DOKUZ EYLÜL UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

MODELING OF THE AGING PROCESS IN STRESS-STRENGTH MODELS

by
Burçin Şeyda ÖZLER

September, 2011
İZMİR

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We have read the thesis entitled **"MODELING OF THE AGING PROCESS IN STRESS-STRENGTH MODELS"** completed by **BURÇİN ŞEYDA ÖZLER** under supervision of **ASSIST. PROF. DR. SELMA GÜRLER** and we certify that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

S. Gürler

Assist. Prof. Dr. Selma GÜRLER

Supervisor

Assist. Prof. Dr. Özlem EGE ORUÇ

(Jury Member)

Assist. Prof. Dr. Engin MERMUT

(Jury Member)

Prof.Dr. Mustafa SABUNCU

Director

Graduate School of Natural and Applied Sciences

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Burçin Şeyda ÖZLER

MODELING OF THE AGING PROCESS IN STRESS-STRENGTH MODELS

ABSTRACT

Modeling of the aging process of a component or a system can be performed in various ways. The reliability function is one of the helpful tools commonly used for such modeling. Under specified conditions, the reliability of a component or a system can be defined as the probability that an item will perform satisfactorily, for a given period time. An important method for improving the reliability of a system is to build redundancy into it. A common structure of redundancy is the k-out-of-n system. Both parallel and series systems are special cases of the k-out-of-n system. A series system is equivalent to a 1-out-of-n system while a parallel system is equivalent to an n-out-of-n system.

In this thesis, the reliability of parallel, series and k-out-of-n: F systems with exchangeable components in the stress-strength model are considered. It is assumed that a random stress common to all the components in the system level. Applications of obtained results to illustrate the reliability for the system consisting of three components using the multivariate FGM and multivariate Marshall-Olkin distributions are given. Also, examples for the series and parallel systems are presented with some bivariate distributions.

Keywords: Reliability, stress-strength model, parallel system, series system, k-out-of-n system, exchangeable components.

STRES-DAYANIKLILIK MODELLERİNDE YAŞLANMA SÜRECİNİN MODELLENMESİ

ÖZ

Bir bileşenin ya da bir sistemin yaşlanma sürecinin modellenmesi çeşitli yollarla gerçekleştirilebilir. Güvenilirlik fonksiyonu böyle bir modelleme için başlıca kullanılan araçlardan biridir. Belirlenmiş şartlarda, verilmiş bir zaman süresi içinde, bir sistemin ya da bir bileşenin güvenilirliği, bir parçanın kifayetli bir şekilde çalışma olasılığıdır. Bir sistemin güvenilirliğini geliştirmek için önemli bir yöntem, sisteme yedekleme yapılmasıdır. Yedeklemenin başlıca yapısı n'den-k-tane sistemidir. Paralel ve seri sistemlerin her ikisi de n'den-k-tane sisteminin özel durumlarıdır. Paralel sistem n'den-n-tane sistemine eşit iken, seri sistem n'den-1-tane sistemine eşittir.

Bu tezde, stres-dayanıklılık modelinde, paralel, seri ve birbiri yerine geçebilen bileşenlere sahip n'den-k-tane: F sistemin güvenilirliği üzerinde duruldu. Rassal bir stresin, sistem düzeyinde tüm bileşenler için ortak olduğu varsayıldı. Çok değişkenli Farlie-Gumbel-Morgenstern ve çok değişkenli Marshall-Olkin dağılımları kullanılarak üç bileşenden oluşan bir sistemin güvenilirliğini incelemek için elde edilen sonuçları verildi. Ayrıca, paralel ve seri sistemler için bazı iki değişkenli dağılımlar ile örnekler sunuldu.

Anahtar kelimeler: Güvenilirlik, paralel sistem, seri sistem, n'den-k-tane sistemi, birbiri yerine geçebilen bileşenler.

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CHAPTER ONE

INTRODUCTION

People are continuously under stress, and not always have the strength to overcome it. Nowadays the stress-strength relationship is used in many branches of science such as engineering, psychology, medicine, pedagogy, pharmaceutical industry, etc. In the reliability theory, the stress-strength model is generally interested in the reliability of a component with strength X, which is under the random stress Y. When it is assumed that X and Y are independent random variables, then the component fails if the stress exceeds the strength of the component, i.e. $X<Y$. There are numerous papers on the reliability of a component in stress-strength models which has been generally concerned with the probability $P(X>Y)$. One can see for more details Kotz et al. (2003).

The stress-strength model originated not in a parametric but rather in a nonparametric set-up in the works of Wilcoxon (1945), Mann & Whitney (1947). They introduced a statistic to compare two random variables X and Y which describe results of two treatments. They also pointed out the connection between the hypothesis $F_X = F_Y$ and $P(X < Y) = 1/2$.

In the area of stress-strength models, there has been a large amount of work as regards estimation of the reliability $R=P(X>Y)$ when X and Y are independent random variables belonging to the same univariate family of distribution. The algebraic form R has been worked out for the majority of the well-known distributions in their standard forms. These include normal, Pareto, exponential, Gumbel, Weibull, Laplace distributions. In the literature, Johnson (1988) has studied estimation of R and has given lots of example about this probability of stress-strength models. Enis & Geisser (1971), Beg & Singh (1979) have studied estimation of R for Pareto distribution. Nadarajah (2003, 2004), Nadarajah & Kotz (2003) have also studied most of distributions about R. Rezaei et al. (2010) have studied estimation of R for generalized Pareto distribution.

In this chapter, we will give a brief introduction to stress-strength models and will present the literature review.

1.1 The Stress-Strength Model

In the context of reliability, the stress-strength model describes the life of a component which has a random strength X and is subjected to random stress Y. The component fails at the instant that the stress applied to it exceeds the strength and the component will function satisfactorily whenever $X>Y$. Thus $R=P(X>Y)$ is a measure of component reliability.

The most prominent examples for applications of $P(X>Y)$ are presented in Johnson's (1988) survey article. These examples which are related to the engineering and medicine applications are as follows:

Rocket Engines: Consider a stress random variable Y which represents the maximal chamber pressure generated by ignition of a solid propellant. X is assumed to be the strength of the rocket chamber so that $P(X>Y)$ is simply a probability of successful firing of an engine.

Two-Treatment Comparisons: There is a close relation between Wilcoxon-type tests and the $P(X>Y)$ models. Wilcoxon (1945) provides results of the fly spray tests on two preparations in terms of the percentage of the mortality. He compares the percent killed in the sample A versus the percent killed in the sample B (each involving 8 observations) concluding by means of this test that sample B provides a lower percent; thus preparation B should be considered less effective.

Response Models: A certain unit- be a receptor in a human eye or ear or any other organ operates only if it is stimulated by source of random magnitude Y and the stimulus exceeds specific a lower threshold for that unit. In this case P (unit function) is equivalent to the familiar $P(X<Y)$, a stress-strength relationship.

Shaft Example: If X represents the diameter of a shaft and Y represents the diameter of a bearing that is to be mounted on the shaft, then R is the probability that the bearing fits without interference.

Mechanical Systems: Let Y and X are the remission times of two chemicals when they are administered two kinds of mechanical system. Inferences about R present a comparison of the effectiveness of the chemicals.

Pyrotechnic Example: If Y represents the distance of pyrotechnic igniter from its adjacent pellet and X represents its ignition distance, then R is the probability that the igniter succeeds to bridge the gap in the pyrotechnic chain.

Military Example: In military warfare, R could be interpreted as the probability that a given round of ammunition will penetrate its target.

1.2 The Reliability of a Stress-Strength Model

Let X, $x > 0$, be the strength of a component which is subjected to the stress Y, $y>0$ and X and Y be two independent random variables with cumulative distribution functions (CDF) $F(x)$ and $G(y)$, respectively.

The reliability of a stress strength model was defined by Johnson (1988) as

$$\begin{aligned} R = P(X > Y) &= \int_0^\infty P(X > Y \mid Y = y) g(y) dy \\ &= \int_0^\infty \overline{F}(y) dG(y) \end{aligned} \tag{1.1}$$

where $\overline{F}(x) = 1 - F(x)$ is the survival function of X.

Some applications for the reliability of a stress-strength model are given in the following examples with exponential, Pareto, normal, Laplace, Gumbel and Weibull distributions.

Example 1.1 (Exponential Model): The exponential distribution is a commonly used distribution in reliability engineering. The amount of time or distance between occurrences of random events like the length of time between emergency arrivals at a hospital, the length of time between breakdowns of manufacturing equipment can often be described by the exponential distribution. It is often used to describe the failure process of electronic equipment. It has also been used as a model for lifetimes of various things. The exponential distribution is used to describe units that have a constant failure rate. This follows as well from the memoryless property. In reliability terms it means that current or future reliability properties of an operating piece of equipment do not change with time and do not depend on the amount of operating time (see for more details, McClave et al., 2009).

The CDF of the exponential distribution is:

$$F(x) = 1 - \exp(-\lambda x), \tag{1.2}$$

where $x > 0$ and the parameter $\lambda > 0$.

Let the strength X and the stress Y be independent exponential random variables. Then the reliability of a component in stress-strength model can be expressed by

$$\begin{aligned} R &= \int_0^\infty \overline{F}(y)\,dG(y) = \int_0^\infty \exp(-\lambda_1 y)\lambda_2 \exp(-\lambda_2 y)\,dy \\ &= \lambda_2 \left[\int_0^\infty \exp(-y(\lambda_1 + \lambda_2))\,dy \right] \\ &= \frac{\lambda_2}{\lambda_1 + \lambda_2}, \end{aligned} \tag{1.3}$$

where $\overline{F}(x) = \exp(-\lambda_1 x)$, $G(y) = 1 - \exp(-\lambda_2 y)$, $\lambda_1, \lambda_2 > 0$ and *x, y>0.*

Since R' exists and is positive, we conclude that R is increasing on $[0, \infty]$ at every level of λ_2 . This result can also be seen from Figure 1.1.

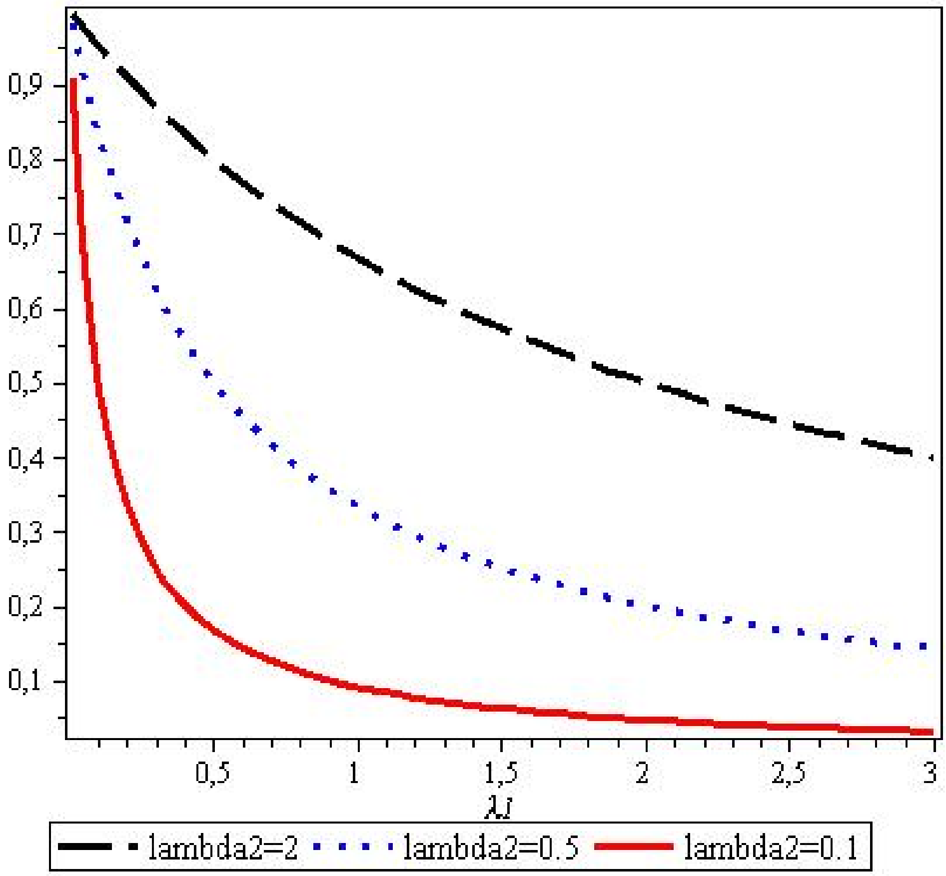


Figure 1.1 Reliability curves of the component for exponential distribution

Example 1.2 (Pareto Model): Pareto distribution is one of importance in both engineering and economics and serves as the most popular model of income distribution. Its applications include using in the analysis of extreme events, in the modeling of large insurance claims, as a failure time distribution in reliability studies and in any situation in which the exponential distribution might be used but in which some robustness is required against heavier tailed or lighter tailed alternatives. It is also well known that this distribution has decreasing failure rate property.

Consider the case for which the strength of a component X and the stress Y are independent generalized Pareto random variables with the CDF's, respectively

$$F(x) = 1 - (1 + \lambda x)^{-\alpha}$$
$$G(y) = 1 - (1 + \lambda y)^{-\beta}, \tag{1.4}$$

where for x, $y>0$, $\lambda>0$, $\alpha>0$ and $\beta>0$. Here α and β are the shape and λ scale parameters, respectively.

Then, the reliability of a stress-strength model is:

$$\begin{aligned} R &= \int_0^\infty \overline{F}(y) dG(y) \\ &= \int_0^\infty \alpha\lambda(1 + \lambda y)^{-(\alpha+1)}\{1 - (1 + \lambda y)^{-\beta}\} dy \\ &= \frac{\beta}{\alpha + \beta}. \end{aligned} \tag{1.5}$$

Example 1.3 (Normal Model): The normal distribution is a basic distribution of statistics. Due to Central Limit Theorem, the distribution is popular. However reliability analysts have seldom used the normal distribution. Because its support is the real line; the normal distribution is also symmetric, whereas failure times tend to exhibit a skewed distribution. This distribution is an appropriate model for practical engineering situations. For example it can be used as a distribution of diameters of manufactured shafts (see, Hamada et al., 2008, p. 106).

If X and Y are independent normal distributed random variables with the means μ_1, μ_2 and the variances σ_1^2, σ_2^2, respectively. Kotz et al. (2003) have given that Y-X is a normal variable with the mean $\mu_2 - \mu_1$ and the variance $\sigma_1^2 + \sigma_2^2$,

$$P(X<Y)=P(Y-X>0)=F\left(\frac{\mu_2-\mu_1}{\sqrt{\sigma_1^{\;2}+\sigma_2^{\;2}}}\right), \tag{1.6}$$

where $F(z)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z}e^{-x^2/2}dx$ is the CDF of the standard normal distribution.

Example 1.4 (Standard Laplace Model): In probability theory and statistics, the Laplace distribution is a continuous probability distribution named after Pierre-Simon Laplace. It has been commonly used as an alternative to the normal distribution. It has been used in the areas of astronomy, biological and environmental sciences, engineering sciences, finance, inventory management and quality control.

Nadarajah (2004) has studied the reliability of a component when the strength X and the stress Y are independent standard Laplace random variables with CDF's as below:

$$F(x)=\begin{cases}\frac{1}{2}\exp\left(\frac{-x-\theta_1}{\varphi_1}\right), x\le\theta_1\\ 1-\frac{1}{2}\exp\left(\frac{-x-\theta_1}{\varphi_1}\right), x>\theta_1\end{cases} \quad\text{and}\quad G(y)=\begin{cases}\frac{1}{2}\exp\left(\frac{-y-\theta_2}{\varphi_2}\right), y\le\theta_2\\ 1-\frac{1}{2}\exp\left(\frac{-y-\theta_2}{\varphi_2}\right), y>\theta_2\end{cases}$$

where $-\infty<x<\infty, -\infty<y<\infty,$ the location parameter $-\infty<\theta_1<\infty, -\infty<\theta_2<\infty$ and the scale parameter $\varphi_1,\varphi_2>0$. Then, the reliability of the component for $\theta_1\le\theta_2$,

$$R_1=\frac{\varphi_1^2}{2(\varphi_1^2-\varphi_2^2)}\exp\left(\frac{\theta_1-\theta_2}{\varphi_1}\right)-\frac{\varphi_2^2}{2(\varphi_1^2-\varphi_2^2)}\exp\left(\frac{\theta_1-\theta_2}{\varphi_2}\right). \tag{1.7}$$

Since $R_1^{'}$ exists and is positive, we conclude that R_1 is increasing on $[-\infty,\infty]$ at every level of θ_1. This result can also be seen from Figure 1.2.

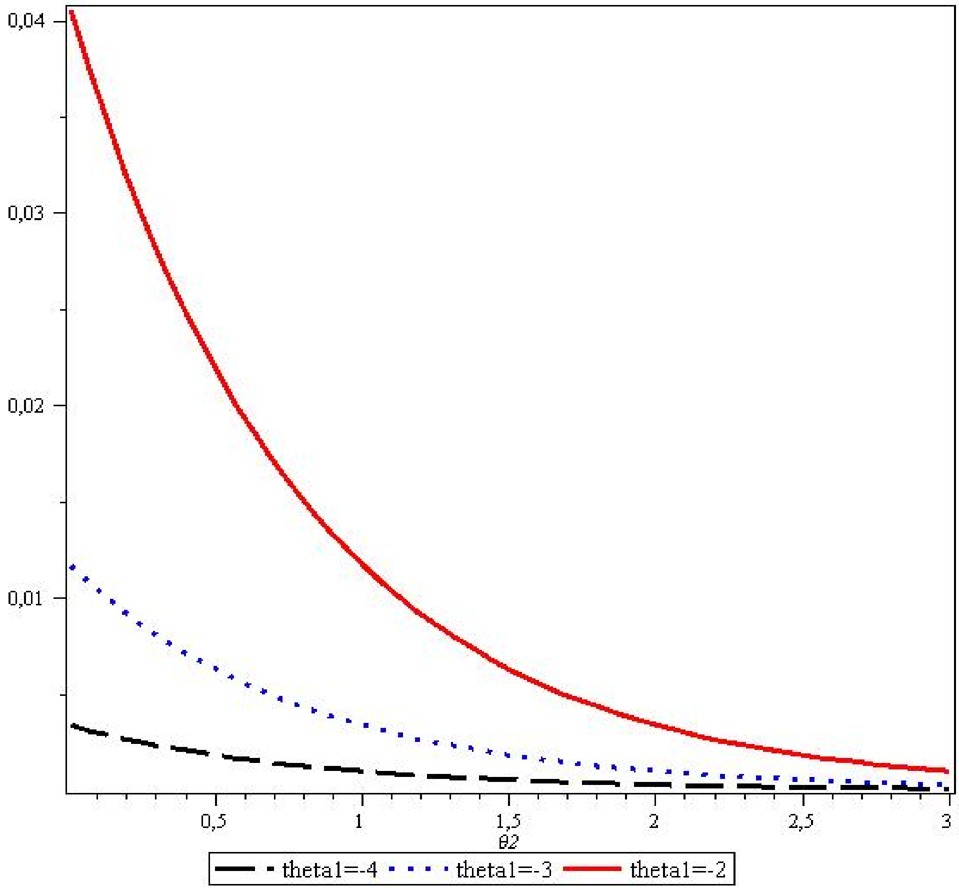


Figure 1.2 Reliability curves of the component for Laplace distribution with $\theta_1 \le \theta_2$, $\varphi_1 = 0.01, \varphi_2 = 0.8.$

The reliability of the component for $\theta_1 > \theta_2$,

$$R_2 = 1 + \frac{\varphi_1^2}{2(\varphi_1^2 - \varphi_2^2)} \exp\left(\frac{\theta_2 - \theta_1}{\varphi_1}\right) - \frac{\varphi_2^2}{2(\varphi_1^2 - \varphi_2^2)} \exp\left(\frac{\theta_2 - \theta_1}{\varphi_2}\right). \tag{1.8}$$

Since $R_2^{'}$ exists and is positive, we conclude that R_2 is increasing on $[-\infty, \infty]$ at every level of θ_2. This result can also be seen from Figure 1.3.

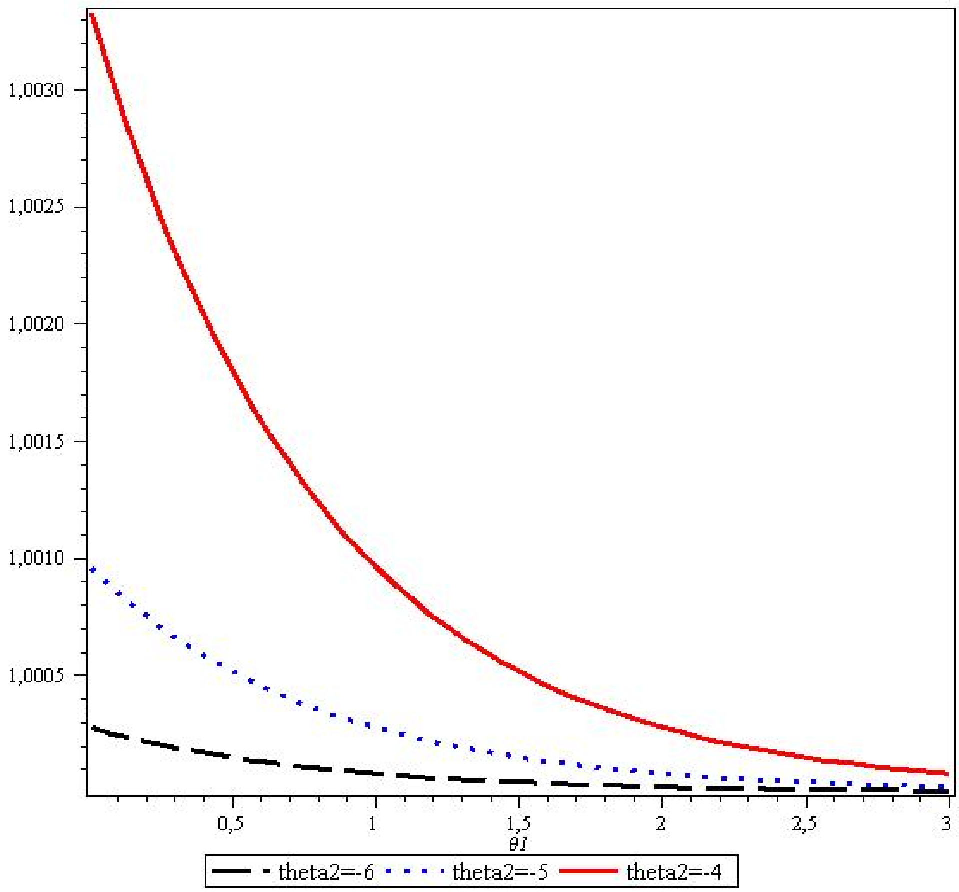


Figure 1.3 Reliability curves of the component for Laplace distribution with $\theta_1 > \theta_2$, $\varphi_1 = 0.01, \varphi_2 = 0.8$.

Example 1.5 (Gumbel Model): In probability theory and statistics, the Gumbel distribution is used to model for predicting the chance that an extreme earthquake, flood or other natural disaster will occur. It has also been used for fire protection and insurance problems, modeling of extremely high temperatures and the prediction high return levels of wind speeds relevant for the design of civil engineering structures. It should also be a good model for the distribution of maxima over fixed periods.

For the Gumbel distribution, the CDF of *X* and the CDF of *Y* are defined as:

$$F(x) = \exp\left\{-\exp\left(-\frac{x-\mu_1}{\sigma_1}\right)\right\} \text{ and } G(y) = \exp\left\{-\exp\left(-\frac{y-\mu_2}{\sigma_2}\right)\right\},$$

where $-\infty \le x \le \infty, -\infty \le y \le \infty$, respectively.

Nadarajah (2003) has given the reliability of the component in the stress-strength setup for Gumbel distribution as below:

$$R = \int_{-\infty}^{\infty} \exp\left\{-\exp\left(-\frac{y-\mu_2}{\sigma_2}\right)\right\} \frac{1}{\sigma_1} \exp\left(-\frac{y-\mu_1}{\sigma_1}\right) \exp\left\{-\exp\left(-\frac{y-\mu_1}{\sigma_1}\right)\right\} dy. \quad (1.9)$$

Substituting $z = \exp(-y/\sigma_1),$ it can be reduced to

$$R = \exp\left(\frac{\mu_1}{\sigma_1}\right) \int_0^{\infty} \exp\left[-\left\{\exp\left(\frac{\mu_1}{\sigma_1}\right) z + \exp\left(\frac{\mu_2}{\sigma_2}\right) z^{\sigma_2/\sigma_1}\right\}\right] dz. \quad (1.10)$$

In the particular case $\sigma_1 = \sigma_2$, the above integral can be obtained as below

$$R = \exp\left(\frac{\mu_1}{\sigma_1}\right) / \left\{\exp\left(\frac{\mu_1}{\sigma_1}\right) + \exp\left(\frac{\mu_2}{\sigma_2}\right)\right\}. \quad (1.11)$$

Since R' exists and is positive, we conclude that R is increasing on $[0, \infty]$ at every level of μ_1. This result can also be seen from Figure 1.4.

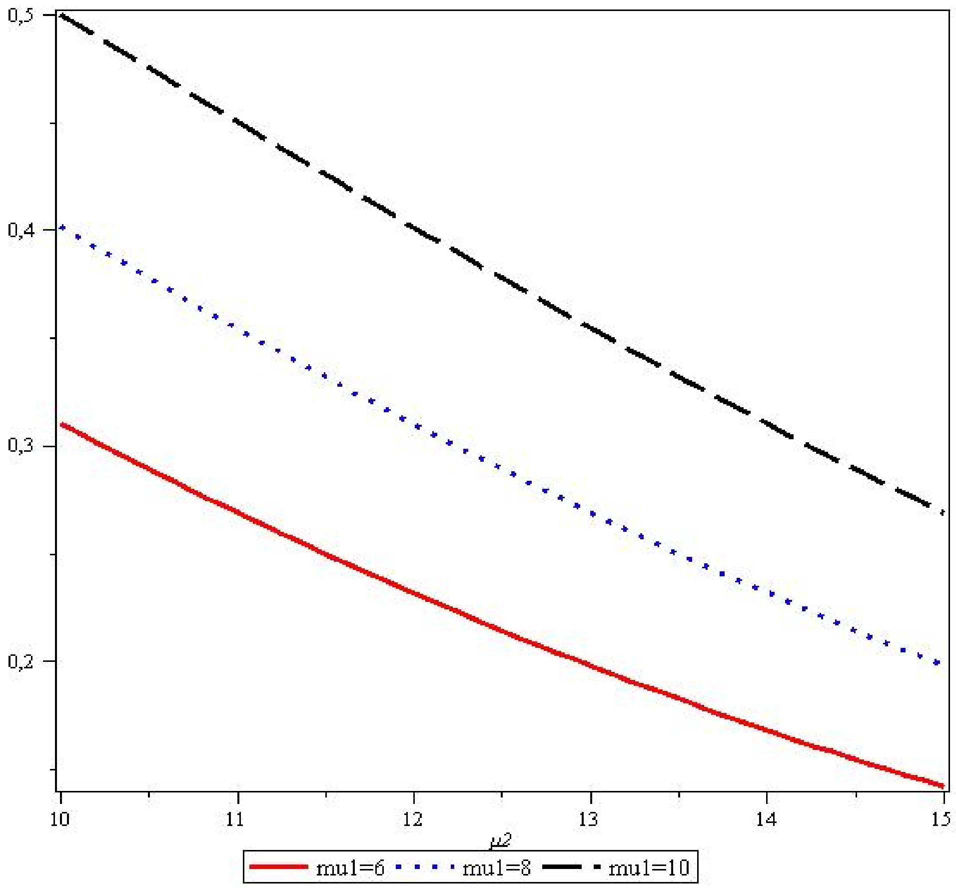


Figure 1.4 Reliability curves of the component for Gumbel distribution with $\sigma_1 = \sigma_2 = 5$.

Example 1.6 (Weibull Model): The Weibull distribution is named after the Swedish physicist Waloddi Weibull, who in 1939 used it to represent the distribution of the breaking strength of materials and in 1951 for a variety of other applications. Applications of this distribution include modeling of failure strengths of load-sharing systems and window glasses, predicting the diameter of crops for growth and yield modeling purposes etc. Because of its tractability and flexibility, it is the most frequently used lifetime model in the reliability literature. As for the exponential distribution, many software packages implement classical statistical methods for the Weibull distribution (see, Hamada et al., 2008, p. 97).

When we assume that the strength X and the stress Y are Weibull distributed, the CDF's of these random variables are as follows for $\alpha_1 > 0$, $\alpha_2 > 0$, $\theta_1 > 0$ and $\theta_2 > 0$.

$$F(x) = 1 - \exp\left\{-\left(x/\theta_1\right)^{\alpha_1}\right\} \text{ and } G(y) = 1 - \exp\left\{-\left(y/\theta_2\right)^{\alpha_2}\right\}. \quad (1.12)$$

The reliability of the Weibull stress-strength model is:

$$\begin{aligned} R = P(X > Y) &= \int_0^\infty \overline{F}(y)\,dG(y) \\ &= \int_0^\infty e^{-(y/\theta_1)^{\alpha_1}} \left[\frac{\alpha_2}{\theta_2^{\alpha_2}} y^{\alpha_2 - 1} e^{-(y/\theta_2)^{\alpha 2}}\right] dy. \end{aligned} \quad (1.13)$$

For common shape parameter α $(\alpha_1 = \alpha_2)$ for stress and strength random variables, we may denote the reliability function

$$R = \int_0^\infty e^{-(y/\theta_1)^{\alpha}} \left[\frac{\alpha}{\theta_2^{\alpha}} y^{\alpha - 1} e^{-(y/\theta_2)^{\alpha}}\right] dy. \quad (1.14)$$

The transformation $y^\alpha = u$ and $\alpha y^{\alpha-1} dy = du$ gives the following result

$$\begin{aligned} R &= \int_0^\infty e^{-u/\theta_1^\alpha} \frac{1}{\theta_2^\alpha} e^{-u/\theta_2^\alpha} du \\ &= \frac{\theta_1^\alpha}{\theta_1^\alpha + \theta_2^\alpha}. \end{aligned} \quad (1.15)$$

Since R exists and is positive, we conclude that R is increasing on $[0, \infty]$ at every level of θ_1. This result can also be seen from Figure 1.5.

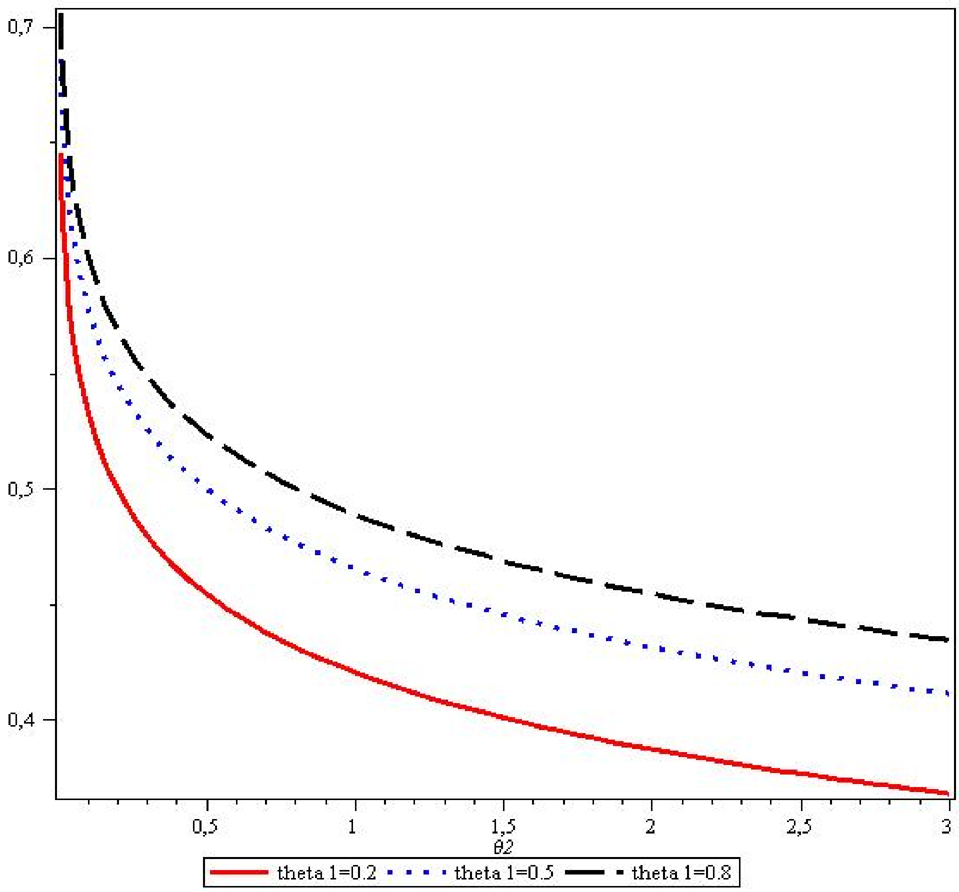


Figure 1.5 Reliability curves of the component for Weibull distribution $(\alpha = 0.2)$.

1.3 Thesis Organization

It is assumed throughout the thesis that the stress and strengths are independent random variables and a random stress common to all the components. Organization of this thesis is as follows:

In Chapter two, the reliability of parallel and series systems are studied in the case of independent and identically distributed (IID) components with a common stress. In Chapter three, the reliability of the *k*-out-of-*n* system is introduced with IID as well as exchangeable components in the stress-strength setup. In addition to the generalized expressions for the system reliability, specific case models such as multivariate FGM and multivariate Marshall-Olkin under the exponential distributed stress Y are also presented in this chapter. Finally, conclusions are discussed in Chapter four.

CHAPTER TWO

THE RELIABILITY OF PARALLEL AND SERIES SYSTEMS WITH INDEPENDENT COMPONENTS IN A STRESS-STRENGTH MODEL

The lifetime of a system is determined by its components and structure. Most of the systems formed in the real life, are parallel and series systems.

2.1 The Reliability of a Parallel System

A parallel system works if and only if at least one component works. It fails if all components are failed. Parallel systems are the most commonly used systems for redundancy. When only one of the components is essential, the others are said to be redundant components. The purpose of using a parallel structure is to increase the system reliability through redundancy. The form of a parallel system is given in Figure 2.1.

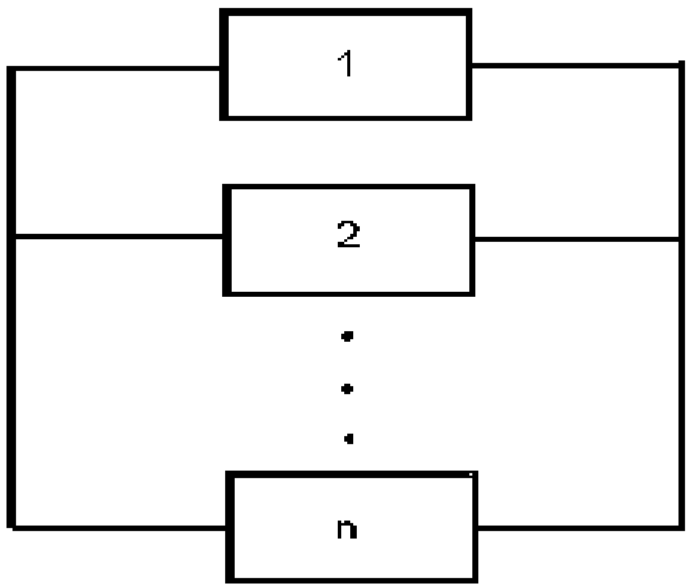


Figure 2.1 A parallel system with *n* components

The reliability of parallel system is widely used in the aerospace industry and generally used in mission critical systems. Other applications include the computer hard drive systems, brake systems and support cables in bridges. A typical example of reliability of the parallel system is an aircraft whose two independent active engines enable it to fly normally when at least one engine is functioning successfully.

Consider a parallel system consisting of n components with lifetimes $X_1, X_2, ..., X_n$. The lifetime of a parallel system is equal to the longest lifetime among the lifetimes of all components. The parallel system can last as long as the best component in the system. If we assume that $X_1, X_2, ..., X_n$ are ordered that is $X_{1:n} < X_{2:n} < ... < X_{n:n}$, the lifetime of the system depends on the $X_{n:n}$.

The reliability function of the parallel system with independent components is given by for $t>0$,

$$R_{n:n}(t) = 1 - \prod_{i=1}^{n} F_i(t). \tag{2.1}$$

This can be easily verified as:

$$\begin{aligned} R_{n:n}(t) &= P(X_{n:n} > t) \\ &= 1 - P(X_{n:n} \leq t) \\ &= 1 - P(X_1 \leq t, X_2 \leq t, ..., X_n \leq t) \\ &= 1 - \prod_{i=1}^{n} F_i(t), \quad t > 0. \end{aligned} \tag{2.2}$$

In the literature, Barlow & Proschan (1975), Kuo & Zuo (2003) have mentioned on the reliability of the parallel system in their books.

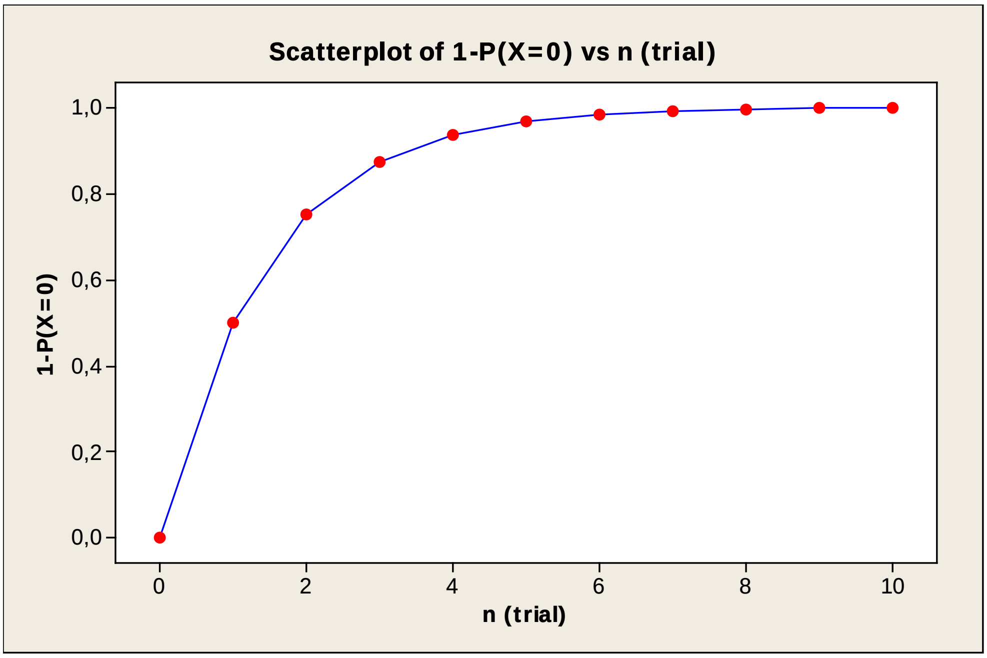


Figure 2.2 The reliability of a parallel system as function of the system size *n* for the survival probability of the component is 0.5.

Figure 2.2 shows the relationship between the reliability of a parallel system and the number of components. From Figure 2.2, we have the following results:

1. The reliability of a parallel system approaches 1 as *n* goes to infinity. We can always achieve very high system reliability through redundancy.

2. As the system size increases, the amount of improvement in the system reliability by each additional component becomes smaller.

2.2 The Reliability of a Parallel System under a Common Stress

Let $X_1, X_2, ..., X_n$ be the independent strengths of the components. We assume that the random variable Y is the common stress and it is independent of $X_1, X_2, ..., X_n$. We also assume that $X_{1:n} < X_{2:n} < ... < X_{n:n}$ are the ordered strengths of the components.

The reliability of the parallel system under a common stress can be expressed as:

$$
\begin{aligned}
R_{n:n}^{Y} &= P(\max(X_1,...,X_n) > Y) \\
&= \int_0^\infty P(X_{n:n} > Y \mid Y = y)\,dG(y) \\
&= \int_0^\infty \left[1 - F^n(y)\right] dG(y), \quad x > 0, \quad y > 0.
\end{aligned}
\tag{2.3}
$$

Dewanji & Rao (2001) have studied on the reliability of the parallel system in the stress-strength setup. One example of the reliability of the parallel system is that two electric bulbs with different voltage capacity in a room. Here the strength of two electric bulbs is the maximum voltage allowable. If the voltage in the current is less than the maximum of the strengths of the two electric bulbs, then there will be light in the room and the reliability is the probability that there is light in the room (Hanagal, 1996, p. 14).

Example 2.1: Consider X_1 and X_2 are IID exponential variables with parameter $\lambda_1 > 0$ and Y is an independent exponential common stress with the parameter $\lambda_2 > 0$. Then the system reliability is expressed by

$$
\begin{aligned}
R_{2:2}^{Y} &= \int_0^\infty [1 - F^2(y)]\,dG(y) \\
&= \int_0^\infty [1 - \{1 - \exp(-\lambda_1 y)\}^2]\lambda_2 \exp(-\lambda_2 y)\,dy \\
&= \lambda_2 \left[\frac{2}{\lambda_1 + \lambda_2} - \frac{1}{2\lambda_1 + \lambda_2}\right] = \frac{\lambda_2(3\lambda_1 + \lambda_2)}{(\lambda_1 + \lambda_2)(2\lambda_1 + \lambda_2)}.
\end{aligned}
\tag{2.4}
$$

Since $(R_{2:2}^{Y})'$ exists and is positive, we conclude that $R_{2:2}^{Y}$ is increasing on $[0,\infty]$ at every level of λ_2. This result can also be seen from Figure 2.3.

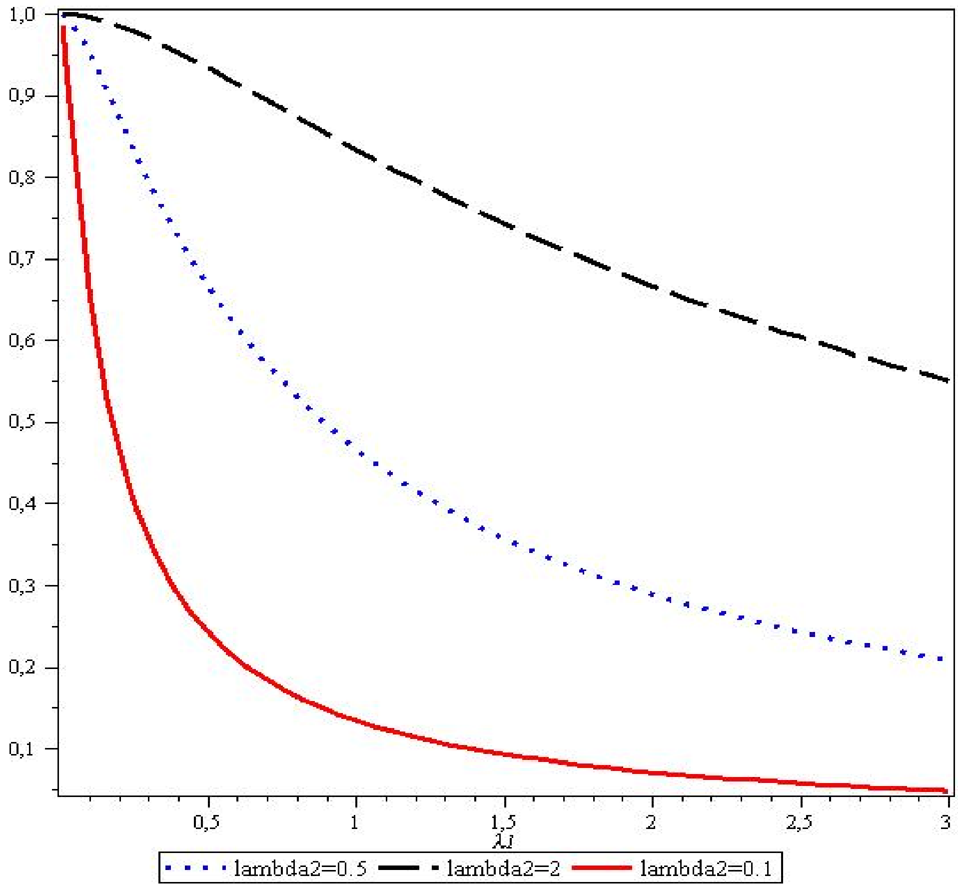


Figure 2.3 Reliability curves of the parallel system for exponential distribution

2.3 The Reliability of a Series System

A series system is a configuration such that, if any one of the system components fails, the entire system fails. The form of a series system is given in Figure 2.4

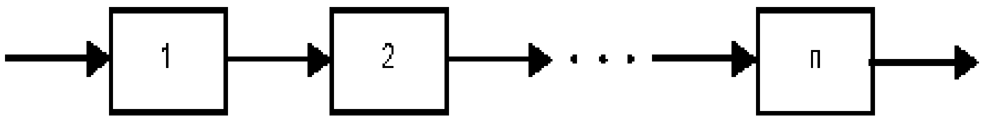


Figure 2.4 Series system with *n* components

Consider a series system consisting of n components with lifetimes $X_1, X_2, ..., X_n$. If we assume that $X_1, X_2, ..., X_n$ are ordered that is $X_{1:n} < X_{2:n} < ... < X_{n:n}$, the lifetime of the system depends on the $X_{1:n}$.

The reliability function of the series system with independent components is given by for t>0,

$$R_{1:n}(t) = P(X_{1:n} > t) = P(\min\{X_1, X_2, ..., X_n\} > t)$$

$$= P(X_1 > t, X_2 > t, ..., X_n > t) \tag{2.5}$$

$$= \prod_{i=1}^{n} \overline{F}_i(t).$$

2.4 The Reliability of a Series System under a Common Stress

In the literature, some researchers have studied on reliability of the series system. Chandra (1975) has estimated the reliability of series system with common stress. Hanagal (2003) estimated the reliability of series system with n components using gamma, Weibull and Pareto distributions.

Assuming that X_i's are IID and they are independent of the common stress Y, the reliability of the series system under a common stress can be expressed by:

$$\begin{aligned} R_{1:n}^{Y} &= P(\min(X_1, ..., X_n) > Y) \\ &= \int_0^\infty P(X_{1:n} > Y \mid Y = y) dG(y) \\ &= \int_0^\infty \overline{F}^n(y) dG(y), \quad x > 0, \quad y > 0. \end{aligned} \tag{2.6}$$

Example 2.2: Consider X_1 and X_2 are IID exponential random variables with the parameter $\lambda_1 > 0$ and Y is an exponential common stress with the parameter $\lambda_2 > 0$. Then the reliability of the two-component series system is

$$\begin{aligned} R_{1:2}^{Y} &= \int_0^\infty \overline{F}^2(y)dG(y) \\ &= \int_0^\infty \exp(-2\lambda_1 y)\lambda_2 . \exp(-\lambda_2 y)dy \\ &= \frac{\lambda_2}{2\lambda_1 + \lambda_2}. \end{aligned} \tag{2.7}$$

Since $(R_{1:2}^{Y})'$ exists and is positive, we conclude that $R_{1:2}^{Y}$ is increasing on $[0,\infty]$ at every level of λ_2. This result can also be seen from Figure 2.5.

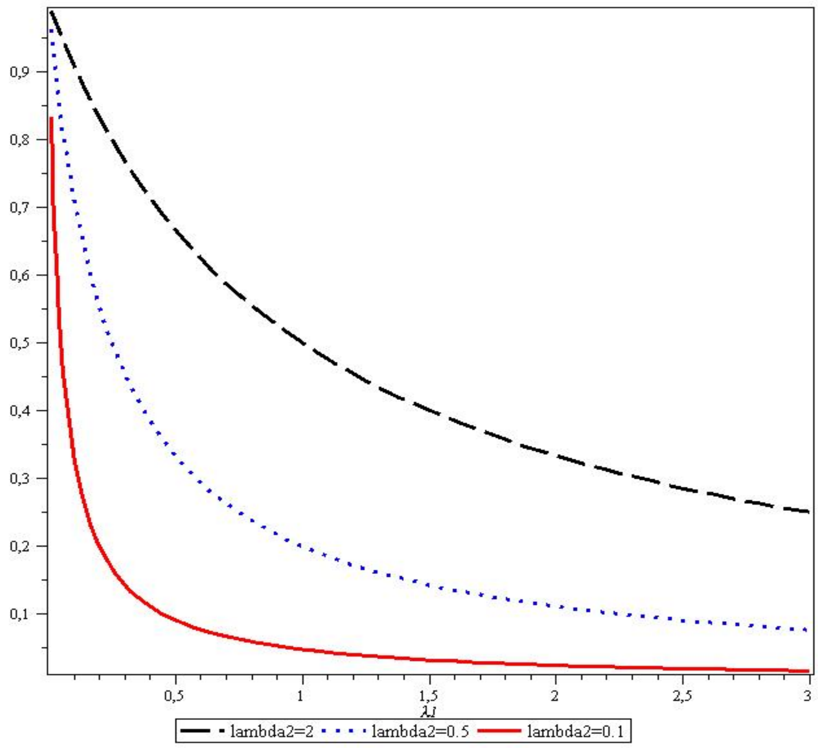


Figure 2.5 Reliability curves of the series system for exponential distribution

CHAPTER THREE

THE RELIABILITY of a k-OUT-OF-n SYSTEM in a STRESS-STRENGTH MODEL

3.1 Introduction

Many technical systems or subsystems have the *k*-out-of-*n* structure. A system fails if and only if at least *k* of the *n* components fails is called a *k*-out-of-*n: F* system. A system works if and only if at least *k* of the *n* components work is called a *k*-out-of-*n: G* system. Based on these two definitions, a *k*-out-of-*n: F* system is equivalent to an *n*-*k*+1-out-of-*n: G* system. Both parallel and series system are special cases of the *k*-out-of-*n* system. A series system is equivalent to a 1-out-of-*n: F* system while a parallel system is equivalent to an *n*-out-of-*n: F* system.

There are many papers related to the *k*-out-of-*n* system. Barlow & Heidtmann (1984) have presented methods to get expressions for reliability methods. Sarhan & Abouammoh (2001) have investigated the reliability of nonrepairable *k*-out-of-*n* systems with nonidentical components subjected to independent and common shocks and the relationship between the failure rate of the system and that of its components. Basu & El Mawaziny (1978) estimated the reliability of *k*-out-of-*n* structures in the independent exponential case. Boland & Proschan (1983), Høyland & Rausand (1994) have also studied *k*-out-of-*n* systems.

Practical examples of *k*-out-of-*n* systems are, e.g., an aircraft with four engines which will not crash if at least two out of its four engines remain functioning (Cramer & Kamps (1996)), a car with a V8 engine will be driven if only four cylinders fire, or satellite which will have enough power to send signals if not more than four out of its ten batteries are discharged. In the case of automobile with four tires, for example, usually one additional spare tire is equipped on the vehicle. Thus, the vehicle can be driven as long as at least 4-out-of-5 tires are in good condition (Kuo & Zuo, 2003).

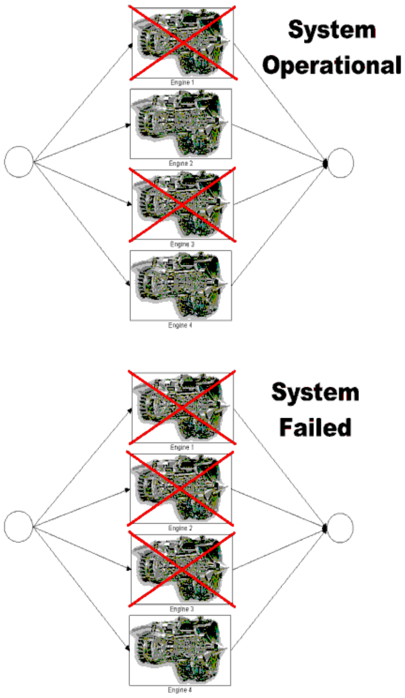


Figure 3.1 A 2-out-of-4 configuration

The form of a 2-out-of-4 system is given in Figure 3.1. It shows that when at least two components work, the system is operational and at least three components fail, the system is failed.

When n components are IID in a k-out-of-n: G system, the number of working components follows the binomial distribution with parameter n and p. Thus, we have

$$P(exactly \quad j \quad components \quad work) = \binom{n}{j} p^j q^{n-j}, \quad j = 0,\ldots k \tag{3.1}$$

where p is reliability of each component when all components are IID and $q=1-p$.

The reliability of the system is equal to the probability that the number of working components is greater than or equal to k:

$$R_{k:n}(t) = \sum_{j=k}^{n} \binom{n}{j} p^{j} q^{n-j} \tag{3.2}$$

where k is the minimum number of components that must work for the k-out-of-n: G.

When the components in a k-out-of-n: F system is IID, the reliability function of the system can be expressed as:

$$R_{k:n}(t) = 1 - \sum_{j=k}^{n} \binom{n}{j} (1 - F(t))^{n-j} F(t)^{j}, \ \ t > 0. \tag{3.3}$$

Example 3.1: Consider a 2-out-of-3: F system with three independent and exponentially distributed components with the parameter $\lambda > 0$. The reliability of the 2-out-of-3: F system is

$$\begin{aligned} R_{2:3}(t) &= 1 - \sum_{j=2}^{3} \binom{3}{j} (e^{-\lambda t})^{3-j} (1 - e^{-\lambda t})^{j} \\ &= 3e^{-2\lambda t} - 2e^{-3\lambda t}, \quad t > 0. \end{aligned} \tag{3.4}$$

3.2 The Reliability of a *k*-out-of-*n* System with Independent Components under a Common Stress

In the statistical approach to the stress-strength model, in some considerations depend on the assumption that the component strengths are independent and identically distributed for k-out-of-n systems. Bhattacharyya & Johnson (1974) have studied on k-out-of-n systems consisting of identical components and independent common stress under parametric model of exponential distributions.

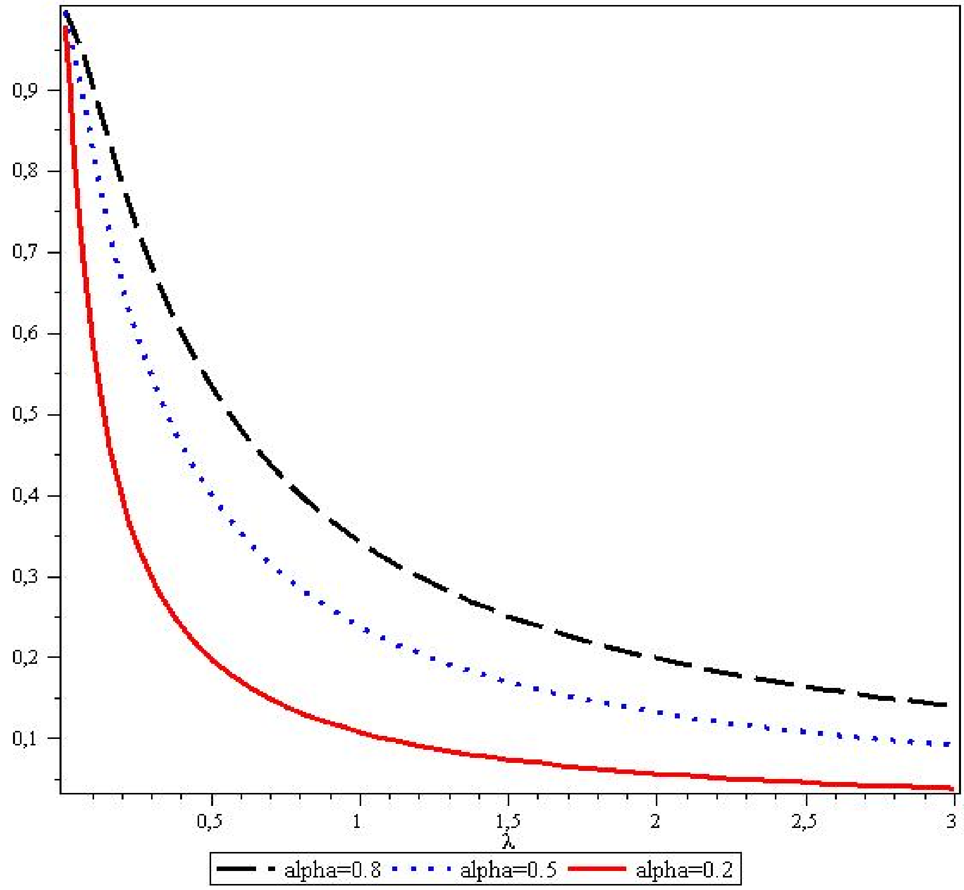
Let the random variables $X_1, X_2, ..., X_n$ independent components and Y is a common stress. The reliability of a k-out–of-n: F system is:

$$\begin{aligned} R_{k:n}^{Y} &= P[at \quad least \quad n-k+1-of - X_1, X_2, ..., X_n \quad exceed \quad Y] \\ &= 1-\sum_{i=k}^{n}\binom{n}{i}\int_0^\infty [1-F(y)]^{n-i} F(y)^i \, dG(y) \\ &= \sum_{i=n-k+1}^{n}\binom{n}{i}\int_0^\infty [1-F(y)]^{i} F(y)^{n-i} \, dG(y), \quad x>0, \quad y>0. \end{aligned} \tag{3.5}$$

Example 3.2: Consider a 2-out-of-4: F system consisting of IID exponential distributed components with the parameter λ>0 and independent common stress with the parameter α>0. The reliability of 2-out-of-4: F system is:

$$\begin{aligned} R_{2:4}^{Y} &= \sum_{i=3}^{4}\binom{4}{i}\int_0^\infty \{\exp(-\lambda y)\}^i \{1-\exp(-\lambda y)\}^{4-i} \alpha \exp(-\alpha y) dy \\ &= \binom{4}{3}\alpha\int_0^\infty \{\exp(-3\lambda y)\}\{1-\exp(-\lambda y)\}\exp(-\alpha y) dy \\ &\quad + \binom{4}{4}\alpha\int_0^\infty \{\exp(-4\lambda y)\}\exp(-\alpha y) dy \\ &= \frac{4\alpha}{3\lambda+\alpha} - \frac{3\alpha}{4\lambda+\alpha}. \end{aligned} \tag{3.6}$$

Since $(R_{2:4}^{Y})'$ exists and is positive, we conclude that $R_{2:4}^{Y}$ is increasing on $[0,\infty]$ at every level of α. This result can also be seen from Figure 3.2.



3.2 Reliability curves of the 2-out-of-4: *F* system for exponential distribution with IID case.

3.3 The Reliability of a *k*-out-of-*n* System with Exchangeable Components under a Common Stress

When the random vector is exchangeable, the components should be similar in the same environment and the IID case is also included, but they affect one another within the system. Systems with exchangeable components have been studied in the literature as Bassan & Spizzichino (2005), Navarro & Rychlik (2007), Navarro & Balakrishnan (2010), Eryılmaz (2011).

Let us consider a system with n exchangeable components whose strengths are represented by a random vector $X_1, X_2, \ldots, X_n$. We assume that the random vector $X_1, X_2, \ldots, X_n$ is exchangeable for each n, if the joint distribution

$$P(X_1 \le x_1, \dots, X_n \le x_n) = P(X_{\pi(1)} \le x_1, \dots, X_{\pi(n)} \le x_n), \tag{3.7}$$

for any finite permutation $\pi(1), \dots, \pi(n)$ of the indices $\{1, 2, \dots, n\}$. Then the joint survival function is represented as

$$\overline{F}_{\mathrm{x}}(x_1, x_2, \dots, x_n) = P(X_1 > x_1, X_2 > x_2, \dots, X_n > x_n). \tag{3.8}$$

The survival function of the k-out-of-$n:F$ system with exchangeable strengths with $P(X_i \le x) = F(x)$ is given by

$$\begin{aligned}
\overline{F}_{k:n}(t) &= P(X_{k:n} > t) \\
&= 1 - \sum_{i=k}^{n} (-1)^{i-k} \binom{i-1}{k-1} \binom{n}{i} F(\underbrace{t, \dots, t}_{i}) \\
&= \sum_{i=n-k+1}^{n} (-1)^{i-n+k-1} \binom{i-1}{n-k} \binom{n}{i} \overline{F}(\underbrace{t, \dots, t}_{i})
\end{aligned} \tag{3.9}$$

for which $F(\underbrace{t, \dots, t}_{i}) = P(\max(X_1, \dots, X_i) \le t)$, $\overline{F}(\underbrace{t, \dots, t}_{i}) = P(\min(X_1, \dots, X_i) > t)$, $1 \le i \le n$, (David & Nagaraja, 2003).

Navarro et al. (2006) have shown that $\overline{F}_{k:n}(t)$ with exchangeable lifetimes is the generalized mixture of series and parallel systems.

Lemma 1 (Navarro et al., 2006): If *(X_1,…, X_n)* is exchangeable random vector, then

$$\overline{F}_{k:n}(t) = \sum_{i=n-k+1}^{n} \sum_{j=0}^{n-i} (-1)^j \frac{n!}{i!\, j!\, (n-i-j)!} P(X_1 \ge t, \dots, X_{i+j} \ge t). \tag{3.10}$$

This expression can be also written for series system as

$$\overline{F}_{k:n}(t) = \sum_{j=n-k+1}^{n} \binom{n}{j} c_{n-k+1,j} \overline{F}_{(1,j)}(t) \tag{3.11}$$

where $c_{k,j} = \sum_{i=k}^{j} (-1)^{j-i} \binom{j}{i}$. In a similar way for parallel system can be written as:

$$\overline{F}_{k:n}(t) = \sum_{j=k}^{n} \binom{n}{j} c_{k,j} \overline{F}_{(j,j)}(t). \tag{3.12}$$

Let us consider a k-out-of-n: F system with n exchangeable strengths and the strength of each component is subjected to a common random stress Y. Let also the random stress be independent of the random vector of the strengths.

Lemma 2: If $X_1, X_2, \ldots, X_n$ are exchangeable strengths, the reliability of the k-out-of-n: F system denoted by $R_{k:n}^{Y}$ under the stress Y is

$$R_{k:n}^{Y} = \int_{y} \left(\sum_{i=n-k+1}^{n} (-1)^{i-n+k-1} \binom{i-1}{n-k} \binom{n}{i} \overline{F} \mathrm{x} (\underbrace{y, \ldots, y}_{i}) \right) dG_Y(y), \tag{3.13}$$

where $G_Y(y)$ is the distribution function of the stress Y for $y > 0$.

Proof: From (3.13), we have

$$P(X_{k:n} > Y) = \int_{0}^{\infty} \left(\sum_{i=n-k+1}^{n} (-1)^{i-n+k-1} \binom{i-1}{n-k} \sum_{1 \le j_1 < \cdots < j_l \le n} P(X_{j_1} > Y, \ldots, X_{j_l} > Y \mid Y = y) \right) dG(y)$$

Then the proof follows noting that $R_{k:n}^{Y} = P(X_{k:n} > Y)$. (3.14)

The results are illustrated for the 2-out-of-3: F system consisting of multivariate Marshall-Olkin and multivariate FGM distributions in Example 3.3 and 3.4.

Example 3.3: Let X_1, X_2, X_3 have Marshall-Olkin multivariate exponential distribution with the joint survival function

$$\overline{F}_{\mathrm{x}}(x_1, x_2, ..., x_n) = \exp(-\sum_{i=1}^{n} \lambda_i x_i - \lambda_0 \max(x_1, x_2, ..., x_n)).$$

where $x_i > 0, \lambda_i > 0$ and $\lambda_0 \geq 0$ (see, Kotz et al., 2000).

The reliability of the 2-out-of-3: F system consisting of three components each have exponential (λ_1, λ_2, λ_3) marginals with the exponential stress Y with the parameter α is given by from Lemma 2:

$$\begin{aligned} R_{2:3}^{Y} &= \int_0^\infty [3\exp(-\lambda_1 y - \lambda_2 y - \lambda_0 y) - 2\exp(-\lambda_1 y - \lambda_2 y - \lambda_3 y - \lambda_0 y)]\alpha e^{-\alpha y} dy \\ &= 3\int_0^\infty e^{-y(\lambda_1+\lambda_2+\lambda_0+\alpha)} dy - 2\int_0^\infty e^{-y(\lambda_1+\lambda_2+\lambda_3+\lambda_0+\alpha)} dy \\ &= \alpha\left(\frac{3}{\lambda_1+\lambda_2+\lambda_0+\alpha} - \frac{2}{\lambda_1+\lambda_2+\lambda_3+\lambda_0+\alpha}\right). \end{aligned} \tag{3.15}$$

When we assume $\lambda_1 = \lambda_2 = \lambda_3$, then we have

$$R_{2:3}^{Y} = \left(\frac{3\alpha}{2\lambda_1+\alpha+\lambda_0} - \frac{2\alpha}{3\lambda_1+\alpha+\lambda_0}\right). \tag{3.16}$$

Since $(R_{2:3}^{Y})'$ exists and is positive, we conclude that $R_{2:3}^{Y}$ is increasing on

$[0,\infty]$, at every level of α . This result can also be seen from Figure 3.3.

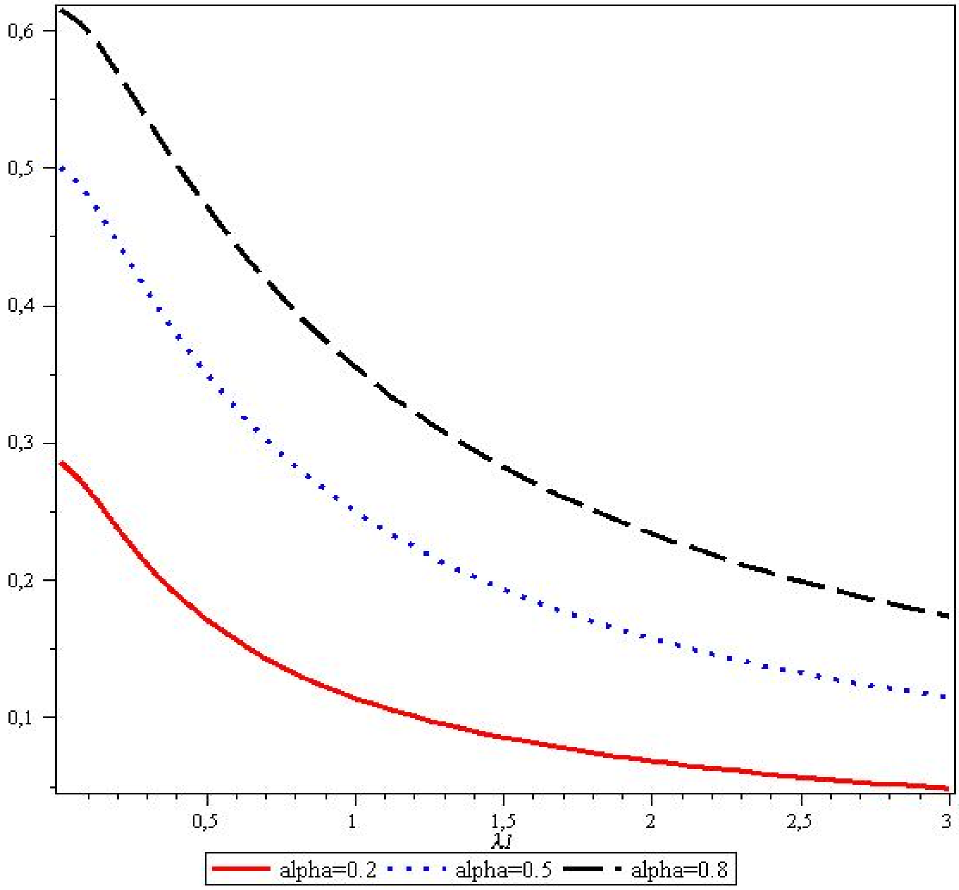


Figure 3.3 Reliability curves of the 2-out-of-3 system for Marshall & Olkin multivariate exponential distribution ($\lambda_0 = 0.5$).

Example 3.4: Let X_1, X_2, X_3 have an exchangeable FGM distribution with the joint survival function

$$\overline{F}_{\mathbf{X}}(x_1,x_2,x_3)=\prod_{i=1}^{3}\overline{F}_X(x_i)\left[1+\alpha\left(\sum_{1\le i_1<i_2\le 3}\left(1-\overline{F}_X(x_{i_1})\right)\left(1-\overline{F}_X(x_{i_2})\right)-\prod_{i=1}^{3}\left(1-\overline{F}_X(x_i)\right)\right)\right],$$

where $x_i > 0$ and α is the common dependence parameter (see, Kotz et al. (1997)). If we assume that X_1, X_2, X_3 have exponential marginals with common parameter λ_1 and the stress Y is also exponential with parameter λ_2, then we have for the reliability of a 2-out-of-3: F system from Lemma 2

$$
\begin{aligned}
R^{Y}_{2:3} &= \int_0^{\infty} \left(3\overline{F}_X(y,y) - 2\overline{F}_X(y,y,y)\right) dG_Y(y) \\
&= \frac{3\lambda_2(1+\alpha)}{2\lambda_1+\lambda_2} - \frac{2\lambda_2(1+5\alpha)}{3\lambda_1+\lambda_2} + \frac{9\lambda_2\alpha}{4\lambda_1+\lambda_2} - \frac{2\lambda_2\alpha}{6\lambda_1+\lambda_2},
\end{aligned}
\tag{3.17}
$$

where $0<\alpha<1/2$, $\lambda_1, \lambda_2 > 0$.

Since $(R^{Y}_{2:3})'$ exists and is positive, we conclude that $R^{Y}_{2:3}$ is increasing on $[0,\infty]$ at every level of λ_2. This result can also be seen from Figure 3.4.

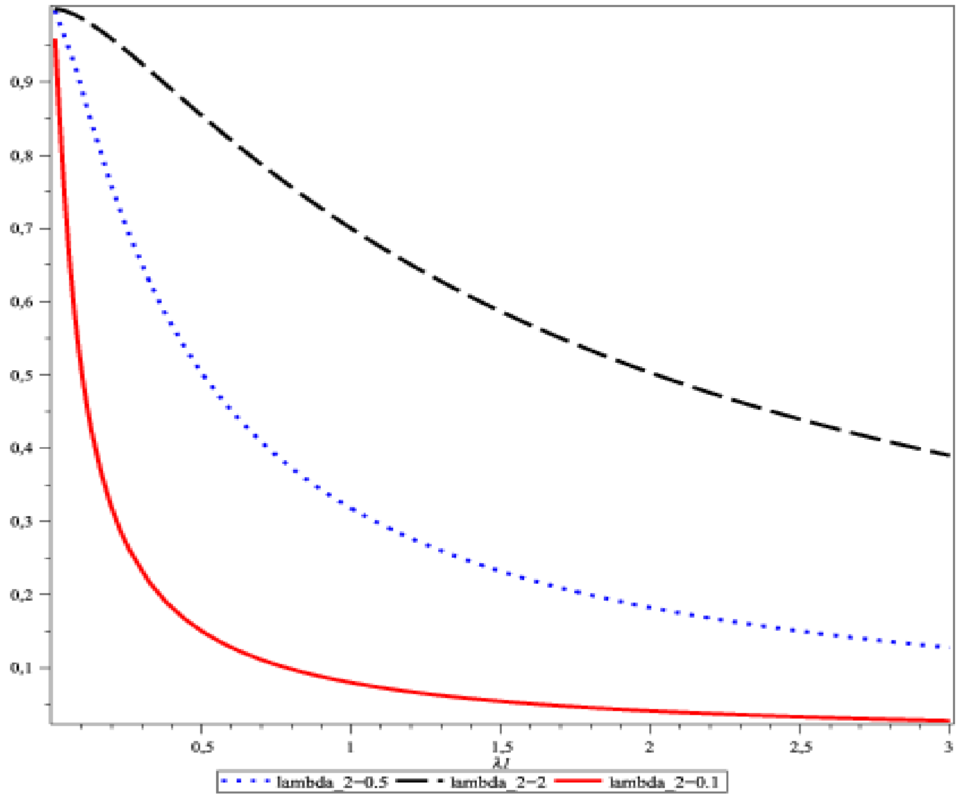


Figure 3.4 Reliability curves of the 2-out-of-3 system multivariate distribution with exponential marginals (α = 0.2).

In a special case some researchers have studied about the dependent components in the parallel system and series system using bivariate distributions. Franco & Vivo

(2002) studied the classification of the generalized mixtures of two or three exponential distributions in parallel system. Mokhlis (2006) considered the problem of the estimating the reliability of a two component parallel system under four different stress models. Ebrahimi (1982) estimated of reliability for a series stress-strength system using four models of series system. Hanagal (1996) obtained the estimation of system reliability in the series system using multivariate Pareto distributions.

Remark 1: A special case for $k = n$ is equivalent to a parallel system. In this case, Lemma 2 reduces to the reliability of the parallel system under the random stress. The reliability of the two-component parallel system which is equivalent to 2-out-of-2*: F*, can be expressed as:

$$
\begin{aligned}
R_{2:2}^{Y} &= P(X_{2:2} > Y) \\
&= 2\int_0^\infty \overline{F}_X(y)\,dG(y) - \int_0^\infty \overline{F}_{\mathbf{X}}(y,y)\,dG(y).
\end{aligned}
\tag{3.18}
$$

Remark 2: A special case for $k = 1$ is equivalent to a series system. In this case, the reliability of the two-component series system which is equivalent to 1-out-of-2: *F*, can be expressed from Lemma 2 as given below:

$$
\begin{aligned}
R_{1:2}^{Y} &= P(X_{1:2} > Y) \\
&= \int_0^\infty \overline{F}_{\mathbf{X}}(y,y)\,dG(y).
\end{aligned}
\tag{3.19}
$$

One example for the dependent case is the welding machine giving stress on both eyes of an operator. If the number of hours of operating the welding machine is less than the maximum of the strengths of two eyes, then an operator can able to work successfully. Here the strength of the two eyes is the maximum possible number of hours per day working with welding machine and the reliability is the probability that an operator can able to work successfully (Hanagal, 1996, p. 14).

The special cases are also illustrated for 2-out-of-2: F system and 1-out-of-2: F system in the following examples.

Example 3.5: Let X_1 and X_2 have Marshall-Olkin bivariate exponential distribution with the joint survival function

$$\bar{F}_{\mathbf{X}}(x_1, x_2) = \exp(-\lambda_1 x_1 - \lambda_2 x_2 - \lambda_0 \max(x_1, x_2)), \tag{3.20}$$

where $x_1 > 0$, $x_2 > 0$, $\lambda_1 > 0$, $\lambda_2 > 0$ and $\lambda_0 \geq 0$.

If we assume that X_1 and X_2 have exponential marginals with parameters λ_1 and λ_2, the stress Y is also exponential with parameter α, then we have for the reliability of a 2-out-of-2: F system from Remark 1:

$$\begin{aligned} R_{2:2}^{Y} &= \int_0^\infty \{2\exp(-\lambda_1 y - \lambda_0 y) - \exp(-\lambda_1 y - \lambda_2 y - \lambda_0 y)\}\alpha \exp(-\alpha y)dy \\ &= \alpha\left(\frac{2}{\lambda_1 + \lambda_0 + \alpha} - \frac{1}{\lambda_1 + \lambda_2 + \lambda_0 + \alpha}\right). \end{aligned} \tag{3.21}$$

Since $(R_{2:2}^{Y})'$ exists and is positive, we conclude that $R_{2:2}^{Y}$ is increasing on $[0, \infty]$ at every level of α. This result can also be seen from Figure 3.5.

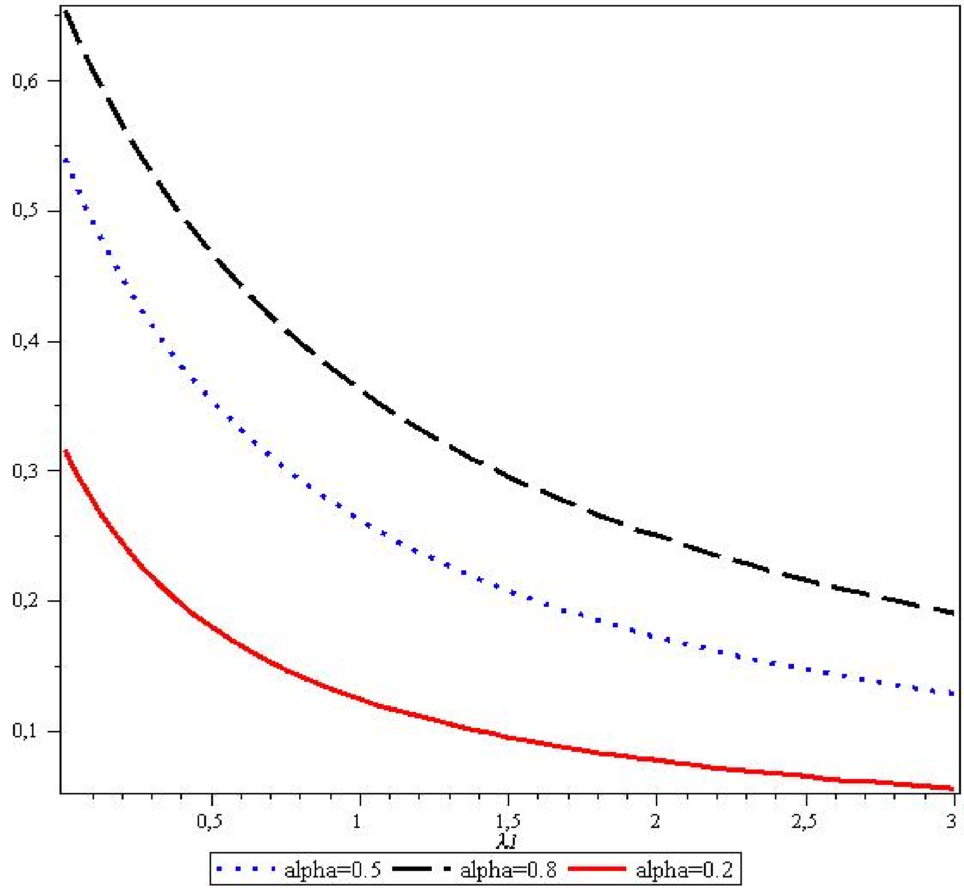


Figure 3.5 Reliability curves of the 2-out-of-2: *F* system for Marshall-Olkin bivarite exponential distribution ($\lambda_2 = 0.1,\ \lambda_0 = 0.5$).

Example 3.6: Let X_1 and X_2 have Marshall-Olkin bivariate exponential distribution with the joint survival function

$$\bar{F}_{\mathbf{X}}(x_1, x_2) = \exp(-\lambda_1 x_1 - \lambda_2 x_2 - \lambda_0 \max(x_1, x_2)), \tag{3.22}$$

where $x_1 > 0$, $x_2 > 0$, $\lambda_1 > 0$, $\lambda_2 > 0$ and $\lambda_0 \geq 0$.

If we assume that X_1 and X_2 have exponential marginals with parameters λ_1 and λ_2 and the stress Y is also exponential with the parameter α, then we have for the reliability of a 1-out-of-2: *F* system from Remark 2

$$R_{1:2}^{Y} = \int_{0}^{\infty} \left[\exp(-\lambda_1 y - \lambda_2 y - \lambda_0 y\right] \alpha \exp(-\alpha y) dy$$

$$= \alpha \left(\frac{1}{\lambda_1 + \lambda_2 + \lambda_0 + \alpha} \right). \tag{3.23}$$

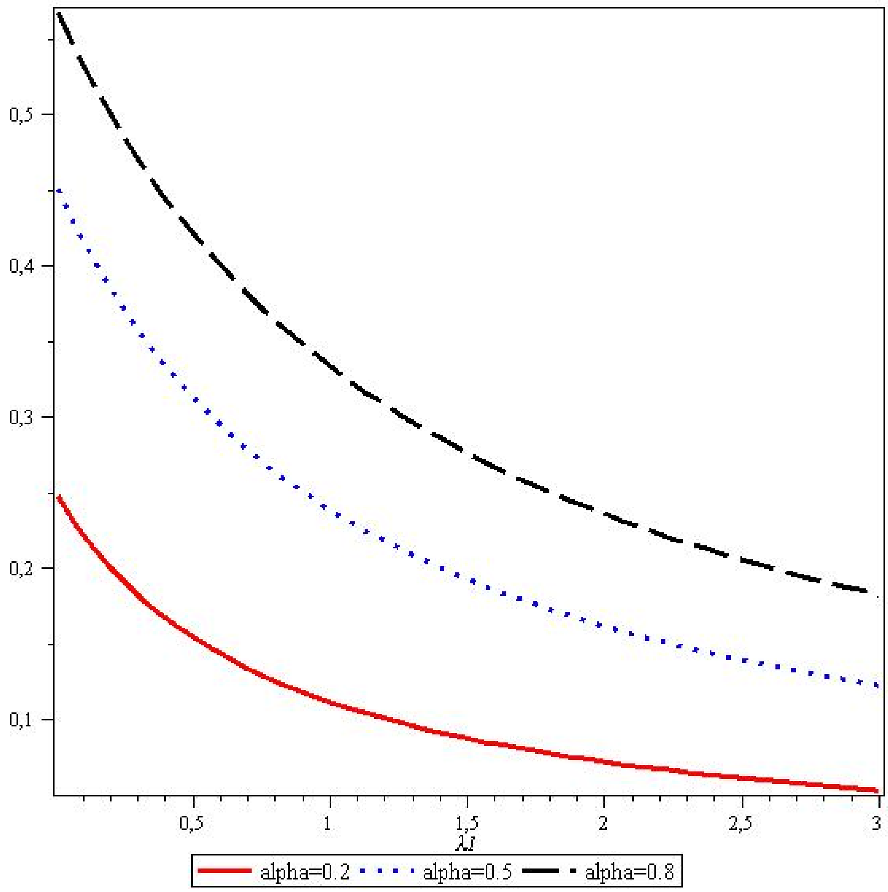


Figure 3.6 Reliability curves of the 1-out-of-2: *F* system for Marshall-Olkin bivarite exponential distribution ($\lambda_2 = 0.1, \lambda_0 = 0.5$).

Since $(R_{1:2}^{Y})'$ exists and is positive, we conclude that $R_{1:2}^{Y}$ is increasing on $[0, \infty]$ at every level of α . This result can also be seen from Figure 3.4.

CHAPTER FOUR

CONCLUSION

There are a large amount of works as regards the reliability $R=P(X>Y)$ when X and Y are independent random variables belonging to the same univariate family of distribution. In this thesis, we focus on the problem of determining the reliability in parallel, series and k-out-of-n systems consisting of n exchangeable components of strengths under the stress. Some examples which are satisfied with the results of several lifetime distribution functions are given. The reliability of systems under different stresses for each component can be considered for further research.

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