DOKUZ EYLÜL UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

INVENTORY POLICIES FOR PERISHABLE ITEMS

by
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INVENTORY POLICIES FOR PERISHABLE ITEMS

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M.Sc. THESIS EXAMINATION RESULT FORM

We have read the thesis entitled **"INVENTORY POLICIES FOR PERISHABLE ITEMS"** completed by **BAHAR YALÇIN** under supervision of **ASSIST. PROF. DR. UMAY UZUNOĞLU KOÇER** and we certify that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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INVENTORY POLICIES FOR PERISHABLE ITEMS

ABSTRACT

Inventory simply means the goods and services that businesses hold in stock. However, there are several different types of inventory. Inventory of perishable product is one of them. Inventory control is the supervision of supply, storage and accessibility of items, in an optimum way for an organization. Inventory control is an important problem for many companies because, it is impossible to have an indefinitely supply on hand and on the other hand holding stock has a big cost.

Perishable products have specific life time and cannot be used after its life is ended. For this reason, inventory control of perishable items requires different methods rather than that of durable products. The usage of perishable products is huge such as food industry and healthcare services (fresh foods, chemicals, blood and blood products etc.). Many inventory systems assume that life time is indefinitely. Hence, for perishable products inventory management unlike the traditional methods, some different methods have been used.

The purpose of this study is to examine the inventory system for perishable products with two different policies; such as continuous review and periodic review. For this purpose, the continuous review approach is examined through a numerical study. Moreover, the system is analyzed by periodic review approach with similar assumptions. Mathematical formulas of cost functions and numerical results are obtained for different positive lead times with using MATLAB. As a result, the optimal order quantity determined which has minimum cost for each policy.

Keywords: Perishable products, continuous review, periodic review, lead time, Poisson demand.

BOZULABİLİR ÜRÜNLER İÇİN ENVANTER POLİTİKALARI

ÖZ

Envanter, en basit tanımıyla şirketlerin stok altındaki tuttukları ürünler ve hizmetler olarak tanımlanabilir. Ancak, birkaç farklı türde envanter vardır. Bozulabilir ürünler envanteri bunlardan biridir. Envanter kontrolü, ürün depolama ve erişilebilirliğinin işletmelere optimum yararı sağlayacak biçimde denetlenmesidir. Envanter kontrolü çoğu işletme için önemli bir problemdir çünkü elde süresiz kaynak tutmak imkânsızdır ve diğer yandan envanter bulundurmak büyük bir maliyettir.

Bozulabilir ürünler belirli bir ömrü olan ve ömrü bittikten sonra kullanılamayan ürünlerdir. Bu nedenle bozulabilir ürünler için envanter kontrolü, dayanıklı ürünler için kullanılan yöntemlerden daha farklı yöntemler gerektirir. Bozulabilir ürünlerin gıda sektöründen sağlık alanına kadar geniş bir kullanım alanı bulunmaktadır (taze yiyecekler, kimyasallar, kan ve kan ürünleri vs.). Bozulabilir envanter sistemleri ile ilgili araştırmalar çok fazla sayıdadır. Çoğu envanter modeli ürünleri süresiz olarak elde tutabildiğini varsayar. Bu nedenle bozulabilir ürünlerin envanter yönetiminde geleneksel modellerden farklı yaklaşımlar kullanılmaktadır.

Bu çalışmanın amacı bozulabilir ürünler için envanter sistemini, sürekli gözden geçirme ve periyodik gözden geçirme gibi iki farklı politika altında incelemektir. Bu amaç doğrultusunda sürekli gözden geçirme yaklaşımı nümerik bir çalışma ile incelenmiştir. Ayrıca periyodik gözden geçirme politikası yaklaşımı kullanılarak ve benzer varsayımlar altında sistem modellenmiştir. Modelden yararlanılarak maliyet fonksiyonlarının matematiksel formülleri elde edilmiş ve MATLAB yardımı ile farklı tedarik süreleri için nümerik sonuçlar elde edilmiştir. Sonuç olarak, her bir yaklaşım için en düşük maliyete sahip uygun sipariş miktarı elde edilmiştir.

Anahtar Kelimeler: Bozulabilir ürünler, sürekli gözden geçirme, periyodik gözden geçirme, tedarik süresi, Poisson talep.

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CHAPTER ONE
INTRODUCTION

Inventory control means keeping the current costs associated with having inventory as low as possible without any problems. It is impossible to have an unlimited supply on hand.

Inventory control involves the optimal supply, care and disposition of material required in a retailing process. They are many different reasons why inventory management is so important. Most business wants to know how many of products sold, how many of them stolen and exactly how much needs to be ordered. It is able to know exactly how many of items you have in stock with control policy.

When studying with perishable products, it is important for businesses to maintain the correct amount of inventory if businesses keep too few items; they are losing profit because customers will not be able to purchase the items they want. However, if they storage too many items, they will have to discard them after the items perish and lose money. For this reason, inventory control is really important about perishable products.

In this study, we examined the perishable items with two different approaches. Firstly, we studied on a continuous (s,S) policy model with respect to Poisson demand, zero lead time, random life time and stock out policy. Secondly, we examined the periodic review policy with a Poisson demand, positive lead time, fixed life time and lost sales policy. We calculated the costs for both policies and will find out which policy is optimal.

The remainder of this study is organized as follows. In Chapter two, we defined the inventory, inventory costs, and deterministic and stochastic inventory models. Later we presented an extensive literature review. In Chapter three, we examined the probability of inventory level in the steady state and cost functions for the continuous model. Then we gave the numerical study and sensitivity analysis for the continuous review policy. For the periodic review policy we defined the cost functions and order quantity for a positive lead time. Then, we examined the experimental results for different level of lead times, different mean demands and different initial order quantities. Also, tables and figures are presented in Chapter three which summarizes the results. Consequently, optimal order quantity and optimal order policies that minimize the expected total cost are obtained for different values. Chapter four includes the conclusions, commentaries and led the way for further researches.

CHAPTER TWO
LITERATURE REVIEW

Inventory is a quantity or store of goods that is stock of items kept to meet future demand. Inventory can refer to both the total amount of goods and the act of counting them. Many companies take an inventory of their supplies on a regular basis in order to avoid running out of popular items. Others take an inventory to insure the number of items ordered matches the actual number of items counted. Inventory Management system provides information to efficiently manage the flow of materials, effectively utilize people and equipment, coordinate internal activities and communicate with customers. The main purpose of inventory management is to determine 'how many units to order' and 'when to order'.

Usually companies need to keep inventory. Why they hold inventories? There are many answers for this question. Companies usually want to balance against uncertainty, ensure a high level of customer service, prevent speculations on future events, meet seasonal or cycling demand and take advantage of price discounts. Also, inventory control provides independence between stages and avoids work stoppages and independence from vendors. Some of the basic notations used in the control of inventory.

- D: Demand
- L: Lead time
- T: Review time

Inventory models usually use cost minimization. All inventories bring with it a number of costs. Some of costs involved in inventory models:

1) Ordering and Setup Cost (C_o): Set-up costs are the costs incurred from getting a machine ready to produce the desired good. In a manufacturing setting this would require the use of a skilled technician who disassembles the tooling that is currently in use on the machine. If the firm purchases the part or raw material, then an order cost, rather than a set-up cost, is incurred. Also, some firms include the cost of shipping the purchased goods in the order cost.

2) Purchasing Cost (*p*): Purchasing cost is simply the cost of the purchased item itself. If the firm purchases a part that goes into its finished product, the firm can determine its annual purchasing cost by multiplying the cost of one purchased unit (p) by the number of finished products demanded in a year (D). The purchasing cost includes the variable labor cost, variable overhead cost and raw material cost associated with purchasing or production a single unit. If goods are ordered from an external source, the unit purchase cost must include shipping cost.

3) Holding or Carrying Cost (C_h): The cost of carrying one unit of inventory for the unit time-period. Holding costs are the costs that result from maintaining the inventory. Inventory in excess of current demand frequently means that its holder must provide a place for its storage when not in use. Storage facilities also require heating, cooling, lighting, and water. The holding cost usually includes storages cost, insurance cost, taxes on inventory, and a cost due to the possibility of spoilage, theft or obsolesce. All of these things add cost to holding or carrying inventory.

4) Stockout or Shortage Cost (C_u): When a customer demands a product and the demand is not met on time, a stockout, or shortage, is said to occur. If customer will accept delivery at a later date, we say that demands may be **back-ordered**. If no customer will accept late delivery, we are in the **lost sales** case.

2.1 Deterministic Inventory Models

This model based on the assumptions that all parameters and variables are known or can be computed with certainty. Demand is assumed to occur at a constant rate and lead time for each order is constant and independent of the demand. Deterministic inventory models can be classified into four groups.

2.1.1 The Basic Economic Order Quantity Model (EOQ)

The EOQ model is one of the oldest and most well known inventory control techniques. The EOQ helps to determining how much to order. This model based on a number of assumptions;

1. Demand rate is known and constant
2. Shortages are not permitted
3. Lead time known and constant
4. The cost include
    a) Order and setup cost C_o per order placed
    b) Holding cost C_h holding inventory per unit time

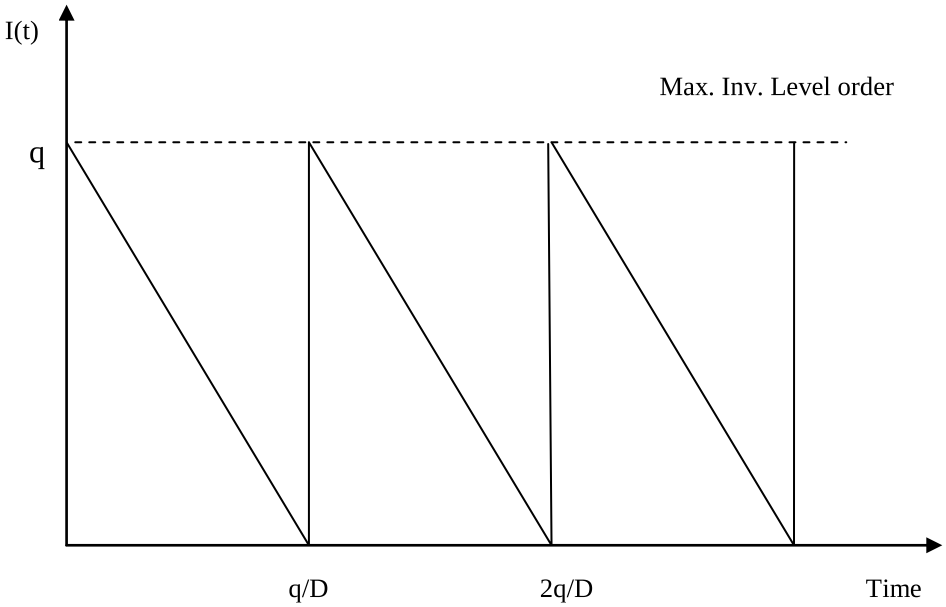

Figure 2.1 Behavior of I (t) in basic EOQ model

In the EOQ model only ordering and holding cost need to be minimized. All other costs are assumed constant. The EOQ model do not depend the purchasing cost. To find the optimal order quantity, first, we determine the annual total cost. Let TC (q) be the total annual cost.

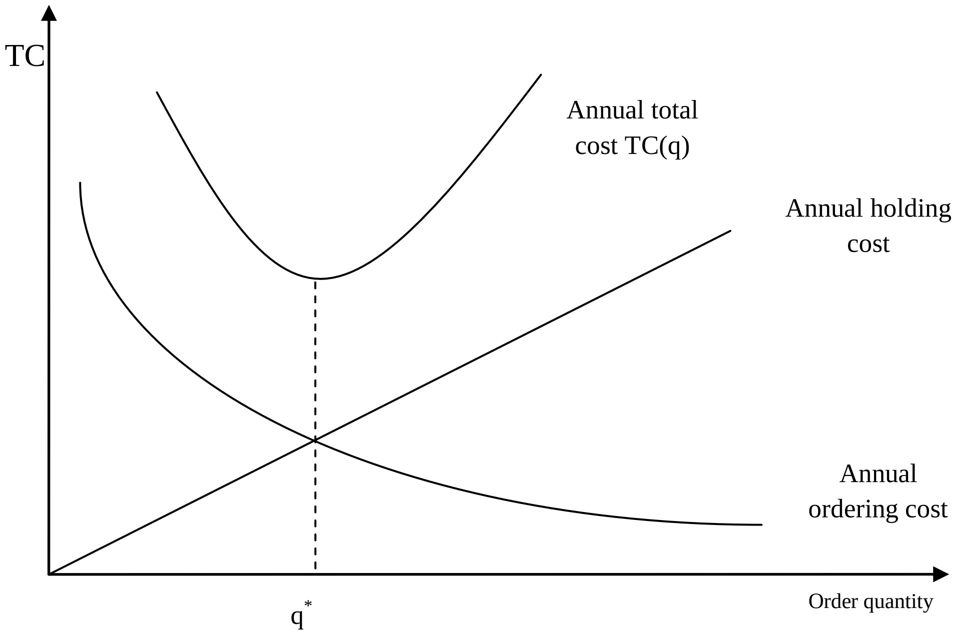

Figure 2.2 Trade-off between holding cost and ordering cost

TC(q) = annual cost of placing orders + annual purchasing cost + annual holding cost

$$TC(q)=\frac{C_oD}{q}+pD+\frac{C_hq}{2} \tag{2.1}$$

To find the value of q that minimizes TC(q), calculate TC(q)=0. And find the EOQ or economic order quantity (q^*)

$$q^*=\left(\frac{2C_oD}{C_u}\right)^{\frac{1}{2}} \tag{2.2}$$

2.1.2 Quantity Discount Model

A quantity discount model is a reduced unit price based on purchasing a large quantity. The general quantity discount model described as follows:

If $q < x_1$, each item costs p_1

If $x_1 \le q \le x_2$, each item costs p_2

If $x_{n-1} \le q \le x_n$, each item costs p_n

x_1 , x_2, … , x_n are price break points. To find the order quantity minimizing total annual costs, we use these steps:

1. Calculate q* for each discount price

2. If q* is too small to qualify for that price, adjust q* upward
3. Calculate total cost for each q*
4. Select the q* with the lowest total cost

2.1.3 The Continuous Rate EOQ Model

The EOQ model assumes inventory is obtained from an outside supplier and arrives instantaneously. But the Continuous rate EOQ assumes inventory is being produced at a rate of 'r' units per time period.

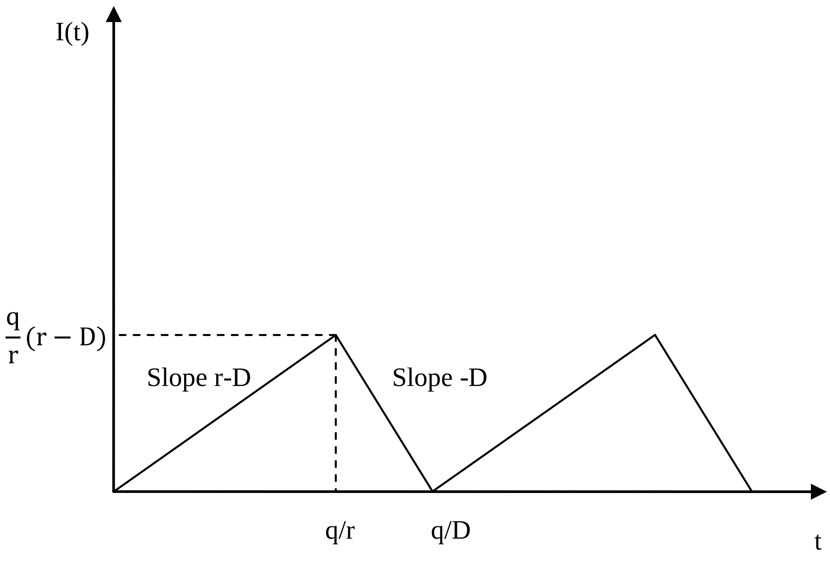

Figure 2.3 Variation of the inventory for continuous rate

Total expected cost function is

$$TC(q) = \frac{C_o D}{q} + pD + \frac{C_h q}{2}$$

Using the EOQ formula and (ordering cost + holding cost) equation, we find,

$$\text{optimal run size} = \left(\frac{2C_o Dr}{C_h(r-D)}\right)^{\frac{1}{2}} = EOQ\left(\frac{r}{r-D}\right)^{\frac{1}{2}} \qquad (2.3)$$

2.1.4 The EOQ Model with Backorders Allowed

In reality, demand is not met on time, and shortages occur. Let C_u be the shortage cost per unit per time. All demand are backlogged, C_o is the setup cost, C_h is the holding cost, *D* is the demand. To determine the order policy, Winston (2004) define

q = order quantity

q-M = maximum shortage that occurs under an ordering policy

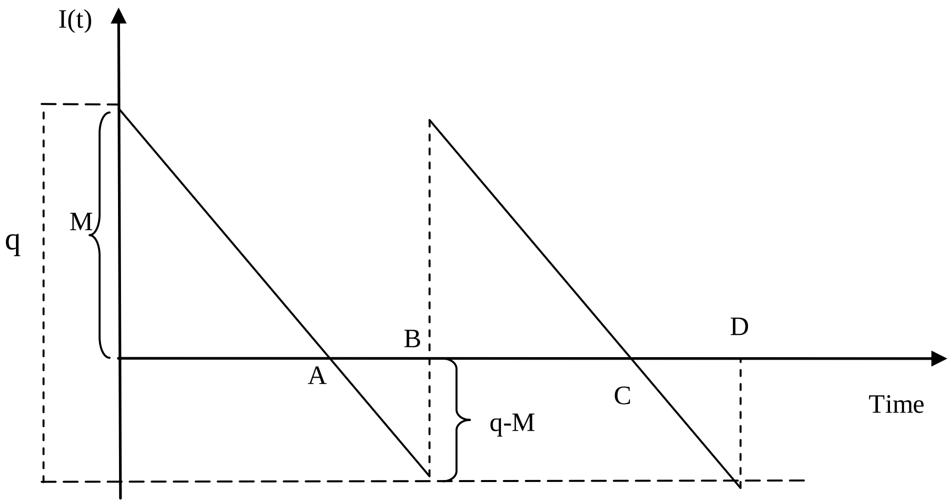

Figure 2.4 EOQ model with backorders allowed

and the annual total cost

$$\mathrm{TC}(q, M) = \frac{M^2 C_h}{2q} + \frac{(q - M)^2 C_u}{2q} + \frac{C_o D}{q} \qquad (2.4)$$

TC (q, M) is minimized for q^* and M^*:

$$q^* = \mathrm{EOQ}\left(\frac{C_h + C_u}{Cu}\right)^{\frac{1}{2}} \qquad (2.5)$$

and

$$M^* = \mathrm{EOQ}\left(\frac{C_u}{Cu + C_h}\right)^{\frac{1}{2}} \qquad (2.6)$$

and maximum shortage is calculated as q^*-M^*.

2.2 Stochastic Inventory Models

This model based on the assumptions that the average for inventory items is reasonably constant over time. It is possible to describe the probability distribution of the demand and lead time for each order is nonzero and random. When demand is assumed to be stochastic, inventory is managed according to two principles; such as continuous review and periodic review policy.

2.2.1 The (r, q) Continuous Review Policy

We consider the (r, q) inventory policy, alternatively called the reorder point, order quantity system. When the level on-hand inventory reaches a reorder point level *r*, place an order for *q* units. The order arrives to replenishment the inventory after a lead time L. During which a stock out might occur, the order received. Figure 2.5 shows the inventory pattern determined by the (r, q) inventory policy.

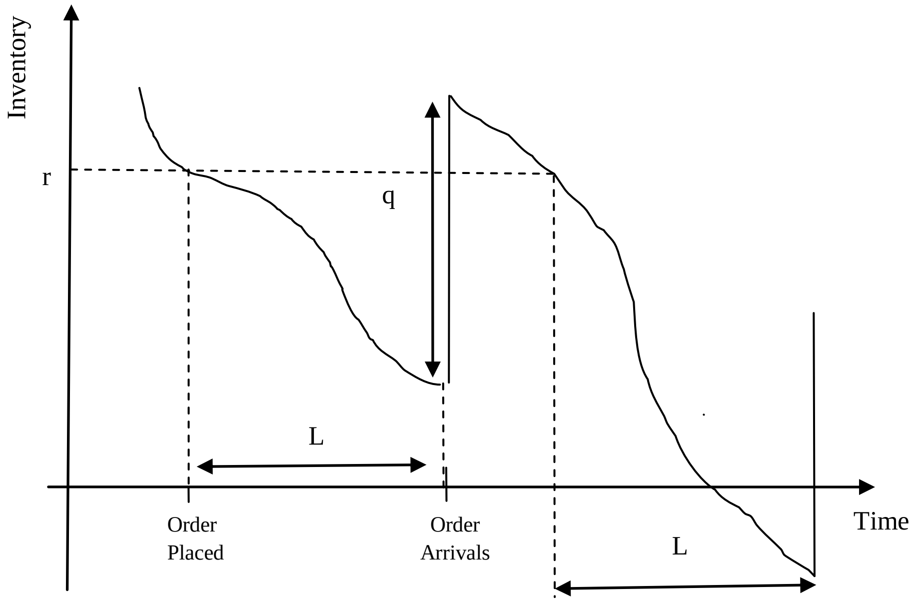

Figure 2.5 Continuous review (r, q) policy

The model assumes;

1. The system is continuous review. That is demands are reordered as they occur, and the level of on-hand inventory is known at all times.
2. Demand is random and stationary.
3. There is a fixed positive lead time *L* for placing an order.
4. The following costs are assumed:

   C_o: ordering cost

   C_h: holding cost

   C_B: cost incurred for each unit short

   C_{LS}: cost incurred for each lost sales

We also require the following definitions:

D: random variable representing annual demand with mean E(D), variance varD, and standard deviation σ_D.

L: lead time for each order

q: quantity ordered each time an order takes place

r: reorder level at which order is placed (reorder point)

X: random variable representing demand during lead time

B_r: random variable representing the number of stock outs or backorders during a cycle if the reorder point is r

If we assume that *L* is relatively small compared to the expected time required to exhaust the quantity *q*, it is likely that only one order is outstanding at any one time. This is the case illustrated in the Figure 2.5. We call the period between two consecutive order arrivals as an order cycle. The cycle begins with the receipt of the lot, it progresses as demand depletes the inventory to the level *s*, and then it continues for the time *L* when the next lot is received. As we see in the Figure (2.5), the inventory level increases instantaneously by the amount *q* with the receipt of an order. We desire to determine the optimal *q* and *r* to minimize the annual expected total cost. In the first case, we assumed all demand must be met and no sales are lost. So, we need the total cost for the backordered case.

Define expected annual total cost

TC(r, q)= (expected annual holding cost)+(expected annual ordering cost)+(expected annual shortage cost). Hence, we calculate the

expected annual holding cost $= C_h\left(\frac{q}{2} + r - E(X)\right)$

expected annual shortage cost $= \frac{C_B E(B_R) E(D)}{q}$

expected annual order cost $= \frac{C_o E(D)}{q}$

and we obtain

$$TC(r, q) = C_h\left(\frac{q}{2} + r - E(X)\right) + \frac{C_B E(B_r) E(D)}{q} + \frac{C_0 E(D)}{q} \quad (2.7)$$

There are two variables in this cost function, *q* and *r*. To find the optimal policy that minimizes total cost, we take the partial derivatives of the expected cost, (2.7), with respect to each variable and set them equal to zero. First, the partial derivative with respect to q^* is

$$q^* = \left(\frac{2C_o E(D)}{C_h}\right)^{\frac{1}{2}} \quad (2.8)$$

and taking the partial derivative with respect to the variable r,

$$P(X \geq r^*) = \frac{C_h q^*}{C_B E(D)} \quad (2.9)$$

As a second case we assume that all stockout results in lost sales. In this case, the optimal order quantity will be

$$q^* = \left(\frac{2C_0 E(D)}{C_h}\right)^{\frac{1}{2}} \quad (2.10)$$

And for the reorder point (2.11) is obtained

$$P(X \geq r^*) = \frac{C_h q^*}{C_h q^* + C_{LS} E(D)} \quad (2.11)$$

In many circumstances, the stockout cost, C_u, is difficult to estimate. For this reason, it is common business practice to set inventory levels to meet a specified service objective instead. The two most common service objectives are:

1. Type 1 service: Choose *r* so that the probability of not stocking out in the lead time is equal to a specified value.
2. Type 2 service: Choose both *q* and *r* so that the proportion of demands satisfied from stock equals a specified value.

2.2.2 The (s, S) Continuous Review Policy

An order could be placed exactly at the point when the inventory level reached the reorder point *r*. We used this policy to compute the expected inventory level at the beginning and the end of a cycle. Inventory level is likely to overshoot the reorder point *r*, making it impossible to place on order the instant the inventory reaches *r*. Then the (r, q) model may not yield a policy that minimizes expected annual cost function. In this situation, is has been shown that an (s, S) policy is optimal. The (s,S) policy is when the level of on-hand inventory is less than or equal to *s*, the size of the order is sufficient to raise the inventory level to *S*.

If *u* is the starting inventory level in any period, then the (s,S) policy is

$$\text{If} \quad \begin{array}{ll} u \leq s & \text{, order S-u} \\ u < s & \text{, do not order} \end{array}$$

Determining optimal values of (s, S) is extremely difficult and several approximation have been suggested. Set S-s = q and s = r. This approximation will give reasonable results. (Nahmias, 1997)

2.2.3 The (R, S) Periodic Review Policy

The inventory level is reviewed periodically at regular time intervals in this policy. A convenient quantity is ordered after each review. A different way to manage a stochastic inventory system is the (R, S) periodic review policy. Every *R* units of time (years, months, etc…), the on-hand inventory level is reviewed and an order is placed to bring up the on-order inventory level *S*. After a lead time interval *L*, the replenishment order is delivered. Figure 2.6 shows the (R, S) inventory policy.

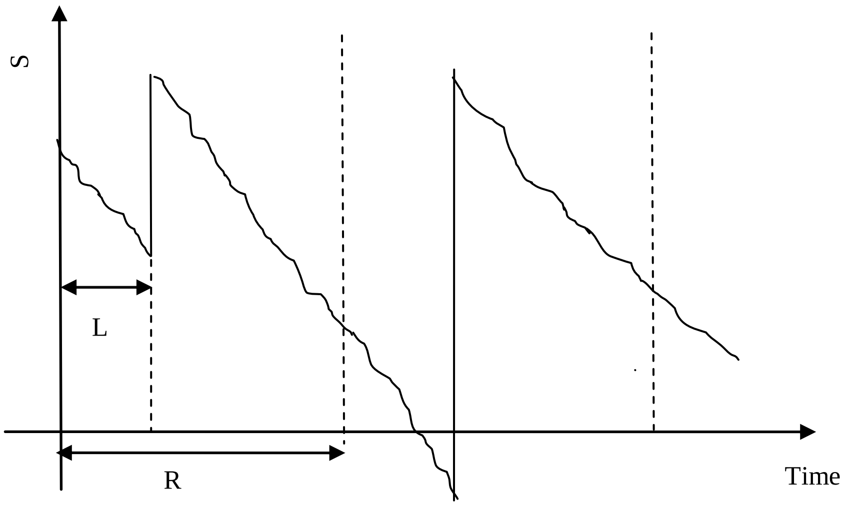

Figure 2.6 The (R, S) periodic review policy

In general (R, S) policy will incur higher holding costs than a cost minimizing (r, q) policy. The analysis of this policy is much like that for the (r, q) policy. The model assumes

1. All shortages are backlogged
2. Demand is a continuous random variable
3. The per-unit purchase price is constant
4. The following costs are assumed:

   C_o: ordering cost

   C_h: holding cost per unit

   J: cost of reviewing inventory level

   C_B: cost per unit short in the backlogged case

   C_{LS}: cost per unit short in the lost sales case

We also require the following definitions:

D: random variable representing annual demand with mean E(D), variance varD, and standard deviation σ_D.

L: lead time for each order

R: time between reviews

D_{L+R}: demand during a time interval of length L+R with mean $E(D_{L+R})$

Define expected annual cost

TC(R, S)= (expected annual holding cost)+(annual ordering cost)+(expected annual shortage cost)+(annual review cost). Hence, each component of the total cost is calculated as:

annual order cost $= \frac{C_o}{R}$

annual review cost $= \frac{J}{R}$

expected annual shortage cost $= \frac{C_M}{R}(C_M : C_B \text{ or } C_{LS})$

expected annual holding cost $= C_h(S - E(D_{L+R}) + \frac{E(D)R}{2})$

$$TC(R,S) = \frac{C_o}{R} + \frac{J}{R} + \frac{C_M}{R} + C_h\left(S - E(D_{L+R}) + \frac{E(D)R}{2}\right) \qquad (2.12)$$

Both order and review costs are independent of S. Thus, the value of *S* that minimizes the sum of the annual expected holding cost and annual expected shortage cost is optimal. In the backlogged case given a value of *R*, the value of *S* is determined from

$$P(D_{L+R} \geq S) = \frac{RC_h}{C_B} \qquad (2.13)$$

and in the lost sale case, *S* is determined from

$$P(D_{L+R} \geq S) = \frac{RC_h}{RC_h + C_{LS}} \qquad (2.14)$$

2.3 Recent Studies

Continuous deteriorating inventory models have so far analyzed extensively either form deterministic or stochastic approach. In deterministic approach the parameters are assumed to be known under fixed constraints. In real world inventory of deterioration items, the information is not always well defined and the mathematical modeling of deterioration items is a significant subject.

Early studies on deteriorating inventory systems assume a periodic review approach. Nahmias (1982) classified both inventory deterioration with fixed lifetime and inventory decaying with a random lifetime. Nahmias (1982) considers both deterministic and stochastic demand for single and multiple products. Later, Rafaat (1991) present a comprehensive and up-to-date survey of inventory literature for the deteriorating inventory models where deterioration was consider as function of the on-hand level of inventory. But he did not touch on the effect of constant decay into the variety of existing inventory models. Furthermore, deteriorating inventory models need to be developed to consider the effects of quality discount and multiple-item stocking. So Goyal and Giri (2001) studied the deteriorating inventory models under these conditions.

Kalpakam and Arrivarignan (1988) studied a continuous review (s, S) model with Poisson demand with zero lead time and an exponential lifetime. By assuming no backorders and instantaneous delivery of orders, the steady state probability distribution of the stock level and mean time between successive reorders are delivered. Besides they calculated that reorder point *s* should be set to zero. Liu (1990) allows backorders for the same model but used an alternative approach which gives the stationary probability distribution of the stock level and suggested that the reorder point *s* would be smaller than one.

When a positive lead time is introduced in the problem the analysis becomes extremely complex. Kalpakam and Sapna (1994) consider extensions of Kalpakam and Arrivarignan (1988) model. They investigate a lost-sales (s,S) system with exponential leadtimes for items with exponential lifetimes. They used to Markov process which satisfied the Kolmogorov’s forward differential equations and derived an exact cost function. Kalpakam and Shanthi (2006) analyzed the same model under renewal demand. They formulated the system using semi-regenerative process which applied to obtain the various operating characteristics.

Lian and Liu (1999) study a continuous review (s,S) model with a fixed shelf life and renewal arrival where degradation is only detected at demands arrival. They used Laplace-Stieltjes transforms and analyzed the structure of the cost function with random batch size. But this method is rather complicated for models with batch demand. So Lian and Liu (2001) used the Markov chain for the same model. They provide a heuristic for positive lead time. Also they studied the effect of the demand rate and the lifetime which is the missing part of the Lian and Liu (1999).

Gürler and Özkaya (2008) assumed a random shelf-life and allowing backorders for the Lian and Liu (1999) and Lian and Liu (2001) models. Also they extensively investigate the impact of the shelf-life distributions and show that the expected cost rate function is quasi-convex in (s,S) for unit demand. Gürler and Özkaya (2008) also provide a heuristic and the heuristic they proposed performs as well as Lian and Liu (2001).

Lian, Liu and Zhao (2009) studied an (s, S) continuous review model for items with exponential lifetime and a general Markovian renewal demand process. By constructing Markovian renewal equations they compared the numerical results of Markovian renewal process (MRD) and renewal process (RD). They approximated on MRD model by an RD model and they found the cost is higher than the minimum cost.

A good summary of fixed life perishability problem can be found in Goyal and Giri(2001), Nahmias(1982) and Uckun, Karaesmen and Savas. (2008). They basically consider continuous time inventory control models where deterioration of inventories. Uckun, Karaesmen and Savas. (2008) review the supply chain management literature of perishable products having fixed or random lifetimes. They classify the literature into periodic and continuous review inventory control. They provide a detailed classification specific model assumption, e.g. replenishment policy and lead time.

Wagner and Within (1958) presented a simply algorithm for solving the dynamic version of the economic lot size model. Veinott (1960) studied periodic review and known demand. Veinott (1960), consider three problems;

1. Determining an optimal ordering policy when the disposal and issuing policies are given,
2. Determining optimal ordering and disposal policies when the issuing policy is given,
3. Determining optimal issuing and disposal policies when the ordering policy is given.

Veinott (1960) shows that when the life time function is non-increasing in the items age at issue and for Problem 1 a FIFO issuing policy is used on optimal order policy will order an amount equal to demand.

Van Zyl (1964) investigate a periodic review problem of a product having a two period life time, zero lead time and FIFO issuing policy with the minimize expected costs. Nahmias and Pierskalla (1973) and Fries (1975) extended van Zyl's model and derive ordering policies for a general life time of n-periods. Nahmias and Pierskalla (1973) consider optimal policy for the multi-period version of van Zyl's model with ordering and holding costs. Nahmias (1975) and Fries (1975) both consider the zero lead time and constant lifetime. By the dynamic programming approach, they show that the base-stock policy is a good approximation of the real optimal policy. Also Nahmias and Pierskalla (1973) considered only shortage and outdate cost of the same model. Pierskalla and Roach (1975) show that FIFO is optimal issuing policy when the objective is minimize total inventory holding costs. Nahmias (1977) suggest to group older on hand items together in order to reduce the state space. And they conclude the order quantity is more sensitive to the fresh inventory rather than older inventory. Nahmias (1977) extend the van Zyl's model to include a positive set-up cost for ordering and derive the solution for the single period.

Nandakumar and Morton (1993) derive near myopic upper and lower bounds on the order quantities for the base-stock inventory policy with fixed lifetime and used the bounds to evaluate the performance of the resulting heuristics. Jain and Silver (1994) developed a stochastic dynamic programming model to determine the optimal order policy for a random life time perishable. They assumed life time as a discrete random variable which follows an arbitrary probability distribution. They also presented two approximate solution methods based on Silver-Meal heuristic and Wagner-Within algorithm.

All previously studies assume lead time zero and have the objective of minimizing cost. Inventory models, including positive finite lead time, lost sales and service level constraints, have little attention in the literature, although these problems are highly relevant in retailing. Williams and Patuwo (1999, 2004) analyzed a periodic review inventory control problem of a single perishable product having two period life time. The lead time is positive and any unmet demand is lost. They derive optimal order quantities based on system recursion for a single-period problem. And optimal order quantities for lead times up to four-periods are computed for different demands distributions. Kapalka (1999) investigate a single-product, periodic review inventory problem with fixed positive lead time under the lost sales assumption which minimizes long-run average cost under a service level constraint. Van Donselaar (1996) present a dynamic replenishment policy for a lost sales inventory control system without perishability and compare the performance of the dynamic method to a base-stock policy. Minner and Transchel (2010) present a numerical approach to dynamically determine replenishment quantities for perishable items with limited life time, positive lead time, FIFO and LIFO issuing policy and multiple service level constraints. They show that a constant order policy might provide good results under stationary demand, short life time and LIFO inventory depletion.

Also, Zipkin(2000) and Kouki, Sahin, Jamei and Dallery (2010) studies are examined.

CHAPTER THREE

TWO INVENTORY MANAGEMENT APPROACHES FOR PERISHABLE PRODUCTS

In this chapter, we study two types of inventory control policies for perishable products, such as continuous review policy and periodic review policy. In this chapter both policies will be explained in detailed with two similar models.

3.1 (s,S) Continuous Review Approach

Liu (1990) consider an (s,S) continuous review inventory system with Poisson demand and exponential lifetime distribution. Backlogs are allowed in the model but lead time is assumed to be zero. Liu (1990) solved this model with two different approaches and presented numerical analysis for these approaches and compare the results. We paraphrase the same model with Liu (1990) in this part.

Previous studies show what the conditions are under which random lifetime for items are equivalent to the corresponding proportional decay of the mean inventory level. Liu (1990) consider steady state behavior of the system in which demands occur in single units following a Poisson process with constant rate μ. The life time of inventory is exponentially distributed with a constant failure rate λ. I(t) is a Markov process with a discrete state space $\{s+1, ..., S\}$, $s \leq -1, S \geq 0.$ Lead time is zero. $P_n(t)$ is probability that inventory level I(t) is n at time t. $\left[P_n(t) = P\{I(t) = n\}\right]$. In Figure 3.1 shows the state transition diagram for Markov process with a discrete state space (0 ≤ n ≤ S). We generated the probability formula (3.1) with the help of Figure 3.1.

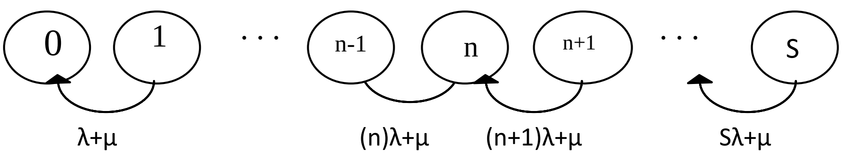

Figure 3.1 State transition diagrams for discrete state space (0≤ n ≤ S).

$$P_n(t+\Delta t)=P_{n+1}(t)[(n+1)\lambda+\mu)\Delta t]+P_n(t)[1-(n\lambda+\mu)\Delta t] \quad (3.1)$$

Rearranging (3.1);

$$P_n(t+\Delta t)=P_{n+1}(t)(n+1)\lambda\Delta+P_n(t)\mu\Delta t+P_n(t)-P_n(t)(n\lambda+\mu)\Delta t$$

$$\frac{P_n(t+\Delta t)-P_n(t)}{dt}=P_{n+1}(t)(n+1)\lambda+P_n(t)\mu-P_n(t)(n\lambda)+P_n(t)\mu$$

$$\frac{dP_n(t)}{dt}=-\lambda[nP_n(t)-(n+1)P_{n+1}(t)]-\mu[P_n(t)-P_{n+1}(t)]$$

$$\frac{dP_n(t)}{dt}+\lambda[nP_n(t)-(n+1)P_{n+1}(t)]=-\mu[P_n(t)-P_{n+1}(t)],\ 0\leq n\leq S \quad (3.2)$$

Similarly in Figure 3.2 and Figure 3.3 shows the state transition diagram for (n=S) and ($0 \leq n \leq S$). The probability formula (3.3) and (3.4) was generated with the Figure 3.2 and Figure 3.3.

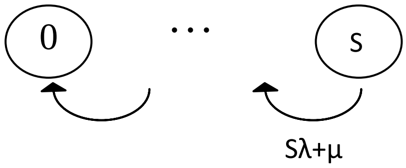

Figure 3.2 State transition diagrams for discrete state space (n = S).

$$P_S(t+\Delta t)=P_{S+1}(t)\mu\Delta t+P_S(t)[1-(S\lambda+\mu)\Delta t]$$

$$P_S(t+\Delta t)=P_{S+1}(t)\mu\Delta t+P_S(t)-P_S(t)(S\lambda+\mu)\Delta t$$

$$\frac{dP_n(t)}{dt}+\lambda SP_S(t)=-\mu[P_S(t)-P_{S+1}(t)],\ n=s \quad (3.3)$$

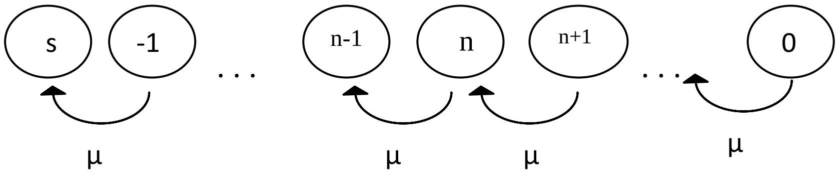

Figure 3.3 State transition diagrams for discrete state space (-s ≤ n ≤ 0).

$$P_n(t+\Delta t) = P_{n+1}(t)\mu\Delta t + P_n(t)[1-(n\lambda+\mu)\Delta t]$$

$$P_n(t+\Delta t) = P_{n+1}(t)\mu\Delta t + P_n(t) - P_n(t)\mu\Delta t$$

$$\frac{dP_n(t)}{dt} = -\mu[P_n(t) - P_{n+1}(t)],\ s<n<0 \qquad (3.4)$$

The time periods any two consecutive reorder points are independent and identically distributed. Liu (1990) defined these time periods reorder cycles in Figure 3.4. Let *T* denote the reorder cycle in the steady state. *T* consists of two distinct periods. The first period T_1 is the inventory level is positive. The second period T_2 is the inventory level is zero or negative.

T(j) : the end of the j-th reorder cycle

T_1(j):the time period in the (j+1)-th cycle in which inventory level drops from *S* to 0.

M(t) : the mean inventory level.

s_1: positive reorder point

s_2: negative reorder point

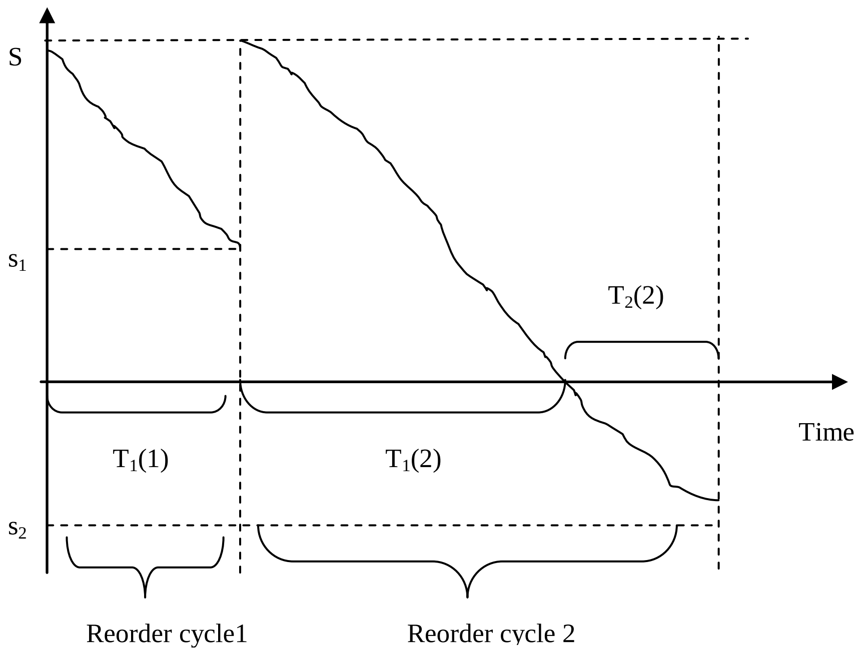

Figure 3.4 (s,S) policy with reorder

Let T(0) =0. Readjusting (3.1)-(3.3) according to the reorder cycles, for $0 \le n \le S$ we have;

$$\frac{dP_n(t)}{dt} + \lambda[nP_n(t) - (n+1)P_{n+1}(t)] = -\mu[P_n(t) - P_{n+1}(t)],\ 0 \le n \le S$$

$$\sum_{n=1}^{S} n\frac{dP_n(t)}{dt} + \lambda\sum_{n=1}^{S} n\,[nP_n(t) - (n+1)P_{n+1}(t)] = -\mu\sum_{n=1}^{S} n\,[P_n(t) - P_{n+1}(t)]$$

$$\frac{dM(t)}{dt} = -\lambda\sum_{n=1}^{S} n\,[nP_n(t) - (n+1)P_{n+1}(t)] - \mu\sum_{n=1}^{S} n\,[nP_n(t) - (n+1)P_{n+1}(t)]$$

$$= -\lambda[[1[1.P_1(t) - 2.P_2(t)] + \cdots + S[S.P_S(t) - (S+1)P_{S+1}(t)]]$$

$$= -\lambda[P_1(t) + 2P_2(t) + \ldots + SP_S(t)] - \mu[P_1(t) + P_2(t) + \cdots + P_S(t)]$$

$$= -\lambda\sum_{n=1}^{S} nP_n - \mu\sum_{n=1}^{S} P_n$$

$$\frac{dM(t)}{dt} + \lambda M(t) = -\mu,\ T(j) \le t \le T(j) + T_1(j) \qquad (3.5)$$

And for $s+1 \le n \le 0$ we have;

$$\sum_{n=s+1}^{-1} n\frac{dP_n(t)}{dt} = -\mu\sum_{n=s+1}^{-1} n\,[P_n(t) - P_{n+1}(t)]$$

$$\frac{dM(t)}{dt} = -\mu[(s+1)[P_{s+1}(t) - P_{s+2}(t)] + \cdots + (-1)[P-1(t) - P_0(t)]]$$

$$\frac{dL(t)}{dt} = -\mu\left[P_0 + P_{-1} + \ldots + P_s\right]$$

$$\frac{dM(t)}{dt} = -\mu\sum_{n=1}^{S} P_n\ ,T(j) + T_1(j) \le t \le T(j+1) \qquad (3.6)$$

Liu (1990) consider the system behavior in the steady state. Let P_n be the probability that the inventory level is *n* in the steady state. Letting $t \to \infty$ and simplifying (3.2)-(3.4);

$$\frac{dP_n(t)}{dt} + \lambda[nP_n(t) - (n+1)\,P_{n+1}(t)] = -\mu[P_n(t) - P_{n+1}(t)]\,,0 \le n \le S$$

$$n \Rightarrow 0 \Longrightarrow \lambda[0 - P_1(t)] = -\mu[P_0(t) - P_1(t)]$$

$$-\lambda P_1(t) = -\mu P_0(t) - \mu P_1(t) \ \Rightarrow\ P_1(t) = \frac{\mu}{\mu + 2\lambda}P_0$$

$$n = 1 \Rightarrow \lambda[P_1(t) - 2P_2(t)] = -\mu[P_1(t) - P_2(t)]$$

$$\lambda P_1(t) - 2\lambda P_2(t) = -\mu P_2(t) \Rightarrow P_2(t) = \frac{\mu}{\mu + 2\lambda} P_0$$

$$\text{for } n = n \Rightarrow P_n = \frac{\mu}{\mu + n\lambda} P_0 \qquad (3.7)$$

$$\frac{dP_n(t)}{dt} = -\mu[P_n(t) - P_{n+1}(t)], \qquad s < n < 0$$

$$n = 0 \Rightarrow \mu P_0(t) = \mu P_1(t) \Rightarrow P_0(t) = P_1(t)$$

$$n = -1 \Rightarrow \mu P_{-1}(t) = \mu P_0(t) \Rightarrow P_{-1}(t) = P_0(t)$$

$$\text{for } n{=}n \Rightarrow P_n(t) = P_0(t) \qquad (3.8)$$

$$\sum_{n=s-1}^{S} P_n = 1$$

$$\sum_{n=s-1}^{-1} P_n + \sum_{n=0}^{S} P_n = 1 \rightarrow \sum_{n=s-1}^{-1} P_0 + \sum_{n=o}^{S} \frac{\mu}{\mu + n\lambda} P_0 = 1$$

$$(-s)P_0 + \sum_{n=1}^{S} \frac{\mu}{\mu + n\lambda} P_0 = 1 \rightarrow P_0 \left[-s + \sum_{n=1}^{S} \frac{\mu}{\mu + n\lambda}\right] = 1$$

$$P_0 \left[-s + \sum_{n=1}^{S} \frac{\mu}{\mu + n\lambda}\right] = 1 \rightarrow P_0 = 1 / \left[-s + \sum_{n=1}^{S} \frac{\mu}{\mu + n\lambda}\right]$$

$$P_0 = \left[\sum_{n=1}^{S} \frac{\mu}{\mu + n\lambda} - s\right]^{-1} = [\Phi(S) - s]^{-1} \qquad (3.9)$$

$\Phi_n = \mu/(\mu+n\lambda)$ is the conditional probability that given n items in stock.

With (3.7)-(3.9) the mean and variance of the steady state inventory level can be obtained. Mean inventory level can be obtained as follows:

$$E[n] = \sum_{n=s}^{S} nP_n = \sum_{n=1}^{S} nP_n + \sum_{n=s}^{0} nP_n = \sum_{n=1}^{S} n \frac{\mu}{\mu + \lambda} P_0 + \sum_{n=s}^{0} nP_0$$

$$= \frac{\mu}{\lambda}\left[\sum_{n=1}^{S} \frac{n}{\frac{\mu}{\lambda} + n} P_0\right] + [-P_0(1 + 2 + \cdots + s)]$$

$$= \frac{\mu}{\lambda}\left[\sum_{n=1}^{S} P_0 - \sum_{n=1}^{S} \frac{\frac{\mu}{\lambda}}{\frac{\mu}{\lambda} + n} P_0\right] - \left[P_0 \frac{1}{2} s(s+1)\right]$$

$$= \frac{\mu}{\lambda}\left[SP_0 - \sum_{n=1}^{S} \frac{\mu}{\mu + n\lambda} P_0\right] - \left[P_0 \frac{1}{2} s(s+1)\right]$$

$$= \left[\frac{\mu}{\lambda}[S - \Phi(S)] - \left[\frac{1}{2} s(s+1)\right]\right] P_0 E[N]$$

$$= \left[\frac{\mu}{\lambda}[S - \Phi(S)] - \left[\frac{1}{2} s(s+1)\right]\right] [\Phi(S) - s]^{-1} \tag{3.10}$$

σ^2: the variance of the inventory level in steady state.

$$\sigma^2 = \frac{\mu}{\lambda}\left(\frac{1}{2} S(S+1) - \frac{1}{3}(s+2)(s+1)s\right)[\phi(S) - s]^{-1} - \left(\frac{\mu}{\lambda} + E[n]\right) E[n] \tag{3.11}$$

Let V_i be the time for the inventory level to drop from i to i-1 ($s < i \leq S$). The distribution function for V_i is given by;

$$F_i(x) = \begin{cases} 1 - e^{-(\mu + i\lambda)x}, & 0 < i \leq S \\ 1 - e^{-\mu x}, & s < i \leq 0 \end{cases} \tag{3.12}$$

Since

$$T = T_1 + T_2 = (V_S + \cdots + V_1) + (V_0 + \cdots + V_{s+1}) \tag{3.13}$$

the Laplace-Stieltjes transform of T is the product of Laplace-Stieltjes transforms of the V_i's,

$$f_T(w) = f_{T_1}(w) f_{T_2}(w) = \left(\prod_{i=1}^{S} \frac{\mu + i\lambda}{w + \mu + i\lambda}\right)\left(\frac{\mu}{w + \mu}\right)^{-s} \tag{3.14}$$

From (3.14), Liu (1990) obtain the mean length and variance of the reorder cycle;

$$E[n]_T = E[n]_{T_1} + E[n]_{T_2} = \mu^{-1}\Phi(S) + \mu^{-1}(-s) \tag{3.15}$$

$$\sigma^2{}_T = \sigma^2_{T_1} + \sigma^2{}_{T_2} = \sum_{n=1}^{S} \frac{1}{(\mu + n\lambda)^2} + \frac{(-s)}{\mu^2} \tag{3.16}$$

$\left(\frac{\mu}{w+\mu}\right)^{i-s}$: The Laplace-Stieltjes transforms of the waiting time of i-th backlog demand.

3.1.1 The Expected Operating Costs

The following notations are used for the corresponding unit costs by Liu (1990).

C_h: the inventory holding cost per unit per unit time;

C_r: the replacement cost per unit decaying;

C_u: the shortage penalty per unit short;

C_s: the shortage penalty per unit per unit time;

C_o: the ordering cost per order.

For convenience, Liu (1990) used *x*=-*s* instead of *s*. *x* can be interpreted as the backlogging level. The inventory holding cost incurs only when the inventory level is positive. The mean length of this period is $E[n]_{T_1}$. Liu (1990) calculated the mean inventory level during this period as

$$E[n]' = \sum_{n=1}^{S} nP_n' = \sum_{n=1}^{S} n\left(\frac{\mu}{\mu+n\lambda}P_0\right)\left(\sum_{n=1}^{S}\frac{\mu}{\mu+n\lambda}P_0\right)^{-1} = \frac{\mu}{\lambda}[S-\Phi(S)]\Phi^{-1}(S) \quad (3.17)$$

The total inventory holding cost in a reorder cycle is

$$C_h E[n]_{T_1} E[n]' = C_h \lambda^{-1}[S-\Phi(S)] \quad (3.18)$$

The deterioration can occur in period T_1. The mean rate of deterioration in this period is $\lambda E[n]'$ and the mean total replenishment cost per cycle is

$$C_r E[n]' E[n]_{T_1} = C_r[S-\Phi(S)] \quad (3.19)$$

The period which backlogs exist has a mean rate $E[n]_{T_1} - \frac{1}{\mu} = \frac{x-1}{\mu}$. The number of backlogs is (x-1), and the mean backlog level is $\frac{1}{2}(x-1)$. The total shortage penalty cost is

$$TC_s = C_s\frac{(x-1)^2}{2\mu} + C_u(x-1) \quad (3.20)$$

Liu (1990) summing up the three costs and ordering cost C_0 and divided $E[n]_T$ and obtain the expected total costs per unit time

$$TC(x,S) = \frac{C_o + C_s\frac{(x-1)^2}{2\mu} + C_u(x-1) + [C_h\lambda^{-1} + C_r][S-\Phi(S)]}{\mu^{-1}[\Phi(S)+x]} \quad (3.21)$$

$$-\frac{C_o + C_u(x-1) + [C_h\lambda^{-1} + C_r][S - \Phi(S)]}{\mu^{-1}[\Phi(S) + x][\Phi(S) + x + 1]}$$

$$\Delta_x TC(x,S) = \frac{C_u(\Phi(S) + 1) - C_o - [C_h\lambda^{-1} + C_r][S - \Phi(S)]}{\mu^{-1}[\Phi(S) + x][\Phi(S) + x + 1]} \qquad (3.22)$$

Thus, Liu (1990) conclude: when

$$C_u(\Phi(S) + 1) - C_o - [C_h\lambda^{-1} + C_r][S - \Phi(S)] > 0$$

$$C_u(\Phi(S) + 1) > C_o + [C_h\lambda^{-1} + C_r][S - \Phi(S)]$$

$$C_u > \frac{C_o + [C_h\lambda^{-1} + C_r][S - \Phi(S)]}{\Phi(S) + 1} \qquad (3.23)$$

TC(x, S) a strictly increasing function of *x* and, as a result, no backlogging will be permitted in the system; when

$$C_u < \frac{C_o + [C_h\lambda^{-1} + C_r][S - \Phi(S)]}{\Phi(S) + 1}$$

TC(x, S) is a strictly decreasing function of *x* and all demands should be backlogged and no feasible optimal solution is available for this system. When

$$C_u = \frac{C_o + [C_h\lambda^{-1} + C_r][S - \Phi(S)]}{\Phi(S) + 1}$$

TC(x, S) is independent of *x*. An artificial limit may be unrealistic for practical systems. Thus a positive C_s is important to the modeling of inventory systems with shortages.

(b) $C_s>0$. TC(x, S) is no longer a simple monotone function of *x*. Letting

$$g_x(x,S) = \frac{1}{2}[TC(x-1,S) + TC(x+1,S)] - TC(x,S) \qquad (3.24)$$

We have from (3.24)

$$g_x(x,S) = \frac{2\mu\{C_o + [C_h\lambda^{-1} + C_r][S - \Phi(S)] - C_u[\Phi(S) + 1]\} + C_s[\Phi(S) + 1]^2}{2[\Phi(S) + x - 1][\Phi(S) + x][\Phi(S) + x + 1]}$$

Thus, Liu (1990) conclude: when

$$2\mu\{C_o + [C_h\lambda^{-1} + C_r][S - \Phi(S)] - C_u[\Phi(S) + 1]\} + C_s[\Phi(S) + 1]^2 \geq 0$$

$$\frac{2\mu C_o + 2\mu[C_h\lambda^{-1} + C_r][S - \Phi(S)] + C_s[\Phi(S) + 1]^2}{2\mu[\Phi(S) + 1]} \geq C_u$$

$$\frac{C_o + [C_h\lambda^{-1} + C_r][S - \Phi(S)]}{[\Phi(S) + 1]} + \frac{[\Phi(S) + 1]}{2\mu} \geq C_u \qquad (3.25)$$

TC(x, S) is a convex function of *x*; otherwise it is a concave function of *x* and the minimum is at *x* =1, that is, no backlogging should be allowed.

Letting $\Delta_x TC(x, S) = 0$ and $[C_h\lambda^{-1} + C_r][S - \Phi(S)] = A$, from (3.21),

$$\frac{C_o + \frac{C_s x^2}{2\mu} + C_u(x) + A}{\mu^{-1}[\Phi(S) + x + 1]} - \frac{C_o + \frac{C_s(x-1)^2}{2\mu} + C_u(x-1) + A}{\mu^{-1}[\Phi(S) + x]} = 0$$

$$x^2 + x\big(1 + 2\Phi(S)\big) = \frac{2\mu}{C_s}(C_o + A) + \left[(\Phi(S) + 1)\left(1 - \frac{2\mu C_u}{C_s}\right)\right]$$

$$x^2 + x\big(1 + 2\Phi(S)\big) - k(S) = 0 \tag{3.26}$$

in which

$$k(S) = \frac{2\mu}{C_s}(C_o + [C_h\lambda^{-1} + C_r][S - \Phi(S)]) + \left[(\Phi(S) + 1)\left(1 - \frac{2\mu C_u}{C_s}\right)\right] \tag{3.27}$$

Thus, when TC(x, S) is a convex function of *x*, the positive solution of (3.26) is given by

$$x^* = \sqrt{[\Phi(S) + 0.5]^2 + k(S)} - [\Phi(S) + 0.5] \tag{3.28}$$

and the cost function is minimized at

$$x = [x^*] + 1 \tag{3.29}$$

where $[x^*]$ is the largest integer which is smaller than or equal to x^*. There exists a finite backlogging level *x* which will minimize the expected inventory costs. In conclusion, when C_u=0, which is a realistic assumption when C_s>0, the TC(x, S) is always a convex function of *x*.

(c) If there exists a local minimum point, then TC(x, S) is a unimodal function of *S* for *S*>0. While fixing *x*, Liu (1990) consider the increment of TC(x, S).Letting

$$\Delta_x TC(x, S) = TC(x, S + 1) - TC(x, S) \tag{3.30}$$

and

$$\Delta_x TC(x, S) = TC(x, S) - TC(x, S - 1) \tag{3.31}$$

Liu (1990) calculate from (3.21),

$$\Delta_1 TC(x, S) = \frac{\mu g_1(x, S)}{[\mu + (S + 1)\lambda][\Phi(S) + x]\left[\Phi(S) + x + \frac{\mu}{\mu + (S + 1)\lambda}\right]} \tag{3.32}$$

and

$$\Delta_2 TC(x, S) = \frac{\mu g_2(x, S)}{[\mu + S\lambda][\Phi(S) + x]\left[\Phi(S) + x - \frac{\mu}{\mu + S\lambda}\right]} \tag{3.33}$$

in which

$$g_1(x,S) = [C_h\lambda^{-1} + C_r][\mu + (S+1)\lambda]\Phi(S)$$
$$-\mu C(x) - [C_h\lambda^{-1} + C_r][S\mu - (S+1)\lambda x] \quad (3.34)$$

$$g_2(x,S) = [C_h\lambda^{-1} + C_r][\mu + S\lambda]\Phi(S) - \mu C(x)$$
$$-[C_h\lambda^{-1} + C_r][\mu - \lambda x]S \quad (3.35)$$

and

$$C(x) = C_o + C_s\frac{(x-1)^2}{2\mu} + C_u(x-1) \quad (3.36)$$

If S^* is a local minimum,

$$[C_h\lambda^{-1} + C_r][\mu + (S^*+1)\lambda]\Phi(S^*) - \mu C(x) - [C_h\lambda^{-1} + C_r][S^*\mu - (S^*+1)\lambda x] \geq 0$$
$$[C_h\lambda^{-1} + C_r][\mu + (S^*+1)\lambda]\Phi(S^*) \geq \mu C(x) + [C_h\lambda^{-1} + C_r][S^*\mu - (S^*+1)\lambda x] \quad (3.37)$$

$$[C_h\lambda^{-1} + C_r][\mu + S^*\lambda]\Phi(S^*) - \mu C(x) - [C_h\lambda^{-1} + C_r][\mu - \lambda x]S^* \geq 0$$
$$[C_h\lambda^{-1} + C_r][\mu + S^*\lambda]\Phi(S^*) \geq \mu C(x) + [C_h\lambda^{-1} + C_r][\mu - \lambda x]S^* \quad (3.38)$$

Considering $\Delta_1 TC(x,S^*+1)$ and $\Delta_2 TC(x,S^*-1)$,

$$g_1(x,S^*+1) = g_1(x,S^*) + [C_h + C_r\lambda]\left[\Phi(S^*) + \frac{\mu}{\mu + (S^*+1)\lambda} + x\right] > 0 \quad (3.39)$$

$$g_2(x,S^*-1) = g_2(x,S^*) - [C_h + C_r\lambda]\left[\Phi(S^*) - \frac{\mu}{\mu + S^*\lambda} + x\right] < 0 \quad (3.40)$$

By the same procedure, Liu (1990) then has $g_1(x,S^*+2) > 0$ and$g_2(x,S^*-2) < 0$. Thus, through induction, the following can be established to complete the proof:

$$g_1(x,S) > 0 \textit{ if } S > S^* \quad (3.41)$$

and

$$g_2(x,S) < 0 \textit{ if } S < S^* \quad (3.42)$$

Noticing from (3.41) and (3.42) that TC(x, S) is strictly unimodal, the existence of a local minimum is almost certain unless the minimum is at S=0. When S^*=0, the optimal policy is to order one unit at a time to meet the demand just received.

3.1.2 Numerical Analysis

In this section we present numerical analysis to show the impact of perishability on the optimal policy with respect to cost parameters. And the model allows to stock out for all analysis. Firstly, we compare the impact of the order cost C_o and reorder level *s* on total cost (TC). Table 3.1 shows the total cost under these assumptions: C_s=15, μ =100, λ =0.1, C_r =15, C_u =15, C_h =4.5 and the same maximum inventory level $S=7$.

Table 3.1 The impact of the order cost (C_o) and reorders level (s) on total cost

C_o	**s**	**TC**
40	-5	-
40	-1	-
40	**1**	**268,9622**
40	5	679,2281
50	-5	-
50	**-1**	**41,46207**
50	1	389,5192
50	5	760,5631
60	-5	-
60	**-1**	**200,3225**
60	1	510,0761
60	5	841,8981

We draw a conclusion from Table 3.1. That when C_o =40, we pick the reorder level *s* =1 for minimizing the total cost. Same as when C_o =50 and C_o =60, the optimal *s* values are -1 and 1, respectively. And for the minimum total cost, the optimal *s* value is -1 and the optimal C_o value is 50, respectively.

Table 3.2 shows the impact of holding cost C_h and shortage cost C_s on total cost under these assumptions: S =7, μ=100, λ =0.1, C_r =15, C_u =15, C_o =50 and a fixed reorder level s = -1.

Table 3.2 The impact of holding cost (C_h) and shortage cost (C_s) on total cost

C_h	C_s	TC
5	**15**	**18,04335**
10	15	-
20	15	-
4.5	**5**	**38,28486**
4.5	10	39,87347
4.5	20	43,05068

In Table 3.2, when shortage cost C_s is fixed, if we increase holding cost C_h, total cost will be meanless. When holding cost C_h is stable, if we increase C_s, total cost will increase. Total cost is minimum, when C_h =5 and C_s =15.

Table 3.3 shows the impact replacement cost C_r and reorder level s on total cost with S =7, μ=100, λ =0.1, C_u =15, C_o =50, C_s =15, C_h =4.5.

Table 3.3 The impact of replacement cost (C_r) and reorders level (s) on TC

C_r	s	TC
5	-5	-
5	**-1**	**88,2995**
5	1	425,0635
5	5	784,5434
10	-5	-
10	**-1**	**64,88079**
10	1	407,2913
10	5	772,5532
15	-5	-
15	**-1**	**41,46207**
15	1	389,5192
15	5	760,5631

In Table 3.3, if we examined variance of reorder level s for different C_r values. For example for C_r =5 the optimal s =-1 and for C_r =10 and C_r =15 the optimal s =-1 too. For this reason s value is independent from C_r and always s =-1. If C_r increases, total cost would decrease. So the optimal C_r value has to be set 15 to find the minimum total cost.

Table 3.4 shows the impact of shortage cost C_u and reorder level s on total cost For S =7, μ=100, λ=0.1, C_o =50, C_s =15, C_h =4.5, C_r =15.

Table 3.4 The impact of shortage cost (C_u) and reorders level (s) on TC

s	**C_u**	**TC**
-10	0	-
-5	0	1525,599
-1	0	518,0434
1	0	389,5192
5	0	272,5532
10	**0**	**221,9447**
-10	15	4569,591
-5	15	-
-1	**15**	**41,46207**
1	15	389,5192
5	15	760,5631
10	15	755,4795

For minimizing the total cost, when C_u =0 and C_u =15 the optimal reorder level s =10 and s= -1, respectively. When *s* increases; if C_u =0, total cost decreases and if C_u =15 total cost is convex. For minimum Total cost, s and C_u should be -1 and 15, respectively.

We fixed the all costs and demand rate μ and reorder level *s*. We just change the failure rate λ to see the variation of total cost.

Table 3.5 The variation of failure rate λ

λ	**TC**
0,1	41,46207
1	231,1537
10	250,1228

When λ increases, the total cost increases too. So we must use the minimum λ to minimize the total cost. In Figure 3.5 shows the variation of failure rate λ on total cost with fixed maximum inventory level S =7, constant Poisson rate μ=100, order cost C_o =50, shortage cost C_s =15, holding cost C_h =4.5, replacement C_r =15 and reorder level s = -1.

3.2 Periodic Review Approach

Williams and Patuwo (1999) consider a single period, periodic review inventory system with different continuous demand distributions. The same model utilized with only a few differences relevant to demand distribution. In this model is similar to Liu (1990) model demand is assumed to be Poisson distribution. But there are some differences. For example, lead time is assumed to be a positive constant, lifetime is known and fixed (*m*=2 periods) and backlogs are not allowed. To determine the optimal incoming quantity for a single product for this model we followed Williams and Patuwo (1999) solutions.

At the start of the period 0, we must make a decision to order quantity, y_L, which will minimizing the total expected costs in period *L* and order will arrive at period *L*. *L* is defined as the lead time. In period 0, the total starting inventory composes the order quantity, y_0, ordered at the period (-*L*) and the aged inventory $X_0^1, X_0{}^2 \ldots X_0{}^{m-1}$

$X_0{}^{m-1}$: the inventory with (m-1) period useful lifetime remaining, in period 0.

In period *L*, the order quantity y_L be used to meet demands to start of the period *L* and the end of the period *L+m-1*. The remainder of the y_L will outdate and must be discarded.

3.2.1 The Model Definition and Assumptions

In this model Williams and Patuwo (1999) consider a single period, periodic review and positive lead time inventory model. Lead time is fixed and *L* period. The lifetime is *m* periods. So the actual lifetime of the items equal to '(m+L) period '. We used the same model but unlike Williams and Patuwo (1999) we assumed the demand distribution is Poisson.

$D_t(.)$: demand in any period *t*, *t*=0, 1, 2 …

$G_t(.)$: CDF of D_t $\Rightarrow G(t) = e^{-\lambda}.\sum_{i=0}^{k}\frac{\lambda^i}{i!}$

$g_t(.)$: pdf of D_t $\Rightarrow g(x) = e^{-\lambda}\frac{\lambda^x}{x!}$ for $x = 0,1,2,\ldots$

C_o: the ordering cost for per unit

C_h: inventory holding cost for per unit per period

C_u: unsatisfied demand or inventory shortage cost for per unit per period.

C_r: outdate cost for per period.

All costs are charged in the period in which the order arrives. Finally, all unsatisfied demands are assumed to be lost. We used the same notations with Williams and Patuwo (1999);

a_t : is the excess demand, $[D_t – X_t^1]^+$ in period t

q_t: density function of a_t

S_t: inventory shortage in period t

X_t^1: starting inventory with one period of life remaining in period t

$f_t(.)$: density function of X_t^1

y_t: incoming order quantity at the beginning period

O_t^{t+m-1}: the quantity of y_t that will outdate at the end of period t+m-1

3.2.2 The two period lifetime problem

Williams and Patuwo (1999) used life time m =2 for this model and determine the optimal quantity, y_L.

Total expected cost = expected order cost + expected holding cost + expected shortage cost + expected outdate cost

$$E[TC_L] = C_o.y_L + C_h.E[X_{L+1}^1] + C_u.\ E[S_L] + C_r.\ E[O_L^{L+1}] \qquad (3.43)$$

The starting inventory in period L+1 with one period of life remaining is;

$$X_{L+1}^1 = [y_L – (D_t – X_t^1)^+]^+ \qquad (3.44)$$

The inventory shortage in period L is;

$$S_L = [D_t – y_L – X_t^1]^+ \qquad (3.45)$$

The portion of the order quantity in period L, which will outdate at the end of the period L+1;

$$O_L^{L+1} = [y_L - D_{L+1} - (D_t - X_t^1)^+]^+ \qquad (3.46)$$

Let $f_t(x_t^1)$ represent the pdf for the starting inventory X_t^1, for t=1,2,3,…

For *t*=1;

- $x_1^1=0$ => the minimum demand is $y_0+x_0^1$ for period 0

  $P(D_0 \geq y_0+x_0^1) = 1-G_0(y_0+x_0^1) = f_1(0)$
- $x_1^1 \geq 0$ => $x_1^1 = y_0+x_0^1-D_0$ and $D_0=0$ => $x_1^1 = y_0+x_0^1$ the range of the x_1^1 equal to $0 \leq x_1^1 \leq y_0+x_0^1$

  $P(0<D_0< y_0-X_0^1- x_1^1) = g_0(X_0^1+y_0-x_1^1) = f_1(x_1^1)$

For *t*=2;

- $x_2^1=0$ => $f_2(0) = [1-G(y_1)].f_1(0) + \sum_{x_1^1=0}^{y_0}\left[1-G\left(y_1+x_1^1\right)\right].\, f_1(x_1^1)$
- $x_2^1 \geq 0$ => $f_2(x_2^1) = g_1(y_1-x_2^1)\, f1(0) + \sum_{x_1^1=0}^{y_0} g_1\left(y_1+x_1^1-x_2^1\right) f_1\left(x_1^1\right)$

At the end for *t*=1,

$$f_1(x_1^1) = \begin{cases} 1-G_0(y_0+x_0^1) & x_1^1=0 \\ g_0(x_0^1+y_0-x_1^1), & 0 \leq x_1^1 \leq X_0^1+y_0 \\ 0 & \text{otherwise} \end{cases} \qquad (3.47)$$

And for *t*=2, 3, 4…

$$f_t(x_t^1) = \begin{cases} [1-G_{t-1}(y_{t-1})].f_{t-1}(0) + \sum_{x_{t-1}^1=0}^{y_{t-2}} \left[1-G_{t-1}(y_{t-1}+x_{t-1}^1)\right] f_{t-1}(x_{t-1}^1) & , \quad x_t^1=0 \\ [g_{t-1}(y_{t-1}-x_t^1)].f_{t-1}(0) + \sum_{x_{t-1}^1=0}^{y_{t-2}} \left[g_{t-1}(y_{t-1}+x_{t-1}^1-x_t^1)\right] f_{t-1}(x_{t-1}^1) & , \quad 0 \leq x_t^1 \leq y_{t-1} \\ 0 & , \quad \text{otherwise} \end{cases} \qquad (3.48)$$

Consequently, the expected inventory holding cost in period t is;

$E[X_{t+1}^1] = E[E[X_{t+1}^1 \mid X_t^1]]$ =>

$$E[X_{t+1}^1] = \left[\left(\sum_{z_t=x_t^1}^{y_t} (y_t - z_t).\, g_t(z_t) \right) f_t(0) + \sum_{x_t^1=1}^{y_t-1} \sum_{z_t=x_t^1+1}^{x_t^1+y_t} (y_t + x_t^1 - z_t).\, g_t(z_t)\, f_t(x_t^1) \right] \qquad (3.49)$$

The expected shortage cost in period t is;

$E[S_t] = E[E[S_t \mid X_t^1]]$

$$E[St] = \left[\sum_{z_t=y_t}^{\infty} (z_t - y_t).\, g_t(z_t) f_t(0) + \sum_{x_t^1=1}^{y_t-1} \sum_{z_t=x_t^1+y_t+1}^{\infty} (z_t - y_t - x_t^1).\, g_t(z_t)\, f_t(x_t^1)\right] \quad (3.50)$$

To determine the expected outdate we need to determine the density function of $[D_t - X_t^1]^+$. q_t be the density function of and a_t be a random variable.

For $a_t=0$;

- $x_t^1=0$ => this is not possible because the demand cannot be negative.
- $x_t^1>0$ => $q_t(0)=\sum_{x_t^1=1}^{y_{t-1}} G\left(x_t^1\right).f_t\left(x_t^1\right)$

For $a_t>0$;

- $x_t^1=0$ => there is no imported product from previous period.
- $x_t^1>0$ => $q_t(a_t)=g(a_t).f_t(0) + \sum_{x_t^1=1}^{y_{t-1}} g\left(x_t^1 + a_t\right).f_t\left(x_t^1\right)$

For t=1, 2, 3 …

$$q_t(a_t) = \begin{cases} \sum_{x_t^1=0}^{y_{t-1}} \left[G_t(x_t^1)\right] f_t(x_t^1) & , \quad x_t^1 = 0 \\ [g_t(a_t)].f_t(0) + \sum_{x_t^1=0}^{y_{t-1}} \left[g_t(x_t^1 + a_t)\right] f_t(x_t^1) & , \quad 0 \le x_t^1 \le y_{t-1} \\ 0 & , \quad \text{otherwise} \end{cases} \quad (3.51)$$

In conclusion, the expected outdate cost is

$E[O_t^{t+1}] = E[E[O_t^{t+1} \mid (D_t - X_t^1)^+ = 0]]$

$$E[O_t^{t+1}] = \left[\sum_{z_{t+1}=0}^{y_t} \left(y_t - z_t\right) g_{t+1}\left(z_{t+1}\right)\right].q_t(0) + \sum_{a_t=0}^{\infty} \left[\sum_{z_{t+1}=0}^{y_t - a_t} \left(y_t - a_t - z_{t+1}\right) g_{t+1}(z_{t+1})\right].q_t(a_t) \quad (3.52)$$

By the substituting Equations (3.49), (3.50) and (3.52) into Equation (3.43) the total expected cost is;

$$E(TC_L)=C_o y_L + C_h\left[\sum_{z_L=0}^{y_L}(y_L - z_L)g(z_L).f(0) + \sum_{x_L^1=1}^{y_L-1}\sum_{z_L=x_L^1+1}^{x_L^1+y_L}(y_L + x_L^1 - z_L)g(z_L).f_L(x_L^1)\right]$$

$$+C_u\left[\sum_{z_L=y_L}^{\infty}(z_L - y_L)g(z_L).f(0) + \sum_{x_L^1=1}^{y_L-1}\sum_{z_L=y_L+x_L^1+1}^{\infty}(z_L - y_L - x_L^1)g(z_L).f_L(x_L^1)\right]$$

$$+C_r\left[\sum_{z_{L+1}=0}^{y_L}(y_L - z_{L+1})g(z_{L+1}).q_L(0) + \sum_{a_L=0}^{\infty}\sum_{z_{L+1}=0}^{y_L-a_L}(y_L - a_L - z_{L+1})g(z_{L+1})q_L(a_L)\right]$$

(3.53)

Williams and Patuwo (1999) used Equation (3.53) to compute the total inventory cost in any period for a perishable item with two period lifetimes, an order lead time of *L* period and C_o, C_h, C_u and C_r are per unit ordering, holding, shortage and outdate cost respectively. Also Williams and Patuwo (1999) determine the optimal order quantity with the same equation.

To determine the order quantity that minimizes the cost function is given by setting the first derivative of the total expected cost function with respect to y_L, equal to zero.

$$E(TC_L)=C_o y_L + C_h\left[\sum_{z_L=0}^{y_L}(y_L - z_L)g(z_L).f(0) + \sum_{x_L^1=1}^{y_L-1}\sum_{z_L=x_L^1+1}^{x_L^1+y_L}(y_L + x_L^1 - z_L)g(z_L).f_L(x_L^1)\right]$$

$$+C_u\left[\sum_{z_L=y_L}^{\infty}(z_L - y_L)g(z_L).f(0) + \sum_{x_L^1=1}^{y_L-1}\sum_{z_L=y_L+x_L^1+1}^{\infty}(z_L - y_L - x_L^1)g(z_L).f_L(x_L^1)\right]$$

$$+C_r\left[\sum_{z_{L+1}=0}^{y_L}(y_L - z_{L+1})g(z_{L+1}).q_L(0) + \sum_{a_L=0}^{\infty}\sum_{z_{L+1}=0}^{y_L-a_L}(y_L - a_L - z_{L+1})g(z_{L+1})q_L(a_L)\right]$$

From Equation (3.49),

$$E[X_{L+1}^1] = \left[\sum_{z_L=0}^{y_L}(y_L - z_L)g(z_L).f(0) + \sum_{x_L^1=1}^{y_L-1}\sum_{z_L=x_L^1+1}^{x_L^1+y_L}(y_L + x_L^1 - z_L)g(z_L).f_L(x_L^1)\right]$$

$$\frac{dE[x_{L+1}^1]}{dy_L} = \frac{d}{dy_L}\left[\left(\sum_{z_L=0}^{y_L}(y_L - z_L)g_L(z_L)f(0) + \frac{\partial}{\partial y_L}\sum_{x_L^1=1}^{y_L-1}\sum_{z_L=x_L^1}^{x_L^1+y_L}(y_L + x_L^1 - z_L)\, f_L(x_L^1)\right)\right]$$

$$= \left\{[g_L(0) + \cdots + g_L(y_L)] + \left[\sum_{z_L=0}^{y_L} g_L^1(z_L)(y_L - z_L)\right]\right\} f(0)$$

$$+ \left\{\sum_{x_L^1=1}^{y_L-1}[g(x_L^1 + 1) + g(x_L^1 + 2) + \cdots + g(x_L^1 + y_L)]f_L(x_L^1)\right\}$$

$$\frac{dE[x_{L+1}^1]}{dy_L} = G_L(y_L)f_L(0) + \sum_{x_L^1=0}^{y_{L-1}} G_L(y_L + x_L^1)f_L(x_L^1) \tag{3.54}$$

$$E[S_L] = \left[\sum_{z_L=y_L}^{\infty}(z_L - y_L)g(z_L).f(0) + \sum_{x_L^1=1}^{y_L-1}\sum_{z_L=y_L+x_L^1+1}^{\infty}(z_L - y_L - x_L^1)g(z_L).f_L(x_L^1)\right]$$

$$\frac{dE[S_L]}{dy_L} = \frac{d}{dy_L}\left[\sum_{z_L=y_L}^{\infty}(z_L - y_L)g(z_L).f(0) + \sum_{x_L^1=1}^{y_L-1}\sum_{z_L=y_L+x_L^1+1}^{\infty}(z_L - y_L - x_L^1)g(z_L).f_L(x_L^1)\right]$$

$$= -(g(y_L + 1) + g(y_L + 2) + \cdots).f(0)$$

$$- \sum_{x_L^1=1}^{y_L-1}(g(y_L + x_L^1 + 1) + g(y_L + x_L^1 + 2) + \cdots).f_L(x_L^1)$$

$$= -[1 - G(y_L)]f(0) - [1 - G(y_L + x_L^1)]f_L(x_L^1)$$

$$\frac{dE[S_L]}{dy_L} = [G(y_L) - 1]f(0) + \sum_{x_L^1=1}^{y_L-1}[G(y_L + x_L^1) - 1].f_L(x_L^1) \tag{3.55}$$

$$E[O_L^{L+1}] = \left[\sum_{z_{L+1}=0}^{y_L}(y_L - z_{L+1})g(z_{L+1}).q_L(0) + \sum_{a_L=0}^{\infty}\sum_{z_{L+1}=0}^{y_L-a_L}(y_L - a_L - z_{L+1})g(z_{L+1})q_L(a_L)\right]$$

$$\frac{dE[O_L^{L+1}]}{dy_L} = \frac{d}{dy_L}\left[\sum_{z_{L+1}=0}^{y_L}(y_L - z_{L+1})g(z_{L+1}).q_L(0) + \sum_{a_L=0}^{\infty}\sum_{z_{L+1}=0}^{y_L-a_L}(y_L - a_L - z_{L+1})g(z_{L+1})q_L(a_L)\right]$$

$$= (g(0) + g(1) + \cdots + g(y_L)).q_L(0)$$

$$+ \sum_{a_L=0}^{\infty}(g(0) + g(1) + \cdots + g(y_L - a_L)).q_L(a_L)$$

$$\frac{dE\left[O_L^{L+1}\right]}{dy_L} = G(y_L).\,q_L(0) + \sum_{a_L=0}^{\infty}\left(G(y_L - a_L)\right).\,q_L(a_L) \qquad (3.56)$$

Substituting Eq(3.54)-(3.56) into Eq(3.53) and combining terms;

$$\frac{dE[TC_L]}{dy_L} = C_o - C_u f_L(0) + (C_h + C_u)\left[G_L(y_L)f_L(0) + \sum_{x_L^1=0}^{y_{L-1}} G_L(\,y_L + x_L^1)f_L(x_L^1)\right]$$

$$-C_h \sum_{x_L^1=0}^{y_{L-1}} G_L(\,x_L^1\,)f_L(x_L^1) - C_u \sum_{x_L^1=0}^{y_{L-1}} f_L(x_L^1)$$

$$+C_r\left[G_{L+1}(y_L)q_L(0) + \sum_{a_L=0}^{\infty} G_{L+1}(\,a_L - y_L\,)q_L(a_L)\right] = 0 \qquad (3.57)$$

The convexity of Eq(3.57) is establish by showing the second derivative with respect to y_L to be positive,

$$\frac{d^2E[TC_L]}{dy_L^2} = (C_h + C_u)\frac{d}{dy_L}\left[G_L(y_L)f_L(0) + \sum_{x_L^1=0}^{y_{L-1}} G_L(\,y_L - x_L^1)\right]f_L(x_L^1)$$

$$- C_h \frac{d}{dy_L}\left[\sum_{x_L^1=0}^{y_{L-1}} G_L(\,x_L^1\,)f_L(x_L^1)\right] - C_u \frac{d}{dy_L}\left[\sum_{x_L^1=0}^{y_{L-1}} f_L(x_L^1)\right]$$

$$+ C_r \frac{d}{dy_L}\left[G_{L+1}(y_L)q_L(0) + \sum_{a_L=0}^{\infty} G_{L+1}(\,y_L - a_L\,)q_L(a_L)\right]$$

$$= (C_h + C_u)\left[g_L(y_L)f_L(0) + \sum_{x_L^1=0}^{y_{L-1}} g_L(\,y_L - x_L^1)\right] - C_h(0) - C_u(0)$$

$$+ C_r\left[g_{L+1}(y_L)q_L(0) + \sum_{a_L=0}^{\infty} g_{L+1}(\,y_L - a_L\,)q_L(a_L)\right]$$

$$= (C_h + C_u)\left[g_L(y_L)f_L(0) + \sum_{x_L^1=0}^{y_{L-1}} g_L(\,y_L - x_L^1)\right]$$

$$+C_r\left[g_{L+1}(y_L)q_L(0) + \sum_{a_L=0}^{\infty} g_{L+1}(\,y_L - a_L\,)q_L(a_L)\right] > 0 \qquad (3.58)$$

$$\frac{d^2E[TC_L]}{dy_L^2} > 0$$

The holding, shortage and outdate costs are all positive and the density function g(.), f_L(.), q_L(.) are all nonnegative. So that the Eq(3.58) must be positive.

3.2.3. Experimental results

The experimental investigation for optimal order quantities will be computed for lead times (L=1, 2, 3), for the Poisson demand distribution two different level of mean demands (λ=5, 10) and for three different starting order quantity (y_0=5, 10, 15) with a fixed lifetime (m=2). Examinations of the experimental results, shown in Tables 3.6 – Tables 3.8, indicate that the optimal incoming quantity is a function of the Lead time. The behavior of the optimal order quantities are differentiated in;

1. When L=1
2. When L=2 (L=m)
3. When L>m

For L=1, the optimal incoming quantity for one period lead time problem, illustrate in Figure 1, which shows y_1^* for L=1, λ=5, 10 and stationary Poisson demand distribution and the starting order quantity y_0=5, 10, 15.

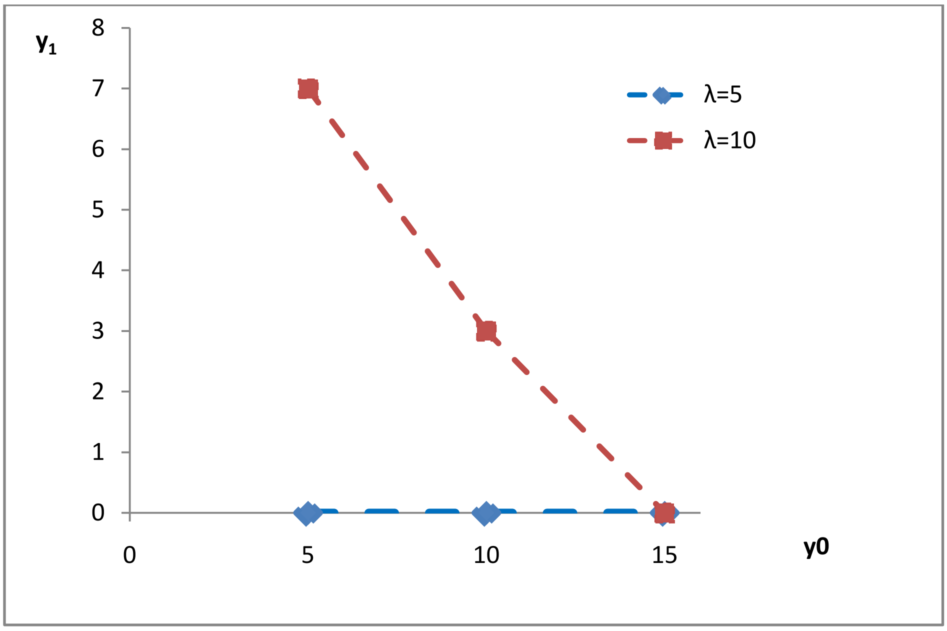

Figure 3.6 Optimal incoming quantity y_1^* for L=1

In Table 3.6, for the mean demand λ=5 units per period, the optimal order quantity in period 1, y_1^*, is equal to 0 when the starting inventory for period 1 is $X_0^1+y_0 \geq k$ (k is equal to 10). This implies that it is optimal not to order under this condition. Where the mean demand is λ=10 units per period, k is increasing to 20. Under these conditions we make a decision by using y_0, because x_0 is a fixed number. For λ=10;

- When $y_0=5$, the optimal order quantity $y_1^*=7$ units.
- When $y_0=10$, the optimal order quantity $y_1^*=3$ units.
- When $y_0=15$, $X_0^1+y_0 \geq 20$, it is optimal not to order $y_1^*=0$.

Table 3.6 Optimal incoming quantity y_1^* for L=1

	λ=5			λ=10		
	y_0=5	y_0=10	y_0=15	y_0=5	y_0=10	y_0=15
y_1=0	**2,20445**	**4,6213**	**4,9872**	-2,7208	-1,6864	**1,4786**
y_1=1	3,0357	4,83255	4,99645	-2,6428	-1,0654	2,3286
y_1=2	3,8202	4,9663	5,00095	-2,4938	-0,32415	3,1316
y_1=3	4,51095	5,05305	5,0037	-2,2408	**0,50835**	3,91685
y_1=4	5,1352	5,11955	5,00595	-1,8433	1,3936	4,77735
y_1=5	5,75845	5,1873	5,0077	-1,2753	2,28785	5,82285
y_1=6	6,40095	5,2848	5,0097	-0,5338	3,1546	7,1721
y_1=7	6,9877	5,4483	5,0132	**0,3517**	3,9686	8,87535
y_1=8	7,42045	5,71405	5,02195	1,3232	4,71535	10,8881
y_1=9	7,6772	6,08955	5,0422	2,3027	5,38385	12,59685
y_1=10	7,81045	6,5428	5,0872	3,21995	5,9656	15,33885

For *L*=2, when we increase the order lead time to two periods, Figure 3.7 shows y_2^* with a Poisson demand distribution and order quantity for period 1, y_1. Also, we used two different mean demands λ=5, 10 to constitute Figure 3.7.

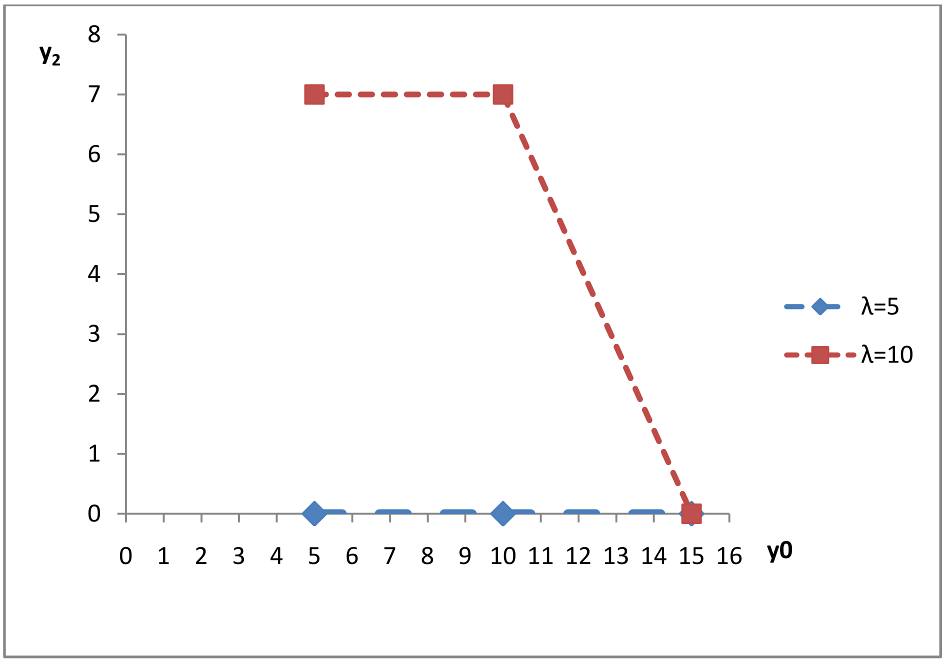

Figure 3.7 Optimal incoming quantity y_2^* for L=2

In Table 3.7, similar to the case where L=1, for the mean demand λ=5 units per period, the optimal order quantity in period 2, y_2^*, is equal to 0 when the starting inventory in period is 2, $X_0^1+y_0+y1\geq k$ (k is not constant) and this implies that it is optimal not to order under this condition. For λ=10;

- When y_0=5, the optimal order quantity y_2^*=7 units.
- When y_0=10, the optimal order quantity y_2^*=7 units.
- When y_0=15, it is optimal not to order y_2^*=0.

Table 3.7 Optimal incoming quantity y_2^* for L=2

	λ =5			λ=10		
	y_0=5	y_0=10	y_0=15	y_0=5	y_0=10	y_0=15
	y_1=0	y_1=0	y_1=0	y_1=7	y_1=3	y_1=0
y_2=0	2,21075	4,476	4,97675	-1,51745	-0,94035	1,4912
y_2=1	2,218	4,4775	4,97775	-1,67795	-1,11585	1,3747
y_2=2	2,44875	4,521	4,97875	-1,7827	-1,25035	1,27095
y_2=3	2,938	4,6135	4,98275	-1,7882	-1,2991	1,2052
y_2=4	3,604	4,7375	4,98825	-1,6352	-1,2036	1,21145
y_2=5	4,29475	4,8675	4,99375	-1,2682	-0,9036	1,33245
y_2=6	4,88225	4,9775	4,99825	-0,65245	-0,3601	1,6052
y_2=7	5,30675	5,0575	5,00225	0,20355	0,4204	2,03595
y_2=8	5,574	5,10725	5,004	1,24205	1,38315	2,5992
y_2=9	5,7245	5,13525	5,005	2,36255	2,4324	3,2397
y_2=10	5,799	5,14875	5,006	3,45805	3,4639	3,8892

Comparing Table 3.6 and Table 3.7 the optimal incoming quantities for period 1 and period 2 when λ=5, is found as y_1^*=0, y_2^*=0 respectively for all values of y_0. This implies that for the mean demand is equal to 5; and when the starting inventory is equal to 5 and the inventory on order is equal to three different values, the optimal incoming quantity is independent of the order lead time. When λ=10, we examined optimal order quantities for different values of y_0. For y_0=5 we find the optimal order quantity for L=1 and L=2 as y_1^*=7 and y_2^*=7, respectively. The optimal order quantities are equal to the same values It shows that the optimal incoming quantity is independent of the order lead time. For y_0=10 we find the optimal order quantity for L=1 and L=2 as y_1^*=3 y_2^*=7, respectively. The optimal order quantities are equal to different values. This shows that; if lead time increases, order quantity would increase too. For y_0=15 we find the optimal order quantity for L=1 y_1^*=0 and for L=2 y_2^*=0. As in the case of y_0=5, the optimal incoming quantity is independent of the order lead time for y_0=15.

For L=3, Figure 3.8 shows y_3^* as a function of y_0 and y_2 for Poisson demand distribution with mean demand λ=5, 10 and three different value of y_0.

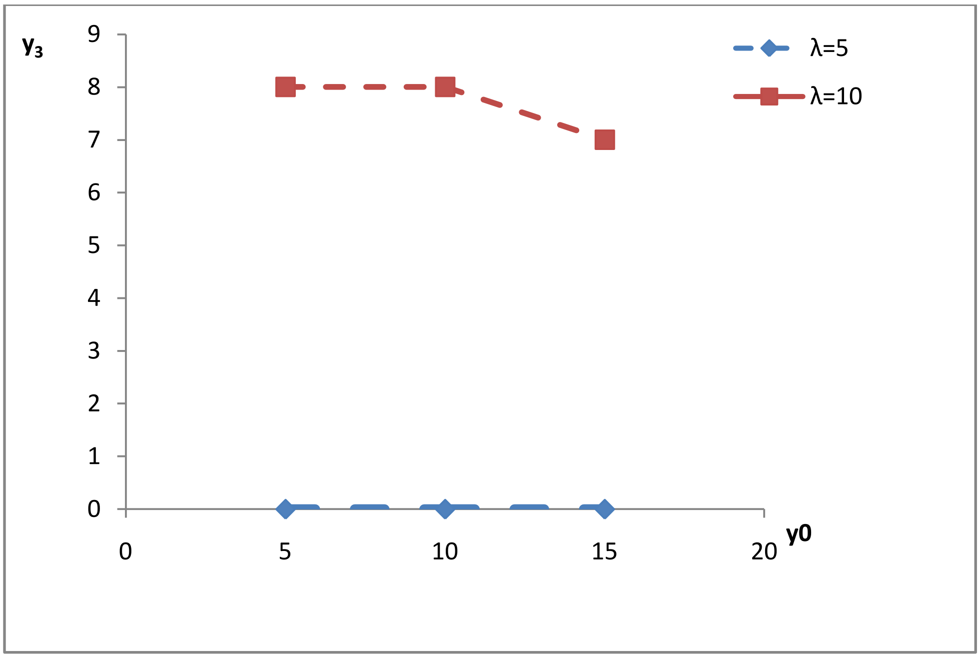

Figure 3.8 Optimal incoming quantity y_3^* for L=3

Table 3.8 shows the optimal order quantity for L=3 similar to L=1 and L=2 problems, for the mean demands λ=5 and λ=10 units per period, the optimal order quantity in period 2, y_3^*.

Table 3.8 Optimal incoming quantity y_3^* for L=3

	λ=5			**λ=10**		
	y_0=5	**y_0=10**	**y_0=15**	**y_0=5**	**y_0=10**	**y_0=15**
	y_2=0	**y_2=0**	**y_2=0**	**y_2=7**	**y_2=7**	**y_2=0**
y_3=0	**2,21175**	**4,4772**	**4,97675**	-7,34915	-2,49365	1,49335
y_3=1	2,219	4,47845	4,97775	-7,39865	-2,67515	1,3776
y_3=2	2,4495	4,5222	4,97875	-7,23965	-2,7834	1,2741
y_3=3	2,93875	4,61345	4,98275	-6,7839	-2,75865	**1,2071**
y_3=4	3,6035	4,73845	4,98825	-5,9244	-2,5224	1,21285
y_3=5	4,294	4,86745	4,99375	-4,5629	-1,99565	1,3331
y_3=6	4,88125	4,9782	4,99825	-2,65065	-1,12665	1,6046
y_3=7	5,3065	5,0582	5,00225	-0,2364	**0,07285**	2,0321
y_3=8	5,57375	5,10795	5,004	**2,52535**	1,52085	2,59285
y_3=9	5,72325	5,13595	5,005	5,4016	3,08135	3,23035
y_3=10	5,79775	5,14945	5,006	8,14085	4,60335	3,87585

Comparing the Table 3.6, Table 3.7, Table 3.8 the optimal incoming quantity when λ=5, y_3^*=0, y_2^*=0 and y_1^*=0 all values of y_0. This implies that order quantity is independent of the order lead time. When λ=10, we examined different values of y_0. Firstly, find optimal order quantities for all the lead time values.

y_0=5 → y_1^*=7, y_2^*=7, y_3^*=8.

y_0=10 → y_1^*=3, y_2^*=7, y_3^*=7

y_0=15 → y_1^*=0, y_2^*=0, y_3^*=3.

This implies that; if lead time increases, order quantity would increase too.

CHAPTER FOUR
CONCLUSION

In this study we considered a periodic review policy for perishable items with Poisson demand, life time of two periods, positive order lead time and a lost sales policy. We determined the single period optimal incoming quantity by the help of total expected cost function.

The purpose of this work is to present the form and properties of the optimal incoming quantity for two different mean demand (λ=5, 10), three different starting order quantity (y_0=5, 10, 15) and three different order lead time (L=1, 2, 3) with Poisson demand and starting inventory X_0=5. For L=1, the optimal incoming quantity y_1^* is obtained as a function of y_0. When the λ=5, we do not need the optimal order policy is not to order any level of y_0. When λ=10, the optimal quantity is depend on y_0. The sum of the initial inventory level and the order quantity at the beginning is greater than k (X_0+ y_0 $\geq$k). When k equal to 20 the optimal policy is not to order. If k is between 15 and 20 the optimal order quantity is y_1^*=3 and when k is between 10 and 15 the optimal order quantity is y_1^*=7. As a result, if X_0+ y_0 increases, the optimal order quantity will also decrease.

For L=2 and L=3, the optimal incoming quantity y_2^* and $y_3^*$are obtained as the functions of y_2, y_1 and y_0. When the λ=5, the optimal policy is not to order at any level of y_0. It is shown that the optimal order quantity and y_0 are independent. When λ=10, the optimal quantity depends on y_0. X_0+ y_0 $\geq$k and when k is equal to 20, the optimal policy is not to order. When k is between 15 and 20 the optimal order quantity is y_2^*=7 and when k is between 10 and 15 the optimal order quantity is y_2^*=7. If X_0+ y_0 increases, the optimal order quantity decreases for L=2. Moreover, for L=3, the same relationship X_0+ $y_0$$\geq$k is relevant. If k equals to 20 the optimal order quantity is y_3^*=8, when k is between 15 and 20 the optimal order quantity is found as y_3^*=7 and when k is between10 and 15 the optimal order quantity is y_3^*=3. If X_0+ y_0 increases, the optimal order quantity would increase, too.

In this work we demonstrate the ability to compute the single period, single item optimal incoming quantity for product with fixed life time of two periods, with Poisson demand and have positive order lead time. Further research needs to be studying on the other discrete distributions and the effect of the characteristic inventory costs on the optimal order quantity as well as the total inventory costs.

REFERENCES

Fries, B. E. (1975). Optimal ordering policy for a perishable commodity with fixed lifetime. *Operations Research, 23,* 46-61.

Goyal, S., & Giri, B. (2001). Recent trends in modeling of deteriorating. *European Journal of Operational Research, 134* (1), 1-16.

Gürler, Ü., & Özkaya, B. Y. (2008). Analysis of the (s,S) policy for perishable with a random shelf life. *IIE Transactions, 40* (8), 759-781.

Jain, K., & Silver, E. A. (1994). Lot-Sizing for a Product Subject to Obsolescence or Perishability. *European Journal of Operational Research, 75.*

Kalpakam, S., & Arivarignan, G. (1988). A continuous Review Perishable inventory model. *Statistics, 19,* 389-398.

Kalpakam, S., & Sapna, K. P. (1994). Continuous review (s,S) inventory system with random lifetimes and positive leadtimes. *Operations Research Letters, 16,* 115-119.

Kalpakam, S., & Shanthi, S. (2006). A continuous review perishable system with renewal demands. *Annals of Operations Research, 143* (1), 211-225.

Kouki, C., Sahin, E., Jemai, Z., & Dallery, Y. (2010). Periodic review inventory policy for perishables with random lifetime. *Eighth International Conference of Modeling and Simulation.*

Lian, Z., & Liu, L. (1999). A discrete-time model for perishable inventory systems. *Annals of Operations Research, 87,* 103-116.

Lian, Z., & Liu, L. (2001). Continuous review perishable inventory systems: models and heuristics. *IIE Transactions, 33* (9), 809-822.

Lian, Z., Liu, L., & Zhao, N. (2009). A perishable inventory model with Markovian renewal demands. *International Journal of Production Economics, 121* (1), 176-182.

Liu, L. (1990). (s,S) continuous review models for inventory with random lifetimes. *Operations Research Letters, 9,* 161-167.

Minner, S., & Transchel, S. (2010). Periodic review inventory control for perishable products under service-level constraints. *OR Spectrum, 32, (4),* 979-996.

Nahmias, S. (1975). Optimal ordering policies for perishable inventory-II. *Operations Research, 23*, 735-749.

Nahmias, S. (1977). Higher order approximations for the perishable inventory problem. *Operations Research, 25*, 630-640.

Nahmias, S. (1982). Perishable inventory theory: A review. *Operations Research, 30* (4), 680-708.

Nahmias, S., & Pierskalla, W. P. (1973). Optimal ordering policies for product that perishes in two periods subject to stochastic demand. *Naval Research Logistics Quarterly, 20* (2), 207-229.

Nandakumar, P., & Morton, T. E. (1993). Near myopic heuristics for the fixed-life perishable inventory problem. *Management science, 39,* 1490-1498.

Raafat, F. (1991). Survey of literature on continuously deteriorating inventory model. *Journal of the Operational Research Society, 42*, 27-37.

Uckun C., F. Karaesmen, & S. Savas. (2008). Investment in Improved Inventory Accuracy in a Decentralized Supply Chain. *International Journal of Production Economics*, *113*, 546-566.

Van Donselaar, K., van Woensel, T., Broekmuelen, R., & Fransoo, J. (2006). Inventory control of perishables in supermarkets. *Operations Research*, *104*, 462-472.

Van Zyl, G. J. J. (1964). Inventory control for perishable commodities. *Ph. D. Dissertation, University of North Carolina.*

Veinott, A. F. (1960). Ordering and Disposal of Inventory with FIFO or LIFO Issuing and Known Demands. *Technical Report No. 5*, *Statistical Engineering Group, Columbia University*.

Wagner, H. M., & Whitin, T. (1958). Dynamic version of the economic lot size model. *Management Science*, *5*, 89-96.

Williams, C. L., & Patuwo, B. E. (1999). A perishable inventory model with positive order lead times. *European Journal of Operational Research*, *116*, 352-373.

Winston, W. L. (2004). *Operation research- Aplications and algorithms*. USA: Duxbury Press.

Zipkin, P. H. (2000). *Foundations of inventory management*. McGraw-Hill: New York.

APPENDIX
SIMULATION CODE FOR THE EXPECTED COSTS WITH POSITIVE LEAD TIME MODELS

A1 Simulation code for density function for the starting inventory with L=1

```
t1=0;
lamda=15;
x0=5; y0=15;
x=x0+y0;
        for i=0:x
           t1=t1+((lamda^i)/factorial(i));
        end
        f1=1-((1/exp(lamda))*t1);
        fprintf('.a.=%4.4f\n', f1);
```

A2 Simulation code for starting inventory in period t with L=1

```
t1=0;
f1=0.0830;
lamda=15;

for y=0:10
for i=0:y
    t1=t1+((1/exp(lamda))*((lamda^i)/factorial(i)));
end

y1(1,(i+1))=(t1*f1);
t1=0;
end

for i=1:11
fprintf('....y......=%4.4f\n', y1(1,i));
end
```

and

```
t1=0;t2=0;t3=0;
lamda=15;
y0=15;
x0=5;

for y=0:10
    for x=1:y0
        for i=0:(x+y)
           t1=t1+((1/exp(lamda))*((lamda^i)/factorial(i)));
```

```
        end
          t2=((1/exp(lamda))*((lamda^((x0+y0)-x))/factorial((x0+y0)-x)));

    t3=t3+(t1*t2);
    t1=0;t2=0;
    end
    y1(1,(y+1))=t3;
 t3=0;
end
for i=1:11
fprintf('....b2................=%4.4f\n', y1(1,i));
end
```

A3 Simulation code for shortage inventory in period t with L=1

```
t1=0;t2=0;t3=0;
lamda=15;
x0=5;y0=15;
        for x=0:y0
             t2=((1/exp(lamda))*((lamda^((x0+y0)-x))/factorial((x0+y0)-x)));
           for i=0:x
             t1=t1+((1/exp(lamda))*((lamda^i)/factorial(i)));

           end
           t3=t3+(t1*t2);
           t1=0;t2=0;
        end

    fprintf('....c......=%4.4f\n', t3);
```

and

```
t1=0;t2=0;
y0=15;x0=5;
lamda=15;

        for x=0:y0
             t1=((1/exp(lamda))*((lamda^((x0+y0)-x))/factorial((x0+y0)-x)));

           t2=t2+t1;
           t1=0;
        end

    fprintf('....d......=%4.4f\n', t2);
```

A4 Simulation code for outdate inventory in period t with L=1

```
t1=0;t2=0;t4=0;
lamda=15;
x0=5;y0=15;
for y=0:10
   for i=0:y
   t1=t1+((1/exp(lamda))*((lamda^i)/factorial(i)));
   end

   for x=0:y
      for j=0:x
      t2=t2+((1/exp(lamda))*((lamda^j)/factorial(j)));
      end
   t3=((1/exp(lamda))*((lamda^((x0+y0)-x))/factorial((x0+y0)-x)));
   t4=t4+(t2*t3);

 t5=t4*t1;
 t2=0;
   end


y1(1,(y+1))=t5;
t1=0;t4=0;

end

for i=1:11
fprintf('....y......=%4.4f\n', y1(1,i));
end
```

and

```
t1=0;t2=0;t3=0;t7=0;t4=0;t5=0;t6=0;t=0;
lamda=15;
x0=5; y0=15;
f1=0.0830;

for y=0:10
  for a=0:(x0+y0)

      for i=0:(a-y)
        t1=t1+((1/exp(lamda))*((lamda^i)/factorial(i)));
      end
      t2=f1*((1/exp(lamda))*((lamda^a)/factorial(a)));


      for x=1:y0
```

```
        t4=((1/exp(lamda))*((lamda^(x+a))/factorial(x+a)));
      t5=((1/exp(lamda))*((lamda^((x0+y0)-x))/factorial((x0+y0)-x)));
      t=t+(t4*t5);
      end


  t6=t+t2;
  t7=t7+(t1*t6);
  t1=0;t=0;

 end
  y1(1,(y+1))=t7;
  t7=0;
end

for i=1:11
fprintf('....y................=%4.4f\n', y1(1,i));
end
```