DOKUZ EYLÜL UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

ON INVESTIGATION OF TOPOLOGY FORMED FROM A GIVEN TOPOLOGY AND IDEAL

by
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İZMİR

ON INVESTIGATION OF TOPOLOGY FORMED FROM A GIVEN TOPOLOGY AND IDEAL

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ON INVESTIGATION OF TOPOLOGY FORMED FROM A GIVEN TOPOLOGY AND IDEAL

ABSTRACT

A nonempty collection of subsets of a set X which is closed under the operations of subset and finite unions defines an ideal on X. On X, a topology named ideal topology has been formed by using a given ideal f and topology τ. The set X is called an ideal topological space with the ideal topology defined on X.

In this study, the relations between ideal topologies obtained by different ideals and original topologies have been examined. Moreover the proof of some theorems has been given with details.

Keywords : Ideal, Ideal topology, Ideal topological space.

VERİLEN BİR TOPOLOJİ VE İDEAL İLE OLUŞTURULAN TOPOLOJİLERİN İNCELENMESİ

ÖZ

Bir X kümesinin alt kümelerinin kalıtsallık ve sonlu birleşim altında kapalılık özelliklerini sağlayan alt kümelerinin boştan farklı bir kolleksiyonu X kümesi üzerinde bir ideal tanımlar. X kümesi üzerinde verilen bir τ topolojisi ve f ideali kullanılarak ideal topolojisi olarak adlandırılan bir topoloji oluşturulmuştur. X kümesi, üzerinde tanımlanmış ideal topolojisi ile birlikte ideal topolojik uzayı olarak adlandırılmaktadır.

Bu çalışmada farklı ideallerden elde edilen ideal topolojileri ile orijinal topolojiler arasındaki ilişkiler incelenmiştir. Ayrıca bazı teoremlerin ispatı detaylı bir şekilde verilmiştir.

Anahtar sözcükler: İdeal, İdeal topolojisi, İdeal topolojik uzayı.

CONTENTS

CHAPTER ONE
INTRODUCTION

1.1 Introduction

Ideals in topological spaces have been taken into consideration since 1930. The tittle of an ideal in general topology was considered by Kuratowski (1966) and Vaidyanathaswamy (1960). Hayashi (1964), defined and studied the notions of dense in itself sets and perfect subsets in ideal topological spaces. The definition of the topology in terms of its derived sets was given by Hayashi, following Hashimoto (1952) and Freud (1958), and also independently by Martin (1961). Samuel (1975) made an extensive research into properties of these topologies without any limited conditions on an ideal I. Furthermore, Hashimoto (1976) examined the relation between the set of the first category and the null sets by introducing the $*$-topology to T_1-space. According to some others, topological ideals have been significant requirements for general topologists. Jankovic & Hamlet (1990) defined the concept of I-open set thanks to local function which was given by Vaidyanathaswamy (1945).

In this research, the studies of Jankovic & Hamlet (1990) were basically considered to be able to obtain ideal topologies with different ideals.

CHAPTER TWO
IDEAL TOPOLOGICAL SPACES

Ideal topological spaces are the structures which have been studied for a long time. Many studies have been made on this issue as of today.

In this chapter, some known definitions are reminded initially, and the concept of ideal, which is the non empty family of subsets of a set, is defined and some ideal examples are given. Ideal topologies are obtained in two different ways by using the given ideals in a topological space. First, they are obtained by using a new defined closure operator, then by a defined base.

Here, some thorems on the subject are noted and some unproved given theorems are proved.

2.1 Basic Definitions

Some definitions used in this study are given as follows. (Kuratowski, 1966)

1. In order that $p \in \overline{A}$ it is necessary and sufficient that each open neighbourhood E of p satisfies the inequality $A \cap E \neq \varnothing$. We may write $cl(A)$ for $\overline{A}$.
2. p is an accumulation point of the set A if $p \in \overline{A - \{p\}}$. The set A^d of accumulation points of A is called the derived set of A. $p \in A^d$ iff each open neighbourhood E of p satisfies $A \cap \left(E - \{p\}\right) \neq \varnothing$.
3. A point p is an isolated point of the set A, if $p \in A - A^d$; in other words, if $\exists U \in N(p)$ such that $U \cap A = \{p\}$.
4. A point p is said to be a condensation point of the set A, if every open neighbourhood of p contains an uncountable set of points of A. The set of condensation points of A will be denoted by $cond(A)$.
5. The boundary of a set A is the set $\partial A = \overline{A} \cap \overline{\left(X - A\right)} = \overline{A} \cap \overline{\left(A^c\right)}$.

6. The interior of a set A is the set $\operatorname{int}(A) = A^{\circ} = X - \overline{(X - A)} = \left(\overline{(A^{c})}\right)^{c}$

7. A is a dense set, if $\overline{A} = X$.

8. A is a boundary set if its complement is dense, i.e if $\overline{X - A} = X$.

9. A is a nowhere dense set if its closure is a boundary set, i.e. if $\overline{\left(X - \overline{A}\right)} = X$.

   Thus A is a nowhere dense set if $\overset{\circ}{\overline{A}} = \varnothing$.

10. A set is said to be of the first category (meager set), if it is the union of a countable sequence of nowhere dense sets.

11. A set composed exclusively of isolated points is said to be discrete, that is, $A \subseteq X$ is called discrete if $is(A) = A$ where $is(A)$ is the set of all isolated points of A.

12. A set A is said to be closed discrete iff the derived set of A is empty set, i.e $A^{d} = \varnothing$.

13. A set A is said to be dense in itself, if A contains no isolated points, i.e. if $A \subseteq A^{d}$.

14. A set A is said to be a scattered set if it contains no dense in itself non-empty subset. Every isolated set is scattered.

15. If A is closed and dense in itself, it is said to be a perfect set.

16. A family B of open sets is called a (open) base of the space if each open set can be represented as the union of elements of a subfamily of B.

2.2 The Local Function of a set

In this part, the definition of ideal and some ideal examples are given. Furthermore, the local function of a set is defined and some properties of this function are noted.

Given a space (X, τ) and a point $x \in X$, $N(x)$ will denote the open neighbourhood system at x, that is, $N(x) = \{U : U \in \tau \text{ and } x \in U\}$.

Definition 2.2.1 (Jankovic & Hamlet, 1990) A nonempty collection f of subsets of a set X is said to be an ideal on X, if it satisfies the following two conditions:

(i) $A \in f$ and $B \subseteq A \Rightarrow B \in f$ (heredity).

(ii) $A \in f$ and $B \in f \Rightarrow A \cup B \in f$ (finite additivity).

From (i), $\varnothing \in f$ for each ideal f.

Example 2.2.2 Some important ideals in a topological space (X, τ) are given by the followings:

$f_{ind} = \{\varnothing\}$

$f_f = \{A \subseteq X \mid A \text{ is finite subset of } X\}$

$f_c = \{A \subseteq X \mid A \text{ is countable subset of } X\}$

$f_{cd} = \{A \subseteq X \mid A \text{ is closed discrete subset of } X\}$

$f_n = \{A \subseteq X \mid A \text{ is nowhere dense subset of } X\}$

$f_m = \{A \subseteq X \mid A \text{ is meager set}\}$

$f_d = P(X)$ where $P(X)$ is a power set of X.

Let's show that some of these collections are ideals on X. Before showing f_{cd} is an ideal on X, we will give the following remark.

Remark 2.2.3 Let (X, τ) be topological space and $A \subseteq X$. A is closed and discrete iff $A^d = \varnothing$.

Proof. Let A be closed and discrete subset of X. Suppose that there exists $x \in X$ such that $x \in A^d$.

$x \in A^d \Rightarrow \forall U \in N(x), (U - \{x\}) \cap A \neq \varnothing,$

$A \subseteq X$ is closed $\Rightarrow \overline{A} = A \Rightarrow A^d \subseteq A \Rightarrow x \in A$ $\left(\text{Since } x \in A^d\right)$

$A \subseteq X$ is discrete $\Rightarrow (A, \tau_A)$ is a discrete subspace

$\Rightarrow \forall y \in A, \{y\} \in \tau_A$

$\Rightarrow \forall y \in A, A-\{y\}$ is τ_A- closed

$\Rightarrow \forall y \in A, A-\{y\}$ is τ- closed (Since $A \subseteq X$ is closed)

$\Rightarrow \forall y \in A,\ X-\left(A-\{y\}\right)=\left(X-A\right)\cup\{y\}\in\tau$

$\Rightarrow U=\left(X-A\right)\cup\{x\}$ is an open set containing x such that $\left(U-\{x\}\right)\cap A=\varnothing.$ This contradicts to $x\in A^d$. Thus, $A^d=\varnothing.$

To prove the converse assume $A^d=\varnothing.$ This implies that $\overline{A}=A\cup A^d=A\cup\varnothing=A.$ Thus, A is closed and $A^d=\varnothing$ also implies that $\forall x\in A,\ x\notin A^d$, that is, $x\in$ is (A). Each point of A is an isolated point of A. So A is closed and discrete set. ■

From above remark,

$f_{cd}=\left\{A\subseteq X \mid A \text{ is closed discrete subset of } X\right\}=\left\{A\subseteq X \mid A^d=\varnothing\right\}.$ Let's show that f_{cd} is an ideal on X :

(i) Let $A\in f_{cd}$ and $B\subseteq A$.

$A\in f_{cd}\Rightarrow A^d=\varnothing$

$B\subseteq A\Rightarrow B^d\subseteq A^d$

$\Rightarrow B^d=\varnothing$

Thus, $B\in f_{cd}$.

(ii) Let $A,B\in f_{cd}$.

$A,B\in f_{cd}\Rightarrow A^d=B^d=\varnothing$

$\Rightarrow\left(A\cup B\right)^d=A^d\cup B^d$

$=\varnothing\cup\varnothing=\varnothing.$

Thus, $A\cup B\in f_{cd}$.

From (i) and (ii), f_{cd} is an ideal on X .

Let's show that the collection $f_n=\left\{A\subseteq X \middle| \overset{\circ}{\overline{A}}=\varnothing\right\}$ is an ideal on X .

(i) Let $A\in f_n$ and $B\subseteq A.$

$A \in f_n \Rightarrow \overset{\circ}{\overline{A}} = \varnothing$

$B \subseteq A \Rightarrow \overset{\circ}{\overline{B}} \subseteq \overset{\circ}{\overline{A}} = \varnothing$

$\Rightarrow \overset{\circ}{\overline{B}} = \varnothing$

Thus, $B \in f_n$.

(ii) Let $A, B \in f_n$.

$A, B \in f_n \Rightarrow \overset{\circ}{\overline{A}} = \overset{\circ}{\overline{B}} = \varnothing$

$\overline{(X - A)^\circ} = \left(X - \overline{A}\right) = X - \overset{\circ}{\overline{A}}$ $\qquad$ $\overline{(X - B)^\circ} = \left(X - \overline{B}\right) = X - \overset{\circ}{\overline{B}}$

$= X - \varnothing = X.$ $\qquad$ $= X - \varnothing = X.$

So $(X - A)^\circ$ and $(X - B)^\circ$ are dense in (X, τ).

$(X - A)^\circ$ is dense in (X, τ) implies that for all $V \in \tau (V \neq \varnothing)$, $V \cap (X - A)^\circ \neq \varnothing$.

Let $W = V \cap (X - A)^\circ$, then $W \in \tau$.

Since $(X - B)^\circ$ is dense in (X, τ), then $W \cap (X - B)^\circ \neq \varnothing$.

$$\begin{aligned}
\varnothing \neq (X - B)^\circ \cap W &= (X - B)^\circ \cap V \cap (X - A)^\circ \\
&= \left[(X - A)^\circ \cap (X - B)^\circ\right] \cap V \\
&= \left[(X - A) \cap (X - B)\right]^\circ \cap V \\
&= \left[X - (A \cup B)\right]^\circ \cap V, \ \forall V \in \tau (V \neq \varnothing).
\end{aligned}$$

So $\left[X - (A \cup B)\right]^\circ$ is dense in (X, τ), that is,

$$\begin{aligned}
\overline{\left(\left[X - (A \cup B)\right]^\circ\right)} = X &\Rightarrow \overline{\left[X - \overline{(A \cup B)}\right]} = X \\
&\Rightarrow \left[X - \overset{\circ}{\overline{A \cup B}}\right] = X \\
&\Rightarrow \overset{\circ}{\overline{A \cup B}} = \varnothing \\
&\Rightarrow (A \cup B) \in f_n.
\end{aligned}$$

Thus, f_n is an ideal on X .

Finally, we show that the collection f_m is an σ -ideal on X . Before this, we give the definition of σ -ideal.

Definition 2.2.4 (Jankovic & Hamlet, 1990) Let (X,τ) be a topological space and f be an ideal on X . Then f is said to be σ -ideal if the statement "$\forall n \in \mathbb{N},\ I_n \in f \Rightarrow \bigcup_{n\in\mathbb{N}} I_n \in f$" is satisfied.

Firstly, let us show that the collection $f_m = \{A \subseteq X \mid A \text{ is meager set}\}$ is an ideal on X . We know that if A is meager set then A is of the first category. Then $A \subseteq \bigcup_{i\in\mathbb{N}} A_i$ such that $\overset{\circ}{\overline{A_i}} = \varnothing, \forall i \in \mathbb{N}.$

So $f_m = \left\{ A \subseteq X \mid A \subseteq \bigcup_{i\in\mathbb{N}} A_i \text{ where } \overset{\circ}{\overline{A_i}} = \varnothing, \forall i \in \mathbb{N} \right\}.$

(i) Let $A \in f_m$ and $B \subseteq A.$

$$A \in f_m \Rightarrow A \subseteq \bigcup_{i\in\mathbb{N}} A_i \text{ such that } \overset{\circ}{\overline{A_i}} = \varnothing, \forall i \in \mathbb{N}.$$

$$B \subseteq A \text{ and } A \subseteq \bigcup_{i\in\mathbb{N}} A_i \Rightarrow B \subseteq \bigcup_{i\in\mathbb{N}} A_i \text{ such that } \overset{\circ}{\overline{A_i}} = \varnothing, \forall i \in \mathbb{N}.$$

Thus, $B \in f_m.$

(ii) Let $A,\ B \in f_m$.

Since $A,\ B \in f_m,$ then $A \subseteq \bigcup_{i\in\mathbb{N}} A_i$ where $\overset{\circ}{\overline{A_i}} = \varnothing$ and $B \subseteq \bigcup_{j\in\mathbb{N}} B_j$ where $\overset{\circ}{\overline{B_j}} = \varnothing$ for all i, j . Thus,

$$\begin{aligned} A \cup B &\subseteq \left(\bigcup_{i\in\mathbb{N}} A_i \right) \cup \left(\bigcup_{j\in\mathbb{N}} B_j \right) \\ &= \bigcup_{k\in\mathbb{N}} (A_k \cup B_k). \end{aligned}$$

$\forall i, j \in \mathbb{N},\ \overline{\overset{\circ}{A_i}} = \varnothing$ and $\overline{\overset{\circ}{B_j}} = \varnothing \Rightarrow \forall k \in \mathbb{N},\ \overline{\overset{\circ}{A_k}} = \varnothing$ and $\overline{\overset{\circ}{B_k}} = \varnothing$. Let $k_0 \in \mathbb{N}$. Since $\overline{\overset{\circ}{A_{k_0}}} = \overline{\overset{\circ}{B_{k_0}}} = \varnothing$ and f_n is an ideal on X, then $\overline{\overset{\circ}{A_{k_0} \cup B_{k_0}}} = \varnothing$. Choosing arbitrary $k_0 \in \mathbb{N}$ implies that $\forall k \in \mathbb{N}$, $\overline{\overset{\circ}{A_k \cup B_k}} = \varnothing$. So $A \cup B \in f_m$. From (i) and (ii), we get that f_m is an ideal on X .

Finally, let us show that f_m is a σ-ideal. Thus, we take $I_n \in f_m$, for all $n \in \mathbb{N}$.

$\forall n \in \mathbb{N},\ I_n \in f_m \Rightarrow \forall n \in \mathbb{N},\ I_n \subseteq \bigcup_{i \in \mathbb{N}} A_{i,n}$ where $\overline{\overset{\circ}{A_{i,n}}} = \varnothing$, for all i.

$$\Rightarrow \bigcup_{n \in \mathbb{N}} I_n \subseteq \bigcup_{n \in \mathbb{N}} \left(\bigcup_{i \in \mathbb{N}} A_{i,n} \right) = \bigcup_{i,n} A_{i,n} \text{ where } \overline{\overset{\circ}{A_{i,n}}} = \varnothing,\ \forall i, n.$$

$$\Rightarrow \bigcup_{n \in \mathbb{N}} I_n \in f_m .$$

So f_{m} is an σ-ideal.

Example 2.2.5 Let (X, τ) be any topological space and $A \subseteq X$, then the collection $f(A) = \{B \subseteq X \mid B \subseteq A\}$ is an ideal on X . Indeed;

(i) Let $B \in f(A)$ and $C \subseteq B$.

$B \in f(A)$ and $C \subseteq B$ imply that $B \subseteq A$ and $C \subseteq B$.

Thus, $C \subseteq A$. This means that $C \in f(A)$.

(ii) Let $B, C \in f(A)$.

$B, C \in f(A)$ implies that $B \subseteq A$ and $C \subseteq A$.

Then $B \cup C \subseteq A$. Thus, $B \cup C \in f(A)$.

From (i) and (ii), we see that $f(A)$ is an ideal on X .

Example 2.2.6 If f_1 and f_2 are two ideals on a given topological space (X, τ), then $f_1 \vee f_2$ and $f_1 \wedge f_2$ are also ideals on this space.

Let's show that $f_1 \vee f_2 = \{I_1 \cup I_2 \mid I_1 \in f_1 \text{ and } I_2 \in f_2\}$ is an ideal on X .

(i) Let $A \in f_1 \vee f_2$ and $B \subseteq A$.

$A \in f_1 \vee f_2$ implies that $\exists I_1 \in f_1$ and $\exists I_2 \in f_2$ such that $A = I_1 \cup I_2$.

$B \subseteq A = I_1 \cup I_2 \Rightarrow \exists J_1 \subseteq I_1$ and $\exists J_2 \subseteq I_2$ such that $B = J_1 \cup J_2$.

$J_1 \subseteq I_1 \in f_1$ and $J_2 \subseteq I_2 \in f_2 \Rightarrow J_1 \in f_1$ and $J_2 \in f_2 \Rightarrow B = J_1 \cup J_2 \in f_1 \vee f_2$.

(ii) Let $A, B \in f_1 \vee f_2$. If $A \cup B = \varnothing$, then $A \cup B = \varnothing = \varnothing \cup \varnothing \in f_1 \vee f_2$.

If $A = \varnothing$ and $B \neq \varnothing$, then $A \cup B = B \in f_1 \vee f_2$.

If $B = \varnothing$ and $A \neq \varnothing$, then $A \cup B = A \in f_1 \vee f_2$. Assume that both A and B are non empty subsets.

$A \neq \varnothing$ and $A \in f_1 \vee f_2 \Rightarrow \exists I_1 \in f_1$ and $\exists I_2 \in f_2$ such that $A = I_1 \cup I_2$

$B \neq \varnothing$ and $B \in f_1 \vee f_2 \Rightarrow \exists J_1 \in f_1$ and $\exists J_2 \in f_2$ such that $B = J_1 \cup J_2$. From this,

$A \cup B = (I_1 \cup I_2) \cup (J_1 \cup J_2)$

$= (I_1 \cup J_1) \cup (I_2 \cup J_2)$ such that $(I_1 \cup J_1) \in f_1$ and $(I_2 \cup J_2) \in f_2$.

Thus $A \cup B \in f_1 \vee f_2$.

From (i) and (ii), $f_1 \vee f_2$ is an ideal on X .

Now we show that the collection $f_1 \wedge f_2 = \left\{ I_1 \cap I_2 \middle| I_1 \in f_1 \text{and } I_2 \in f_2 \right\}$ is an ideal on X .

(i) Let $A \in f_1 \wedge f_2$ and $B \subseteq A$.

$A \in f_1 \wedge f_2$ implies that $\exists I_1 \in f_1$ and $\exists I_2 \in f_2$ such that $A = I_1 \cap I_2$.

$B \subseteq A$ implies that $B \subseteq I_1 \cap I_2 \subseteq I_1$. We have $B \in f_1$ since $B \subseteq I_1$ and $I_1 \in f_1$.

$B \subseteq A$ also implies that $B \subseteq I_1 \cap I_2 \subseteq I_2$. We have $B \in f_2$ since $B \subseteq I_2$ and $I_2 \in f_2$. So $B \in f_1 \wedge f_2$.

(ii) Let $A, B \in f_1 \wedge f_2$. If $A \cup B = \varnothing$, then $A \cup B = \varnothing \cap \varnothing \in f_1 \wedge f_2$.

If $A = \varnothing$ and $B \neq \varnothing$, then $A \cup B = B \in f_1 \wedge f_2$.

If $B = \varnothing$ and $A \neq \varnothing$, then $A \cup B = A \in f_1 \wedge f_2$. Assume that both A and B are nonempty subsets.

$A \neq \varnothing$ and $A \in f_1 \wedge f_2 \Rightarrow \exists I_1 \in f_1$ and $\exists I_2 \in f_2$ such that $A = I_1 \cap I_2$.

$B \neq \varnothing$ and $B \in f_1 \wedge f_2 \Rightarrow \exists J_1 \in f_1$ and $\exists J_2 \in f_2$ such that $B = J_1 \cap J_2$. Then

$$A \cup B = (I_1 \cap I_2) \cup (J_1 \cap J_2) = \left[(I_1 \cap I_2) \cup J_1\right] \cap \left[(I_1 \cap I_2) \cup J_2\right]$$

$$= \underbrace{(I_1 \cup J_1)}_{\in f_1} \cap \underbrace{(I_2 \cup J_1)}_{\in f_1 \vee f_2} \cap \underbrace{(I_1 \cup J_2)}_{\in f_1 \vee f_2} \cap \underbrace{(I_2 \cup J_2)}_{\in f_2} \subseteq \underbrace{(I_1 \cup J_1)}_{\in f_1} \cap \underbrace{(I_2 \cup J_2)}_{\in f_2}.$$

Since $(I_1 \cup J_1) \cap (I_2 \cup J_2) \in f_1 \wedge f_2$ and $A \cup B \subseteq (I_1 \cup J_1) \cap (I_2 \cup J_2)$, we get $A \cup B \in f_1 \wedge f_2$ by (i).

So $f_1 \wedge f_2$ is an ideal on X.

Definition 2.2.7 (Jankovic & Hamlet, 1990) Let (X, τ) be a space with an ideal f on X and $A \subseteq X$. Then $A^*(f, \tau) = \{x \in X \mid \forall U \in N(x),\ A \cap U \notin f\}$ is called the local function of A with respect to f and τ.

We may write $A^*(f)$ simply A^* for $A^*(f, \tau)$. From definition 2.2.7, $x \in A^*(f, \tau)$ implies that "$\forall U \in N(x),\ U \cap A \neq \varnothing$".

Example 2.2.8 Let (X, τ) be any topological space and let f be an ideal on X. If $I \in f$, then

$$I^* = \{x \in X \mid \forall U \in N(x),\ U \cap I \notin f\}$$

$$\subseteq \{x \in X \mid \forall U \in N(x),\ I \notin f\} = \varnothing.$$

Thus, $\mathrm{I}^* = \varnothing$.

Example 2.2.9 Let (X, τ) be a space with the ideals $f_{ind} = \{\varnothing\}$ and $f_d = P(X)$. We have $A^*(\{\varnothing\}) = \overline{A}$ and $A^*(P(X)) = \varnothing$, for every $A \subseteq X$. Indeed;

$$A^*(\{\varnothing\}) = \{x \in X \mid \forall U \in N(x), U \cap A \notin \{\varnothing\}\}$$

$$= \{x \in X \mid \forall U \in N(x), U \cap A \neq \varnothing\} = \overline{A}$$

$$A^*(P(X)) = \{x \in X \mid \forall U \in N(x), U \cap A \notin P(X)\}$$
$$= \{x \in X \mid \forall U \in N(x), U \cap A \not\subset X\} = \varnothing.$$

Theorem 2.2.10 (Jankovic & Hamlet, 1990) Let (X,τ) be a space with ideals f_1 and f_2 on X, and let A and B be subsets of X. Then

(a) $A \subseteq B \Rightarrow A^* \subseteq B^*$,

(b) $f_1 \subseteq f_2 \Rightarrow A^*(f_2) \subseteq A^*(f_1)$,

(c) $A^* = cl(A^*) \subseteq cl(A)$ $(A^*$ is a closed subset of $cl(A))$,

(d) $(A^*)^* \subseteq A^*$,

(e) $(A \cup B)^* = A^* \cup B^*$,

(f) $A^* - B^* = (A-B)^* - B^* \subseteq (A-B)^*$,

(g) $U \in \tau \Rightarrow U \cap A^* = U \cap (U \cap A)^* \subseteq (U \cap A)^*$, and

(h) $(A \cup I)^* = A^* = (A-I)^*$ where I belongs to an arbitrary ideal on X.

Proof.

(a) We will show that " if $A \subseteq B$, then $A^* \subseteq B^*$ ". Let $A \subseteq B$ and $x \in A^*$, then

$$x \in A^* \Rightarrow \forall U \in N(x), U \cap A \notin f$$
$$\Rightarrow \forall U \in N(x), U \cap B \notin f$$
$$\Rightarrow x \in B^*.$$

Since x is arbitrary, $A^* \subseteq B^*$.

(b) Let $f_1 \subseteq f_2$ and $x \in A^*(f_2)$. We must show that $x \in A^*(f_1)$.

$$x \in A^*(f_2) \Rightarrow \forall U \in N(x), U \cap A \notin f_2$$
$$\Rightarrow \forall U \in N(x), U \cap A \notin f_1$$
$$\Rightarrow x \in A^*(f_1).$$

So $A^*(f_2) \subseteq A^*(f_1)$.

(c) Firstly, we prove that A^* is closed. So we must show that $X - A^*$ is open. Let $x \in X - A^*$, that is, $x \notin A^*$.

$x \notin A^* \Rightarrow \exists U \in N(x)$ such that $U \cap A \in f$.

Assume that $U \cap A^* \neq \emptyset$ for $U \in N(x)$.

$U \cap A^* \neq \emptyset$ for $U \in N(x) \Rightarrow \exists y \in X$ such that $y \in U \cap A^*$

$$\Rightarrow y \in U \text{ and } y \in A^*.$$

Since $U \in N(y)$ and $y \in A^*$, then $U \cap A \notin f$. This contradicts to $U \cap A \in f$. Thus, $U \cap A^* = \emptyset$, for $U \in N(x)$. Then $x \in U \subseteq (A^*)^c$ for $U \in N(x)$. This shows that $(A^*)^c$ is an open set. And so A^* is closed, that is, $cl(A^*) = A^*$.

Finally, we prove that $A^* \subseteq \overline{A}$, for any $A \subseteq X$.

Let $x \in X - \overline{A}$. It's clear that $X - \overline{A} \in N(x)$. Since $(X - \overline{A}) \cap A = \emptyset$ and $\emptyset \in f$, then $x \notin A^*$. So $X - \overline{A} \subseteq X - A^*$, *i.e,* $A^* \subseteq \overline{A}$ for any $A \subseteq X$.

We may also prove it in another way:

Let $x \in A^*$. This implies that for all $U \in N(x)$, $U \cap A \notin f$. Since $\emptyset \in f$, $U \cap A \neq \emptyset$. This means that $x \in \overline{A}$. Since x is arbitrary, then we say that $A^* \subseteq \overline{A}$.

$A^* \subseteq \overline{A}$ and A^* is closed $\Rightarrow A^* = \overline{A^*} \subseteq \overline{A}$, that is, $A^* = cl(A^*) \subseteq cl(A)$.

(d) By (c), $A^* \subseteq \overline{A}$, for all $A \subseteq X$. Let $C = A^*$ be a subset of X, then

$$C^* \subseteq \overline{C} \Rightarrow (A^*)^* \subseteq \overline{(A^*)} = A^* \quad \text{(Since } A^* \text{ is closed)}$$

$$\Rightarrow (A^*)^* \subseteq A^*.$$

(e) Let's show that $(A \cup B)^* = A^* \cup B^*$.

Since $A \subseteq A \cup B$ and $B \subseteq A \cup B$, then $A^* \subseteq (A \cup B)^*$ and $B^* \subseteq (A \cup B)^*$ from (a).

So $A^* \cup B^* \subseteq (A \cup B)^*$. (1)

We show that $(A \cup B)^* \subseteq A^* \cup B^*$. Let $x \notin A^* \cup B^*$, that is, $x \notin A^*$ and $x \notin B^*$. Thus, $\exists U_1 \in N(x)$ such that $U_1 \cap A \in f$ and $\exists U_2 \in N(x)$ such that $U_2 \cap B \in f$. Let $V = U_1 \cap U_2$. Then

$$V \cap (A \cup B) = (U_1 \cap U_2) \cap (A \cup B)$$

$$= (U_1 \cap U_2 \cap A) \cup (U_1 \cap U_2 \cap B).$$

Since $U_1 \cap A \in f$ and $U_1 \cap U_2 \cap A \subseteq U_1 \cap A$, then $U_1 \cap U_2 \cap A \in f$. And similarly, since $U_2 \cap B \in f$ and $U_1 \cap U_2 \cap B \subseteq U_2 \cap B$, then $U_1 \cap U_2 \cap B \in f$. Thus, we have $(U_1 \cap U_2 \cap A) \cup (U_1 \cap U_2 \cap B) = V \cap (A \cup B) \in f$.

Since $V \in N(x)$ and $V \cap (A \cup B) \in f$, then $x \notin (A \cup B)^*$.

So $(A \cup B)^* \subseteq A^* \cup B^*$. (2)

By (1) and (2), we have $(A \cup B)^* = A^* \cup B^*$.

(f) We will show that $A^* - B^* = (A - B)^* - B^* \subseteq (A - B)^*$. Firstly, let's show that $A^* - B^* \subseteq (A - B)^*$. Let $x \in A^* - B^*$ and we assume that $x \notin (A - B)^*$. Since $x \notin (A - B)^*$, then $\exists V_1 \in N(x)$ such that $V_1 \cap (A - B) \in f$. Also, $\exists V_2 \in N(x)$ such that $V_2 \cap B \in f$ since $x \notin B^*$. Let $U = V_1 \cap V_2$. We know $U \in N(x)$ and

$$U \cap (A - B) = (V_1 \cap V_2) \cap (A - B) \subseteq V_1 \cap (A - B) \in f \Rightarrow U \cap (A - B) \in f$$

$U \cap B = (V_1 \cap V_2) \cap B \subseteq V_2 \cap B \in f \Rightarrow U \cap B \in f$. Then

$$[U \cap (A - B)] \cup [U \cap B] = U \cap (A \cup B) = (U \cap A) \cup (U \cap B) \in f \Rightarrow U \cap A \in f$$

where $U \in N(x)$. Thus, $x \notin A^*$. This contradicts to $x \in A^* - B^*$. Therefore, $x \in (A - B)^*$, then we have $A^* - B^* \subseteq (A - B)^*$. Now we will prove that $(A^* - B^*) = (A - B)^* - B^*$.

$A^* - B^* \subseteq (A - B)^*$ implies that $[A^* - B^*] - B^* \subseteq (A - B)^* - B^*$.

Thus, $(A^* - B^*) \subseteq (A - B)^* - B^*$. (3)

Let $x \in (A-B)^* - B^*$. Then for all $U \in N(x), U \cap (A-B) \notin f$ and $x \notin B^*$. Since $U \cap (A-B) \subseteq U \cap A$ and $U \cap (A-B) \notin f$, then $U \cap A \notin f$. This implies that $x \in A^*$. So $x \in A^* - B^*$.

From this, we have $(A-B)^* - B^* \subseteq A^* - B^*$. (4)

By (3) and (4), we get $(A-B)^* - B^* = A^* - B^*$.

We can show the relation (4) in this way:

$$\begin{aligned}(A-B)^* - B^* &= \{x \in X \mid x \in (A-B)^* \text{ and } x \notin B^*\} \\ &= \{x \in X \mid \forall U \in N(x), U \cap (A-B) \notin f \text{ and } x \notin B^*\} \\ &\subseteq \{x \in X \mid \forall U \in N(x), U \cap A \notin f \text{ and } x \notin B^*\} \\ &= \{x \in X \mid x \in A^* \text{ and } x \notin B^*\} \\ &= A^* - B^*.\end{aligned}$$

Thus, $(A-B)^* - B^* \subseteq A^* - B^*$. Since the inverse of this relation is clear, we say that $(A-B)^* - B^* = A^* - B^*$.

(g) We will show that for $U \in \tau$, $U \cap A^* = U \cap (U \cap A)^* \subseteq (U \cap A)^*$. For this we take an arbitrary open set $U \in \tau$ and $x \in U \cap A^*$.

$$\begin{aligned}x \in U \cap A^* &\Rightarrow x \in U \text{ and } x \in A^* \\ &\Rightarrow x \in U \text{ and } \forall V \in N(x),\ V \cap A \notin f.\end{aligned}$$

Since $U \cap V \in N(x)$ and $x \in A^*$, $(U \cap V) \cap A \notin f$. Thus, $V \cap (U \cap A) \notin f$ for all $V \in N(x)$. This implies that $x \in (U \cap A)^*$. So $U \cap A^* \subseteq (U \cap A)^*$.

$x \in U \cap A^*$ implies that $x \in U$ and $x \in (U \cap A)^*$, then $x \in U \cap (U \cap A)^*$.

So $U \cap A^* \subseteq U \cap (U \cap A)^*$. (5)

Let $x \in U \cap (U \cap A)^*$. This implies that $x \in U$ and $x \in (U \cap A)^*$. Since $(U \cap A)^* \subseteq A^*$, then $x \in A^*$. So $x \in U \cap A^*$.

From this, we get $U \cap (U \cap A)^* \subseteq U \cap A^*$. (6)

By (5) and (6), we have $U \cap (U \cap A)^* = U \cap A^*$.

(h) Now we prove that for an arbitrary element I which is taken from an ideal f, $(A \cup I)^* = A^* = (A - I)^*$.

Let's show that $(A \cup I)^* = A^*$ for arbitrary $I \in f$. By (e),

$$
\begin{aligned}
(A \cup I)^* &= A^* \cup I^* \\
&= A^* \cup \varnothing \quad (\text{Since } \forall I \in f,\ I^* = \varnothing) \\
&= A^*.
\end{aligned}
$$

Finally, we show that $(A - I)^* = A^*$ for an arbitrary $I \in f$.

Since $A - I \subseteq A$, then

$(A - I)^* \subseteq A^*$. (7)

By (f), $(A - I)^* \supseteq A^* - I^*$

$= A^* - \varnothing$

$= A^*$.

Thus, $(A - I)^* \supseteq A^*$. (8)

By (7) and (8), we obtain $(A - I)^* = A^*$. ■

2.3 Kuratowski Closure Operator

In this part, a new Kuratowski closure operator is given followed from Jankovic & Hamlet (1990) and some properties of this operator are emphasized.

Teorem 2.2.10 clearly shows that the local function $* : P(X) \to P(X)$ satisfies the following axioms:

(1) $\varnothing^* = \varnothing$

(2) $(A \cup B)^* = A^* \cup B^*$

(3) $A^{**} \subseteq A^*$.

Now let's show that the operator $cl^* : P(X) \to P(X)$ defined as $cl^*(A) = A \cup A^*$ is a Kuratowski closure operator. Before giving the definition of this operator, let's remember how Kuratowski build a topological structure.

Definition 2.3.1 (Kuratowski, 1966) A topological space is a set X and a function (called closure) assigning to each set $A \subseteq X$ a set $\overline{A} \subseteq X$ satisfying the following four axioms:

Axiom 1. $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

Axiom 2. $A \subseteq \overline{A}$.

Axiom 3. $\overline{\varnothing} = \varnothing$.

Axiom 4. $\overline{\overline{A}} = \overline{A}$.

If, moreover, the following axiom is satisfied:

Axiom 5. $\overline{(p)} = (p)$ where $p \in X$,

the space is called a T_1-space.

From above definition, Kuratowski closure operator is a function assigning to each set $A \subseteq X$ a set $\alpha(A) \subseteq X$ satisfying the following four axioms (Kuratowski closure axioms) :

Axiom 1. $\alpha(A \cup B) = \alpha(A) \cup \alpha(B)$.

Axiom 2. $A \subseteq \alpha(A)$.

Axiom 3. $\alpha(\varnothing) = \varnothing$.

Axiom 4. $\alpha(\alpha(A)) = \alpha(A)$. And

$\Re = \{A \subseteq X \mid \alpha(A) = A\}$ is a collection of closed sets for a topology on X .

Lemma 2.3.2 (Jankovic & Hamlet, 1990) If $d : P(X) \to P(X)$ is a function satisfying

(1) $d(\varnothing) = \varnothing$,

(2) $d(A \cup B) = d(A) \cup d(B)$, and

(3) $d\left(d\left(A\right)\right) \subseteq d\left(A\right)$

then $\alpha : P\left(X\right) \to P\left(X\right)$ defined by $\alpha(A) = A \cup d\left(A\right)$ is a Kuratowski closure operator on $P\left(X\right)$ where d does not necessarily coincide with the derived set operator in the generated topology.

Proof.

Axiom 1. $\alpha\left(A \cup B\right) = \left(A \cup B\right) \cup d\left(A \cup B\right) = \left(A \cup B\right) \cup d(A) \cup d(B) = \alpha(A) \cup \alpha(B).$

Axiom 2. $A \subseteq A \cup d(A) = \alpha(A).$

Axiom 3. $\alpha\left(\varnothing\right) = \varnothing \cup d\left(\varnothing\right) = \varnothing.$

Axiom 4.
$$\begin{aligned} \alpha\left(\alpha(A)\right) &= \alpha\left(A \cup d(A)\right) \\ &= \left(A \cup d(A)\right) \cup d\left(A \cup d(A)\right) \\ &= \left(A \cup d(A)\right) \cup \left(d(A) \cup d\left(d(A)\right)\right) \\ &= A \cup d(A) \\ &= \alpha(A). \end{aligned}$$
■

The local function $* : P\left(X\right) \to P\left(X\right)$ satisfies the axioms $\left((1), (2), (3)\right)$ similar to the function $d : P\left(X\right) \to P\left(X\right)$, which is used while defining Kuraratowski closure operator, this gives that the operator $cl^{*} : P\left(X\right) \to P\left(X\right)$ defined as $cl^{*}\left(A\right) = A \cup A^{*}$ is a Kuratowski closure operator. Also, the collection $\left\{A \subseteq X \middle| cl^{*}\left(A\right) = A\right\}$ is a collection of closed sets for a topology on X. However, the topology from which we obtain the closed ones via this new closure operator is the ideal topology $\tau\left(f\right)$, where the topology τ is the original topology on X, and the collection f is an ideal on X. The obtained topology $\tau\left(f\right)$ can be given as $\tau^{*}\left(f\right)$ or τ^{*}. The topological space $\left(X, \tau\right)$ with a given ideal f is called an ideal topological space. This new space is shown as $\left(X, \tau, f\right)$ $\big($or $\left(X, \tau\left(f\right)\right)$ or

$\left(X,\tau^*(f)\right)$. Since the operator cl^* is a closure operator in $\left(X,\tau(f)\right)$, then $A\subseteq X$ is $\tau(f)$-closed $\Leftrightarrow cl^*(A)=A \Leftrightarrow X-A\in\tau(f)$ for any $A\subseteq X$.

So the ideal topology $\tau(f)$ obtained by a given ideal f and a given topology τ on X is defined as

$$\tau(f)=\left\{X-A\subseteq X \,\middle|\, cl^*(A)=A\right\}.$$

Also, the operator $cl^*:P(X)\to P(X)$, which is Kuratowski closure operator, can be shown as $\widetilde{A}=A\cup A^*$. Some properties of this operator are given by the following theorem: (We can also use the representation $\widetilde{A}$ in place of $cl^*(A)$.)

Theorem 2.3.3 (Hayashi, 1964) Let the subsets $A,B\subseteq X$ be given, then

(1) $A\subseteq B\Rightarrow \widetilde{A}\subseteq\widetilde{B}$,

(2) $\widetilde{A\cup B}=\widetilde{A}\cup\widetilde{B}$,

(3) $A\subseteq\widetilde{A}$,

(4) $\widetilde{\widetilde{A}}=\widetilde{A}$, and

(5) $\widetilde{\varnothing}=\varnothing$.

Proof. By lemma 2.3.2, we know that the operator $cl^*:P(X)\to P(X)$ defined as $cl^*(A)=A\cup A^*$ is a Kuratowski closure operator. Thus, this operator satisfies Kuratowski closure axioms. Because of this, it’s enough to check that the condition “$A\subseteq B\Rightarrow\widetilde{A}\subseteq\widetilde{B}$” holds for any $A,B\subseteq X$. But we will prove all.

(1) $\widetilde{A}=A\cup A^*$.

$\widetilde{B}=B\cup B.$

$A\subseteq B$ implies that $A^*\subseteq B^*$.

Since $A\subseteq B$ and $A^*\subseteq B^*$, then $A\cup A^*\subseteq B\cup B^*$. So $\widetilde{A}\subseteq\widetilde{B}.$

(2) $\widetilde{A\cup B}=(A\cup B)\cup(A\cup B)^*$

$$=(A\cup B)\cup\left(A^*\cup B^*\right)=\left(A\cup A^*\right)\cup\left(B\cup B^*\right)=\widetilde{A}\cup\widetilde{B}.$$

(3) $\widetilde{A} = A \cup A^* \supseteq A \Rightarrow A \subseteq \widetilde{A}.$

(4) $\widetilde{\widetilde{A}} = cl^*\left(cl^*(A)\right)$

$$= cl^*\left(A \cup A^*\right)$$

$$= \left(A \cup A^*\right) \cup \left(A \cup A^*\right)^*$$

$$= \left(A \cup A^*\right) \cup \left(A^* \cup A^{**}\right)$$

$$= A \cup A^* \quad \left(\text{Since } A^{**} \subseteq A^*\right)$$

$$= cl^*(A)$$

$$= \widetilde{A}.$$

Consequence of the condition (4) is that $\tau(f) = [\tau(f)](f)$.

(5) $\widetilde{\varnothing} = \varnothing \cup \varnothing^*$

$$= \varnothing.$$

■

We must always assume that $X \notin f$. If $X \in f$, for all $A \subseteq X$, $A \in f$ and then f is equal to $P(X)$. Now let us give some basic examples.

Example 2.3.4 If (X, τ) is any topological space, then $\tau(f_{ind}) = \tau$ and $\tau(f_d)$ is discrete topology where $f_{ind} = \{\varnothing\}$ and $f_d = P(X)$. Actually;

$f_{ind} = \{\varnothing\} \Rightarrow$ We showed that $A^*(\{\varnothing\}) = \overline{A}, \forall A \subseteq X$.

$$\tau(f_{ind}) = \tau^* = \left\{A \subseteq X \middle| cl^*(X - A) = X - A\right\}$$

$$= \left\{A \subseteq X \middle| (X - A) \cup (X - A)^* = X - A\right\}$$

$$= \left\{A \subseteq X \middle| (X - A) \cup \overline{(X - A)} = X - A\right\}$$

$$= \left\{A \subseteq X \middle| \overline{(X - A)} = (X - A)\right\}$$

$$= \left\{A \subseteq X \middle| A \in \tau\right\}$$

$$= \tau.$$

Thus, $\tau(f_{ind}) = \tau$ where $f_{ind} = \{\varnothing\}$.

$f_d = P(X) \Rightarrow$ We showed that $A^*(P(X)) = \varnothing, \forall A \subseteq X$.

$$\tau(f_d) = \tau^* = \{A \subseteq X \mid cl^*(X - A) = X - A\}$$

$$= \{A \subseteq X \mid (X - A) \cup (X - A)^* = (X - A)\}$$

$$= \{A \subseteq X \mid (X - A) \cup \varnothing = (X - A)\}$$

$$= \{A \subseteq X \mid (X - A) = (X - A)\}.$$

From this, $\tau(f_d)$ is a discrete topology where $f_d = P(X)$.

Example 2.3.5 If we take the topology τ whose base is $\mathrm{B} = \{\{2n-1,\ 2n\} : n \in \mathbb{N}\}$ and take the ideal f_f on $\mathbb{N}$, then $A^*(f_f) = \varnothing$ for all $A \subseteq \mathbb{N}$. Therefore $\tau(f_f)$ is discrete.

We assume that $A^*(f_f) \neq \varnothing$, for any $A \subseteq \mathbb{N}$. If $A^*(f_f) \neq \varnothing$, then $\exists x \in \mathbb{N}$ such that $x \in A^*(f_f)$. This implies that for all $U \in N(x)$, $U \cap A \notin f_f$. Since $\{x, x+1\}$ or $\{x-1, x\} \in \mathrm{B} \subseteq \tau$, then $\{x, x+1\}$ or $\{x-1, x\} \in N(x)$. $x \in A^*(f_f)$ and $\{x, x+1\}$ or $\{x-1, x\} \in N(x)$ imply that $\{x, x+1\} \cap A \notin f_f$ (or $\{x-1, x\} \cap A \notin f_f$) where $x \in \mathbb{N}$. Thus, $\{x, x+1\} \cap A$ (or $\{x-1, x\} \cap A$) is not finite subset of $\mathbb{N}$, which is a contradiction. So $A^*(f_f) = \varnothing$, for all $A \subseteq \mathbb{N}$. Then

$$cl^*(A) = A \cup A^*(f_f)$$

$$= A \cup \varnothing$$

$= A$, for all $A \subseteq \mathbb{N}$. And

$$\tau(f_f) = \tau^* = \{U \subseteq X \mid cl^*(X - U) = X - U\}$$

$= \{U \subseteq X \mid X - U = X - U\}$. Thus, $(\mathbb{N}, \tau(f_f))$ is a discrete space. ■

As a result of above examples, if $f = P(X)$, then $\tau(f)$ is a discrete topology. But it is not necessary that the ideal f is equal to $P(X)$ in order that the ideal topology which is formed from the ideal f is a discrete topology.

Theorem 2.3.6 Let (X, τ) be any topological space with a given ideal f. If the ideal f has the property that $\{x\} \in f$ for each $x \in X$, then the ideal topological space $(X, \tau(f))$ is T_1-space.

Proof. Let f be an ideal on X which has the property that $\{x\} \in f$ for each $x \in X$. From example 2.2.8, for each $x \in X$, $\{x\} \in f$ implies that $\{x\}^* = \varnothing$. Then $cl^*(\{x\}) = \{x\}$. This means that $\{x\}$ is $\tau(f)$-closed. Thus, $(X, \tau(f))$ is T_1-space. ■

2.4 The Base of Ideal Topology

Now, here a family is defined by using a given topology and ideal on X. Also, it is shown that this family is a topological basis on X. Furthermore, it is found out that the topology which is generated by this family coincide with the ideal topology. Later, the proof of some theorems are noted by using the basis of ideal topology. Additionally, it is concluded that the ideal topologies, which are formed from a given topology (or a given ideal) and two given ideals (or two given topologies) in the same space, are comparable within themselves while these two ideals (or these two topologies) can be compared within themselves.

Let X be a non empty set and let τ, f be a given topology and ideal on X, respectively. The non empty collection of subsets of X which is defined as $\beta(f, \tau) = \{V - I \mid V \in \tau \text{ and } I \in f\}$ is a base for any topology on X. (We will simply write β when no ambiguity is present). Indeed;

i) Since τ is a topology on X, then $X \in \tau$. f is an ideal on X, then $\varnothing \in f$. By definition of β, $X - \varnothing = X \in \beta$. Thus $\bigcup_{\alpha}(V_{\alpha} - I_{\alpha}) = X$ where $V_{\alpha} \in \tau$ and $I_{\alpha} \in f$, for all α.

ii) Let B_1 and B_2 be two sets in β, then

$B_1 \cap B_2 = \varnothing$ or $B_1 \cap B_2 \neq \varnothing$.

If $B_1 \cap B_2 = \varnothing$, then $B_1 \cap B_2 \in \beta$.

If $B_1 \cap B_2 = \varnothing$, then $B_1 \cap B_2$ can be written as a union of sets which belong to β on an empty indexing set.

Let $B_1 \cap B_2 \neq \varnothing$ where $B_1, B_2 \in \beta$.

$B_1 \in \beta$ implies that $\exists V_1 \in \tau$ and $\exists I_1 \in f$ such that $B_1 = V_1 - I_1$.

$B_2 \in \beta$ implies that $\exists V_2 \in \tau$ and $\exists I_2 \in f$ such that $B_2 = V_2 - I_2$. Then

$$
\begin{aligned}
B_1 \cap B_2 &= (V_1 - I_1) \cap (V_2 - I_2) \\
&= \left(V_1 \cap (I_1)^c\right) \cap \left((V_2 \cap (I_2)^c)\right) \\
&= (V_1 \cap V_2) \cap \left((I_1)^c \cap (I_2)^c\right) \\
&= (V_1 \cap V_2) \cap (I_1 \cup I_2)^c \\
&= (V_1 \cap V_2) - (I_1 \cup I_2) \text{ where } (V_1 \cap V_2) \in \tau \text{ and } (I_1 \cup I_2) \in f.
\end{aligned}
$$

So $B_1 \cap B_2 \in \beta$.

It's clear that β is a base for any topology on X, and the topology generated by β is τ_{β}, that is,

$$\tau_{\beta} = \left\{ \bigcup_{\alpha}(V_{\alpha} - I_{\alpha}) \mid V_{\alpha} \in \tau \text{ and } I_{\alpha} \in f, \text{ for all } \alpha \right\}.$$

Lemma 2.4.1 Let τ and f be a given topology and ideal on a non empty set X, respectively. The ideal topology $\tau(f)$ formed from a given topology and ideal is the same as with the topology τ_{β} which is generated by the collection $\beta = \{V - I \mid V \in \tau \text{ and } I \in f\}$.

Proof. We know that $\tau(f)=\{U\subseteq X \mid cl^*(X-U)=X-U\}$. Let $U\in\tau(f)$ and $x\in U$. $U\in\tau(f)$ iff $X-U$ is $\tau(f)$-closed iff $cl^*(X-U)=X-U$. This implies that $(X-U)^*\subseteq X-U$, then $U\subseteq X-(X-U)^*$. Therefore $x\in U$ implies that $x\in X-(X-U)^*$. From this, $x\notin(X-U)^*$. Since $x\notin(X-U)^*$, then $\exists V\in\tau$ $(x\in V)$ such that $V\cap(X-U)\in f$. Thus, there exists $I\in f$ such that $V\cap(X-U)=I$. Then $V\cap(X-U)\cap(I)^c=\varnothing$, that is, $(V-I)\cap(X-U)=\varnothing$ implies that $(V-I)\subseteq U$. Then $x\in(V-I)\subseteq U$ such that $(V-I)\in\beta$. Since $\forall x\in U$, $\exists B_i\in\beta$ such that $x\in B_i\subseteq U$, then $U=\bigcup_{B_i\in\beta}B_i$, that is, $U\in\tau_\beta$.

Therefore, $\tau(f)\subseteq\tau_\beta$. (9)

Let $V\in\tau_\beta$ and $x\in V$. Since β is a base for τ_β, then $\exists G\in\tau$ and $\exists Z\in f$ such that $x\in G-Z\subseteq V$. Then we have $G\cap(Z)^c\subseteq V$. This implies that $G\cap(Z)^c\cap(V)^c=\varnothing$. So $\left(G\cap(V)^c\right)\subseteq Z$ and $Z\in f$. Since f is an ideal on X, then $\left(G\cap(V)^c\right)\in f$. Hence $\left(G\cap(X-V)\right)\in f$ such that $x\in G\in\tau$. Then $x\notin(X-V)^*$, and $x\in X-(X-V)^*$. So $V\subseteq X-(X-V)^*$ and $(X-V)^*\subseteq(X-V)$. This means that $(X-V)$ is $\tau(f)$-closed. Thus, $V\in\tau(f)$.

From this, $\tau_\beta\subseteq\tau(f)$. (10)

By (9) and (10), we obtain $\tau(f)=\tau_\beta$. ■

Result 2.4.2 Let (X,τ) be any topological space and f be an ideal on X. The collection $\beta=\{V-I \mid V\in\tau \text{ and } I\in f\}$ is a base for the ideal topology $\tau(f)$.

Example 2.4.3 Let $X=\{a,b,c\}$, $\tau=\{X,\varnothing,\{a\},\{a,b\},\{a,c\}\}$ and $f=\{\varnothing,\{a\}\}$. Let's find the ideal topology $\tau(f)$ using two distinct ways:

First way: We know that the collection

$$\beta=\{V-I \mid V\in\tau \text{ and } I\in f\}$$
$$=\{X,\varnothing,\{a\},\{a,b\},\{a,c\},\{b,c\},\{b\},\{c\}\}$$

is a base for $\tau(f)$. Since for all $x \in X$, $\{x\} \in \beta \subseteq \tau(f)$, then $\tau(f)$ is a discrete topology.

Second way: $P(X) = \{\varnothing, X, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}\}$.

$\varnothing^* = \varnothing \Rightarrow cl^*(\varnothing) = \varnothing \Rightarrow X - \varnothing = X \in \tau(f)$.

$X^* = \{b,c\} \Rightarrow cl^*(X) = X \Rightarrow X - X = \varnothing \in \tau(f)$.

Let us find $\{a\}^*$;

$a \in U \in \tau \Rightarrow U = X$ or $U = \{a\}$ or $U = \{a,b\}$ or $U = \{a,c\}$

$\Rightarrow U \cap \{a\} = \{a\} \in f \Rightarrow a \notin \{a\}^*$

$b \in U \in \tau \Rightarrow U = X$ or $U = \{a,b\}$

$\Rightarrow U \cap \{a\} = \{a\} \in f \Rightarrow b \notin \{a\}^*$

$c \in U \in \tau \Rightarrow U = X$ or $U = \{a,c\}$

$\Rightarrow U \cap \{a\} = \{a\} \in f \Rightarrow c \notin \{a\}^*$.

So $\{a\}^* = \varnothing$.

$\{a\}^* = \varnothing \Rightarrow cl^*(\{a\}) = \{a\} \Rightarrow X - \{a\} = \{b,c\} \in \tau(f)$.

Let us find $\{b\}^*$;

$a \in U \in \tau \Rightarrow U = X$ or $U = \{a\}$ or $U = \{a,b\}$ or $U = \{a,c\}$

$\Rightarrow U \cap \{b\} = \{b\}$ or $U \cap \{b\} = \varnothing \Rightarrow a \notin \{b\}^*$ (Since $\varnothing \in f$)

$b \in U \in \tau \Rightarrow U = X$ or $U = \{a,b\}$

$\Rightarrow U \cap \{b\} = \{b\} \notin f \Rightarrow b \in \{b\}^*$

$c \in U \in \tau \Rightarrow U = X$ or $U = \{a,c\}$

$\Rightarrow U \cap \{b\} = \{b\}$ or $U \cap \{b\} = \varnothing \Rightarrow c \notin \{b\}^*$ (Since $\varnothing \in f$).

So $\{b\}^* = \{b\}$.

$\{b\}^* = \{b\} \Rightarrow cl^*(\{b\}) = \{b\} \Rightarrow X - \{b\} = \{a,c\} \in \tau(f)$.

Let us find $\{c\}^*$;

$a \in U \in \tau \Rightarrow U = X$ or $U = \{a\}$ or $U = \{a,b\}$ or $U = \{a,c\}$

$\Rightarrow U \cap \{c\} = \{c\}$ or $U \cap \{c\} = \varnothing \Rightarrow a \notin \{c\}^*$ (Since $\varnothing \in f$)

$b \in U \in \tau \Rightarrow U = X$ or $U = \{a,b\}$

$\Rightarrow U \cap \{c\} = \{c\}$ or $U \cap \{c\} = \varnothing \Rightarrow b \notin \{c\}^*$ (Since $\varnothing \in f$)

$c \in U \in \tau \Rightarrow U = X$ or $U = \{a, c\}$

$\Rightarrow U \cap \{c\} = \{c\} \notin f \Rightarrow c \in \{c\}^*$.

So $\{c\}^* = \{c\}$.

$\{c\}^* = \{c\} \Rightarrow cl^*(\{c\}) = \{c\} \Rightarrow X - \{c\} = \{a, b\} \in \tau(f)$.

Let us find $\{a, b\}^*$;

$a \in U \in \tau \Rightarrow U = X$ or $U = \{a\}$ or $U = \{a, b\}$ or $U = \{a, c\}$

$\Rightarrow U \cap \{a, b\} = \{a, b\}$ or $U \cap \{a, b\} = \{a\} \Rightarrow a \notin \{a, b\}^*$ (Since $\{a\} \in f$)

$b \in U \in \tau \Rightarrow U = X$ or $U = \{a, b\}$

$\Rightarrow U \cap \{a, b\} = \{a, b\} \notin f \Rightarrow b \in \{a, b\}^*$

$c \in U \in \tau \Rightarrow U = X$ or $U = \{a, c\}$

$\Rightarrow U \cap \{a, b\} = \{a, b\}$ or $U \cap \{a, b\} = \{a\} \Rightarrow c \notin \{a, b\}^*$ (Since $\{a\} \in f$).

So $\{a, b\}^* = \{b\}$.

$\{a, b\}^* = \{b\} \Rightarrow cl^*(\{a, b\}) = \{a, b\} \Rightarrow X - \{a, b\} = \{c\} \in \tau(f)$.

Let us find $\{a, c\}^*$;

$a \in U \in \tau \Rightarrow U = X$ or $U = \{a\}$ or $U = \{a, b\}$ or $U = \{a, c\}$

$\Rightarrow U \cap \{a, c\} = \{a, c\}$ or $U \cap \{a, c\} = \{a\} \Rightarrow a \notin \{a, c\}^*$ (Since $\{a\} \in f$)

$b \in U \in \tau \Rightarrow U = X$ or $U = \{a, b\}$

$\Rightarrow U \cap \{a, c\} = \{a, c\}$ or $U \cap \{a, c\} = \{a\} \Rightarrow b \notin \{a, c\}^*$ (Since $\{a\} \in f$)

$c \in U \in \tau \Rightarrow U = X$ or $U = \{a, c\}$

$\Rightarrow U \cap \{a, c\} = \{a, c\} \notin f \Rightarrow c \in \{a, c\}^*$.

So $\{a, c\}^* = \{c\}$.

$\{a, c\}^* = \{c\} \Rightarrow cl^*(\{a, c\}) = \{a, c\} \Rightarrow X - \{a, c\} = \{b\} \in \tau(f)$.

Let us find $\{b, c\}^*$;

$a \in U \in \tau \Rightarrow U = X$ or $U = \{a\}$ or $U = \{a, b\}$ or $U = \{a, c\}$

$\Rightarrow U \cap \{b, c\} = \{b, c\}$ or $\varnothing$ or $\{b\}$ or $\{c\} \Rightarrow a \notin \{b, c\}^*$ (Since $\varnothing \in f$)

$b \in U \in \tau \Rightarrow U = X$ or $U = \{a, b\}$

$\Rightarrow U \cap \{b, c\} = \{b, c\}$ or $U \cap \{b, c\} = \{b\} \Rightarrow b \in \{b, c\}^*$

$c \in U \in \tau \Rightarrow U = X$ or $U = \{a, c\}$

$$\Rightarrow U \cap \{b,c\} = \{b,c\} \text{ or } U \cap \{b,c\} = \{c\} \Rightarrow c \in \{b,c\}^{*}.$$

So $\{b,c\}^{*} = \{b,c\}$.

$\{b,c\}^{*} = \{b,c\} \Rightarrow cl^{*}(\{b,c\}) = \{b,c\} \Rightarrow X - \{b,c\} = \{a\} \in \tau(f)$.

We know that $\tau(f) = \{U \subseteq X \mid cl^{*}(X - U) = X - U\}$. From this

$\tau(f) = \{X, \varnothing, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}\}$ is a discrete topology.

The following theorem shows that the ideal topologies obtained by two given ideals in the same space are comparable within themselves while these two ideals can be compared within themselves.

Theorem 2.4.4 (Jankovic & Hamlet, 1990) Let (X, τ) be any topological space and let f_1, f_2 be two ideals on X. If $f_1 \subseteq f_2$, then $\tau(f_1) \subseteq \tau(f_2)$.

Proof.

First way: Let $f_1 \subseteq f_2$ and $U \in \tau(f_1)$. Since $U \in \tau(f_1)$, then U has a representation as $U = \bigcup_{\alpha} (V_\alpha - I_\alpha)$ where $V_\alpha \in \tau$ and $I_\alpha \in f_1$, for all α. If $f_1 \subseteq f_2$ and $I_\alpha \in f_1$, then $I_\alpha \in f_2$ for all α. So $U = \bigcup_{\alpha} (V_\alpha - I_\alpha) \in \tau(f_2)$. Since the subset U is arbitrary, then $\tau(f_1) \subseteq \tau(f_2)$.

Second way: Let f_1, f_2 be two ideals on X with $f_1 \subseteq f_2$ and let the collections

$\beta(f_1, \tau) = \{V_\alpha - I_\alpha \mid V_\alpha \in \tau \text{ and } I_\alpha \in f_1\}$

$\beta(f_2, \tau) = \{V'_\alpha - I'_\alpha \mid V'_\alpha \in \tau \text{ and } I'_\alpha \in f_2\}$ be bases for $\tau(f_1)$ and $\tau(f_2)$, respectively. To prove $\tau(f_1) \subseteq \tau(f_2)$, we must show that for all $B \in \beta(f_1, \tau)$ and for all $x \in B$, $\exists B' \in \beta(f_2, \tau)$ such that $x \in B' \subseteq B$.

Let $B \in \beta(f_1, \tau)$ and $x \in B$. This implies that $\exists V_\alpha \in \tau$ and $\exists I_\alpha \in f_1$ such that $B = V_\alpha - I_\alpha$, then $x \in V_\alpha - I_\alpha$ where $V_\alpha \in \tau$ and $I_\alpha \in f_1$. Let $\theta_\alpha \subseteq I_\alpha$. Since $\theta_\alpha \subseteq I_\alpha \in f_1$ and $f_1 \subseteq f_2$, then $I_\alpha, \theta_\alpha \in f_2$. Since f_2 is an ideal on X, then $(I_\alpha \cup \theta_\alpha) \in f_2$.

$x \in (V_\alpha - I_\alpha) \Rightarrow x \in V_\alpha$ and $x \in (I_\alpha)^c$

$\Rightarrow x \in V_\alpha$ and $x \notin I_\alpha$

$\Rightarrow x \in V_\alpha$ and $x \notin \theta_\alpha$ (Since $\theta_\alpha \subseteq I_\alpha$)

$\Rightarrow x \in V_\alpha$ and $x \notin (I_\alpha \cup \theta_\alpha)$.

Then $x \in V_\alpha - (I_\alpha \cup \theta_\alpha) \subseteq (V_\alpha - I_\alpha)$ where $V_\alpha \in \tau$ and $(I_\alpha \cup \theta_\alpha) \in f_2$. Thus, we have $B' = \left[V_\alpha - (I_\alpha \cup \theta_\alpha)\right] \in \beta(f_2, \tau)$ such that $x \in B' \subseteq B$. So $\tau(f_1) \subseteq \tau(f_2)$. ■

Corollary 2.4.5 Let (X, τ) be any topological space and let f be a given ideal on X, then $\tau \subseteq \tau(f)$.

Proof. Let f be any ideal on X. Since $f_{ind} = \{\varnothing\} \subseteq f$, then by theorem 2.4.3 $\tau(f_{ind}) \subseteq \tau(f)$, that is, $\tau \subseteq \tau(f)$. ■

The following theorem shows that the ideal topologies formed from a given ideal f and two given topologies in the same space are comparable within themselves while these two topologies can be compared within themselves.

Theorem 2.4.6 Let τ_1, τ_2 be two topologies on a non empty set X. If $\tau_1 \subseteq \tau_2$, then $\tau_1(f) \subseteq \tau_2(f)$ for a given ideal f on X.

Proof. We assume $x \in A^*(f, \tau_2)$ for any $A \subseteq X$ and let $x \in U \in \tau_1$. If $\tau_1 \subseteq \tau_2$ and $x \in U \in \tau_1$, then $x \in U \in \tau_2$. Since $x \in A^*(f, \tau_2)$, $U \cap A \notin f$. Thus, for any $U \in \tau_1$ with $x \in U$ we have $U \cap A \notin f$. This implies that $x \in A^*(f, \tau_1)$. Since x is arbitrary, $A^*(f, \tau_2) \subseteq A^*(f, \tau_1)$, $\forall A \subseteq X$.

$A^*(f, \tau_2) \subseteq A^*(f, \tau_1) \Rightarrow A \cup A^*(f, \tau_2) \subseteq A \cup A^*(f, \tau_1)$, that is, $\widetilde{A}(f, \tau_2) \subseteq \widetilde{A}(f, \tau_1)$.

So $\tau_1(f) \subseteq \tau_2(f)$. ■

Theorem 2.4.7 (Jankovic & Hamlet, 1990) Let $(X, \tau(f))$ be any ideal topological space and $A \subseteq X$. Then for all $A \subseteq X$, $A^{d^*} \subseteq A^d$ where A^{d^*} is the derived set of A

in the ideal topological space $(X,\tau(f))$ and A^d is the derived set of A in the topological space (X,τ).

Proof. Since $\tau \subseteq \tau(f)$, then $A^{d^*} \subseteq A^d$ for any $A \subseteq X$. Indeed;

$x \in A^{d^*}$ implies that for all $U \in \tau(f)$ with $x \in U$, $(U-\{x\}) \cap A \neq \emptyset$, then for all $V \in \tau$ with $x \in V$, $(V-\{x\}) \cap A \neq \emptyset$. This implies that $x \in A^d$, and so $A^{d^*} \subseteq A^d$ for any $A \subseteq X$. ■

Theorem 2.4.8 (Jankovic & Hamlet, 1990) Let (X,τ) be any topological space and let f be a given ideal on X, then $x \in A^{d^*}$ iff $x \in cl^*(A-\{x\})$ for any $A \subseteq X$.

Proof.

$(\Rightarrow)$ Let $x \in A^{d^*}$ and we assume that $\exists G \in N(x)$ such that $G \cap (A-\{x\}) \in f$. This implies that $\exists I \in f$ such that $G \cap (A-\{x\}) = I$. Thus, $[G \cap (A-\{x\})] \cap I^c = \emptyset$ where $G \in \tau$ and $I \in f$. Then $(G-I) \cap (A-\{x\}) = \emptyset$ where $x \in (G-I) \in \beta \subseteq \tau(f)$. So $x \notin A^{d^*}$. We get a contradiction. Thus, for all $G \in N(x)$, $G \cap (A-\{x\}) \notin f$, that is, $x \in (A-\{x\})^*$, then $x \in cl^*(A-\{x\})$.

$(\Leftarrow)$ Let $x \in cl^*(A-\{x\})$. Then $x \in (A-\{x\})^*$. This means that for all $U \in N(x)$, $U \cap (A-\{x\}) \notin f$. We assume that $T \cap (A-\{x\}) = \emptyset$ for $T \in \tau(f)$ with $x \in T$. Since $x \in T \in \tau(f)$, then $\exists G \in \tau$ and $\exists Z \in f$ such that $x \in G-Z \subseteq T$. Since $(G-Z) \cap (A-\{x\}) \subseteq T \cap (A-\{x\}) = \emptyset$, then $(G-Z) \cap (A-\{x\}) = \emptyset$. This implies that $G \cap (A-\{x\}) \subseteq Z \in f$. Since f is an ideal on X, then $G \cap (A-\{x\}) \in f$ where $G \in \tau$ $(x \in G)$, that is, $G \in N(x)$. Thus, $x \notin (A-\{x\})^*$. From this, $x \notin cl^*(A-\{x\})$. This contradicts to $x \in cl^*(A-\{x\})$. So for all $T \in \tau(f)$ with $x \in T$, $T \cap (A-\{x\}) \neq \emptyset$, that is, $x \in A^{d^*}$. This ends the proof. ■

By using this theorem, we obtain the following relation:

$x \in A^{d^*}$ if and only if $x \in cl^*(A-\{x\})$ if and only if $x \in (A-\{x\})^*$, and also $x \in (A-\{x\})^*$ implies that $x \in A^*$. Thus $A^{d^*} \subseteq A^*$ for all $A \subseteq X$.

Theorem 2.4.9 (Jankovic & Hamlet, 1990) Let (X,τ) be any topological space and let f be a given ideal on X . If $\{x\} \in f$ for each $x \in X$, then $A^{d^*} = A^*$ for all $A \subseteq X$.

Proof. Let f be a given ideal on X and $\{x\} \in f$ for each $x \in X$, then for any $A \subseteq X$,

$$x \in A^* \Leftrightarrow x \in \left(A-\{x\}\right)^* \quad \left(\forall x \in X, \{x\} \in f \text{ implies that } \left(A-\{x\}\right)^* = A^*\right)$$

$$\Leftrightarrow x \in cl^*\left(A-\{x\}\right)$$

$$\Leftrightarrow x \in A^{d^*}.$$

Thus, $A^* = A^{d^*}$ for all $A \subseteq X$. ■

Theorem 2.4.10 Let (X,τ) be topological space and let f be an ideal on X . If $\{x\} \in f$ for each $x \in X$, then $A^*(f) \subseteq A^d$, for all $A \subseteq X$.

Proof. Let $\{x\} \in f$ for each $x \in X$ and assume $y \notin A^d$ where $y \in X$ and $A \subseteq X$. $y \notin A^d$ implies that $\exists U \in N(y)$ such that $\left(U-\{y\}\right) \cap A = \varnothing$. Then $U \cap A \subseteq \{y\} \in f$. Since f is an ideal on X , then $U \cap A \in f$. Therefore $y \notin A^*(f)$. Then $A^*(f) \subseteq A^d$ for all $A \subseteq X$ if $\{x\} \in f$, for each $x \in X$. ■

From theorem 2.4.9 and theorem 2.4.10, if $\{x\} \in f$ for each $x \in X$, then the following relation holds for all $A \subseteq X$:

$$A^{d^*} = A^*(f) \subseteq A^d .$$

Example 2.4.11 Let (X,τ) be a discrete space and let $f_n = \{A \subseteq X \mid \overset{\circ}{\overline{A}} = \varnothing\}$ be a given ideal on X . We take $B = \{x\}$, then

$$B^*(f_n) = \{y \in X \mid \forall U \in N(y), U \cap B \notin f_n\}$$

$$= \{y \in X \mid \forall U \in N(y), \overline{\overset{\circ}{U \cap B}} \neq \varnothing\}$$

$$= \{x\}.$$

$$B^d = \{x\}^d = \left\{y \in X \mid \forall U \in N(y), \left(U - \{y\}\right) \cap \{x\} \neq \varnothing\right\} = \varnothing.$$

So we see that $B^*(f_n) = \{x\} \not\subset \varnothing = B^d$ for any subset B. From this example, if the ideal f does not have the property that $\{x\} \in f$ for each $x \in X$, then we cannot say that $A^*(f) \subseteq A^d$, for all $A \subseteq X$.

From above theorems, we have the following relations for all $A \subseteq X$:

(i) $A^{d^*} \subseteq A^d$,

(ii) $x \in A^{d^*}$ iff $x \in cl^*\left(A - \{x\}\right)$,

(iii) $A^{d^*} \subseteq A^*$,

(iv) If $\{x\} \in f$ for each $x \in X$, then $A^{d^*} = A^*$,

(v) If $\{x\} \in f$ for each $x \in X$, then $A^*(f) \subseteq A^d$.

Corollary 2.4.12 (Jankovic & Hamlet, 1990) Let $\left(X, \tau(f)\right)$ be any ideal topological space where f is an ideal on X. If $I \in f$, then $I^{d^*} = \varnothing$.

Proof. Let f be an ideal on X and $I \in f$. From example 2.2.8, $I^* = \varnothing$. Since $I^{d^*} \subseteq I^* = \varnothing$, then $I^{d^*} = \varnothing$. ■

Theorem 2.4.13 (Jankovic & Hamlet, 1990) Let (X, τ) be a space with f_1 and f_2 ideals on X, and $A \subseteq X$, then

a) $A^*(f_1 \wedge f_2) = A^*(f_1) \cup A^*(f_2)$.

b) $A^*(f_1 \vee f_2, \tau) = A^*(f_1, \tau(f_2)) \cap A^*(f_2, \tau(f_1))$.

Proof.

a) $A^*(f_1 \wedge f_2) = \{x \in X \mid \forall U \in N(x), U \cap A \notin f_1 \wedge f_2\}$

$$= \{x \in X \mid \forall U \in N(x), U \cap A \notin f_1 \text{ or } U \cap A \notin f_2\}$$

$$= \{x \in X \mid \forall U \in N(x), U \cap A \notin f_1\} \cup \{x \in X \mid \forall U \in N(x), U \cap A \notin f_2\}$$

$$= A^*(f_1) \cup A^*(f_2).$$

b) Let $x \in A^*(f_1, \tau(f_2)) \cap A^*(f_2, \tau(f_1))$, then $x \in A^*(f_1, \tau(f_2))$ and $x \in A^*(f_2, \tau(f_1))$. We assume that $\exists V \in \tau$ with $x \in V$ such that $V \cap A \in f_1 \vee f_2$. By the definition of $f_1 \vee f_2$, $\exists B_1 \in f_1$ and $\exists B_2 \in f_2$ such that $V \cap A = B_1 \cup B_2$.

(i) Let $x \in B_1$ and $x \notin B_2$.

$x \in B_1$ implies that $x \in B_1 \cup B_2 = V \cap A$.

$$V \cap A = B_1 \cup B_2 \Rightarrow (V \cap A) - \{x\} = (B_1 \cup B_2) - \{x\} \subseteq B_1 \cup B_2$$

$$\Rightarrow \left[(V \cap A) - \{x\}\right] - B_2 \subseteq B_1$$

$$\Rightarrow \left[\left[(V \cap A) \cap \{x\}^c\right] \cap (B_2)^c\right] \cap (B_1)^c = \varnothing.$$

Then $\left[(V \cap A) - B_2\right] \cap \left(\{x\} \cup B_1\right)^c = \varnothing$. This implies that $\left[(V \cap A) - B_2\right] \subseteq \{x\} \cup B_1 = B_1 \in f_1$. Then $\left[(V \cap A) - B_2\right] \in f_1$.

Since $x \in \left(V - B_2\right) \in \tau(f_2)$ and $\left[(V - B_2) \cap A\right] \in f_1$, then $x \notin A^*(f_1, \tau(f_2))$. This is a contradiction.

(ii) Let $x \in B_2$ and $x \notin B_1$. Similarly it can be shown that $x \notin A^*(f_2, \tau(f_1))$. Thus we get contradiction.

(iii) Let $x \in B_1$ and $x \in B_2$.

We take $B_3 = B_1 - \{x\}$, then $x \notin B_3$ and $B_3 \subseteq B_1 \in f_1$. Thus, $x \notin B_3 \in f_1$ and $x \in B_2 \in f_2$.

$x \in B_2$ implies that $x \in B_1 \cup B_2 = V \cap A$.

$$V \cap A = B_1 \cup B_2 \Rightarrow (V \cap A) - \{x\} = (B_1 \cup B_2) - \{x\}$$

$$= (B_1 \cup B_2) \cap \{x\}^c$$

$$= \left(B_1 \cap \{x\}^c\right) \cup \left(B_2 \cap \{x\}^c\right)$$

$$\subseteq \left(B_1 \cap \{x\}^c\right) \cup B_2$$

$$= \left(B_1 - \{x\}\right) \cup B_2 = B_3 \cup B_2.$$

Thus, $(V \cap A) - \{x\} \subseteq B_3 \cup B_2$.

$$(V \cap A) - \{x\} \subseteq B_3 \cup B_2 \Rightarrow \left[(V \cap A) - \{x\}\right] - B_3 \subseteq B_2$$

$$\Rightarrow \left[\left[(V \cap A) \cap \{x\}^c\right] \cap (B_3)^c\right] \cap (B_2)^c = \varnothing$$

$$\Rightarrow \left[(V - B_3) \cap A\right] \subseteq \{x\} \cup B_2 = B_2 \in f_2.$$

This implies that $\left[(V - B_3) \cap A\right] \in f_2$ where $(V - B_3) \in \tau(f_1)$ and $x \in (V - B_3)$.

Thus, $x \notin A^*(f_2, \tau(f_1))$, which is a contradiction.

(iv) Let $x \notin B_1$ and $x \notin B_2$. Then

$$V \cap A = B_1 \cup B_2 \Rightarrow (V \cap A) - B_1 \subseteq B_2 \in f_2 \text{ and } (V \cap A) - B_2 \subseteq B_1 \in f_1$$

$$\Rightarrow (V - B_1) \cap A \in f_2 \text{ and } (V - B_2) \cap A \in f_1$$

where $x \in (V - B_1) \in \tau(f_1)$ and $x \in (V - B_2) \in \tau(f_2)$

$$\Rightarrow x \notin A^*(f_2, \tau(f_1)) \text{ and } x \notin A^*(f_1, \tau(f_2))$$

$$\Rightarrow x \notin A^*(f_2, \tau(f_1)) \cap A^*(f_1, \tau(f_2)).$$

This contradicts to $x \in A^*(f_1, \tau(f_2)) \cap A^*(f_2, \tau(f_1))$.

From (i), (ii), (iii) and (iv), we get all contradiction. Thus, for all $V \in \tau$ with $x \in V$, $V \cap A \notin f_1 \vee f_2$, that is, $x \in A^*(f_1 \vee f_2, \tau)$. Since x is arbitrary, then

$$A^*(f_1, \tau(f_2)) \cap A^*(f_2, \tau(f_1)) \subseteq A^*(f_1 \vee f_2, \tau). \qquad (11)$$

Conversely, let's show that $A^*(f_1 \vee f_2, \tau) \subseteq A^*(f_1, \tau(f_2)) \cap A^*(f_2, \tau(f_1))$. Let $x \notin A^*(f_1, \tau(f_2))$. This implies that $\exists U \in \tau(f_2)$ with $x \in U$ such that $(U \cap A) \in f_1$. Since $x \in U \in \tau(f_2)$, then $\exists U' \in \tau$ and $\exists V' \in f_2$ such that $x \in U' - V' \subseteq U$. Then we have $(U' - V') \cap A \subseteq U \cap A \in f_1$. Since f_1 is an ideal on X, then $(U' - V') \cap A \in f_1$ where $U' \in \tau$ and $V' \in f_2$. Thus, $\exists I \in f_1$ such that $(U' - V') \cap A = I$. This implies that $U' \cap A \subseteq V' \cup I$ where $V' \in f_2$ and $I \in f_1$. By the definition of $f_1 \vee f_2$, $U' \cap A \subseteq V' \cup I \in f_1 \vee f_2$. Since $f_1 \vee f_2$ is an ideal on X, then $U' \cap A \in f_1 \vee f_2$ where $x \in U' \in \tau$. From this, $x \notin A^*(f_1 \vee f_2, \tau)$. Thus, $A^*(f_1 \vee f_2, \tau) \subseteq A^*(f_1, \tau(f_2))$. Similarly it can be shown that $A^*(f_1 \vee f_2, \tau) \subseteq A^*(f_2, \tau(f_1))$.

Thus, we get $A^*(f_1 \vee f_2, \tau) \subseteq A^*(f_1, \tau(f_2)) \cap A^*(f_2, \tau(f_1))$. (12)

By (11) and (12), we have $A^*(f_1 \vee f_2, \tau) = A^*(f_1, \tau(f_2)) \cap A^*(f_2, \tau(f_1))$. ■

Corollary 2.4.14 (Jankovic & Hamlet, 1990) Let (X, τ) be any topological space and let f be an ideal on X. Then $\tau(f) = [\tau(f)](f)$, that is, $\tau^* = \tau^{**}$.

Proof. Let $A \subseteq X$ and let f be an ideal on X.

$A^*(f, \tau) = A^*(f \vee f, \tau) = A^*(f, \tau(f)) \cap A^*(f, \tau(f)) = A^*(f, \tau(f))$.

$A^*(f, \tau) = A^*(f, \tau(f)) \Rightarrow A \cup A^*(f, \tau) = A \cup A^*(f, \tau(f))$, then $\widetilde{A} = \widetilde{\widetilde{A}}$ for all $A \subseteq X$.

Thus, $\tau(f) = [\tau(f)](f)$, that is, $\tau^* = \tau^{**}$. ■

Theorem 2.4.15 (Jankovic & Hamlet, 1990) Let (X, τ) be a space with the ideals f_1 and f_2 on X, then

(a) $\tau(f_1 \vee f_2) = [\tau(f_1)](f_2) = [\tau(f_2)](f_1)$

(b) $\tau(f_1 \vee f_2) = \tau(f_1) \vee \tau(f_2)$

(c) $\tau(f_1 \wedge f_2) = \tau(f_1) \cap \tau(f_2)$.

Proof.

(a) Let $V \in \tau(f_1 \vee f_2)$.

$V \in \tau(f_1 \vee f_2) \Rightarrow V = \bigcup_{\alpha} (G_\alpha - \theta_\alpha)$ where $G_\alpha \in \tau$ and $\theta_\alpha \in f_1 \vee f_2$ for all α.

$\theta_\alpha \in f_1 \vee f_2$ implies that $\exists Z_\alpha \in f_1$ and $\exists Z_\alpha^{'} \in f_2$ such that $\theta_\alpha = Z_\alpha \cup Z_\alpha^{'}$ for all α.

$V = \bigcup_{\alpha} (G_\alpha - \theta_\alpha) = \bigcup_{\alpha} \left[G_\alpha - \left(Z_\alpha \cup Z_\alpha^{'} \right) \right]$ where $G_\alpha \in \tau$ and $Z_\alpha \cup Z_\alpha^{'} \in f_1 \vee f_2$, then

$V = \bigcup_{\alpha} \left[(G_\alpha - Z_\alpha) - Z_\alpha^{'} \right]$ where $(G_\alpha - Z_\alpha) \in \tau(f_1)$ and $Z_\alpha^{'} \in f_2$. So $V \in [\tau(f_1)](f_2)$.

This implies that $\tau(f_1 \vee f_2) \subseteq [\tau(f_1)](f_2)$. (13)

Similarly V can be written as $V = \bigcup_{\alpha} \left[\left(G_\alpha - Z_\alpha^{'} \right) - Z_\alpha \right]$ where $\left(G_\alpha - Z_\alpha^{'} \right) \in \tau(f_2)$ and $Z_\alpha \in f_1$. So $V \in [\tau(f_2)](f_1)$, then we have

$\tau(f_1 \vee f_2) \subseteq [\tau(f_2)](f_1)$. (14)

Let $U \in [\tau(f_1)](f_2)$.

$U \in [\tau(f_1)](f_2) \Rightarrow U = \bigcup_{\alpha} (G_\alpha - Z_\alpha)$ where $G_\alpha \in \tau(f_1)$ and $Z_\alpha \in f_2$ for all α.

$\forall \alpha,\ G_\alpha \in \tau(f_1) \Rightarrow G_\alpha = \bigcup_{i} (H_{\alpha_i} - I_{\alpha_i})$ where $H_{\alpha_i} \in \tau$ and $I_{\alpha_i} \in f_1$ for all i. Thus,

$$U = \bigcup_{\alpha} (G_\alpha - Z_\alpha) = \bigcup_{\alpha} \left(\left[\bigcup_{i} (H_{\alpha_i} - I_{\alpha_i}) \right] - Z_\alpha \right) = \bigcup_{\alpha,i} \left(H_{\alpha_i} - (I_{\alpha_i} \cup Z_\alpha) \right)$$

where $H_{\alpha_i} \in \tau$ and $(I_{\alpha_i} \cup Z_\alpha) \in f_1 \vee f_2$. This implies that $U \in \tau(f_1 \vee f_2)$.

So $[\tau(f_1)](f_2) \subseteq \tau(f_1 \vee f_2)$. (15)

By (13) and (15), we get $\tau(f_1 \vee f_2) = [\tau(f_1)](f_2)$.

Similarly it can be shown that $\tau(f_1 \vee f_2) = [\tau(f_2)](f_1)$ using (14).

(b) Since $\tau(f_1 \vee f_2) = [\tau(f_1)](f_2)$ and $\tau(f) = \tau \vee \psi(f)$ where ψ is indiscrete topology, then $\tau(f_1 \vee f_2) = \tau(f_1) \vee \psi(f_2) = (\tau \vee \psi(f_1)) \vee \psi(f_2) = \tau(f_1) \vee \tau(f_2)$. The equation "$\tau(f) = \tau \vee \psi(f)$" will be shown in theorem 2.5.2.4.

(c) Let $U \in \tau(f_1 \wedge f_2)$.

$U \in \tau(f_1 \wedge f_2) \Rightarrow U = \bigcup_{\alpha} (G_\alpha - Z_\alpha)$ where $G_\alpha \in \tau$ and $Z_\alpha \in f_1 \wedge f_2$, for all α.

Since $f_1 \wedge f_2 \subseteq f_1$ and $f_1 \wedge f_2 \subseteq f_2$, then $Z_\alpha \in f_1$ and $Z_\alpha \in f_2$. Thus, $U \in \tau(f_1)$ and $U \in \tau(f_2)$. This implies that $U \in \tau(f_1) \cap \tau(f_2)$, then we get

$\tau(f_1 \wedge f_2) \subseteq \tau(f_1) \cap \tau(f_2)$. (16)

Let $V \in \tau(f_1) \cap \tau(f_2)$ and $x \in V$. Then $x \in V \in \tau(f_1)$ and $x \in V \in \tau(f_2)$. Since $x \in V$ and $V \in \tau(f_1)$, then $\exists G_1 \in \tau$ and $\exists Z_1 \in f_1$ such that $x \in G_1 - Z_1 \subseteq V$. And $x \in V \in \tau(f_2)$ implies that $\exists G_2 \in \tau$ and $\exists Z_2 \in f_2$ such that $x \in G_2 - Z_2 \subseteq V$.

$x \in G_1 - Z_1 \subseteq V \Rightarrow (G_1 - Z_1) \cap (V)^c = \varnothing$. Then

$G_1 \cap (Z_1)^c \cap (V)^c = \varnothing$.

$x \in G_2 - Z_2 \subseteq V \Rightarrow (G_2 - Z_2) \cap (V)^c = \varnothing$. Then

$G_2 \cap (Z_2)^c \cap (V)^c = \varnothing$.

Thus, $\left(G_1 \cap (Z_1)^c \cap (V)^c\right) \cup \left(G_2 \cap (Z_2)^c \cap (V)^c\right) = \varnothing$. From this,

$(V)^c \cap \left[\left(G_1 \cap (Z_1)^c\right) \cup \left(G_2 \cap (Z_2)^c\right)\right] = \varnothing,$

$(V)^c \cap \left[\left(\left(G_1 \cap (Z_1)^c\right) \cup G_2\right) \cap \left(\left(G_1 \cap (Z_1)^c\right) \cup (Z_2)^c\right)\right] = \varnothing,$

$(V)^c \cap (G_1 \cup G_2) \cap \left((Z_1)^c \cup G_2\right) \cap \left(G_1 \cup (Z_2)^c\right) \cap \left((Z_1)^c \cup (Z_2)^c\right) = \varnothing.$

This implies that $(V)^c \cap (G_1 \cup G_2) \cap G_2 \cap G_1 \cap \left((Z_1)^c \cup (Z_2)^c\right) = \varnothing,$ an $(V)^c \cap (G_2 \cap G_1) \cap (Z_1 \cap Z_2)^c = \varnothing.$ Then $x \in (G_2 \cap G_1) - (Z_1 \cap Z_2) \subseteq V$ where $(G_2 \cap G_1) \in \tau$ and $(Z_1 \cap Z_2) \in f_1 \wedge f_2$. From this, $V \in \tau(f_1 \wedge f_2)$.

So we have $\tau(f_1) \cap \tau(f_2) \subseteq \tau(f_1 \wedge f_2)$. (17)

By (16) and (17), we obtain $\tau(f_1 \wedge f_2) = \tau(f_1) \cap \tau(f_2)$. ■

2.5 Topologies from different Ideals

In this part, ideal topologies obtained by some given ideals are emphasized and some theorems related to these ideals are investigated.

2.5.1 The Ideal f_f

f_f is an ideal which contains all finite subsets of a set X, that is,

$f_f = \{A \subseteq X \mid A \text{ is a finite subset of } X\}$.

In example 2.3.5, we have showed that the ideal topology can be discrete topology when the ideal is different from $P(X)$. The following example also shows that the ideal topology can be discrete topology when $f \neq P(X)$ by using the base of ideal topology.

Example 2.5.1.1 Let $X = \mathbb{N}$ and $\tau = \{\varnothing, \mathbb{N}\} \cup \{\{1,...,n\} \mid n \in \mathbb{N}\}$,

$f_f = \{A \subseteq \mathbb{N} \mid A \text{ is a finite subset of } \mathbb{N}\}$, then $\tau(f_f)$ is a discrete topology. Actually;

$\{1\}-\{\varnothing\}=\{1\}\in\tau(f_f)$

$\{1,2\}-\{1\}=\{2\}\in\tau(f_f)$

.

.

.

$\{1,\ldots,n\}-\{1,\ldots,n-1\}=\{n\}\in\tau(f_f)$

.

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Repeating this, we have $\{n\}\in\tau(f_f)$ for all $n\in\mathbb{N}$. So $\tau(f_f)$ is a discrete topology.

Lemma 2.5.1.2 Let (X,τ) be a topological space with the ideal f_f, then $A^*(f_f)\subseteq A^d$ for all $A\subseteq X$.

Proof. Since $\{x\}\in f_f$ for each $x\in X$, then $A^*(f_f)\subseteq A^d$ for all $A\subseteq X$. Indeed, if we take an arbitrary subset $A\subseteq X$, then we can see that $A^*(f_f)\subseteq A^d$. Let $x\in A^*(f_f)$, then for every $U\in N(x)$, $U\cap A\notin f_f$. Thus $U\cap A$ is infinite. This implies that $(U-\{x\})\cap A\neq\varnothing$. So $x\in A^d$. Since x is arbitrary, then $A^*(f_f)\subseteq A^d$ for all $A\subseteq X$, but the inverse of this relation may or may not be satisfied for all $A\subseteq X$ with a given topological space (X,τ) and the given ideal f_f. ■

Example 2.5.1.3 Let (X,τ) be an indiscrete topological space where X is infinite and let f_f be a given ideal on X. Then for any $A\subseteq X$,

$$A^d=\begin{cases}\varnothing & \text{if } A=\varnothing\\ X-\{x\} & \text{if } A=\{x\}\\ X & \text{if } A \text{ has at least one element which is different from } x\end{cases}$$

We take $A\subseteq X$ which is equal to $\{x,y\}$. So $A^d=X$.

$$A^*(f_f)=\{x\in X\mid \forall U\in N(x), U\cap A\notin f_f\}$$

$$=\{x\in X\mid X\cap A=A\notin f_f\}$$

$$=\begin{cases}\varnothing & \text{if } A \text{ is finite}\\ X & \text{if } A \text{ is infinite}\end{cases}$$

A is finite, then $A^*(f_f)=\varnothing$. So $A^d = X \not\subset \varnothing = A^*(f_f)$. There exists $A\subseteq X$ such that $A^d \not\subset A^*(f_f)$. So we cannot say that $A^*(f_f)=A^d$, for all $A\subseteq X$ in any topological space (X,τ) with the ideal f_f.

Since (X,τ) is not T_1-space, we cannot say that $A^*(f_f)=A^d$ for all $A\subseteq X$. We will show that if (X,τ) is a T_1-space, then $A^*(f_f)=A^d$ for all $A\subseteq X$. In the previous studies, to be able to satisfy the existence of this equation, T_1-space has been taken.

We know that p is an accumulation point of the set A if $p\in\overline{A-\{p\}}$. So $p\in A^d$ iff each open neighbourhood E of p satisfies $A\cap(E-\{p\})\neq\varnothing$. In a T_1-space, the statement " $A\cap(E-\{p\})\neq\varnothing$ " can be replaced by the condition " $A\cap E$ is infinite" . This means that $A^d = A^*(f_f), \forall A\subseteq X$ if (X,τ) is a T_1-space.

Let's show that $A^d \subseteq A^*(f_f)$ for all $A\subseteq X$ where (X,τ) is a T_1-space. Let $p\in A^d$. We assume that E is an open neighbourhood of p such that $A\cap E$ is finite. Then $A\cap\left(E-\{p\}\right)$ is also finite and $A\cap\left(E-\{p\}\right)$ is closed. So $E-\left[A\cap\left(E-\{p\}\right)\right]$ is an open set containing p.

Let $B=E-\left[A\cap\left(E-\{p\}\right)\right]$, then

$$\begin{aligned}A\cap\left(B-\{p\}\right) &= A\cap\left(\left(E-\left[A\cap\left(E-\{p\}\right)\right]\right)-\{p\}\right)\\ &= A\cap\left(E\cap\left(A^c\cup E^c\cup\{p\}\right)\right)\cap\{p\}^c\\ &= \left(A\cap E\right)\cap\left(\left(A\cap E\right)^c\cup\{p\}\right)\cap\{p\}^c\\ &= \left(\left[(A\cap E)\cap(A\cap E)^c\right]\cup\left[(A\cap E)\cap\{p\}\right]\right)\cap\{p\}^c\end{aligned}$$

$$=\left(\varnothing\cup\left[(A\cap E)\cap\{p\}\right]\right)\cap\{p\}^c$$

$=(A\cap E)\cap\varnothing=\varnothing$ for any $B\in N(p)$. This contradicts that $p\in A^d$.

Thus, for all $E\in N(p)$, $A\cap E$ is infinite, that is, $p\in A^*(f_f)$. Then we say that $A^d\subseteq A^*(f_f)$ for all $A\subseteq X$. The inverse of this relation is always true. So the statement " $A\cap(E-\{p\})\neq\varnothing$ " can be replaced by the condition " $A\cap E$ is infinite" , that is, $A^d=A^*(f_f)$ for all $A\subseteq X$ if (X,τ) is T_1-space.

Theorem 2.5.1.4 (Jankovic & Hamlet, 1990) Let (X,τ) be topological space, then the following equivalences hold:

$$A^*(f_f)=A^d,\forall A\subseteq X \text{ iff } \tau=\tau(f_f) \text{ iff } (X,\tau) \text{ is } T_1\text{-space.}$$

Proof. Assume that $\tau=\tau(f_f)$. Then $A^{d^*}=A^d$ for every $A\subseteq X$. Since $\{x\}\in f_f$ for each $x\in X$, then $A^{d^*}=A^*(f_f)$. Therefore $A^*(f_f)=A^d,\forall A\subseteq X$. To prove the converse assume that $A^*(f_f)=A^d$ for every $A\subseteq X$, then $A\cup A^*(f_f)=A\cup A^d$. Thus, $cl^*(A)=cl(A)$. This implies that $\tau(f_f)=\tau$.

Thus, $A^*(f_f)=A^d,\forall A\subseteq X$ iff $\tau=\tau(f_f)$. (18)

Now we assume that (X,τ) is T_1-space, then we have $A^*(f_f)=A^d,\forall A\subseteq X$. Hence, $A\cup A^*(f_f)=A\cup A^d$, that is, $cl^*(A)=cl(A)$. This implies that $\tau(f_f)=\tau$. Conversely we assume that $\tau(f_f)=\tau$.

$\tau(f_f)=\tau\Rightarrow\tau(f_f)\subseteq\tau$

$\Rightarrow\forall U\in\tau(f_f),U\in\tau$

Since for each $x\in X, X-\{x\}\in\tau(f_f)$, then $X-\{x\}\in\tau$. And so $\{x\}$ *is* τ-closed. This means that (X,τ) is T_1-space.

So $\tau=\tau(f_f)$ iff (X,τ) is T_1-space. (19)

By (18) and (19), we conclude that $A^*(f_f)=A^d,\forall A\subseteq X$ iff (X,τ) is T_1-space. This completes the proof. ■

Example 2.5.1.5 Let (X,ψ) be an indiscrete topological space and f_f be a given ideal on X. Then the ideal topology $\psi(f_f)$ is finite complement topology.

Let $A \subseteq X$. Then it's easily check that

$$A^*(f_f) = \begin{cases} X & \text{if} \quad A \notin f_f \\ \varnothing & \text{if} \quad A \in f_f \end{cases}$$

$$\psi(f_f) = \{U \subseteq X \mid cl^*(X-U) = X-U\}$$

$$= \{U \subseteq X \mid (X-U)^* \subseteq X-U\}$$

$$= \{U \subseteq X \mid (X-U) \in f_f \text{ or } X-U = X\}$$

$$= \{U \subseteq X \mid (X-U) \text{ is finite}\} \cup \{\varnothing\}.$$

So $\psi(f_f)$ is a finite complement topology.

2.5.2 The Ideal f_c

f_c is an ideal which contains all countable subsets of a set X, that is, $f_c = \{A \subseteq X \mid A \text{ is a countable subset of } X\}$.

Example 2.5.2.1 Let f_c be a given ideal on X and let A be a given subset of X. Then $A^*(f_c) = cond(A)$.

$$A^*(f_c) = \{x \in X \mid \forall U \in N(x), U \cap A \notin f_c\}$$

$$= \{x \in X \mid \forall U \in N(x), U \cap A \text{ is uncountable}\}$$

$$= cond(A).$$

Since $\{x\} \in f_c$ for each $x \in X$, then $A^{d^*} = A^*(f_c)$, $\forall A \subseteq X$. Thus, $A^{d^*} = cond(A)$.

In theorem 2.5.1.4, it has been shown that the following equivalences hold: $A^*(f_f) = A^d$, $\forall A \subseteq X \Leftrightarrow \tau = \tau(f_f) \Leftrightarrow (X,\tau)$ is T_1-space.

However, theorem 2.5.1.4 is not always true when an ideal f which has the property that $\{x\} \in f$ for each $x \in X$ and any topological space (X,τ) are given. In the following example, it was indicated that for any subset A, the equation of "$A^*(f) = A^d$" is not satisfied when a T_1-space and an ideal f which has the property that $\{x\} \in f$ for each $x \in X$ are given.

Example 2.5.2.2 Let $(\mathbb{R}, U)$ be usual space and $S = \left\{ \frac{1}{n} \mid n \in \mathbb{N} \right\}$ be a subset of $\mathbb{R}$ and f_c be a given ideal on $\mathbb{R}$. Then $S^d \not\subset S^*(f_c)$.

We know that $S^d = \{0\}$. So $0 \in S^d$. Let $0 \in U \in \tau$. Since $U \cap S \subseteq S$ and S is a countable subset of $\mathbb{R}$, then $U \cap S$ is also countable subset of $\mathbb{R}$.

$U \in N(0), U \cap S$ is countable $\Rightarrow U \in N(0),\ U \cap S \in f_c$

$$\Rightarrow 0 \notin S^*(f_c).$$

This implies that $S^d \not\subset S^*(f_c)$.

Under the following conditions:

(i) $(\mathbb{R}, U)$ is T_1-space,

(ii) $\forall x \in \mathbb{R}, \{x\} \in f_c$

we showed that there exists $S \subseteq \mathbb{R}$ such that $S^d \not\subset S^*(f_c)$, that is, $S^d \neq S^*(f_c)$. So if (X, τ) is T_1-space and $\{x\} \in f$ for each $x \in X$, then it is not necessary that $A^*(f) = A^d, \forall A \subseteq X$. However, if an ideal f which has the property that $\{x\} \in f$ for each $x \in X$ is given, then

$$A^*(f) = A^d,\ \forall A \subseteq X \Leftrightarrow \tau = \tau(f) \Rightarrow (X, \tau) \text{ is } T_1\text{-space}.$$

If the statement "(X, τ) is T_1-space implies that $\tau = \tau(f)$ when $\{x\} \in f$ for each $x \in X$." is true, then theorem 2.5.1.4 would be true. In this case, for all $A \subseteq X$, $A^*(f) = A^d$. It will contradict with the example 2.5.2.2. So, if $\{x\} \in f$ for each $x \in X$, then it is not necessary that "(X, τ) is T_1-space implies that $\tau = \tau(f)$". Therefore, theorem 2.5.1.4 may or may not be true when $\{x\} \in f$ for each $x \in X$.

Theorem 2.5.2.3 (Karaçay, 1982) Let (X, τ) be topological space and δ be a subbase of τ, then τ is the coarsest topology containing δ.

From above theorem, the supremum of the set $\{\tau_i \mid i \in I\}$ is the topology such that it's subbase is the family which is the union of given topologies. The topology $\tau \vee \psi(f)$, which is the supremum of τ and $\psi(f)$, is the smallest topology containing $\tau \cup \psi(f)$. So the supremum $\tau \vee \psi(f)$ of τ and $\psi(f)$ is the topology of which subbase is the family $\delta = \tau \cup \psi(f)$.

Theorem 2.5.2.4 (Jankovic & Hamlet, 1990) Let (X, τ) be any topological space and let f be an ideal on X. Then $\tau(f) = \tau \vee \psi(f)$ where ψ denotes the indiscrete topology.

Proof. We know that $\beta = \{G - Z \mid G \in \tau \text{ and } Z \in f\}$ is a base for $\tau(f)$,

$\beta' = \left\{ \bigcap_{j \in J} S_j \mid J \text{ is a finite indexing set and } S_j \in \delta, \text{ for } j \in J \right\}$ is a base for $\tau \vee \psi(f)$ where $\delta = \tau \cup \psi(f)$ and $\beta'' = \{X - M \mid M \in f\} \cup \{\varnothing\}$ is a base for $\psi(f)$. Let $V \in \tau \vee \psi(f)$ and $x \in V$. Then $\exists V' \in \beta'$ such that $x \in V' \subseteq V$.

$V' \in \beta' \Rightarrow$ For a finite indexing set J, $V' = \bigcap_{j \in J} S_j$ where $S_j \in \delta$, for $j \in J$. Then

$V' = \bigcap_{j \in J} S_j$ where $S_j \in \tau \cup \psi(f)$, for $j \in J$,

$V' = \bigcap_{j \in J} S_j$ where $S_j \in \tau$ or $S_j \in \psi(f)$, for $j \in J$.

i) Let $J = \{1, ..., n\}$ and $S_j \in \tau$, for all $j = 1, ..., n$ where $n \in \mathbb{N}$. Then

$V' = \bigcap_{j \in J} S_j = S_1 \cap ... \cap S_n$ where $S_j \in \tau$, for all $j = 1, ..., n$.

$x \in V' \subseteq V \Rightarrow x \in S_1 \cap ... \cap S_n \subseteq V \quad (S_1, ..., S_n \in \tau \Rightarrow S_1 \cap ... \cap S_n \in \tau)$

$\Rightarrow x \in \left[(S_1 \cap ... \cap S_n) - \varnothing\right] \subseteq V$ where $\left[(S_1 \cap ... \cap S_n) - \varnothing\right] \in \beta$

Since β is a base for $\tau(f)$, then V is an element of $\tau(f)$.

ii) Let $J = \{1, ..., n\}$ and $S_j \in \psi(f)$, for all $j = 1, ..., n$ where $n \in \mathbb{N}$. Then

$V' = \bigcap_{j \in J} S_j = S_1 \cap ... \cap S_n$ where $\mathrm{S_j} \in \psi(f)$, for all $j = 1, ..., n$.

$x \in V'$ implies that $x \in S_1 \cap ... \cap S_n$ where $\mathrm{S_j} \in \psi(f)$, for all $j = 1, ..., n$. Thus,

$x \in S_j$ where $S_j \in \psi(f)$, for all $j = 1, ..., n$. Then

$\exists M_j \in f$ such that $x \in X - M_j \subseteq S_j$, for all $j = 1, ..., n$. This implies that

$x \in (X - M_1) \cap (X - M_2) \cap ... \cap (X - M_n) \subseteq S_1 \cap S_2 \cap ... \cap S_n$,

$x \in X \cap (M_1 \cup M_2 \cup ... \cup M_n)^c \subseteq S_1 \cap S_2 \cap ... \cap S_n$,

$x \in X - (M_1 \cup M_2 \cup ... \cup M_n) \subseteq S_1 \cap S_2 \cap ... \cap S_n$,

$x \in X - (M_1 \cup M_2 \cup ... \cup M_n) \subseteq S_1 \cap S_2 \cap ... \cap S_n = V' \subseteq V$,

$x \in \left[X - (M_1 \cup M_2 \cup ... \cup M_n)\right] \subseteq V$ where $[X - (M_1 \cup M_2 \cup ... \cup M_n)] \in \beta$.

Then V is an element of $\tau(f)$.

iii) Let $J = \{1, ..., n\}$ and $S_1, S_2, ..., S_{n_0} \in \tau$ and $S_{n_0+1}, S_{n_0+2}, ..., S_n \in \psi(f)$ where $n_0 \in \mathbb{N}$ such that $n_0 \le n \in \mathbb{N}$. Then

$V' = \bigcap_{j \in J} S_j = S_1 \cap S_2 \cap ... \cap S_{n_0} \cap S_{n_0+1} \cap S_{n_0+2} \cap ... \cap S_n$ and $x \in V' \subseteq V$.

$x \in V'$ implies that $x \in S_1 \cap S_2 \cap ... \cap S_{n_0} \cap S_{n_0+1} \cap S_{n_0+2} \cap ... \cap S_n$. Thus,

$x \in S_{n_0+1}, x \in S_{n_0+2}, ..., x \in S_n$ where $S_{n_0+1}, S_{n_0+2}, ..., S_n \in \psi(f)$. Then

$\exists M_i \in f$ such that $x \in X - M_i \subseteq S_i$, $i = n_0 + 1, n_0 + 2, ..., n$, that is,

$x \in (X - M_{n_0+1}) \subseteq S_{n_0+1}$, $x \in (X - M_{n_0+2}) \subseteq S_{n_0+2}, ...,$ $x \in (X - M_n) \subseteq S_n$

where $M_{n_0+1}, M_{n_0+2}, ..., M_n \in f$. Then

$x \in S_1, x \in S_2, ..., x \in S_{n_0}, x \in (X - M_{n_0+1}), x \in (X - M_{n_0+2}), ..., x \in (X - M_n)$ implies

$x \in S_1 \cap S_2 \cap ... \cap S_{n_0} \cap (X - M_{n_0+1}) \cap (X - M_{n_0+2}) \cap ... \cap (X - M_n) \subseteq V' \subseteq V$,

$x \in S_1 \cap S_2 \cap ... \cap S_{n_0} \cap (X - M_{n_0+1}) \cap (X - M_{n_0+2}) \cap ... \cap (X - M_n) \subseteq V$,

$x \in (S_1 \cap S_2 \cap ... \cap S_{n_0}) \cap (M_{n_0+1} \cup M_{n_0+2} \cup ... \cup M_n)^c \subseteq V$,

$x \in (S_1 \cap S_2 \cap ... \cap S_{n_0}) - (M_{n_0+1} \cup M_{n_0+2} \cup ... \cup M_n) \subseteq V$

where $(S_1 \cap S_2 \cap ... \cap S_{n_0}) \in \tau$ and $(M_{n_0+1} \cup M_{n_0+2} \cup ... \cup M_n) \in f$. Thus,

$x \in \left[(S_1 \cap S_2 \cap ... \cap S_{n_0}) - (M_{n_0+1} \cup M_{n_0+2} \cup ... \cup M_n)\right] \subseteq V$

where $\left[(S_1 \cap S_2 \cap ... \cap S_{n_0}) - (M_{n_0+1} \cup M_{n_0+2} \cup ... \cup M_n)\right] \in \beta$.

From this, V is an element of $\tau(f)$.

By i), ii), iii), we get $\tau \vee \psi(f) \subseteq \tau(f)$. (20)

Let's show that $\tau(f) \subseteq \tau \vee \psi(f)$.

Let $U \in \tau(f)$ and $x \in U$. Then $\exists G \in \tau$ and $\exists Z \in f$ such that $x \in G - Z \subseteq U$. $Z \in f$ implies that $X - Z \in \beta^{''} \subseteq \psi(f)$. Then $X - Z \in \psi(f) \subseteq \tau \cup \psi(f)$. From this, $X - Z \in \tau \cup \psi(f)$.

$G \in \tau \subseteq \tau \cup \psi(f)$ implies that $G \in \tau \cup \psi(f)$.

Then $X - Z, G \in \tau \cup \psi(f) = \delta$. By the definition of $\beta^{'}$, $(X - Z) \cap G \in \beta^{'}$.

$x \in G - Z \subseteq U$ then $x \in G \cap (X - Z) \subseteq U$ where $G \cap (X - Z) \in \beta^{'}$. This implies that $U \in \tau \vee \psi(f)$.

Thus, $\tau(f) \subseteq \tau \vee \psi(f)$. (21)

By (20) and (21), we obtain $\tau(f) = \tau \vee \psi(f)$. ■

Example 2.5.2.5 Let (X, ψ) be an indiscrete topological space with the given ideal f_c. Then the ideal topology $\psi(f_c)$ is countable complement topology. Actually;

Let $A \subseteq X$. Then it's clear that

$$A^*(f_c) = \begin{cases} \varnothing & \text{if} \quad A \in f_c \\ X & \text{if} \quad A \notin f_c \end{cases}$$

So

$$\begin{aligned} \psi(f_c) &= \{U \subseteq X \mid cl^*(X - U) = X - U\} \\ &= \{U \subseteq X \mid (X - U)^* \subseteq X - U\} \\ &= \{U \subseteq X \mid (X - U) \in f_c \text{ or } X - U = X\} \\ &= \{U \subseteq X \mid (X - U) \text{ is countable}\} \cup \{\varnothing\} \end{aligned}$$

is a countable complement topology.

Example 2.5.2.6 Let (X, τ) be topological space and f_c be a given ideal on X, then $\tau(f_c) = \tau \vee \psi(f_c)$ where ψ is indiscrete topology on X.

If we take the ideal f_c in theorem 2.5.2.4, then we have $\tau(f_c) = \tau \vee \psi(f_c)$. It's shown in the following:

We know that $\beta = \{G - Z \mid G \in \tau \text{ and } Z \in f_c\}$ is a base for $\tau(f_c)$,

$\beta' = \left\{ \bigcap_{j\in J} S_j \mid J \text{ is a finite indexing set and } S_j \in \delta,\ \text{for } j \in J \right\}$ is a base for $\tau \vee \psi(f_c)$

where $\delta = \tau \cup \psi(f_c)$.

Let $V \in \tau \vee \psi(f_c)$ and $x \in V$, then $\exists V' \in \beta'$ such that $x \in V' \subseteq V$.

$V' \in \beta'$ implies that for a finite indexing set J, $V' = \bigcap_{j\in J} S_j$ where $S_j \in \delta$ for $j \in J$,

then $V' = \bigcap_{j\in J} S_j$ where $S_j \in \tau \cup \psi(f_c),$ for $j \in J$,

$V' = \bigcap_{j\in J} S_j$ where $S_j \in \tau$ or $S_j \in \psi(f_c),$ for $j \in J$.

i) Let $J = \{1, \ldots, n\}$ and $S_j \in \tau$, for all $j = 1, \ldots, n$ where $n \in \mathbb{N}$, then

$V' = \bigcap_{j\in J} S_j = S_1 \cap \ldots \cap S_n$ where $S_j \in \tau,$ for all $j = 1, \ldots, n.$

$x \in V' \subseteq V \Rightarrow x \in S_1 \cap \ldots \cap S_n \subseteq V \quad (S_1, \ldots, S_n \in \tau \Rightarrow S_1 \cap \ldots \cap S_n \in \tau)$

$\Rightarrow x \in [(S_1 \cap \ldots \cap S_n) - \varnothing] \subseteq V$ where $[(S_1 \cap \ldots \cap S_n) - \varnothing] \in \beta$

Since β is a base for $\tau(f_c),$ then V is an elemen of $\tau(f_c).$

ii) Let $J = \{1, \ldots, n\}$ and $S_j \in \psi(f_c)$, for all $j = 1, \ldots, n$ where $n \in \mathbb{N}$, then

$V' = \bigcap_{j\in J} S_j = S_1 \cap \ldots \cap S_n$ where $S_j \in \psi(f_c),$ for all $j = 1, \ldots, n$.

$x \in V'$ implies that $x \in S_1 \cap \ldots \cap S_n$ where $S_j \in \psi(f_c),$ for all $j = 1, \ldots, n$. Then $x \in S_j$ where $S_j \in \psi(f_c),$ for all $j = 1, \ldots, n$. Since $\psi(f_c)$ is cocountable topology, then $X - S_j$ is countable for all $j = 1, \ldots, n$. Thus,

$(X - S_1) \cup (X - S_2) \cup \ldots \cup (X - S_n)$ is countable, that is,

$X - (S_1 \cap S_2 \cap \ldots \cap S_n) = X - V'$ is countable. This implies that $X - V$ is countable.

Let $U \in N(x)$. Since $U \cap (X - V) \subseteq (X - V)$ and $X - V$ is countable, then $U \cap (X - V)$ is also countable, then $U \cap (X - V) \in f_c$. Thus, $x \notin (X - V)^*$. This implies that $x \in X - (X - V)^*$. From this, $V \subseteq X - (X - V)^*$. Since

$(X-V)^{*}\subseteq(X-V)$, then $(X-V)$ is $\tau(f_c)$-closed. Therefore, V is open in $\tau(f_c)$.

iii) Let $J=\{1,...,n\}$ and $S_1,S_2,...,S_{n_0}\in\tau$ and $S_{n_0+1},S_{n_0+2},...,S_n\in\psi(f_c)$ where $n_0\in\mathbb{N}$ such that $n_0\le n\in\mathbb{N}$. We can easily show that $V\in\tau(f_c)$ as in the theorem 2.5.2.4.

So $\tau\vee\psi(f_c)\subseteq\tau(f_c)$. (22)

Let's show that $\tau(f_c)\subseteq\tau\vee\psi(f_c)$.

Let $U\in\tau(f_c)$ and $x\in U$, then $\exists G\in\tau$ and $\exists Z\in f_c$ such that $x\in G-Z\subseteq U$.

$Z\in f_c$ implies that $X-Z\in\psi(f_c)\subseteq\tau\cup\psi(f_c)$. Then $X-Z\in\tau\cup\psi(f_c)$.

$G\in\tau\subseteq\tau\cup\psi(f_c)$ implies that $G\in\tau\cup\psi(f_c)$.

Thus $X-Z,G\in\tau\cup\psi(f_c)=\delta$.

$X-Z,G\in\tau\cup\psi(f_c)=\delta$ implies that $(X-Z)\cap G\in\beta'$.

Since $x\in G-Z\subseteq U$, then $x\in G\cap(X-Z)\subseteq U$ where $G\cap(X-Z)\in\beta'$.

This implies that $U\in\tau\vee\psi(f_c)$, then $\tau(f_c)$ is coarser than $\tau\vee\psi(f_c)$, that is, $\tau(f_c)\subseteq\tau\vee\psi(f_c)$. (23)

By (22) and (23), we get $\tau(f_c)=\tau\vee\psi(f_c)$. ■

2.5.3 The ideal f_{cd}

f_{cd} is a family of all closed discrete subsets of a set X which is defined as $f_{cd}=\{A\subseteq X \mid A^d=\varnothing\}$.

Lemma 2.5.3.1 (Jankovic & Hamlet, 1990) Let (X,τ) be a topological space with the given ideal f_{cd}. Then $A^d\subseteq A^{*}(f_{cd})$, for all $A\subseteq X$.

Proof. Let $A\subseteq X$ and $x\in A^d$. This implies that $U\cap(A-\{x\})\neq\varnothing$ for all $U\in N(x)$. Since $V\cap U\in N(x)$ for all $V\in N(x)$ and $x\in A^d$, then $(V\cap U)\cap(A-\{x\})\neq\varnothing$. And so $V\cap[(U\cap A)-\{x\}]\neq\varnothing$, for all $V\in N(x)$. From this, $x\in(U\cap A)^d$, that is,

$(U \cap A)^d \neq \varnothing$. Thus, $U \cap A \notin f_{cd}$ for all $U \in N(x)$. This means that $x \in A^*(f_{cd})$. Since x is arbitrary, then $A^d \subseteq A^*(f_{cd})$ for all $A \subseteq X$. ■

$A^d \subseteq A^*(f_{cd}), \forall A \subseteq X \Rightarrow A \cup A^d \subseteq A \cup A^*(f_{cd})$, that is, $\overline{A} \subseteq \widetilde{A}$

$$\Rightarrow \widetilde{A} = \overline{A},\ \forall A \subseteq X. \text{ Therefore } \tau(f_{cd}) = \tau.$$

As a result of this lemma, the ideal topology $\tau(f_{cd})$ is always equal to the original topology τ.

Example 2.5.3.2 Let (X, τ) be an indiscrete space where X has at least two different elements and f_{cd} be a given ideal on X. Take $A = \{x\}$, for $x \in X$. Then $A^*(f_{cd}) \not\subset A^d$. Indeed;

$A^d = \{x\}^d = X - \{x\} \neq \varnothing$. From this, $\{x\} \notin f_{cd}$ for each $x \in X$. And

$$\begin{aligned} A^*(f_{cd}) = \{x\}^* &= \{y \in X \mid \forall U \in N(y),\ U \cap \{x\} \notin f_{cd}\} \\ &= \{y \in X \mid X \cap \{x\} = \{x\} \notin f_{cd}\} \\ &= X. \end{aligned}$$

Thus, $A^*(f_{cd}) = X \not\subset X - \{x\} = A^d$.

From this example, the statement " $A^*(f_{cd}) = A^d,\ \forall A \subseteq X$ " is not always true.

Theorem 2.5.3.3 (Jankovic & Hamlet, 1990) Let (X, τ) be any topological space and f_{cd} be a given ideal on X, then (X, τ) is T_1-space iff $A^*(f_{cd}) = A^d$ for all $A \subseteq X$.

Proof. ($\Leftarrow$) Let $A^*(f_{cd}) = A^d$ for all $A \subseteq X$, then for each $x \in X$, $\{x\}^* = \{x\}^d$. There are four cases:

i) $x \in \{x\}^*$ if $\{x\} \notin f_{cd}$

ii) $y \neq x$ and $y \in \{x\}^*$ if $\{x\} \notin f_{cd}$ and for all $U \in N(y), x \in U$

iii) $x \notin \{x\}^*$ if $\{x\} \in f_{cd}$

iv) $y \neq x$ and $y \notin \{x\}^*$ if $\exists U \in N(y)$ such that $x \in U$ and $\{x\} \in f_{cd}$ or if $\exists U \in N(y)$ such that $x \notin U$.

Since $x \notin \{x\}^d$, then $\{x\}^d \subseteq X - \{x\}$. Then there are two cases:

a) $y \neq x$ and $y \in \{x\}^d$ if for all $U \in N(y), x \in U$

b) $y \neq x$ and $y \notin \{x\}^d$ if $\exists U \in N(y)$ such that $x \notin U$.

$\forall x \in X$, $\{x\}^* = \{x\}^d \Rightarrow x \notin \{x\}^d$ implies that $x \notin \{x\}^*$

$\Rightarrow \{x\} \in f_{cd}$

$\Rightarrow X - \{x\} \in \tau(f_{cd})$

Since $\tau(f_{cd})=\tau$, then $X - \{x\} \in \tau$. And therefore $\{x\}$ is τ-closed. This means that (X,τ) is T_1-space.

$(\Rightarrow)$ Let (X,τ) be T_1-space. Then for every $x \in X$, $\{x\}$ is τ-closed. And since $x \notin \{x\}^d$, then $x \in is(\{x\})$. Hence $\{x\}$ is discrete set in (X,τ). And so $\{x\}$ is closed and discrete set in (X,τ), that is, $\{x\} \in f_{cd}$ for every $x \in X$.

We know that $A^d \subseteq A^*(f_{cd})$ for all $A \subseteq X$. Thus, we need only show that $A^*(f_{cd}) \subseteq A^d$. Let $x \in A^*(f_{cd})$, for $A \subseteq X$. Then for all $U \in N(x)$ $U \cap A \notin f_{cd}$. Assume that there exists $V \in N(x)$ such that $(V - \{x\}) \cap A = \varnothing$.

$\exists V \in N(x)$ such that $(V - \{x\}) \cap A = \varnothing \Rightarrow (V \cap \{x\}^c) \cap A = \varnothing,$

$\Rightarrow V \cap A \subseteq \{x\} \in f_{cd}$

This implies that $V \cap A \in f_{cd}$. Hence, $x \notin A^*(f_{cd})$, which is a contradiction. Thus, for all $V \in N(x)$, $(V - \{x\}) \cap A \neq \varnothing$, that is, $x \in A^d$. Then $A^*(f_{cd}) \subseteq A^d$, for all $A \subseteq X$. Since the inverse of this relation is always true, then $A^*(f_{cd}) = A^d$ for all $A \subseteq X$. ■

Lemma 2.5.3.4 (Jankovic & Hamlet, 1990) Let (X,τ) be any topological space and let f be a given ideal on X. If $I \in f$, then I is closed and discrete set in the ideal topological space $\left(X,\tau(f)\right)$.

Proof. Since I^* is empty for every I in f, then $I^{d^*} = \varnothing$. So I is closed and discrete set in $\left(X,\tau(f)\right)$, from the remark 2.2.3. ■

Corollary 2.5.3.5 (Jankovic & Hamlet, 1990) Let (X,τ) be a topological space, then the ideal f_{cd} is the largest ideal on X with the property $\tau(f_{cd})=\tau$.

Proof. Let f be an ideal on X which satisfying $\tau(f)=\tau$. From lemma 2.5.3.4, we know that every $I\in f$ is closed and discrete in $\left(X,\tau\left(f\right)\right)$. Since $\tau(f)=\tau$, then every $I\in f$ is closed and discrete in (X,τ). This means that $I\in f_{cd}$ for every I in f. Thus, for every ideal f with the property $\tau(f)=\tau$, we have $f\subseteq f_{cd}$. This shows that f_{cd} is the largest ideal with the property $\tau(f_{cd})=\tau$. ■

Theorem 2.5.3.6 (Jankovic & Hamlet, 1990) Let (X,τ) be topological space with a given ideal f. Then $\tau(f)=\tau$ iff every $I\in f$ is closed in (X,τ).

Proof. Let $\tau(f)=\tau$ and $I\in f$. Then $X-I\in\tau(f)$. Since $\tau(f)=\tau$, then $X-I\in\tau$. Thus, I is τ-closed, then every $I\in f$ is closed in (X,τ). Conversely assume that every $I\in f$ is closed in (X,τ) and let $V\in\tau(f)$.

$V\in\tau(f)\Rightarrow V=\bigcup_{\alpha}\left(G_\alpha-Z_\alpha\right)$ where $G_\alpha\in\tau$ and $Z_\alpha\in f$, $\forall\alpha$,

$V=\bigcup_{\alpha}\left(G_\alpha\cap(Z_\alpha)^c\right)$ where $G_\alpha\in\tau$ and Z_α is τ-closed,

$V=\bigcup_{\alpha}\left(G_\alpha\cap(Z_\alpha)^c\right)$ where $G_\alpha\cap(Z_\alpha)^c\in\tau$. And so $V\in\tau$. This shows that $\tau(f)$ is coarser then τ. Thus, $\tau(f)=\tau$. ■

From theorem 2.5.3.6, we can say that $\tau(f_{cd})=\tau$ and from corollary 2.5.3.5, we can also say that f_{cd} is the largest ideal with the property $\tau(f_{cd})=\tau$. We conclude that f_{cd} is the largest ideal on X with the property $\tau(f_{cd})=\tau$ and $f_{ind}=\{\varnothing\}$ is the smallest ideal on X with the property $\tau(f_{ind})=\tau$.

2.5.4 The Defined Ideal $f(A)$ for a subset $A\subseteq X$

Let (X,τ) be any topological space and $f(A)$ be an ideal on X which defines for a given subset $A\subseteq X$. Now we obtain new topologies by using this ideal.

Example 2.5.4.1 Let (X,τ) be an indiscrete topological space and $p\in X$. Then the ideal topology $\tau\left(f\left(X-\{p\}\right)\right)$ is the particular point topology where $f\left(X-\{p\}\right)$ is an ideal which defines for $X-\{p\}\subseteq X$, that is;

$$f\left(X-\{p\}\right)=\{A\subseteq X \mid A\subseteq X-\{p\}\}$$
$$=\{A\subset X \mid p\notin A\}$$

Let $A\subseteq X$. Then

$$A^*\left(f\left(X-\{p\}\right)\right)=\left\{x\in X \mid \forall U\in N(x), U\cap A\notin f\left(X-\{p\}\right)\right\}$$
$$=\{x\in X \mid A\notin f(X-\{p\})\}$$
$$=\begin{cases} X & \text{if} \quad p\in A \\ \varnothing & \text{if} \quad p\notin A \end{cases}$$

$$\tau\left(f\left(X-\{p\}\right)\right)=\{U\subseteq X \mid cl^*(X-U)=X-U\}$$
$$=\{U\subseteq X \mid (X-U)^*\subseteq(X-U)\}$$
$$=\{U\subseteq X \mid p\notin(X-U) \text{ or } (X-U)=X\}$$
$$=\{U\subseteq X \mid p\in U\}\cup\{\varnothing\}$$

So $\tau\left(f\left(X-\{p\}\right)\right)$ is the particular point topology.

(Given a point $p\in X$, the collection $\tau(p)=\{U\subseteq X \mid p\in U\}\cup\{\varnothing\}$ is called the particular point topology on X.)

Example 2.5.4.2 Let $\tau(p)$ be the particular point topology on a set X where $p\in X$ and $X\neq\{p\}$. Then the ideal topology obtained by the given topology $\tau(p)$ and the given ideal $f\left(X-\{p\}\right)$ is equal to the original topology. Actually;

Let $A\subseteq X$. Then

$$A^*\left(f\left(X-\{p\}\right)\right)=\left\{x\in X \mid \forall U\in N(x), U\cap A\notin f\left(X-\{p\}\right)\right\}$$
$$=\begin{cases} X & \text{if} \quad p\in A \\ \varnothing & \text{if} \quad p\notin A \end{cases}$$

$$\tau(p)\left(f\left(X-\{p\}\right)\right)=\{U\subseteq X \mid cl^*(X-U)=X-U\}$$
$$=\{U\subseteq X \mid (X-U)^*\subseteq X-U\}$$
$$=\{U\subseteq X \mid p\notin(X-U) \text{ or } X-U=X\}$$

$$=\{U\subseteq X \mid p\in U\}\cup\{\varnothing\}$$

Thus, the ideal topology is equal to the original topology while the given ideal is different from $\{\varnothing\}$.

$\left(X\neq\{p\}\right.$ implies that $\exists y\neq p$ such that $y\in X$. Thus, $\left.\{y\}\in f\left(X-\{p\}\right)\neq\{\varnothing\}.\right)$

2.5.5 The ideal f_n

Let (X,τ) be any topological space and $f_n=\{A\subseteq X \mid \overline{A}^{\circ}=\varnothing\}$ be a given ideal on X. Now let's show that $A^*(f_n)=\overline{\mathring{A}}$ for any subset A in the ideal topological space $\left(X,\tau(f_n)\right)$. Before showing this, we will give following theorem.

Theorem 2.5.5.1 (Kuratowski, 1966) Let (X,τ) be any topological space. If U is open, then $U\cap\overline{A}\subseteq\overline{U\cap A}$, for every $A\subseteq X$.

Proof. $U\cap\overline{A}=\overline{A}\cap(X-\overline{X-U})=\overline{A}\cap X\cap\left(\overline{X-U}\right)^c$

$$=\overline{A}-(\overline{X-U})$$

$$\subseteq\overline{A-(X-U)}=\overline{A\cap U}.$$

■

From this theorem, if U is open, then $U\cap\overline{A}^{\circ}\subseteq\overline{U\cap A}^{\circ}$ for every $A\subseteq X$. (24)

Let's show that $\overline{\mathring{A}}\subseteq A^*(f_n)$ for any $A\subseteq X$.

Let $x\notin A^*(f_n)$. This implies that $\exists U\in N(x)$ such that $U\cap A\in f_n$, that is, $\overline{U\cap A}^{\circ}=\varnothing$. From (24), $U\cap\overline{A}^{\circ}\subseteq\overline{U\cap A}^{\circ}=\varnothing$. Then $U\cap\overline{A}^{\circ}=\varnothing$, for any $U\in N(x)$. This means that, $x\notin\overline{\mathring{A}}$. Since x is arbitrary, then

$\overline{\mathring{A}}\subseteq A^*(f_n)$ for any $A\subseteq X$. (25)

Now we will show that $A^*(f_n)\subseteq\overline{\mathring{A}}$, for any $A\subseteq X$. Let $x\notin\overline{\mathring{A}}$. And we assume that for all $U\in N(x)$, $U\cap A\notin f_n$. Then for all $U\in N(x)$, $U\cap A$ is not nowhere dense set. We know that $B \subseteq X$ is a nowhere dense set, if its closure is a boundary set, i.e. if $\overline{X - \overline{B}} = X$ (Kuratowski, 1966). Thus, if for all $U \in N(x)$, $U \cap A$ is not nowhere dense set, then $\overline{U \cap A}$ is not boundary set. Also $\overline{B}$ is a boundary set iff $\overline{B} \subseteq \overline{X - \overline{B}}$ or, iff $B \subseteq \overline{X - \overline{B}}$ (Kuratowski, 1966). Thus, if for all $U \in N(x)$, $\overline{U \cap A}$ is not boundary set, then $U \cap A \not\subset \overline{X - \overline{U \cap A}}$.

$$\forall U \in N(x),\ U \cap A \not\subset \overline{X - \overline{U \cap A}} \Rightarrow \forall U \in N(x),\ U \cap A \not\subset \left(\overset{\circ}{\overline{U \cap A}}\right)^{c}$$

$$\Rightarrow \forall U \in N(x),\ (U \cap A) \cap (\overset{\circ}{\overline{U \cap A}}) \neq \varnothing.$$

Since $\varnothing \neq (U \cap A) \cap (\overset{\circ}{\overline{U \cap A}}) \subseteq U \cap \overset{\circ}{\overline{A}}$, then $U \cap \overset{\circ}{\overline{A}} \neq \varnothing$ for all $U \in N(x)$. This contradicts to $x \notin \overline{\overset{\circ}{\overline{A}}}$. Thus, $\exists V \in N(x)$ such that $V \cap A \in f_n$, that is, $x \notin A^{*}(f_n)$.

Then we say that $A^{*}(f_n) \subseteq \overline{\overset{\circ}{\overline{A}}}$, for any $A \subseteq X$. (26)

By (25) and (26), we obtain $A^{*}(f_n) = \overline{\overset{\circ}{\overline{A}}}$, for any $A \subseteq X$. ∎

Now we will show that $\tau(f_n) = \tau^{\alpha}$ where

$$\tau^{\alpha} = \left\{U \subseteq X \mid U \subseteq \left(\overline{U^{\circ}}\right)^{\circ}\right\} = \{U \subseteq X \mid U \text{ is } \alpha\text{-open}\}.$$

Let $X - U \in \tau(f_n)$. This implies that $U^{*}(f_n) \subseteq U$, $\overline{\overset{\circ}{\overline{U}}} \subseteq U$. So U is α-closed. Then $X - U \in \tau^{\alpha}$.

From this, $\tau(f_n) \subseteq \tau^{\alpha}$. (27)

Let $X - V \in \tau^{\alpha}$. This implies that V is α-closed, $\overline{\overset{\circ}{\overline{V}}} \subseteq V$, $V^{*}(f_n) \subseteq V$. Thus, V is $\tau(f_n)$-closed. Because of this, $X - V \in \tau(f_n)$.

Then $\tau^{\alpha} \subseteq \tau(f_n)$. (28)

By (27) and (28), we have $\tau(f_n) = \tau^{\alpha}$. ∎

So the ideal topology $\tau(f_n)$ which formed from a given topology τ and the given ideal f_n is the topology τ^{α}, that is, $\tau \subseteq \tau^{\alpha} = \tau(f_n)$.

Example 2.5.5.2 Let (X,τ) be an indiscrete topological space and f_n be a given ideal on X. For any $\varnothing \neq A \subseteq X$,

$$A^*(f_n) = \{x \in X \mid \forall U \in N(x), U \cap A \notin f_n\}$$
$$= \{x \in X \mid A \notin f_n\}$$
$$= X \quad \text{(Since for all } \varnothing \neq A \subseteq X,\ \mathring{\overline{A}} = X \neq \varnothing \text{)}$$
$$= \overline{\mathring{\overline{A}}}.$$

$$\tau(f_n) = \{U \subseteq X \mid cl^*(X-U) = X-U\}$$
$$= \{U \subseteq X \mid (X-U)^* \subseteq X-U\}$$
$$= \{U \subseteq X \mid (X-U) = X \text{ or } (X-U) = \varnothing\}$$
$$= \{U \subseteq X \mid U = \varnothing \text{ or } U = X\} = \{\varnothing, X\} = \tau.$$

Example 2.5.5.3 Let (X,τ) be a discrete topological space and let f_n be a given ideal. For any $A \subseteq X$,

$$A^*(f_n) = \{x \in X \mid \forall U \in N(x), U \cap A \notin f_n\}$$
$$= \{x \in X \mid \forall U \in N(x), \mathring{\overline{U \cap A}} \neq \varnothing\}$$
$$= \{x \in X \mid \forall U \in N(x), U \cap A \neq \varnothing\}$$
$$= \overline{A}$$
$$= A.$$

Since each subset of X is both open and closed, then $\overline{\mathring{\overline{A}}} = A$ for any $A \subseteq X$. Thus,

$$A^*(f_n) = \overline{\mathring{\overline{A}}}.$$

$$\tau(f_n) = \{U \subseteq X \mid cl^*(X-U) = X-U\}$$
$$= \{U \subseteq X \mid (X-U) \cup (X-U)^* = (X-U)\}$$
$$= \{U \subseteq X \mid (X-U) \cup (X-U) = (X-U)\}.$$

Then $\tau(f_n)$ is a discrete topology.

Example 2.5.5.4 Let $\tau = \{U \subseteq X \mid X - U \text{ is finite}\} \cup \{\varnothing\}$ be a given topology and let f_n be a given ideal on X where X is infinite set. Then for any $A \subseteq X$,

$$\overline{A} = \begin{cases} A & \text{if } A \text{ is finite} \\ X & \text{if } A \text{ is infinite} \end{cases}$$

$$\overline{\overset{\circ}{\overline{A}}} = \begin{cases} A^\circ & \text{if } A \text{ is finite} \\ X & \text{if } A \text{ is infinite} \end{cases}$$

Let's show that $A^*(f_n) = \overline{\overset{\circ}{\overline{A}}}$, for any $A \subseteq X$.

(i) Let $A \subseteq X$ be a finite subset of X. Then $U \cap A \subseteq A$ is also finite, for all $U \in N(x)$. Thus, $\overline{U \cap A} = U \cap A$. From this, $\overset{\circ}{\overline{U \cap A}} = (U \cap A)^\circ = U \cap A^\circ$. And so $A^*(f_n) = \{x \in X \mid \forall U \in N(x), U \cap A \notin f_n\}$

$$= \left\{ x \in X \mid \forall U \in N(x), \overset{\circ}{\overline{U \cap A}} \neq \varnothing \right\}$$

$$= \{x \in X \mid \forall U \in N(x), U \cap A^\circ \neq \varnothing\}$$

$$= \overline{A^\circ} = A^\circ.$$

If A is finite, then $A^*(f_n) = \overline{\overset{\circ}{\overline{A}}}$.

(ii) Let $A \subseteq X$ be an infinite subset of X. If $U \in N(x)$, then $U \cap \overline{A} \subseteq \overline{U \cap A}$ for an infinite subset A of X. So $U \cap \overset{\circ}{\overline{A}} \subseteq \overset{\circ}{\overline{U \cap A}}$. Since A is an infinite subset of X, then $U \cap X \subseteq \overset{\circ}{\overline{U \cap A}}$. From this, $x \in U \subseteq \overset{\circ}{\overline{U \cap A}}$. Thus, $\overset{\circ}{\overline{U \cap A}} \neq \varnothing$, that is, $U \cap A \notin f_n$. Since we take arbitrary $U \in N(x)$, then for all $U \in N(x)$, $U \cap A \notin f_n$, that is, $x \in A^*(f_n)$. Because of choosing arbitrary $x \in X$, then $A^*(f_n) = X$ where A is an infinite subset of X. Thus, $A^*(f_n) = X = \overline{\overset{\circ}{\overline{A}}}$ if A is an infinite subset of X.

From (i) and (ii), $A^{*}(f_{n}) = \overline{\overset{\circ}{\overline{A}}}$ for all $A \subseteq X$. ■

CHAPTER THREE

CONCLUSION

It is possible to construct a topological structure in different ways. We have used Kuratowski's method for this aim. With this method, we have obtained a topology named ideal topology. It has been seen that ideal topologies are finer than original ones. It was concluded that this would be an advantage related to the continuity of functions and also some mathematical studies. Since the number of continuous functions on the space is increasing as the topological structure is getting finer.

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