DOKUZ EYLÜL UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

GEOMETRY OF TORIC VARIETIES

by
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İZMİR

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We have read the thesis entitled **"GEOMETRY OF TORIC VARIETIES"** completed by **ÖZLEM UĞURLU** under supervision of **ASST. PROF. DR. MURAT ALTUNBULAK** and we certify that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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GEOMETRY OF TORIC VARIETIES

ABSTRACT

Toric varieties admit a computable description that arise from combinatorial objects, so-called cones and fans. On the other hand the whole deformation theory of an isolated singularity is encoded in its semi-universal deformation. More generally, for a complete intersection singularity, deformation is a family over a smooth base space that is obtained by perturbations of the defining equations. In this thesis, we want to investigate a description of deformation of affine toric varieties, which was studied in Altmann (1995a). It follows that, by the geometric properties of a cone, the semi-universal deformation, or the total spaces over the components can be described by completely combinatorial methods. Key points for all our investigations are the geometric properties of a cone and the notion of a Minkowski summand of some polyhedron that comes from an affine cross cut of the cone.

Keywords: Toric variety, toric deformations, complete intersection singularity, cyclic quotient singularity, Minkowski sum.

SİMİTSİ ÇEŞİTLEMLERİN GEOMETRİSİ

ÖZ

Bu tezde, afin simitsi çeşitlemlerinin deformasyonunun tanımını incelemek istiyoruz. Simitsi çeşitlemler kombinasyonal nesneler olan koni ve fanlarla ifade edilebildiğinden daha kolay ve hesaplanabilir bir tanımlamaya olanak sağlar. Diğer taraftan yalıtılmış tekilliklerin bütün deformasyon teorisi onların yarı-evrensel deformasyonları ile ifade edilir. Genel olarak tam kesişim tekillikleri için bu aile pürüzsüz bir taban uzayı üzerinde tanım denklemlerinin perturbasyonundan elde edilir. Bundan dolayı yarı-evrensel deformasyon ya da her bir bileşen üzerindeki tüm uzay koninin geometrik özelliklerinden faydalanarak sadece kombinasyonal methodlarla ifade edilebilir. Bu tez için yapacağımız tüm araştırmalarımız için asıl kilit noktalar ise koninin geometrik özellikleri ve konilerin afin çapraz kesiminden elde ettiğimiz bazı çok yüzlülerin Minkowski toplamıdır.

Anahtar Sözcükler : Simitsi çeşitlem, simitsi deformasyonlar, tam kesişim tekilliği, devirli bölüm tekilliği, Minkowski toplamı.

CONTENTS

Page

CHAPTER ONE
INTRODUCTION

Since 1970's, the study of torus actions has become increasingly important in several areas. The main force of this progress was provided by the theory of toric varieties in algebraic geometry. A toric variety is an irreducible normal algebraic variety which contains an algebraic torus $(\mathbb{C}^*)^n$, as a dense open subset together with a torus action on itself extended to an action on the whole variety. It provide an alternative way to see many problems in algebraic geometry.

Up to today, a lot of results and applications related to toric varieties have been obtained by using different approaches. In particular, combinatorial approach is a mixture of principles from combinatorics and principles from geometry. The most basic and elementary object in combinatorial geometry is called fan. This notion allows us to describe toric varieties by combinatorial tools, that is, algebraic objects can be translated into combinatorics. It follows that, toric varieties relates algebraic geometry to the geometry of convex objects in real affine space. Then we obtain more impressive and computable description of toric varieties. The benefit of the theory lies in the fact that the geometric properties of toric varieties are constructed in terms of the elementary geometry of fans. The standard textbooks on the theory of toric variety are Fulton (1993), Ewald (1996), Cox et al. (2011) and Danilov (1978) with analytic approach.

Deformation theory is as old as algebraic geometry and is one of the fundamental techniques in algebraic geometry and in many other disciplines. We can deform various kinds of objects, for example algebraic varieties, complex spaces, or singularities. The main idea of the deformation is to perturb a given object by suitably varying the coefficients of its defining equations. The whole deformation theory is encoded in the concept of flatness, which preserves the information of the original objects after deformation. For example, flatness implies continuity of certain invariants. Good references for details about deformation theory are Artin (1976), Sernesi (2006) and Stevens (2003), the first two of them are in algebraic sense and the latter one is in analytic sense.

In recent years, the value of using the idea of toric deformation has emerged as a promising tool. Toric deformation allows us to replace a complicated object by a simpler one that still carries most or all of the numerical and combinatorial information. This gives rise to a theory with a geometric concept which is described by cones and fans.

The base point of our investigation is Christophersen's observation which states that deforming a two-dimensional cyclic quotient singularity yields the total spaces over the components of the reduced base space are also toric varieties. Based on this observation, we will investigate the deformation theory of toric singularities that occur in toric varieties. Their semi-universal deformations are analysed by using combinatorial data, after the method was first introduced in Altmann (1995a). The main result of Altmann (1995a) is that the toric deformations can be obtained from homogeneous toric regular sequences which comes from Minkowski decomposition of affine slices of the cone.

Now we give a more detail about how this thesis is organized:

In Chapter 2, we will try to provide a basic terminology for varieties and schemes. We will construct an affine variety V in $\mathbb{C}^n$ and its coordinate ring $\mathbb{C}[V]$. By using the gluing axiom, we will investigate the notion of algebraic variety. Then we will examine the generalization of these notions, i.e., over a commutative ring. Theory of schemes is introduced by Grothendick in late 1950's.

In Chapter 3, we will introduce our main concept, toric varieties. As stated before, toric varieties can be described in terms of combinatorial object, a strongly convex rational polyhedral cone $\sigma \subset N_{\mathbb{R}}$. The procedure of the construction of affine toric varieties associates to a cone σ: the dual cone $\check{\sigma}$, a semigorup S_{σ}, a finitely generated reduced $\mathbb{C}$-algebra R_{σ} and eventually an affine variety X_{σ}. By the gluing method, in the same manner given in Chapter 2, we will construct general toric varieties X_{Σ} that correspond to the compatible collection of strongly convex rational polyhedral cones, so-called fan $\Sigma \subset N_{\mathbb{R}}$. We will end this chapter by investigating some topological and geometric properties of toric varieties.

In chapter 4, we will give a brief introduction to deformation theory in general case. The main point of this theory is the existence of a semi-universal deformation. Because of this we will especially introduce the deformation theory of isolated singularities of affine schemes. More generally, the deformation of a complete intersection singularities is obtained by perturbations of the defining equations over the smooth base space. If we change the class of singularities, then the structure of the deformation family or the base space will become more complicated.

In chapter 5, we will investigate the deformation of toric singularities, which occurs in toric varieties, by combinatorial methods. Our aim is to understand the following fact: a semi-universal deformation of a toric variety is also a toric variety. The first step is always to look at the vector space of infinitesimal deformations T^1. In addition, toric deformations are existing deformations, i.e., admits reduced (smooth) base spaces. In Section 5.3 we will explicitly construct homogeneous toric regular sequences. Each toric regular sequence can be regarded as a flat map $\mathcal{X} \to \mathbb{C}^m$ by itself. It follows that, toric deformations always comes from homogeneous toric regular sequences. Then, we will investigate the Kodaira-Spencer map $\varrho : \mathbb{C}^m \to T^1$ corresponding to toric deformations. Finally, we will end our work by giving some examples to illustrate all statements and methods completely. Basic references for this notion are Altmann (1995a), Altmann (2009) and Altmann (1995b).

CHAPTER TWO
PRELIMINARIES

In this chapter we will give a brief information about some fundamental notions of algebraic geometry which are necessary to understand the more deeper theory. This chapter is based on Cox et al. (1997), Fulton & Weiss (1969), Hartshorne (1977) and Reid (1988).

2.1 Affine Variety

Studying with polynomials gives us some conveniences in terms of geometry. More explicitly, the solution set of polynomials gives us a geometric object. In this section we will investigate this geometric object in the affine sense.

Let $\mathbb{k}$ be a field and $\mathbb{k}[x_1, \dots, x_n]$ denote the ring of polynomials with n variables, $x_1, \dots, x_n$. Monomials form a basis for $\mathbb{k}[x_1, \dots, x_n]$ as a $\mathbb{k}$-vector space.

Definition 2.1.1. An *n-dimensional affine space* over $\mathbb{k}$ is defined to be the set:

$$\mathbb{A}^n_{\mathbb{k}} := \mathbb{k}^n = \mathbb{k} \times \cdots \times \mathbb{k} = \{(a_1, \dots, a_n) \mid a_i \in \mathbb{k},\ \forall\ i = 1, \dots, n\}.$$

For example, $\mathbb{A}^n_{\mathbb{k}}$ is $\mathbb{C}^n$, if we take $\mathbb{k} = \mathbb{C}$, and $\mathbb{R}^n$ if $\mathbb{k} = \mathbb{R}$.

The fundamental theorem of algebra states that every nonzero polynomial in one variable over $\mathbb{C}$ is determined up to a scalar factor by its roots. Hilbert extends this fact to the multi-variable polynomials over $\mathbb{C}$. It follows that this idea works best for an algebraically closed field $\mathbb{k}$. An algebraically closed field means a field for which every non-constant polynomial has a root in $\mathbb{k}$. In this thesis, unless otherwise stated we will always work over the algebraically closed field $\mathbb{C}$. Now, we have enough tools to construct the relation between polynomials and affine space.

Definition 2.1.2. Let $S = \{f_1, \dots, f_s\}$ be a set of polynomials in $\mathbb{C}[x_1, \dots, x_n]$. Then the set $\mathbb{V}(S) = \{(a_1, \dots, a_n) \in \mathbb{C}^n \mid f_i(a_1, \dots, a_n) = 0,\ 1 \leq i \leq s\}$ is called an *affine variety* defined by $f_1, \dots, f_s$.

Remark 2.1.3. Note that, since more equations gives fewer solutions, we have $S \subset S'$ implies $\mathbb{V}(S) \supset \mathbb{V}(S')$.

Every affine variety can be defined by an ideal with the following construction. Let $I := \langle S \rangle = \langle f_1, \ldots, f_s \rangle$ be the ideal generated by the polynomials $f_i \in \mathbb{C}[x_1, \ldots, x_n]$, $i = 1, \ldots, s$. The elements of I are in the form as $\sum g_i f_i$, $g_i \in \mathbb{C}[x_1, \ldots, x_n]$ by the definition of an ideal. If f_i are all zero at a point, then such a sum is zero at that point. This means that $\mathbb{V}(S) \subset \mathbb{V}(I)$ and conversely since $S \subset I$, by Remark 2.1.3 we have $\mathbb{V}(S) \supset \mathbb{V}(I)$. Thus, $\mathbb{V}(S) = \mathbb{V}(I)$.

Note that, an affine variety V is a hypersurface in $\mathbb{C}^n$ if it can be given as roots of a single polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]$. For example, the set $\mathbb{V}(y^2 - x^3)$ is an affine variety in $\mathbb{C}^2$. Since it is defined by only one polynomial, $\mathbb{V}(y^2 - x^3)$ is a hypersurface.

The Hilbert Basis Theorem states that the ring $\mathbb{C}[x_1, \ldots, x_n]$ is Noetherian. A Noetherian ring means that every ascending chain of ideals $I_1 \subset I_2 \subset \cdots$ in a ring R eventually becomes constant, or equivalently every ideal is finitely generated. So, for a given affine variety there exists a finite set of polynomials defining the variety. In other words all varieties in $\mathbb{C}^n$ are of the form $\mathbb{V}(I)$.

Proposition 2.1.4. *(Reid, 1988, page 50) The following properties are true:*

i) $\mathbb{V}(\{0\}) = \mathbb{C}^n$ *and* $\mathbb{V}(\mathbb{C}[x_1, \ldots, x_n]) = \emptyset$*,*

ii) $\mathbb{V}(I \cap J) = \mathbb{V}(I) \cup \mathbb{V}(J)$*,*

iii) $\mathbb{V}(\sum I_\alpha) = \bigcap \mathbb{V}(I_\alpha)$*, for any family of ideals* $\{I_\alpha\}_{\alpha \in \Lambda}$.

These properties show that the affine variety of $\mathbb{C}^n$ satisfy the axioms for the closed sets of a topology of $\mathbb{C}^n$. This topology is called the *Zariski topology* on $\mathbb{C}^n$. One can show that this is a cofinite topology on $\mathbb{C}^n$. The induced topology on a subset V of $\mathbb{C}^n$ is called the *Zariski topology* on V.

On the other hand, given any affine variety V in $\mathbb{C}^n$, we can associate it with an ideal as follows:

Definition 2.1.5. The set $\mathbb{I}(V) = \{f \in \mathbb{C}[x_1,\dots,x_n] \mid f(a_1,\dots,a_n) = 0,\ \forall(a_1,\dots,a_n) \in V\}$ is called the *ideal of* V.

Note that, $V \subset W$ implies $\mathbb{I}(V) \supset \mathbb{I}(W)$. Moreover, $\mathbb{I}(\emptyset) = \mathbb{C}[x_1,\dots,x_n]$ and $\mathbb{I}(\mathbb{C}^n) = 0$.

Consider the point $P = (a_1,\dots,a_n) \in \mathbb{C}^n$, then $\{P\} = \mathbb{V}(x_1 - a_1,\dots,x_n - a_n)$. Hence, every singleton of $\mathbb{C}^n$ is an affine variety and thus closed in Zariski topology. Denote the ideal $\mathbb{I}(\{P\})$ by

$$\mathcal{M}_P := \mathbb{C}[x](x_1 - a_1) + \cdots + \mathbb{C}[x](x_n - a_n). \tag{2.1.1}$$

At this stage, a natural question arises;

"What is the relation between the ideal I and $\mathbb{I}(V)$, where $V = \mathbb{V}(I)$?"

To investigate this relation, we need some notions from algebra. A *radical of an ideal* I is defined as to be a set $\{f \mid f^r \in I$, for some $r \in \mathbb{Z}_{\geq 0}\} = \sqrt{I}$ and the ideal I is called *radical* if $\sqrt{I} = I$. In addition, an ideal I is radical in a ring R if and only if R/I is a reduced ring, i.e., a ring without nonzero nilpotent elements. The Nullstellensatz states that if I is an ideal in $\mathbb{C}[x_1,\dots,x_n]$, then $\mathbb{I}(\mathbb{V}(I)) = \sqrt{I}$. Therefore, $\mathbb{I}(V)$ is a radical ideal for any affine variety $V \subset \mathbb{C}^n$. Now we are ready to define the notion of the (reduced) coordinate ring of an affine variety V in $\mathbb{C}^n$. The Hilbert's Nullstellensatz theorem shows that V, endowed with the Zariski topology, is determined by its coordinate ring. So we need to determine a regular mapping and to define a map between varieties.

Definition 2.1.6. Let $V \subset \mathbb{C}^m$ and $W \subset \mathbb{C}^n$ be two varieties. A function $\phi : V \to W$ is said to be a *regular mapping* (or *polynomial mapping*) if there exist polynomials $f_1,\dots,f_n \in \mathbb{C}[x_1,\dots,x_m]$ such that $\phi(a_1,\dots,a_m) = (f_1(a_1,\dots,a_m),\dots,f_n(a_1,\dots,a_m))$ for all $(a_1,\dots,a_m) \in V$. We say that the n-tuple of polynomials $(f_1,\dots,f_n) \in (\mathbb{C}[x_1,\dots,x_m])^n$ *represents* ϕ.

Example 2.1.7. Consider the varieties $V = \mathbb{V}(y - x^2, z - x^3) \subset \mathbb{C}^3$ (*the twisted cubic*) and $W = \mathbb{V}(y^3 - z^2) \subset \mathbb{C}^2$ (*the cusp*). Let $\pi : \mathbb{C}^3 \to \mathbb{C}^2$ be the projection map defined by $(x,y,z) \mapsto (y,z)$. Since every point in $\pi(V) = \{(x^2,x^3)|x \in \mathbb{C}\}$ satisfies the defining equation of W, then π is a regular mapping $\pi : V \to W$.

Now consider the simple case $W = \mathbb{C}$. For any variety $V \subset \mathbb{C}^n$ a mapping $\phi : V \to \mathbb{C}$ is a *regular function* (or *polynomial function*) if there exists a polynomial $f \in \mathbb{C}[x_1,\ldots,x_n]$ representing ϕ. The polynomials $f,g \in \mathbb{C}[x_1,\ldots,x_n]$ represent the same regular function on $V \subset \mathbb{C}^n$ if and only if $f - g \in \mathbb{I}(V)$. Thus, there exists a one-to-one correspondence between polynomials in $\mathbb{C}[x_1,\ldots,x_n]$ and regular functions. This means that the polynomial ring $\mathbb{C}[x_1,\ldots,x_n]$ is also coordinate ring of $\mathbb{C}^n$. For an arbitrary affine variety $V \subset \mathbb{C}^n$, we define the *coordinate ring* of V as follows:

$$\mathbb{C}[V] := \mathbb{C}[x_1,\ldots,x_n]/\mathbb{I}(V).$$

In particular, we can identify the coordinate ring $\mathbb{C}[V]$ with the regular functions on V.

Notice that, since $\mathbb{I}(V)$ is a radical ideal, the coordinate ring $\mathbb{C}[V]$ is finitely generated reduced $\mathbb{C}$-algebra. This means that $\mathbb{C}[V]$ is a vector space over $\mathbb{C}$. Furthermore, the homomorphism of $\mathbb{C}$-algebras is a linear transformation, i.e., $\phi(afg) = a\phi(fg) = a\phi(f)\phi(g)$, for all $a \in \mathbb{C}$, $f,g \in \mathbb{C}[V]$.

Example 2.1.8. Consider the affine variety $V = \mathbb{V}(x)$ in $\mathbb{C}^2$. Then the coordinate ring of V is the ideal $< y >$. Indeed, $\mathbb{C}[V] = \mathbb{C}[x,y]/\mathbb{I}(V) = \mathbb{C}[x,y]/ < x > \cong < y >$.

Now we are going to introduce the notion of the irreducibility of an affine variety V in $\mathbb{C}^n$. Some authors say that 'affine variety' instead of our 'irreducible affine variety'. There is no confusion, because we want to especially emphasize the notion of irreducibility.

Definition 2.1.9. An affine variety $V \subset \mathbb{C}^n$ is *irreducible* if there exist no decomposition of subvarieties V_1, V_2 such that $V = V_1 \cup V_2$. Otherwise, V is called *reducible*.

Since the polynomial ring $\mathbb{C}[x_1,\ldots,x_n]$ is Noetherian, an ascending chain of ideals

$$\mathbb{I}(V_1) \subset \cdots \subset \mathbb{I}(V_r) \subset \ldots$$

must stabilizes. Then the corresponding varieties satisfy the descending chain conditions of varieties, by the fact $\mathbb{V}(\mathbb{I}(V)) = V$. Thus, we obtain the following structure of an affine variety.

Theorem 2.1.10. *(Cox et al., 1997, Theorem 2, page 204) An affine variety $V \subset \mathbb{C}^n$ can be written in the form $V = V_1 \cup \cdots \cup V_r$, where each V_i is an irreducible variety.*

For example, the variety $\mathbb{V}(xz, yz)$ is a reducible variety, since $\mathbb{V}(xz, yz) = \mathbb{V}(z) \cup \mathbb{V}(x, y)$.

On the other hand, irreducibility can be tought in algebraic terms. To do this we need some fundamental notions of algebra. A proper ideal $I \subset \mathbb{C}[x_1, \ldots, x_n]$ is called *prime* if $fg \in I$ for $f, g \in \mathbb{C}[x_1, \ldots, x_n]$, then either $f \in I$ or $g \in I$. A proper ideal $I \subset \mathbb{C}[x_1, \ldots, x_n]$ is called *maximal* if $I \neq \mathbb{C}[x_1, \ldots, x_n]$ and any proper ideal $J \supset I$ implies $J = I$.

For any point $P \in \mathbb{C}^n$ the ideal $\mathcal{M}_P$, see Equation (2.1.1), is maximal in $\mathbb{C}[x_1, \ldots, x_n]$, since one can show that the quotient $\mathbb{C}[x_1, \ldots, x_n]/\mathcal{M}_P$ is a field. On the other hand, any maximal ideal in $\mathbb{C}[x_1, \ldots, x_n]$ is prime, since the polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$ is a commutative ring. Furthermore, all maximal ideals of $\mathbb{C}[x_1, \ldots, x_n]$ are in the form $\mathcal{M}_P$. Thus, $\mathcal{M}_P$ is a prime ideal, and all maximal ideals of $\mathbb{C}[x_1, \ldots, x_n]$ are prime.

Proposition 2.1.11. *(Cox et al., 1997, Proposition 4, page 218) Let $V \subset \mathbb{C}^n$ be an affine variety. Then the followings are equivalent:*

i) V is irreducible

ii) $\mathbb{I}(V)$ is a prime ideal

iii) $\mathbb{C}[V]$ is an integral domain

Therefore, the following one-to-one correspondences are valid.

$$\{\text{Irreducible varieties of } \mathbb{C}^n\} \longleftrightarrow \{\text{Prime ideals of } \mathbb{C}[x_1, \ldots, x_n]\}$$

$$\{\text{Points of } \mathbb{C}^n\} \longleftrightarrow \{\text{Maximal ideals of } \mathbb{C}[x_1, \ldots, x_n]\}$$

$$\{\text{Points of affine variety } V\} \longleftrightarrow \{\text{Maximal ideals of its coordinate ring } \mathbb{C}[V]\}.$$

Definition 2.1.12. Two affine varieties $V_1 \subset \mathbb{C}^n$ and $V_2 \subset \mathbb{C}^m$ are *isomorphic* if there are polynomial maps $F : \mathbb{C}^n \to \mathbb{C}^m$ and $G : \mathbb{C}^m \to \mathbb{C}^n$ such that $F(V_1) = V_2$, $G(V_2) = V_1$ and $F \circ G = \mathrm{id}_{V_2}$, $G \circ F = \mathrm{id}_{V_1}$.

As a result, we obtain the relation between V and $\mathbb{C}[V]$. Furthermore, the coordinate ring $\mathbb{C}[V]$ of an affine variety V can be characterized as follows.

Proposition 2.1.13. *(Cox, 2000a) A $\mathbb{C}$-algebra R is isomorphic to the coordinate ring of an affine variety if and only if R is reduced finitely generated $\mathbb{C}$-algebra.*

Now, we describe another function, so-called rational function, on a variety.

Definition 2.1.14. A *rational function* in $x_1, \dots, x_n$ with coefficients in $\mathbb{C}$ is a quotient f/g of two polynomials where g is not the zero polynomial. Two rational functions f/g and h/k are equal if $fk = gh$ in $\mathbb{C}[x_1, \dots, x_n]$. The set of all rational functions in $x_1, \dots, x_n$ with coefficients in $\mathbb{C}$ is denoted $\mathbb{C}(x_1, \dots, x_n)$. It is a field with classical addition and multiplication operations, and called *quotient field* (or *field of fractions*).

Given $f/g \in \mathbb{C}(V)$, $g = 0$ gives a subvariety $W \subset V$ and $f/g : V \setminus W \to \mathbb{C}$ is a well-defined function, denoted by $f/g : V \dashrightarrow \mathbb{C}$. If an affine variety V is irreducible, then its coordinate ring $\mathbb{C}[V]$ is an integral domain. So, $\mathbb{C}[V]$ has a field of fractions. For example, in the case of $V = \mathbb{C}^n$, its field of rational functions $\mathbb{C}(V)$ is $\mathbb{C}(x_1, \dots, x_n)$.

Finally, we introduce some topological properties of an affine variety V. Given an affine variety $V \subset \mathbb{C}^n$, a subset W of V is called a *subvariety* if W is also an affine variety. Then by the property of $\mathbb{I}$, we have $\mathbb{I}(W) \supset \mathbb{I}(V)$. Given a subvariety $W \subset V$, the complement $V - W$ is called a *Zariski open subset* of V. Some Zariski open subsets of an affine variety V are themselves affine varieties. Given $f \in \mathbb{C}[V] \setminus \{0\}$, define

$$D(f) = V_f := \{P \in V \,|\, f(P) \neq 0\} \subset V.$$

Indeed, if $\mathbb{I}(V) =< f_1, \dots, f_r >$ for an affine variety V, then for any $g \in \mathbb{k}[x_1, \dots, x_n]$, we can write f in the form $g + \mathbb{I}(V)$. Thus, $V_f = V - \mathbb{V}(f_1, \dots, f_r, g)$. This means that V_f

is a Zariski open in V. And if we take $W = \mathbb{V}(f_1,\dots,f_r,1-gy) \subset \mathbb{C}^n \times \mathbb{C}$, then we can identify this variety with V_f. The sets V_f are bases for the topology on V and called the *principal open subsets* of V.

If V is irreducible and $f \in \mathbb{C}[V]$, then denote by $\mathbb{C}[V]_f$ the localization of $\mathbb{C}[V]$ at the multiplicative set $S = \{f^r \mid r \geq 0\}$. Thus, we obtain $\mathbb{C}[V]_f = \{g/f^r \in \mathbb{C}(V) \mid g \in \mathbb{C}[V]\,,\, r \geq 0\}$.

2.1.1 Spectrum

The identification of the points of an affine space $\mathbb{C}^n$ with the maximal ideals in the polynomial ring $\mathbb{C}[x_1,\dots,x_n]$ gives us a useful object which is called spectrum. We will define the spectrum as a set, for more detail, we direct the reader to Eisenbud & Harris (2000) and Ueno (1997).

Definition 2.1.15. Let R be a commutative ring. The *spectrum* of R, denoted Spec(R), is the set of all prime ideals of R.

Example 2.1.16. Let $R = \mathbb{Z}$. Since $\mathbb{Z}$ is a principal ideal domain, every prime is generated by only one element. Thus, we have $\text{Spec}(\mathbb{Z}) = \{0,2,3,\dots\}$.

Example 2.1.17. Consider the polynomial ring $\mathbb{C}[x]$ in one variable. Since prime ideals are also maximal ideals in $\mathbb{C}[x]$, we have maximal ideals of the form $< x-a >$ for any $a \in \mathbb{C}$. Thus, $\text{Spec}(\mathbb{C}[x]) \cong \mathbb{C}$. More generally, $\text{Spec}(\mathbb{C}[x_1,\dots,x_n]) \cong \mathbb{C}^n$.

The notion of spectrum gives us the close relationship between V and $\mathbb{C}[V]$. Because of this relation we can write $V \cong \text{Spec}(\mathbb{C}[V])$. Since the principal open set V_f has a natural affine structure, we have $V_f \cong \text{Spec}\mathbb{C}[V]_f$.

2.1.2 Normal Affine Variety

Normality is an important tool for us because a toric variety, which we will define in Chapter 3, are always normal. Let R be an integral domain with the field of fractions

K. R is integrally closed if every element of a field of fractions K which is integral over R, means that it is a root of a monic polynomial in $R[x]$, lies in R.

Definition 2.1.18. An irreducible affine variety V is *normal* if its coordinate ring $\mathbb{C}[V]$ is integrally closed.

Example 2.1.19. $\mathbb{C}^n$ is normal since its coordinate ring $\mathbb{C}[x_1,\dots,x_n]$ is integrally closed.

Example 2.1.20. Consider the irreducible variety $V = \mathbb{V}(x^3 - y^2) \subset \mathbb{C}^2$. Then its coordinate ring is $\mathbb{C}[V] = \mathbb{C}[x,y]/ < x^3 - y^2 >$. Assume that X and Y be the cosets of x and y in $\mathbb{C}[V]$, respectively. Since $(Y/X)^2 = X$, Y/X is not integral over $\mathbb{C}[V]$. Thus, V is not a normal variety.

We will end this section with another important tool, the dimension, since dimension is an important invariant in algebraic geometry and we will especially use in Chapter 4 and 5.

Definition 2.1.21. The *dimension* of an affine variety V, denoted by $\dim V$, is the supremum of all integers n for which there exists a chain $\emptyset \neq V_0 \subset V_1 \subset \cdots \subset V_n = V$ of distinct irreducible sets.

For example, the dimension of $V = \mathbb{C}$ is 1, since we have $\{P\} = V_0 \subset V_1 = V$ for $P \in \mathbb{C}$.

Definition 2.1.22. By the *height*, we mean the supremum of all integers n for which there exists a chain $\mathfrak{p}_0 \subset \cdots \subset \mathfrak{p}_n = \mathfrak{p}$ of distinct prime ideals. The supremum of heights of all prime ideals is called the *Krull dimension* of a ring.

Remark 2.1.23. Let V be an irreducible affine variety. We have identified any irreducible subvariety of V with the prime ideals in $\mathbb{C}[x_1,\dots,x_n]$ which contains $\mathbb{I}(V)$. Thus, we obtain the following fact:

$$\dim V = \dim\Big(\mathbb{C}[x_1,\dots,x_n]/\mathbb{I}(V)\Big) = \dim \mathbb{C}[V].$$

This fact allows us to apply results from the dimension theory of rings to the algebraic geometry.

Proposition 2.1.24. *Let R be an integral domain. Then for any prime ideal $\mathfrak{p}$ in R we have:* $\operatorname{height}\mathfrak{p} + \dim R/\mathfrak{p} = \dim R$.

2.2 Projective Variety

Let M be an $(n+1)$-dimensional vector space over a field $\Bbbk$. The projective space $\mathbb{P}(M)$ is the parameter space of one-dimensional subspaces of the $\Bbbk$-vector space M, i.e., $\mathbb{P}(M) := \{1\text{-dimensional vector subspaces of } M\}$.

Define an equivalence relation $\sim$ on the nonzero points of $\Bbbk^{n+1}$ by setting $(a_0, \ldots, a_n) \sim (b_0, \ldots, b_n)$ if there is a nonzero scalar $\lambda \in \Bbbk$ such that $(a_0, \ldots, a_n) = \lambda(b_0, \ldots, b_n)$. Let $\mathbf{0}$ denote the origin $(0, \ldots, 0) \in \Bbbk^{n+1}$. Then we can give an equivalent definition for a projective space as follows:

Definition 2.2.1. The set of equivalence classes of $\sim$ on $\Bbbk^{n+1} \setminus \{\mathbf{0}\}$ is called an n-dimensional *projective space* over $\Bbbk$, i.e.,

$$\mathbb{P}^n_{\Bbbk} = \mathbb{P}^n := (\Bbbk^{n+1} \setminus \{\mathbf{0}\})/\sim \, = \{(a_0, \ldots, a_n) \in \Bbbk^{n+1} \,|\, (a_0, \ldots, a_n) \neq \mathbf{0}\}.$$

For simplicity, assume $\Bbbk = \mathbb{C}$. Each nonzero $(n+1)$-tuple $(a_0, \ldots, a_n) \in \mathbb{C}^{n+1}$ defines a line through the origin and a point $(a_0, \ldots, a_n)$. But there are many points $(b_0, \ldots, b_n)$ in $\mathbb{C}^{n+1}$ defining the same lines. By the equivalence relation $\sim$, the ratios $a_0 : \ldots : a_n$ and $b_0 : \ldots : b_n$ are the same. So, the notation $[a_0 : \ldots : a_n]$ can be used to describe the equivalence class of $(a_0, \ldots, a_n)$, and it denotes a point P in $\mathbb{P}^n$. In other words, one can view $\mathbb{P}^n$ as the space of lines through the origin. In this notation, the coordinates $[a_0 : \ldots : a_n]$ are called *homogeneous coordinates*.

At once, we will describe the projective varieties in terms of affine varieties follows: Let $U_j = \{[a_0 : \ldots : a_n] \in \mathbb{P}^n \,|\, a_j \neq 0\} \subset \mathbb{P}^n$. For all j, one can define a map $\psi_j : U_j \to \mathbb{C}^n$ by $P = [a_0 : \ldots : a_n] \in U_j \mapsto P = \left[\frac{a_0}{a_j} : \ldots : 1 : \ldots : \frac{a_n}{a_j}\right]$. Then the set $\psi(P) = \left[\frac{a_0}{a_j} : \ldots :\right.$

$\frac{a_{j-1}}{a_j} : \frac{a_{j+1}}{a_j} : \ldots : \frac{a_n}{a_j}\Big]$ is contained in $\mathbb{C}^n$. Since the j-th component is nonzero, we get an inverse map $\phi : \mathbb{C}^n \to U_j$ given by $\phi((b_0, \ldots, b_n)) \to [b_1 : \ldots : 1 : \ldots : b_n]$. Thus, there exists a one-to-one correspondence between $\mathbb{C}^n$ and $U_j \subset \mathbb{P}^n$.

Definition 2.2.2. A *homogeneous polynomial of degree d* is a polynomial in $\mathbb{C}[x_0, \ldots, x_n]$ whose all terms has total degree d or equivalently, $F[\lambda x_0 : \ldots : \lambda x_n] = \lambda^d F[x_0 : \ldots : x_n]$, $\lambda \in \mathbb{C}^*$.

Given $P \in \mathbb{P}^n$, $F(P) = F([a_0 : \ldots : a_n])$ is not equal to $F(\lambda P) = F(\lambda[a_0 : \ldots : a_n]) = F([\lambda a_0 : \ldots : \lambda a_n]) = \lambda^d F([a_0 : \ldots : a_n])$. It follows that, we cannot define $F(P)$. But, the equation $F(P) = 0$ is well-defined since $\lambda \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$. Let $F \in \mathbb{C}[x_0, \ldots, x_n]$ be a homogeneous polynomial of degree d. The polynomial ring is an important example of a graded ring, because

$$\mathbb{C}[x_0, \ldots, x_n] = \bigoplus_{d \geq 0} \mathbb{C}^d[x_0, \ldots, x_n],$$

where $\mathbb{C}^d[x_0, \ldots, x_n] \coloneqq \{f \in \mathbb{C}[x_0, \ldots, x_n] \mid f \text{ is homogeneoues of degree } d\} \cup \{0\}$. So if F vanishes on any one set of homogeneous coordinates for a point $P \in \mathbb{P}^n$, then F vanishes for all homogeneous coordinates of P as in affine case. Thus a projective variety can be described in the following sense.

Definition 2.2.3. Let S be the set of homogeneous polynomials in $\mathbb{C}[x_0, \ldots, x_n]$. The set $\mathbb{V}(S) = \{P \in \mathbb{P}^n \mid F(P) = 0, \forall F \in S\}$ is called a *projective variety*.

As in affine case if I is the ideal generated by S, then $\mathbb{V}(S) = \mathbb{V}(I)$. An ideal I in $\mathbb{C}[x_0, \ldots, x_n]$ is called *homogeneous* if it is generated by homogeneous polynomials, i.e., any $F \in I$ can be written as $F = \sum_{d=0}^{m} F_d$, $F_d \in I$ where F_d denotes the homogeneous polynomials of degree d.

Definition 2.2.4. Given any projective variety $V = \mathbb{V}(I) \subset \mathbb{P}^n$ we define the *ideal* as to be a set, $\mathbb{I}(V) = \{F \in \mathbb{C}[x_0, \ldots, x_n] \mid F(P) = 0, \forall P \in V\}$.

This ideal is a homogeneous ideal. And by the same reason given in Section 2.1 this ideal is finitely generated. If I is a homogeneous ideal, then $\sqrt{I}$ is also homogeneous.

Furthermore, it is known that $\mathbb{V}(<1>) = \emptyset$ in the affine case, but in projective case there is an another homogeneous ideal, $m =< x_0, \ldots, x_n >$ such that $\mathbb{V}(m) = \emptyset$.

Theorem 2.2.5 (Projective Nullstellensatz)**.** *For any homogeneous ideal I, we have the following:*

i) $\mathbb{V}(I) = \emptyset$ *if and only if* $\sqrt{I} \supset m$.

ii) If $\mathbb{V}(I) \neq \emptyset$, *then* $\mathbb{I}(\mathbb{V}(I)) = \sqrt{I}$.

Thus we have the following one-to-one correspondence:

$$\{\text{Homogeneous Prime Ideals}\} \longleftrightarrow \{\text{Irreducible Projective Varities}\}.$$

Remark 2.2.6. We can define the topological notions on the projective variety as in the affine case. If $V \in \mathbb{P}^n$ is a projective variety, then $\mathbb{P}^n \setminus V$ is called a *Zariski open subset* of $\mathbb{P}^n$. The *Zariski topology* is the topology on $\mathbb{P}^n$ whose open sets are Zariski open sets. The subset $W \subset V$ is called a *subvariety* of $V \subset \mathbb{P}^n$ if W is a projective variety in $\mathbb{P}^n$.

At the end of this section we discuss the rational function on a projective variety. We have seen that a homogeneous polynomial in $x_0, \ldots, x_n$ does not give a function on $\mathbb{P}^n$. However the quotient of two such polynomials does if they have the same degree. Now, suppose that $F, G \in \mathbb{C}[x_0, \ldots, x_n]$ homogeneous polynomials of degree d and that $G \neq 0$. Then we obtain a well-defined function $\frac{F}{G} : \mathbb{P}^n \setminus \mathbb{V}(G) \to \mathbb{C}$. As in Section 2.1, we can write this as $\frac{F}{G} : \mathbb{P}^n \dashrightarrow \mathbb{C}$ and it is a rational function on $\mathbb{C}$. Thus, for an irreducible projective variety V we define $\mathbb{C}(V) := \left\{ \frac{F}{G} \;\middle|\; F, G \text{ homogeneous and } \deg F = \deg G,\ G \notin \mathbb{I}(V) \right\} \Big/ \sim$ where the relation is defined as $\frac{F}{G} \cong \frac{F'}{G'}$ if and only if $FG' - GF' \in \mathbb{I}(V)$.

2.3 Algebraic (Abstract) Variety

Recall that in Proposition 2.1.13 we have identified affine varieties with reduced finitely generated, $\mathbb{C}$-algebras. If we remove these restrictions we obtain a new object of an algebraic geometry, called an affine scheme. This means that an affine scheme is a tool obtained from a commutative ring R. Because all of the differences between the schemes theory and the theory of abstract varieties are clashed in the affine case, we will focus on the notion of an affine scheme to define an affine variety, which parallels our construction of an affine variety in Section 2.1. As in Section 2.1 there is a one to one correspondence between a ring and an affine scheme. Studying with schemes admits global constructions in our process, so will describe an abstract variety by using an affine scheme. All statements can be found in Eisenbud & Harris (2000) and Hartshorne (1977). To construct a scheme we need to define a sheaf, which includes more local data on a topological space.

Definition 2.3.1. Let X be a topological space. A family with the following properties:

i) $\mathcal{F}(U)$ is an abelian group, for all open subset U of X,

ii) For any inclusion $V \subset U$ of open subsets of X, there is a morphism of abelian groups $\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$ such that

  a) $\mathcal{F}(\emptyset) = 0$,

  b) $\rho_{UU} : \mathcal{F}(U) \to \mathcal{F}(U)$ is the identity map,

  c) If $W \subset V \subset U$ are open, then $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$

is called a *presheaf* $\mathcal{F}$ of abelian groups on X.

Remark 2.3.2. For an open set $U \subset X$, elements of $\mathcal{F}(U)$ are called *sections*, denoted by $\Gamma(U,\mathcal{F})$. Elements of $\Gamma(X,\mathcal{F})$ are called *global sections*. The maps ρ_{UV} are called *restrictions* and denoted by $s\,|_V$ for simplicity.

Let $\mathcal{F}$ and $\mathcal{G}$ be two presheaves on X. We can define a *morphism of presheaves*, $\varphi : \mathcal{F} \to \mathcal{G}$, as a morphism of an abelian groups $\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ for any open set U with the commutative diagram,

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\varphi(U)} & \mathcal{G}(U) \\ \downarrow & & \downarrow \\ \mathcal{F}(V) & \xrightarrow[\varphi(V)]{} & \mathcal{G}(V) \end{array}$$

for each inclusion $V \subset U$.

Remark 2.3.3. If $\mathcal{F}$ is a presheaf on X and U is an open subset of X, we can define a presheaf $\mathcal{F}|_U$ on U by setting $\mathcal{F}|_U(V) = \mathcal{F}(V)$ for any open subset V of U, which is called the restriction of $\mathcal{F}$ to U.

A sheaf $\mathcal{F}$ on X is a presheaf that satisfies the gluing axiom.

Definition 2.3.4. A presheaf $\mathcal{F}$ on a topological space X is a *sheaf* if it satisfies the following additional conditions:

i) If $U = \bigcup V_i$ is an open covering, and $s \in \mathcal{F}(U)$ such that $s|_{V_i} = 0$ for all i, then $s = 0$.

ii) If $U = \bigcup V_i$ is an open covering, and $s_i \in \mathcal{F}(V_i)$ for each i such that $s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$ for all j, then there exist $s \in \mathcal{F}(U)$ such that $s|_{V_i} = s_i$ for each i, (this guarantees that s is unique).

Note that we can define a *morphism of sheaves* to be the same as a morphism of presheaves.

Definition 2.3.5. A *subsheaf* of a sheaf $\mathcal{F}$ is a sheaf $\mathcal{F}'$ such that for every open set $U \subset X$, $\mathcal{F}'(U)$ is a subgroup of $\mathcal{F}(U)$, and the restriction maps of the sheaf $\mathcal{F}'$ are induced by those of $\mathcal{F}$.

On the other hand, there is another way to describe sheaf; sheaf by its stalks.

Definition 2.3.6. If $\mathcal{F}$ is a presheaf on X, and P is a point of X, we define the *stalk* $\mathcal{F}_P$ of $\mathcal{F}$ at P to be the direct limit of the groups $\mathcal{F}(U)$ for all open set U containing P, via the restriction maps ρ, i.e., $\mathcal{F}_P = \varinjlim \mathcal{F}(U) = \bigsqcup_{P \in U \subset X} \mathcal{F}(U) \Big/ \sim$.

An element of $\mathcal{F}_P$ is represented by a pair $< U, s >$ where U is an open neighbourhood of P, and s is an element of $\mathcal{F}(U)$. We can define an equivalence relation $\sim$ as follows: $< U, s >$ and $< V, t >$ define the same element if and only if there exists a neighbourhood W containing P with $W \subset U \cap V$ such that $s|_W = t|_W$. Thus we have equivalence classes on $\mathcal{F}(U)$. Therefore, one may speak of elements of the stalk $\mathcal{F}_P$ as *germs* of sections of $\mathcal{F}$ at the point P.

So far we have talked about a presheaf of abelian groups and their basic properties, but we can define a presheaf (or sheaf) of rings. Now, we are able to describe affine schemes: to any coordinate ring $\mathbb{C}[V]$ of an affine variety V we associate a topological space together with a structure sheaf on it, $\mathrm{Spec}\mathbb{C}[V]$.

Firstly, we need to construct a space $\mathrm{Spec}\mathbb{C}[V]$ as a set. We have defined the spectrum of a commutative ring as a set in Subsection 2.1.1, but in this case we take a coordinate ring $\mathbb{C}[V]$ instead of a commutative ring R. In particular, points of $\mathrm{Spec}\mathbb{C}[V]$ were identified points of the affine variety V, maximal ideals of $\mathbb{C}[V]$, and also irreducible subvarieties of V.

The next step is to define a topology on a space $\mathrm{Spec}\mathbb{C}[V]$. We can consider a regular function on $\mathrm{Spec}\mathbb{C}[V]$ as an element of $\mathbb{C}[V]$. By using regular functions, we transform $\mathrm{Spec}\mathbb{C}[V]$ into a topological space; this topology is called the *Zariski topology* with closed sets: $\mathbb{V}(S) = \{P \in \mathrm{Spec}\mathbb{C}[V] \mid f(P) = 0, \forall f \in S\} = \{\mathfrak{p} \in \mathrm{Spec}\mathbb{C}[V] \mid \mathfrak{p} \supset S\}$, for each subset $S \subset R$. If $f \in \mathbb{C}[V]$, we define the principal open subset of $V = \mathrm{Spec}\mathbb{C}[V]$ associated with f to be $V_f = \mathrm{Spec}\mathbb{C}[V] \setminus \mathbb{V}(f)$.

Finally, to complete the definition of $\mathrm{Spec}\mathbb{C}[V]$, we have to describe the structure sheaf $\mathcal{O}_V = \mathcal{O}_{\mathrm{Spec}\mathbb{C}[V]}$. The structure sheaf of an irreducible affine variety $V = \mathrm{Spec}\mathbb{C}[V]$ is the sheaf of $\mathbb{C}$-algebras in the Zariski topology which is defined as follows: given a Zariski open $U \subset V$, a function $f : U \to \mathbb{C}$ is regular if for every $P \in V$, there is $f_P \in \mathbb{C}[V]$ such that $P \in V_{f_P} \subset U$ and $\phi\,|_{V_{f_P}} \in \mathbb{C}[V]_{f_P}$. Then

$$\mathcal{O}_V(U) = \{f : U \to \mathbb{C} \mid f \text{ is a regular function}\}$$

is a sheaf of $\mathbb{C}$-algebras. Let us establish an important property of the structure sheaf $\mathcal{O}_V$.

Theorem 2.3.7. *Let* $V = \mathrm{Spec}\mathbb{C}[V]$ *be an irreducible affine variety. Then the structure sheaf* $\mathcal{O}_V$ *has the following properties:*

i) $\mathcal{O}_V(U) = \mathbb{C}[V]$.

ii) If $f \in \mathbb{C}[V]$*, then* $\mathcal{O}_V |_{V_f} = \mathcal{O}_{V_f}$.

This theorem tells us that $\mathcal{O}_V(V_f) = \mathcal{O}_V |_{V_f} (V_f) = \mathcal{O}_{V_f}(V_f) = \mathbb{C}[V]_f$ when $V = \mathrm{Spec}\mathbb{C}[V]$ and $f \in \mathbb{C}[V]$.

Definition 2.3.8. A *ringed space* is a pair $(X, \mathcal{O}_X)$ consisting of a topological space X and a sheaf of rings $\mathcal{O}_X$ on X. The ringed space $(X, \mathcal{O}_X)$ is a *locally ringed space* if the stalk of X is a local ring for each point $P \in X$.

Now, we are ready to define our main concept, affine scheme, in this section.

Definition 2.3.9. An *affine scheme* is a locally ringed space $(X, \mathcal{O}_X)$ which is isomorphic to the spectrum of some ring. An *abstract variety* $(X, \mathcal{O}_X)$, say simply X, is a ringed space over $\mathbb{C}$ where each $P \in X$ has a neighbourhood U such that the restriction $(U, \mathcal{O}_X |_U)$ is isomorphic to $(V, \mathcal{O}_V)$ for some affine variety V.

Remark 2.3.10. If X is an affine scheme, then the dimension of X is the same as the Krull dimension of $\mathbb{C}[X]$.

Given an abstract variety X, an open $U \subset X$ is called a *Zariski open* if $(X, \mathcal{O}_X |_U)$ is isomorphic to the ringed space of an affine variety. Two rational functions are equivalent if they agree on some nonempty Zariski open. The set of equivalence classes is denoted by $\mathbb{C}(X)$ and is called the function field of X. Thus one can define a local ring:

Definition 2.3.11. The *local ring* of V at P is $\mathcal{O}_{X,P} = \{\phi \in \mathbb{C}(X) \mid \phi \text{ is defined at } P\}$ with maximal ideal $\mathcal{M}_{X,P} = \{\phi \in \mathcal{O}_{X,P} \mid \phi(P) = 0\}$.

Example 2.3.12. Consider the projective space $\mathbb{P}^n$. Now we will show that $\mathbb{P}^n$ is an abstract variety. Let $U \subset \mathbb{P}^n$ be a Zariski open and $\phi : U \to \mathbb{C}$ be a regular function such that for each $P \in U$ there exists $f/g \in \mathbb{C}(\mathbb{P}^n)$ with $g(P) \neq 0$ and $\phi |_{U \cap \mathbb{V}(g)} = (f/g) |_{U \cap \mathbb{V}(g)}$. Then we obtain a structure sheaf on $\mathbb{P}^n$ as follows:

$$\mathcal{O}_{\mathbb{P}^n}(U) = \{\phi : U \to \mathbb{C} \mid \phi \text{ is a regular function}\}.$$

In Section 2.2, we defined the affine open sets U_i, and obtained $U_i \cong \mathbb{C}^n$. This gives an isomorphism $\mathbb{C}(\mathbb{P}^n) \cong \mathbb{C}(x_0/x_i, \ldots, x_{i-1}/x_i, x_{i+1}/x_i, \ldots, x_n/x_i)$. Thus, the ringed space $(U_i, \mathcal{O}_{\mathbb{P}^n}|_{U_i})$ is isomorphic to $(\mathbb{C}^n, \mathcal{O}_{\mathbb{C}^n})$ for an affine variety $\mathbb{C}^n$.

At last, we describe the morphism of abstract varieties. A *morphism of abstract varieties* from X to Y is a pair of a continuous map $f : X \to Y$ and a map $f^{\#} : \mathcal{O}_Y(U) \to \mathcal{O}_X(f^{-1}(U))$ of sheaves of rings on W for each open set U such that $f^{\#}$ is compatible with restriction maps and the induced map $f^{\#} : \mathcal{O}_{Y,f(P)} \to \mathcal{O}_{X,P}$ satisfies $\mathcal{M}_{Y,f(P)} = (f^{\#}_P)^{-1}(\mathcal{M}_{X,P})$. Let R and S be any two commutative rings. If $X = \mathrm{Spec}R$ and $Y = \mathrm{Spec}S$ are irreducible affine varieties then a morphism is equivalent to $\mathbb{C}$-algebra homomorphism.

2.3.1 Gluing with Affine Varieties

The definition of an abstract variety implies that X has an affine cover U_α, so that $U_\alpha \underset{f_\alpha}{\cong} V_\alpha$ where V_α is an affine variety. Then the set $V_{\alpha,\beta} = f_\alpha(U_\alpha \cap U_\beta) \subset V_\alpha$ is a Zariski open in V_α and the map $g_{\alpha,\beta} = f_\beta \circ f_\alpha^{-1} : V_{\alpha,\beta} \to V_{\beta,\alpha}$ is an isomorphism of Zariski open subsets for any α,β. Moreover, these maps have the following properties, called *compatibility conditions*:

i) $g_{\alpha,\alpha} = 1_{V_\alpha}$, for all α,

ii) $g_{\beta,\alpha}\big|_{V_{\beta,\alpha}\cap V_{\beta,\gamma}} \circ g_{\alpha,\beta}\big|_{V_{\alpha,\beta}\cap V_{\alpha,\gamma}} = g_{\alpha,\gamma}\big|_{V_{\alpha,\beta}\cap V_{\alpha,\gamma}}$, for all α,β,γ.

Now, suppose we have a collection $\{\{V_\alpha\}_\alpha, \{V_{\alpha,\beta}\}_{\alpha,\beta}, \{g_{\alpha,\beta}\}_{\alpha,\beta}\}$ where each V_α is an affine variety, $V_{\alpha,\beta} \subset V_\alpha$ is Zariski open and $g_{\alpha,\beta} : V_{\alpha,\beta} \to V_{\beta,\alpha}$ are isomorphisms which satisfy the compatibility conditions. Then we get the topological space $X = \bigsqcup_\alpha V_\alpha / \sim$ where the relation is defined as; $(a \in V_\alpha) \sim (b \in V_\beta)$ if $a \in V_{\alpha,\beta}$ and $b = g_{\alpha,\beta}(a)$. And the structure sheaves $\mathcal{O}_{V_\alpha}$ patch to give a sheaf $\mathcal{O}_V$. So, X is a variety with an affine open cover U_α such that $U_\alpha \cong V_\alpha$ for every α. This means that, a variety X is constructed by gluing together affine varieties along Zariski open subsets $V_{\alpha,\beta}$ by the map $g_{\alpha,\beta}$, see Cox (2000b).

Example 2.3.13. Let $V_0 = V_1 = \mathbb{C}$, $V_{0,1} = V_{1,0} = \mathbb{C}^*$ and $g_{0,1}(x) = g_{1,0}(x) = x^{-1}$. Then we take the disjoint union of V_0 and V_1 and the equivalence relation which identifies points under the gluing. Thus, we obtain

$$\begin{aligned} X &= V_0 \sqcup V_1 \Big/ (x \in V_{0,1} \sim g_{0,1}(x) \in V_{1,0}) \\ &= \{(a_0, a_1) \mid a_i \neq 0,\ i = 1,2\} \\ &\cong \mathbb{P}^1. \end{aligned}$$

2.3.2 Sheaves on Modules

One of another most important constructions of presheaf is that of a presheaf of modules $\mathcal{F}$ over a presheaf of rings $\mathcal{O}$ on a space X and also sheaf. The notion of smoothness, we will especially introduce in Section 3.3, is related with the notion of differentiability. So we need to investigate the notion of differentiability. All statements can be found in Eisenbud & Harris (2000), Hartshorne (1977).

Definition 2.3.14. Let $(X, \mathcal{O}_X)$ be a ringed space. A *sheaf of $\mathcal{O}_X$-modules* is a sheaf $\mathcal{F}$ on X, such that the group $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module for each open set $U \subset X$ and for each inclusion of open sets $V \subset U$, the restriction homomorphism $\mathcal{F}(U) \to \mathcal{F}(V)$ is compatible with the module structures by the ring homomorphism $\mathcal{O}_X(U) \to \mathcal{O}_X(V)$.

A *morphism* $\mathcal{F} \to \mathcal{G}$ of sheaves of $\mathcal{O}_X$-modules is defined as the morphism of sheaves, such that for each open set $U \subset X$, the map $\mathcal{F}(U) \to \mathcal{G}(U)$ is a homomorphism of $\mathcal{O}_X(U)$-modules.

The direct sum $\bigoplus_{i \in I} \mathcal{F}_i$ of sheaves, is defined by the presheaf $U \mapsto \bigoplus_{i \in I} \Gamma(U, \mathcal{F}_i)$ for open subset $U \subset X$. In particular, if the index set I is finite, then it is a sheaf.

Definition 2.3.15. An $\mathcal{O}_X$-module $\mathcal{F}$ is *free*, if it is isomorphic to a direct sum of copies of $\mathcal{O}_X$. It is *locally free* if X can be covered by open sets U for which $\mathcal{F}|_U$ is a free $\mathcal{O}_X|_U$-module. In the case of I is finite, its number of elements is called the *rank* of $\mathcal{F}$. A locally free sheaf of rank 1 is also called an *invertible sheaf*.

The general notion of a sheaf of modules on a ringed space is a sheaf associated which is defined on Spec$R$.

Definition 2.3.16. Let R be a ring and let M be an R-module. For each prime $\mathfrak{p} \subset R$, let $\mathcal{M}_{\mathfrak{p}}$ be the localization of $\mathfrak{p}$. For any open set $U \subset \text{Spec}R$ define the group $\tilde{M}(U)$ to be the set of functions $s : U \to \bigsqcup_{\mathfrak{p} \in U} M_{\mathfrak{p}}$ such that for each $\mathfrak{p} \in U$ there is a neighbourhood V of $\mathfrak{p}$ in U, and there are elements $\mathfrak{m} \in M$ and $f \in R$ such that for each $\mathfrak{q} \in V$, $f \in \mathfrak{q}$ and $s(\mathfrak{q}) = \mathfrak{m}/f$ in $M_{\mathfrak{q}}$. Such $\tilde{M}$ is called a *sheaf associated* to M on Spec$R$.

Definition 2.3.17. A sheaf of O_X-modules $\mathcal{F}$ is *quasicoherent* if X can be covered by open affine subsets $U_i = \text{Spec}R_i$, such that for each i there is an R_i-module M_i with $\mathcal{F}|_{U_i} \cong \tilde{M}_i$. $\mathcal{F}$ is called *coherent* if additionally each M_i is finitely generated R_i-module.

Example 2.3.18. Let X be an any affine scheme. The structure sheaf O_X is coherent.

2.3.3 Differentials and Applications

Firstly we will introduce the module of differentials of one ring over another. And then we generalize this idea. Let R be a commutative ring with identity and let B be an R-algebra and let M be a B-module.

Definition 2.3.19. An *R-derivation* of B into M is a map $d : B \to M$ such that

i) d is additive,

ii) $d(bb') = bdb' + b'db$ (Leibniz's Rule),

iii) $dr = 0$ for all $r \in R$.

Definition 2.3.20. The *module of relative differential forms* of B over R is defined to be a B-module $\Omega_{B/R}$, with an R-derivation $d : B \to \Omega_{B/R}$ defined as $b \mapsto db$, which satisfies the following property: for any B-module M and R-derivation $d' : B \to M$, there exists a unique B-module homomorphism $f : \Omega_{B/R} \to M$ such that $f \circ d = d'$. It follows that $\Omega_{B/R}$ is generated as a B-module by $\{db \mid b \in B\}$.

Proposition 2.3.21. *(Hartshorne, 1977, Proposition 8.1A, page 173) Let $f : B \otimes_R B \to B$ be the diagonal homomorphism defined by $f(b \otimes b') = bb'$, and let $I = \text{Ker}(f)$.*

Consider $B \otimes_R B$ as B-module by multiplication on the left. Then I/I^2 inherits a structure of B-module. Let a map $d : B \to I/I^2$ defined by $db = 1 \otimes b - b \otimes 1$. Then $< I/I^2, d >$ is a module of relative differentials for B/R.

Example 2.3.22. Let $B = \mathbb{C}[x_1, \ldots, x_n]$ be a polynomial ring over $\mathbb{C}$. Then $\Omega_{B/\mathbb{C}}$ is the free B-module of rank n generated by $dx_1, \ldots, dx_n$, where $x_1, \ldots, x_n$ are affine coordinates of $\mathbb{C}^n$.

Definition 2.3.23. Let Y be any subscheme of a scheme X.

1) The quotient $I/I^2 = I \otimes_{\mathcal{O}_X} \mathcal{O}_Y$ can be regarded as a coherent sheaf on Y, and called *conormal sheaf.* Its dual $\mathcal{N}_{Y/X} \coloneqq \mathrm{Hom}_{\mathcal{O}_Y}((I/I^2)|_Y, \mathcal{O}_Y)$ is called the *normal sheaf* of the embedding $Y \subset X$.

2) The *tangent sheaf* of Y is $\Theta_Y \coloneqq \mathrm{Hom}_{\mathcal{O}_Y}(\Omega^1_{Y/\mathbb{C}}, \mathcal{O}_Y)$.

Let X be a scheme over Y. Then *the sheaf of relative differentials* of X over Y is the conormal sheaf to the diagonal in $X \times_Y X$, and denoted by $\Omega^1_{X/Y}$. This sheaf is a coherent sheaf on X. Additionally, a sheaf $\Omega^r_{X/Y}$ is a higher order differential and computed by an exterior powers. Note that, for each coherent $\mathcal{O}_Y$-sheaf M, there is a canonical isomorphism of $\mathcal{O}_X$-modules $\mathrm{Hom}(\Omega_Y, M) \xrightarrow{\cong} \mathrm{Der}_{\mathbb{C}}(\mathcal{O}_Y, M)$ defined by $\varphi \mapsto \varphi \circ d$, where $d : \mathcal{O}_Y \to \Omega_Y$ is the exterior derivation and $\mathrm{Der}_{\mathbb{C}}(\mathcal{O}_Y, M)$ is the sheaf of $\mathbb{C}$-derivations of $\mathcal{O}_Y$ with values in M. In particular, we have $\Theta_Y \cong \mathrm{Der}_{\mathbb{C}}(\mathcal{O}_Y, \mathcal{O}_Y)$.

Furthermore, the sheaf Ω_X is locally free with $\Omega_X = \bigoplus_{i=1}^{n} \mathcal{O}_X \cdot dx_i$ where $x_1, \ldots, x_n$ are local coordinates of X. As a consequence Θ_X is a locally free of rank n and

$$\Theta_X = \bigoplus_{i=1}^{n} \mathcal{O}_X \cdot \frac{\partial}{\partial x_i},$$

where $\partial x_1, \ldots, \partial x_n$ is the dual basis of $dx_1, \ldots, dx_n$.

Let $f \in \mathcal{O}_X$. Then in local coordinates, we have $df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i$. In particular, we can define an $\mathcal{O}_X$-linear map $\alpha : I \to \Omega^1_X$ defined as $f \mapsto df$. By the Leibniz rule, α induces a map $\alpha : I/I^2 \to \Omega_X \otimes_{\mathcal{O}_X} \mathcal{O}_Y$, gives the exact sequence $I/I^2 \xrightarrow{\alpha} \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{O}_Y \to \Omega^1_X \to 0$. Taking its dual, we obtain the exact sequence

$$0 \to \Theta_Y \to \Theta_X \otimes_{\mathcal{O}_X} \mathcal{O}_Y \xrightarrow{\beta} \mathcal{N}_{Y/X}, \tag{2.3.1}$$

where β is the dual of α.

In the local coordinates, we have $\Theta_{X,P} \otimes_{\mathcal{O}_{X,P}} \mathcal{O}_{Y,y} = \bigoplus_{i=1}^{n} \mathcal{O}_{Y,y} \cdot \dfrac{\partial}{\partial x_i}$, and the image $\beta(\frac{\partial}{\partial x_i}) \in \mathrm{Hom}_{\mathcal{O}_{Y,y}}\left(I_Y/I_Y^2, \mathcal{O}_{Y,y}\right)$ sends a residue class $[h] \in I_Y/I_Y^2$ to $\left[\frac{\partial h}{\partial x_i}\right] \in \mathcal{O}_{Y,y}$, where I_Y is subsheaf of ideals of $\mathcal{O}_X$ consisting of the sections that vanish on Y.

CHAPTER THREE
TORIC VARIETIES

Toric varieties are special type in the scheme theory. The reason for this, toric varieties allow a more simple and impressive description that uses objects from elementary convex and combinatorial geometry. These objects are “convex polyhedral cones” and their compatible collection so called “fans”, in a real vector space of dimension equal to complex dimension of a variety. It follows that there is a one-to-one correspondence between toric varieties and combinatorial objects. Thus, this makes everything more computable than the usual one. The fundamental references for this chapter are Cox et al. (2011), Ewald (1996) and Fulton (1993).

3.1 Affine Toric Variety

In this section we describe rational polyhedral cones and then explain how they relate to affine toric varieties. We start by giving some fundamental notions from convex geometry, see Oda (1985), Grünbaum & Ziegler (2003).

A set $\sigma \subset \mathbb{R}^n$ is *convex* if and only if for each pair of distinct points $a, b \in \sigma$ the closed segment with end points a and b is contained in σ. We can consider any linear subspace of $\mathbb{R}^n$ as a convex set. A set $\sigma \subset \mathbb{R}^n$ is *cone* if and only if for all $u \in \sigma$ and $\lambda \in \mathbb{R}$ implies that $\lambda u \in \sigma$. A set $\sigma \subset \mathbb{R}^n$ is *polyhedral* if for all $x \in \sigma$ are written as a linear combination of only finite elements. Now we are ready to give our main combinatoric objects, called a convex polyhedral cones.

Definition 3.1.1. Let $S = \{u_1, \ldots, u_r\}$ be a finite set of vectors in $\mathbb{R}^n$. The set

$$\sigma = \left\{ u \in \mathbb{R}^n \,\middle|\, u = \sum_{i=1}^{r} \lambda_i u_i \ , \ \lambda_i \in \mathbb{R}_{\geq 0} \right\}$$

is called *convex polyhedral cone* and the vectors u_i’s are called *generators* of σ, denoted by $\sigma =< u_1, \ldots, u_r >$.

In particular, if we take $S = \emptyset$, then $\sigma = \{0\}$. To understand better we will investigate the below examples. Let $\{e_i\}$ be the standart basis of $\mathbb{R}^n$, for $i = 1,\ldots,n$.

Example 3.1.2. Let $S = \{u_1 = e_1, u_2 = e_2\}$. Then applying Definition 3.1.1, we obtain

$$\begin{aligned} \sigma &= \{u \in \mathbb{R}^n \mid u = \lambda_1(1,0) + \lambda_2(0,1)\ ,\ \lambda_{1,2} \in \mathbb{R}_{\geq 0}\} \\ &= \text{First quadrant of } \mathbb{R}^2. \end{aligned}$$

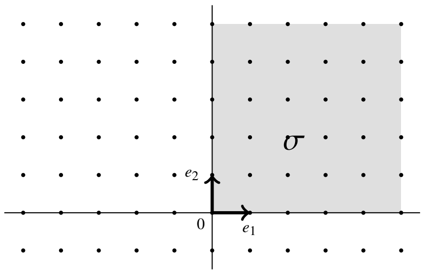


Figure 3.1 The cone σ generated by e_1 and e_2

Example 3.1.3. The largest possible convex polyhedral cone is $\mathbb{R}^n$, generated by $u_1 = \pm e_1,\ldots,u_n = \pm e_n$, while the smallest is the trivial cone $o = \{0\}$.

Definition 3.1.4. Let N be a subgroup in $\mathbb{R}^n$ containing the origin. N is called a *lattice* if it is a discrete group with respect to addition. N is a discrete group means that for all $x \in N$ there exists a neighbourhood U containing x such that $U \cap N = \{x\}$.
Let $S = \{v_1,\ldots,v_n\}$ be a linearly independent subset of $\mathbb{R}^n$. A lattice in $\mathbb{R}^n$, generated by S, can be described as follows: $N = \{z_1v_1 + \cdots + z_nv_n \mid z_i \in \mathbb{Z},\ 1 \leq i \leq n\}$. An element v in the lattice N is called a *lattice point* and v_i's are called a *basis* for the lattice N.

We will study with the *standard lattice* $N \cong \mathbb{Z}^n = \mathbb{Z}\times\cdots\times\mathbb{Z}$. In particular, a lattice N is a finitely generated free abelian group such that $N = \mathbb{Z}\cdot e_1 \oplus\cdots\oplus \mathbb{Z}\cdot e_n$, where $\{e_i\}_{i=1}^n$ is a standard basis of $\mathbb{R}^n$. If we want to talk about vectors we must pass to real vector space $N_{\mathbb{R}} = \mathbb{R}\cdot e_1 \oplus\cdots\oplus \mathbb{R}\cdot e_n \cong \mathbb{R}^n$. Thus, we can consider the convex polyhedral cone as a subset of $N_{\mathbb{R}}$, i.e., we can write $\sigma \subset N_{\mathbb{R}}$. Now we will define our main tool, so-called strongly convex rational polyhedral cone, to construct an affine toric variety.

Definition 3.1.5. A cone σ is a *rational (or lattice) cone* if all generators $u_i \in S$ of σ belongs to N.
A cone σ is *strongly convex* if it does not contain any straight line going through the origin. In other words, $\sigma \cap (-\sigma) = \{0\}$.
The *dimension* of a cone σ is the dimension of the smallest linear space containing σ, and denoted by $\dim(\sigma)$. Note that, $\dim(\sigma) = \dim(\sigma + (-\sigma))$.

Example 3.1.6. Consider the cone $\sigma =< e_1, e_1 + e_2 >$ in $N_{\mathbb{R}} \cong \mathbb{R}^2$, see Figure 3.2. Since the generators of σ are in N this cone is rational and since $\sigma \cap (-\sigma) = \{0\}$, it is strongly convex. The dimension is 2, because the smallest linear space containing σ is $\mathbb{R}^2$.

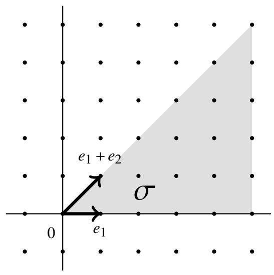


Figure 3.2 The cone σ in $N_{\mathbb{R}}$ with a lattice $N = \mathbb{Z}^2$

Definition 3.1.7. The *dual lattice* of a lattice N is defined by

$$M = \mathrm{Hom}(N, \mathbb{Z}) = \{v : N \to \mathbb{Z} \mid v(u) = (u, v),\ \forall u \in N\}.$$

If we take $\pm e_1^*, \ldots, \pm e_n^*$ as a basis for M, then

$$(e_i, e_j^*) = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

is satisfied.

In this definition (,) coincides with the usual inner product $\langle , \rangle$ in $\mathbb{R}^n$. On the dual level, we will work over a real vector space corresponding to M such that

$$M_{\mathbb{R}} = \mathbb{R} \cdot e_1^* \oplus \cdots \oplus \mathbb{R} \cdot e_n^* \cong (\mathbb{R}^n)^* = \mathbb{R}^n.$$

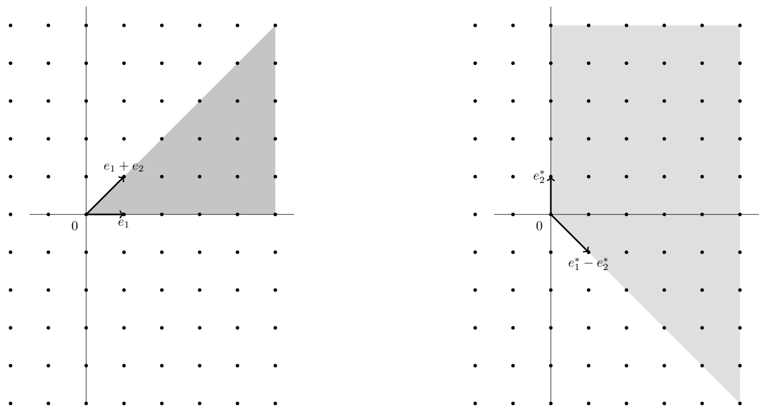
Definition 3.1.8. The *dual cone* of a cone σ is the subset of $M_{\mathbb{R}}$ defined by

$$\check{\sigma} = \{v \in M_{\mathbb{R}} \mid \langle u, v\rangle \geq 0,\ \forall u \in \sigma\}.$$

Example 3.1.9. Consider the cone σ given in Example 3.1.6, (see Figure 3.3a) where $u_1 = e_1$ and $u_2 = e_1 + e_2$. The generators of $\check{\sigma}$ are of the form $v_1 = ae_1^* + be_2^*$ and $v_2 = ce_1^* + de_2^*$ where $a, b, c, d \in \mathbb{R}$, since M is generated by $\pm e_1^*, \pm e_2^*$. Then by Definition 3.1.8, we have to find vectors in $M_{\mathbb{R}}$ such that they are perpendicular to a vector in $N_{\mathbb{R}}$ and $\langle\ ,\ \rangle \geq 0$ for other elements in $N_{\mathbb{R}}$. In other words, $\langle u_i, v_j\rangle = 0$ if $i = j$ and $\langle u_i, v_j\rangle > 0$ if $i \neq j$, for $i, j = 1, 2$. Then we have two systems of inequalities such that:

$$\begin{array}{ll} a+b>0 & c+d=0 \\ a=0 & c>0 \end{array}$$

Thus, we get $v_1 = be_2^*$ and $v_2 = ce_1^* - de_2^*$. This means that $\check{\sigma} =< e_2^*, e_1^* - e_2^* >$, see Figure 3.3b.



(a) $\sigma =< e_1, e_1 + e_2 >$ (b) $\check{\sigma} =< e_2^*, e_1^* - e_2^* >$

Figure 3.3 Cone with its dual

Proposition 3.1.10 (Duality Theorem). $\check{\check{\sigma}} = \sigma$ *for any cone* $\sigma \subset N_{\mathbb{R}}$.

Definition 3.1.11. Let v be a nonzero vector in $M_{\mathbb{R}}$. The set $v^{\perp} = H_v = \{u \in N_{\mathbb{R}} \mid \langle u, v\rangle = 0\}$ is called a *hyperplane* and the set $H_v^+ = \{u \in N_{\mathbb{R}} \mid \langle u, v\rangle \geq 0\}$ is called a *closed half space*.

We can define a face of a cone σ by using hyperplanes and closed half-spaces.

Definition 3.1.12. Let σ be a cone and let $v \in \check{\sigma} \cap M$. A *face* of a cone σ is defined to be a set $\tau = \sigma \cap H_v = \tau \cap v^{\perp} := \{u \in \sigma \mid \langle u, v\rangle = 0\}$ for some $v \in \check{\sigma}$ and denoted by $\tau \leq \sigma$. An *edge* (*ray*) of a cone is a one-dimensional face and faces different from σ are called *proper faces*. A face of codimension one is called a *facet* of σ. We can consider the cone σ as a face of itself.

Let us investigate some fundamental properties of a convex polyhedral cone σ and its faces.

Lemma 3.1.13. *There is an inclusion reversing relation between a cone σ and its face τ such that if $\tau \leq \sigma$, then $\check{\tau} \supset \check{\sigma}$.*

Remark 3.1.14. $\sigma = \sigma_1 + \sigma_2$ implies $\check{\sigma} = \check{\sigma}_1 \cap \check{\sigma}_2$.

Proposition 3.1.15. *(Cox et al., 2011, Lemma 1.2.6, page 25) Let σ be a convex polyhedral cone and τ be its face. Then we have following properties:*

i) τ is also a convex polyhedral cone,

ii) Every intersection of faces of σ is again a face of σ,

iii) The face ρ of τ is also a face of σ.

Proposition 3.1.16. *(Fulton, 1993, Property 8, page 11) Suppose that $\sigma \in N_{\mathbb{R}}$ is an n-dimensional convex polyhedral cone such that $\sigma \neq N_{\mathbb{R}}$. Let the facets of σ be $\tau_i = v_i^{\perp} \cap \sigma$, where $\sigma \subset H_{v_i}^{+}$ for $i = 1, \ldots, s$. Then σ is an intersection of closed-half spaces, that is $\sigma = H_{v_1}^{+} \cap \cdots \cap H_{v_s}^{+}$.*

Proposition 3.1.17 (Farkas' Lemma)**.** *The dual of a convex polyhedral cone is a convex polyhedral cone.*

In particular, the dual of a rational cone is also rational. But if σ is a strongly convex cone, then $\check{\sigma}$ need not to be a strongly convex. For example, consider the cone $\sigma =< e_2 >$ and it's dual cone $\check{\sigma} =< e_1, -e_1 >$ in $N_{\mathbb{R}} \cong \mathbb{R}^2$. The dual cone $\check{\sigma}$ is not a strongly convex cone while σ is, since $\check{\sigma} \cap (-\check{\sigma}) = \check{\sigma} \neq \{0\}$. As an end, our aim is to identify the faces of a cone σ and the faces of its dual cone $\check{\sigma}$. To do this we need to define the relative interior of a cone.

Definition 3.1.18. The topological interior of the space $\mathbb{R}\cdot\sigma$ generated by σ is called the *relative interior* of a cone σ, denoted by Relint(σ).

Remark 3.1.19. We can take the positive linear combination of m linearly independent vectors among the generators of σ to obtain the relative interior of a cone σ, where $m = \dim(\sigma)$. If σ is a lattice cone, then these points can be in lattice N.

Now, we define a set $\{v \in \check{\sigma} \mid \langle u, v\rangle = 0, \text{ for all } u \in \tau \preceq \sigma\} = \check{\sigma} \cap \tau^{\perp}$ to describe a face of a dual cone $\check{\sigma}$.

Theorem 3.1.20. *(Fulton, 1993, Property 10, page 12) If $\tau \preceq \sigma$, then $\check{\sigma} \cap \tau^{\perp}$ is a face of $\check{\sigma}$ with the property $dim(\tau) + dim(\check{\sigma} \cap \tau^{\perp}) = n = dim(\sigma)$. This gives a one-to-one inclusion-reversing correspondence between the faces of σ and the faces of $\check{\sigma}$.*

3.1.1 Semigroup and Semigroup Algebras

This part is a second step to construct a toric variety. More explicitly, we will construct a semigroup by using the elements of a dual cone.

Definition 3.1.21. A *monoid* S is a non-empty set with an associative binary operation $+ : S \times S \to S$. If it has an identity element, it is called a *semigroup*. In a semigroup, every element need not has an inverse. A semigroup S is said to be *commutative* if the operation + is commutative. Now suppose that a semigroup S satisfies the *cancelation property*: $s + x = t + x \Rightarrow s = t$, for all $s, t, x \in S$ then S is called *cancellative*.

Remark 3.1.22. Let S and T be two semigroups. A map $f : S \to T$ is called a semigroup homomorphism if $f(a + b) = f(a) + f(b)$ for every a and b in S and $f(0_S) = 0_T$.

Definition 3.1.23. A semigroup S is said to be *finitely generated* if there exist $a_1, \dots, a_r \in S$, such that $\forall s \in S$, $s = \lambda_1 a_1 + \cdots + \lambda_r a_r$ with $\lambda_i \in \mathbb{Z}_{\geq 0}$. The elements $a_1, \dots, a_r$ are called *generator* of the semigroup.

Let S be a finitely generated semigroup with generators $\{a_1, \dots, a_r\}$. S can be embedded as a semigroup into a group $G(S)$ which has $a_1, \dots, a_r$ as group generators (coefficients in $\mathbb{Z}$) such that $G(\sigma \cap N) = (\sigma + (-\sigma)) \cap \mathbb{Z}^n$ where $N \cong \mathbb{Z}^n$.

Theorem 3.1.24 (Gordon's Lemma)**.** *If σ is a rational polyhedral cone, then $\sigma \cap N$ is a finitely generated semigroup.*

Proof. By the definition of a cone σ, $x, y \in \sigma \Rightarrow x + y \in \sigma$. And so, $x + y \in \sigma \cap N$ if $x, y \in \sigma \cap N$. The zero vector in σ gives the identity of $\sigma \cap N$. Then $\sigma \cap N$ is a semigroup. For the second part, let $S = \{u_1, \ldots, u_t\}$ be the set of vectors defining the cone σ. Each u_i is an element of $\sigma \cap N$. Consider the set

$$K = \left\{ \sum \alpha_i u_i \,\middle|\, 0 \le \alpha_i \le 1 \right\}.$$

Then, K is compact in $N_{\mathbb{R}}$, in the usual sense. Since N is discrete, the intersection $K \cap N$ has only finitely many elements. Now, we show that it generates $\sigma \cap N$. Take $u \in \sigma \cap N$. Then u can be written $v = a_1 u_1 + \cdots + a_t u_t$, $a_i \in \mathbb{Z}_{\ge 0}$. Let $\lfloor a_i \rfloor$ be the largest integer less than or equal to a_i. Then for each of the a_i's we have $a_i = \lfloor a_i \rfloor + b_i$, where $b_i = a_i - \lfloor a_i \rfloor$, and so $b_i \in [0, 1]$. Then u can be written as

$$\begin{aligned} u = a_1 u_1 + \cdots + a_t u_t &= (\lfloor a_1 \rfloor + b_1) u_1 + \cdots + (\lfloor a_t \rfloor + b_t) u_t \\ &= \lfloor a_1 \rfloor u_1 + \cdots + \lfloor a_t \rfloor u_t + b_1 u_1 + \cdots + b_t u_t. \end{aligned}$$

If we set $w = b_1 u_1 + \cdots + b_t u_t$, then each u_i's are in $K \cap N$ and w is also in $K \cap N$, so that u is a combination with integer coefficients of elements of $K \cap N$. This means that $\sigma \cap N$ is generated as a semigroup by the elements of $K \cap N$. Since $K \cap N$ is finite, $\sigma \cap N$ is finitely generated. □

Remark 3.1.25. By Proposition 3.1.17, we can apply this lemma to the dual of rational cone $\check{\sigma}$ and so we obtain a semigroup $\check{\sigma} \cap M$, which is denoted by S_σ. Furthermore, S_σ is saturated, i.e., $cm \in S_\sigma$ implies $m \in S_\sigma$ for $m \in M$ and $c \in \mathbb{Z}^+$. There is a close relation between the notion of saturation and being normal, we will study in Subsection 3.3.3.

Remark 3.1.26. Lemma 3.1.13 implies that $\check{\sigma} \cap M \subset \check{\tau} \cap M$, in other words $S_\sigma \subset S_\tau$.

Example 3.1.27. Consider the cone $\sigma =< e_2, 2e_1 - e_2 >$ in $\mathbb{R}^2$. Then $S_\sigma = \check{\sigma} \cap M$ can not be generated by the vectors e_1^* and $e_1^* + 2e_2^*$, since we cannot write $e_1^* + e_2^*$ in terms of e_1^* and $e_1^* + 2e_2^*$. To obtain a set of generators, one has to add $e_1^* + e_2^*$. Thus, S_σ is generated by the set $\{e_1^*, e_1^* + e_2^*, e_1^* + 2e_2^*\}$, see Figure 3.4.

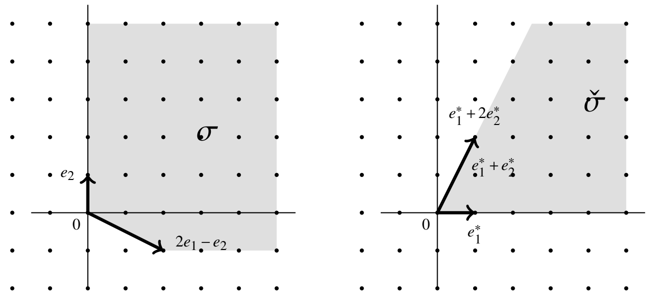


Figure 3.4 The cone σ and its dual cone

Proposition 3.1.28. *(Fulton, 1993, Proposition 2, page 13) Let σ be a rational cone and $\tau = \sigma \cap v^{\perp}$ be a face of σ, with $v \in \check{\sigma}$, then $S_{\tau} = S_{\sigma} + \mathbb{Z}_{\geq 0}(-v)$.*

Proof. Let $v' \in S_{\tau} = \check{\tau} \cap M$. Firstly, we have to show that there exists an element $\lambda \in \mathbb{R}_{\geq 0}$ such that $v' + \lambda v \in \check{\sigma}$, or in other words

$$\langle v' + \lambda v, u \rangle \geq 0 \tag{3.1.1}$$

for all $u \in \sigma$. Suppose that for each generator v_i, there exists a real number λ_i satisfying the inequality (3.1.1). Set $\lambda := \max\{\lambda_i\}_{i=1}^{n}$. Then for every vector $v' \in S_{\tau}$, the inequality (3.1.1) is satisfied, by the property of inner product. Let v_i be one of the generators of S_{σ}. Suppose that $\langle v, v_i \rangle = 0$. Then $v_i \in \tau = \sigma \cap v^{\perp}$. Since $v' \in \check{\tau}$, we get $\langle v', v_i \rangle \geq 0$. Indeed

$$0 \leq \langle v' + \lambda v, v_i \rangle = \langle v', v_i \rangle + \lambda \langle v, v_i \rangle = \langle v', v_i \rangle \geq 0. \tag{3.1.2}$$

Now, suppose that $\langle v, v_i \rangle \geq 0$, define $\lambda_i = \dfrac{\langle v', v_i \rangle}{\langle v, v_i \rangle}$. Then

$$\begin{aligned} \langle v' + \lambda_i v, v_i \rangle &= \langle v', v_i \rangle + \lambda_i \langle v, v_i \rangle \\ &= \langle v', v_i \rangle + \frac{\langle v', v_i \rangle}{\langle v, v_i \rangle} \langle v, v_i \rangle = 2\langle v', v_i \rangle \geq 0. \end{aligned}$$

Thus, we have shown that there exists a real number $\lambda \in \mathbb{R}_{\geq 0}$ satisfying the inequality (3.1.1), i.e., $\check{\tau} = \check{\sigma} + \mathbb{R}_{\geq 0}(-v)$. For such λ, let $\lceil p \rceil = l$ be the smallest integer greater than or equal to p. Then $v' + lv \in \check{\sigma} \cap M = S_{\sigma}$ and $v' = (v' + lv) + l(-v) \in S_{\sigma} + \mathbb{Z}_{\geq 0}(-v)$.

For the converse inclusion, let $v' \in S_{\sigma} + \mathbb{Z}_{\geq 0}(-v)$. Then $v' = u + l(-v)$ for some $l \in \mathbb{Z}_{\geq 0}$ and for any $w \in \tau$ we have $\langle v', w \rangle = \langle u + l(-v), w \rangle = \langle u, w \rangle - l \langle v, w \rangle$. Since $w \in \tau = \sigma \cap v^{\perp}$, then $\langle v, w \rangle = 0$ and since $u \in S_{\sigma}$, then $\langle u, w \rangle \geq 0$. Thus, $\langle v', w \rangle \geq 0$, i.e., $v' \in \check{\tau}$. Since $v' \in M$, we obtain $v' \in \check{\tau} \cap M = S_{\tau}$. □

Example 3.1.29. Let us consider $\sigma =< e_1, e_2 >$ in $N_{\mathbb{R}} \cong \mathbb{R}^2$. For the face $\tau = e_2$ of σ the vector $v = e_1^*$ satisfies the inequality (3.1.1) for $\lambda \in \mathbb{R}_{\geq 0}$, see Figure 3.5.

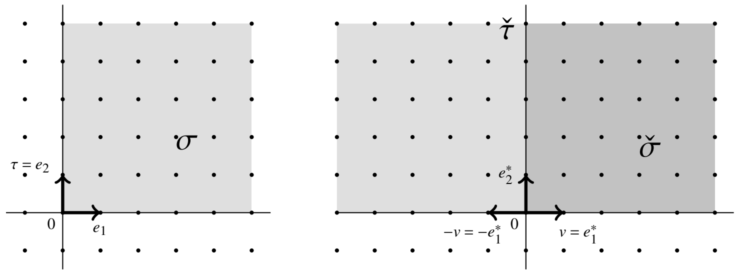


Figure 3.5 Cone and face relation

Our main point is associate a semigroup S to a finitely generated reduced $\mathbb{C}$-algebra, to obtain the coordinate ring of some affine variety. We construct this by the following way: consider $\mathbb{C}[S]$ as a vector space with basis S such that the basis vector is defined as a power χ^s of the corresponding element $s \in S$. Every element in $\mathbb{C}[S]$ can be written as finite formal linear combination with coefficients in $\mathbb{C}$, that is $\sum_{s \in S} a_s \chi^s$, $a_s \in \mathbb{C}$. A binary operation, multiplication, on $\mathbb{C}[S]$ is determined by the addition in S; $\chi^s \cdot \chi^{s'} = \chi^{s+s'}$. This is also a $\mathbb{C}$- algebra with identity $\chi^0 = 1$, and if an element $s \in S$ is invertible, then χ^s is a unit in $\mathbb{C}[S]$.

For example, if $S = \mathbb{N}$, then the $\mathbb{C}$-algebra $\mathbb{C}[\mathbb{N}]$ is the set of all formal expressions $\sum_{n \in \mathbb{N}} a_n n$, where $a_n \in \mathbb{C}$ for all n and $a_n = 0$ for sufficiently large $n > N$. Thus, we can write elements as in the form $\sum_{n=1}^{N} a_n n$. Then the map which sends $\sum_{n=1}^{N} a_n n$ to $\sum_{n=1}^{N} a_n \chi^n$ gives us an isomorphism of $\mathbb{C}$-algebras between $\mathbb{C}[N]$ and the polynomial ring $\mathbb{C}[x]$. More generally, $\mathbb{C}[\mathbb{N}^n] \cong \mathbb{C}[x_1, \ldots, x_n]$.

3.1.2 Description of an Affine Toric Variety

Now, we are able to see a connection of semigroup algebras with algebraic geometry. We will use only "cone" instead of strongly convex rational polyhedral cone in $N_{\mathbb{R}}$.

A Laurent polynomial is defined as the finite formal linear combinations with coefficients in $\mathbb{C}$. If we set $S = \mathbb{Z}^n$, then there is a natural isomorphism of $\mathbb{C}$-algebras between $\mathbb{C}[\mathbb{Z}^n]$ and the algebra $\mathbb{C}[z_1, \ldots, z_n, z_1^{-1}, \ldots, z_n^{-1}]$ of Laurent polynomials in the variables $z_1, \ldots, z_n$. This isomorphism is given on the basis $\{\chi^\alpha\}_{\alpha \in \mathbb{Z}^n}$ of $\mathbb{C}[\mathbb{Z}^n]$ by

$$\chi^\alpha \mapsto z_1^{\alpha_1} \cdots z_n^{\alpha_n},$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$. For simplicity, denote the set of all Laurent polynomials by $\mathbb{C}[\mathrm{z}, \mathrm{z}^{-1}]$, where $\mathrm{z} = (z_1, \ldots, z_n)$.

In Section 3.1.1 we have defined the semigroup algebra, now in a similar way we will define the $\mathbb{C}$-algebra $\mathbb{C}[S_\sigma]$ for a cone as follows:

Definition 3.1.30. For any cone $\sigma \subset N_{\mathbb{R}}$, the ring R_σ is defined as

$$R_\sigma = \mathbb{C}[S_\sigma] := \left\{ \sum a_v \chi^v \;\middle|\; v \in S_\sigma,\, a_v \in \mathbb{C} \right\}.$$

Example 3.1.31. Consider the cone $\mathrm{o} = \{0\}$ in $N_{\mathbb{R}}$. Then the dual cone $\check{\sigma}$ is all of $M_{\mathbb{R}}$, the associated semigroup is nothing but the group $M \cong \mathbb{Z}^n$ which is generated by $\pm e_1^*, \ldots, \pm e_n^*$. By setting $\chi^{e_i^*} = X_i$ and $\chi^{-e_i^*} = X_i^{-1}$ we have $R_{\mathrm{o}} := \mathbb{C}[M] = \mathbb{C}[X_1, \ldots, X_n, X_1^{-1}, \ldots, X_n^{-1}]$. For any cone $\sigma \in N_{\mathbb{R}}$, the semigroup S_σ is a subsemigroup of S_{o}, so the semigroup algebra R_σ is a subalgebra of R_{o}.

Remark 3.1.32. It follows from this fact that for any $\tau \leq \sigma$, we have $R_\sigma \subset R_\tau$.

Since S_σ is finitely generated by Theorem 3.1.24, we have obtained a finitely generated $\mathbb{C}$-algebra, $\mathbb{C}[S_\sigma]$. Moreover, since S_σ has no torsion element, i.e., if $n \cdot s = n \cdot t$ implies $n = 0$ or $s = t$, we have identified $\mathbb{C}[S_\sigma]$ with an algebra of Laurent polynomials. Thus, $\mathbb{C}[S_\sigma]$ is an integral domain and it has no nonzero nilpotents. Hence, $\mathbb{C}[S_\sigma]$ is the coordinate ring of some irreducible affine variety $\mathrm{Spec}\,\mathbb{C}[S_\sigma]$.

Let $\{v_1, \ldots, v_m\}$ be a generator set of S_σ. Since $\mathbb{C}[S_\sigma]$ is finitely generated, we can define a map $f : \mathbb{C}[Z_1, \ldots, Z_m] \to \mathbb{C}[S_\sigma]$ by using $Z_i = \chi^{v_i}$ for $i = 1, \ldots, m$. Then the kernel of this map gives an ideal I in $\mathbb{C}[Z_1, \ldots, Z_m]$, so that

$$R_\sigma = \mathbb{C}[Z_1, \ldots, Z_m]/I.$$

Definition 3.1.33. The affine variety $X_\sigma := \mathrm{Spec}\mathbb{C}[S_\sigma] = \mathrm{Spec}\mathbb{C}[Z_1,\ldots,Z_m]/I$ associated to a cone σ in $N_\mathbb{R} \cong \mathbb{R}^n$ is called the *affine toric variety* corresponding to σ. The dimension of an affine toric variety X_σ is n.

Remark 3.1.34. It is known that, there is a bijective correspondence between points of an affine variety V and maximal ideals of its coordinate ring $\mathbb{C}[V]$. By this fact we will construct a correspondence between points of an affine variety $X_\sigma = \mathrm{Spec}\mathbb{C}[S_\sigma]$ and semigroup homomorphisms $S_\sigma \to \mathbb{C}$, where $\mathbb{C}$ considered as a multiplicative semigroup. Let P be a point in X_σ. Define a map $P : S_\sigma \to \mathbb{C}$ such that $v \in S_\sigma$ maps to $\chi^v(P) = P(v) \in \mathbb{C}$, where $\chi^v \in \mathbb{C}[S_\sigma]$. Since $P(m_1+m_2) = P(\chi^{m_1+m_2}) = P(\chi^{m_1}\chi^{m_2}) = P(\chi^{m_1})P(\chi^{m_2}) = P(m_1)P(m_2)$ and $P(0) = P(\chi^0) = P(1) = 1$, this map is really a semigroup homomorphism. For the converse, consider a semigroup homomorphism $S_\sigma \to \mathbb{C}$. Then this homomorphism can arise a $\mathbb{C}$-algebra homomorphism $\mathbb{C}[S_\sigma] \to \mathbb{C}$. Since the kernel of this homomorphism gives us a maximal ideal, we obtain a one-to-one correspondence between points of an affine toric variety and semigroup homomorphisms. This correspondence is special in the case of toric.

Theorem 3.1.35. *(Ewald, 1996, Theorem 2.7, page 217) Let σ be a cone in $N_\mathbb{R} \cong \mathbb{R}^n$ and let I be the ideal generated by the relations between the generators of S_σ. Then, $X_\sigma = \mathbb{V}(I)$.*

It follows that, the height of the ideal I is $m-n$, where $\dim\mathbb{C}[Z_1,\ldots,Z_m] = m$.

Lemma 3.1.36. *If $\tau \leq \sigma$, then the map $X_\tau \to X_\sigma$ embeds X_τ as a principal open subset of X_σ.*

Proof. For any $\tau \leq \sigma$ we have $S_\tau = S_\sigma + \mathbb{Z}_{\geq 0}(-v)$ where $v \in S_\sigma$ and $\tau = \sigma \cap v^\perp$. This implies that if $v' \in S_\tau$, then $v' = w + l(-v)$ for some $l \in \mathbb{Z}_{\geq 0}$, and $w \in S_\sigma$. If we pass to $\mathbb{C}$-algebra $\mathbb{C}[S_\sigma]$, then

$$\chi^{v'} = \chi^{w+l(-v)} = \frac{\chi^w}{(\chi^v)^l}.$$

This means that R_τ is a localization of R_σ at χ^v, i.e., $X_\tau \hookrightarrow X_\sigma$. □

Example 3.1.37. Consider the cone given in Example 3.1.31. We have shown that its semigroup S_o is generated by the vectors $\pm e_1^*,\ldots,\pm e_n^*$. Then its $\mathbb{C}$-algebra R_o is given by $R_{\text{o}} = \mathbb{C}[M] = \mathbb{C}[X_1, \ldots, X_n, X_1^{-1}, \ldots, X_n^{-1}]$. Let $X_i = Z_i$ and $X_i^{-1} = Z_{n+i}$ for $i = 1, \ldots, n$. Then we obtain a natural isomorphism

$$\mathbb{C}[S_{\text{o}}] = \mathbb{C}[M] = \mathbb{C}[Z_1, \ldots, Z_{2n}]/I$$

There are n relations between the variables $Z_1, \ldots, Z_{2n}$ for the ideal I, because $\dim(X_\sigma) = n$: $Z_1Z_{n+1} = 1, \ldots, Z_nZ_{2n} = 1$. Then, $\text{Spec}\mathbb{C}[S_{\text{o}}] = \text{Spec}R_{\text{o}} \cong \mathbb{V}(Z_1Z_{n+1} - 1, \ldots, Z_nZ_{2n} - 1)$. Assume that $u_i \neq 0 \in \mathbb{C}$ for all $i = 1, \ldots, n$. Then by the projection $\mathbb{C}^{2n} \to \mathbb{C}^n$ we have

$$\begin{aligned} X_{\text{o}} &= \text{Spec}R_{\text{o}} = \{(u_1, \ldots, u_n) \in \mathbb{C}^n \mid u_i \neq 0,\ \forall 1 \leq i \leq n\} \\ &= (\mathbb{C} \setminus \{0\})^n = (\mathbb{C}^*)^n. \end{aligned}$$

Remark 3.1.38. As stated before, we have different choices of generator elements to obtain the semigroup S_σ for the cone σ in $N_{\mathbb{R}}$. And we can represent the finitely generated $\mathbb{C}$-algebra R_σ as a coordinate ring $\mathbb{C}[\xi_1, \ldots, \xi_n]/I$ in a different ways, i.e., we have different representations for affine varieties $\mathbb{V}(I)$ in $\mathbb{C}^n$. But, $\text{Spec}R_\sigma$ is identified with these subvarieties $\mathbb{V}(I)$ in $\mathbb{C}^n$. This means that, $\mathbb{V}(I)$ are all homeomorphic to the variety $\text{Spec}R_\sigma$. For example, another representation of $(\mathbb{C}^*)^n$ is obtained by using the generator set $S_{\text{o}} = \{e_1^*, \ldots, e_n^*, -e_1^* - \cdots - e_n^*\}$.

Definition 3.1.39. The set $T_N := (\mathbb{C} \setminus \{0\})^n := (\mathbb{C}^*)^n$ is called an *affine (complex algebraic) n-torus*.

Remark 3.1.40. Since the set of all semigroup homomorphisms $\text{Hom}(S_{\text{o}}, \mathbb{C}) = \text{Hom}_{\mathbb{Z}}(M, \mathbb{C}^*)$, we can write T_N in the form:

$$T_N := \text{Hom}_Z(S_{\text{o}}, \mathbb{C}^*) = N \otimes_{\mathbb{Z}} \mathbb{C}^* \cong \mathbb{C}^*.$$

Given any cone $\sigma \in N_{\mathbb{R}}$, we have $X_{\text{o}} \subset X_\sigma$. So, every n-dimensional affine toric variety contains $T_N := (\mathbb{C}^*)^n$ as a Zariski open subset.

Example 3.1.41. In the case of Example 3.1.27 the generators of S_σ are $v_1 = e_1^*, v_2 = e_1^* + e_2^*$ and $v_3 = e_1^* + 2e_2^*$. Then the monic Laurent monomials $Z_1 = X_1$, $Z_2 = X_1X_2$ and $Z_3 = X_1X_2^2$. The corresponding $\mathbb{C}$-algebra is

$$\begin{aligned} R_\sigma &= \mathbb{C}[S_\sigma] = \mathbb{C}[X_1, X_1X_2, X_1X_2^2] \\ &= \mathbb{C}[Z_1, Z_2, Z_3]/I, \end{aligned}$$

where the relation $v_1 + v_3 = 2v_2$ between the generators of S_σ implies the relation $Z_1Z_3 = Z_2^2$ in $\mathbb{C}[S_\sigma]$. Thus, $X_\sigma = \mathrm{Spec}\mathbb{C}[S_\sigma] \cong \mathbb{V}(Z_1Z_3 - Z_2^2)$. This affine toric variety corresponds to the quadric cone in $\mathbb{C}^3$, see Figure 3.6.

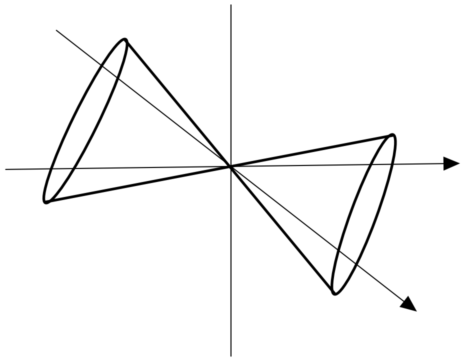


Figure 3.6 Real part of a quadratic cone

Now, we will give more information about the ideal I. We have defined a map from $\mathbb{C}[Z_1,\ldots,Z_m]$ to $\mathbb{C}[S_\sigma]$ by $\chi^{v_i} \mapsto Z_i$ with the ideal I. This map can be identified with an isomorphism given in semigroup algebra construction. Hence, the generator of the semigroup S_σ are related with the ideal I. More explicitly, we have a correspondence

$$a_1v_1 + \cdots + a_mv_m = b_1v_1 + \cdots + b_mv_m \longleftrightarrow (\chi^{v_1})^{a_1} \cdots (\chi^{v_m})^{a_m} = (\chi^{v_1})^{b_1} \cdots (\chi^{v_m})^{b_m}$$

where $a_i, b_i \in \mathbb{Z}_{\geq 0}$. This means that the polynomial ring $\mathbb{C}[Z_1,\ldots,Z_m]$ is obtained by the relation $Z_1^{a_1} \cdots Z_m^{a_m} = Z_1^{b_1} \cdots Z_m^{b_m}$.

Definition 3.1.42. A polynomial with at most two monomials, say $\alpha Z^a + \beta Z^b$ where $\alpha, \beta \in \mathbb{C}$ and $a, b \in \mathbb{Z}^n_{\geq 0}$, is called a *binomial*. A *binomial ideal* is an ideal of $\mathbb{C}[Z_1,\ldots,Z_m]$ generated by binomials.

From the definition, we can say that our ideal I is generated by the finite binomials of the form $Z_1^{a_1} \cdots Z_m^{a_m} - Z_1^{b_1} \cdots Z_m^{b_m}$. This ideal is called a *toric ideal*.

We will conclude this section by defining a morphism of affine toric varieties.

Definition 3.1.43. A morphism between affine toric varieties $\psi : X_{\sigma'} \to X_\sigma$ is *toric morphism* if the corresponding map of coordinate ring $\mathbb{C}[S_\sigma] \to \mathbb{C}[S_{\sigma'}]$ is induced by a semigroup homomorphism $S_{\sigma'} \to S_{\sigma}$. If ψ is a bijective and its inverse is also a toric morphism, then ψ is called a *toric isomorphism*.

Let N, N' be lattices and σ, σ' be strongly convex rational polyhedral cone with respect to lattices, respectively. Consider the lattice homomorphism $\varphi : N \to N'$ with the property $\varphi(\sigma \cap N) \subset \sigma'$. Its dual map $\check{\varphi} : M' \to M$ is defined as $\check{\varphi}(S_{\sigma'}) = S_{\sigma}$. Then we have an algebra homomorphism $\mathbb{C}[S_{\sigma'}] \to \mathbb{C}[S_{\sigma}]$, since our semigroups are finitely generated as stated before. By Lemma 3.1.36, we obtain a morphism $X_\sigma \to X_{\sigma'}$. Therefore, a toric morphism can be described by using lattice homomorphism. As a result we obtain the following proposition.

Proposition 3.1.44. *(Barthel, 2000a) Let $X_{\sigma'}$ and X_σ be affine toric varieties given by cones $\sigma' \in N'_{\mathbb{R}}$ and $\sigma \in N_{\mathbb{R}}$. Then a lattice homomorphism $N \to N'$ mapping $N \cap \sigma$ to σ' determines a morphism $X_\sigma \to X_{\sigma'}$. That is, this map is equivariant with respect to the induced homomorphism $T_N \to T_{N'}$ of torus.*

Proposition 3.1.45. *(Cox et al., 2011, Proposition 1.3.14, Page 41) Let V_1, V_2 be affine toric varieties with tori T_{N_1}, T_{N_2}, respectively. Then:*

i) A morphism $\varphi : V_1 \to V_2$ is toric *if and only if $\varphi(T_{N_1}) \subset T_{N_2}$ and $\varphi|_{T_{N_1}} : T_{N_1} \to T_{N_2}$ is a group homomorphism.*

ii) A toric morphism is equivariant,that is, $\varphi(t \cdot P) = \varphi(t) \cdot \varphi(P)$ for all $t \in T_{N_1}$ and $P \in V_1$.

3.2 General Toric Variety

3.2.1 Fans and Toric Variety

Now, we will generalize the idea given in Section 3.1 to obtain a general toric variety. So, start by defining the set of strongly convex rational polyhedral cones.

Definition 3.2.1. A *fan* Σ in a lattice N is a finite set of strongly convex rational polyhedral cones such that:

i) Every face of a cone of Σ is a cone of Σ,

ii) If σ and σ' are cones of Σ, then $\sigma \cap \sigma'$ is a common face of σ and σ'.

In particular, the zero cone o belongs to every fan since o is a face of any cone.

Example 3.2.2. Given cones in Figure 3.7 gives a fan in $N_{\mathbb{R}} \cong \mathbb{R}^2$.

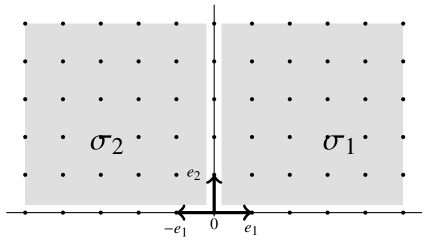


Figure 3.7 The fan $\Sigma = \{\sigma_1, \sigma_2\}$

Now, we will construct a general toric variety. A general toric variety is obtained by taking the disjoint union of an affine toric variety, for each cone σ in the fan Σ, and gluing them. We will explain the way of gluing, as follows: let $\tau \leq \sigma$ and the semigroup S_σ generated by $\{v_1, \ldots, v_k\}$. Then, by Proposition 3.1.28, the semigroup S_τ is obtained from S_σ by adding one generator $v_{k+1} = -v$. That is the generators of S_τ is $v_1, \ldots, v_k, -v$. We can assume $v_k = v$ since $v_k \in S_\sigma$. In this case to obtain a relationships between the generators of S_τ, we have to use relationships between the generators $v_1, \ldots, v_k$ of S_σ and additionally $v_k + v_{k+1} = 0$. In terms of $\mathbb{C}$-algebra $\mathbb{C}[S_\tau]$, this relation gives $u_k u_{k+1} = 1$, since $u_i = \chi^{v_i}$ for all $i = 1, \ldots, k+1$. Thus, we obtain

$$X_\sigma = \{(u_1, \ldots, u_k) \in \mathbb{C}^k\}$$

$$X_\tau = \{(u_1, \ldots, u_k, u_{k+1}) \in \mathbb{C}^{k+1} \mid u_k u_{k+1} = 1\}$$

This means that the projection $\mathbb{C}^{k+1} \to \mathbb{C}^k$, given by $(x_1, \ldots, x_k, x_{k+1}) \mapsto (x_1, \ldots, x_k)$ identifies X_τ with the open subset of X_σ defined by $x_k \neq 0$. As a result:

Proposition 3.2.3. *(Ewald, 1996, Lemma 3.1, page 225) There is a natural isomorphism* $X_\tau \cong X_\sigma \backslash \{u_k = 0\}$.

For two cones, $\sigma, \sigma' \in \Sigma$, let $\tau = \sigma \cap \sigma'$ be the common face. Let's take any compatible coordinate system $v_1, \ldots, v_l$ for $X_{\sigma'}$, then we have

$$X_\sigma \backslash \{u_k = 0\} \cong X_\tau \cong X_{\sigma'} \backslash \{v_l = 0\}.$$

We have a toric isomorphism $\varphi : X_\sigma \backslash \{u_k = 0\} \to X_{\sigma'} \backslash \{v_l = 0\}$ defined by

$$\varphi(u_1, \ldots, u_k, u_{k+1}) = (v_1, \ldots, v_l, v_{l+1}).$$

Thus, φ glues together X_σ and $X_{\sigma'}$ along X_τ and it is called the *gluing map*.

Now, we are ready to define a general toric varieties.

Definition 3.2.4. Let Σ be a fan in $N_{\mathbb{R}}$. A quotient space of the disjoint union $\bigsqcup_{\sigma \in \Sigma} X_\sigma$ modulo the equivalence relation that defined by identifying two points $x \in X_\sigma$ and $x' \in X_{\sigma'}$ by the gluing map φ is called a *general toric variety*. A toric variety corresponding to a fan Σ is denoted by X_Σ.

It follows that the toric variety X_Σ is a topological space endowed with an open covering by affine complex varieties that intersect (Zariski) open subvarieties.

Example 3.2.5. Let $\Sigma \subset N_{\mathbb{R}} \cong \mathbb{R}$ be a fan obtained by the cones $\sigma_1 =< e_1 >$ and $\sigma_2 =< -e_1 >$. Then the corresponding affine toric varieties are $X_{\sigma_1} \cong \mathbb{C}$ and $X_{\sigma_2} \cong \mathbb{C}$.

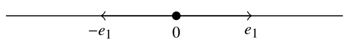


Figure 3.8 The fan $\Sigma = \{\sigma_1, \sigma_2\}$

Now, consider the common face of σ_1 and σ_2, we have $\tau = \sigma_1 \cap \sigma_2 = \{0\}$. Then the semigroup S_τ is generated by $\{e_1^*, -e_1^*\}$. So, the $\mathbb{C}$-algebra $\mathbb{C}[S_\tau] = \mathbb{C}[\mathrm{x}, \mathrm{x}^{-1}]$, and hence $X_\tau = \mathrm{Spec}\mathbb{C}[\mathrm{x}, \mathrm{x}^{-1}] \cong \mathbb{C}^*$. Consider the projection map $(u_1, u_1^{-1}) \mapsto u_1$ for $u_1 \neq 0$. Then we have identifications

$$X_\tau \cong X_{\sigma_1} \setminus \{u_1 = 0\} \qquad \text{and} \qquad X_\tau \cong X_{\sigma_2} \setminus \{u_1^{-1} = 0\}.$$

This means that, the gluing map is defined by $\mathrm{x} \mapsto \mathrm{x}^{-1}$. Assume that $\mathrm{x} = \frac{t_1}{t_0}$ and $\phi_i : \mathbb{C} \to \mathbb{P}^1$ such that $\phi(t) = (1 : t)$. And we get

$$\mathrm{x} = \frac{t_1}{t_0} \mapsto (1 : \frac{t_1}{t_0}) = U_0 \qquad \mathrm{x}^{-1} = \frac{t_0}{t_1} \mapsto (\frac{t_0}{t_1} : 1) = U_1$$

This gives us a correspondence between the affine toric varieties $X_{\sigma_1}, X_{\sigma_2}$ and the open subsets U_0, U_1 of $\mathbb{P}^1$, respectively. Thus, gluing X_{σ_1} and X_{σ_2} by the map $\mathrm{x} \mapsto \mathrm{x}^{-1}$ is the same thing with taking the union of open subsets U_0 and U_1 of $\mathbb{P}^1$. Therefore, $X_\Sigma = \mathbb{P}^1$.

We can generalize Example 3.2.5 to $\mathbb{P}^n$ by taking the fan Σ in $N_\mathbb{R} \cong \mathbb{R}^n$ generated by all proper subsets of $(e_1, \ldots, e_n, -e_1 - \cdots - e_n)$. The affine toric varieties X_{σ_i} are copies of $\mathbb{C}^n$, corresponding to affine open subsets of $\mathbb{P}^n$ and glued together to obtain $\mathbb{P}^n$.

Proposition 3.2.6. *(Fulton, 1993, Lemma at page 21) If σ and τ are cones that intersect in a common face, then the diagonal map $X_{\sigma\cap\tau} \to X_\sigma \times X_\tau$ is a closed embedding, i.e., it maps $X_{\sigma\cap\tau}$ injectively onto a closed subvariety.*

In fact this proposition states that every toric variety is determined by a fan.

Remark 3.2.7. Generalization of the affine case implies that, toric morphisms between general toric varieties $X_{\Sigma'}$ and X_Σ are defined by the fans $\Sigma' \in N'_\mathbb{R}$ and $\Sigma \in N_\mathbb{R}$. Then we have the following fact:

Proposition 3.2.8. *(Barthel, 2000b) Let $X_{\Sigma'}$ and X_Σ be toric varieties given by fans $\Sigma' \in N'_\mathbb{R}$ and $\Sigma \in N_\mathbb{R}$. Then a lattice homomorphism $N \to N'$ mapping $N \cap \sigma$, for each cone $\sigma \in \Sigma$ to some cone $\sigma' \in \Sigma'$ determines a morphism $X_\Sigma \to X_{\Sigma'}$. That is, this map is equivariant with respect to the induced homomorphism $T_N \to T_{N'}$ of torus.*

3.2.2 Polytopes and Toric Varieties

There are a lot of different methods to describe a toric variety. One of these ways comes from a polytope. Firstly, we will define this geometric object to understand the description and details can be found in Grünbaum & Ziegler (2003).

A *polyhedron* in a finite dimensional real vector (or affine) space with a lattice is any set obtained as the intersection of finitely many halfspaces. If additionally bounded, it is called a *polytope*. There is an equivalent way to define a polytope, which we will use in this section.

Definition 3.2.9. A *polytope* in $M_{\mathbb{R}}$ is a set of the form

$$P = \mathrm{Conv}(S) = \Big\{ \sum_{v \in S} \lambda_v v \;\Big|\; \lambda_v \geq 0,\ \sum_{v \in S} \lambda_v = 1 \Big\},$$

where $S \subset M_{\mathbb{R}}$ is finite.

In other words, a polytope is the convex hull of a finite set in a finite dimensional vector space.

Remark 3.2.10. For a given polytope P in $M_{\mathbb{R}}$, we obtain a polyhedral cone defined by $C(P) = \Big\{ \lambda \cdot (v, 1) \in M_{\mathbb{R}} \times \mathbb{R} \;\Big|\; v \in P,\ \lambda \geq 0 \Big\}$ and is called the *cone of* P, see Figure 3.9.

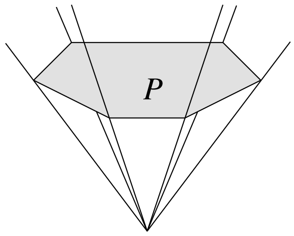


Figure 3.9 The cone of a polytope P

Definition 3.2.11. The *dimension* of a polytope $P \subset M_{\mathbb{R}}$ is the dimension of the smallest affine subspace of $M_{\mathbb{R}}$ containing P.

Definition 3.2.12. A (proper) *face* F of P is the intersection with a supporting affine hyperplane, i.e., $F = \{v \in P \mid \langle u, v \rangle = r\}$ where $u \in N_{\mathbb{R}}$ is a function with $\langle u, v \rangle \geq r$ for all $v \in P$; P is usually included as an improper face.

Every face of P is again a polytope. We call facets and vertices, faces of P with dimension $\dim P - 1$ and 0, respectively. By the definition of the polytope, we can say that P is the convex hull of its vertices.

Definition 3.2.13. Let $P \subset M_{\mathbb{R}}$ be a polytope of dimension d. P is a *simplex* if it has $d+1$ vertices, for example a tetrahedron is a 3-simplex. P is *simplical* if every facet of P is a simplex, for example octahedron. P is *simple* if every vertex is the intersection of d facets.

Definition 3.2.14. The *Minkowski sum* of subsets $A_1, A_2 \subset \mathbb{R}^n$ is

$$A_1 + A_2 = \{m_1 + m_2 \mid m_1 \in A_1,\ m_2 \in A_2\}.$$

For two polytopes P_1 and P_2, their Minkowski sum $P_1 + P_2 = \mathrm{Conv}(S_1 + S_2)$ is again a polytope, where S_1, S_2 are generator sets of polytopes, respectively. For simplicity, we will assume that P is n-dimensional and P contains the origin in its interior point.

Definition 3.2.15. The *polar (or dual) set* of P is defined to be the set

$$P^\circ = \{u \in N_\mathbb{R} \mid \langle u, v\rangle \geq -1 \text{ for all } v \in P\} \subset N_\mathbb{R}$$

Example 3.2.16. Let P be the square with vertices $(\pm 1, \pm 1)$ in $N_\mathbb{R} \cong \mathbb{R}^2$. Then the polar set P° of P is given by the inequality $|x| + |y| \leq 1$ for $(x, y) \in N_\mathbb{R}$, see Figure 3.10.

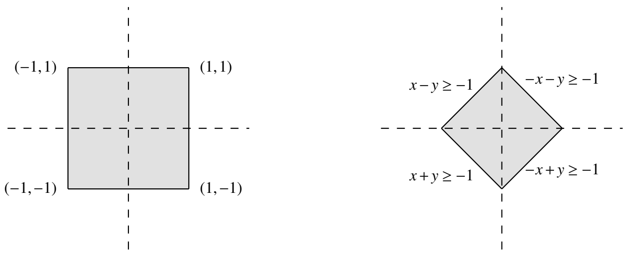


Figure 3.10 The polytope P and its polar polytope P°

Definition 3.2.17. A polytope P is called *rational* if its vertices lie in a lattice in $M_\mathbb{R}$.

Lemma 3.2.18. *The polar set P° of P has the following properties:*

i) *P° is a convex polytope.*

ii) *If P is rational, then P° is a lattice polytope.*

We have a relation between faces of P and faces of P°. To construct this relation define the face F^* of P° as follows: $F^* = \{u \in P^\circ \mid \langle u, v\rangle = -1,\ \forall\ v \in F\}$ for every face F of P.

Proposition 3.2.19. *(Fulton, 1993, Proposition at page 24)*

i) *There is a one-to-one correspondence between faces of P and faces of $P^\circ : F \leftrightarrow F^*$ reversing order*

ii) $\dim F + \dim F^* = n - 1$

3.2.2.1 Fan Associated to a Polytope

Assume that P is n-dimensional but it is not necessary that it contains the origin.

Definition 3.2.20. Let P be a polytope, we construct a *cone* σ_F to each face F of the polytope P in the following way: $\sigma_F = \{v \in N_{\mathbb{R}} \mid \langle u, v\rangle \leq \langle u', v\rangle, \ \forall\, u \in F\}$. A compatible collection of cones σ_F give a fan Σ_F which is called an *inner normal fan*.

Remark 3.2.21. If one rewrites the cone associated to a face F of a lattice polytope P as $\sigma_F = \{v \in N_{\mathbb{R}} \mid \langle u - u', v\rangle \geq 0, \ \forall\, u \in F\}$, it can be easily seen that the construction in Definition 3.2.20 is a translation and dilation invariant. For example, if we consider another lattice polytope P' of the form $P' = u + P$ for $u \in M$, that is, $P' = \{u + w \mid w \in P\}$ then the fans Σ_P and $\Sigma_{P'}$ coincide. Therefore, all translations and dilations of a polytope P gives the same toric variety.

Example 3.2.22. Let $P \subset M_{\mathbb{R}} \cong \mathbb{R}^2$ be a polytope with vertices 0, e_1 and e_2. Then by the construction in Definition 3.2.20 we have the cone and fan given in Figure 3.11.

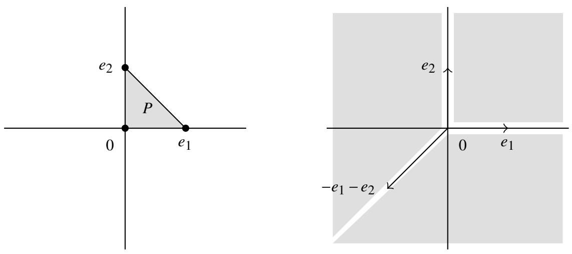


Figure 3.11 A polytope P and corresponding fan

Proposition 3.2.23. *(Fulton, 1993, Proposition at page 26) If* $\{0\} \in \operatorname{Int}(P)$*, then* Σ_P *is made of the cones based on the faces of the polar polytope* P°.

We have known that for each lattice cones σ_F in Σ_P there exist an affine toric variety X_{σ_F} over $\mathbb{C}$. The structure of a polytope is such that these varieties satisfy the conditions needed to glue them and obtain a new variety, X_P.

Example 3.2.24. Let P be a polytope in $\mathbb{R}^2$ with vertices at $\pm e_1^* \pm e_2^*$. Then, the fan $\Sigma_P = \{\sigma_0 =< e_1, e_2 >, \ \sigma_1 =< -e_1, e_2 >, \ \sigma_2 =< -e_1, -e_2 >, \ \sigma_3 =< e_1, -e_2 >\}$. Thus, the corresponding toric variety is $X_P \cong \mathbb{P}^1 \times \mathbb{P}^1$.

Proposition 3.2.25. *Let P be a polytope in $M_{\mathbb{R}}$ and let Σ_P be the fan corresponds to P. Then:*

i) $\dim F + \dim \sigma_F = n$ *for all faces* $F \preceq P$.

ii) $N_{\mathbb{R}} = \bigcup_{\sigma_F \in \Sigma_F} \sigma_F$.

Definition 3.2.26. A fan is *polytopal* if there exist a polytope P such that $0 \in P$ and Σ is spanned by the faces of P. A fan satisfying the condition (*ii*) in Proposition 3.2.25 is called *complete*.

Thus, a fan of a lattice polytope is always complete. Note that, the notion of completeness corresponds to the notion of compactness in the classical topology. This means that if Σ is complete, then X_Σ is a compactification of $T_N = (\mathbb{C}^*)^n$. It follows that the fan Σ is polytopal if and only if X_σ is projective.

3.3 Torus Action and Orbit Structure

We want to generalize a natural action of a torus T_N on itself to a toric variety X_Σ corresponding to a fan Σ in a real vector space $N_{\mathbb{R}}$. So, we can recover the definition of a toric variety by using a torus action. Actually, the origin of the name "toric variety (originally, torus embedding)" depends on this action. The basic references for this section are Ewald (1996), Fulton (1993) and Kempf et al. (1973).

3.3.1 The Torus Action

A torus action helps us to understand combinatorial results topologically. We can easily interpret some properties of toric variety by using combinatorial structure of a torus orbit. We begin by defining some fundamental definitions about an action in terms of algebra.

Definition 3.3.1. An *action* of a group G on a set X is a mapping $G \times X \to X$ defined by $(g, x) \mapsto g \cdot x$ that satisfies the following two conditions:

i) $g \cdot (h \cdot x) = (gh) \cdot x$ and

ii) $e \cdot x = x$ for all $g, h \in G$, $x \in X$.

Here, $e \in G$ is the identity element of G.

Note that we always have $g^{-1} \cdot (g \cdot x) = x$ for fixed $g \in G$ so the mapping $X \to X$, $x \mapsto g \cdot x$ is a bijection.

Definition 3.3.2. For a fixed element $x \in X$, the subset $G \cdot x := \{g \cdot x \mid g \in G\} \subset X$ is called the *orbit* of x, denoted by O_x.

Each point of X lies in a unique orbit. If x and y are in the same orbit, then $g \cdot x = y$ for all $x, y \in X$. This means that, x is equivalent to y and denoted by $x \sim y$. Thus X can be written as a disjoint union of its orbits, that is $X = \bigsqcup_{x \in X} \mathrm{O}_x$.

Now we return our main issue, toric varieties. First of all, we consider the affine case. Any affine toric varieties can be considered as an affine variety whose coordinate ring is determined by a strongly convex rational polyhedral cone. And it is known that there is a one-to-one correspondence between points of an affine toric variety X_σ corresponding to a strongly convex rational polyhedral cone σ and semigroup homomorphism $S_\sigma \to \mathbb{C}$. For a given n-dimensional lattice N, we have $T_N = N \otimes_{\mathbb{Z}} \mathbb{C}^* = \mathrm{Hom}(M, \mathbb{C}^*) = \mathrm{Spec}(\mathbb{C}[M]) = X_{\mathrm{o}}$, and it is isomorphic to $(\mathbb{C}^*)^n$. This means that T_N has a group structure and the group operation given by regular functions. The elements of the torus T_N are identified with group homomorphism $\mathrm{t} : M \to \mathbb{C}^*$, ($M$ is a group), so the group structure is just the multiplication, that is, for $\mathrm{t}_1, \mathrm{t}_2 : M \to \mathbb{C}^*$ we have $(\mathrm{t}_1\mathrm{t}_2) \cdot (u) = \mathrm{t}_1(u) \cdot \mathrm{t}_2(u) \in \mathbb{C}^*$ for all $u \in M$, with an identity element t satisfying $\mathrm{t}(u) = 1$ for all $u \in M$.

At this point one may naturally ask:

"Can we generalize this action to an affine toric variety?"

To do this we will take S_σ instead of M. But there is a little bit works because S_σ is just a semigroup, not a group. Let $v_i = \lambda_{i1} e_1^* + \cdots + \lambda_{in} e_n^*$ be generators of S_σ, for

$\lambda_i = (\lambda_{i1}, \ldots, \lambda_{in}) \in \mathbb{Z}^n$, $i = 1, \ldots, r$, and let $X_\sigma \cong \mathbb{V}(I) \subset \mathbb{C}^r$. Since the ideal I is generated by the relations between $\chi^{v_1}, \ldots, \chi^{v_r}$, we have $(\chi^{v_1}(\mathrm{t}), \ldots, \chi^{v_r}(\mathrm{t})) = \mathrm{t}^{\lambda_1}, \ldots, \mathrm{t}^{\lambda_r} \in \mathbb{V}(I) \cap (\mathbb{C}^*)^r$. In this case, there is an isomorphism φ such that $\varphi(\mathrm{t}) = (\chi^{v_1}(\mathrm{t}), \ldots, \chi^{v_r}(\mathrm{t})) = (\mathrm{t}^{\lambda_1}, \ldots, \mathrm{t}^{\lambda_r}) \in \mathbb{V}(I) \cap (\mathbb{C}^*)^r$. Thus, we say that the torus T_N can be embedded in the affine toric variety X_σ by an isomorphism φ, Brasselet (2001).

Example 3.3.3. Consider the affine toric variety $X_\sigma \cong \mathbb{V}(x_2^2 - x_1x_3)$ given in Example 3.1.41. Since S_σ is generated by $v_1 = e_1^*$, $v_2 = e_1^* + e_2^*$ and $v_3 = e_1^* + 2e_2^*$ we get $\lambda_1 = (1,0)$, $\lambda_2 = (1,1)$ and $\lambda_3 = (1,2)$, respectively. For $\mathrm{t} \in T_N$, we have $\mathrm{t}^{\lambda_1} = \mathrm{t}_1$, $\mathrm{t}^{\lambda_2} = \mathrm{t}_1\mathrm{t}_2$, $\mathrm{t}^{\lambda_3} = \mathrm{t}_1\mathrm{t}_2^2$. Hence, the map $\varphi : T_N \to X_\sigma$ given by $(\mathrm{t}_1, \mathrm{t}_2) \mapsto (\mathrm{t}_1, \mathrm{t}_1\mathrm{t}_2, \mathrm{t}_1\mathrm{t}_2^2)$ is an embedding of T_N into X_σ.

Since the points of the affine toric variety X_σ and semigroup homomorphisms $S_\sigma \to \mathbb{C}$ have been identified, we can restrict the group homomorphism $\mathrm{t} : M \to \mathbb{C}^*$ to S_σ and we obtain a natural product $\mathrm{t} \cdot x : S_\sigma \to \mathbb{C}$, $u \mapsto \mathrm{t}(u) \cdot x(u)$ for all $u \in S_\sigma$, that is a semigroup homomorphism and hence an element of X_σ. The map $T_N \times X_\sigma \to X_\sigma$ defines an *action of the torus* on an affine toric variety. There is an important result contains a relation between toric morphisms and torus actions.

For general toric varieties, we use the fact that each affine toric variety X_σ is embedded in the toric variety X_Σ as an open subset, for a cone $\sigma \in \Sigma$. Thus, we have an action $T_N \times X_\Sigma \to X_\Sigma$.

3.3.2 The Orbit of a Cone

We will discuss the correspondence between orbits and cones, and show how $\tau \in \Sigma$ determines an T_N-orbit of X_Σ. Our main point is that every toric variety can be written as a disjoint union of T_N-orbits.

By Remark 3.1.34 we can define a special point in X_σ. Let $x_\sigma : S_\sigma \to \mathbb{C}$ denote the semigroup homomorphism defined by the rule;

$$x_\sigma(u) = \begin{cases} 1, & \text{if } \ -u \in S_\sigma \\ 0, & \text{otherwise.} \end{cases}$$

The point x_σ is called a *distinguished point* of X_σ. Note that this map is well-defined, since for $u \in S_\sigma$, $-u \in S_\sigma$ if and only if $u \in \sigma^\perp \cap M$. Indeed, if u and $-u$ are in S_σ, then by Definition 3.1.8 we have $\langle u, v\rangle \geq 0$ and $\langle -u, v\rangle \geq 0$ for $v \in \sigma$. Then $\langle u, v\rangle = \langle -u, v\rangle = 0$, since $\langle u, v\rangle + \langle -u, v\rangle = \langle u - u, v\rangle = 0$, by the property of inner product. This means that $u \in \sigma^\perp \cap M$.

Example 3.3.4. Let σ be a cone given in Example 3.1.27. The generators of S_σ are $v_1 = e_1, v_2 = e_1 + e_2$ and $v_3 = e_1 + 2e_2$.

If we take σ as a face of itself, $\tau_1 = \sigma$, then $\sigma^\perp = \{0\}$. This means that, $v_i \notin \tau_1^\perp \cap M$. Thus, the distinguished point of τ_1 is given by $x_{\tau_1} = (0, 0, 0)$.

Let $\tau_2 = \{0\}$. Since $\{0\}^\perp = \mathbb{R}^2$, all v_1, v_2, v_3 are in $\tau_2^\perp \cap M$. Thus, $x_{\tau_2} = (1, 1, 1)$.

Let $\tau_3 = 2e_1 - e_2$. Then only $v_3 \in \tau_3^\perp \cap M$. This gives us the distinguished point such as $x_{\tau_3} = (0, 0, 1)$.

Finally take $\tau_4 = e_2$. By the similar way we obtain $x_{\tau_4} = (1, 0, 0)$.

The affine toric variety $X_\tau \subset X_\Sigma$ has the distinguished point x_τ. Then we define the orbit of a distinguished point as follows:

Definition 3.3.5. Let x_τ be a distinguished point corresponds to a face $\tau \preceq \sigma$. Then the *T_N-orbit* of x_τ is defined as $\mathrm{O}_\tau = T_N \cdot x_\tau$. The *closure* of an orbit of x_τ is defined by $V(\tau) = \mathrm{O}_\tau \cup \{0\}$.

Remark 3.3.6. If τ is an n-dimensional, then O_τ is the point x_τ. If $\dim(\tau) = k$ in $N_\mathbb{R} \cong \mathbb{R}^n$, then $\mathrm{O}_\tau \cong (\mathbb{C}^*)^{n-k}$. Moreover, if $\tau = \{0\}$, then $\mathrm{O}_\tau \cong (\mathbb{C}^*)^n = T_N$. Then there is a one-to-one correspondence between the points of T_N and the points of X_σ except the origin. This comes from the fact that, the action of T_N on X_τ is just an extension of the action of T_N on X_σ for any face $\tau \preceq \sigma$. So, $V(\tau) = T_N \cup \{0\}$. Thus, T_N is an open dense subset of X_σ.

Example 3.3.7. Now, we will determine the orbits corresponding to distinguished points constructed in Example 3.3.4. By applying Definition 3.3.5, we obtain

$$\begin{aligned} O_{\tau_1} &= T_N \cdot x_{\tau_1} = \{(0,0,0)\} \in (\mathbb{C}^*)^3, \\ O_{\tau_2} &= T_N \cdot x_{\tau_2} = \{(t_1, t_1 t_2, t_1 t_2^2)\} \cong T_N = (\mathbb{C}^*)^2, \\ O_{\tau_3} &= T_N \cdot x_{\tau_3} = \{(0,0,t_1 t_2^2)\} \cong \{0\} \times \{0\} \times \mathbb{C}^*, \\ O_{\tau_4} &= T_N \cdot x_{\tau_4} = \{(t_1,0,0)\} \cong \mathbb{C}^* \times \{0\} \times \{0\}. \end{aligned}$$

Example 3.3.8. Consider the affine toric variety $X_\sigma \cong (\mathbb{C}^*)^n$. Since X_σ corresponds to the cone $\sigma = \{0\}$ in $N_\mathbb{R} \cong \mathbb{R}^n$, there is only one distinguished point $x_\tau = (1, \dots, 1)$. Thus, $O_\sigma = T_N$.

Now, we will describe the orbits as the torus of some toric variety and then show how to embed them in that toric variety. In order to do this we must describe its fan by the following way. Let Σ be a fan in $N_\mathbb{R}$ and σ be a cone in Σ. For each face τ of a cone σ we set $M(\tau) = \tau^\perp \cap M$ to be a sublattice of M of rank $n - \dim(\tau)$. On the other hand, in general $\tau \cap N$ does not determine a sublattice in N. So we define N_τ to be the sublattice of N generated by $\tau \cap N$ and $N_\tau = (\tau \cap N) + (-\tau \cap N)$. The quotient $N(\tau) = N/N_\tau$ is also a lattice, called the *quotient lattice*, and its dual lattice is $M(\tau)$.

Proposition 3.3.9. *(Oda, 1985)* $O_\tau \cong T_{N(\tau)} = \mathrm{Hom}(M(\tau), \mathbb{C}^*) = \mathrm{Spec}(\mathbb{C}[M(\tau)])$*, is a torus whose dimension is* $n - \dim(\tau)$*.*

Example 3.3.10. Now by using Proposition 3.3.9 we will construct orbit of the toric variety given in Example 3.1.41.

If we take $\tau_1 = \sigma$, then $\sigma^\perp = \{0\}$. This implies $\mathbb{C}[\sigma^\perp \cap M]$ includes only zero polynomial. Thus, $O_{\tau_1} = \mathrm{Spec}(\mathbb{C}[M(\sigma)])$ corresponds to the origin.

Take $\tau_2 = \{0\}$. Then since $\tau_2^\perp = M_\mathbb{R}$, we have $\mathbb{C}[\tau_2^\perp \cap M] = \mathbb{C}[x^{\pm 1}, y^{\pm 1}]$. Thus, $O_{\tau_2} = \mathrm{Spec}(\mathbb{C}[M(\tau_2)]) \cong (\mathbb{C}^*)^2$.

For $\tau_3 = 2e_1 - e_2$, since $\tau_3^\perp = \langle \pm e_1^* \pm e_2^* \rangle$, we obtain $O_{\tau_3} = \mathrm{Spec}(\mathbb{C}[xy^2, x^{-1}y^{-2}]) \cong \mathbb{C}^*$.

For $\tau_4 = e_2$, since $\tau_4^\perp = \langle \pm e_1^* \rangle$, we obtain $O_{\tau_4} = \mathrm{Spec}(\mathbb{C}[x, x^{-1}]) \cong \mathbb{C}^*$.

Definition 3.3.11. Let $\Sigma \subset N_{\mathbb{R}}$ be a fan and τ be a cone in Σ. The *star of* τ is the set of cones which contains τ as a face. By using the cone σ in a star of τ, we can obtain the quotient cone $\bar{\sigma}$ as follows:

$$\bar{\sigma} = (\sigma + (N_\tau)_{\mathbb{R}})/(N_\tau)_{\mathbb{R}} \subset N_{\mathbb{R}}/(N_\tau)_{\mathbb{R}} = N(\tau)_{\mathbb{R}}.$$

Then we set $\{\bar{\sigma} | \tau \preceq \sigma \in \Sigma\}$ in $N(\tau)$, and denoted by $\Sigma(\tau)$, see Figure 3.12.

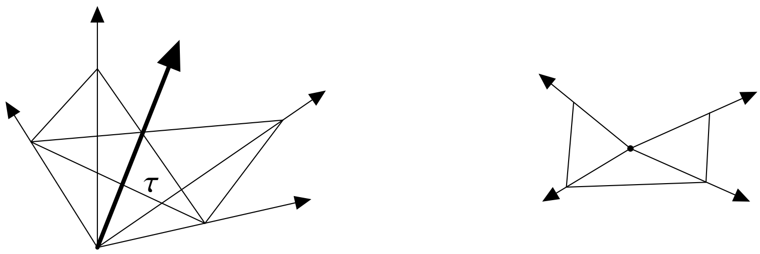


Figure 3.12 A face τ in a lattice $N = \mathbb{Z}^3$ and $\Sigma(\tau)$

Proposition 3.3.12. *i) For any* $\sigma \in \mathrm{Star}(\tau)$*,* $\bar{\sigma}$ *is a strongly convex rational polyhedral cone in* $N(\tau)_{\mathbb{R}}$.

ii) $\Sigma(\tau)$ *form a fan in* $N(\tau)_{\mathbb{R}}$.

Let τ be a face of $\sigma \in \Sigma$. Then the corresponding affine variety is defined by

$$X_\sigma(\tau) = \mathrm{Spec}(\mathbb{C}[\check{\bar{\sigma}} \cap M(\tau)]) = \mathrm{Spec}(\mathbb{C}[\check{\sigma} \cap \tau^\perp \cap M]).$$

In particular, since for $\sigma = \tau$ we have $X_\tau(\tau) = \mathrm{Spec}(\mathbb{C}[\tau^\perp \cap M(\tau)])$, we have $X_\tau(\tau) = \mathrm{O}_\tau$. For each $i = 1, \ldots, r$, if there exist σ_i in Σ, which have a face τ, then by gluing the corresponding affine toric varities $X_{\sigma_i}(\tau)$ we obtain the toric variety

$$V(\tau) = X(\Sigma(\tau)) = X_{\sigma_1}(\tau) \cup \cdots \cup X_{\sigma_r}(\tau),$$

i.e., the toric variety $V(\tau)$ is covered by the affine toric varieties. Considering points as semigroup homomorphisms, the embedding

$$X_\sigma(\tau) = \mathrm{Hom}(\check{\sigma} \cap \tau^\perp \cap M, \mathbb{C}^*) \hookrightarrow \mathrm{Hom}(\check{\sigma} \cap M, \mathbb{C}) = X_\sigma$$

is given by the map $u \mapsto u$ if $u \in \check{\sigma} \cap \tau^\perp \cap M$, $u \mapsto 0$ otherwise, i.e., zero extension. Since $\check{\sigma} \cap \tau^\perp$ is a face of $\check{\sigma}$, the extension by the zero of a semigroup homomorphism is also a semigroup homomorphism. Then the corresponding surjection $\mathbb{C}[\check{\sigma} \cap M] \to \mathbb{C}[\check{\sigma} \cap \tau^{\perp} \cap M]$ is defined by $\chi^u \mapsto \chi^u$ if $u \in \check{\sigma} \cap \tau^{\perp}$ and $\chi^u \mapsto 0$ otherwise, see Fulton (1993).

Let $\tau \leq \sigma \leq \sigma'$. Then we have a commutative diagram

$$\begin{array}{ccc} X_{\sigma}(\tau) & \hookrightarrow & X_{\sigma'}(\tau) \\ \downarrow & & \downarrow \\ X_{\sigma} & \hookrightarrow & X_{\sigma'} \end{array}$$

and gluing these maps gives an embedding $V(\tau) \hookrightarrow X(\Sigma(\tau))$ as a closed subvariety.

In particular, if $\tau \leq \tau'$, then we have closed embedding $V(\tau') \hookrightarrow V(\tau)$. Thus, we have an order-reversing correspondence

$$\{\text{cones}\, \tau \in \Sigma\} \longleftrightarrow \{\text{Orbit closures}\, V(\tau) \in X(\Sigma(\tau))\}$$

By this construction, the ideal of $V(\tau) \cap X_{\sigma}$ in R_{σ} is $\bigoplus \mathbb{C} \cdot \chi^u$, the sum over all $u \in S_{\sigma}$ such that $\langle u, v \rangle > 0$ for $v \in \text{Relint}(\tau)$.

Example 3.3.13. Let $X_{\Sigma} = \mathbb{P}^1$, where $\Sigma = \{\sigma_1 =< e_1 >, \sigma_2 =< -e_1 >\}$.

Let $\tau_1 =< e_1 >$. Then τ_1 is a face of only σ_1, so $\text{Star}(\tau_1) = \{\sigma_1\}$. Then the toric variety $X_{\sigma_1}(\tau_1) = \{0\}$. By the morphism $0 \mapsto (1:0)$, we have an embedding $X_{\sigma_1}(\tau_1) \hookrightarrow \mathbb{P}^1$, while $X_{\sigma_1} \hookrightarrow \mathbb{P}^1$ given by $x \mapsto (1:x)$. Thus $V(\tau_1) = \{(1:0) \in \mathbb{P}^1\}$.

Let $\tau_2 =< -e_1 >$. Then τ_2 is a face of only σ_2, so $\text{Star}(\tau_2) = \{\sigma_2\}$. Thus, by the similar construction we have $V(\tau_2) = \{(0:1) \in \mathbb{P}^1\}$.

Let $\tau_3 = \{0\}$. Then τ_3 is a face of both σ_1 and σ_2, so $\text{Star}(\tau_3) = \{\sigma_1, \sigma_2\}$. Then, the toric varieties $X_{\sigma_1}(\tau_3) = X_{\sigma_1} = \mathbb{C}_{(x)}$ and $X_{\sigma_2}(\tau_3) = X_{\sigma_2} = \mathbb{C}_{(x^{-1})}$. By gluing these varieties we obtain $V(\tau_3) = \mathbb{P}^1$.

Let us conclude this section by constructing a general toric variety X_{Σ} from a T_N-orbit O_{τ}.

Proposition 3.3.14. *(Fulton, 1993) The following relations are true:*

i) $X_\sigma = \bigsqcup_{\tau \leq \sigma} \mathrm{O}_\tau$

ii) $V(\tau) = \bigsqcup_{\gamma \geq \tau} \mathrm{O}_\gamma$

iii) $\mathrm{O}_\tau = V(\tau) \setminus \bigcup_{\gamma \geq \tau} V(\gamma)$

3.3.3 Characters and One-Parameter Subgroups

Now our main concept is to define the one-parameter subgroups and characters of the torus and their limit points in toric varieties. Then we will show how to define a fan from the torus action.

By the fact that every algebraic group endomorphism of the algebraic one-torus $T_N = \mathbb{C}^*$ is of the form $t \mapsto t^k$ with a unique integer $k \in \mathbb{Z}$, we obtain the canonical group isomorphism $\mathrm{Hom}(\mathbb{C}^*, \mathbb{C}^*) \cong \mathbb{Z}$ sending $\mathrm{id}_{\mathbb{C}^*}$ to 1.

Definition 3.3.15. A homomorphism of algebraic groups $\lambda : \mathbb{C}^* \to T_N$ is defined by $\lambda^u(t) = u \otimes z$ for $u \in N$ and this is called a *one-parameter subgroup* of T_N.

If an isomorphism $N \cong \mathbb{Z}^n$ sends u to $(a_1, \ldots, a_n)$, then $\lambda^u(t) = (t^{a_1}, \ldots, t^{a_n})$ under the induced isomorphism $T_N \cong (\mathbb{C}^*)^n$.

Definition 3.3.16. A homomorphism of algebraic groups $\chi^v : T_N \to \mathbb{C}^*$is defined as $\chi^v(t) = \prod_{i=1}^{l} t^{\langle u_i, v_i \rangle}$ for a given $v \in M$ and it is called a *character* of T_N.

In particular, $\lambda_v(t)(u) = \chi^u(\lambda_v(t)) = t^{\langle u,v \rangle}$, see Barthel (2000b). So that M is its character group with the dual pairing N. If $M \cong \mathbb{Z}^n$ sends v to $(b_1, \ldots, b_n)$, then $\chi^v(t_1, \ldots, t_n) = (t_1^{b_1}, \ldots, t_n^{b_n})$ under the isomorphism $T_N \cong (\mathbb{C}^*)^n$. The character corresponding to v can be identified with the function χ^v in the coordinate ring $\mathbb{C}[M] = \Gamma(T_N, \mathcal{O}^*)$. It turns out that the possible limit points are necessarily the images of distinguished points x_σ under the embedding $X_\sigma \hookrightarrow X_\Sigma$. Then we have a one-to-one correspondence between faces, orbits and limits.

Proposition 3.3.17. *i) If a lattice vector v is contained in the relative interior of some cone $\sigma \in \Sigma$, then* $\lim_{z \to 0} \lambda_v(z) = x_\sigma$ *exists.*

ii) If a lattice vector v is not contained in any cone of Σ, then $\lim_{z \to 0} \lambda_v(z)$ *does not exist in X_Σ.*

This Proposition gives an idea why we prefer a cone in $N_\mathbb{R}$ rather than $M_\mathbb{R}$. To define toric varieties more explicitly we may consider the relation between normality and saturation. This relation is given as follows:

Proposition 3.3.18. *(Barthel, 2000a) A finitely generated subsemigroup S of M is saturated if and only if the algebra $\mathbb{C}[S]$ is normal.*

In our case, we have focused on strongly convex rational polyhedral cones in $N_\mathbb{R}$ to construct semigroups S_σ, our semigroups are all saturated. Thus, we obtain the following fact:

Proposition 3.3.19. *Every toric varieties is normal.*

After all that we can recover the definition of a toric variety as follows:

Definition 3.3.20. An *n-dimensional toric variety* is an irreducible normal variety X that contains a torus $T_N = (\mathbb{C}^*)^n$ as a dense open subset, together with an action $T_N \times X \to X$ of T_N on X that extends the natural action of the torus T_N itself.

It is a natural thing to think about the converse part. Let X be a normal variety endowed with a torus action that has an open orbit. Then X is also a toric variety.

3.4 Properties of Toric Varieties

3.4.1 Smoothness

Our aim is to give a combinatorial criterion for smoothness of the toric variety. Since smoothness is a local property, we will study on an affine toric variety X_σ for a cone σ. Then we will generalize the idea of being smooth to general toric varieties X_Σ. Basic references for this notion are Cox et al. (2011), Fulton (1993), Kempf et al. (1973) and Ewald (1996).

Before proceeding, we need to recall some facts from algebraic geometry. Let X be any variety and let P be a point in X. To describe the notion of smoothness we will define the tangent space in terms of algebraic geometry. The *Zariski tangent space* is defined to be $T_P(X) := \mathrm{Hom}_{\mathbb{C}}(\mathcal{M}_{X,P}/\mathcal{M}^2_{X,P}, \mathbb{C})$ where $\mathcal{M}_{X,P}$ is a maximal ideal of the local ring $\mathcal{O}_{X,P}$. Since $\mathcal{O}_{X,P}/\mathcal{M}_{X,P} \cong \mathbb{C}$, $\mathcal{M}_{X,P}/\mathcal{M}^2_{X,P}$ has a natural structure as a vector space over $\mathbb{C}$.

For the affine case we compute the Zariski tangent space as follows: Let $V \subset \mathbb{C}^n$ be an affine variety, $P \in V$ and $\mathbb{I}(V) = < f_1, \ldots, f_s >$. Let

$$d_P(f_i) = \frac{\partial f_i}{\partial x_1}(P)x_1 + \cdots + \frac{\partial f_i}{\partial x_n}(P)x_n, \tag{3.4.1}$$

for each i. The tangent space $T_P(V)$ is isomorphic to the subspace of $\mathbb{C}^n$ defined by $d_P(f_1) = \cdots = d_P(f_s) = 0$.

Example 3.4.1. Let $V = \mathbb{V}(x^3 - y^2) \subset \mathbb{C}^2$ be an affine variety and let $P \in V$. The Zariski tangent space of V is computed by using the formula given in Equation (3.4.1).

$$d_P(f) = \frac{\partial f}{\partial x}(P)x + \frac{\partial f}{\partial y}(P)y = 3x^2(P)x - 2y(P)y.$$

Thus, the Zariski tangent space is isomorphic to subspace of $\mathbb{C}^2$ defined by the line equation $ax + by = 0$, where $a, b \in \mathbb{C}$.

Definition 3.4.2. A variety V is *smooth* (or *nonsingular*) at $P \in V$ if $\dim_{\mathbb{C}} T_P(V) = \dim_P(V)$. The point P is called a *singular* point of V if it is not a smooth point. The set of singular points of V is denoted by $\mathrm{Sing}(V)$. A point $P \in V$ is called *isolated singular point* if $\mathrm{Sing}(V) \cap (V \setminus \{P\}) = \emptyset$.

After all that we can characterize the smoothness of an affine toric variety as in the following cases:

Let $\dim(\sigma) = n$. Let $\mathbb{C}[S_\sigma] \to \mathbb{C}$ be the $\mathbb{C}$-algebra homomorphism which sends χ^u to 0 for all $u \in S_\sigma \setminus \{0\}$. Then the kernel of this homomorphism gives a maximal ideal $\mathcal{M}$ that corresponds to a point x_σ. It follows that

$$\mathcal{M} = \langle \chi^u \mid u \in S_\sigma \setminus \{0\}\rangle \quad \text{and} \quad \mathcal{M}^2 = \langle \chi^u \mid u \in S_\sigma \setminus \{0\} \text{ is reducible}\rangle.$$

Thus, the irreducible elements of $S_\sigma \setminus \{0\}$ give a basis of $\mathcal{M}/\mathcal{M}^2$ as a vector space over $\mathbb{C}$. Since $\mathcal{M}_{X_\sigma,x_\sigma}$ is a maximal ideal of O_{X_σ,x_σ}, we get $\mathcal{M}/\mathcal{M}^2 \cong \mathcal{M}_{X_\sigma,x_\sigma}/\mathcal{M}^2_{X_\sigma,x_\sigma}$, and since $x_\sigma \in X_\sigma$ is smooth, it follows that $T_{x_\sigma}(X_\sigma) \cong \mathrm{Hom}_{\mathbb{C}}(\mathcal{M}_{X_\sigma,x_\sigma}/\mathcal{M}^2_{X_\sigma,x_\sigma})$ has dimension n as a vector space over $\mathbb{C}$, we have $\dim_{\mathbb{C}}\mathcal{M}/\mathcal{M}^2 = n$. This implies that, the dual cone $\check{\sigma}$ cannot have more than n edges, and that the minimal generators, i.e., primitive elements, along these edges must generate S_σ. Since S_σ generates M as a group, the minimal generators for S_σ must be a basis for M. Thus, $\check{\sigma}$ must be generated by a basis for N. Hence, $X_\sigma \cong \mathbb{C}^n$.

Let $\dim(\sigma) \leq d$. Assume that N_σ be the sublattice of rank d generated by $\sigma \cap N$, i.e., $N_\sigma = \sigma \cap N + (-\sigma \cap N)$. Since σ is saturated, N_σ is also saturated, so that we can find a splitting $N = N_\sigma \oplus N_1$, where $\sigma = \sigma' \times \{0\}$. By duality and $\langle M_1, N_\sigma\rangle = 0$, we have $M = (N_\sigma)^* \oplus M_1$ and $\check{\sigma} = (\check{\sigma'}) \oplus M_1$. This gives an isomorphism $\mathbb{C}[\check{\sigma}] \cong \mathbb{C}[(\check{\sigma'})] \otimes_{\mathbb{C}} \mathbb{C}[M_1]$. Thus, $X_\sigma \cong X_{\sigma'} \times (\mathbb{C}^*)^{n-d}$. By the previous case, σ' must be generated by a basis for N_σ.

Proposition 3.4.3. *(Fulton, 1993, Proposition at page 29) An affine toric variety X_σ is smooth if and only if σ is generated by part of a basis for the lattice N, in which case* $X_\sigma \cong \mathbb{C}^d \times (\mathbb{C}^*)^{n-d}$, $d = \dim(\sigma)$.

Therefore, a cone σ is called a *regular* (or *nonsingular*) if it is generated by part of a basis for the lattice N, and we call a fan *regular* (or *nonsingular*) if all of its cones are regular, i.e., if the corresponding toric variety is smooth.

Remark 3.4.4. In general case, we can characterize smooth toric varieties as follows: A toric variety X_Σ is smooth if and only if every cone σ in a fan Σ is regular.

Definition 3.4.5. A cone σ in $N_{\mathbb{R}}$ is *simplicial* if its minimal generators are linearly independent over $\mathbb{R}$, i.e., generated by numbers of $\dim(\sigma)$ edges. A toric variety X_Σ is *simplicial* if every cone in Σ is simplicial.

A simplest example of a simplical cone is any two dimensional cone.

Before going further, we will investigate an idea of changing the lattice that allows us to construct singularities of toric varieties. Let N be a lattice with a sublattice N' of finite index. Now consider a cone σ given in a lattice N. We write σ' instead of σ if we consider it in a lattice N'. It can be seen that, for a cone $\sigma \subset N_{\mathbb{R}} = N'_{\mathbb{R}}$, the property being strongly convex, rational and polyhedral is equivalent with respect to both lattices. Then the morphism of affine toric varieties $X' = X_{\sigma'} \to X_\sigma = X$ induced by the inclusion $N' \to N$. Let N/N' be a group denoted by G, where

$$N/N' = \mathrm{Hom}(M'/M, \mathbb{Q}/\mathbb{Z}) = \mathrm{Hom}(M'/M, \mathbb{C}) \subset \mathrm{Hom}(M', \mathbb{C}^*).$$

This group can be identified with the kernel of the homomorphism

$$T_{N'} = \mathrm{Hom}(M', \mathbb{C}^*) \longrightarrow T_N = \mathrm{Hom}(M, \mathbb{C}^*)$$

induced by the inclusion $N' \hookrightarrow N$. Therefore, an affine toric variety X_σ can be identified by the quotient of $X_{\sigma'}$ under the action of N/N'. More generally, a toric variety X_Σ is identified with the quotient of $X_{\Sigma'}$ under the action of N/N'.

More explicitly, suppose that σ is a cone in $N_{\mathbb{R}}$ which is simplicial, and suppose that $u_1, \ldots, u_k \in \sigma \cap N$ are the primitive elements, along the edges of σ. Let N_σ denote the subgroup of N generated by the u_i. Then N_σ can be extended to a lattice $N' \subset N$ such that $\sigma \cap N'_{\mathbb{R}} = N_\sigma$, i.e., σ is nonsingular with respect to N', denoted by σ'. On the other hand, we have

$$\mathbb{C}[X'] = R_{\sigma'} = \mathbb{C}[\chi^u \mid u \in S_{\sigma'}] \supset \mathbb{C}[\chi^u \mid u \in S_\sigma] = R_\sigma = \mathbb{C}[X]$$

for $S_\sigma \subset S_{\sigma'}$. If $g \in G$, $u \in S_\sigma$, $x' \in X'$, then

$$(g \cdot \chi^u)(x) = \chi^u(g^{-1}u') = g^{-1}(u) \cdot x'(u) = x'(u) = \chi^u(x').$$

This means that $R_\sigma \subset R^G_{\sigma'}$. Conversely, suppose $f \notin R_\sigma$. Write $f = \sum c_j \chi^{u_j}$ for some $c_j \in \mathbb{C}, u_j \in M'$. Since $f \notin R_\sigma$ at least one $u_j \notin M$, say u_1. Choose $g \in G$ such that $g(u_1) \neq 1$. Then, $g(f) \neq f$ and hence $R_\sigma = R^G_{\sigma'}$. It follows that, $X = X'/G$. Thus, we obtain the main result about simplicial toric varieties.

Theorem 3.4.6. *A toric variety X given by a simplicial fan Σ has only quotient singularities.*

Suppose that Σ is a fan in N in which every cone is simplicial. Then X_Σ is covered by affine open sets X_σ each of which is a quotient of affine space by a finite group. Thus, X_Σ is a quotient of a smooth variety by the action of a finite groups locally. Such a space is called an *orbifold*.

Before specizaliting our case, we will formulate two-dimensional affine toric varieties as follows:

Lemma 3.4.7. *Any two-dimensional affine toric variety comes from a cone σ generated by $u_1 = e_2 = (0,1)$ and $u_2 = pe_1 - qe_2 = (p,-q)$ with $0 \leq q < p$ and* $\gcd(p,q) = 1$.

Example 3.4.8. Assume that the cone $\sigma \subset N_{\mathbb{R}} \cong \mathbb{R}^2$ is generated by $pe_1 - qe_2$ and e_2 where $0 \leq q \leq p$ and $\gcd(p,q) = 1$.

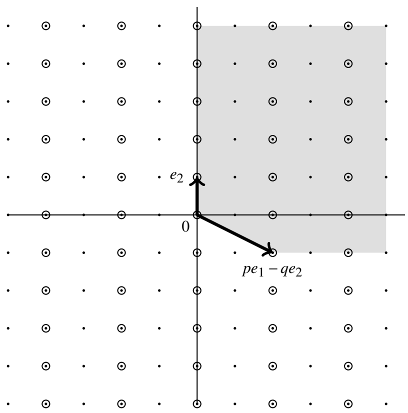


Figure 3.13 σ and σ' in lattices N and N', respectively

Let N be a lattice generated by dots, and let $N' = \{\alpha e_1 + \beta e_2 \mid \beta \in q\mathbb{Z}\} =< pe_1 - qe_2, e_2 >$, that is generated by circles, see Figure 3.13. Then $N/N' \to \mathbb{Z}/q\mathbb{Z}$. The dual lattice M' is generated by $e_2^* + \frac{p}{q}e_1^*$ and $\frac{1}{q}e_1^*$, i.e., $\frac{1}{q}e_1^*$ and e_2^* corresponding to monomials U and Y, and the corresponding σ' has $S_{\sigma'} = \langle \frac{1}{q}e_1^*, \frac{p}{q}e_1^* + e_2^* \rangle$. So we get $V = U^qY$ and then

$$R_{\sigma'} = \mathbb{C}[U,V] = \mathbb{C}[U,U^qY].$$

Thus, $X_{\sigma'} \cong \mathbb{C}^2$. The group $G = \mathbb{Z}/p\mathbb{Z}$ acts on $X_\sigma = \mathbb{C}^2$ by $\zeta(u,v) = (\zeta u, \zeta^q v)$ and $X_\sigma = X_{\sigma'}/G = \mathbb{C}^2/G$, i.e., $R_\sigma = R_{\sigma'}^G = \mathbb{C}[U,V]^G$. In this case, X_σ is a *cyclic quotient singularity*.

It follows that, any two-dimensional affine toric variety has only cyclic quotient singularities.

Example 3.4.9. Apply the above construction to the cone $\sigma =< pe_1 - e_2, e_2 >\subset N_\mathbb{R} = \mathbb{R}^2$, where $q = 1$. Then $R_\sigma = \mathbb{C}[S_\sigma] = \mathbb{C}[X, XY, \ldots, XY^p]$. Let $X = U^p$ and $Y = V/U$. Thus, $R_\sigma = \mathbb{C}[U^p, U^{p-1}V, \ldots, V^p] \subset \mathbb{C}[U,V]$ so $X_\sigma = \mathrm{Spec}(R_\sigma)$ is the cone over the rational normal curve of degree p. Now assume that N' be the sublattice in N generated by pe_1 and e_2, and let σ' be the same cone as σ, but considered in N'. $N' \subset N$ implies that $M' \supset M$ and M' is generated by $\frac{1}{p}e_1^*$ and e_2^*, corresponding to monomials U and Y with $U^p = X$. Since $S_\sigma = \langle \frac{1}{p}e_1^*, \frac{1}{p}e_1^* + e_2^* \rangle$, we get $R_{\sigma'} = \mathbb{C}[U, UY] = \mathbb{C}[U,V]$ with $V = UY$. Thus, $X_{\sigma'} \cong \mathbb{C}^2$. The inclusion $N' \subset N$ gives us $X_{\sigma'} \to X_\sigma$. Therefore, $X_\sigma \cong \mathbb{C}^2/p\mathbb{Z}$.

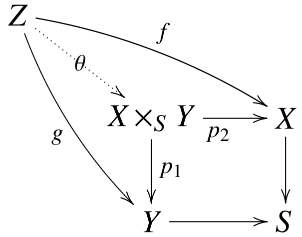
CHAPTER FOUR
DEFORMATION THEORY

Deformation theory is the fundamental technique in many branches of mathematics. We will develop the deformation theory of affine schemes, and their singularities. This theory gives us methodical ways in which these schemes can be perturbed. In particular, deformation theory allows to better understand some properties of an original object on a simpler tool, which comes from an algebraic notion, so-called flatness. Flatness preserves certain invariants of an original object and so we will especially introduce these properties of flatness in the present chapter. The standard textbooks on the theory of deformations are Artin (1976), Stevens (2003) and Greuel et al. (2007).

4.1 Definitions and Examples

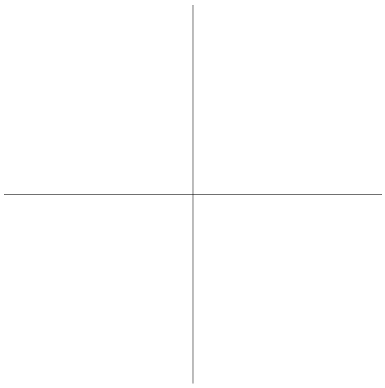
Our main point is to obtain a description of an affine scheme X, it is useful to investigate the characteristics of X under deformations. Then we will try to reinforce our description on some basic examples.

To talk about a morphism for the notion of deformations firstly we need to define the fibre of a morphism. Let X and Y be schemes over S. A *fibred product* (or *pullback*) of X and Y over S is a scheme $X \times_S Y$ with morphisms $p_1 : X \times_S Y \to X$ and $p_2 : X \times_S Y \to Y$ such that given any scheme Z with morphism $f : Z \to X$ and $g : Z \to Y$, there exists a unique morphism $\theta : Z \to X \times_S Y$ such that $f = p_1 \circ \theta$ and $g = p_2 \circ \theta$ which makes a commutative diagram:

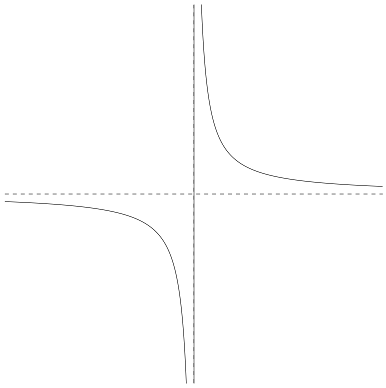


We can use the fibred product to define the fibre of a morphism. Let $f : X \to Y$ be a morphism of schemes and let $y \in Y$ be a point. Let $\mathbb{C}(y) = \mathcal{O}_{Y,y}/\mathcal{M}_{Y,y}$ be the residue field of y, and let $\mathrm{Spec}\mathbb{C}(y) \to Y$ be the natural morphism. Then we define the *fibre* of the morphism f over the point y to be the scheme $X_y = X \times_y \mathrm{Spec}\mathbb{C}(y)$. In other words, the fibre X_y is a scheme over $\mathbb{C}(y)$, and one can show that $X_y = f^{-1}(y)$. If $y = 0$, then the fibre $X_0 = f^{-1}(0)$ is called a *special fibre*. The fibre admits to consider a morphism as a family of schemes that parametrized by the points of the image scheme. Conversely, this family gives a useful way to vary a family of schemes algebraically. Thus, intuitively a deformation of X is actually a variation of X in a family.

Example 4.1.1. Let $X = X_0 = \mathbb{V}(xy) \subset \mathbb{C}^2$. Now we perturb the defining equation to deform X_0. $X_t = \mathbb{V}(xy - t) \subset \mathbb{C}^2$ gives us a family of smooth plane curves. This family degenerates to X_0 as $t \to 0$. In other words, we can lift the relation of the defining equation. Since X_0 is given by only one defining equation, every perturbation is a deformation of X_0.



(a) X_0, (for $t = 0$)



(b) X_t, (for $t \neq 0$)

Figure 4.1 Deformations of X

We define a family with special fibre X_0 over a base space S to be a morphism $\pi : \mathcal{X} = \bigcup_t X_t \to S$ such that X_0 is isomorphic to $\pi^{-1}(0)$. This means that each X_t arises as fibres of the morphism $\pi : \mathcal{X} \to S$. In terms of locality, we have equations $\tilde{f}_1(\underline{x}, \underline{t}), \dots, \tilde{f}_k(\underline{x}, \underline{t})$ whose restrictions $\tilde{f}_1(\underline{x}, \underline{0}), \dots, \tilde{f}_k(\underline{x}, \underline{0})$ generate the ideal, isomorphic to the ideal of X_0. But, in general this family is not nice enough to define deformations. Investigating the following example we will find a necessary condition to obtain a good description of deformation. The good deformation means that discrete invariants stable with respect to $\underline{t}$.

Example 4.1.2. Let $X = X_0 = \mathbb{V}(xy, xz, yz) \subset \mathbb{C}^3$ which consists of the three coordinate axes in $\mathbb{C}^3$.

Firstly, consider the one-parameter family given by $X_t = \mathbb{V}(xy - t, xz - t, yz - t) \subset \mathbb{C}^3$. Then the equations of X_t gives us just two points $(-\sqrt{t}, -\sqrt{t}, -\sqrt{t})$ and $(\sqrt{t}, \sqrt{t}, \sqrt{t})$ when $t \neq 0$. Then clearly this family cannot be a deformation of X_0, since X_t does not gives X_0 as $t \to 0$. One can show that the dimension changes with respect to this family. Something is WRONG!

Now, we investigate what the problem is here. For $t = 0$, we have three linearly independent equations $f_1 = xy$, $f_2 = xz$ and $f_3 = yz$. But we have non-trivial relations such that

$$f_1 \cdot z - f_2 \cdot y = 0, \quad f_1 \cdot z - f_3 \cdot x = 0 \quad \text{and} \quad f_2 \cdot y - f_3 \cdot x = 0.$$

If we try to extend this relations to be include the variable t, we obtain

$$F \cdot z - G \cdot y = t(z - y) \quad \text{and} \quad F \cdot z - H \cdot x = t(z - x),$$

where $F = xy - t$, $G = xz - t$ and $H = yz - t$. For $t \neq 0$, we can divide by t and take new generators of the ideal $< z - y, z - x, x^2 - t >$ describes the lines $(\pm\sqrt{t}, \pm\sqrt{t}, \pm\sqrt{t})$. Thus, we cannot lift the relations.

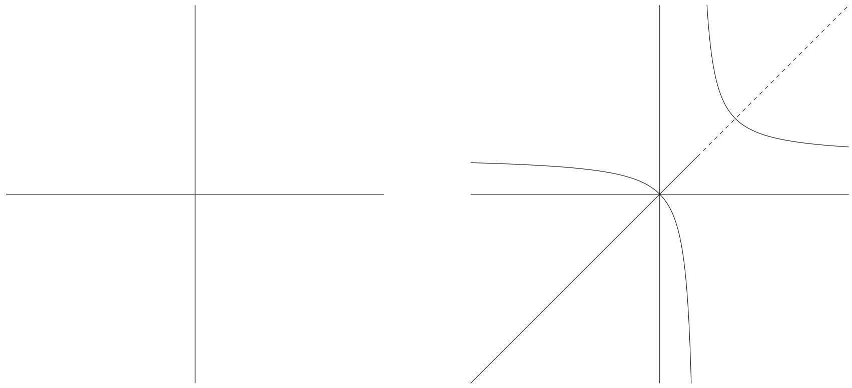
On the other hand, consider a different family such that

$$F = xy, \quad G = xz \quad \text{and} \quad H = yz + ty + tz = yz + t(y + z).$$

For $t \neq 0$, the space X_t consists of the x-axis and the smooth hyperbola passing through the origin.

Now we can lift the relations as follows:

$$F \cdot z - G \cdot y = 0 \quad \text{and} \quad F \cdot (z + t) + G \cdot t - H \cdot x = 0.$$



(a) X_0, (for $t = 0$) (b) X_t, (for $t \neq 0$)

Figure 4.2 Deformations of X

Thus, we obtain a true deformation of X, the key point is that the relation between the defining equations $f_i(\underline{x})$ lift to some relations, depending on t, between the perturbed equations $\tilde{f}_i(\underline{x}, \underline{t})$. The corresponding algebraic tool for this notion is that of flatness.

A module M over a commutative ring R (with unity) is said to be *flat* if for every short exact sequence of R-modules $0 \to N' \to N \to N'' \to 0$, the induced sequence $0 \to N' \otimes_R M \to N \otimes_R M \to N'' \otimes_R M \to 0$ is again exact. Flatness is exactly an algebraic tool, but it can be regarded as continuous behaviours of the fibres in view of geometry.

Now, we define a ring of deformation parameters $\mathbb{C}[t_1, \ldots, t_r]$ and $g_1, \ldots, g_k \in \mathbb{C}[t_1, \ldots, t_r]$ generate an ideal in $\mathbb{C}[t_1, \ldots, t_r]$. They also define an affine scheme $S = \text{Spec}\mathbb{C}[t_1, \ldots, t_r]/ < g_i >$. This will be our base space of deformations.

Let $I =< f_1(\underline{x}), \ldots, f_m(\underline{x}) > \subset \mathcal{O}_{\mathbb{C}^n}$ be an ideal, and let $\tilde{I} =< \tilde{f}_1(\underline{x}, \underline{t}), \ldots, \tilde{f}_m(\underline{x}, \underline{t}) > \subset \mathcal{O}_{\mathbb{C}^n \times S}$ a lifting of I, which define schemes $X \subset \mathbb{C}^n$ and $\mathcal{X} \subset \mathbb{C}^n \times S$, respectively. Now we will construct the lifting relations in terms of flatness, under these notation.

Definition 4.1.3. The map $\pi : \mathcal{X} \to S$ is *flat* if every relations between the f_i lifts to some relations between the $\tilde{f}_i$.

The criterion for flatness can be given as follows:

Proposition 4.1.4 (Lifting Relations)**.** *With the above notations the followings are equivalent:*

i) The map $\pi : \mathcal{X} \to S$ *is flat*

ii) $O_{\mathcal{X},0}$ *is a flat* $O_{S,0}$*-module.*

iii) Every exact sequences $\cdots \to O^k_{X_0,0} \to O^r_{X_0,0} \to O_{X_0,0} \to O_{X_0,0}/I \to 0$ *lifts to an exact sequence* $\cdots \to O^k_{\mathcal{X},0} \to O^r_{\mathcal{X},0} \to O_{\mathcal{X},0} \to O_{\mathcal{X},0}/\tilde{I} \to 0.$

To further proceed we recall that some properties of flat morphism:

i) Flatness is preserved under base change: For a given morphism diagram

$$\begin{array}{ccc} X' & \xrightarrow{f} & X \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ S' & \xrightarrow[g]{} & S \end{array}$$

Take any $p' \in X'$ and let $f(p') = p$. If π is flat at p, then π' is flat at p'.

ii) If $\pi : X \to S$ is flat, then for every $P \in X$ the dimension formula $\dim_P X = \dim_{\pi(P)} S + \dim_P X_0$ holds.

iii) Every flat morphism is open.

Now, we are able to give our main definition of deformations as follows.

Definition 4.1.5. A *deformation* of an affine scheme X is a flat family of schemes $\pi : \mathcal{X} \to S$, such that X is isomorphic to the fibre $\pi^{-1}(0)$. A scheme $\mathcal{X} \subset \mathbb{C}^n \times S$ is called the *total space*, S the *base space* of the deformation. We call π an *r-parameter* deformation for an open subset S of $\mathbb{C}^r$.

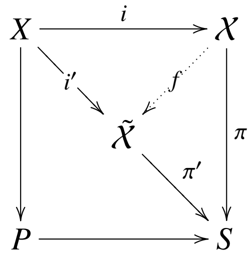
We can write a deformation in the deformation language

$$\begin{array}{ccc} X & \xrightarrow{i} & \mathcal{X} \\ \downarrow & & \downarrow{\scriptstyle \pi} \\ \{0\} & \longrightarrow & S \end{array}$$

where i is a called embedding mapping X_0 isomorphically onto $\pi^{-1}(0)$. Note that, we can define a deformation of X for any $s \in S$. We just prefer using 0 for simplicity.

The simplest example is the hypersurfaces, because every perturbations gives a deformation, see Example 4.1.1.

Definition 4.1.6. A *morphism* between two deformations $\pi : \mathcal{X} \to S$ and $\pi' : \tilde{\mathcal{X}} \to S$ of X over the same base S is a morphism $f : \mathcal{X} \to \tilde{\mathcal{X}}$ over S, i.e., $\pi' \circ f = \pi$, compatible with the embedding $i : X \to \mathcal{X}$ and $i' : X \to \tilde{\mathcal{X}}$ such that $f \circ i = i'$. In other words, the diagram



is commutative.

Two deformations are isomorphic if there exists an isomorphism $f : \mathcal{X} \to \tilde{\mathcal{X}}$.

Definition 4.1.7. Let $\pi : \mathcal{X} \to S$ be a deformation of X, and let $f : S' \to S$ be a morphism. The *induced deformation* is the flat map $f^*(\pi) : (\mathcal{X} \times_S S') \to S'$. Sometimes, called a *pull-back*.

By the property of flatness (i), $f^*(\pi)$ is really a deformation of X over $(S', 0)$. More explicitly, consider a deformation of X. We can induce other deformations of X by applying changes of coordinates to the variables x_i and substituting in new deformation parameters for the t_i. A simplest example of an induced deformation is the restriction to subspace in the parameter space S.

Example 4.1.8. Let $X = X_0 := \mathbb{V}(x^2 + y^2 - z^2)$. We perturb this with a parameter t to get $f' = x^2 + y^2 - z^2 - t$. The fibre over 0 is just X and the fibre over $t \neq 0$ is smooth. Thus, this is a deformation of X. If we substitute $t = -\frac{1}{4}s^2$ and take the change of coordinates $z \mapsto (z + \frac{1}{2}s)$, then we obtain another deformation given by $x^2 + y^2 - z^2 - sz$.

Definition 4.1.9. Given any schemes X and S, we always have a deformation of X, namely, the product family $\pi : X \times S \to S$. Any deformation isomorphic to the product family is called *trivial*.

The main point is that we can obtain a deformation from other ones, by making a base change. There is no new informations during this process, and hence we can restrict our searching for “all possible” deformations of a given variety to the looking for selected deformations including all other ones with base change.

Definition 4.1.10. A deformation $\pi : \mathcal{X} \to S$ of X is called *semi-universal* (or *mini-versal*) if every deformation $\rho : \tilde{\mathcal{X}} \to S'$ of X is isomorphic to a deformation $f^*(\pi)$, for some $f : S' \to S$.

In particular, the map f may not be unique, but its derivative df is uniquely determined by π and π'. From the definition, we can say that if we know a semi-universal deformation of X, then we will know all other deformations. So we will know all nearby fibres and hence all nearby singularities for an arbitrary deformation of X.

Example 4.1.11 (Cone over Rational Normal Curve)**.** In Example 3.4.9, we have described the cone over rational normal curve of degree p by using combinatorial object cone. On the other hand, we can define by using the Veronese map. Consider the map $\upsilon_p : \mathbb{P}^n \to \mathbb{P}^{\binom{n+p}{p}-1}$ defined by $[a_0 : \ldots : a_n] \mapsto [a_0^p : a_0^{p-1}a_1 : \ldots : a_1^p]$. This map is called the Veronese embedding of $\mathbb{P}^n$ of degree p. In particular case $n = 1$, the Veronese variety is called rational normal curve of degree p. Considered as map between affine varieties, this map arises the affine cone X over the rational normal curve.

Let $n = 1$ and $p = 4$. Then the image of the map $[a_0 : a_1] \mapsto [a_0^4 : a_0^3 a_1 : a_0^2 a_1^2 : a_0^1 a_1^3 : a_1^4]$ gives the cone over rational normal curve X of degree 4 in $\mathbb{C}^5$ whose defining equations comes from the matrix

$$\operatorname{rank}\begin{pmatrix} y_0 & y_1 & y_2 & y_3 \\ y_1 & y_2 & y_3 & y_4 \end{pmatrix} = 1 \qquad (4.1.1)$$

where we identify $a_0^{4-i}a_1^i = y_i$ for $i = 0,1,2,3,4$. The 2×2-minors generate the ideal $\mathbb{I}(X)$ of X with the binomials $f_{ij} = y_i y_{j+1} - y_{i+1} y_j$ for $0 \leq i, j \geq 3$.

How can we find the flat deformations?:

To obtain relations between of f_{ij} we look at the matrix

$$\operatorname{rank}\begin{pmatrix} y_0 & y_1 & y_2 & y_3 \\ y_0 & y_1 & y_2 & y_3 \\ y_1 & y_2 & y_3 & y_4 \end{pmatrix} \leq 2. \tag{4.1.2}$$

The maximal minors, 3×3-minors, vanish identically: on the other hand, row expansion of a minor yields a linear combination of equations.

$$\begin{aligned} y_0 f_{12} - y_1 f_{02} + y_2 f_{01} &= 0 \\ y_0 f_{13} - y_1 f_{03} + y_3 f_{01} &= 0 \\ y_0 f_{23} - y_2 f_{03} + y_3 f_{02} &= 0 \\ y_1 f_{23} - y_2 f_{13} + y_3 f_{12} &= 0 \end{aligned} \tag{4.1.3}$$

Under deformation this property should be preserved. By the relations (4.1.3) gives the following for $y_1 \neq 0$ and $y_2 \neq 0$

$$\begin{aligned} f_{02} &= \tfrac{y_0}{y_1} f_{12} + \tfrac{y_2}{y_1} f_{01} \\ f_{03} &= \tfrac{y_0}{y_1} f_{13} + \tfrac{y_3}{y_1} f_{01} \\ f_{03} &= \tfrac{y_0}{y_2} f_{23} + \tfrac{y_3}{y_2} f_{02} \\ f_{13} &= \tfrac{y_1}{y_2} f_{23} + \tfrac{y_3}{y_2} f_{12} \end{aligned} \tag{4.1.4}$$

This means that f_{01}, f_{12}, f_{23} determine the other f_{ij} away from the coordinate hyperplanes. Therefore we obtain flat deformations

$$\operatorname{rank}\begin{pmatrix} y_0 & y_1 + t_1 & y_2 + t_2 & y_3 + t_3 \\ y_1 & y_2 & y_3 & y_4 \end{pmatrix} = 1 \tag{4.1.5}$$

which is three-dimensional.

On the other hand, we can also write the six equations as 2×2-minors of a symmetric matrix, and we obtain another deformation

$$\operatorname{rank}\begin{pmatrix} y_0 & y_1 & y_2 \\ y_1 & y_2 + s & y_3 \\ y_2 & y_3 & y_4 \end{pmatrix} \leq 1. \tag{4.1.6}$$

which is one-dimensional.

Thus, the semi-universal deformation of X equals the union of these two families. Its base space is the union of hyperplane and a line in $\mathbb{C}^4$. In particular, it is not possible to find any flat family over a smooth parameter space containing both deformations $X_t \to \mathbb{C}^3$ and $X_s \to \mathbb{C}$.

To further proceed we will establish the notion of deformations of complete intersections space which is the key point for the next chapter. To do this we need some preparations.

Definition 4.1.12. Let R be a ring and M an R-module. An ordered sequence of elements $x_1, \ldots, x_n \in R$ is called an *M-regular sequence* if and only if

i) $< x_1, \ldots, x_n > M \neq M$, or equivalently $M/ < x_1, \ldots, x_n > M \neq 0$,

ii) x_i is a non-zerodivisor of $M/ < x_1, \ldots, x_{i-1} > M$.

A typical example of a regular sequence is $x_1, \ldots, x_n$ in the polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$.

Example 4.1.13. Let $R = \Bbbk[x, y, z]$, where $\Bbbk$ is a field. Now consider the sequence $y(1-x), z(1-x), x$. Then $z(1-x)y = zy - zxy = zy - zy = 0$ since $y = yx$ in $R/ < y(1-x) > R$. This means that this sequence is not a regular sequence in $R = \Bbbk[x, y, z]$.

Definition 4.1.14. An algebraic set X is called a *set-theoretically complete intersection*, if it is the intersection of r hypersurface $\{f_{i=0}\}$ in the n-space. If f_i's can be chosen so that $\mathbb{I}(X) =< f_1, \ldots, f_r >$, then we say that X is *ideal-theoretically complete intersection*.

It follows that the ideal of X in $\mathbb{C}^n$ is generated from a regular sequence, i.e., from as many equations as the codimension from X in $\mathbb{C}^n$. Furthermore, let X contains Y. Y is called *relatively complete intersection*, if the ideal of Y in X is generated from as many equations as the codimension from Y in X.

Let $X \subset \mathbb{C}^n$ be a complete intersection, and let $f_1, \ldots, f_k$ be a minimal set of generators of the ideal of X in $\mathcal{O}_{\mathbb{C}^n}$. Since $f_1, \ldots, f_k$ is a regular sequence, any relation among $f_1, \ldots, f_k$ can be generated by the trivial relations $(0, \ldots, 0, -f_j, 0, \ldots, 0, f_i, 0 \ldots, 0)$ with $-f_j$ sitting in the i-th place and f_i in j-th place. Then, for any base space S and any lifting $\tilde{f}_i \in \mathcal{O}_{\mathbb{C}^n \times S}$ of f_i for $i = 1, \ldots, k$ the diagram $X \hookrightarrow \mathcal{X} \xrightarrow{\text{pr}} S$ with $\mathcal{X} \subset \mathbb{C}^n \times S$

defined by $\tilde{f}_1 = \cdots = \tilde{f}_k = 0$, and pr projection onto the second factor, is a deformation of X over S.

Remark 4.1.15. In terms of geometry, if $S = \mathbb{C}^m$, then flatness of a map $\pi : \mathcal{X} \to \mathbb{C}^m$ is equivalent to the fact that $f_1, \ldots, f_k$ is an $O_{\mathcal{X}}$-regular sequence.

After all we can say that the deformation theory of complete intersections is relatively simple. For hypersurface singularities or more generally complete intersection singularities we can compute the semiuniversal deformation by the following theorem:

Theorem 4.1.16. *(Greuel et al., 2007) Let $X \subset \mathbb{C}^n$ be an isolated complete intersection singularity, and let $f = (f_1, \ldots, f_k)$ be a minimal set of generators for the ideal of X. Let $g_1, \ldots, g_r$ be in $O^k_{\mathbb{C}^n,0}$, $g_i = (g_i^1, \ldots, g_i^k)$, represent a basis for the finite dimensional $\mathbb{C}$-vector space*

$$T^1_X := O^n_{\mathbb{C}^n,0} \Big/ \Big\langle Df \cdot O^n_{\mathbb{C}^n,0} + < f_1, \ldots, f_k > \cdot O^k_{\mathbb{C}^n,0} \Big\rangle$$

where Df is a Jacobian matrix, and set $\tilde{f} = (\tilde{f}_1, \ldots, \tilde{f}_k)$, $\tilde{f}_i(\underline{x}, \underline{t}) = \sum_{j=1}^{r} t_j g^i_j(\underline{x})$, $\mathcal{X} = \mathbb{V}(\tilde{f}_1, \ldots, \tilde{f}_k) \subset \mathbb{C}^n \times \mathbb{C}^r$. Then $\pi : \mathcal{X} \to \mathbb{C}^r$ obtained by the inclusion $\mathbb{C}^n \subset \mathbb{C}^n \times \mathbb{C}^r$ and the projection $\mathbb{C}^n \times \mathbb{C}^r \to \mathbb{C}^r$, is a semiuniversal deformation of X.

In Theorem 4.1.16, $Df \cdot O^n_{\mathbb{C}^n,0}$ is a submodule of $O^k_{\mathbb{C}^n,0}$ generated by columns of the Jacobian matrix of f. Note that T^1_X is an O_X-module, called the *Tjurina module* of the complete intersection X. If X is a hypersurface, then T^1_X is an algebra and called the *Tjurina algebra* of X.

Corollary 4.1.17. *Let $X \subset \mathbb{C}^n$ be an isolated singularity defined by $f \in O_{\mathbb{C}^n,0}$, and $g_1, \ldots, g_r \in O_{\mathbb{C}^n,0}$, a $\mathbb{C}$-basis of Tjurina algebra*

$$T^1_X := O^k_{\mathbb{C}^n,0} \Big/ \Big\langle f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \Big\rangle. \tag{4.1.7}$$

If we set $\tilde{f}(\underline{x}, \underline{t}) = f(\underline{x}) + \sum_{j=1}^{r} t_j g_j(\underline{x})$, $\mathcal{X} := \mathbb{V}(\tilde{f}) \subset \mathbb{C}^n \times \mathbb{C}^r$, then $\pi : \mathcal{X} \to \mathbb{C}^r$ is a semiunversal deformation of X.

Example 4.1.18. Let X be the cone in $\mathbb{C}^3$ defined by the equation $f = z^2 - xy$. Then by the formula (4.1.7)

$$T^1_X = \mathbb{C}[x,y,z]\Big/ < f,-y,-x,2z > \cong \mathbb{C}.$$

Thus, the semiuniversal deformation is given by $z^2 - xy + t = 0$.

We will end this section with an important result in our context.

Theorem 4.1.19 (Grauert,1972)**.** *Any $X \subset \mathbb{C}^n$ with an isolated singularity has a semiuniversal deformation $\pi : \mathcal{X} \to S$.*

4.2 Infinitesimal Deformation

Definition 4.2.1. The space consists of one point with local ring $\mathbb{C}[\epsilon] = \mathbb{C} + \epsilon \cdot \mathbb{C}$, $\epsilon^2 = 0$, that is $\mathbb{C}[\epsilon] = \mathbb{C}[t]/ < t^2 >$ where t is an indetermined. Then $\mathbb{D} = \mathrm{Spec}\mathbb{C}[\epsilon]$ is called the *double point*. An *infinitesimal or (first-order) deformation* of X is a deformation over $\mathbb{D}$.

Our aim is to generalize the above idea. Let $X \subset \mathbb{C}^n$ and let $\mathcal{X} \to \mathbb{D}$ be a deformation of X. Suppose that $\mathbb{I}(X) =< f_1, \ldots, f_k > \subset \mathcal{O}_{\mathbb{C}^n}$, $\mathcal{O}_X = \mathcal{O}_{\mathbb{C}^n}/\mathbb{I}(X)$. Then we have

$$\mathcal{O}^l_{\mathbb{C}^n} \xrightarrow{r} \mathcal{O}^k_{\mathbb{C}^n} \xrightarrow{f} \mathcal{O}_{\mathbb{C}^n} \to \mathcal{O}_X \to 0$$

where r is an $(l \times k)$-matrix and $f = [f_1, \ldots, f_k]$ such that $fr = 0$. Lifting everything, we obtain:

$$\mathcal{O}^l_{\mathbb{C}^n \times \mathbb{D}} \xrightarrow{R} \mathcal{O}^k_{\mathbb{C}^n \times \mathbb{D}} \xrightarrow{F} \mathcal{O}_{\mathbb{C}^n \times \mathbb{D}} \to \mathcal{O}_{\mathcal{X}} \to 0$$

with $F = f + \epsilon f'$ and $R = r + \epsilon r'$. Since $\epsilon^2 = 0$, the condition $FR = 0$ implies

$$\begin{aligned} FR &= (f + \epsilon f')(r + \epsilon r') \\ &= fr + \epsilon(fr' + f'r) + \epsilon^2 f'r' \\ &= fr + \epsilon(fr' + f'r) = 0. \end{aligned}$$

Since $fr = 0$, we obtain $fr' + f'r = 0$ in $\mathcal{O}_{\mathbb{C}^n}$.

The infinitesimal deformations form an $\mathcal{O}_X$-module:

i) $(f+\epsilon f_1')(r+\epsilon r_1')=0$ and $(f+\epsilon f_2')(r+\epsilon r_2')=0$ implies

$$(f+\epsilon(f_1'+f_2'))(r+\epsilon(r_1'+r_2')) = fr+\epsilon(f(r_1'+r_2')+(f_1'+f_2')r)=0$$

ii) For $\phi \in \mathcal{O}_{\mathbb{C}^n}$, $(f+\epsilon\phi f')(r+\epsilon\phi r')=0$.

Thus, if $f' \in I^k \subset \mathcal{O}^k_{\mathbb{C}^n}$, then there exists a matrix $M \in M_k(\mathcal{O}_{\mathbb{C}^n})$ with $f+\epsilon f' = f(\mathrm{Id}+\epsilon M)$ where each column of M is a relation of $f_1,\dots,f_k$.

As $\mathrm{Id}+\epsilon M$ is invertible, the ideals generated by f and by $f+\epsilon f'$ are equal.

Proposition 4.2.2. *The $\mathcal{O}_X$-module of first-order deformations is isomorphic to the normal module $\mathcal{N}_X = \mathrm{Hom}_{\mathcal{O}_X}(I/I^2, \mathcal{O}_X)$.*

An infinitesimal deformation $f+\epsilon f'$ is trivial, if there is an automorphism $\varphi(x,\epsilon) = (x+\epsilon\delta(x),\epsilon) \in \mathbb{C}^n \times \mathbb{D}$ such that $f+\epsilon f'$ and $f\circ\varphi$ determine the same ideal.

$$\frac{d}{d\epsilon} f\circ\varphi(x,\epsilon)\Big|_{\epsilon=0} = \frac{d}{d\epsilon} f(x+\epsilon\delta(x))\Big|_{\epsilon=0} = \sum_j \frac{\partial f}{\partial x_j}\delta_j(x)$$

this gives us

$$\Theta_{\mathbb{C}^n}\Big|_X = \Theta_{\mathbb{C}^n}\otimes\mathcal{O}_X \to \mathrm{Hom}_{\mathcal{O}_{\mathbb{C}^n}}(I,\mathcal{O}_X) = \mathcal{N}_X$$

that is, the trivial deformations are the image of the above natural map. The kernel of this map is the $\mathcal{O}_X$-module $\Theta_X = \{\delta|_X \mid \delta(I)\subset I\}$.

Definition 4.2.3. The module T^1_X of isomorphism classes of first-order deformations is $T^1_X = \mathrm{Coker}\{\Theta_{\mathbb{C}^n}|_X \to \mathcal{N}_X\}$.

In particular, if X is smooth, then $T^1_X = 0$.

Example 4.2.4. Let $X = [f=0] \subset \mathbb{C}^n$ be a hypersurface. Then the ideal $\mathbb{I}(X) = < f >$ is a principal ideal generated by a function f, and $\mathcal{N}_X = \mathrm{Hom}(f/f^2, \mathcal{O}_X)$ is a free $\mathcal{O}_X$-module with generators $f \mapsto 1$. Therefore, $T^1_X = \mathcal{O}_{\mathbb{C}^{n+1}} \Big/ \Big\langle f, \dfrac{\partial f}{\partial x_0}, \dots, \dfrac{\partial f}{\partial x_n}\Big\rangle$.

CHAPTER FIVE

TORIC DEFORMATIONS

Our source of inspiration is the Christophersen's observation, in Christophersen (1991), which states that deforming of a two-dimensional cyclic quotient singularity (i.e., two-dimensional affine toric varieties), total spaces over components of the reduced base space are again toric. It follows from this fact, we will try to answer the following question: "Is it possible to describe the total spaces over the component just by combinatorial objects?" More explicitly, our aim is to find the semi-universal deformation of X with toric total space by using combinatorial data of a cone. This chapter based on Altmann (2009) and Altmann (1995a).

5.1 Infinitesimal Deformations

We will compute the vector space T^1 of infinitesimal deformations for affine toric varieties X_σ corresponding a cone $\sigma \subset N_{\mathbb{R}}$, by using the combinatorial data of a cone σ. All statements and proofs can be found in Altmann (1994).

Let us begin defining a useful object as follows: the minimal set of generators of this semigroup is defined as

$$E := \{v \in S_\sigma \mid v \neq 0 \text{ and } v = v_1 + v_2 \text{ implies } v_1 = 0 \text{ or } v_2 = 0\} \subset S_\sigma.$$

This means that E only consists of the irreducible elements of S_σ.

It is known that the ring $\mathbb{C}[S_\sigma]$ itself an M-grading. So is $\mathbb{C}[S_\sigma]$-module $T^1_{X_\sigma}$, which is important for describing infinitesimal deformations. It follows that, we will give description of the homogeneous piece $T^1_X(-\bar{r}_i)$. Assume that a cone $\sigma \subset \mathbb{R}^n$ is given by its fundamental generators, $\sigma =< u_1, \ldots, u_k >\subset N_{\mathbb{R}} \cong \mathbb{R}^n$ and its dual cone

$$\check{\sigma} = \{v \in M_{\mathbb{R}} \cong \mathbb{R}^n \mid \langle u_i, v\rangle \geq 0, \text{ for } i = 1, \ldots, k\}.$$

Now, we choose and fix an element $\bar{r}_i \in M$. Then we define the following sets to describe the module T^1:

$$K_i = \{v \in S_\sigma \mid \langle u_i, v\rangle < \langle u_i, \bar{r}_i\rangle,\ i = 1,\ldots,k\}; \tag{5.1.1}$$

$$E_i = E \cap K_i; \tag{5.1.2}$$

$$E' = \bigcup_{i=1}^{k} E_i. \tag{5.1.3}$$

Note that, one can say that these sets depend on the choice of $\bar{r}_i \in M$.

Theorem 5.1.1. *(Altmann, 1994, Theorem 2.3) Let $L(E')$ be a vector space of all linear dependences between elements of E'. Then*

$$T^1_{X_\sigma}(-\bar{r}_i) = \Big(L(E')/\sum_{i=1}^{k} L(E_i)\Big)^* \otimes_{\mathbb{R}} \mathbb{C}.$$

Corroborating the theorem, we will examine the following example which is taken from the class of two-dimensional affine toric varieties.

Example 5.1.2. Consider any two-dimensional affine toric variety comes from a cone σ generated by $u_1 = e_2 = (0,1)$ and $u_2 = 5e_1 - 3e_2 = (5,-3)$. Using the method of continued fraction, we want to obtain the minimal generating set $E = \{v_0,\ldots,v_{r+1}\} \subset S_\sigma$.

$$\frac{5}{5-3} = 3 - \frac{1}{2},$$

so we have $a_1 = 3$, $a_2 = 2$ and $r = 2$. Now, we set $v_0 = [1,0]$, $v_1 = [1,1]$, $v_3 = [3,5]$ and $v_{i+1} = a_i v_i - v_{i-1}$, for $i = 1,2$. So we have $v_2 = a_1 v_1 - v_0 = [2,3]$. Then, the minimal set of generators is $E = \{v_0, v_1, v_2, v_3\}$ with elements $v_i \in M \cong \mathbb{Z}^2$, see Figure 5.1.

Firstly, consider the case $\bar{r}_1 = v_1$. By using the formulas given in Equation (5.1.1), we find $K_1 = \{v \in S_\sigma \mid \langle u_1, v\rangle < \langle u_1, [1,1]\rangle = 1\}$ and $K_2 = \{v \in S_\sigma \mid \langle u_2, v\rangle < \langle u_2, [1,1]\rangle = 2\}$, see Figure 5.1. From Equations (5.1.2) and (5.1.3), we obtain the sets as follows:

$$E_1 = E \cap K_1 = \{v_0\}, \tag{5.1.4}$$

$$E_2 = E \cap K_2 = \{v_2, v_3\}, \tag{5.1.5}$$

$$E' = E_1 \cup E_2 = \{v_0, v_2, v_3\}. \tag{5.1.6}$$

It follows that, applying Theorem 5.1.1 gives us $T^1_X(-\bar{r}_i) = \mathbb{C}$.

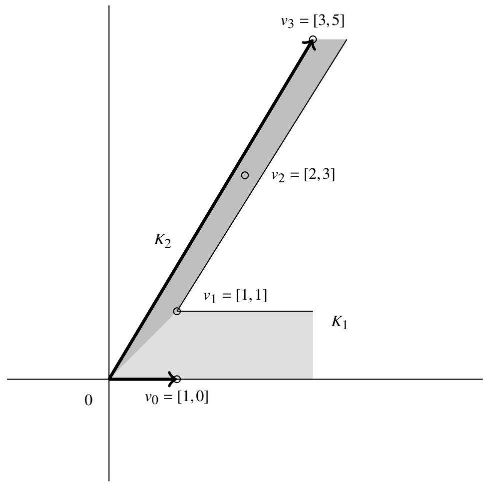


Figure 5.1 The points in E and K_1, K_2 for $\bar{r}_1 = [1,1]$

Now, consider $\bar{r}_2 = v_2$: We obtain $E_1 = \{v_0, v_1\}$ and $E_2 = \{v_3\}$, and Theorem 5.1.1 implies $T^1_X(-\bar{r}_2) = \mathbb{C}$.

Consider the cases $\bar{r}_0 = v_0$ and $\bar{r}_3 = v_3$: Since $E_1 = \emptyset \subset E_2 = \{v_1, v_2, v_3\}$ and $E_1 = \{v_0, v_1, v_2\} \supset E_2 = \emptyset$, respectively, the theorem yields $T^1_X(-\bar{r}_0) = T^1_X(-\bar{r}_3) = 0$.

5.2 Toric Deformations

By Theorem 4.1.19, there exists a semi-universal deformation at least for an isolated singularities, which induces all other ones by specialization of parameters, and more generally we have showed that if X is a complete intersection, then each perturbation of equations gives a deformation with smooth base space. It follows that, in this section our aim is to investigate deformations of X with toric total space by embedding it into higher dimensional toric variety as a relative complete intersection.

Definition 5.2.1. A deformation $f : \mathcal{X} \to S$ of X is said to be *toric* if

i) $\mathcal{X}$ is an affine toric variety,

ii) A morphism $i : X \hookrightarrow \mathcal{X}$ induces an algebraic group homomorphism $T_X \hookrightarrow T_{\mathcal{X}}$ between the embedded toric which makes i equivariant,

iii) i(closed $T_X -$orbit in X) isomorphically onto (closed $T_{\mathcal{X}} -$orbit in $\mathcal{X}$).

Note that, if the deformation $\mathcal{X} \to S$ satisfies the conditions (i) and (ii) of Definition 5.2.1, then $i : X \hookrightarrow \mathcal{X}$ corresponds to a lattice embedding $i : \bar{N} \hookrightarrow N$. On the dual level it gives us a surjection of the semigroups $i^* : S_\sigma \twoheadrightarrow S_{\bar{\sigma}}$. Then we define the m-dimensional lattice $L := \mathrm{Ker}(i^*) \subset M$ and so we have $n-m$ dimensional lattice $\bar{N} = N \cap L^\perp$ and $\bar{\sigma} = \sigma \cap L^\perp \subset \bar{N}_{\mathbb{R}}$. Let we define affine toric varieties corresponding to cones σ and $\bar{\sigma}$ as follows: $X = \mathrm{Spec}\mathbb{C}[S_{\bar{\sigma}}]$ and $\mathcal{X} = \mathrm{Spec}\mathbb{C}[S_\sigma]$, respectively.

Proposition 5.2.2. *(Altmann, 1995b, Proposition 7.1.3) Let $f : \mathcal{X} \to S$ be a toric deformation of X. Then S is smooth, and the ideal $I = \mathrm{Ker}(\mathbb{C}[S_\sigma] \to \mathbb{C}[S_{\bar{\sigma}}])$ defining $X \subset \mathcal{X}$ can be generated by m binomials of the form $x^{v_i} - x^{w_i} \in \mathbb{C}[S_\sigma]$ where $v_i, w_i \in S_\sigma$, $v_i - w_i \in L$, and $i = 1, \ldots, m$. In particular, they form a binomial regular sequence, and X is a relative complete intersection in $\mathcal{X}$.*

Remark 5.2.3. Nakayama Lemma can be stated as follows: let I be an ideal in the Jacobson radical of a commutative ring R and M is finitely generated. If $m_1, \ldots, m_n$ have images in M/IM that generate it as an R-module, then $m_1, \ldots, m_n$ also generate M as an R-module.

Proof. Firstly we will show that the base space $0 \in S$ is smooth. To prove this, consider the deformation diagram

$$\begin{array}{ccc} X & \overset{i}{\hookrightarrow} & \mathcal{X} \\ \downarrow & \otimes & \downarrow f \\ \{0\} & \hookrightarrow & S. \end{array}$$

This gives a deformation of the corresponding torus T_X:

$$\begin{array}{ccc} T_X & \hookrightarrow & T_{\mathcal{X}} \\ \downarrow & \otimes & \downarrow f \\ \{0\} & \hookrightarrow & S. \end{array}$$

We know that that the corresponding torus T_X and $T_{\mathcal{X}}$ are smooth and $T_{\mathcal{X}} \cong T_X \otimes S$. Thus the base space S is also smooth.

In terms of local ring, we obtain the following diagram:

$$\begin{array}{ccc} \mathcal{O}_{X,0} & \xrightarrow{i^*} & \mathcal{O}_{X,0} \\ {\scriptstyle \text{flat}}\uparrow & \otimes & \uparrow \\ \mathcal{O}_{S,0} & \twoheadrightarrow & \mathcal{O}_{S,0}/\mathcal{M}_{S,0} = \mathbb{C}. \end{array}$$

Thus, $I \cdot \mathcal{O}_{X,0} = \mathcal{M}_{X,0} \cdot \mathcal{O}_{X,0}$ is generated by m elements $g_1, \ldots, g_m$ and by the Nakayama lemma, we can choose these generators among the elements of the form $x^{v_i} - x^{w_i}$, with $v_i, w_i \in S_\sigma; v_i - w_i \in L$.

Let $\tilde{I} :=< g_1, \ldots, g_m >\subset \mathbb{C}[S_\sigma]$. Then, $\tilde{I} \subset I$ are ideals in $\mathbb{C}[S_\sigma]$ which satisfies the properties:

i) $\tilde{I}$ and I are homogeneous with respect to the $\bar{M}$-grading,

ii) $\tilde{I} = I$ in the local ring $\mathcal{O}_{X,0}$.

Now take any $\bar{M}$-homogeneous element g from I. By (ii), there exists an $h \in \mathbb{C}[S_\sigma]$ such that $h \cdot g \in \tilde{I}$ and $h \notin \mathcal{M}_0 := \bigoplus_{v_i \in S_\sigma} \mathbb{C} \cdot \chi^{v_i}$. And by (i), we can assume h to be $\bar{M}$-homogeneous. Hence h is a monomial of $\mathbb{C}[\sigma^\perp \cap M]$. This means that h is invertible. □

Note that, if we define $L' := span(v_i - w_i) \subset L$, then the ideal I is homogeneous in R_σ. Now, for each $l \in L$ there are $v, w \in S_\sigma$ such that $l = v - w$. Hence $x^v - x^w \in I$. Since I does not contain monomials at all, $x^v - x^w$ has to be homogeneous itself, i.e., $v - w \in L'$. It follows that the m-vectors $v_i - w_i$ that correspond to the generators of I are free generators of the sublattice $L \subset M$.

Definition 5.2.4. Let $\mathcal{X}$ be an affine toric variety. The sequence $f_1, \ldots, f_m \in \Gamma(\mathcal{X}, \mathcal{O}_\mathcal{X})$ is called a *toric regular sequence* if and only if f_i are all binomials in $\Gamma(\mathcal{X}, \mathcal{O}_\mathcal{X})$, for $i = 1, \ldots, m$ and $X := [f_1 = \cdots = f_m = 0] \subset \mathcal{X}$ is an affine toric variety of codimension m in $\mathcal{X}$. It follows that toric regular sequence form a regular sequence in $\mathcal{X}$.

In particular, all toric regular sequences can be considered as a flat map $\mathcal{X} \to \mathbb{C}^m$ by itself.

Let $X \hookrightarrow \mathcal{X}$ defined by a regular sequence $f_1,\ldots,f_m \in \Gamma(\mathcal{X},\mathcal{O}_{\mathcal{X}})$. Perturbing the equations $f_1,\ldots,f_m$ over a parameter space S in an arbitrary way gives a deformation of X in $\mathcal{X}$, and this deformation is called a *relative deformation* of X. That is relative deformation can be given by the following diagram:

$$\begin{array}{ccc} X & \hookrightarrow & \tilde{X} \\ \downarrow & \otimes & \downarrow \\ \{0\} & \hookrightarrow & S \end{array} \qquad \begin{array}{ccc} X & \hookrightarrow & X \times S \\ \downarrow & \otimes & \downarrow \\ \{0\} & \hookrightarrow & S. \end{array}$$

Note that, this deformation is comparable with the deformations of complete intersections in $\mathbb{C}^n$. It follows that the notion of relative complete intersection can be given by a toric regular sequence, since there is a close relation between these objects. As a result, toric deformations always obtain from relative deformations of X inside a higher dimensional affine toric variety $\mathcal{X}$ containing X as a relatively complete intersection. Furthermore, $X \subset \mathcal{X}$ is defined by a toric regular sequence $f_1,\ldots,f_m \in \Gamma(\mathcal{X},\mathcal{O}_{\mathcal{X}})$. Before further proceeding, we will try to better understand this relation and its results by an example.

Example 5.2.5. In Example 3.4.9 we have described the cone over rational normal curve of degree 4, X in $\mathbb{C}^5$ with the cone $\bar{\sigma} =< (1,0);(-1,4) >$ and its dual $\check{\bar{\sigma}} =< [0,1];[4,1] >$. On the other hand two affine toric varieties are isomorphic if the corresponding cones are equivalent under $\mathrm{SL}(n,\mathbb{Z})$ action. Because of that we can take the cone $\bar{\sigma} =< (-1,2);(1,2) >$ and its dual $\check{\bar{\sigma}} =< [2,1];[-2,1] >$ for X to be more easily in affine space. And alternatively in Example 4.1.11 we have defined X by the equations:

$$\operatorname{rank}\begin{pmatrix} y_0 & y_1 & y_2 & y_3 \\ y_1 & y_2 & y_3 & y_4 \end{pmatrix} = 1.$$

Now, we will identified the points of the semigroup $S_{\check{\bar{\sigma}}}$ with the variables of X. Let $y_0 = [-2,1], y_1 = [-1,1], y_2 = [0,1], y_3 = [1,1], y_4 = [2,1]$.

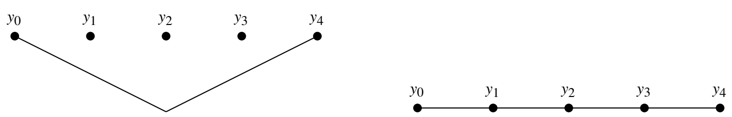


Figure 5.2 Affine Slice of $\check{\bar{\sigma}}$

Let $\mathcal{X} \subset \mathbb{C}^6$ be three-dimensional affine variety given by the equations

$$\operatorname{rank}\begin{pmatrix} y_0 & y_1 & \tilde{y}_2 := y_2 + t & y_3 \\ y_1 & y_2 & y_3 & y_4 \end{pmatrix} = 1.$$

Then we use the relations between the monomials to obtain a linear system of coordinates and the solutions of this system realise the relations between the generators of the semigroup S_σ. Thus, we have the cone $\sigma =< (-1,2,0); (0,0,1); (1,0,2); (0,1,0) >$ with dual cone $\check{\sigma} =< [0,0,1]; [-2,0,1]; [0,1,0]; [2,1,0] >$ which is given in figure: The

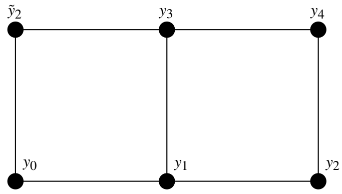


Figure 5.3 Affine Slice of $\check{\sigma}$

special fibre $t = 0$ of $\mathcal{X}$ is isomorphic to X and its codimension is one in $\mathcal{X}$. This means that X is relatively complete intersection in $\tilde{X}$ with the regular sequence $\tilde{y}_2 - y_2$ ($\operatorname{codim}_{\mathcal{X}} X = 1$). By Definition 5.2.4, $\tilde{y}_2 - y_2$ is also a toric regular sequence in $\mathcal{X}$.

More explicitly, we can obtain the closed embedding $X \hookrightarrow \mathcal{X}$ by identifying the variables y_2 and $\tilde{y}_2$. Then, we define a group homomorphism $\operatorname{pr} : N \cong \mathbb{Z}^3 \to \mathbb{Z}^2 \cong \bar{N}$ by considering the standard basis of the lattices, i.e, defined by $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$. The kernel of this map is generated by the vector $[0,-1,1] = [0,0,1] - [0,1,0]$ and it is a surjection. This makes the closed embedding equivariant.

In the dual case we have the group homomorphism $\mathbb{Z}^2 \hookrightarrow \mathbb{Z}^3$ defined by the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}$. Then $\bar{\sigma} = \sigma \cap \mathbb{R}^2$. The lattice points $y_2 = [0,0,1]$ and $\tilde{y}_2 = [0,1,0]$ with the corresponding facets become parallel in the affine slice of σ which is defined by $\langle \bullet, [0,1,1] \rangle = 1$.

On the other hand, consider another group homomorphism $\operatorname{pr}' : N \cong \mathbb{Z}^3 \to \mathbb{Z}^2 \cong \bar{N}$ defined by identfiying the variables $y_1 = \tilde{y}_2$ and $y_2 = y_3$. This means that the special fiber X' of $\mathcal{X}$ is not a relatively complete intersection in $\mathcal{X}$, actually it is a cone over rational normal curve of degree 3.

5.3 Homogeneous Toric Regular Sequences

Now our aim is to describe the conditions for $\bar{\sigma}, \sigma$ which makes X a relatively complete intersection in $\mathcal{X}$ for the pair of affine toric varieties $(X, \mathcal{X})$. To do this firstly we will define an important notion which is known as homogeneous degree in the following sense.

Definition 5.3.1. Let $g = (x^{v_1} - x^{w_1}, \dots, x^{v_m} - x^{w_m})$ be a toric regular sequence which defines $X \hookrightarrow \mathcal{X}$. Then the common images $\bar{r}_i \in \bar{M}$ of $v_i, w_i \in M$ are called the *degrees* of g. The sequence g is said to be *homogeneous of degree* $\bar{r}$, if $\bar{r} = \bar{r}_1 = \cdots = \bar{r}_m$.

Note that, a homogeneous toric regular sequence is of the form $g = (x^{v_1} - x^{v_0}, \dots, x^{v_m} - x^{v_0})$ up to $\mathbb{Z}$-linear transformations. If g is given as in this form, then

$$L = \sum_{i=0}^{m} \mathbb{Z} \cdot (v_i - v_0) = \sum_{i,j=0}^{m} \mathbb{Z} \cdot (v_i - v_j) = \operatorname{Ker}\Big(\deg : \bigoplus_{i=0}^{m} \mathbb{Z} \cdot v_i \twoheadrightarrow \mathbb{Z}\Big),$$

where $\deg(v_i) = 1$. The elements $v_0, \dots, v_m$ are linearly independent in $M_{\mathbb{R}}$.

Now, we are able to construct homogeneous toric regular sequences, which is an important tool to describe the pair $(X, \mathcal{X})$.

Definition 5.3.2. Let $(\mathbb{A}, \mathbb{L})$ be a pair of a k-dimensional real vector space $\mathbb{A}$ and a lattice $\mathbb{L} \in \mathbb{A}$. A *deformation element* of size m is a tuple $(R_0, R_1, \dots, R_m; C; p)$ satisfying the following properties:

i) $C \subset \mathbb{A}$ is a rational polyhedral cone with apex, i.e., $0 \in \mathbb{A}$, and $p \geq 1$ is a natural number.

ii) $R_0, R_1, \dots, R_m \subset \mathbb{A}$ are rational polyhedra with cone C as their common cone of unbounded directions.

Note that, taking the convex hull of the vertices of R_i gives us compact polytopes $\bar{R}_i$ such that $R_i = \bar{R}_i + C$.

Let $t \in C^\vee \subset \mathbb{A}^*$. We define the face of a polyhedron $P \subset \mathbb{A}$ to be

$$F(P,t) := \{a \in P \mid \langle a,t\rangle = \mathrm{Min}\langle P,t\rangle\}.$$

Definition 5.3.3. A deformation element $(R_0, R_1, \ldots, R_m; C; p)$ is said to be *admissible*, if

i) In the case $p = 1$. For each $t \in C^\vee \subset \mathbb{A}^*$ at least m of the $m+1$ faces $F(R_i,t)$ of R_i, $i = 0,\ldots,m$, contain lattice points.

ii) In the case $p \geq 2$. $R_1,\ldots,R_m$ are lattice polyhedra.

Remark 5.3.4. A deformation element $(R_0, R_1, \ldots, R_m; C)$ is admissible if and only if for each $t \in \mathbb{L}^* \cap C^\vee$ the values of t on at least m of the $m+1$ faces $F(R_i,t)$ of R_i are integers for $i = 0,\ldots,m$. We can use this as an alternative description for admissible deformation elements.

How can we define the correponding objects $X, \mathcal{X}$ and $\sigma, \bar{\sigma}$?

Constructing X: One can define the polyhedron Q to be the Minkowski sum $Q := R_0 + \cdots + R_m = C + (\bar{R}_0 + \cdots + \bar{R}_m) \subset \mathbb{A}$. We can embed the whole space as an affine hyperplane in a higher-dimensional space:

i) $\bar{N}_R := \mathbb{A} \times \mathbb{R}$ is a vector space containing the lattice $\bar{N} := \mathbb{L} \times \mathbb{Z}$, with the dual space $\bar{M} := \bar{N}^*_{\mathbb{R}}$ and dual lattice $\bar{M} := \bar{N}^*$;

ii) $\psi_1 : \mathbb{A} \hookrightarrow \bar{N}_{\mathbb{R}};\, u \mapsto (u, p^{-1})$.

In particular, Q turns out to be a polyhedron in $\bar{N}_{\mathbb{R}}$ via $Q := \psi_1(Q)$. So, the associated linear embedding $\psi : \mathbb{A} \hookrightarrow \bar{N}_R$ defined as $u \mapsto (u,0)$. Thus, the $(k+1)$-dimensional affine toric variety $X = \mathrm{Spec}\mathbb{C}[\bar{\sigma}^\vee \cap \bar{M}]$ that is given by the cone

$$\bar{\sigma} := \overline{\mathbb{R}_{\geq 0} \cdot \psi_1(Q)} = \psi(C) \cup \mathbb{R}_{\geq 0} \cdot \psi_1(Q) \subset \bar{N}_{\mathbb{R}}. \tag{5.3.1}$$

Constructing $\mathcal{X}$: We put the polyhedra $R_0,\ldots,R_m$ into parallel affine planes of a vector space:

i) $N_R := \mathbb{A}\times\mathbb{R}^{m+1}$ is a vector space containing the lattice $N := \mathbb{L}\times\mathbb{Z}^{m+1}$, with the dual space $M := N_{\mathbb{R}}^*$ and dual lattice $M := N^*$;

ii) $\Phi : N_{\mathbb{R}} \to \mathbb{R}^{m+1}$ via the projection onto the second factor;

iii) $\phi_i : \mathbb{A} \hookrightarrow N_{\mathbb{R}};\ u \mapsto \begin{cases} (u, p^{-1}e_0), & i = 0 \\ (u, e_i), & i = 1\ldots, m, \end{cases}$ and these maps correspond to an embedding $\phi : \mathbb{A} \hookrightarrow \bar{N}_{\mathbb{R}}$ defined by $u \mapsto (u, 0)$.

iv) $\bar{R}_i := \phi_i(R_i) \subset \Phi^{-1}(e_i)$, $i = 0,\ldots,m$.

Now we set $P := \operatorname{conv}(\cup_{i=0}^m \bar{R}_i) \subset N_{\mathbb{R}}$. Then we can define the $(k+1)+m$-dimensional affine toric variety $\mathcal{X} = \operatorname{Spec}\mathbb{C}[\sigma^\vee \cap M]$ given by the cone

$$\sigma := \overline{\mathbb{R}_{\geq 0}\cdot P} = \phi(C) \cup \mathbb{R}_{\geq 0}\cdot P \subset N_{\mathbb{R}}. \tag{5.3.2}$$

Constructing Regular Functions: Let $pr_i : \mathbb{R}^{m+1} \to \mathbb{R}$ be the projection onto the i-th factor, we can define linear maps $v_1,\ldots,v_m : N \to \mathbb{Z}$ by $v_i := \begin{cases} p\cdot(pr_0\circ\Phi), & i = 0 \\ pr_i\circ\Phi, & i = 1,\ldots,m. \end{cases}$ These maps correspond to the elements $v_i \in S_\sigma$.

On the other hand, we can consider $\bar{N}$ as a sublattice of N by the inclusion map $\bar{N} \hookrightarrow N$; $(u;1) \mapsto (u;1,p,\ldots,p)$. This implies that $\bar{N} = N \cap \bigcap_{i=1}^m (v_i - v_0)^\perp$ and $\bar{\sigma} = \sigma \cap \bar{N}_{\mathbb{R}}$.

Therefore, we obtain a map $X \to \mathcal{X}$ which sends X into the special fiber of the morphism $\mathcal{X} \to \mathbb{C}^m$ defined by the regular functions $x^{v_1} - x^{v_0},\ldots,x^{v_n} - x^{v_0} \in \mathbb{C}[S_\sigma]$.

After all these constructions we have the following theorem:

Theorem 5.3.5. *(Altmann, 1995a, Theorem 3.5) Let $(R_0,\ldots,R_m;C;p)$ be an admissible deformation element. The above construction gives a pair $(X,\mathcal{X})$ of affine toric varieties such that $X \subset \mathcal{X}$ is given by a homogeneous toric regular sequence $x^{v_1} - x^{v_0},\ldots,x^{v_m} - x^{v_0}$. Furthermore, all those pairs $(X,\mathcal{X})$ arise in this way.*

Now, we change our point of view and let the affine toric variety $X = \mathrm{Spec}\mathbb{C}[S_{\bar{\sigma}}]$ be given. We want to construct toric deformation of X. Note that, the homogeneous toric regular sequences of degree $\bar{r} \in \bar{M}$ can be found by looking for admissible deformation elements such that $R_0 + \cdots + R_m = \bar{\sigma} \cap [\langle\ ,\bar{r}\rangle = 1]$. To do this fix some degree $\bar{r}$ corresponds to the choice of an affine cross cut Q of the cone $\bar{\sigma} \subset \bar{N}_{\mathbb{R}}$. Then, Theorem 5.3.5 shows that toric deformations of X comes from the decompositions of Q into a Minkowski sum. Our method is given as follows:

i) Let us define the vector space $\mathbb{A}_0 := \{u \in \bar{N}_{\mathbb{R}} \mid \langle u, \bar{r}\rangle = 0\}$, with the lattice $\mathbb{L}_0 := \mathbb{A}_0 \cap \bar{N}$.

ii) Let p be the greatest common divisor of the coordinates of $\bar{r}$. In other words, $p^{-1}\bar{r}$ is a primitive element of $\bar{M}$.

iii) Define the affine space $\mathbb{A} := \{u \in \bar{N}_{\mathbb{R}} \mid \langle u, \bar{r}\rangle = 1\}$. We fix some point $0 \in \mathbb{A} \cap p^{-1}\bar{N}$, and we obtain the lattice $\mathbb{L} := 0 + \mathbb{L}_0$. Furthermore, we use this point to identity $(\mathbb{A}, \mathbb{L})$ with the pair $(\mathbb{A}_0, \mathbb{L}_0)$ providing a linear structure.

iv) Let set $C := \bar{\sigma} \cap \mathbb{A}_0$ to be a cone and $Q := \bar{\sigma} \cap \mathbb{A}$ to be a polyhedron. Then, by Theorem 5.3.5, homogeneous regular sequences of degree $\bar{r}$ correspond to admissible decompositions of Q into a Minkowski sum $Q = R_0 + \cdots + R_m$.

Finally, we assume that the projective toric variety X corresponds to a lattice polytope $P \subset M_{\mathbb{R}}$. It is known that from Chapter 3, the cone $C(P)$ of a polytope P defines the dual cone $\check{\sigma}$ of a cone σ. The corresponding affine toric variety is called the cone over X. In this case, Minkowski sums occur in connection with affine slices of the cone $\bar{\sigma}$ by itself, not of the dual cone.

5.4 The Kodaira-Spencer Map

Consider the elements $v_0, \ldots, v_m \in S_{\sigma}$. We have defined the map $\mathrm{Spec}\mathbb{C}[S_{\sigma}] \to \mathbb{C}^m$ by the regular functions $x^{v_1} - x^{v_0}, \ldots, x^{v_m} - x^{v_0} \in \mathbb{C}[S_{\sigma}]$. It is known that this gives a deformation of the special fiber $X = \mathrm{Spec}\mathbb{C}[S_{\bar{\sigma}}]$. Now we will define the Kodaira-Spencer map $\varrho : \mathbb{C}^m \to T^1_X$ corresponding to this deformation.

Let $(R_0, R_1, \ldots, R_m; C; p)$ be an admissible deformation elements. Recall that we have defined the cone $\bar{\sigma}$ as the cone over the polyhedron Q embedded in to the hyperplane in $\mathbb{A} \times \mathbb{R}$. Thus, the elements of E can be written in the form $v_a = [c_a, \eta_a]$ with $c_a \in \mathbb{L}^* \cap \check{C}$ and $\langle q, -pc_a \rangle \leq \eta_a$ for $q \in Q$. Now, let take an any lifting $\{\bar{v}_0, \ldots, \bar{v}_m\}$ of E to $S_\sigma \subset M$. This means that $\bar{v}_a = [c_a; \eta_{0a}, \ldots, \eta_{ma}]$ where $\eta_{0a} + p\eta_{1a} + \cdots + p\eta_{ma} = \eta_a$ and

$$\langle q, -pc_a \rangle \leq \begin{cases} \eta_{0a} & \text{for } q \in R_0 \\ \eta_{ia} & \text{for } q \in R_i \, (i \geq 1). \end{cases}$$

Note that, since the given deformation elements is admissible, there exist integer denoted by η_{ia}.

Now we are able to give the main result for this section.

Theorem 5.4.1. *i) The Kodaira-Spencer map sends the whole space $\mathbb{C}^m$ into the homogeneous summand $T^1_X(-\bar{r}_i)$.*

ii) The Kodaira-Spencer map given as

$$\varrho : \mathbb{C}^m \to \Big(L(E') / \sum_{i=1}^{k} L(E_i) \Big)^* \otimes_{\mathbb{R}} \mathbb{C}$$

which induced by the bilinear map $\mathbb{R}^m \times L(E) \to \mathbb{R}$ defined by the matrix

$$\begin{pmatrix} \eta_{ia} & \cdots & \eta_{ia} \\ \vdots & \vdots & \vdots \\ \eta_{ia} & \cdots & \eta_{ia} \end{pmatrix}. \tag{5.4.1}$$

5.5 Examples

We will end this chapter by investigating some examples. Now try to figure out on some examples of what we have done up here.

Example 5.5.1 (Cone Over Rational Normal Curve of Degree 4)**.** Let X be given by the cones $\bar{\sigma} =< (-1,2);(1,2) >\subset \bar{N}_{\mathbb{R}} \cong \mathbb{R}^2$ with dual $\bar{\sigma}^{\vee} =< [2,1];[-2,1] >\subset \bar{M}_R \cong \mathbb{R}^2$. On the other hand we can define $X \subset \mathbb{C}^5$ by the equations

$$\operatorname{rank}\begin{pmatrix} y_0 & y_1 & y_2 & y_3 \\ y_1 & y_2 & y_3 & y_4 \end{pmatrix} = 1 \tag{5.5.1}$$

Then the homogeneous coordinates are $[0,1],[1,1],[-1,1]$. Since our example is taken from a class of two-dimensional cyclic quotient singularities, (i.e., two-dimensional affine toric varieties), we take $p = 1$. We will investigate these coordinates case by case:

$\bar{r} = [0,1]$: Define $\mathbb{A}_0 := \{u \in \bar{N}_{\mathbb{R}} \mid \langle u,\bar{r}\rangle = 0\}$ with the lattice $\mathbb{L}_0$. And define $\mathbb{A} := \{a \in \bar{N}_{\mathbb{R}} \mid \langle u,\bar{r}\rangle = 1\}$ with the lattice $\mathbb{L} := \{u \in \mathbb{Z}^2 \mid \langle u,\bar{r}\rangle = 1\} = \mathbb{A} \cap \bar{N}$. Then we have $Q := \bar{\sigma} \cap \mathbb{A}$ and $C := \bar{\sigma} \cap \mathbb{A}_0$.

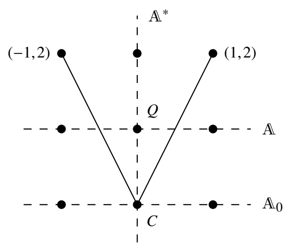


Figure 5.4 The cone C and the polyhedron Q with $\bar{r} = [0,1]$

The pair $(\mathbb{A},\mathbb{L})$ can be identified with $(\mathbb{R},\mathbb{Z})$ by $(u,1) \mapsto u$. Then the line segment corresponds to the closed interval $[\frac{-1}{2}, \frac{-1}{2}]\subset \mathbb{R}$. Now, the one-dimensional polyhedron $Q(C = 0)$ can be split into $Q = [\frac{-1}{2}, \frac{1}{2}] = [\frac{-1}{2}, 0] + [0, \frac{1}{2}] = R_0 + R_1$. We will check that $(R_0,R_1;C)$ is a deformation element of size 1. Since $C = 0 \in \mathbb{A}$, C is a rational polyhedral cone with apex, and since we take $R_0 = [\frac{-1}{2}, 0]$ and $R_1 = [0, \frac{1}{2}]$ with cone $C = 0$, $(R_0,\ldots,R_1;C)$ is a deformation element. To check admissibility take $t \in C^{\vee} \subset \mathbb{A}^*$:

ii) $t > 0$: $F(R_0,t) := \{a \in R_0 \,|\, \langle a,t\rangle = Min\langle R_0,t\rangle\} = \frac{-1}{2}$
$F(R_1,0) := \{a \in R_1 \,|\, \langle a,t\rangle = Min\langle R_1,t\rangle\} = 0$

iii) $t < 0$: $F(R_0, 0) := \{a \in R_0 \mid \langle a, 0\rangle = Min\langle R_0, t\rangle\} = 0$

$F(R_1, 0) := \{a \in R_1 \mid \langle a, 0\rangle = Min\langle R_1, t\rangle\} = \frac{-1}{2}$

they contain lattice points.

Then, we set $N_R := \mathbb{A} \times \mathbb{R}^2 \cong \mathbb{R}^3$ and $N := \mathbb{L} \times \mathbb{Z}^2 \cong \mathbb{Z}^3$. Let $\Phi : \mathbb{R}^3 \to \mathbb{R}^2$ be a projection, $\phi_i : \mathbb{A} \hookrightarrow \mathbb{R}^3$ defined by $u \mapsto (u; e_i)$ for $i = 0, 1$ and let $\bar{R}_i := \phi(R_i) \subset \Phi^{-1}(e_i)$ for $i = 0, 1$.

More explicitly,

$$\phi_0(\frac{-1}{2}) = (\frac{-1}{2}; e_0) = (\frac{-1}{2}, 1, 0) \quad (5.5.2)$$

$$\phi_0(0) = (0; e_0) = (0, 1, 0) \quad (5.5.3)$$

$$\phi_1(0) = (0; e_1) = (0, 0, 1) \quad (5.5.4)$$

$$\phi_1(\frac{1}{2}) = (\frac{1}{2}; e_1) = (\frac{1}{2}, 0, 1) \quad (5.5.5)$$

Then we have $P := \mathrm{conv}(\bar{R}_0 \cup \bar{R}_1)$ and $\sigma :=< (\frac{-1}{2}, 1, 0); (0, 1, 0); (0, 0, 1); (\frac{1}{2}, 0, 1)) >$. Therefore we obtain the 3-dimensional affine toric variety $\mathcal{X}_1 = \mathrm{Spec}\mathbb{C}[S_\sigma]$. The dual cone of σ is $\sigma^\vee :=< [0, 0, 1]; [-2, 0, 1]; [0, 1, 0]; [2, 1, 0] >$. By the relations between the generators of $S_\sigma =< [0, 0, 1]; [-1, 0, 1]; [-2, 0, 1]; [0, 1, 0]; [1, 1, 0]; [2, 1, 0] >$ we can define $\mathcal{X}_1$ by the equations

$$\mathrm{rank}\begin{pmatrix} y_0 & y_1 & \tilde{y}_2 := y_2 + t & y_3 \\ y_1 & y_2 & y_3 & y_4 \end{pmatrix} = 1 \quad (5.5.6)$$

It follows that the equation $t = 0$ gives us Y as a closed subvariety in $\mathcal{X}_1$ that is given by the equation $y_2 = \tilde{y}_2$. Since $\dim(X) = 2$ and $\dim(\mathcal{X}_1) = 3$, X is a relatively complete intersection in $\mathcal{X}_1$. By Definition 5.2.4, $y_2 - \tilde{y}_2$ is a toric regular sequence of length one in $\mathcal{X}_1$.

$\bar{r} = [-1, 1]$: Now we set $\mathbb{A}_0 := \{u \in \bar{N}_\mathbb{R} \mid \langle u, \bar{r}\rangle = 0\}$ with the lattice $\mathbb{L}_0$ and set $\mathbb{A} := \{u \in \bar{N}_\mathbb{R} \mid \langle u, \bar{r}\rangle = 1\}$ with the lattice $\mathbb{L} := \{u \in \mathbb{Z}^2 \mid \langle u, \bar{r}\rangle = 1\} = \mathbb{A} \cap \bar{N}$. Then we have $Q := \bar{\sigma} \cap \mathbb{A} = [\frac{-1}{3}, 1] = R_0 + R_1 = [\frac{-1}{3}, 0] + [0, 1]$ and $C := \bar{\sigma} \cap \mathbb{A}_0 = \{0\}$.

Again by similar construction we obtain a cone $\sigma =< (-1, 0, 3); (0, 0, 1); (0, 1, 0); (1, 1, 0) >$ which defines 3-dimensional affine toric variety $\mathcal{X}_2 = \mathrm{Spec}\mathbb{C}[S_\sigma]$. By the same reason

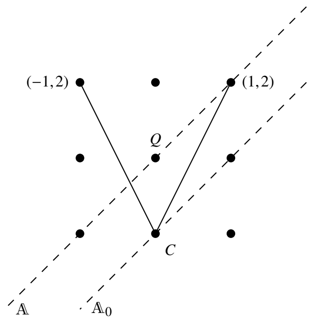


Figure 5.5 The cone C and the polyhedron Q with $\bar{r} = [-1,1]$

we can define $\mathcal{X}_2$ by the equations

$$\operatorname{rank}\begin{pmatrix} y_0 & \tilde{y}_1 := y_1 + t & y_2 & y_3 \\ y_1 & y_2 & y_3 & y_4 \end{pmatrix} = 1 \tag{5.5.7}$$

$\bar{r} = [1,1]$: Now we set $\mathbb{A}_0 := \{u \in \bar{N}_{\mathbb{R}} \mid \langle u, \bar{r}\rangle = 0\}$ with the lattice $\mathbb{L}_0$ and set $\mathbb{A} := \{u \in \bar{N}_{\mathbb{R}} \mid \langle u, \bar{r}\rangle = 1\}$ with the lattice $\mathbb{L} := \{u \in \mathbb{Z}^2 \mid \langle u, \bar{r}\rangle = 1\} = \mathbb{A} \cap \bar{N}$. Then we have $Q := \bar{\sigma} \cap \mathbb{A} = [-1, \frac{1}{3}] = R_0 + R_1 = [-1, 0] + [0, \frac{1}{3}]$ and $C := \bar{\sigma} \cap \mathbb{A}_0 = \{0\}$.

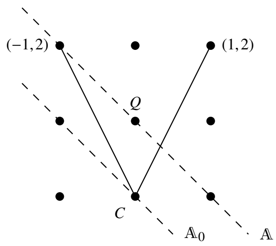


Figure 5.6 The cone C and the polyhedron Q with $\bar{r} = [1,1]$

Again by similar construction we obtain a cone $\sigma = \langle (-1,0,1); (0,0,1); (0,1,0); (1,3,0) \rangle$ which defines 3-dimensional affine toric variety $\mathcal{X}_3 = \operatorname{Spec}\mathbb{C}[S_\sigma]$. By the same reason we can define $\mathcal{X}_3$ by the equations

$$\operatorname{rank}\begin{pmatrix} y_0 & y_1 & y_2 & \tilde{y}_3 := y_3 + t \\ y_1 & y_2 & y_3 & y_4 \end{pmatrix} = 1 \tag{5.5.8}$$

Additionally the interval $Q = [\frac{-1}{2}, \frac{1}{2}]$ has another decomposition $Q = \{\frac{-1}{2}\} \cup [0,1]$ which is an admissible deformation element. Then we have 3-dimensional affine toric variety $\mathcal{X}'$ with the cone $\sigma' =< (-1,2,0);(0,0,1);(1,0,1) >$. By the relations between the lattice points of $(\sigma')^{\vee}$, we can define $\mathcal{X}' \subset \mathbb{C}^6$ by the equations

$$\operatorname{rank}\begin{pmatrix} y_0 & y_1 & y_2 \\ y_1 & \tilde{y}_2 := y_2 + s & y_3 \\ y_2 & y_3 & y_4 \end{pmatrix} = 1 \tag{5.5.9}$$

Therefore, we obtain the semiuniversal deformation of X with the total space $\mathcal{X}$ which is defined by the equations

$$\operatorname{rank}\begin{pmatrix} y_0 & \tilde{y}_1 := y_1 + t_1 & \tilde{y}_2 := y_2 + t_2 & \tilde{y}_3 := y_3 + t_3 \\ y_1 & y_2 & y_3 & y_4 \end{pmatrix} = 1$$

and

$$\operatorname{rank}\begin{pmatrix} y_0 & y_1 & y_2 \\ y_1 & \tilde{y}_2 := y_2 + s & y_3 \\ y_2 & y_3 & y_4 \end{pmatrix} = 1$$

and with the reducible base space S. In particular, S consists of two smooth components with dimensions three and one respectively.

Example 5.5.2. Consider $\psi_i : X \to \mathbb{P}^1$ for $i = 0,1,2$ defined as

$$\begin{aligned} \psi_0([x_0 : x_1 : x_2; y_0 : y_1 : y_2]) &= [x_1 : x_2] \text{ or } [y_2 : y_1] \\ \psi_0([x_0 : x_1 : x_2; y_0 : y_1 : y_2]) &= [x_2 : x_0] \text{ or } [y_0 : y_2] \\ \psi_0([x_0 : x_1 : x_2; y_0 : y_1 : y_2]) &= [x_0 : x_1] \text{ or } [y_1 : y_0]. \end{aligned}$$

Each ψ_i is well-defined and a morphism of verieties. These morphisms define a morphism $\psi : X \to \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. The map ψ sends X isomorphically onto the hypersurface of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ with the defining equation $X_0Y_0Z_0 = X_1Y_1Z_1$, where $X_0, Y_0, Z_0, X_1, Y_1, Z_1$ are the coordinates of the product variety $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. In this case X is called *Del Pezzo Surface of degree* 6, and the corresponding fan given in Figure 5.7.

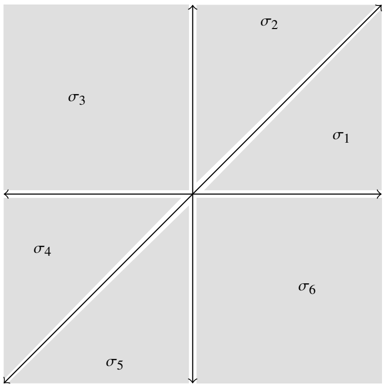


Figure 5.7 Fan of Del Pezzo surface

Let $Q := \text{conv}((0,0),(1,0),(2,1),(2,2),(1,2),(0,1)) \subset \mathbb{R}^2$ be the hexagon. Then, we obtain the corresponding cone

$$\bar{\sigma} = \text{cone}(Q) =< (0,0,1),(1,0,1),(2,1,1),(2,2,1),(1,2,1),(0,1,1) > \subset N_{\mathbb{R}} \cong \mathbb{R}^3,$$

by putting H into the affine hyperplane $z = 1 \subset \mathbb{R}^3$. Thus, $X \subset \mathbb{C}^7$ is a three-dimensional affine toric variety. There is only one homogenous coordinate which is $\bar{r} = [0,0;1]$. This implies that the splitting of the base space S into two irreducible components corresponds to the existence of two different Minkowski decomposition of Q.

Firstly, we consider the decomposition of Q in Figure 5.8, that is

$$Q = \text{conv}((0,0),(1,0),(1,1)) + \text{conv}((0,0),(0,1),(1,1)).$$

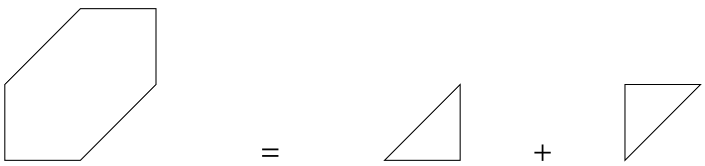


Figure 5.8 $Q = \text{conv}((0,0),(1,0),(1,1)) + \text{conv}((0,0),(0,1),(1,1))$

We put into two parallel planes contained in $\mathbb{R}^3$. This gives an octahedron which corresponds to a 4- dimensional cone σ such that

$$\sigma =< (0,0;1,0),(1,0;1,0),(1,1;0,1),(0,0;0,1),(0,1;0,1),(1,1,0,1) > .$$

This gives us one-parameter deformation $X_t \to \mathbb{C}$.

Now, we consider the decomposition of Q in Figure 5.9, that is

$$Q = \text{conv}((0,0),(1,0)) + \text{conv}((0,0),(0,1)) + \text{conv}((0,0),(1,1)).$$

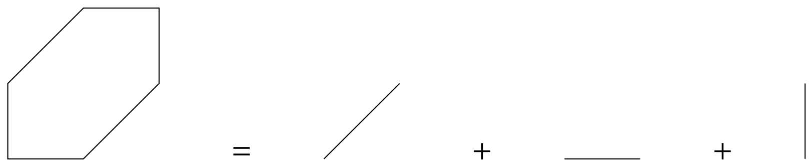


Figure 5.9 $Q = \text{conv}((0,0),(1,0)) + \text{conv}((0,0),(0,1)) + \text{conv}((0,0),(1,1))$

We put into three parallel 2-planes in general position in $\mathbb{R}^4$. This gives a 4-dimensional polytope which corresponds to a 5- dimensional cone σ such that

$$\sigma = < (0,0;1,0,0),(1,0;1,0,0),(0,0;0,1,0),(0,1;0,1,0),(0,0;0,0,1),(1,1,0,0,1) > .$$

This gives us a two-dimensional deformation $X_s \to \mathbb{C}^2$.

Therefore, we have computed the semi-universal deformation of the cone over the Del Pezzo surface of degree 6 and its base space S consists of two smooth components with dimension 1 and 2, respectively.

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