ are known functions, while g is to be determined; λ is nonzero real or complex numerical parameter. In practical applications, λ is usually composed of physical quantities. The function $\kappa(x,t)$ is called the kernel. Using this classification, we can give the following special cases of equation (2.5).

2.2.1 Fredholm Integral Equations

In all Fredholm integral equations the limits of integration are finite and the upper limit of integration b is fixed.

i. First Kind Fredholm Integral Equation

$h(x) = 0$

$$f(x) + \lambda \int_a^b \kappa(x,t) g(t) dt = 0 \tag{2.6}$$

ii. Second Kind Fredholm Integral Equation

$h(x) = 1$

$$g(x) = f(x) + \lambda \int_a^b \kappa(x,t) g(t) dt \tag{2.7}$$

iii. The Homogeneous Fredholm Integral Equation of the Second Kind

$f(x) = 0$

$$g(x) = \lambda \int_a^b \kappa(x,t) g(t) dt \tag{2.8}$$

2.2.2 Volterra Integral Equations

In all Volterra Equations, the upper limit of integration b is variable, b=x.

i. ***First Kind Volterra Integral Equation***

$h(x) = 0$

$$f(x) + \lambda \int_a^x \kappa(x,t) g(t) dt = 0 \tag{2.9}$$

ii. Second Kind Volterra Integral Equation

$h(x) = 1$

$$g(x) = f(x) + \lambda \int_a^x \kappa(x,t) g(t) dt \tag{2.10}$$

iii. The Homogeneous Volterra Integral Equation of the Second Kind

$f(x)=0$

$$g(x)=\lambda\int_a^x \kappa(x,t)g(t)dt \tag{2.11}$$

Equation (2.7) itself called Volterra equation of the third kind.

2.2.3 Singular Integral Equations

When one or both limits of integration become infinite or when the kernel becomes infinite at one or more points within the range of integration, the integral equation is called singular.

For example, the integral equations

$$g(x)=f(x)+\lambda\int_{-\infty}^{\infty} e^{-|x-t|}g(t)dt \tag{2.12}$$

and

$$f(x)=\int_0^x \frac{1}{(x-t)^{\alpha}}g(t)dt \qquad 0<\alpha<1 \tag{2.13}$$

are singular equations.

2.2.4 Linear and Nonlinear Integral Equations

The linearity is related to the degree of the unknown function $g(t)$ in an integral equation the degree of $g(t)$ must be one.

The second kind linear and nonlinear nonhomogeneous Fredholm integral equations, respectively are:

$$g(x)=f(x)+\int_a^b \kappa(x,t)g(t)dt \tag{2.14}$$

$$g(x)=f(x)+\int_a^b \kappa(x,t,g(t))dt \tag{2.15}$$

If $f(x)=0$, equations (2.14) and (2.15) are called as the homogeneous.

$$g(x)=\int_0^{\pi}\cos(x-t)g(t)dt$$

is linear homogeneous and

$$g(x)=\int_0^{\pi}\kappa(x,t)\sin(g(t))dt$$

is nonlinear homogeneous but

$$g(x)=f(x)+\int_0^{\pi}\kappa(x,t)g(t)dt$$

is linear nonhomogeneous and

$$g(x)=f(x)+\int_0^{\pi}\kappa(x,t)\sin(g(t))dt$$

is nonlinear nonhomogeneous Fredholm integral equations.

The second kind linear and nonlinear nonhomogeneous Volterra integral equations, respectively, are

$$g(x)=f(x)+\int_a^{x}\kappa(x,t)g(t)dt \tag{2.16}$$

$$g(x)=f(x)+\int_a^{x}\kappa(x,t,g(t))dt \tag{2.17}$$

If $f(x)=0$, equations (2.16) and (2.17) are called as homogeneous.

$$g(x)=\int_0^{x}e^{x-t}g(t)dt$$

is linear homogeneous and

$$g(x)=\int_0^{x}\kappa(x,t)\sin(g(t))dt$$

is nonlinear homogeneous but

$$g(x)=f(x)+\int_0^{x}e^{x-t}g(t)dt$$

is linear nonhomogeneous and

$$g(x) = f(x) + \int_0^x \kappa(x,t)\sin(g(t))dt$$

is nonlinear nonhomogeneous Volterra integral equations.

2.2.5 Regularity Conditions

In integral equations theory, the functions are either continuous or integrable or square integrable. By a square integrable function $g(t)$, we mean that

$$\int_a^b | g(t) |^2 dt < \infty$$

This is called an L_2 function.

The regularity conditions on the kernel $\kappa(x,t)$ as a function of two variables are similar.

$\kappa(x,t)$ is an L_2 function if,

a) for each set of values x, t in the square $a \le x \le b$, $a \le t \le b$,

$$\int_a^b \int_a^b | \kappa(x,t) |^2 dt < \infty$$

b) for each set of value of x in $a \le x \le b$,

$$\int_a^b | \kappa(x,t) |^2 dt < \infty$$

c) for each set of value of t in $a \le t \le b$,

$$\int_a^b | \kappa(x,t) |^2 dt < \infty$$

2.2.6 Special Types of Kernels

i) Separable or Degenerate Kernels

Let K(x, t) be a kernel defined on the square [a, b]x[a, b] and let there are finitely many functions $a_1, a_2, \ldots, a_n; b_1, b_2, \ldots, b_n$ on [a, b] such that

$$\kappa(x,t) = \sum_{i=1}^{n} a_i(x) b_i(t) \qquad a \le x,t \le b \tag{2.18}$$

In this case the kernel $\kappa(x,t)$ is said to be separable or degenerate. The functions $a_i(s)$ can be assumed linearly independent; otherwise the number of terms in the expression of $\kappa(x,t)$ can be reduced.

ii) Symmetric Kernels

A complex-valued function $\kappa(x,t)$ is called symmetric (or Hermitian) if

$$\kappa(x,t) = \kappa^*(x,t)$$

for almost all x and t; where $\kappa^*(x,t)$ is the complex conjugate of $\kappa(x,t)$. For a real - valued kernel this property reduces to

$$\kappa(x,t) = \kappa(t,s)$$

CHAPTER THREE

NONLINEAR INTEGRAL EQUATIONS

3.1 Introduction

The theory of nonlinear integral equations is very important in pure and applied mathematics. The nonlinear integral equations arise in many problems of physics and technology especially in the theory of elasticity and the theory of aircraft wing, is played by singular integral equations with Cauchy type kernels.

The initial-value problems for ordinary differential equations can be reduced to a nonlinear Volterra integral equation. The theory of Volterra integral equations incorporates the problem of the growth of populations the influences of heredity. The problem of the growth of a single population in which the growth as influenced

- by a generative factor proportional to the population,
- an inhibiting influence proportional to the square of the population,
- a heredity component composed of the sum of individual factors encountered in the past (Davis, 1962).

This problem lead to an integro-differential equation of the form

$$\frac{1}{y}\frac{dy}{dt} = a + by + \int_0^t \kappa(t,s)y(s)ds \qquad (3.1.1)$$

In the case of two competing populations, one preying on the other, Volterra introduced the following system:

$$\frac{1}{x}\frac{dx}{dt} = a - by - \int_{-\infty}^t \kappa_1(t-s)y(s)ds \qquad (3.1.2a)$$

$$\frac{1}{y}\frac{dy}{dt} = -\alpha y + \beta x + \int_{-\infty}^t \kappa_2(t-s)y(s)ds \qquad (3.1.2b)$$

where a, b, α and β are positive constants.

The existence theorems of Picards for the differential equation

$$\frac{dy}{dx} = f(x, y) \tag{3.1.3}$$

and for the system

$$\frac{dy}{dx} = f(x, y, z), \qquad \frac{dz}{dx} = g(x, y, z) \tag{3.1.4}$$

depends upon expressing Equation (3.1.3) as the integral equation

$$y = y_0 + \int_{x_0}^{x} f(x, y)dx \tag{3.1.5}$$

and system (3.1.4) in the following form

$$y = y_0 + \int_{x_0}^{x} f(x, y, z)dx, \qquad z = z_0 + \int_{x_0}^{x} g(x, y, z)dx \tag{3.1.6}$$

A generalization of (3.1.5) can be written as

$$y(x) = f(x) + \int_{a}^{x} \kappa[x, s, y(s)]ds \tag{3.1.7}$$

which includes as a special case the linear Volterra equation of the second kind, (given in section 2.2.2 as equation (2.10)), namely

$$y(x) = f(x) + \int_{a}^{x} \kappa(x, s)y(s)ds \tag{3.1.8}$$

Unlike linear integral equations we can not, in general, solve nonlinear integral equations; we can do so only for sufficiently small values of the diameter of the region of integration by employing the method of successive approximations, the topological Schauder method, Adomian’ s method and He’ s Homotopy Perturbation Method.

Existence theorems for equation

$$y(x) = f(x) + \int_{a}^{x} \kappa[x, s, y(s)]ds \qquad (3.1.7)$$

have been given by T. Lalesco, E. Cotton, M. Picone, and others in which the essential idea is an adaptation of a Lipschitz condition to the more general problem

(Davis, 1962). These proofs can be extended to a functional equation sufficiently general to include integro-differential equations such as equation (3.1.1).

Lalesco has given an existence proof under general conditions for the Fredholm equation

$$y(x) = f(x) + \int_a^b \kappa[x, s, y(s)]ds \tag{3.1.9}$$

and Bratu (1914) has studied the following special cases:

$$y(x) = f(x) + \int_0^1 \kappa(x, s) y^2(s) ds \tag{3.1.10}$$

and

$$y(x) = f(x) + \int_0^1 \kappa(x, s) e^{y(s)} ds \tag{3.1.11}$$

3.2 Classification of Nonlinear Integral Equations

3.2.1 Nonlinear Fredholm Integral Equations of the Second Type

The nonlinear integral Fredholm equation of the second kind, after the Swedish mathematician I. Fredholm, has the form

$$y(x) = f(x) + \lambda \int_a^b \kappa[x, t, y(t)]dt \tag{3.2.1}$$

where $y(x)$ is the unknown function of x in the domain D which is assumed to be a bounded open set.

We make the following assumptions under which a solution exists for the equation (3.2.1);

a) $f(x)$ is a known real function which is defined continuous and bounded in the interval: $a \le x \le b$,

$$| f(x) | < f$$

b) The kernel $\kappa(x, y, z)$ is integrable and bounded,

$$\kappa(x, y, z) < M$$

in the domain D: $a \le x$, $y \le b$, $| z |< c$.

c) The kernel $\kappa(x, y, z)$ satisfies the Lipschitz condition with respect to z in D, namely

$$| \kappa(x, y, z_1) - \kappa(x, y, z_2) | \le K | z_1 - z_2 |$$

K being a positive constant.

d) Moreover, let m_1 and m_2 denote the lower and upper bounds of $f(x)$, respectively, that is,

$$m_1 \le f(x) \le m_2$$

and assume that

$$a < m_1 \le m_2 < b$$

For instance,

$$y(x) = 1 + \lambda \int_0^1 y^2(t)dt$$

$$y(x) = x + \int_0^1 ty^3(t)dt$$

are nonlinear second kind Fredholm integral equations.

3.2.2 Nonlinear Volterra Integral Equations of the Second Type

The nonlinear Volterra integral equation of the second type, after the Italian mathematician Vito Volterra, has the form

$$y(x) = f(x) + \int_a^x \kappa[x, t, y(t)]dt \qquad (3.2.2)$$

where $y(x)$ in the unknown function of x in the region D which is assumed to be a bounded open set.

We consider conditions under which a solution exists for the equation (3.2.2).We make the following assumptions;

a) $f(x)$ is a known real function which is defined integrable and bounded,

$$|f(x)| < f$$

in the interval: $a \leq x \leq b$,

b) The following Lipschitz condition is satisfied by $f(x)$ in the interval (a,b):

$|f(x_1) - f(x_2)| \leq k \, |x_1 - x_2|$

K being a positive constant.

c) The function $\kappa(x, y, z)$ is integrable and bounded,

$$\kappa(x, y, z) < M$$

in the domain D: $a \leq x$, $y \leq b$, $|z| < c$.

d) The kernel $\kappa(x, y, z)$ satisfies the Lipschitz condition with respect to z in its domain of definition:

$$|\kappa(x, y, z_1) - \kappa(x, y, z_2)| \leq K \, |z_1 - z_2|$$

K being a positive constant

e) Moreover, let m_1 and m_2 denote the lower and upper bounds of $f(x)$, respectively, that is,

$$m_1 \leq f(x) \leq m_2$$

and assume that

$c_1 < m_1 \leq m_2 < c_2$

where |z|<c that is $c_1 < z < c_2$.

For instance,

$$y(x) = x - \frac{1}{4}x^4 + \int_0^x ty^2(t)dt$$

$$y(x) = 2x + \frac{1}{6}x^5 - \int_0^x ty^3(t)dt$$

are nonlinear second kind Volterra integral equations.

Equations (3.2.1), (3.2.2) are called homogeneous integral equations if $f(x) = 0$ and nonhomogeneous integral equations if $f(x)$ is not vanish in the interval $[a,b]$.

3.3 Some Examples of Physical Problems Leading to Integral Equations

3.3.1 Duffing's Variation Problem

The forced vibrations of finite amplitude of a pendulum are governed by the differential equation

$$\frac{d^2y}{dt^2} + \alpha^2 \sin y = f(t). \tag{3.3.1}$$

Assuming driving function f is an odd-periodic function of period 2, then the problem of finding an odd-periodic solution with the same period can be easily reduced to finding a solution on the interval $0 \le t \le 1$ which satisfies the boundary conditions

$y(0) = y(1) = 0$

This boundary value problem is equivalent to the integral equation.

$$y(t) = -\int_0^1 \kappa(t,s)\left[f(s) - \alpha^2 \sin y(s)\right]ds \tag{3.3.2}$$

where the kernel $\kappa(t,s)$ is given by

$$\kappa(x,t) = t(1-x) \qquad 0 \le t \le x \le 1$$

$$\kappa(x,t) = x(1-t) \qquad 0 \le x \le t \le 1$$

3.3.2 Bending of a Rod by a Longitudinal Force

When a thin uniform rod is hinged at one end and acted upon by a longitudinal compressive force P at the other end, the equation for the bending moment is $\mu = -Py$, where y is the deflection of the rod from its original straight-line position and the bending moment μ is given by

$\mu = EIk$

where E is Young's Modulus, I is the moment of inertia of the cross-section and k is the curvature at the point under consideration. Let the arc length s be measured from the hinged end as the independent variable. Then the curvature is

$$k = \frac{y''(s)}{\left(1 - y'(s)^2\right)^{1/2}}$$

and the equation for the bending moment $\mu = -Py$ takes the form

$$y''(s) + \lambda y\sqrt{1 - \left(y'(s)\right)^2} = 0 \qquad (3.3.3)$$

where $\lambda = \dfrac{p}{EI}$ is a positive parameter, $\lambda > 0$.

The boundary conditions appropriate to this problem are

$y(0) = y(1) = 0$

if the length of the rod is taken to be unity. Taking

$$x(s) = y'(s) \qquad (3.3.4)$$

we obtain

$$y(s) = \int_0^1 \kappa(s,t)y'(t)dt$$

or

$$y(s) = \int_0^1 \kappa(s,t)x(t)dt$$

where the kernel $\kappa(s,t)$ is given by

$$\kappa(x,t) = t(1-x) \quad 0 \le t \le x \le 1$$

$$\kappa(x,t) = x(1-t) \quad 0 \le x \le t \le 1$$

Differentiation gives

$$y'(s) = \int_0^1 \frac{\partial \kappa(s,t)}{\partial s} x(t)dt$$

where

$$\frac{\partial \kappa(s,t)}{\partial s} = 1 - t \ , \ s < t$$

$$\frac{\partial \kappa(s,t)}{\partial s} = -t \ , \ s > t$$

Then the problem given by the differential equation for y together with the boundary conditions can be reduced to the following Fredholm integral equation of the first kind

$$x(s) = \lambda \int_0^1 \kappa(s,t) x(t) \sqrt{1 - \left[\int_0^1 \frac{\partial \kappa(s,u)}{\partial s} x(u) du \right]^2} dt \qquad (3.3.5)$$

for the unknown function x.

CHAPTER FOUR

THEORETICAL BACKGROUND

4.1 Definitions and Theorems on Nonlinear Integral Equations

Definition 4.1.1 (Normed Space)

Let X be a linear space on a field K. The mapping $\|.\|: X \to R_+,\ x \to \|x\|$ is called a norm on X if satisfies the following properties:

- $\|x\| \geq 0$
- $\|x\| = 0 \Leftrightarrow x = 0$
- $\|\alpha x\| = |\alpha|\|x\|$
- $\|x + y\| \leq \|x\| + \|y\|$ (Triangle inequality)

for all $x, y \in X$ and for all $\alpha \in K$. Hence, a norm on X is real-valued function on X. The normed space is denoted by $(X, \|.\|)$.

A norm on X defines a metric d which is given by

$$d(x, y) = \|x - y\|\ ,\ x, y \in X$$

and is called metric induced by the norm.

Definition 4.1.2 (Metric Space)

A metric space is a pair (X, d), where X is a set and d is a metric on X (or distance function on X), that is, a function defined on the Cartesian product X×X such that for all $x, y, z \in X$ we have:

(M1) d is a real-valued, finite and nonnegative,

(M2) d(x, y)=0 if and only if x=y

(M3) d(x, y)=d(y, x) (symmetry)

(M4) $d(x, y) \leq d(x, z) + d(z, y)$ (Triangle inequality)

A metric space X is called compact if every sequence in x has a convergent subsequence.

Definition 4.1.3 (Function Space)

The space on which the metric is defined by

$d(x,y) = \max_{t \in J} |x(t) - y(t)|$

where $J = [a,b]$ is a closed interval and max denotes the maximum is called $C'[a,b]$ function space.(because every point of $C'[a,b]$ is a function)

Definition 4.1.4 (Cauchy Sequence)

Let $(X, \|\cdot\|)$ be a normed space and $\{f_n\}$ be a sequence in X. A sequence $\{f_n\}$ is said to be Cauchy (or fundamental) if for every $\varepsilon > 0$ and for every $m,n > N$ there is an $N = N(\varepsilon) \in IN$ such that

$$\|f_n - f_m\| < \varepsilon .$$

Definition 4.1.5 (Complete Metric Space)

Let $(X, \|\cdot\|)$ be a normed space and $\{f_n\}$ be a Cauchy sequence in X. If for every $\varepsilon > 0$ there is an $N(\varepsilon) > 0$ such that

$$\|f_n - f_m\| < \varepsilon \text{ for every } m,n > N$$

then X is said to be complete. In other words, if

$$Lim_{n,m \to \infty} \|f_n - f_m\| = 0,$$

i.e. every Cauchy sequence in X converges and X is said to be complete metric space.

Ordinary Euclidean space and the space L^2 of functions quadratically integrable are complete normed (metric) spaces.

Definition 4.1.6 (Banach Space)

If a space is linear, normed, metric and complete then it is called a Banach space.

Definition 4.1.7 (Contraction Operator)

Let X be a Banach space and t is a bounded operator (not necessarily linear) in X. The operator T is called a contraction operator in X if for every functions f_1 and f_2 in X there is a positive constant $\alpha < 1$ such that

$$\|Tf_1 - Tf_2\| \leq \|f_1 - f_2\|$$

In his case there exists a unique point f of the space X which satisfies the equation

$$f = Tf$$

that is, point fixed with respect to the operator T.

4.1.2 Schauder's Fixed Point Theorem

The proofs of existence of solutions of nonlinear integral equations where the classical methods are useless are based on the fixed point theorem proved by the Polish mathematician Schauder (1942).

The geometrical nature of the problem of solving the nonlinear integral equation

$$f(x) = \int_{\Omega} F[x, y, f(y)]dy \tag{4.1.2a}$$

is finding a point f^x of the function space $C'[a,b]$ which corresponds to itself under the transformation (the functional operation)

$$f(x) = Tf(y) = \int_{\Omega} F[x, y, f(y)]dy \tag{4.1.2b}$$

This point f^x is called the fixed point of the function space with respect to the operation (4.1.2b).

Theorem 4.1.2(Schauder's Theorem)

Let T be a contraction operator in the Banach space X. Then the equation

$$Tf = f$$

has a unique solution in X.

The Schauder Fixed Point Theorem makes it possible to prove the existence of solutions of nonlinear integral equations under very general considerations, where the classical theory is inapplicable.

4.2 An Existence Theorem for Nonlinear Integral Equations of Volterra Type

We give the conditions under which a solution exists for the nonlinear Volterra integral equation

$$y(x) = f(x) + \int_a^x \kappa[x, s, y(s)]ds \tag{4.2.1}$$

making the following assumptions: (Davis, 1962)

a) The function f(x) is integrable and bounded, $|f(x)| < f$, in the interval $a \le x \le b$.

b) The following Lipschitz condition is satisfied by f(x) in the interval (a, b):

$$| f(x) - f(x') | \le k \, | x - x' | \tag{4.2.2}$$

c) The function $\kappa(x, y, z)$ is integrable and bounded,

$$|\kappa(x, y, z)| < K$$

in the domain $a \le x$, $y \le b$, $| z | < c$.

d) The following Lipschitz condition is satisfied by $\kappa(x, y, z)$ within its domain of definition

$$| \kappa(x, y, z) - \kappa(x, y, z') | \le M \, | z - z' | \tag{4.2.3}$$

By the method of successive approximations we have

$$y_0(x) = f(x) - f(a), \text{ (as the first approximation)}$$

from which we get

$$y_1(x) = f(x) + \int_a^x \kappa[x, s, y_0(s)]ds \tag{4.2.4}$$

and in general

$$y_n(x) = f(x) + \int_a^x \kappa[x, s, y_{n-1}(s)]ds \tag{4.2.5}$$

Using our assumptions given above, we can obtain a bound (4.2.7) for the approximation $y_1(x)$:

From (4.2.4), we have

$$|y_1(x)| = \left| f(x) + \int_a^x \kappa(x,s,y_0(s))ds \right|$$

$$\leq |f(x)| + \int_a^x |\kappa(x,s,y_0(s))|ds$$

$$\leq |f(x)| + |\kappa(x,s,y_0(s))||x-a|$$

$$\leq |f(x) - f(a) + f(a)| + |\kappa(x,s,y_0(s))||x-a|$$

From this inequality and the Lipschitz condition on f we get,

$$|y_1(x)| < |f(x) - f(a)| + |f(a)| + |\kappa(x,s,y_0(s))||x-a|$$

$$\leq k|x-a| + |f(a)| + K|x-a|$$

so

$$|y_1(x)| \leq k|x-a| + |f(a)| + K|x-a|$$

$$\leq k|x-a| + K|x-a| + f(a)$$

$$\leq (f+K)|x-a| \qquad (4.2.6)$$

If f is the larger of the two numbers K and $f(a)$ and $|x-a| < a'$. If x is so limited that

$$|x-a| < \frac{f}{f+K}$$

then

$$|y_1(x)| \leq (f+K)\frac{f}{f+K}$$

or

$$|y_1(x)| < f \qquad (4.2.7)$$

$$f = \max(k, f(a))$$

Next let h be the smallest of the numbers a' and $\frac{f}{f+K}$, that is $h = \min\left(a', \frac{f}{f+K}\right)$.

Then, for each approximation we have the following inequality

$$|y_n(x)| < f, \tag{4.2.8}$$

where

$$|x-a| < h = \min\left(a', \frac{f}{f+K}\right)$$

Let us now first construct the series

$$y(x) = y_0(x) + [y_1(x) - y_0(x)] + [y_2(x) - y_1(x)] + \ldots + [y_n(x) - y_{n-1}(x)] + \ldots \tag{4.2.9}$$

and then using (4.2.5) we can obtain the desired solution of given integral equation (4.2.1), provided the series (4.2.9) converges uniformly.

Uniform Convergence of Series (4.2.9)

Since we have

$$y_n(x) - y_{n-1}(x) = \int_0^x \{\kappa[x, s, y_{n-1}(s)] - \kappa[x, s, y_{n-2}(s)]\} ds \tag{4.2.10}$$

it follows from the Lipschitz condition (4.2.3) on κ that we have the inequality

$$|y_n(x) - y_{n-1}(x)| = \left|\int_0^x \{\kappa[x, s, y_{n-1}(s)] - \kappa[x, s, y_{n-2}(s)]\} ds\right|$$

$$|y_n(x) - y_{n-1}(x)| < M\left|\int_0^x [y_{n-1}(s) - y_{n-2}(s)] ds\right| \tag{4.2.11}$$

Letting $n = 2,3...$ in (4.2.11), we obtain the following sequence of inequalities

$$|y_2(x) - y_1(x)| = \left|\int_0^x \{\kappa[x, s, y_1(s)] - \kappa[x, s, y_0(s)]\} ds\right|$$

$$|y_2(x) - y_1(x)| \le M\left|\int_0^x [y_1(s) - y_0(s)] ds\right|$$

$$= M\left|\int_0^x M\left[s-a\right]ds\right| = M^2 \left.\frac{(s-a)^2}{2}\right|_0^x$$

$$\left|y_2(x) - y_1(x)\right| \le M^2 \frac{|x-a|^2}{2!}$$

$$\left|y_3(x) - y_2(x)\right| \le M\left|\int_0^x \left[y_2(s) - y_1(s)\right]ds\right|$$

$$= M\left|\int_0^x M^2 \frac{(s-a)^2}{2!}ds\right| = M^3 \left.\frac{(s-a)^3}{3.2!}\right|_0^x$$

$$\left|y_3(x) - y_2(x)\right| \le M^3 \frac{|x-a|^3}{3!}$$

and in general,

$$\left|y_n(x) - y_{n-1}(x)\right| \le M^n \frac{|x-a|^n}{n!} \quad (4.2.12)$$

Since we have $|x-a| < h$, then

$$\left|y_n(x) - y_{n-1}(x)\right| \le M^n \frac{h^n}{n!} \quad (4.2.13)$$

A majorante for the series (4.2.9) is given by the sum

$$Y = f + Mh + \frac{(Mh)^2}{2!} + \frac{(Mh)^3}{3!} + \ldots + \frac{(Mh)^n}{n!} + \ldots \quad (4.2.14)$$

or by the sum

$$Y = f + \sum_{n=1}^{\infty} \frac{(Mh)^n}{n!}$$

This majorant series converges and therefore the series (4.2.9) converges uniformly.

4.3 An Existence Theorem for Nonlinear Integral Equations of Fredholm Type

We consider the problem of establishing criteria for the existence of solutions for the nonlinear Fredholm integral equation

$$y(x) = f(x) + \lambda\int_a^b \kappa\left[x, s, y(s)\right]ds \quad (4.3.1)$$

where λ is a parameter.

From the theory of linear Volterra and Fredholm equations, we know that the parameter λ plays a significant role. The most essential difference between Volterra and Fredholm equations for bounded kernels, integrable functions and a finite range of integration is as follows:

We can establish criteria under which a solution exists for (4.3.1), making the following assumptions similar for equations of Volterra type given in Sec 4.2 (Davis, 1962):

a) The function $f(x)$ is bounded in the interval $a \le x \le b$, that is, $|f(x)| < f$.

b) The kernel $\kappa(x, y, z)$ is integrable and bounded,

$$|\kappa(x, y, z)| < K \tag{4.3.2}$$

in the domain D: $a \le x \le b$, $|z| < c$.

c) $\kappa(x, y, z)$ satisfies the Lipschitz condition in D, namely,

$$|\kappa(x, y, z) - \kappa(x, y, z')| < M |z - z'| \tag{4.3.3}$$

By successive approximations we have

$y_0(x) = f(x) - f(a)$, (as the first approximation)

from which we get

$$y_1(x) = f(x) + \lambda\int_a^b \kappa[x, s, y_0(s)]ds \tag{4.3.4}$$

and, in general,

$$y_n(x) = f(x) + \lambda\int_a^b \kappa[x, s, y_{n-1}(s)]ds$$

From these we obtain

$$y_1 - y_0 = \lambda\int_a^b \kappa[x, s, y_0(s)]ds + f(a)$$

$$y_2 - y_1 = \lambda\int_a^b \{\kappa[x, s, y_1(s)] - \kappa[x, s, y_0(s)]\}ds$$

$$y_n - y_{n-1} = \lambda \int_a^b \{\kappa[x, s, y_{n-1}(s)] - \kappa[x, s, y_{n-2}(s)]\} ds$$

Using the conditions given above, we have

$$|y_1 - y_0| < |\lambda|\kappa(b-a) + |f(a)| \leq |\lambda|(b-a)K\left[1 + \frac{f}{|\lambda|K(b-a)}\right] \tag{4.3.5}$$

$$|y_1 - y_0| \leq |\lambda| m(b-a),$$

where

$$m = K\left[1 + \frac{f}{|\lambda|K(b-a)}\right] \tag{4.3.6}$$

From this inequality and the Lipschitz condition on κ, we get

$$|y_2 - y_1| < |\lambda| M \int_a^b |y_1 - y_0| ds < |\lambda|^2 Mm(b-a)^2 < |\lambda|^2 k^2 (b-a)^2 \tag{4.3.7}$$

where k is the LARGER of the two numbers M and m.

Similarly we obtain the inequalities:

$$|y_3 - y_2| < |\lambda|^3 k^3 (b-a)^3, \tag{4.3.8}$$

.

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$$|y_n - y_{n-1}| < |\lambda|^n k^n (b-a)^n \tag{4.3.9}$$

A majorante for the series

$$y(x) = y_0(x) + [y_1(x) - y_0(x)] + [y_2(x) - y_1(x)] + \ldots + [y_n(x) - y_{n-1}(x)] + \ldots \tag{4.3.10}$$

is given by the sum

$$Y = f + \sum_{n=1}^{\infty} |\lambda|^n k^n (b-a)^n, \tag{4.3.11}$$

and thus the series converges uniformly for all values of λ for which we have

$$|\lambda| < \frac{1}{k(b-a)} \tag{4.3.12}$$

Although the condition (4.3.12) is equivalent to that obtained when equation (4.3.1) is linear, the role played by λ in the case where $f(x) \equiv \lambda$ is quite different in nonlinear equations from that which it has in the linear case (Davis, 1962).

CHAPTER FIVE

HOMOTOPY PERTURBATION METHOD

5.1 Introduction

The homotopy perturbation method (HPM) which was firstly presented by Liao(1995) and by He (1999) in 1998 and was further developed and improved by He (2000; 2003; 2004) provides an effective procedure for explicit and numerical solutions of a wide and general class of (linear and nonlinear) differential and integral systems representing real physical problems. The essential of this method is to continuously deform a simple problem which is easy to solve into the under study problem which is difficult to solve.

This method is based on both homotopy in topology and the Maclauren series and yields a very rapid convergence of the solution series in most cases. It is a new perturbation technique coupled with the homotopy technique (He, 2003).

The nonlinear analytical methods most widely applied are perturbation techniques (Nayfeh, 1981). In perturbation methods, a nonlinear equation is transformed into an infinite number of linear equations by means of the small parameter assumption. But perturbation methods have some limitations:

- perturbation techniques are based on small or large parameters but not every nonlinear equation has such a small parameter. (The homotopy perturbation method has been proposed to eliminate the small parameter.)
- even if there exists such a parameter, the results given by the perturbation methods are valid, in most cases, only for small values of the parameter.
- mostly, the simplified linear equations have different properties from the original nonlinear equation.
- sometimes some initial and boundary conditions are superfluous for the simplified linear equations.

Liao (1995) has described a nonlinear analytic technique does not require small parameters and thus can be applied to solve nonlinear problems without small or large parameters. This technique is based on homotopy.

Using one interesting property of homotopy which is given in Section 5.2, we can transform any nonlinear problem into an infinite number of linear problems, no matter whether or not there exists a small or large parameter.

This is in opposition to classical perturbation techniques the homotopy perturbation method have some advantages (He, 2003):

- it does not require small or large parameters in the equations, so the limitations of the classical perturbation methods can be eliminated.
- the initial approximations can be freely selected with possible unknown constants.
- the approximations obtained by this method are valid not only for small parameters, but also for very large parameters,
- it may give better approximations which are uniformly valid for both small or large parameters or variables. Because this method is based on the simple property of homotopy in topology, that is, the kth-order deformation equations are linear.

As a result, in this method the solution of functional equations is considered as the summation of an infinite series usually converging to the solution. Using homotopy technique in topology, a homotopy is constructed with an embedding parameter $p \in [0,1]$ which is considered as a small parameter. The approximations obtained by the proposed method are uniformly valid not only for small parameters, but also for very large parameters (Biazar, 2009).

Then He's homotopy perturbation method has been also used by many mathematicians and engineers to solve the linear or nonlinear systems of Fredholm and Volterra type integral equations (Biazar, 2009; Yusufoğlu, 2008).

5.2 What is Homotopy?

The idea in homotopy is: we should consider two functions to be equivalent or homotopic, if one can be deformed into the other.

5.2.1 Example

Let $f:[0,2]\to IR$ be the function

$$f(x)=1+x^2(x-2)^2$$

shown in Figure a. This is almost a constant function to 1, but with a small deviation around x=1. If we take the function

$$f_1(x)=1+\frac{1}{2}x^2(x-2)^2$$

then this has a similar shape, but with a small deviation. Similarly

$$f_2(x)=1+\frac{1}{3}x^2(x-2)^2$$

has the same shape but with an even smaller deviation in Figure b.

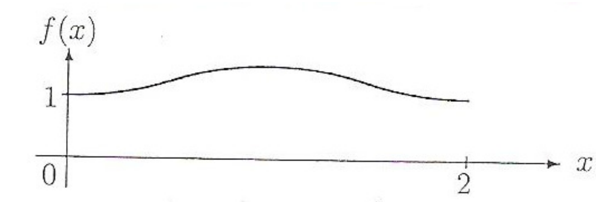

Figure a

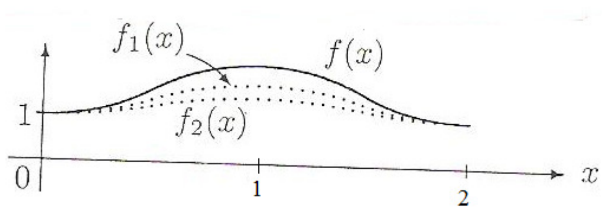

Figure b

Generally, for each $n \geq 1$, we can define

$$f_n(x)=1+\frac{1}{n+1}x^2(x-2)^2$$

and thus we obtain a family of functions interpolating between f and the constant function provided that these interpolating functions to provide a continuous deformation of the one function into the other. This can be done by indexing the interpolating functions $f_1, f_2, ..., f_n, ...$ by real numbers in some fixed range, say between 0 and 1. So we have a family of functions $\{f_t\}_{t\in[0,1]}$, such that $f_0 = f$ and f_1 is the constant function 1.

In this example, we can set

$f_t(x) = 1 + (1-t)x^2(x-2)^2$ for each $t \in [0,1]$

Then

$f_0(x) = 1 + x^2(x-2)^2$

and

$f_1(x) = 1$ is the constant function.

Such a deformation then assigns a function to each point in [0, 1], so the deformation is a function from [0, 1] to the set of continuous maps $[0,2] \to IR$ which takes $t \in [0,1]$ to the function f_t. That is, the family $\{f_t\}_{t\in[0,1]}$ assigns, to each point $t \in [0,1]$, a function

$$f_t : [0,2] \to IR.$$

And this assigns to each point $x \in [0,2]$ a value $f_t(x) \in IR$. Thus we can think of this family as assigning to each pair $(x,t) \in [0,2]\times[0,1]$ the value $f_t(x) \in IR$, Figure c.

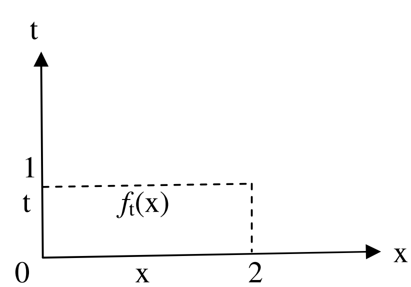

Figure c

In other words, we have a function

$[0,2] \times [0,1] \rightarrow IR$,

where we have a topology on $[0,2]$, and we know a topology on $[0,1]$, so we can use the product topology to topologize $[0,2] \times [0,1]$ and therefore the interpolating family corresponds to a function between two topological spaces. And the family to be continuous if the corresponding function is continuous. Hence we have;

Definition (Homotopy between two functions)

Two maps $f, g : S \rightarrow T$ are homotopic if there is a continuous function

$$F : S \times [0,1] \rightarrow T$$

such that

$F(s,0) = f(s)$ for all $s \in S$ and

$F(s,1) = g(s)$ for all $s \in S$

In this case, F is homotopy between f and g, and we write $f \cong g$.

In the example 5.2.1,

$f : [0,2] \rightarrow IR$

is given by

$f(x) = 1 + x^2 (x-2)^2$,

the function

$F : [0,2] \times [0,1] \rightarrow IR$

is given by

$F(x,t) = 1 + (1-t)(x-2)^2 = f(x)$ and

$F(x,1) = 1$.

Thus, F is a homotopy from f to the constant function 1.

5.3 He's Homotopy Perturbation Method

In He's homotopy perturbation method the solution of the functional equation is considered as a summation of an infinite series (which converges rapidly to accurate solutions) usually converging to the solution. Using homotopy technique of topology given in Section 5.2, a homotopy is constructed with an embedding parameter $p \in [0,1]$ which is considered as a small parameter.

Consider a nonlinear functional equation

$$A(u) = f(r)\ ,\ r \in \Omega \qquad (5.3.1)$$

with the boundary conditions

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0,\ r \in \partial\Omega = s \qquad (5.3.2)$$

where A is a general integral operator, B is a boundary operator, f(r) is a known analytic function on a Banach space $s = \partial\Omega$ is the boundary of the domain Ω. The operator A generally can be divided into two parts L and N, where L is a functional operator with known solution v_0, which can be obtained easily and satisfies the boundary conditions, whereas N is the nonlinear part. Therefore equation (5.3.1) can be rewritten as follows:

$$L(u) + N(u) = f(r) \qquad (5.3.3)$$

We define a homotopy H(v, p) by

$$H(v,0) = L(v) - L(v_0) = 0, \qquad H(v,1) = A(v) - f(r) = 0$$

v_0 is an initial approximation of Eqn.(5.3.1). By the homotopy technique (He, 2003) we can construct a convex homotopy $v(r, p) : \Omega \times [0,1] \to IR$ which satisfies

$$H(v, p) = (1 - p)[L(v) - L(v_0)] + p[A(v) - f(r)] = 0 \qquad (5.3.4)$$

or equivalently

$$H(v, p) = L(v) - L(v_0) + pL(v_0) + p[N(v) - f(r)] = 0 \qquad (5.3.5)$$

and continuously trace an implicitly defined curve from H(v, 0) to a solution function H(g, 1) where g is a solution of Eqn.(5.3.1). The embedding parameter p monotonically changes from zero to unity as the trivial problem $L(v) - L(v_0)$ is continuously deformed to the original problem $A(v) - f(r)$. In topology, this is called deformation, $L(v) - L(v_0)$ and $A(v) - f(r)$ are called homotopic. If the embedding parameter p is considered as a small parameter applying the classical perturbation technique, we can assume that the solution of Eqn. (5.3.5) can be given by a power series in p, that is,

$$v = \sum_{i=0}^{\infty} p^i v_i = v_0 + pv_1 + p^2 v_2 + ... \quad (5.3.6)$$

and p=1 results in the approximate solution of Eqn.(5.3.1) as

$$u = Lim_{p\to 1} v = v_0 + v_1 + v_2 + ... \quad (5.3.7)$$

A combination of the perturbation method and the homotopy method is called the homotopy perturbation method (HPM), which has eliminated the limitations of classical perturbation methods.

The series (5.3.7) is convergent for most cases. The convergence rate depends on the nonlinear operator $A[u]$ which has been given by He(1999):

1) The second derivative of $N(v)$ with respect to v must be small, because the parameter p may be relatively large, i.e. $p \to 1$.

2) The norm of $L^{-1}\frac{\partial N}{\partial v}$ must be smaller than one so that the series converges. We have the following theorem (He, 1999):

Theorem 5.3.1

Suppose that X and Y be Banach spaces and $N : X \to Y$ is a contraction nonlinear mapping, which satisfy the following condition

$$\|N(v) - N(\tilde{v})\| \le \gamma \|v - \tilde{v}\|$$

for all $v, \tilde{v} \in X$ and $0 < \gamma < 1$. With according to Banach's fixed point theorem, having the fixed point u, that is $N(u) = u$.

The sequence generated by the homotopy perturbation method will be taken as

$$V_n = N(V_{n-1}) \ , \ V_{n-1} = \sum_{i=0}^{n-1} u_i \ , \ n = 1,2,...$$

and suppose that $V_0 = v_0 = u_0 \in B_r(u)$,where

$B_r(u) = \{u^* \in X \| \|u^* - u\| < r\}$, then V_n satisfies the following statements:

i) $\|V_n - u\| \le \gamma^n \|v_0 - u\|$

ii) $V_n \in B_r(u)$

iii) $Lim_{n\to\infty} V_n = u$

(Biazar, Ghazvini, 2009)

Proof

i) By the induction method on n, for n=1 we have

$\|V_1 - u\| = \|N(V_0) - N(u)\| \leq \gamma \|v_0 - u\|$

Assume that $\|V_{k-1} - u\| \leq \gamma^{k-1} \|v_0 - u\|$ as an induction hypothesis, then n=k gives

$\|V_k - u\| = \|N(V_{k-1}) - N(u)\| \leq \gamma \|V_{k-1} - u\| \leq \gamma\gamma^{k-1} \|v_0 - u\|$

$= \gamma^k \|v_0 - u\|$

Thus, it is true for any integer n.

ii) Using (i) and the hypothesis

$\|V_n - u\| \leq \gamma^n \|v_0 - u\| \leq \gamma^n r < r$

implies $V_n \in B_r(u)$.

iii) Because of (i) we have

$\|V_n - u\| \leq \gamma^n \|v_0 - u\|$,

and

$Lim_{n\to\infty} \|V_n - u\| = 0$,

that is,

$Lim_{n\to\infty} V_n = u$.

5.4 Homotopy Perturbation Method for Nonlinear Fredholm Integral Equations of the Second Kind

We consider the following Fredholm integral equation of the second kind

$$u(x) = f(x) + \int_a^b \kappa(x, y) F(u(y)) dy$$

or equivalently

$$u(x) = f(x) + \int_a^b \kappa(x,y)[R(u(y)) + N(u(y))]dy \qquad (5.4.1)$$

where u(x) is an unknown function that will be determined, $\kappa(x,y)$ is the kernel of the integral equation, f(x) is a known analytic function, R(u) and N(u) are linear and nonlinear functions of u, respectively.(Ganji, Afrouzi, 2007)

To illustrate the homotopy perturbation method (HPM), we rewrite Eqn. (5.4.1) as

$$L(u) = u(x) - f(x) - \int_a^b \kappa(x,y)[R(u(y)) + N(u(y))]dy = 0 \qquad (5.4.2)$$

In this case we construct a homotopy

$$H(u,0) = F(u), \qquad H(u,1) = L(u) \qquad (5.4.3)$$

where F(u) is an integral operator with known solution u_0, which can be obtained easily, we then choose a convex homotopy by

$$H(u,p) = (1-p)F(u) + pL(u) = 0 \qquad (5.4.4)$$

The changing process of p from zero to unity is just that of $H(u,p)$ from a starting point $H(u_0,0)$ to a solution function $H(u,1)$ that is the known problem $F(u)$ is transformed continuously to the original problem $L(u) = 0$.

Setting

$$F(u) = u(x) - f(x) \qquad L(u) = u(x) - f(x) - \int_a^b \kappa(x,y)[R(u(y)) + N(u(y))]dy = 0 \qquad (5.4.5)$$

the homotopy takes the form

$$H(u,p) = (1-p)[u(x) - f(x)] + p\left[u(x) - f(x) - \int_a^b \kappa(x,y)[R(u(y)) + N(u(y))]dy\right] = 0 \qquad (5.4.6)$$

Substituting

$$u(x) = \sum_{i=0}^{\infty} p^i u_i(x) = u_0(x) + pu_1(x) + p^2 u_2(x) + \ldots \qquad (5.4.7)$$

into Eqn. (5.4.6) and equating the coefficients of p with the same powers leads to

$$p^0: \quad u_0(x) - f(x) = 0 \quad \Rightarrow \quad u_0(x) = f(x) \qquad (5.4.8a)$$

$$p^1: \quad u_1(x) - \int_a^b \kappa(x,y)[R(u_0(y)) + N(u_0(y))]dy = 0 \quad \Rightarrow$$

$$u_1(x) = \int_a^b \kappa(x,y)[R(u_0(y)) + N(u_0(y))]dy \qquad (5.4.8b)$$

$$p^2: \quad u_2(x) = \int_a^b \kappa(x,y)[R(u_1(y)) + N(u_1(y))]dy \qquad (5.4.8c)$$

and in general,

$$u_0(x) = f(x) \qquad (5.4.9a)$$

$$u_{n+1}(x) = \int_a^b \kappa(x,y)[R(u_n) + N(u_n)]dy \ . \qquad (5.4.9b)$$

The approximated solution of Eqn. (5.4.1) therefore, can be obtained by setting p=1.

$$U(x) = Lim_{p\to 1} \sum_{i=0}^{\infty} p^i u_i(x). \qquad (5.4.10)$$

Example

Consider the nonlinear Fredholm integral equation of the second kind (Wazwaz, 1997)

$$u(x) = \sinh x - 1 + \int_0^1 \left(\cosh^2(t) - u^2(t)\right)dt \qquad (5.4.11)$$

where

$$u_0(x) = \sinh x \qquad (5.4.12)$$

He's homotopy perturbation method can be constructed as follows:

$$H(u,p) = (1-p)F(u) + pL(u) = 0$$

Taking

$$F(u) = u(x) - \sinh x\,,$$

$$L(u) = u(x) - \sinh x + 1 - \int_0^1 \left(\cosh^2(t) - u^2(t)\right)dt$$

We have

$$H(u,p)=(1-p)[u(x)-\sinh x]+p\left[u(x)-\sinh x+1-\int_0^1\left(\cosh^2(t)-u^2(t)\right)dt\right]=0$$

(5.4.13)

This gives

$$H(u,p)=u(x)-\sinh x-pu(x)+p\sinh x+pu(x)-p\sinh x+p$$
$$-p\int_0^1\left(\cosh^2(t)-u^2(t)\right)dt=0$$

or

$$H(u,p)=u(x)-\sinh x+p-p\left[\int_0^1\left(\cosh^2(t)-u^2(t)\right)dt\right]=0 \qquad (5.4.14)$$

Suppose the solution of Eqn. (5.4.14) have the following form

$$v(x)=v_0(x)+pv_1(x)+p^2v_2(x)+\ldots+p^nv_n(x)+\ldots \qquad (5.4.15)$$

where $v_i(x)$ $i=0,1,2,\ldots$ are functions yet to be determined. According to Eqn. (5.4.15) the initial approximation is

$$v_0(x)=\sinh x \qquad (5.4.16)$$

Substituting equations (5.4.15), (5.4.16) into Eqn. (5.4.14) and equating the terms with the coefficients of the identical powers of p yields:

$$v_0(x)+pv_1(x)+p^2v_2(x)+\ldots=\sinh x-p+p\int_0^1\left[\cosh^2(t)-\left(v_0(t)+pv_1(t)+p^2v_2(t)+\ldots\right)^2\right]dt$$

(5.4.17)

$$p^0: \quad v_0(x)=\sinh x \qquad (5.4.18a)$$

$$p^1: \quad v_1(x)=-1+\int_0^1\left(\cosh^2(t)-v_0^2(t)\right)dt$$

$$v_1(x)=-1+\int_0^1\left(\cosh^2(t)-\sinh^2(t)\right)dt$$

$$v_1(x)=0 \qquad (5.4.18b)$$

$$p^2: \quad v_2(x)=\int_0^1 2v_0(t)v_1(t)dt$$

$$v_2(x) = 0 \tag{5.4.18c}$$

In the same manner, the rest of components can be obtained:

$$v_{k+2}(x) = \int_0^1 \left(\cosh^2(t) - v_{n+1}^2(t)\right) dt \qquad \Rightarrow \qquad v_{k+2}(x) = 0\,, \quad k \geq 0 \tag{5.4.19}$$

Hence, according to the homotopy perturbation method the solution will be as follows:

$$u(x) = Lim_{p\to 1} v(x) = v_0(x) + v_1(x) + v_2(x) + \ldots \tag{5.4.20}$$

Therefore

$$u(x) = v(x) = \sinh x$$

This is the exact solution of the integral equation. The success of obtaining the exact solution by using two iterations is a result of the proper selection of $u_0(x)$. The plot of the solution is given in Figure 5.4.

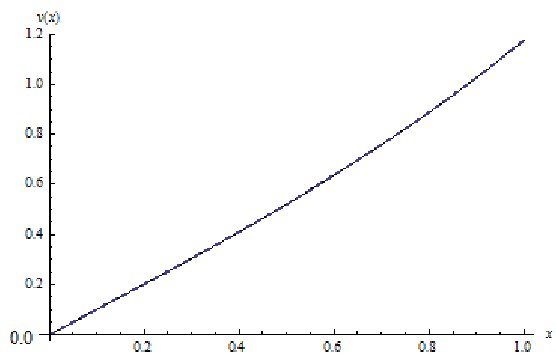

Figure 5.4 The numerical results and exact solutions of example

……v(homotopy) ——v(exact)

5.5 Homotopy Perturbation Method for Nonlinear Volterra Integral Equations of the Second Type

We consider the following Volterra integral equation of the second type

$$u(x) = f(x) + \int_a^x \kappa(x, y) F(u(y)) dy$$

or equivalently

$$u(x) = f(x) + \int_a^x \kappa(x, y)[R(u(y)) + N(u(y))] dy \tag{5.5.1}$$

where u(x) is an unknown function that will be determined, $\kappa(x, y)$ is the kernel of the integral equation, f(x) is a known analytic function, R(u) and N(u) are linear and nonlinear functions of u, respectively and $a \leq x$.

To illustrate the homotopy perturbation method (HPM), we rewrite Eqn. (5.5.1) as

$$L(u) = u(x) - f(x) - \int_a^x \kappa(x, y)[R(u(y)) + N(u(y))] dy = 0 \tag{5.5.2}$$

with $a = 0$.

For solving Eqn. (5.5.2), by He's HPM, we construct a homotopy

$$H(u,0) = F(u), \qquad H(u,1) = L(u) \tag{5.5.3}$$

where F(u) is an integral operator with known solution u_0, which can be obtained easily, we then choose a convex homotopy by

$$H(u, p) = (1 - p)F(u) + pL(u) = 0 \tag{5.5.4}$$

The changing process of p from zero to unity is just that of $H(u, p)$ from a starting point $H(u_0,0)$ to a solution function $H(u,1)$ that is, the known problem $F(u)$ is transformed to the original problem $L(u) = 0$.

Setting

$$F(u) = u(x) - f(x) \ , \quad L(u) = u(x) - f(x) - \int_a^x \kappa(x, y)[R(u(y)) + N(u(y))] dy = 0 \tag{5.5.5}$$

the homotopy takes the form

$$H(u,p) = (1-p)[u(x) - f(x)] + p\left[u(x) - f(x) - \int_a^x \kappa(x,y)[R(u(y)) + N(u(y))]dy\right] = 0$$

(5.5.6)

Substituting

$$u(x) = \sum_{i=0}^{\infty} p^i u_i(x) = u_0(x) + pu_1(x) + p^2 u_2(x) + \ldots \tag{5.5.7}$$

into Eqn. (5.5.6) and equating the coefficients of p with the same powers leads to

$$p^0: \quad u_0(x) = f(x) \tag{5.5.8a}$$

$$p^1: \quad u_1(x) = \int_a^x \kappa(x,y)[R(u_0(y)) + N(u_0(y))]dy \tag{5.5.8b}$$

$$p^2: \quad u_2(x) = \int_a^x \kappa(x,y)[R(u_1(y)) + N(u_1(y))]dy \tag{5.5.8c}$$

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and in general,

$$u_0(x) = f(x) \tag{5.5.9a}$$

$$u_{n+1}(x) = \int_a^x \kappa(x,y)[R(u_n) + N(u_n)]dy \tag{5.5.9b}$$

The approximated solution of Eqn. (5.5.1) therefore, can be obtained by setting p=1.

$$U(x) = Lim_{p\to 1} \sum_{i=0}^{\infty} p^i u_i(x) \tag{5.5.10}$$

Example

Let us solve the following nonlinear Volterra integral equation of the second kind, with the exact solution $u(x) = \sec x$ by the homotopy perturbation method (Wazwaz, 1997)

$$u(x) = \sec x + \tan x + x - \int_0^x \left(1 + u^2(t)\right)dt, \qquad x \le \frac{\pi}{2} \tag{5.5.11}$$

$$u_0(x) = \sec x \tag{5.5.12}$$

To solve equation (5.5.11) by homotopy perturbation method we can construct a homotopy as follows:

$H(u,p)=(1-p)F(u)+pL(u)=0$

Taking

$F(u)=u(x)-\sec x$

$L(u)=u(x)-\sec x-\tan x-x+\int_0^x (1+u^2(t))dt$

We have

$$H(u,p)=(1-p)(u(x)-\sec x)+p\left(u(x)-\sec x-\tan x-x+\int_0^x (1+u^2(t))dt\right)=0 \tag{5.5.13}$$

This gives

$$H(u,p)=u(x)-\sec x-pu(x)+p\sec x+pu(x)-p\sec x-p\tan x-px+p\int_0^x (1+u^2(t))dt=0$$

or

$$H(u,p)=u(x)-\sec x-p\tan x-px+p\int_0^x (1+u^2(t))dt=0 \tag{5.5.14}$$

Let

$$v(x)=v_0(x)+pv_1(x)+p^2v_2(x)+\ldots \tag{5.5.15}$$

be a solution of Eqn. (5.5.14).Here $v_i(x)$ $i=0,1,2,\ldots$ are functions to be determined. According to Eqn. (5.5.1.5) the initial approximation is

$$v_0(x)=\sec x \tag{5.5.16}$$

Substituting equations (5.5.15), (5.5.16) into Eqn. (5.5.14) and comparing the coefficients of the powers of p yields the following scheme:

$$v_0(x)+pv_1(x)+p^2v_2(x)+\ldots=\sec x+p\tan x+px-p\int_0^x \left[1+\left(v_0(t)+pv_1(t)+p^2v_2(t)+\ldots\right)^2\right]dt \tag{5.5.17}$$

$$p^0: \quad v_0(x)=\sec x \tag{5.5.18a}$$

$$p^1: \quad v_1(x) = \tan x + x - \int_0^x \left(1 + v_0^2(t)\right) dt$$

$$v_1(x) = \tan x + x - \int_0^x \left(1 + \sec^2 t\right) dt \qquad \Rightarrow$$

$$v_1(x) = 0 \tag{5.5.18b}$$

$$p^2: \quad v_2(x) = -\int_0^1 2x v_0(t) v_1(t) dt$$

$$v_2(x) = -\int_0^x 2x \sec x.0.dt$$

$$v_2(x) = 0 \tag{5.5.18c}$$

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In general

$$v_{k+2}(x) = -\int_0^x \left(1 + -v_{k+1}^2(t)\right) dt \qquad \Rightarrow$$

$$v_{k+2}(x) = 0, \quad k \geq 0. \tag{5.5.19}$$

And according to the HPM, we can conclude

$$u(x) = Lim_{p \to 1} v(x) = v_0(x) + v_1(x) + v_2(x) + \ldots \tag{5.5.20}$$

Therefore

$$u(x) = v(x) = \sec x \tag{5.5.21}$$

Here we used two iterations only to obtain the exact solution. The plot of the exact solution is presented in Figure 5.5.

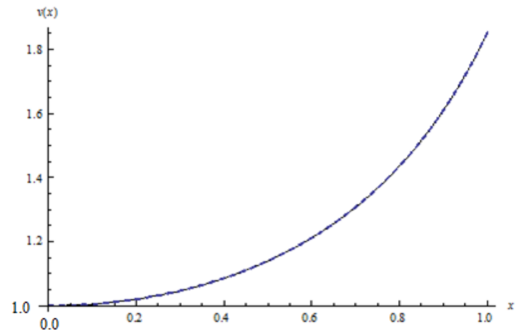

Figure 5.5 The plot of the exact solution of this example

……v(homotopy) ——— v (exact)

5.6 Application of Homotopy Perturbation Method to Integro-Differential Equations

We can apply the He's homotopy perturbation method for the solution of integro-differential equations of the form

$$f'(x) = g(x) + \int_0^x \kappa(t, f(t), f'(t))dt \tag{5.6.1}$$

To construct a convex homotopy, we write the integro-differential equation (5.6.1) as

$$L(u) = u'(x) - \int_0^x \kappa(t, f(t), f'(t))dt - g(x) = 0 \tag{5.6.2}$$

with solution $f(x)$. By using homotopy technique, we can construct a homotopy as given in Chapter 5.

Example (The Nonlinear Volterra Integro-Differential Equation)

Consider the nonlinear Volterra integro-differential equation (Alizadeh, Seyed S. R., Domairy, G. G. and Karimpour, S. (2008))

$$\frac{du(x)}{dx} = 10u(x) - 10u^2(x) - 10u(x)\int_0^x u(t)dx \tag{5.6.3}$$

where the initial condition $u(0)=0$.

In order to solve nonlinear Volterra integral equation (5.6.3) using He's homotopy perturbation method we construct a homotopy $H(v,p)$ such that

$$H(v,p)=(1-p)\left[\frac{d}{dx}v(x)\right]+p\left[\frac{d}{dx}v(x)-10v(x)+10v^2(x)+10v(x)\int_0^x v(t)dx\right]=0 \tag{5.6.4}$$

Substituting

$$v=v_0+pv_1+p^2v_2+\ldots \tag{5.6.5}$$

into Eqn. (5.6.4) we obtain

$$H(v,p)=(1-p)[\frac{d}{dx}(v_0+pv_1+p^2v_2+\ldots)]$$
$$+p[\frac{d}{dx}(v_0+pv_1+p^2v_2+\ldots)-10(v_0+pv_1+p^2v_2+\ldots)+10(v_0+pv_1+p^2v_2+\ldots)^2$$
$$+10(v_0+pv_1+p^2v_2+\ldots)\int_0^x(v_0(t)+pv_1(t)+p^2v_2(t)+\ldots)dx]=0$$

$$H(v,p)=(1-p)[\left(\frac{dv_0}{dx}+p\frac{dv_1}{dx}+p^2\frac{dv_2}{dx}+\ldots\right)]$$
$$+p[\frac{dv_0}{dx}+p\frac{dv_1}{dx}+p^2\frac{dv_2}{dx}+\ldots-10v_0-10pv_1-10p^2v_2+\ldots$$
$$+10(v_0^2+p^2v_1^2+2pv_0v_1+2p^2v_0v_2+\ldots)$$
$$+10v_0\int_0^x v_0(t)dx+10pv_1\int_0^x pv_1(t)dx+10p^2v_2\int_0^x p^2v_2(t)dx+\ldots]=0 \tag{5.6.6}$$

Rearranging based on powers of p terms, we find that:

p^0 : $\quad \frac{dv_0(x)}{dx}=0 \quad \Rightarrow$

$$v_0(x)=0 \tag{5.6.7a}$$

p^1 : $\quad \frac{dv_1(x)}{dx}+0.1x-0.9=0 \quad \Rightarrow$

$$v_1(x)=-0.5x^2+0.9x \tag{5.6.7b}$$

p^2 : $\quad \frac{dv_2(x)}{dx}+0.85x-7.2x+x(-0.05x^2+0.9x)-\frac{1}{60}=0 \quad \Rightarrow$

$$v_2(x)=\frac{1}{60}x^4-\frac{7}{12}x^3+3.6x^2 \tag{5.6.7c}$$

Thus, the approximate solution given by He's HPM with three iterations have the following form

$$u_{HPM} = \sum_{i=0}^{2} v_i(x) = v_0(x) + v_1(x) + v_2(x) \tag{5.6.8}$$

$$u_{HPM} = 0 + \left(-0.05x^2 + 0.9x\right) + \left(\frac{1}{60}x^4 - \frac{7}{12}x^3 + 3.6x^2\right)$$

$$u_{HPM} = \frac{1}{60}x^4 - \frac{7}{12}x^3 + 3.55x^2 + 0.9x \tag{5.6.9}$$

The approximation (5.6.9) is in full agreement with the approximation (5.6.10) obtained by using the ADM solution method with two iterations

$$u_{ADM} = \sum_{i=1}^{2} v_i(x) = \frac{1}{60}x^4 - \frac{7}{12}x^3 + 3.55x^2 + 0.9x \tag{5.6.10}$$

CHAPTER SIX

SYSTEMS OF NONLINEAR INTEGRAL EQUATIONS

6.1 Introduction

In this section we give an application of He's homotopy perturbation method (HPM) to solve nonlinear systems of Fredholm and Volterra integral equations.

The Adomian decomposition method was being used to solve linear and nonlinear systems of Volterra integral equations of the first kind and Fredholm integral equations of the first kind. The He's homotopy perturbation method is applied to solve the nonlinear Volterra-Fredholm integral equations of the second kind [M.Ghasemi,T.Tavassoli].

In this chapter, we extend the homotopy perturbation method to solve nonlinear integral equations such that Fredholm and Volterra integral equations. Then we compare this method with the analytic approximation methods such as Adomian Decomposition Method (ADM) and Taylor-Series Expansion Method. The results reveal that the Homotopy Perturbation Method is very effective and simple.

A system of Fredholm and Volterra type integral equations can be presented, respectively, as the following:

$$f_i(x) = g_i(x) + z_i(x, f(x)) + \int_a^b v_i(x, s, f(s))ds \quad i = 1,2,\ldots,n \qquad (6.1.1)$$

$$f_i(x) = g_i(x) + z_i(x, f(x)) + \int_a^x v_i(x, s, f(s))ds \quad i = 1,2,\ldots,n \qquad (6.1.2)$$

where $f(x) = (f_1(x), f_2(x), \ldots, f_n(x))$, and $f_i(x)$ and $g_i(x)$ are known functions, $x \in [a,b]$. We suppose that the systems (6.1.1) and (6.1.2) have a unique solution. The necessary and sufficient conditions for the existence and uniqueness of the solution of the system (6.1.1) and (6.1.2) is given by Delves.

Concerning the system of Fredholm and Volterra type integral equations of (6.1.1) and (6.1.2) the solution would be taken in the following form:

$$f_1(x) = \sum_{i=0}^{\infty} p^i f_{1i} = f_{10} + pf_{11} + p^2 f_{12} + \dots$$

$$\tag{6.1.3}$$

$$f_2(x) = \sum_{i=0}^{\infty} p^i f_{2i} = f_{20} + pf_{21} + p^2 f_{22} + \dots$$

In practice, all terms of the series (6.1.3) can not be determined and so we use an approximation of the solution by the truncated series:

$$\varphi_{1,m}(x) = \sum_{i=0}^{m-1} f_{1i}(x), \ \varphi_{2,m}(x) = \sum_{i=0}^{m-1} f_{2i}(x) \tag{6.1.4}$$

6.2 System of Volterra Integral Equations of the First Kind

A system of integral equations of the first kind can be presented as:

$$\int_0^x \kappa_i(x,t) g_i(u_1(t), u_2(t), \dots, u_n(t)) dt = f_i(x) \qquad i = 1,2,\dots,n \tag{6.2.1}$$

where f_i are known functions, $\kappa_i(x,t)$ are the kernels of the ith integral equation, g_i are linear and nonlinear functional of the unknown functions u_i. In the system (6.2.1) the equations are not in the canonical form. To derive this form, we differentiate of the both sides of equation (6.2.1), with respect to x, and according to the Leibnitz generalized formula and we obtain (if $\kappa_i(x,x) \neq 0$)

$$\kappa_i(x,x) g_i(u_1(x), u_2(x), \dots, u_n(x)) + \int_0^x \frac{\partial \kappa_i(x,t)}{\partial x} g_i(u_1(t), u_2(t), \dots, u_n t) dt = f_i'(x)$$

$$\tag{6.2.2}$$

$i = 1,2,\dots,n$

And then

$$g_i(u_1(x), u_2(x), \dots, u_n(x)) + \int_0^x \frac{\partial \kappa_i(x,t) / \partial x}{\kappa_i(x,x)} g_i(u_1(t), u_2(t), \dots, u_n t) dt = \frac{f_i'(x)}{\kappa_i(x,x)} \tag{6.2.3}$$

or

$$g_i(u_1(x),u_2(x),...,u_n(x)) = \frac{f_i'(x)}{\kappa_i(x,x)} - \int_0^x \frac{\partial\kappa_i(x,t)/\partial x}{\kappa_i(x,x)} g_i(u_1(t),u_2(t),...,u_n t)dt \quad (6.2.4)$$

The nonlinear system of integral of integral equations (6.2.4) can be reduced into a simpler system of integral equations of the second type. There are two procedures. (Biazar, Babolian and Islam, 2003)

First Approach

If we can recognize invertible functions $g_i(u_i(x))$ for each unknown $u_i(x)$ and we set $v_i = g_i(u_i(x))$ then the nonlinear system of integral equations reduces to a simpler system of integral equations and can be easily solved by the Homotopy Perturbation Method.

Second Approach

Let

$$h_i(x) = g_i(u_1(x),u_2(x),...,u_n(x)) \quad (6.2.5)$$

Then the system of integral equations (6.2.4) can be rewritten as:

$$h_i(x) = \frac{f_i'(x)}{\kappa_i(x,x)} - \int_0^x \frac{\partial\kappa_i(x,t)/\partial x}{\kappa_i(x,x)} h_i(x)dt \quad i = 0,1,2,... \quad (6.2.6)$$

which is a system of linear integral equations of the second kind, which can be solved easily by the Homotopy Perturbation Method.

6.2.1 Analysis of the Homotopy Perturbation Method for Systems of Volterra Integral Equations of the First Kind

To apply the homotopy perturbation method (HPM) we consider Eqn. (6.1.4) as

$$f_i(g_i(u_i(x))) = g_i(u_i(x)) - \frac{f_i'(x)}{\kappa_i(x,x)} + \int_0^x \frac{\partial\kappa_i(x,t)/\partial x}{\kappa_i(x,x)} g_i(u_i(t))dt \quad (6.2.7)$$

where $i = 1,2,...,n$ and $j = 1,2,...,n$

with the solution $g_i(u_i)$, where $f_i(g_i(u_i(x))) = f_i(\varphi_i(x))$.

We can define homotopy $H(g(u_i), p)$ by

$$H(g(u_i), p) = (1-p)L_i(g_i(u_i)) + pf_i(g_i(u_i)) = 0 \quad (6.2.8)$$

where $L(g_i(u_i))$ is a functional operator with known solution $g_i(u_{i0})$, which can be obtained easily. From Eqn. (6.2.8) we have

$$H(g(u_i),0) = L_i(g_i(u_i)), \quad H(g(u_i),1) = f_i(g_i(u_i)) \quad (6.2.9)$$

that is,

$$H_i(g(u_i),0) = L_i(g_i(u_i)) = g_i(u_1(x),\ldots,u_n(x)) - \frac{f_i'(x)}{\kappa_i(x,x)} \quad i = 1,2,\ldots,n \quad (6.2.10a)$$

$$H(g(u_i),1) = f_i(g_i(u_i)) = g_i(u_i(x)) - \frac{f_i'(x)}{\kappa_i(x,x)} + \int_0^x \frac{\partial\kappa_i(x,t)/\partial x}{\kappa_i(x,x)} g_i(u_i(t))dt \quad (6.2.10b)$$

The changing process of p from zero to unity is just that of $H(g(u_i), p)$ from a starting point $H(g(u_i),0)$ to a solution function $H(g_i(\varphi_i(x)),1)$.

We can assume that the solution of Eqn. (6.2.7) can be expressed as a series in p using the perturbation technique:

$$g_i(u_1(x),u_2(x),\ldots,u_n(x)) = g_i(u_{i,0}) + pg_i(u_{i,1}) + p^2 g_i(u_{i,2}) + \ldots \quad (6.2.11)$$

The initial approximations to the solutions $g_i(u_{i,0})$ are taken to be

$$g_i(u_{i0}(t)) = \frac{f_i'(x)}{\kappa_i(x,x)} \quad i = 1,2,\ldots,n. \quad (6.2.12)$$

Substituting (6.2.11) into (6.2.8) and equating the coefficients of p with the same power leads to the following equations:

$$p^0: \quad g_i(u_{i0}(t)) = \frac{f_i'(x)}{\kappa_i(x,x)} \quad i = 1,2,\ldots,n \quad (6.2.13a)$$

$$p^1: \quad g_i(u_{i1}(t)) = \int_0^x \frac{\partial\kappa_i(x,t)/\partial x}{\kappa_i(x,x)} g_i(u_{i0}(t))dt \quad i = 1,2,\ldots,n \quad (6.2.13b)$$

$$p^2: \quad g_i(u_{i2}(t)) = \int_0^x \frac{\partial\kappa_i(x,t)/\partial x}{\kappa_i(x,x)} g_i(u_{i1}(t))dt \quad i = 1,2,\ldots,n \quad (6.2.13c)$$

and in general by the initial approximations

$$g_i(u_{i0}) = \frac{f_i'(x)}{\kappa_i(x,x)}$$

we have

$$g_i(u_{im}) = \int_0^x \frac{\partial \kappa_i(x,t) / \partial x}{\kappa_i(x,x)} g_i(u_{i,m-1}(t))dt \qquad (6.2.14)$$

That is

$$g_1(u_i(x)) = \sum_{i=0}^{\infty} g_1(u_{1i}(x)) = g_1(u_{10}(x)) + g_1(u_{11}(x)) + g_1(u_{12}(x)) + \ldots \qquad (6.2.15a)$$

$$g_2(u_2(x)) = \sum_{i=0}^{\infty} g_2(u_{2i}(x)) = g_2(u_{20}(x)) + g_2(u_{21}(x)) + g_2(u_{22}(x)) + \ldots \qquad (6.2.15b)$$

As $p \to 1$, Eqn. (6.2.8) tends to Eqns.(6.2.7) and (6.2.11) to the solution of Eqn. (6.2.7). Therefore the approximated solutions of (6.2.7), can be obtained by setting $p = 1$.

$$g_i(u_i) = Lim_{p\to 1} f_i(g(u_i(x))) = \sum_{j=0}^{\infty} f_{i,j}(g(u_i(x))) \ \ i = 1,2,\ldots,n\ . \qquad (6.2.16)$$

In practice some terms of this series solution will serve as an approximation solution

$$g_i(\varphi_i^m(x)) = \sum_{j=0}^{m} g(u_{ij}) \qquad (6.2.17)$$

is a m+1 terms approximated solution. This series is convergent for most cases, and the rate of convergence depends on $f_i(g_i(u_i))$ (Adomian, 1986).

6.2 Systems of Volterra Integral Equations of the Second Type

A system of Volterra integral equations of the second kind can be presented as:

$$h_i(x) + \int_0^x \kappa_i(x,t) g_i(u_1(t), u_2(t), \ldots, u_n(t))dt = f_i(x) \quad i = 1,2,\ldots,n$$

where f_i and h_i are known functions, $\kappa_i(x,t)$ are the kernels of the ith integral equation, g_i are linear and nonlinear functional of the unknown functions u_i.The procedure for the Volterra integral equations of the second type as in Section 6.2.1.

6.2.2 Analysis of the Homotopy Perturbation Method for Systems of Fredholm Integral Equations of the Second Type

Consider the following system of Fredholm integral equations of the second kind

$$\underline{f}(x) = \underline{g}(x) + \int_0^1 \underline{\kappa}(x,t)\underline{f}(t)dt\ ,\ 0 \le x \le 1 \qquad (6.2.17)$$

where

$$\underline{\kappa}(x,t) = [k_{ij}(x,t)] = [(x-t)^{q_{ij}}]\ \ i, j = 1,2,\ldots,n \qquad (6.2.18a)$$

$$\underline{f}(x) = [f_1(x), f_2(x),\ldots, f_n(x)]^T \qquad (6.2.18b)$$

$$\underline{g}(x) = [g_1(x), g_2(x),\ldots, g_n(x)]^T \qquad (6.2.18c)$$

In Eqn. (6.2.17) the functions $\underline{\kappa}(x,t)$ and $\underline{g}(x)$ are given, and $\underline{f}(x)$ is the solution to be determined (Delves, Mohamed, 1985). We assume that Eqn. (6.2.17) has the unique solution. The necessary and sufficient conditions for existence and uniqueness of the solution of the system (6.2.17) given in (Delves, Mohamed, 1985).

Let us consider the ith equation of (6.2.17)

$$f_i(x) = g_i(x) + \int_0^1 \sum_{j=0}^{n} k_{ij}(x,t) f_j(x)dt\ ,\ i = 1,2,\ldots,n \qquad (6.2.19)$$

By the homotopy, we construct a convex homotopy for Eqn. (6.2.19) which satisfies

$$H_i\left(\underline{f}, p\right) = (1-p) f_i\left(\underline{f}\right) + pL_i\left(\underline{f}\right) = 0 \qquad (6.2.20)$$

$$H_i\left(\underline{f},0\right) = f_i\left(\underline{f}\right), \quad H_i\left(\underline{f},1\right) = L_i\left(\underline{f}\right) \qquad (6.2.21)$$

where

$$f_i\left(\underline{f}\right) = f_i - g_i$$

$$L_i\left(\underline{f}\right) = f_i(x) - g_i(x) - \int_0^1 \sum_{j=0}^{n} k_{ij}(x,t) f_j(x)dt \qquad (6.2.22)$$

and $p \in [0,1]$ is an embedding parameter. The embedding parameter p monotonically increases from zero to unit as $f_i\left(\underline{f}\right)$ is continuously deformed to the $L_i\left(\underline{f}\right)$.

According to the homotopy perturbation method, we assume that the solution of Eqn. (6.2.19) can be expressed in a series of p

$$f_i(x) = f_{i0} + pf_{i1} + p^2 f_{i2} + \ldots \qquad (6.2.23)$$

Substituting Eqn. (6.2.23) into Eqn. (6.2.20), we find that

$$H_i\left(\underline{f}, p\right) = (1-p)(f_i(x) - g_i(x)) + p\left[(f_i(x) - g_i(x)) - \int_0^1 \sum_{j=0}^{n} k_{ij}(x,t) f_j(x)dt\right] = 0$$

or equivalently

$$H_i\left(\underline{f}, p\right) = (1-p)\left(f_{i0} + pf_{i1} + p^2 f_{i2} + \ldots - g_i\right) +$$

$$p[f_{i0} + pf_{i1} + p^2 f_{i2} + \ldots - g_i - \int_0^1 \sum_{j=0}^{n} (x-t)^{q_{ij}} \left(f_{j0} + pf_{j1} + p^2 f_{j2} + \ldots\right)dt] = 0$$

$$(6.2.24)$$

Eqn. (6.2.24) can be rewritten in the form:

$$H_i\left(\underline{f}, p\right) = (1-p)\left(f_{i0} + pf_{i1} + p^2 f_{i2} + \ldots - g_i\right) +$$

$$p[f_{i0} + pf_{i1} + p^2 f_{i2} + \ldots - g_i -$$

$$\int_0^1 \sum_{j=0}^{n} \sum_{k=1}^{q_{ij}} (-1)^k C(q_{ij}, k) x^{q_{ij}-k} t^k \left(f_{j0}(t) + pf_{j1}(t) + p^2 f_{j2}(t) + \ldots\right)dt] = 0$$

$$(6.2.25)$$

where $C(q_{ij}, k)$ states the Binomial coefficients in the Binomial series expansion.

Eqn. (6.2.25) again can be rewritten in terms of the powers of p as the following form:

$$p^0(f_{i0} - g_i) + p^1\left[f_{i1} - \sum_{j=0}^{n} \sum_{k=1}^{q_{ij}} (-1)^k C(q_{ij}, k) x^{q_{ij}-k} \int_0^1 t^k f_{j0}(t)dt\right]$$

$$+ p^2\left[f_{i2} - \sum_{j=0}^{n} \sum_{k=1}^{q_{ij}} (-1)^k C_{ij}(q_{ij}, k) x^{q_{ij}-k} \int_0^1 t^k f_{j1}(t)dt\right] \qquad (6.2.26)$$

$$+ p^3\left[f_{i3} - \sum_{j=0}^{n} \sum_{k=1}^{q_{ij}} (-1)^k C_{ij}(q_{ij}, k) x^{q_{ij}-k} \int_0^1 t^k f_{j2}(t)dt\right] + \ldots = 0$$

Equating the coefficients of like powers of p yields

$$p^0: \quad f_{i0} - q_i = 0 \quad f_{i0} = q_i \tag{6.2.27a}$$

$$p^1: \quad f_{i1} = \sum_{j=0}^{n}\sum_{k=1}^{q_{iij}} (-1)^k C(q_{ij}, k) x^{q_{ij}-k} \int_0^1 t^k f_{j0}(t)dt \tag{6.2.27b}$$

$$p^2: \quad f_{i2} = \sum_{j=0}^{n}\sum_{k=1}^{q_{iij}} (-1)^k C(q_{ij}, k) x^{q_{ij}-k} \int_0^1 t^k f_{j1}(t)dt \tag{6.2.28a}$$

$$p^3: \quad f_{i3} = \sum_{j=0}^{n}\sum_{k=1}^{q_{iij}} (-1)^k C(q_{ij}, k) x^{q_{ij}-k} \int_0^1 t^k f_{j2}(t)dt \tag{6.2.28b}$$

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and in general

$$f_{i0} = q_i$$

$$f_{im} = \sum_{j=0}^{n}\sum_{k=1}^{q_{iij}} (-1)^k C(q_{ij}, k) x^{q_{ij}-k} \int_0^1 t^k f_{j,m-1}(t)dt \qquad m = 1,2,3,... \tag{6.2.29}$$

CHAPTER SEVEN

PROBLEMS ON NONLINEAR INTEGRAL EQUATIONS AND ON NONLINEAR SYSTEMS OF INTEGRAL EQUATIONS

In this chapter some problems of nonlinear integral equations, systems of nonlinear integral equations, and an integro-differential equation are provided to illustrate the ability of the homotopy perturbation method.

Problem 7.1

Consider the following nonlinear Fredholm integral equation of the second type (Wazwaz, 1997)

$$u(x) = \sin x + \cos x - \frac{\pi + 2}{8} + \frac{1}{4}\int_0^{\frac{\pi}{2}} u^2(t)dt \qquad (7.1.1)$$

$$u_0(x) = \sin x + \cos x \qquad (7.1.2)$$

He's homotopy perturbation method consists of the following scheme:

$$H(v, p) = (1 - p)[L(v) - L(v_0)] + p[A(v) - f(\underline{r})] = 0$$

That is

$$H(v, p) = (1 - p)[v(x) - \sin x - \cos x] + p\left[v(x) - \sin x - \cos x + \frac{\pi + 2}{8} - \frac{1}{4}\int_0^{\frac{\pi}{2}} v^2(t)dt\right] = 0$$

This gives

$$v(x) = \sin x + \cos x - \frac{\pi + 2}{8}p + \frac{1}{4}p\int_0^{\frac{\pi}{2}} v^2(t)dt \qquad (7.1.3)$$

We can try to obtain a solution of equation (7.1.3) in the form:

$$v(x) = v_0(x) + pv_1(x) + p^2 v_2(x) + \ldots \qquad (7.1.4)$$

where $v_i(x)$, $i = 0,1,2,\ldots$ are functions yet to be determined. According to equation (7.1.4) the initial approximation is

$$v_0(x) = \sin x + \cos x \qquad (7.1.5)$$

Substituting equations (7.1.4) and (7.1.5) into Eqn. (7.1.3) and equating the terms with the coefficients of the identical powers of p, we will have

$$v_0(x)+pv_1(x)+p^2v_2(x)+\ldots=\sin x+\cos x-\frac{\pi+2}{8}p+\frac{1}{4}p\int_0^{\frac{\pi}{2}}\left(v_0(t)+pv_1(t)+p^2v_2(t)+\ldots\right)^2dt$$

$$p^0:\quad v_0(x)=\sin x+\cos x$$

$$p^1:\quad v_1(x)=-\left(\frac{\pi+2}{8}\right)+\frac{1}{4}\int_0^{\frac{\pi}{2}}v_0^2(t)dt$$

$$v_1(x)=-\left(\frac{\pi+2}{8}\right)+\frac{1}{4}\int_0^{\frac{\pi}{2}}(\sin t+\cos t)^2dt$$

$$v_1(x)=0$$

$$p^2:\ v_2(x)=\frac{1}{4}\int_0^{\frac{\pi}{2}}2v_0(t)v_1(t)dt$$

$$v_2(x)=0$$

In the same manner, the rest of components will be obtained by using the Mathematica 7 package.

According to the HPM we can conclude

$$u(x)=Lim_{p\to 1}v(x)=v_0(x)+v_1(x)+v_2(x)+\ldots$$

And therefore the exact solution

$$u(x)=v(x)=\sin x+\cos x$$

is readily obtained. Here the only two iterations are cased to obtain this exact solution. The plot of the solution is given in Figure 7.1.

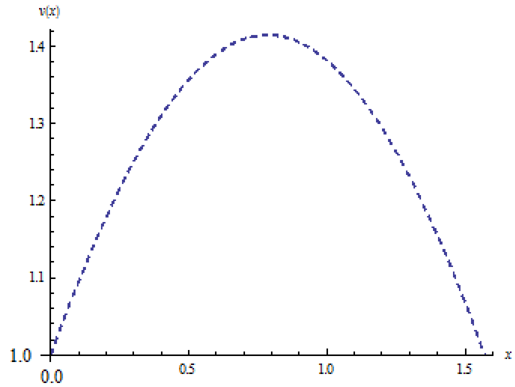

Figure 7.1 The numerical results and exact solutions of Problem 7.1

……v(homotopy)

—— v(exact)

Problem 7.2

Consider the following nonlinear Volterra integral equation of the second type (Wazwaz, 1997):

$$u(x) = e^x - \frac{1}{3}xe^{3x} + \frac{1}{3}x + \int_0^x xu^3(t)dt \qquad (7.2.1)$$

with the exact solution $u(x) = e^x$.

A homotopy can be readily constructed as follows:

$$H(v,p) = (1-p)[L(v) - L(v_0)] + p[A(v) - f(\underline{r})] = 0$$

or

$$H(v,p) = (1-p)[v(x) - e^x] + p\left[v(x) - e^x + \frac{1}{3}xe^{3x} - \frac{1}{3}x - \int_0^x xv^3(t)dt\right] = 0 \qquad (7.2.2)$$

or

$$v(x) - e^x - pv(x) + pe^x + pv(x) - pe^x + p\frac{1}{3}xe^{3x} - \frac{1}{3}px - p\int_0^x xv^3(t)dt = 0$$

This gives

$$v(x) = e^x + \frac{1}{3}px(1 - e^{3x}) + p\int_0^x xv^3(t)dt \qquad (7.2.3)$$

Let

$$v(x) = v_0(x) + pv_1(x) + p^2v_2(x) + \ldots \qquad (7.2.4)$$

be a solution of Eqn. (7.2.3). Here $v_i(x),\ i = 0,1,2,\ldots$ are functions to be determined. According to equation (7.2.4) the initial approximation is

$$v_0(x) = e^x \qquad (7.2.5)$$

Substituting equations (7.2.4) and (7.2.5) into Eqn. (7.2.3) and equating the terms with identical powers of p, we find Eqn. (7.2.6):

$$v_0(x) + pv_1(x) + p^2v_2(x) + \ldots = e^x + \frac{1}{3}px(1 - e^{3x}) + p\int_0^x x(v_0(t) + pv_1(t) + p^2v_2(t) + \ldots)^3 dt$$

$$\begin{aligned} v_0(x) + pv_1(x) + p^2v_2(x) + \ldots &= e^x + \frac{1}{3}px(1 - e^{3x}) \\ &+ p\int_0^x x(v_0^3(t) + 3pv_0^2(t)v_1(t) + 3p^2v_0(t)v_1^2(t) + p^3v_1^3(t) + \ldots)dt \end{aligned} \qquad (7.2.6)$$

$p^0: \quad v_0(x) = e^x$

$$p^1: \quad v_1(x) = \frac{1}{3}x(1 - e^{3x}) + \int_0^x xv_0^3(t)dt$$

$$v_1(x) = \frac{1}{3}x(1 - e^{3x}) + \int_0^x xe^{3t}dt$$

$$v_1(x) = \frac{1}{3}x(1 - e^{3x}) + \frac{1}{3}x(e^{3x} - 1) \ \Rightarrow\ v_1(x) = 0$$

$$p^2: \quad v_2(x) = 3\int_0^x x(v_0^2(t)v_1(t))dt$$

$$v_2(x) = 3\int_0^x x(e^{2t}.0)dt \quad \Rightarrow \quad v_2(x) = 0$$

In general

$$v_{k+2}(x) = \int_0^x xv_{k+1}^3(t)dt\ ,\ k \geq 0$$

According to the HPM, when $p \to 1$, Eqn. (7.2.4) corresponds to Eqn. (7.2.2) and becomes the approximate solution of Eqn. (7.2.1), that is,

$$u(x) = Lim_{p \to 1} v(x) = v_0(x) + v_1(x) + v_2(x) + ...$$

and

$$u(x) = v(x) = e^x$$

Problem 7.3

Consider the following nonlinear Volterra integral equation (Wazwaz, 1997)

$$u(x) = \cos x - \frac{1}{2}x - \frac{1}{4}\sin 2x + \int_0^x u^2(t)dt \qquad (7.3.1)$$

$$u_0(x) = \cos x \qquad (7.3.2)$$

By homotopy perturbation method we may choose a convex homotopy such that

$$H(v,p) = (1-p)[L(v) - L(v_0)] + p[A(v) - f(\underline{r})] = 0$$

That is

$$H(v,p) = (1-p)(v(x) - \cos x) + p\left(v(x) - \cos x + \frac{1}{2}x + \frac{1}{4}\sin 2x + \int_0^x v^2(t)\right)dt = 0$$

or

$$v(x) - \cos x - pv(x) + p\cos x + pv(x) - p\cos x + p\frac{1}{2}x + p\frac{1}{4}\sin 2x + p\int_0^x v^2(t)dt = 0$$

or

$$v(x) = \cos x - \frac{1}{2}px - \frac{1}{4}p\sin 2x - p\int_0^x v^2(t)dt \qquad (7.3.3)$$

Let

$$v(x) = v_0(x) + pv_1(x) + p^2 v_2(x) + ... \qquad (7.3.4)$$

be a solution of Eqn.(7.3.3). Here $v_i(x)$, $i = 0,1,2,...$ are functions to be determined. According to equation (7.3.4) the initial approximation is

$$v_0(x) = \cos x \qquad (7.3.5)$$

Substituting equations (7.3.4) and (7.3.5) into Eqn. (7.3.3) and equating the terms with identical powers of p, we have

$$v_0(x)+pv_1(x)+p^2v_2(x)+\ldots=\cos x-\frac{1}{2}px-\frac{1}{4}p\sin 2x-p\int_0^x\left(v_0(t)+pv_1(t)+p^2v_2(t)+\ldots\right)^2dt$$

that is

$$p^0:\quad v_0(x)=\cos x$$

$$p^1:\quad v_1(x)=-\frac{1}{2}x-\frac{1}{4}\sin 2x-\int_0^x v_0^2(t)dt$$

$$v_1(x)=-\frac{1}{2}x-\frac{1}{4}\sin 2x-\int_0^x \cos^2 t dt \quad\Rightarrow\quad v_1(x)=0$$

$$p^2:\quad v_2(x)=-\int_0^x 2v_0(t)v_1(t)dt \quad\Rightarrow\quad v_2(x)=0$$

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or in general

$$v_k(x)=-\int_0^x 2v_k(t)v_{k+1}(t)dt \qquad\Rightarrow\qquad v_k(x)=0\,,\ k\geq 0$$

Therefore, the approximate solution of example can be readily obtained by

$$u(x)=Lim_{p\to 1}v(x)=v_0(x)+v_1(x)+v_2(x)+\ldots$$

or

$$u(x)=\sum_{k=0}^{\infty}u_k(x)$$

Therefore

$$u(x)=v(x)=\cos x$$

Here we used two iterations only to obtain the exact solution $\cos x$. The plot of the solution is given in Figure 7.2 for $0\leq x\leq 1$.

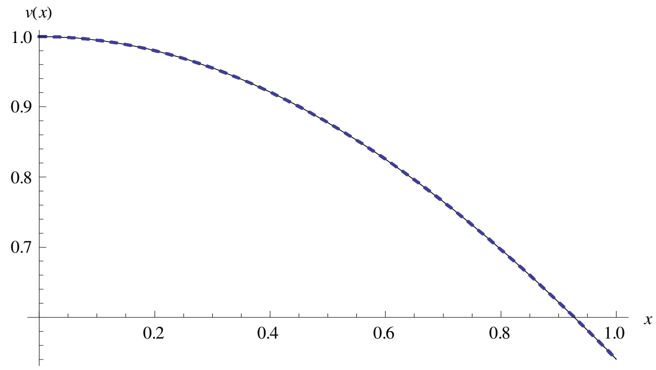

Figure 7.2 The numerical results and exact solutions of Problem 7.3

……v(homotopy)

_____ v(exact)

Problem 7.4

Let us solve the following nonlinear Volterra integral equation of the second type:

$$u(x) = e^x + \frac{1}{2}x\left(e^{2x} - 1\right) - \int_0^x xu^2(t)dt \qquad (7.4.1)$$

$u_0(x) = e^x$.

He's homotopy perturbation method states that

$$H(u,p) = (1-p)[L(u) - L(u_0)] + p\left[u(x) - e^x - \frac{1}{2}x\left(e^{2x} - 1\right) + \int_0^x xu^2(t)dt\right] = 0$$

(7.4.2)

that is

$$H(u,p) = (1-p)\left[u(x) - e^x\right] + p\left[u(x) - e^x - \frac{1}{2}x\left(e^{2x} - 1\right) + \int_0^x xu^2(t)dt\right] = 0$$

(7.4.3)

This gives

$$u(x) - e^x - pu(x) + pe^x + pu(x) - pe^x - p\frac{1}{2}x\left(e^{2x} - 1\right) + p\int_0^x xu^2(t)dt = 0$$

This simplifies as

$$u(x)=e^{x}+\frac{1}{2}px\left(e^{2x}-1\right)-p\int_{0}^{x}xu^{2}(t)dt \qquad (7.4.4)$$

Suppose the solution of Eqn. (7.4.4) has the following form:

$$v(x)=v_0(x)+pv_1(x)+p^2v_2(x)+\dots \qquad (7.4.5)$$

where $v_i(x),\ i=0,1,2,\dots$ are functions yet to be determined. According to equation (7.4.5) the initial approximation is

$$v_0(x)=e^{x}$$

Substituting equations (7.4.5) and (7.4.6) into Eqn. (7.4.4) and equating the terms with identical coefficients of the identical powers of p, we have

$$v_0(x)+pv_1(x)+p^2v_2(x)+\dots=e^{x}+\frac{1}{2}px\left(e^{2x}-1\right)-p\int_{0}^{x}x\left(v_0(t)+pv_1(t)+p^2v_2(t)+\dots\right)^2dt$$

$$v_0(x)+pv_1(x)+p^2v_2(x)+\dots=e^{x}+\frac{1}{2}px\left(e^{2x}-1\right)$$

$$-p\int_{0}^{x}x(v_0^2(t)+2pv_0(t)v_1(t)+p^2v_1^2(t)+\dots)dt \qquad (7.4.6)$$

this gives

$$p^0:\quad v_0(x)=e^{x}$$

$$p^1:\quad v_1(x)=\frac{1}{2}x\left(e^{2x}-1\right)-\int_{0}^{x}xv_0^2(t)dt$$

$$v_1(x)=\frac{1}{2}x\left(e^{2x}-1\right)-\int_{0}^{x}xe^{2t}dt$$

$$v_1(x)=\frac{1}{2}x\left(e^{2x}-1\right)-\frac{1}{2}x\left(e^{2x}-1\right)\ \Rightarrow\ v_1(x)=0$$

$$p^2:\quad v_2(x)=-2\int_{0}^{x}xv_0(t)v_1(t)dt \qquad \Rightarrow \quad v_2(x)=0$$

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In general

$$v_{k+2}(x)=\int_{0}^{x}xv_{k+1}^2(t)dt\ ,\ k\geq 0$$

And according to HPM, we obtain the exact solution as

$u(x) = Lim_{p\to 1} v(x) = v_0(x) + v_1(x) + v_2(x) + \ldots$

And

$u(x) = v(x) = e^x$

with two iterations only.

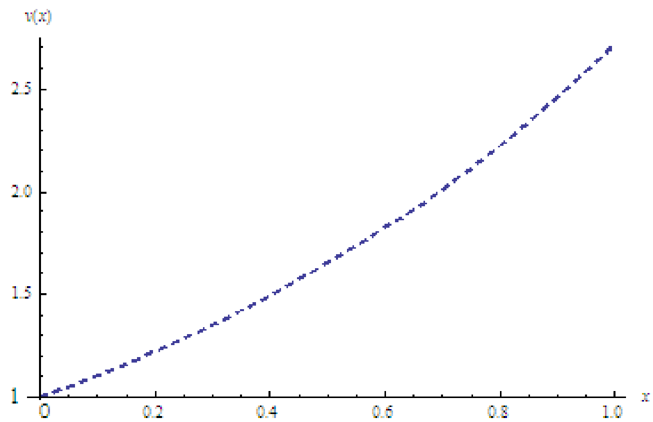

Figure 7.3 The numerical results and exact solutions of Problem 7.4

……v(homotopy)

______ v(exact)

Problem 7.5

Consider the following nonlinear integro-differential equation (Ghasemi, Tavassoli, and Babolian, 2007):

$$u'(x) = -1 + \int_0^x u^2(t)dt \qquad (7.5.1)$$

for $x \in [0,1]$ with the boundary solution

$$u(0) = 0\,. \qquad (7.5.2)$$

We apply the homotopy perturbation method to solve Eqn. (7.5.1).

Let

$L(u) = u'(x) - g(x) = 0\,,\ f(\underline{r}) = 0$

By homotopy perturbation method we may choose a convex homotopy such that

$$H(v,p)=L(v)-L(v_0)+pL(v_0)+p\left[N(v)-f(\underline{r})\right]=0$$

that is

$$H(v,p)=v'(x)-g(x)-p\int_0^x \kappa(x,t,v(t),v'(t))dt=0 \qquad (7.5.3)$$

and continuously trace an implicitly defined curve from a starting point $H(v_0,0)$ to a solution function $H(v,1)$:

$$H(v,0)=v'(x)-g(x)=0 \qquad \Rightarrow \qquad v'(x)=g(x) \qquad (7.5.4)$$

$$H(v,1)=v'(x)-g(x)-\int_0^x \kappa(x,t,v(t),v'(t))dt=0 \qquad \Rightarrow$$

$$H(v,1)=v'(x)-g(x)-\int_0^x v^2(t)dt=0 \qquad (7.5.5)$$

Eqn. (7.5.3) takes the form

$$H(v,p)=v'(x)+1-p\int_0^x v^2(t)dt=0 \qquad (7.5.6)$$

Let us try to obtain a solution of Eqn. (7.5.1) in the form

$$v(x)=v_0(x)+pv_1(x)+p^2v_2(x)+\ldots \qquad (7.5.7)$$

where $v_i(x)$, $i=0,1,2,\ldots$ are functions yet to be determined. According to equation (7.5.7) the initial approximation is

$$v_0(x)=g(x)=-1 \qquad (7.5.8)$$

Substituting equations (7.5.6) and (7.5.7) into Eqn. (7.5.6) gives

$$v_0'(x)+pv_1'(x)+p^2v_2'(x)+\ldots=-1+p\int_0^x\left(v_0(t)+pv_1(t)+p^2v_2(t)+\ldots\right)^2dt$$

or

$$v_0'(x)+pv_1'(x)+p^2v_2'(x)+\ldots=-1+p\int_0^x\left(v_0^2(t)+2pv_0(t)v_1(t)+p^2v_1^2(t)+2p^2v_0(t)v_2(t)+\ldots\right)dt$$

(7.5.9)

Equating the terms with identical powers of p in Eqn. (7.5.9), we have

$$p^0: \quad v_0'(x)=g(x)=-1 \qquad \Rightarrow$$

$$v_0(x)=-x \qquad (7.5.10a)$$

$$p^1: \quad v_1'(x) = \int_0^x v_0^2(t)dt = \int_0^x t^2 dt$$

$$v_1'(x) = \frac{x^3}{3} \quad \Rightarrow \quad v_1(x) = \frac{1}{12}x^4$$

$$p^2: \quad v_2'(x) = \int_0^x 2v_0(t)v_1(t)dt = \int_0^x 2(-t)\left(\frac{1}{12}t^4\right)dt$$

$$v_2'(x) = -\frac{1}{36}x^6 \quad \Rightarrow \quad v_2(x) = -\frac{1}{252}x^7$$

$$p^3: \quad v_3'(x) = \int_0^x \left(2v_0(t)v_2(t) + v_1^2(t)\right)dt$$

$$v_3'(x) = \int_0^x \left[2(-t)\left(-\frac{1}{252}t^7\right) + \left(\frac{1}{12}t^4\right)^2\right]dt$$

$$v_3'(x) = \int_0^x \left(-\frac{1}{126}t^8 + \frac{1}{12}t^8\right)dt = -\frac{1}{126.9}x^9 + \frac{1}{12.9}x^9$$

$$v_3(x) = -\frac{1}{6048}x^{10}$$

$$p^4: \quad v_4'(x) = \int_0^x 2\left(v_0(t)v_3(t) + v_1(t)v_2(t)\right)dt$$

$$v_4'(x) = \int_0^x 2\left[(-t)\left(\frac{1}{6048}t^{10}\right) + \left(\frac{1}{12}t^4\right)\left(-\frac{1}{252}t^7\right)\right]dt$$

$$v_4'(x) = \int_0^x \left(-\frac{1}{3024}t^{11} - \frac{1}{252.6}t^{11}\right)dt$$

$$v_4(x) = -\frac{1}{157248}x^{13}$$

Therefore, the approximate solution of this problem is

$$u(x) = \sum_{n=0}^{\infty} v_n(x) = v_0(x) + v_1(x) + v_2(x) + v_3(x) + v_4(x) + \ldots$$

$$u(x) = -x + \frac{1}{12}x^4 - \frac{1}{252}x^7 + \frac{1}{6048}x^{10} - \frac{1}{157248}x^{13} + \ldots$$

Table 7.1 contains the numerical comparison between our solution using HPM and the exact solution of the problem at some points.

Table 7.1 Numerical results of problem 7.5

x	Exact solution	Homotopy perturbation method
0	0	0
0.0625	-0.0625	-0.0625
0.125	-0.12498	-0.12498
0.250	-0.24968	-0.24968
0.375	-0.37336	-0.37336
0.500	-0.49482	-0.49482
0.625	-0.61243	-0.61243
0.750	-0.72415	-0.72415
0.875	-0.82767	-0.82767
1	-0.92048	-0.92048

The table shows that HPM minimize the computational calculus and supplies quantitatively reliable results. (Appendix 1)

Problem 7.6

Consider the following system of nonlinear Fredholm integral equations of the second type (Babolian, Biazar and Vahidi, 2004)

$$f_1(x) = x - \frac{5}{18} + \frac{1}{3}\int_0^1 \left(f_1(y) + f_2(y)\right)dy \qquad (7.6.1a)$$

$$f_2(x) = x^2 - \frac{2}{9} + \frac{1}{3}\int_0^1 \left(\left(f_1(y)\right)^2 + f_2(y)\right)dy \qquad (7.6.1b)$$

with the exact solutions $f_1(x) = x$, $f_2(x) = x^2$.

By homotopy perturbation method a convex homotopy such that $H(f_1, f_2, p)$ with the components $H_1(f_1, f_2, p)$ and $H_2(f_1, f_2, p)$:

$$H_1(f_1, f_2, p) = f_1(x) - g_1(x) - \frac{1}{3} p \int_0^1 (f_1(y) + f_2(y)) dy = 0 \tag{7.6.2a}$$

$$H_2(f_1, f_2, p) = f_2(x) - g_2(x) - \frac{1}{3} p \int_0^1 ((f_1(y))^2 + f_2(y)) dy = 0 \tag{7.6.2b}$$

Here,

$H(u,0) = F(u)$ and $H(u,1) = L(u)$

corresponds to the following expressions respectively:

$H(u,0) = F(u) = f_1(x) - g_1(x)$ and $H(u,0) = f_2(x) - g_2(x)$

$$H(u,1) = L(u) = f_1(x) - g_1(x) - \frac{1}{3} \int_0^1 (f_1(s) + f_2(s)) ds = 0$$

$$H(u,1) = L(u) = f_2(x) - g_2(x) - \frac{1}{3} \int_0^1 ((f_1(s))^2 + f_2(s)) ds = 0$$

For each equation we construct a homotopy $\Omega \times [0,1] \to IR$ with the following properties:

$$H_1(f_1, f_2, p) = (1-p)(f_1(x) - g_1(x)) + p\left[f_1(x) - g_1(x) - \frac{1}{3} \int_0^1 (f_1(s) + f_2(s)) ds \right] = 0 \tag{7.6.3a}$$

$$H_2(f_1, f_2, p) = (1-p)(f_2(x) - g_2(x)) + p\left[f_2(x) - g_2(x) - \frac{1}{3} p \int_0^1 ((f_1(s))^2 + f_2(s)) ds \right] = 0 \tag{7.6.3b}$$

or equivalently

$H_1(f_1, f_2, p) = 0$, $H_2(f_1, f_2, p) = 0$ gives, respectively,

$$f_1(x) = g_1(x) + \frac{1}{3} p \int_0^1 (f_1(s) + f_2(s)) ds = 0 \tag{7.6.4a}$$

$$f_2(x) = g_2(x) + \frac{1}{3} p \int_0^1 ((f_1(s))^2 + f_2(s)) ds = 0 \tag{7.6.4b}$$

Consider the ith equation of the system, take

$$f_1(x) = \sum_{i=0}^{\infty} p^i f_{1i} = f_{10} + pf_{11} + p^2 f_{12} + \ldots \tag{7.6.5a}$$

$$f_2(x) = \sum_{i=0}^{\infty} p^i f_{2i} = f_{20} + pf_{21} + p^2 f_{22} + \ldots \tag{7.6.5b}$$

Substituting (7.6.5) into (7.6.4) gives

$$f_{10} + pf_{11} + p^2 f_{12} + \ldots = g_1(x) + \frac{1}{3} p \int_0^1 [(f_{10}(s) + pf_{11}(s) + p^2 f_{12}(s) + \ldots) + (f_{20}(s) + pf_{21}(s) + p^2 f_{22}(s) + \ldots)]ds$$

(7.6.6a)

$$f_{20} + pf_{21} + p^2 f_{22} + \ldots = g_2(x) + \frac{1}{3} p \int_0^1 [(f_{10}(s) + pf_{11}(s) + p^2 f_{12}(s) + \ldots)^2 + (f_{20}(s) + pf_{21}(s) + p^2 f_{22}(s) + \ldots)]ds$$

(7.6.6b)

and equating the terms with identical powers of p, we have

$$p^0: \quad f_{10}(x) = g_1(x) \quad \Rightarrow \quad f_{10}(x) = x - \frac{5}{18} \cong x - 0.2778$$

(7.6.7a)

$$f_{20}(x) = g_2(x) \quad \Rightarrow \quad f_{20}(x) = x^2 - \frac{2}{9} \cong x^2 - 0.2222$$

(7.6.7b)

$$p^1: \quad f_{11}(x) = \frac{1}{3}\int_0^1 (f_{10}(s) + f_{20}(s))ds$$

$$f_{11}(x) = \frac{1}{3}\int_0^1 \left(s - \frac{5}{18} + s^2 - \frac{2}{9} \right) ds = \frac{1}{3}\int_0^1 \left(s^2 + s - \frac{1}{2} \right) ds$$

$$f_{11}(x) = \frac{1}{9} \cong 0.1111 \qquad (7.6.8a)$$

$$f_{21}(x) = \frac{1}{3}\int_0^1 ((f_{10}(s))^2 + f_{20}(s))ds$$

$$f_{21}(x) = \frac{1}{3}\int_0^1 \left[\left(s - \frac{5}{18} \right)^2 + s^2 - \frac{2}{9} \right] ds$$

$$f_{21}(x) = \frac{79}{972} \cong 0.0813 \qquad (7.6.8b)$$

$$p^k: \quad f_{1k}(x) = \frac{1}{3}\int_0^1 (f_{1k-1}(s) + f_{2k-1}(s))ds \qquad (7.6.9a)$$

$$f_{2k}(x) = \frac{1}{3}\int_0^1 \left((f_{1k-1}(s))^2 + f_{2k-1}(s)\right)ds \qquad (7.6.9b)$$

Therefore, the approximate solution can be readily obtained by

$$f_1(x) = \sum_{k=0}^{\infty} f_{1k}(x), \qquad f_2(x) = \sum_{k=0}^{\infty} f_{2k}(x) \qquad (7.6.10)$$

For the first iteration, we have:

$$f_{11}(x) = \frac{1}{9} \cong 0.1111$$

$$f_{21}(x) = \frac{79}{972} \cong 0.0813$$

In practice, all terms of series (7.6.10) can not be determined and so we can use an approximation of the solution by the following truncated series

$$\varphi_{1,m}(x) = \sum_{k=0}^{m-1} f_{1k}(x), \qquad \varphi_{2,m}(x) = \sum_{k=0}^{m-1} f_{2k}(x) \qquad (7.6.11)$$

Using the truncated series (7.6.11), the solutions with two terms are

$$\varphi_{1,2}(x) = \sum_{k=0}^{1} f_{1k}(x) = f_{10}(x) + f_{11}(x) = x - \frac{5}{18} + \frac{1}{9} = x - \frac{3}{18} \cong x - 0.1667$$

$$\varphi_{2,2}(x) = \sum_{k=0}^{1} f_{2k}(x) = f_{20}(x) + f_{21}(x) = x^2 - \frac{2}{9} + \frac{79}{792} \cong x^2 - 0.1409$$

For the second iteration we have

$$f_{12}(x) = \frac{187}{2916} \cong 0.0641$$

$$f_{22}(x) = \frac{91}{2916} \cong 0.0312$$

Considering (7.6.11), the solutions with three terms are

$$\varphi_{1,3}(x) = f_{10}(x) + f_{11}(x) + f_{12}(x) = x - \frac{5}{18} + \frac{1}{9} + \frac{187}{2916} \cong x - 0.1025$$

$$\varphi_{2,3}(x) = f_{20}(x) + f_{21}(x) + f_{22}(x) = x^2 - \frac{2}{9} + \frac{79}{792} + \frac{91}{2916} \cong x^2 - 0.0974$$

and so on the rest of components of the iteration formula (7.6.9) can be obtained in a similar way.

The solutions with ten terms are given as

$$\varphi_{1,10}(x)=\sum_{k=0}^{9} f_{1k}(x)=f_{10}(x)+f_{11}(x)+f_{12}(x)+f_{13}(x)+\ldots+f_{19}(x)$$

$$\varphi_{2,10}(x)=\sum_{k=0}^{9} f_{2k}(x)=f_{20}(x)+f_{21}(x)+f_{22}(x)+f_{23}(x)+\ldots+f_{29}(x)$$

That is

$$\varphi_{1,10}(x)\cong x-0.0458$$

$$\varphi_{2,10}(x)\cong x^{2}-0.0915$$

The values of the ten terms approximations to the solutions at some points with the corresponding absolute errors for HPM at various values of x in Table 7.2.

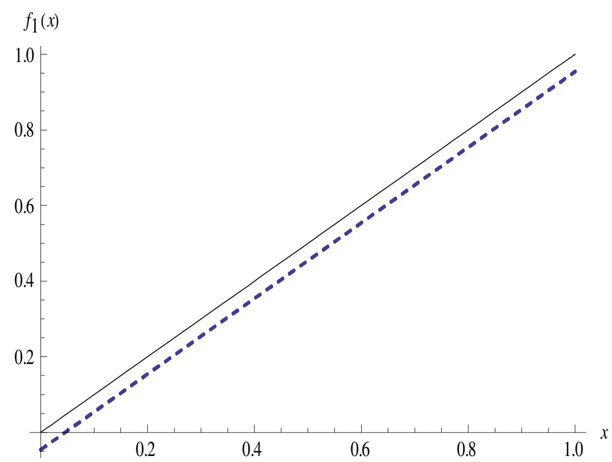

Figure 7.4 The numerical results and exact solutions of problem 7.6

—— $f_1(exact)$

…. f_1(homotopy)

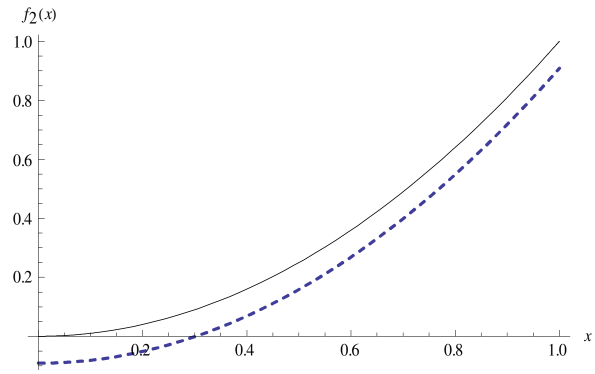

Figure 7.5 The numerical results and exact solutions of problem 7.6

—— $f_2(exact)$

.... f_2 (homotopy)

As the results in Table 7.2 show the more terms in approximations would cause the more accuracy in solutions.

More iteration will reduce the error. Obviously, the maximum absolute error for $x \in [0,1]$ is 0.954 for $f_1(x)$ and 0.908 for $f_2(x)$. (Appendix 2)

Table 7.2 The values of ten terms approximations with the related errors

x	$f_1(x_i)$	$f_{1HPM}(x_i)$	$e_1(x_i)$	$f_2(x_i)$	$f_{2HPM}(x_i)$	$e_2(x_i)$
0	0	-0.046	0.046	0	-0.09	0.09
0.1	0.1	0.054	0.046	0.01	-0.08	0.09
0.2	0.2	0.154	0.046	0.04	-0.05	0.09
0.3	0.3	0.254	0.046	0.09	-0.001	0.091
0.4	0.4	0.354	0.046	0.14	0.07	0.09
0.5	0.5	0.454	0.046	0.25	0.16	0.09
0.6	0.6	0.554	0.046	0.36	0.27	0.09
0.7	0.7	0.654	0.046	0.49	0.40	0.09
0.8	0.8	0.754	0.046	0.64	0.55	0.09
0.9	0.9	0.854	0.046	0.81	0.72	0.09
1	1	0.954	0.046	1	0.91	0.09

Problem 7.7

Consider the following system of Volterra integral equations of the first kind with the exact solution $f_1(x) = x^2$ and $f_2(x) = x$ (Biazar, Babolian, and Islam, 2003).

$$\int_0^x \left(1 - x^2 + y^2\right)\left(f_1(y) + f_2^3(y)\right)dy = -\frac{1}{12}x^6 - \frac{2}{15}x^5 + \frac{1}{4}x^4 + \frac{1}{3}x^3 \qquad (7.7.1a)$$

$$\int_0^x (5 + x - y)\left(f_1^3(y) - f_2(y)\right)dy = -\frac{5}{2}x^2 - \frac{1}{6}x^3 + \frac{5}{7}x^7 + \frac{1}{56}x^8 \qquad (7.7.1b)$$

We use the homotopy perturbation method to solve system (7.7.1). First we put system (7.7.1) into the canonical form. To derive this form, differentiate of the both sides of Eqn. (7.7.1), with respect to x, and according to the Leibnitz generalized rule as in Section 6.2. This process changes the system of integral equations (7.7.1) to the second kind:

$$f_1(x) + f_2^3(x) - 2x\int_0^x \left(f_1(y) + f_2^3(y)\right)dy = -\frac{1}{2}x^5 - \frac{2}{3}x^4 + x^3 + x^2 \qquad (7.7.2a)$$

$$f_1^3(x) - f_2(x) + \frac{1}{5}\int_0^x \left(f_1^3(y) - f_2(y)\right)dy = -x - \frac{1}{10}x^2 + x^6 + \frac{1}{35}x^7 \qquad (7.7.2b)$$

Let's choose

$$g_1(x) = -\frac{1}{2}x^5 - \frac{2}{3}x^4 + x^3 + x^2 \qquad g_2(x) = -x - \frac{1}{10}x^2 + x^6 + \frac{1}{35}x^7$$

$$h_1(x) = f_1(x) + f_2^3(x) \qquad h_2(x) = f_1^3(x) - f_2(x)$$

$$h_1(x) - 2x\int_0^x h_1(y)dy = g_1(x) \qquad h_2(x) + \frac{1}{5}\int_0^x h_2(y)dy = g_2(x)$$

$$h_1(x) = g_1(x) + 2x\int_0^x h_1(y)dy \qquad h_2(x) = g_2(x) - \frac{1}{5}\int_0^x h_2(y)dy$$

Using homotopy perturbation method;

$$H_1(h_1, h_2, p) = (1-p)(h_1(x) - g_1(x)) + p\left[h_1(x) - g_1(x) - 2x\int_0^x h_1(y)dy\right] = 0$$

$$h_1(x) - g_1(x) - ph_1(x) + pg_1(x) + ph_1(x) - pg_1(x) - 2xp\int_0^x h_1(y)dy = 0$$

$$h_1(x) = g_1(x) + 2xp\int_0^x h_1(y)dy \qquad (7.7.3a)$$

$$H_2(h_1, h_2, p) = (1-p)(h_2(x) - g_2(x)) + p\left[h_2(x) - g_2(x) + \frac{1}{5}\int_0^x h_2(y)dy\right] = 0$$

$$h_2(x) - g_2(x) - ph_2(x) + pg_2(x) + ph_2(x) - pg_2(x) + \frac{1}{5}p\int_0^x h_2(y)dy = 0$$

$$h_2(x) = g_2(x) - \frac{1}{5}p\int_0^x h_2(y)dy \qquad (7.7.3b)$$

$$h_i = h_{i0} + ph_{i1} + p^2 h_{i2} + \ldots$$

$$h_1 = h_{10} + ph_{11} + p^2 h_{12} + \ldots \qquad h_2 = h_{20} + ph_{21} + p^2 h_{22} + \ldots$$

$$h_{10}(x) + ph_{11}(x) + p^2 h_{12}(x) + \ldots = g_1(x) + 2xp\int_0^x \left[h_{10}(y) + ph_{11}(y) + p^2 h_{12}(y) + \ldots\right]dy$$

(7.7.4a)

$$h_{20}(x)+ph_{21}(x)+p^2h_{22}(x)+\ldots=g_2(x)-\frac{1}{5}p\int_0^x\left[h_{20}(y)+ph_{21}(y)+p^2h_{22}(y)+\ldots\right]dy \tag{7.7.4b}$$

p^0 :

$$h_{10}(x)=g_1(x) \qquad \Rightarrow \qquad h_{10}(x)=-\frac{1}{2}x^5-\frac{2}{3}x^4+x^3+x^2$$

$$h_{20}(x)=g_2(x) \qquad \Rightarrow \qquad h_{20}(x)=-x-\frac{1}{10}x^2+x^6+\frac{1}{35}x^7$$

p^1 :

$$h_{11}(x)=2x\int_0^x h_{10}(y)dy$$

$$h_{11}(x)=2x\int_0^x\left(-\frac{1}{2}y^5-\frac{2}{3}y^4+y^3+y^2\right)dy$$

$$h_{11}(x)=-\frac{1}{6}x^7-\frac{4}{15}x^6+\frac{1}{2}x^5+\frac{2}{3}x^4$$

$$h_{21}(x)=-\frac{1}{5}\int_0^x h_{20}(y)dy$$

$$h_{21}(x)=-\frac{1}{5}\int_0^x\left[-y-\frac{1}{10}y^2+y^6+\frac{1}{35}y^7\right]dy$$

$$=-\frac{1}{5}\left[-\frac{y^2}{2}-\frac{1}{10}\frac{y^3}{3}+\frac{y^7}{7}+\frac{1}{35}\frac{y^8}{8}\right]_0^x$$

$$h_{21}(x)=\frac{x^2}{10}+\frac{x^3}{150}-\frac{x^7}{35}-\frac{x^8}{1400}$$

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and in general

$$h_{1(n+1)}(x)=2x\int_0^x h_{1n}(y)dy \qquad n=0,1,2,\ldots$$

$$h_{2(n+1)}(x)=-\frac{1}{5}\int_0^x h_{2n}(y)dy$$

$$\varphi_{1,m}(x)=\sum_{n=0}^{m-1}h_{1n}(x) \qquad \varphi_{2,m}(x)=\sum_{n=0}^{m-1}h_{2n}(x)$$

Considering the solutions with two terms, we obtain

$$\varphi_{1,2}(x) = h_{10}(x) + h_{11}(x)$$

$$\varphi_{1,2}(x) = x^2 + x^3 - \frac{4}{15}x^6 - \frac{1}{6}x^7$$

$$\varphi_{2,2}(x) = h_{20}(x) + h_{21}(x)$$

$$\varphi_{2,2}(x) = -x + \frac{x}{150}^{3} + x^6 + -\frac{x^8}{1400}$$

$$h_1(x) = h_{10}(x) + ph_{11}(x) + \ldots$$

$$h_1(x) = -\frac{1}{2}x^5 - \frac{2}{3}x^4 + x^3 + x^2 + p\left(-\frac{1}{6}x^7 - \frac{4}{15}x^6 + \frac{1}{2}x^5 + \frac{2}{3}x^4\right) + \ldots$$

$$h_2(x) = -x - \frac{1}{10}x^2 + x^6 + \frac{1}{35}x^7 + p\left(\frac{x^2}{10} + \frac{x}{150}^{3} - \frac{x}{35}^{7} - \frac{x^8}{1400}\right) + \ldots$$

The numerical results obtained by Adomian Decomposition Method and the HPM are represented in Table 7.3 and Table7.4.

Table 7.3 Numerical results for problem 7.7

n	x_i	$f_1(x_i)$	$f_{1HPM}(x_i)$	$e_1(x_i)$	$f_2(x_i)$	$f_{2HPM}(x_i)$	$e_2(x_i)$
1	0	0	0	0	0	0	0
2	0.1	0.01	0.01	0	0.1	-0.10	0.2
3	0.3	0.09	0.14	-0.05	0.3	-0.29	0.59
4	0.5	0.25	0.45	-0.20	0.5	-0.48	0.98

Table 7.4 Numerical results for problem 7.7

n	x_i	$f_1(x_i)$	$f_{1ADM}(x_i)$	$e_1(x_i)$	$f_2(x_i)$	$f_{2ADM}(x_i)$	$e_2(x_i)$
1	0	0	0	0	0	0	0
2	0.1	0.01	0.0100	0	0.1	0.1	0
3	0.3	0.09	0.0877	0.0023	0.3	0.3006	-0.0006
4	0.5	0.25	0.2440	0.006	0.5	0.5162	-0.0162

The computations associated with in this example were performed using MATHEMATICA 7. (Appendix 3)

Problem 7.8

Let us solve the following nonlinear system of three integral equations of the second type with the exact solutions $f_1(x) = \ln x$, $f_2(x) = x$ and $f_3(x) = x^2$ (Biazar and Ghazvini, 2009):

$$f_1(x) = \ln x - 2x^2 \ln x + x^2 + 4\int_0^x f_1(s) f_2(s) ds \tag{7.8.1a}$$

$$f_2(x) = x - \frac{1}{6}x^6 \ln x + \frac{1}{36}x^6 + \int_0^x s f_1(s) f_3^2(s) ds \tag{7.8.1b}$$

$$f_3(x) = x^2 + \frac{1}{15}x^5 - \frac{1}{3}\int_0^x s f_3(s) f_2(s) ds \tag{7.8.1c}$$

For solving this system by He's homotopy perturbation method (HPM) a convex homotopy such that $H(f_1, f_2, f_3, p): \Omega \times [0,1] \to IR^3$ is constructed for each equation with the following properties:

$$H_1(f_1, f_2, f_3, p) = (1-p)(f_1(x) - g_1(x)) + p\left[f_1(x) - g_1(x) - 4\int_0^x f_1(s) f_2(s) ds \right] = 0$$

$$H_2(f_1, f_2, f_3, p) = (1-p)(f_2(x) - g_2(x)) + p\left[f_2(x) - g_2(x) - \int_0^x s f_1(s) f_3^2(s) ds \right] = 0$$

$$H_3(f_1, f_2, f_3, p) = (1-p)(f_3(x) - g_3(x)) + p\left[f_3(x) - g_3(x) + \frac{1}{3}\int_0^x sf_3(s)f_2(s)ds \right] = 0$$

(7.8.2)

or equivalently

$H_1(f_1, f_2, f_3, p) = 0$, $\qquad H_2(f_1, f_2, f_3, p) = 0$, $\qquad H_3(f_1, f_2, f_3, p) = 0$

gives respectively

$$f_1(x) = g_1(x) + 4p\int_0^x f_1(s)f_2(s)ds,$$

$$f_2(x) = g_2(x) + p\int_0^x sf_1(s)f_3^2(s)ds, \qquad (7.8.3)$$

$$f_3(x) = g_3(x) - \frac{1}{3}p\int_0^x sf_3(s)f_2(s)ds.$$

Suppose the solutions of the system (7.8.2) have the form

$$F_i(x) = f_{i,0}(x) + pf_{i,1}(x) + p^2 f_{i,2}(x) + \ldots \qquad i = 1,2,3 \qquad (7.8.4)$$

where $f_{ij}(x)$ $i = 1,2,3$ and $j = 1,2,3$ are functions to be determined.

The initial approximations to the solutions $f_{i,0}(x)$ are taken to be $g_i(x)$.

$F_{i,0}(x) = f_{i,0}(x) = g_i(x) \qquad i = 1,2,3$

That is

$$F_{10}(x) = f_{1,0}(x) = \ln x - 2x^2 Inx + x^2$$

$$F_{20}(x) = f_{2,0}(x) = x - \frac{1}{6}x^6 Inx + \frac{1}{36}x^6 \qquad (7.8.5)$$

$$F_{30}(x) = f_{3,0}(x) = x^2 + \frac{1}{15}x^5$$

Substituting (7.8.4) into (7.8.3) and comparing the coefficients of the powers of p yields the following schemes:

$$f_1(x) = g_1(x) + 4p\int_0^x (f_{10}(s) + pf_{11}(s) + p^2 f_{12}(s) + \ldots)(f_{20}(s) + pf_{21}(s) + p^2 f_{22}(s) + \ldots)ds$$

$$f_2(x) = g_2(x) + p\int_0^x s(f_{10}(s) + pf_{11}(s) + p^2 f_{12}(s) + \ldots)(f_{30}(s) + pf_{31}(s) + p^2 f_{32}(s) + \ldots)^3 ds$$

$$f_3(x) = g_3(x) - \frac{1}{3}p\int_0^x \left(f_{30}(s) + pf_{31}(s) + p^2 f_{32}(s) + \ldots\right)\left(f_{20}(s) + pf_{21}(s) + p^2 f_{22}(s) + \ldots\right)ds$$

and in general

$$f_{1,j}(x) = 4\int_0^x \sum_{k=0}^{j-1} f_{1,k}(s) f_{2,j-k-1}(s)ds \qquad j = 1,2,3$$

$$f_{2,j}(x) = \int_0^x s\sum_{i=0}^{j-1}\sum_{k=0}^{j-i-1} f_{1,i}(s) f_{3,k}(s) f_{3,j-k-i-1}(s)ds \quad j = 1,2,3 \qquad (7.8.6)$$

$$f_{3,j}(x) = \frac{1}{3}\int_0^x s\sum_{k=0}^{j-1} f_{2,k}(s) f_{3,j-k-1}(s)ds \qquad j = 1,2,3$$

From (7.8.6), if the first six terms approximations are sufficient, we find that

$$f_1(x) = Lim_{p\to 1} F_1(x) = \sum_{k=0}^{6} F_{1,k}(x)$$

$$f_1(x) = Lim_{p\to 1} F_2(x) = \sum_{k=0}^{6} F_{2,k}(x)$$

$$f_3(x) = Lim_{p\to 1} F_3(x) = \sum_{k=0}^{6} F_{3,k}(x)$$

The values of six terms approximations with the related errors are given in the following Table 7.5, Table 7.6 and Table7.7. The computations with in this example were performed using MATHEMATICA 7. (Appendix 4)

Table 7.5 Numerical results for problem 7.8 with m=6

x	$f_1(x_i)$	$f_{1HPM}(x_i)$	$e_1(x_i)$
0.1	-2.302585093	-2.302585093	0
0.2	-1.609437912	-1.609437911	-10^{-9}
0.3	-1.203972804	-1.203972712	-9.2×10^{-8}
0.4	-0.916290731	-0.9162887123	-2.0187×10^{-6}
0.5	-0.69314718	-0.6931287262	-1.84538×10^{-5}
0.6	-0.510825623	-0.5107359296	-8.96934×10^{-5}

Table 7.6 Numerical results for problem 7.8 with m=6

x	$f_2(x_i)$	$f_{2HPM}(x_i)$	$e_2(x_i)$
0.1	0.1	0.1	0
0.2	0.2	0.2	0
0.3	0.3	0.3000000002	-2×10^{-10}
0.4	0.4	0.4000000120	-1.2×10^{-8}
0.5	0.5	0.5000003436	-3.436×10^{-7}
0.6	0.6	0.6000052560	-5.256×10^{-6}

Table 7.6 Numerical results for problem 7.8 with m=6

x	$f_3(x_i)$	$f_{3HPM}(x_i)$	$e_3(x_i)$
0.1	0.01	0.01	0
0.2	0.04	0.04	0
0.3	0.09	0.09000000001	-10^{-11}
0.4	0.16	0.1599999998	2×10^{-10}
0.5	0.25	0.2499999953	4.7×10^{-9}
0.6	0.36	0.3599998863	1.137×10^{-7}

The table shows that the HPM give very good approximation to the solutions.

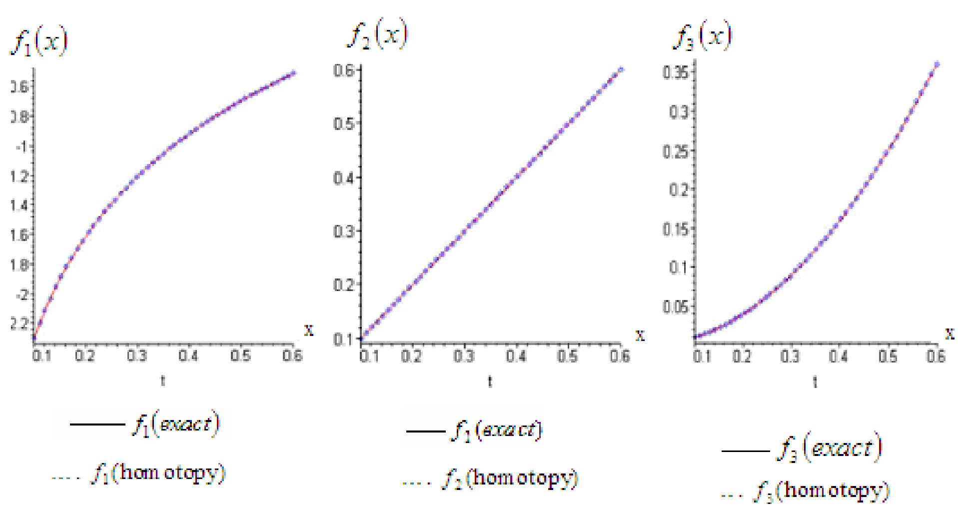

Figure 7.6 The plots of approximation and exact solutions

Problem 7.9

Consider the following nonlinear Fredholm system of integral equations (Maleknejad, Aghazadeh, and Rabbani, ,2006)

$$f_1(x)+\int_0^1 t\cos(xf_1(t))dt+\int_0^1 x\sin(tf_2(t))dt=g_1(x)$$

$$(7.9.1)$$

$$f_2(x)+\int_0^1 e^{xt^2} f_1(t)dt+\int_0^1 (x+t)f_2(t)dt=g_2(x)$$

with $g_1(x)=\dfrac{\cos x}{3}+\dfrac{x\sin^2 1}{2}+x$

$$g_2(x)=\frac{e^x-1}{2x}+\cos x+(x+1)\sin 1+\cos 1-1$$

and with exact solutions $f_1(x)=x$ and $f_2(x)=\cos x$.

For solving this system by He's homotopy perturbation method a convex homotopy is constructed for each equation with the following properties:

$$(1-p)(F_1(x)-F_{1,0}(x))+p\left(F_1(x)-\int_0^1 t\cos(xF_1(t))dt+\int_0^1 x\sin(tF_2(t))dt-g_1(x)\right)=0$$

$$(7.9.2)$$

$$(1-p)(F_2(x)-F_{2,0}(x))+p\left(F_2(x)-\int_0^1 e^{xt^2} F_1(t)dt+\int_0^1 (x+t)F_2(t)dt-g_2(x)\right)=0$$

Suppose the solutions of the system (7.9.2) have the form

$F_i(x)=F_{i,0}(x)+pF_{i,1}(x)+p^2F_{i,2}(x)+\ldots \qquad i=1,2,\ldots,n$

The initial approximations to the solutions $f_{i,0}(x)$ are taken to be $g_i(x)$.

$F_{i,0}(x)=f_{i,0}(x)=g_i(x)$

That is

$$F_{1,0}(x) = f_{1,0}(x) = \frac{\cos x}{3} + \frac{x \sin^2 1}{2} + x$$

(7.9.3)

$$F_{2,0}(x) = f_{2,0}(x) = \frac{e^x - 1}{2x} + \cos x + (x+1)\sin 1 + \cos 1 - 1$$

Substituting (7.9.3) into (7.9.2) and equating the terms with the coefficients of the identical powers of p, we have

$$F_{1,j}(x) = -\int_0^1 t \cos(xF_{1,j-1}(t))dt - \int_0^1 x \sin(tF_{2,j-1}(t))dt \qquad j = 1,2,3,\ldots$$

(7.9.4)

$$F_{2,j}(x) = -\int_0^1 e^{xt^2} F_{1,j-1}(t)dt - \int_0^1 (x+t)F_{2,j-1}(t)dt \qquad j = 1,2,3,\ldots$$

Suppose $f_1(x) \approx \sum_{j=0}^{5} F_{1,j}(x)$ and $f_2(x) \approx \sum_{j=0}^{5} F_{2,j}(x)$

The numerical results obtained by the modified Taylor series expansion method and the HPM are represented in Table 7.7 and Table 7.8.

Table 7.7 Numerical results for problem 7.9 with m=10

x_i	$f_{1HPM}(x_i)$	$f_{1taylor}(x_i)$	$e_1(x_i)$	$f_{2HPM}(x_i)$	$f_{2taylor}(x_i)$	$e_2(x_i)$
0	0.047592	0.047592	-0.047592	0.98839	0.98839	0.01161
0.1	0.138096	0.138096	-0.038096	1.00141	1.00141	-0.051369
0.2	0.227486	0.227486	-0.027486	0.984326	0.984326	-0.004259
0.3	0.317454	0.317454	-0.017454	0.95576	0.95576	-0.000424
0.4	0.408999	0.408999	-0.008999	0.918226	0.918226	0.002835
0.5	0.502914	0.502914	-0.002914	0.873939	0.873939	0.003644
0.6	0.599623	0.599623	0.000377	0.824824	0.824824	0.000512
0.7	0.69903	0.69903	0.00097	0.772597	0.772597	-0.007755
0.8	0.800329	0.800329	-0.000329	0.718871	0.718871	-0.022164
0.9	0.901818	0.901818	-0.00181811	0.66528	0.66528	-0.04367
1	1.00072	1.00072	-0.00072	0.613598	0.613598	-0.073296

Table 7.8 The exact values of f_1 and f_2 for problem 7.9 with m=10

$f_1(x_i)$	$f_2(x_i)$
0	1.0
0.1	0.95004
0.2	0.980067
0.3	0.955336
0.4	0.921061
0.5	0.877583
0.6	0.825336
0.7	0.764842
0.8	0.696707
0.9	0.62161
1	0.540302

The obtained solutions in comparison with the modified Taylor series expansion method and exact solutions admit a remarkable accuracy. (Appendix 5)

CHAPTER EIGHT

CONCLUSION

In this study, the Homotopy Perturbation Method (HPM) has been successfully used for finding the solution of nonlinear problems for integral equations and systems for integral equations. The absolute error, exact and numerical results are presented and compared each other in table, for some values of x or t. The analytic approximate solution shows that the HPM gives efficient results closer to the accurate solutions in bounded domains.

The advantage of this method is that it does need a small parameter in the system, leading to wide approximation in nonlinear integral equations. With the help of some mathematical software, such as MATHEMATICA, MATLAP, the method provides a powerful mathematical tool to more complex nonlinear systems.

The study shows that the HPM is simple and easy to use. Moreover, it minimizes the computational calculus and supplies quantitatively reliable results and can be considered an alternative method for solving a wide class of nonlinear problems which arise in various fields of pure and applied sciences.

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APPENDICES

Appendix 1

Taking the boundary condition $u(0)=0$ the solution of integro-differential equation (7.5.1) is as follows:

$$u'(x)=-1+\int_0^x u^2(t)dt \qquad (7.5.1)$$

$$u'(x)=-1+\int_0^x 0dt=-1+0=-1$$

$$\frac{du(x)}{dx}=-1$$

$$u(x)=-x+c$$
$$u(0)=0$$
$$c=0$$
$$u(x)=-x$$

At the point $x=0.0625$ the solution of integro-differential equation (7.5.1) is,

$$u(0.0625)=-0.0625$$

At the point $x=0.125$ the solution of integro-differential equation (7.5.1) is

$$u'(x)=-1+\int_0^x \frac{1}{12}t^4 dt$$

$$u'(x)=-1+\frac{1}{12}\frac{t^5}{5}\Big|_0^x$$

$$u'(x) = -1 + \frac{1}{60}x^5$$

$$\frac{du(x)}{dx} = -1 + \frac{1}{60}x^5$$

$$u(x) = -x + \frac{1}{360}x^6$$

$$u(0.125) = -0.125 + \frac{1}{360}(-0.125)^6$$

$$u(0.125) = -0.12498$$

At the point $x = 0.250$ the solution of integro-differential equation (7.5.1) is

$$u'(x) = -1 - \int_0^x \frac{1}{252}t^7 dt$$

$$u'(x) = -1 - \frac{1}{252}\frac{t^8}{8}\Bigg|_0^x$$

$$u'(x) = -1 - \frac{1}{2016}x^8$$

$$u(x) = -x - \frac{1}{18144}x^9$$

$$u(0.250) = -0.250 - \frac{1}{18144}(0.250)^9$$

$$u(0.250) = -0.24968$$

Appendix 2

$$h[1,0][x\_] := x - \frac{5}{18};$$

$$h[2,0][x\_] := x^2 - \frac{2}{9};$$

$$For[i = 0, i < 10, i++, h[1, i+1][x\_] = \frac{1}{3} * \int_0^1 (h[1, i][s] + h[2, i][s]) ds;$$

$$h[2, i+1][x\_] = \frac{1}{3} * \int_0^1 ((h[1, i][s])^2 + h[2, i][s]) ds];$$

$$a := Plot[x, \{x, 0, 1\}, PlotStyle \to Black];$$

$$b := Plot\left[\sum_{n=0}^{9} h[1, n][x], \{x, 0, 1\}, PlotStyle \to Dashed\right];$$

$$Show[a, b, PlotRange \to All]$$

Appendix 3

$$f1[x\_] := x^2;$$

$$f2[x\_] := x;$$

$$h[1, 0][x\_] := -\frac{1}{2}x^5 - \frac{2}{3}x^4 + x^3 + x^2;$$

$$h[2, 0][x\_] := -x - \frac{1}{10}x^2 + x^6 + \frac{1}{35}x^7;$$

$$For[i = 0, i < 10, i++, h[1, i+1][x\_] = 2 * \int_0^x h[1, i][s] ds;$$

$$h[2, i+1][x\_] = -\frac{1}{5} * \int_0^x h[2, i][s] ds];$$

$$c := Plot[x, \{x, 0, 1\}, PlotStyle \to Hue[0.5]];$$

$$d := Plot\left[\sum_{n=0}^{9} h[2, n][x], \{x, 0, 1\}, PlotStyle \to Hue[1.0]\right];$$

$$Show[c, d, PlotRange \to All]$$

Appendix 4

$$F[1][[x\_, j\_]] := If[\, j = 0, \ln[x] - 2 * x^2 * \ln[x] + x^2,$$
$$NIntegrate\left[\sum_{k=0}^{j-1} F[1,k][[s]] * F[2,\, j-k-1][[s]]\right], \{s,0,x\}];$$

$$F[2][[x\_, j\_]] := If[\, j = 0, x - \frac{1}{6} x^6 \ln[x] + \frac{1}{36} x^6,$$
$$NIntegrate\left[\sum_{i=0}^{j-1} \sum_{k=0}^{j-i-1} F[1,i][[s]] * F[3,i][[s]] * F[3,\, j-k-i-1][[s]]\right], \{s,0,x\}];$$

$$F[3][[x\_, j\_]] := If[\, j = 0, x^2 + \frac{1}{15} x^5,$$
$$\frac{1}{3} * s * NIntegrate\left[\sum_{k=0}^{j-1} F[2,k][[s]] * F[3,\, j-k-1][[s]]\right], \{s,0,x\}];$$

$$f_1[t\_] := \sum_{j=0}^{5} F[1][[t, j]];$$

$$f_2[t\_] := \sum_{j=0}^{5} F[2][[t, j]];$$

$$f_3[t\_] := \sum_{j=0}^{5} F[3][[t, j]];$$

$$Plot[\{\ln x, t\}, \{t,0,1\}]$$

$$a := Plot[\ln x, \{x,0,1\}, PlotStyle \to Black];$$

$$b := Plot\left[\sum_{j=0}^{5} F[1][[t, j]], \{x,0,1\}, PlotStyle \to Dashed\right];$$

$$Show[a, b, PlotRange \to All]$$

Appendix 5

$$F[1][x\_, j\_] := If[\, j = 0, \frac{Cos[x]}{3} + x\frac{(Sin[1.])^2}{2} + x,$$
$$-NIntegrate[t * Cos[x * F[1][t, j-1]], \{t,0,1\}] -$$
$$x * NIntegrate[Sin[t * F[2][t, j-1]], \{t,0,1\}]];$$

$$F[2][x\_, j\_] := If[\, j = 0, \frac{e^x - 1}{2x} + Cos[x] + (x+1)Sin[1.] + Cos[1.] - 1,$$
$$-NIntegrate\left[e^{xt^2} F[1][t, j-1], \{t,0,1\}\right] - NIntegrate[(x+t)F[2][t, j-1], \{t,0,1\}]];$$

$$f_1[t\_] := \sum_{j=0}^{5} F[1][t, j];$$

$$f_2[t\_] := \sum_{j=0}^{5} F[2][t, j];$$

$$Plot[\{f_1[t], t\}, \{t,0,1\}]$$

$$Plot\left[\{f_2[t], Cos[t]\}, \{t,0,1\}, AspectRatio \to 5/2, PlotStyle \to \{Black, Dashed\}\right]$$