DOKUZ EYLÜL UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

HANDLE DECOMPOSITIONS OF 4-DIMENSIONAL SMOOTH MANIFOLDS

by

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İZMİR

HANDLE DECOMPOSITIONS OF 4-DIMENSIONAL SMOOTH MANIFOLDS

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M.Sc. THESIS EXAMINATION RESULT FORM

We have read the thesis entitled **"HANDLE DECOMPOSITIONS OF 4-DIMENSIONAL SMOOTH MANIFOLDS"** completed by **EYLEM ZELİHA YILDIZ** under supervision of **ASST. PROF. DR. AHMET Z.ÖZÇELİK** and **PROF. DR. BURAK ÖZBAĞCI** and we certify that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

The topology of manifold theory relates to many diverse fields of mathematics such as abstract algebra, differential and algebraic geometry, and analysis. Therefore there are many approaches to manifold theory among mathematicians.

Our study based on topological viewpoint relating algebraic topology and geometric topology. In this thesis, we study handlebodies of four dimensional closed connected smooth manifolds. In order to work with smooth handles we use handle operations. These operations are basically handle sliding, handle cancelling and carving. Then by using these operations we investigate Gluck twisting.

four dimensional smooth manifolds has its own significance among other manifolds, its importance is related to the classification problem. All techniques that are used in this thesis historically have been developed over the years to classify smooth four dimensional manifolds.

Keywords: Four dimensional smooth manifolds, handlebody, handle sliding, handle cancelling, carving, Gluck twist.

4-BOYUTLU PÜRÜZSÜZ MANİFOLDLARIN KULP DAĞILIMLARI ÜZERİNE

ÖZ

Manifold teorisi, cebir, differansiyel, cebirsel geometri ve analiz gibi matematiğin birçok alanı ile ilişkilidir. Bu sebeple matematikçiler arasında manifold teorisine birçok yaklaşım bulunmaktadır.

Biz manifold teorisine topolojik bir bakış açısıyla yaklacsacağız dolayısıyla bu çalışma cebirsel ve geometrik topoloji ile ilişkili olacak. Bu tezde dört boyutlu, pürüzsüz, kapalı ve baglantılı manifoldların kulp yapıları incelenecek. Kulplarla çalışmanın bazı avantajları bulunmaktadır. Örneğin kulplar üzerinde çeşitli operasyonlar tanımlanabilir bunlar kulpların kaydırılması, iptali ve oyulması işlemleridir. Ayrıca bu calışmada Gluck twist operasyonu incelenecektir.

Bilindiği gibi dört boyutlu pürüzsüz manifoldların sınıflandırılma açısından özel bir önemi bulunmaktadır. Bu tezde çalışılan bütün teknikler aslında bu sınıflandırma problemine olası çözüm yaklaşımları olarak geliştirilmişlerdir.

Anahtar sözcükler : Dört boyutlu pürüzsüz manifoldlar, kulpların kaydırılması, kulpların oyulması, kulpların iptali, Gluck twist.

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CHAPTER ONE
INTRODUCTION

The manifold theory goes back to the late 1800's. It is first introduced by Riemann as higher dimensional analogues of surfaces and curves which are 2 and 1-dimensional manifolds, respectively. Shortly after Henri Poincare studied 3 and 4-dimensional manifolds, he conjectured some classification problems of the low dimensional manifolds. His main conjecture states: Every simply connected, closed 3-manifold is homeomorphic to the 3-sphere. Later his conjecture have been generalized to the conjecture "Every homotopy n-sphere is homeomorphic to n-sphere". In 1961 Stephan Smale and John Stallings proved this conjecture for the manifolds of dimension grater than 4. In 1982 Michael Freedman proved this conjecture in dimension four. Finally, 3-dimensional case was resolved by Grigori Perelman in 2003.

There is also a smooth version of this question. It is known that in dimensions greater than four there could be smoothly exotic spheres which were first discovered by John Milnor and Michael Kerveire. But in dimension four this is still not known, more specifically it is not known whether there is a smooth 4-manifold which homemorphic but not diffeomorphic to the 4-sphere. Until now, a lot of results related to this conjecture have been proven by using various approaches. But the main problem remains open. This fact makes the smooth 4-manifolds special among others.

There are many books available for understanding manifold theory in general, but our main goal here is to understand 4-dimensional smooth manifolds in terms of handlebody theory which is related to the Morse theory. The advantage of the handlebody theory lies in the fact that in 4-dimension it is a powerful method to visualise them. Moreover, the tools of handle sliding, cancelling and carving makes this approach particularly useful in understanding 4-manifolds. The standard textbooks on the theory of handlebody are Akbulut (2012), Gompf & Stipsicz (1999). Milnor et al. (1965), and Matsumoto (2002).

In recent years, the value of using handlebody theory has emerged as a promising tool. Indeed it provides an alternative way to see many problems in geometric topology and allows to use impressive and computable techniques.

This thesis has 5 chapters. Next we discuss the content of these chapters.

In Chapter 2, as a motivation we start with some basic definitions about general notion of the manifold theory. Then, we discuss handles in general dimension. After that by using Morse theory and notion of framing we examine visualisation of 4-dimensional smooth manifolds. Afterwards, we discuss intersection form of smooth 4-manifolds, and give examples of some smooth 4-dimensional handlebodies.

In Chapter 3, we investigate the general notion of the surgery operations. Then we introduce our main concepts, namely sliding, cancelling and carving handles. As an application we construct the diffeomorphism $S^2 \times S^2 \# \mathbb{CP}^2 \cong \mathbb{CP}^2 \# \overline{\mathbb{CP}^2} \# \mathbb{CP}^2$, which was originally proved by Hirzebruch.

In chapter 4, we will give a brief introduction to Gluck construction operation. Gluck construction is an important technique in 4-dimensional smooth manifold theory. For example many candidates of exotic 4-sphere were obtained by Gluck twisting to S^2 in S^4. After introduction we will investigate the Gluck construction operation by using handles. Finally, we end our work by giving some examples to demonstrate the methods. The basic reference for this notion is Akbulut (2012).

CHAPTER TWO
PRELIMINARIES

Here we give a brief introduction to some fundamental notions of handle decomposition of 4-manifolds which are necessary to understand the deeper handle theory.

2.1 Basic Definitions

Definition 2.1.1. A second countable, Hausdorff topological space X is *a n-dimensional topological manifold* if
$(\forall p \in X)(\exists U^{open} \subset X)(p \in U)$ such that $\exists f : U^{open} \rightarrow V^{open} \subset \mathbb{R_+}^n$ is an homeomorphism where $\mathbb{R_+}^n = \{(x_1, \dots, x_n) \mid x_n \geq 0\}$ is the upper half space of $\mathbb{R}^n$

Example 2.1.2. Here we give a simple example
$S^4 = \{(x_1, x_2, x_3, x_4, x_5)|x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 = 1\} \subset \mathbb{R}^5$ as a topological manifold. It can be generalise to n-sphere. It is obvious that S^4 Hausdorff and second countable since it is subspace of R^5. It is locally Euclidean since we can cover it with open sets U_i^+, U_i^- where $i = 1, ..., 4$
$U_i^+ = \{(x_1, x_2, x_3, x_4, x_5) \in S^4|x_i > 0\}$,$U_i^- = \{(x_1, x_2, x_3, x_4, x_5) \in S^4|x_i < 0\}$ and define the homeomorphisms $\phi_i^{\pm}; U_i^{\pm} \rightarrow B^4$ given by $\phi_i^{\pm}(x_1, \dots, x_5) = (x_1, \dots, \hat{x}_i, \dots, x_5)$. Where $B^4 = \{x \in \mathbb{R}^4 | \mid x \mid < 1\}$

Definition 2.1.3. A pair (U_α, ϕ_α) of such neighborhood and homeomorphism is called a *chart*.

Definition 2.1.4. A collection $\{(U_\alpha, \phi_\alpha) \mid \alpha \in A\}$ of charts is an *atlas* if it is a cover of X.

Definition 2.1.5. The map $\phi_\alpha \circ {\phi_\beta}^{-1}$ on $\phi_\beta(U_\alpha \cap U_\beta)$ is the *transition functions* between the charts (U_α, ϕ_α) and (U_β, ϕ_β).

Definition 2.1.6. A topological manifold is a *a smooth manifold* if the transition functions are C^∞.

Example 2.1.7. As we show in Example 2.1.2 S^4 is a 4-dimensional topological manifolds. Here we show the standard smooth structure on it using same charts in the Example 2.1.2.
Indeed, $\forall i, j \in \{1, \ldots, 5\}$, $(\phi_i^{\pm}) \circ (\phi_j^{\pm})^{-1}(x_1, \ldots, x_4) = (x_1, \ldots, \hat{x}_i, \ldots, \pm\sqrt{1- \mid a \mid^2}, \ldots, x_4)$ where $a = (x_1, \ldots, x_4)$. It is easily can be seen that $(\phi_i^{\pm}) \circ (\phi_j^{\pm})^{-1} = id_{B^4}$. Therefore S^4 is a smooth manifold and the atlas $\{U_i^{\pm}, \phi_i^{\pm}\}$ is called standard smooth structure on S^4.

After that according to our terminology with a manifold we always mention smooth manifold.

Definition 2.1.8. *The boundary of the n-dimensional manifolds X* defined $\partial X = \{x \in X \mid x \text{ corresponding to points in} \{(x_1, \ldots, x_n) \mid x_n = 0\}\}$. The boundary manifold form a submanifold with dimension $n - 1$.

Definition 2.1.9. A *closed manifold* is a compact manifold without boundary.

Definition 2.1.10. A *diffeomorphism* between two manifolds is a homeomorphism $f : X \to X'$ such that f and f^{-1} are both C^∞ on any chart of the given atlas.

Definition 2.1.11. Let N and M be manifolds. The *isotopy* from N to M is a map $H : N \times I \to M$ such that $\forall t \in I$ the map

$$H_t : N \to M$$

is a diffeomorphism. So we call H_0 and H_1 are *isotopic*.
In addition if $N = M$ with $H_0 = Id_M$ then the isotopy is called a *ambient isotopy (or diffeotopy)*.

2.2 Gluing

Suppose X_1 and X_2 are n-dimensional smooth manifolds, we can obtain a new manifold from given two manifolds by using gluing operation. Here we discuss some of gluing operations; the boundary sum which corresponds to attaching handles and the connected sum which is also an important tool in smooth category. These two operations are special since when we take connected sum or boundary connected sum of two smooth manifolds we get again a smooth manifold as resulting manifold. For this section we just give the definition of this operations and the main reference book for this section is Kosinski (1993).

Definition 2.2.1. Suppose X_1 and X_2 are n-dimensional smooth manifolds and $D_i^n \subset X_i$ is an embedded disc for $i = 1, 2$.
Let $\phi : D_1^n \to D_2^n$ be an orientation-reversing diffeomorphism.
The connected sum of X_1 *and* X_2 is constructed by deleting interior of the embedding balls and identifying resulting boundary ∂D_1 and ∂D_2 by diffeomorphism.

$$X_1 \# X_2 = (X_1 - intD_1) \cup_{\phi|_{\partial D_1}} (X_2 - intD_2)$$

,

Here $\#mX$ denotes connected sum of m copies of X for $m \geq 0$ if $m = 0$ then $\#mX = S^n$ by definition.

Example 2.2.2. $X = S^1 \times S^1$

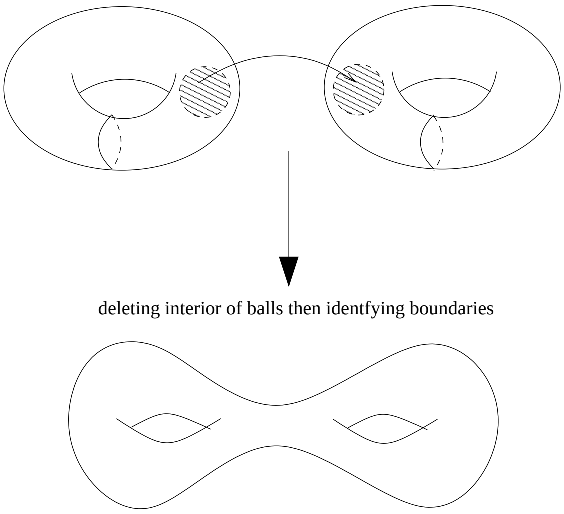

Figure 2.1 $X\#X$ Connected sum

Definition 2.2.3. Suppose X_1 and X_2 are n-dimensional smooth manifolds with boundaries ∂X_1 and ∂X_2. Suppose $Z_i \subseteq X_i$ are co-dimension zero compact submanifold of the boundaries with Z_i homeomorphic to D^{n-1}.

Let $\phi : Z_1 \to Z_2$ be an orientation-reversing diffeomorphism.

The boundary sum of X_1 *and* X_2 is constructed by identifying Z_1 with $\overline{Z_2}$ by diffeomorphism.

$$X_1 \natural X_2 = X_1 \cup_\phi X_2,$$

Here $\natural mX$ denotes boundary sum of m copies of X for $m \geq 0$ if $m = 0$ then $\natural mX = D^n$ by definition.

Example 2.2.4. $X = S^1 \times D^2$

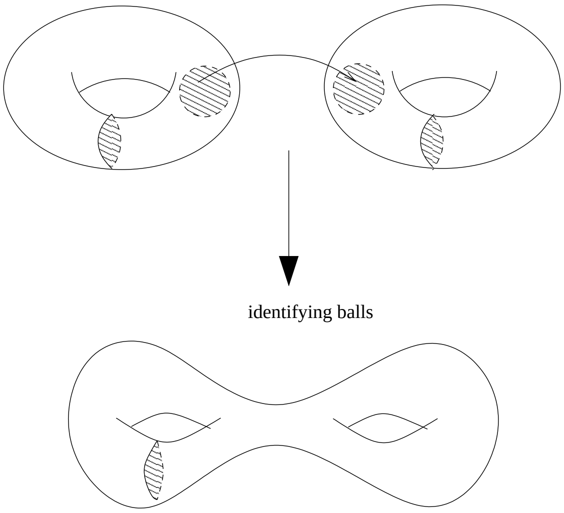

Figure 2.2 $X \natural X$ Boundary connected sum

2.3 Handles in n-dimension

Studying on manifolds with handles gives us some conveniences in terms of classification of manifolds as smooth topological objects. In this section we will consider handles abstractly and introduce these objects in n-dimension.

Definition 2.3.1. For $0 \leq k \leq n$, an *n-dimensional k-handle* denoted by h^k, is defined to be a homeomorphic copy of

$$D^k \times D^{n-k}$$

where $D^n = \{(x_1, \ldots, x_n) \mid x_1^2 + \cdots + x_n^2 \leq 1\} \subseteq \mathbb{R}^n$.

Definition 2.3.2. *Attaching k-handle* h^k *to the n-dimensional monifold* M *by an embedding*; A k-handle h^k is attached to the boundary of n-manifold M along $\partial D^k \times$

D^{n-k} if there is an embedding $\varphi : S^{k-1} \times D^{n-k} \hookrightarrow \partial M$ such that we attach the handle by identifying
$x \in S^{k-1} \times D^{n-k}$ with $\varphi(x) \in \partial M$ therefore we obtain a new manifold $M^{'}$ from M by attaching h^k.

$$M^{'} = [D^k \times D^{n-k} \sqcup M]/x \sim \varphi(x)$$

here φ is called *the attaching map* of handle. Furthermore,
$\partial D^k \times D^{n-k}$ is called *the attaching region* of handle,
$D^k \times 0$ is called *the core* of handle,
$0 \times D^{n-k}$ is called *the cocore* of handle,
$\partial D^k \times 0$ is called *the attaching sphere* of handle,
$0 \times \partial D^{n-k}$ is called *the belt sphere* of handle,

Example 2.3.3. 3-dimensional handles are given below
0-handle is $h^0 = D^0 \times D^3$,
1-handle is $h^1 = D^1 \times D^2$,
2-handle is $h^2 = D^2 \times D^1$,
3-handle is $h^3 = D^3 \times D^0$,

Now let us look at h^1 closer

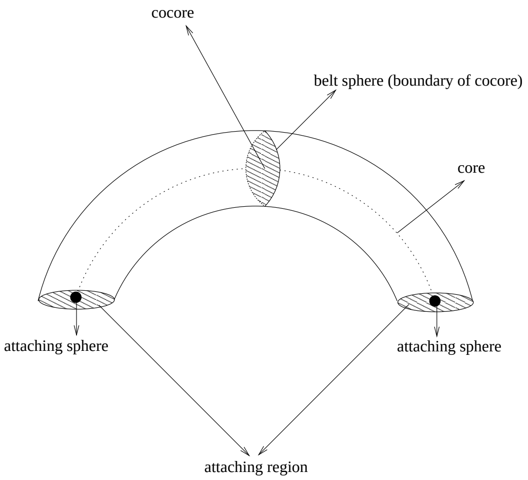

Figure 2.3 Anatomy of 1-handle in 3-dimension

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2.4 Handle Decomposition of 4-Manifolds

Handle decompositions of manifolds are based on Morse theory. In this study we discuss handle decomposition of any arbitrary 4-manifold deeply. So, some essential theorems and definitions must be given here.

Definition 2.4.1. With the *n- handlebodies* of a m-manifold M, we illustrate attaching m-dimensional n-handles to the boundary of D^m :

$$D^m \cup h^n \cup \cdots \cup h^n, \tag{2.4.1}$$

Theorem 2.4.2. *(Matsumoto, 2002, Theorem 3.4) When a Morse function $f : M \to \mathbb{R}$ is given on a closed manifold M, a structure of a handlebody on M is determined by f. The handles of this handlebody correspond on to the critical points of f, and the indices of the handles coincide with the indices of the corresponding critical point. In other words, M can be expressed as a handlebody.*

Theorem 2.4.3. *(Matsumoto, 2002, page 47) Let M be a closed m-manifold and $g : M \to \mathbb{R}$ be a smooth function defined on M. Then there exists a Morse function $f : M \to \mathbb{R}$ arbitrarily close to $g : M \to \mathbb{R}$.*

Therefore any closed 4-manifold can be obtained from D^4 by attaching 4-dimensional handles.

Definition 2.4.4. When a manifold is expressed as a handlebody, it is called a *handle decomposition*.

Theorem 2.4.5. *(Matsumoto, 2002, page 128) Let M be a closed m-dimensional manifold. If M is connected, then there is a Morse function $f : M \to \mathbb{R}$ on M with only one critical point of index* 0 *and one critical point of index m*

So, if M^4 is closed, connected 4-dimensional manifold then it has handlebody consist of only one h^0, some h^1 handles, some h^2 handles, some h^3 handles and only one h^4.

$$M^4 = h^0 \cup k_1.h^1 \cup k_2.h^2 \cup k_3.h^3 \cup h^4$$

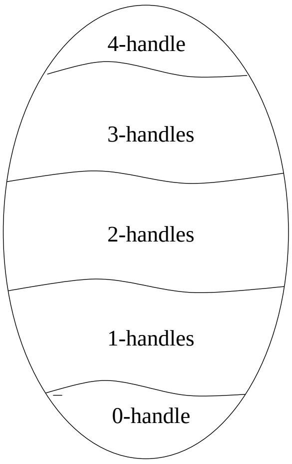

Figure 2.4 Handlebody of a closed-connected 4-manifold

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If a handle decomposition of M^4 is obtained from any Morse function $f : M \to \mathbb{R}$ then the attaching maps of handles determined by a gradient-like vector field of f. Therefore, there are various of choices for attaching map according to gradient-like vector fields of f.

2.5 Visualize Handlebody of a 4-Manifold and Framing

The main purpose of this section is to understand attaching maps of the handles. Hence firstly, we discuss framing deeply , because framing is an important tool for 4-manifolds and must be understood well. Let us start with some necessary definitions.

Definition 2.5.1. *A smooth vector bundle* is a triple (π, E, B) where

i) E and B are smooth manifolds

ii) $\pi : E \to B$ surjective smooth map

iii) $\forall U^{open} \subset B$, $\pi^{-1}(U) \cong U \times F$ where F has finite dimensional vector space structure.

Such that

1. the diagram commutes,

$$\begin{tikzcd}U \times F \arrow[r, "\phi:\cong"] \arrow[d, "p_1"'] & \pi^{-1}(U) \arrow[dl, "\pi"] \\ U & \end{tikzcd}$$

   in other words ϕ is *fiberwise* $\pi(\phi(u, f)) = u$, $u \in U$ and $f \in F$,

2. $\phi : \pi^{-1}(b) \to \{b\} \times F$ is an isomorphism.

The third condition means that it is *locally trivial*. In the above structure
E is called as *total space*
B is called as *base space*
F is called as *fiber of bundle*

we often show this bundle with the notation $F \longrightarrow E \xrightarrow{p_1} B$

As a simple example of the vector bundle we can consider the product space of any smooth manifold B and $\mathbb{R}^n$ over B with projection map onto first factor. Obviously this vector bundle is *trivial bundle*.

$$\mathbb{R}^n \longrightarrow B \times \mathbb{R}^n \xrightarrow{p_1} B$$

Definition 2.5.2. *A local section* of $U^{open} \subset B$ of the bundle $F \longrightarrow E \xrightarrow{p_1} B$ is a continuous map $s : U \to E$ satisfies $\pi \circ s = Id_U$. Thus we can conclude that s maps every point of the $b \in U$ to the vector in $\pi^{-1}(b) \times F$.

Example 2.5.3. Any smooth n-manifold M has canonical vector bundle which is tangent bundle TM.

$$TM = \bigsqcup_{p \in M} T_pM = \bigcup_{p \in M} \{(p, v) \mid v \in T_pM\}$$

with projection $\pi_1 : TM \to M$, $\pi(x, v) = x$ obviously it maps the vector space T_pM to a single point.

If $h : K^n \to M^m$ is an embedding then we can see tangent bundle $R^n \longrightarrow TK \xrightarrow{\pi_1} K$ in M. Then the *normal bundle* is given by $NK = TM/TK$;

$$NK = \bigsqcup_{p \in K} N_p K = \bigsqcup_{p \in K} T_{h(p)} M / T_p K$$

Definition 2.5.4. Generally *n-frame* in n-dimensional vector space E is an ordered set of n linearly independent vectors in E.

With this notion we define a framing of a vector bundle as section of the associated vector bundle such that these sections form a basis for the fibers at any point of the base space. With a normal framing we mention choice of isotopy class of sections of the normal bundle. Such two framings are isotopic if they are isotopic as bundle maps so it coincides with the choice of isotopy class of a trivialisation.

Proposition 2.5.5. *For a vector bundle $F \longrightarrow E \xrightarrow{\pi} B$ is trivial over $U \subset B$ iff there exists a frame $\{s_1, \ldots, s_n\}$.*

Proof. Let $\{e_1, \ldots, e_n\}$ be a standard basis for the vector space F and assume
$\phi : U \times F \to \pi_{-1}(U)$ be a trivialisation then define
$s_i : U \to \pi^{-1}(U) \qquad b \mapsto (b, e_i)$.
Conversely Let $\{s_1, \ldots, s_n\}$ be a given frame on U then we can form trivialisation
$\phi : U \times F \to \pi_{-1}(U) \qquad \phi(b, \sum a_i e_i) \mapsto \sum a_i s_i(b)$.

□

The below discussion will be helpful to understand what framing is and why it is important.

An embedding $\phi : \partial D^k \times D^{n-k} \to \partial M^m$ is constructed by
$\phi : S^{k-1} \to \partial M^m$ and gluing $\phi : S^{k-1} \times D^{n-k}$ to a tubular neighbourhood of the embedded sphere. That is saying normal bundle of embedded sphere is trivial in ∂M.

In addition, the diffeomorphism type of the space $D^k \times D^{n-k} \bigsqcup_\phi M$ is determined by the ϕ up to isotopy. Since if ϕ and $\tilde{\phi}$ are isotopic, then we get

$$D^k \times D^{n-k} \bigsqcup_{\phi} \partial M \cong D^k \times D^{n-k} \bigsqcup_{\tilde{\phi}} \partial M$$

Therefore, the diffeomorhism type of a space $D^k \times D^{n-k} \bigsqcup_{\phi} \partial M$ is determined by the two pieces of data;

1. an embedding $\phi : S^{k-1} \to \partial M$.
2. a framing of $\phi(S^{k-1})$ in ∂M.

What is the relation between the isotopy classes of framings of the normal bundle $\nu\phi(S^{k-1})$ and homotopy group of the orthogonal group $\pi_{k-1}O(n-k)$. The goal is now to identify the difference of framing as an element of $\pi_{k-1}O(n-k)$. As we will show, that we do not identify a framing, indeed we identify difference of two framings with an element of homotopy group. The question is what is the difference of two framings? The difference of two framings is $f \circ {f_0}^{-1}$ as a convention.

Let f , f_0 : $S^{k-1} \times \mathbf{R}^{n-k} \to \nu\phi(S^{k-1})$ two framings where $S^{k-1} \times \mathbf{R}^{n-k}$ is the trivial $n-k$ bundle over S^{k-1}.

$$\begin{tikzcd}
S^{k-1} \times \mathbf{R}^{n-k} \arrow[r, "f"] \arrow[d, "p_1"] & \nu\phi(S^{k-1}) \arrow[d, "\pi"] & S^{k-1} \times \mathbf{R}^{n-k} \arrow[l, "f_0"'] \arrow[d, "p_1"] \\
S^{k-1} \arrow[r, "\phi"] & \phi(S^{k-1}) & S^{k-1} \arrow[l, "\phi"']
\end{tikzcd}$$

Then we have a diffeomorphism given as follows.

$$\begin{aligned} f \circ {f_0}^{-1} : S^{k-1} \times \mathbf{R}^{n-k} &\to S^{k-1} \times \mathbf{R}^{n-k} \\ (x, y) &\mapsto (x, \theta(x, y)) \end{aligned}$$

Then, $\forall x \in S^{k-1}$, we have a map

$$\begin{aligned}\theta_x : \mathbf{R}^{n-k} &\to \mathbf{R}^{n-k}\\ y &\mapsto \theta(x, y)\end{aligned}$$

Therefore $\forall x \in S^{k-1}$ we obtain a self diffeomorphism of $\mathbf{R}^{n-k}$ and so an element of $GL(n-k)$. Now we can construct a map from S^{k-1} to $GL(n-k)$ this map is exactly what we are looking for;

$h : S^{k-1} \to GL(n-k)$ such that $h(p) = \theta_p \in GL(n-k)$ realise that if we fixed any framing f then for any other framing f_0 we identify f_0 as an element of $\pi_{k-1}GL(n-k)$ we are almost done. As a last step using Gram-Schmidt orthogonalisation process it can be shown that $O(n)$ is a deformation retract of $GL(n)$ so $\pi_i GL(n) \approx \pi_i O(n)$.

Conversely, it is much easer to show that for every element of $\pi_{k-1}O(n-k)$ we can find a bundle map. Indeed, if $A \in \pi_{k-1}(O(n-k))$ then A is in the form $A : S^{(k-1)} \to O(n-k)$. Then $\forall x \in S^{(k-1)}$, we have a self diffeomorphism of $\mathbf{R}^{n-k}$ say $A(x)$.

$$\begin{aligned}A(x) : \mathbf{R}^{n-k} &\to \mathbf{R}^{n-k}\\ y &\mapsto A(x).y\end{aligned}$$

then construct a self diffeomorphism of $S^{k-1} \times \mathbf{R}^{n-k}$ as follows:

$$\begin{aligned}\hat{A} : S^{k-1} \times \mathbf{R}^{n-k} &\to S^{k-1} \times \mathbf{R}^{n-k}\\ (x, y) &\mapsto (x, A(x).y)\end{aligned}$$

For a fixed framing f we can obtain the desired framing :

$f \circ \hat{A} : S^{k-1} \times \mathbf{R}^{n-k} \to \nu\phi(S^{k-1})$.

Main goal of this section; visualizing handlebody of a 4-manifold by drawing their attaching regions. Here we only consider 4-dimensional connected closed smooth manifolds.

4-dimensional handles are given as below

0-handle is $h^0 = D^0 \times D^4$

1-handle is $h^1 = D^1 \times D^3$

2-handle is $h^2 = D^2 \times D^2$

3-handle is $h^3 = D^3 \times D^1$

4-handle is $h^4 = D^4 \times D^0$

Using Theorem 2.4.5 we know that any closed connected 4-manifolds can be obtain from one 1-handle , one 4-handle , some 2- and 3-handles.

Firstly let us visualise the one-handle attachment to the boundary of D^4;

In 4-dimension a 1-handle is $D^1 \times D^3$ with the attaching region $\partial D^1 \times D^3$ is given in the figure 2.5 :

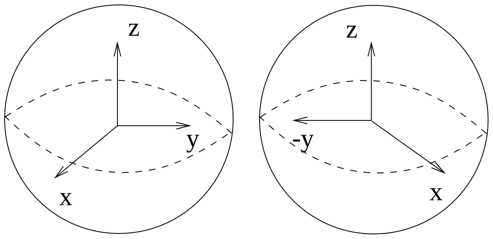


Figure 2.5 Attaching region of 1-handle $\partial D^1 \times D^3$

.

It is attached by an embedding $\phi : \partial D^1 \times D^3 \rightarrow \partial D^4$. As we discuss above it is determined by $\phi_0(S^0)$ with trivial normal bundle and a framing of $\phi_0(S^0)$. So there is only two embedding since $\pi_0(O(3)) = \mathbb{Z}_2$. This means there are exactly two manifolds which can be obtained from 1- handle attachment to D^4. If we consider the orientation there is only one orientable manifold. We picture it as above or as a circle with dot.

Secondly, let us visualise the two-handle attachment to the boundary of D^4. Attaching 2-handle is given by an embedding $\phi(\partial D^2 \times D^2) \rightarrow S^3$ is determined by

i) $\phi_0(S^1) = K$ which is a knot in S^3 and

ii) normal framing of K

So we visualise 2-handle attachment by a knot and a normal framing, such two data gives us *Framed Knot*. In Matsumoto (2002) there is a figure to illustrate a framed close curve as below. Also note that by orienting K and using this orientation, one normal vector field u determines the other normal vector field v.

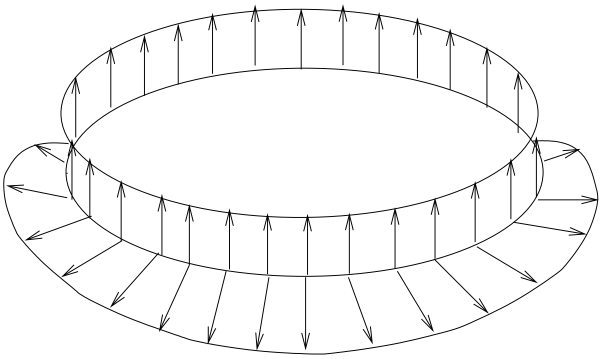


Figure 2.6 A framed closed curve

The question is how many embeddings we can write which are not isotopic. We easily see that there are infinitely many embeddings and they are not isotopic. Indeed, $\pi_1(O(2)) = \mathbb{Z}$. Our next goal is to identify every framing by an integer.

Now time to specialise notion of framing for a knot K in S^3. Let $K = \phi(S^1)$ be an embedding knot , together with a normal framing $e = \{u, v\}$ of its normal bundle in R^3 where u and v normal vector field of K. This framing determines the embedding ;

$$\phi : \partial D^2 \times D^2 \to S^3 \text{ by}$$

$$\phi(x, \lambda, \nu) = (x, \lambda u + \nu v)$$

So here is the definition of zero framing;

Definition 2.5.6. Since every closed curve in S^3 bounds an oriented surface which is called Seifert surface. *Zero framing* is induced from the Seifert surface of the knot by pushing the knot K into the Seifert surface. Namely, taking a tangent vector of the surface which is pointing inwards of the surface and perpendicular to the knot.

When we use the term knot it make sense but let us give some basic definition related knot theory.

Definition 2.5.7. Rolfsen (1976) Let X be any topological space, a subset K of X is a *knot* if K is homeomorphic copy of S^p. More generally, K is a *link* if K is homeomorphic copy of disjoint union of spheres $S^{p_1} \sqcup \cdots \sqcup S^{p_r}$.

In this thesis we always consider a knot as an embedding of S^1 in S^3. In the same manner a link as an embedding of disjoint union of spheres $S^1 \sqcup \cdots \sqcup S^1$.

Definition 2.5.8. The *linking number* is defined for links let K be a link with component and K_1, K_2 the *linking number* given by the formula:

$$Lk(K_1, K_2) = \text{ (Positive crossing number - negative crossing number) /2}$$

To make convention positive and negative crossing illustrated in the below picture

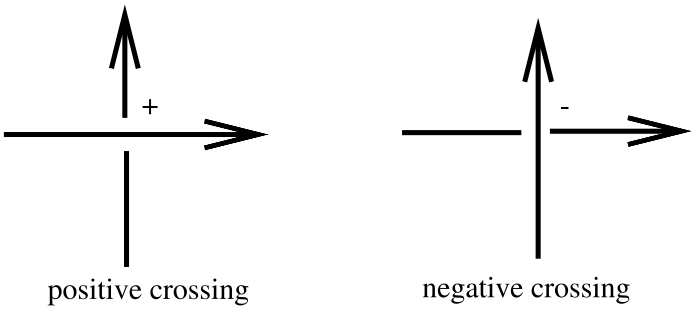


Figure 2.7 Signed of crossing

Definition 2.5.9. The *writhe* is defined for knots, let K be a knot the *writhe* given by the formula:

$$w(K) = \sum_{p \in C(K)} \epsilon(p)$$

where $C(K)$ is the set of crossing point and $\epsilon(p)$ is the sign of the crossing.

Assume K' obtained from K by pushing it in the direction of the vector u where $\{u, v\}$ is given framing. If $\{u, v\}$ is obtained from the Seifert surface then linking number of K and K' is always zero.

With this convention, we assign an integer to any framing as linking number of C and $C^{'}$ where C is a simple closed curve and $C^{'}$ is obtained from C by pushing it along the direction of the one component of the framing .

Definition 2.5.10. Let K be a knot and $K^{'}$ is the paralel copy of K then the *framing coefficient of the blackboard framing* of a knot K is given by $BB(K) = Lk(K, K^{'})$ and it can be seen that it satisfy the equation:

$$BB(K) = w(K)$$

Example 2.5.11. Here we show that, writhe of right hand trefoil is equal to blackboard framing.

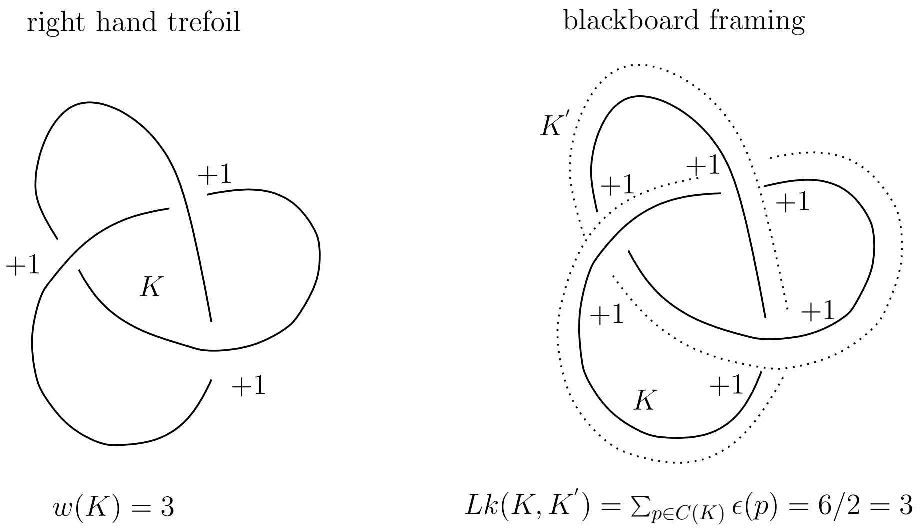


Figure 2.8 Blackboard framing of right hand trefoil

As a conclusion, 2-handle attachment is visualised by framed link and we illustrate it by a knot and corresponding integer.

What about 3-handles? A 3-handle $D^3 \times D^1$ is attached by an embedding $\phi : S^2 \times D^1$. Unfortunately, it is not easy to visualise embedding of S^2 in the boundary of the manifold. On the other hand we do not really need to deal with 3-handles and 4-handle since they do not effect the diffeomorphism type of the closed connected manifold. We will give this fact here as a theorem with sketch of the proof.

Theorem 2.5.12. *Let M be closed connected orientable 4-manifold. So it is in the form* $M^4 = h^0 \cup k_1.h^1 \cup k_2.h^2 \cup k_3.h^3 \cup h^4$. *Then the diffeomorphism type of M determined by N. More explicitly if M and* $M^{'}$ *are two closed connected orientable manifolds obtained from N by attaching 3-handles and a 4-handle then they are diffeomorphic.*
Where $N = h^0 \cup k_1.h^1 \cup k_2.h^2$.

Proof. The complement $M - int(N) = N^*$ is the 4-manifold consist of 3-handles and a 4-handle. Using up-side down method (changing the Morse function f by $-f$) we can see, that it is in the handle decomposition of a 0-handle and 1-handles. Obviously diffeomorphism type of N^* determined by 1-handles. So there is unique oriented manifold which is obtained from 0-handle by attaching 1-handles which is $N^* = \natural k_3(S^1 \times D^3)$. It is not hard to see that $\partial N^* \cong \# k_3(S^1 \times S^2)$. Also we have $\partial N^* = \partial N$. Therefore the self diffeomorphims $h : \partial N \to \partial N^*$ is exactly $h : \# k_3(S^1 \times S^2) \to \# k_3(S^1 \times S^2)$. By the Laudenbach (1972) any self diffeomorphism of $\# k_3(S^1 \times S^2)$ extents over $\natural k_3(S^1 \times D^3)$ uniquely. Then we conclude that, if a 4-manifold $M^{'}$ constructed from N by attaching k_3 3-handles and a 4-handle then the self diffeomorphism of N extends over $M^{'}$. □

This useful fact gives us an efficient tool to see relation between 4-manifolds. Indeed, if we consider diffeomorphism type of closed connected oriented 4-manifolds then we will only consider only 0-handle through 2-handles.

2.6 Homology of Handles

There is an easy definition of homology of handlebody using cellular homology. This definition quoted from Scorpan (2005).

Remark 2.6.1. Here we need to emphasise that, attaching $k - handle$ to the $0 - handle$ can be seen as attaching $k - cell$ to the $0 - cell$ since any $k - handle$, $D^k \times D^{n-k}$ in $n - dimension$ is the thickened of the $k - cell = D^k$ so they have same homotopy type.

Homology of handlebody is defined by using cellular homology. n-chains are defined free abelian groups generated by n-handles. And the boundary map $\partial : C_k \to C_{k-1}$ is defined by $\partial h^k_\alpha = d_{(\alpha,\beta)}.h^{k-1}_\beta$ where $d(\alpha,\beta)$ is the *intersection number* of attaching sphere of h^k_α and the belt sphere of h^{k-1}_β.

Definition 2.6.2. The n^{th} *homology* of M is given by the formula $H_n(M) = ker\partial_n / im\partial_{n+1}$. Notice that we use same notation with singular homology since they are identical.

Example 2.6.3. Let us compute the homology groups of $\mathbb{CP}^n$ complex projective n-space. It is well known $\mathbb{CP}^n$ has handle decomposition as $\mathbb{CP}^n = h^0 + h^2 + \cdots + h^{2n}$, so we can compute its homology groups easily;

$$C_{2n}(\mathbb{CP}^n) \xrightarrow{\partial_{2n}} C_{2n-1}(\mathbb{CP}^n) \xrightarrow{\partial_{2n-1}} \dots C_1(\mathbb{CP}^n) \xrightarrow{\partial_1} C_0(\mathbb{CP}^n)$$

$$\mathbb{Z} \xrightarrow{\partial_{2n}} 0 \xrightarrow{\partial_{2n-1}} \dots 0 \xrightarrow{\partial_1} \mathbb{Z} \text{ equaly } \mathbb{Z} \xrightarrow{0} 0 \xrightarrow{id} \dots 0 \xrightarrow{id} \mathbb{Z}$$

$$\text{Therefore } H_m(\mathbb{CP}^n) = ker\partial_m / im\partial_{m+1} = \begin{cases} \mathbb{Z} & \text{if m is even} \\ 0 & \text{if m is odd} \end{cases}$$

2.7 Intersection Form of 4-Manifold

Finally, we introduce the intersection form of 4-dimensional closed oriented smooth manifolds then we will be able to see picture of some important manifolds.

Definition 2.7.1. Let $L = (L_1^{c_1}, L_2^{c_2}, \dots L_n^{c_n})$ be a framed link in S^3. The *linking matrix* of L is defined as $m \times m$ symmetric matrix $[a_{ij}]_{m\times m}$ where the components of the matrix are:

$$a_{ij} = \begin{cases} lk(L_i, L_j) & \text{if } i \neq j \\ \text{framing coefficient of } L_i & \text{if } i = j \end{cases}$$

Definition 2.7.2. Let M be a closed, oriented, smooth 4-manifold and let [M] denotes its fundamental class.

Define $Q_M : H^2(M) \otimes H^2(M) \to \mathbb{Z}$ by $Q_M(a,b) = (a \smile b, [M])$ This symmetric bilinear form is called the *intersection form* of M.

We will argue here a more geometric way of defining this bilinear form in terms of intersections of embedded surfaces.

Let M be a 4-manifold obtained by attaching 2-handles to D^4 and one 4-handle. We represent M be a framed link $L = (L_1^{c_1}, L_2^{c_2}, \dots L_n^{c_n})$ where every component of the L corresponds to the attaching sphere of 2-handle.

Say $H_2(M) = \langle \alpha_1, \dots \alpha_n \rangle$, and $F_i = \partial L_i$ be a Seifert surface of L_i. We obtain closed oriented smoothly embedded surface $\hat{F}_i$ from F_i by pushing $intF_i$ into D^4 and attaching core of 2-handle along L_i. We may assume that $\hat{F}_i$ and $\hat{F}_j$ intersect transversely in M. Assign each intersection point by a number ±1. The sign depends on whether the induced orientation of $T_x(\hat{F}_i) \oplus T_x(\hat{F}_j)$ agree with $T_x(M)$ or not. The sum of this numbers is called *intersection product* of α_i and α_j and it is denoted by $\alpha_i.\alpha_j$. Let us give the formal definition of it.

Definition 2.7.3. The *intersection product* of α_i and α_j is defined by

$$\alpha_i.\alpha_j = PD(D(\alpha_i) \smile D(\alpha_j))$$

where $H^2(M) = \langle D(\alpha_i) \rangle$ where $D = PD^{-1} : H_2(M, \mathbb{Z}) \to H^2(M)$, $a = D(a) \frown [M]$.

Here the another interpretation of the intersection form using intersection product :

$$\begin{aligned} \alpha_i.\alpha_j &= (D(\alpha_i) \frown [M]).(D(\alpha_j) \frown [M]) \\ &= (D(\alpha_i) \smile (D(\alpha_j)) \frown [M] \\ &= (D(\alpha_i) \smile (D(\alpha_j), [M]) = Q_M(D(\alpha_i), D(\alpha_j)). \end{aligned}$$

As any reader could notice that we gave geometric interpretation of intersection product and we use formal definition to define intersection form. Details can be found in (Bredon, 1993, Intersection Theory).

Therefore we can compute the cup product of co-homology class of M which are dual to the orientation class of sub-manifolds $\hat{F}_i$ and $\hat{F}_j$ of M by looking at the intersection of $\hat{F}_i$ and $\hat{F}_j$. The geometric interpretation of $[\hat{F}_i \cap \hat{F}_j]$ is sum of sign of intersection points. So $[\hat{F}_i \cap \hat{F}_j] = \alpha_i.\alpha_j$.

More generally;

Proposition 2.7.4. *(Gompf & Stipsicz, 1999, Proposition 1.2.3) Let X be oriented closed smooth 4-manifold then every element of* $H_2(X,\mathbb{Z})$ *can be represented by an embedded surface.*

Proposition 2.7.5. *(Gompf & Stipsicz, 1999, Proposition 1.2.5) For* $a, b \in H^2(X,\mathbb{Z})$ *and* $\alpha, \beta \in H_2(X,\mathbb{Z})$ *be Poincare duals of a and b then* $Q_X(a,b)$ *is the number of points in the intersection of representative surfaces* $F_\alpha \cap F_\beta$ *counted with sign.*

Definition 2.7.6. If we fix the basis $H_2(M) = \langle \alpha_1, \ldots \alpha_n \rangle$ then we call the $[Q_M] = [\alpha_i.\alpha_j]$ *matrix form of the intersection form* or *intersection matrix.*

We conclude from above discussion for a 2-handlebody the intersection matrix is given by linking matrix; $\{\alpha_i, \alpha_j\} = [\hat{F}_i \cap \hat{F}_j] = lk(L_i, L_j)$.

Example 2.7.7. Let $M = S^2 \times S^2$ so $H_2(M)$ generated by $\alpha_1 = \{x\} \times S^2$ and $\alpha_2 = S^2 \times \{y\}$ for a base point $(x,y) \in M$. It is obvious that α_1 and α_2 intersect transversely in one point. Then $\alpha_i.\alpha_i = 1$ by choosing orientation agree with M. Since $\alpha_i^2 = 0$ then the intersection matrix is

$$Q_M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Therefore, we can draw the picture

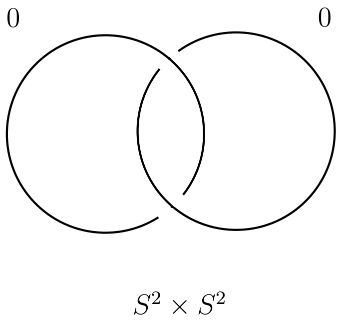
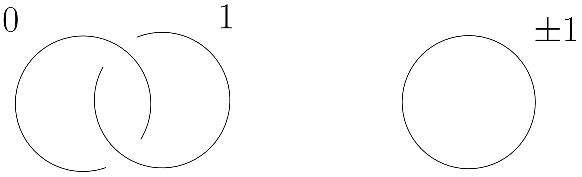


Figure 2.9 Handlebody description of $S^2 \times S^2$

Example 2.7.8. Other well known spaces are $\mathbb{CP}^2$, $\overline{\mathbb{CP}}^2$ and $S^2 \tilde{\times} S^2$. Their intersection forms and pictures are below.



$$Q_{S^2 \tilde{\times} S^2} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \qquad Q_{\mathbf{CP}^2} = (\pm 1)$$

Figure 2.10 Examples of handlebody descriptions

CHAPTER THREE
CALCULATION WITH HANDLES

The main purpose of this chapter is define some methods which is called Handle Calculus includes handle sliding, handle cancelling, and carving operations. These operations can be seen as tools to change handle decomposition of a manifold without changing its smooth structure. Therefore this operations quite useful methods to show relation between two smooth manifolds. The reason for using this calculations that it allows more simple and impressive way of description of being same between two given manifolds. Thus, this makes everything more computable. In this chapter these methods will be described and end of the chapter some examples will be given. The fundamental references for this chapter are Akbulut (2012), Milnor et al. (1965), Gompf & Stipsicz (1999) and Matsumoto (2002).

3.1 Surgery

Dehn Surgery is an important method to construct 3-manifolds. It was introduced by Max Dehn in 1910 to construct homology sphere. In the early 1960 Lickorish and Wallace proved independently that; any closed orientable 3-manifolds can be obtained by Dehn surgery operation on a framed link in S^3 with ± 1 surgery coefficient. Its importance comes after this theorems, as we will see in this section all closed orientable 3-manifold bounds a simply-connected compact 4-manifolds. The first two definitions help us to understand the general idea of the surgery theory. After that we will give the definition of the Dehn surgery. In this section we only consider the connected and orientable manifold and this section base on Rolfsen (1976).

Definition 3.1.1. The *surgery operation* on a manifold generally can be defined by cutting out part of a manifold and replacing it with another manifold. The point here this two manifold must have same boundary.

Definition 3.1.2. Assume that $\phi : S^k \to M^m$ be an embedding with a normal framing. This define the embedding $\phi : S^k \times D^{m-k} \to M^m$ uniquely up to isotopy. The *surgery* *on* S^k is defined removing $\phi(S^k \times D^{m-k})$ and replacing it by $D^{k+1} \times S^{m-k-1}$ with a map ϕ.

In same manner the idea of the Dehn Surgery is defined as surgery operation on a knot in S^3.

Definition 3.1.3. Let K be a knot in S^3 and νK be a closed tubular neighbourhood of K which is solid torus. Then one can define *Dehn surgery* on a knot K as removing $int \nu K$ from the manifold and gluing in $S^1 \times D^2$ along boundary of the solid torus by any diffeomorphism. This is exactly same to say that, removing interior of the solid torus and glue it back by a any boundary diffeomorphism.

Definition 3.1.4. Let V_1 and V_2 two manifold homeomorphic to solid torus and $h : \partial V_1 \to \partial V_2$ be a homeomorphism.

Define a space

$$M^3 = V_1 \coprod_h V_2 = (V_1 \coprod V_2)/x \sim h(x)$$

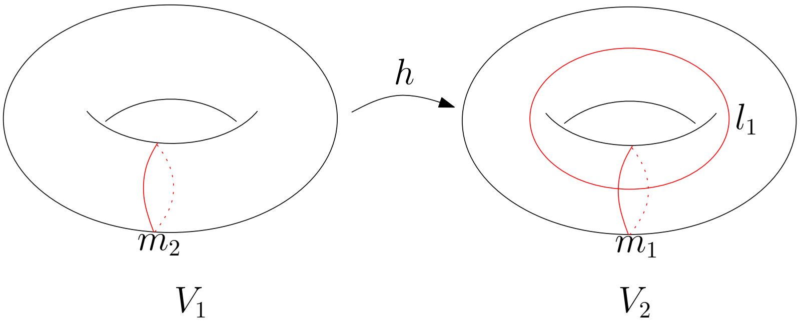


Figure 3.1 Construction of lens space

$< l_1, m_1 >$ is generator for $\pi_1(\partial V_1)$ and $h_*(m_2) = pl_1 + qm_1$ where $gcd(p, q) = 1$. The resulting manifold M^3 is called the *lens space of type* (p, q)

$$M^3 = L(p, q)$$

3.2 Handle Sliding

In this section we describe handle sliding in terms of Morse theory and explain how it is related framing coefficient handle attachment in 4-dimension and end of this section we will be able to use some technique related handle sliding. We start by giving some fundamental notions from Morse Theory and H-Cobordism, see Milnor et al. (1965) , Matsumoto (2002).

Firstly remember that when a Morse function $f : M \to \mathbf{R}$ on a $m-$dimensional closed manifold M and a vector field X of f are given then the handle decomposition of the manifold M defined by f and X. And also we can arrange the Morse function such that different critical points have different critical values.

Let $p_0, \dots, p_n$ be $n + 1$ critical points of ascending order according to their critical values.

To simplify the notation for a handle decomposition of this manifold

$$M = h^0 \cup_{\phi_{\alpha_i}} h^{\alpha_i} \cup_{\phi_{\alpha_{i+1}}} h^{\alpha_{i+1}} \cdots \cup_{\phi_n} h^n \tag{3.2.1}$$

we use the notation

$$M = (D^m; \phi_1, \dots, \phi_n)$$

By M_i we denote the subhandlebody obtained by attahcing handles from $0-$handle truogh $i^{th}-$handle. So we write $M_i = (D^m; \phi_1, \dots, \phi_i)$

After clarify notations we can give main theorem of this section.

Theorem 3.2.1. *(Matsumoto, 2002, Theorem 3.21, page 106) Given an isotopy h_t of the boundary ∂M_{i-1}, the attaching map ϕ_i of the α_i-handle $= D^{\alpha_i} \times D^{m-\alpha_i}$ can be replaced by $h_1 \circ \phi_i$. Also by this replacement of the i^{th} attaching map, the diffeomorphism type of each subhandlebodies does not change.*

Proof. The proof is too long and include many details so we will skip some of details and discuss here just the main ideas of the proof.

Start with the handlebody (3.2.1), let c_i be the critical value of f corresponding critical point p_i. Let us look at closely to the i^{th} hanle and its attaching map ϕ_{α_i}

$$\phi : D^{\alpha_i} \times D^{m-\alpha_i} \to \partial M_{c_i-\epsilon} \tag{3.2.2}$$

notice that here we identify M_i with $M_{c_i-\epsilon}$ for sufficiently small ϵ where

$$M_{c_i-\epsilon} = \{x \in M | f(x) \leq c_i - \epsilon\} \tag{3.2.3}$$

Lemma 3.2.2. *(Milnor et al., 1965, Theorem 3.4, page 21)*

If the Morse number μ *of the triad* $(W; V_0, V_1)$ *is zero, then* $(W; V_0, V_1)$ *is a product cobordism.*

As a consequence of the above theorem we can conclude that

For the given Morse function f has no critical value in the interval $[c_{i-1} + \epsilon, c_i - \epsilon]$ then we have diffeomorphism

$$\psi : f^{-1}([c_{i-1} + \epsilon, c_i - \epsilon]) \to \partial M_{c_{i-1}+\epsilon} \times [0, 1] \tag{3.2.4}$$

The interval $\{p\} \times I$ in the right hand side corresponds to the integral curve $\gamma_p(t)$ on the left hand side. Using above theorem we can show that
$f^{-1}([c_{i-1} + \epsilon/2, c_{i-1} + \epsilon]) \simeq \partial M_{c_{i-1}+\epsilon} \times [0, 1]$ also
$f^{-1}([c_{i-1} + \epsilon/2, c_i - \epsilon]) \simeq \partial M_{c_{i-1}+\epsilon} \times [0, 1]$ then we can conclude that there is a diffeomorphism $h : M_{[c_{i-1}+\epsilon/2, c_{i-1}+\epsilon]} \to M_{[c_{i-1}+\epsilon/2, c_i-\epsilon]}$ so define a diffeomorphism
$\Phi = id \cup h : M_{c_{i-1}+\epsilon/2} \cup M_{[c_{i-1}+\epsilon/2, c_{i-1}+\epsilon]} \to M_{c_{i-1}+\epsilon/2} \cup M_{[c_{i-1}+\epsilon/2, c_i-\epsilon]}$

$$\Phi : M_{c_{i-1}+\epsilon} \to M_{c_i-\epsilon} \tag{3.2.5}$$

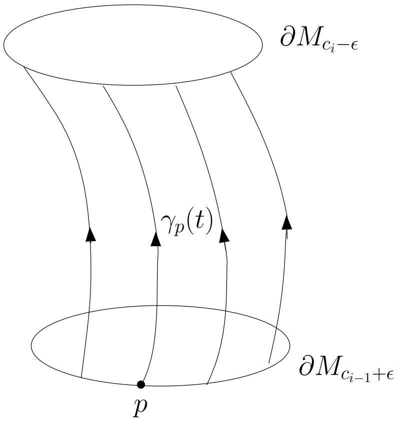


Figure 3.2 Flow diffeomorphism

Geometrically $M_{c_{i-1}+\epsilon}$ flow along a gradient like vector field X of f and coincide with $M_{c_i-\epsilon}$. It is time for last step of proof, let us illustrate given isotopy with $\{h_t\}_{t\in I}$. Using this isotopy we have smooth level-preserving embedding $H : \partial M_{c_{i-1}+\epsilon}\times I \to M_{c_{i-1}+\epsilon}\times I$

$$H(x,t) = (h_t(x), t)$$

Also we can define another level-preserving embedding
$\tilde{H} : \partial M_{c_{i-1}+\epsilon} \times I \to M_{c_{i-1}+\epsilon} \times I$ by $\tilde{H}(x,t) = (h_{1-t}(x), t)$

Just perturb the gradient like vector field X to Y using $\tilde{H}$.

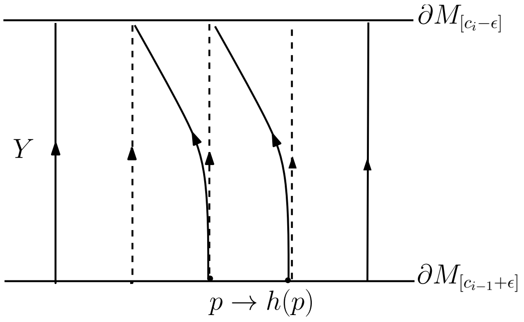


Figure 3.3 New vector field

A new diffeomorphism can be defined by a new vector field Y and f

$$\Psi : M_{c_{i-1}+\epsilon} \to M_{c_i-\epsilon} \tag{3.2.6}$$

so the i^{th} handle attached to $M_{i-1} = M_{c_{i-1}+\epsilon}$ by the attaching map

$$\Psi^{-1} \circ \phi = h_1 \circ \Phi^{-1} \circ \phi \tag{3.2.7}$$

It is obvious that the handlebody up to $i-1$ handle does not change since the vector field Y differs from X only after $M_{[c_i-1+\epsilon]}$ then the handlebody of $M_{[c_i-1+\epsilon]}$ remains unchanged. Also we can easly conclude that the diffeomorphism type of $M_{[c_j+\epsilon]}$ does not change for any j since the definition of $M_{c_j+\epsilon} = \{x \in M|f(x) \leq c_j + \epsilon\}$ does not involve the gradient like vector field.

□

3.2.1 *Visualise Handle Sliding*

In this section, our main goal to understand handle slide in diagrammatic language. We have already seen that, how to draw picture of smooth closed connected 4-dimensional manifolds with framed links. Now we will discuss handle slide in this way. The main reference book for this section will be Akbulut (2012) and Scorpan (2005).

Let $K_1^{r_1}$ and $K_2^{r_2}$ be knots in S^3 they are allowed to be linked and let $K_1^{'}$ be a push of K_1 in the direction of its frame r_1. We define the *connected sum along a band b* $K_1^{'}\#_b K_2$ then replace K_2 by $K_1^{'}\#_b K_2$ this move corresponds to handle sliding namely sliding h_2 over h_1 where they are corresponding handles . Why?

As we define in the previous section handle slide is defined as changing the attaching map of the handle by an isotopy.

When one slide any 2-handle $h_2 = D^2 \times D^2$ over the boundary of the rest of manifolds then it might slide over boundary of another 2-handle $h_1 = D^2 \times D^2$. Means the attaching sphere of h_2 (S^1) is goes over $\partial h_1 = D^2 \times pt$ obviously this process might change the attaching map of h_2 therefore the boundary operator. 2-handle sliding over another two handle has two possible resulting movements. These are *addition* and *subtraction*. To understand better we will examine the below picture.

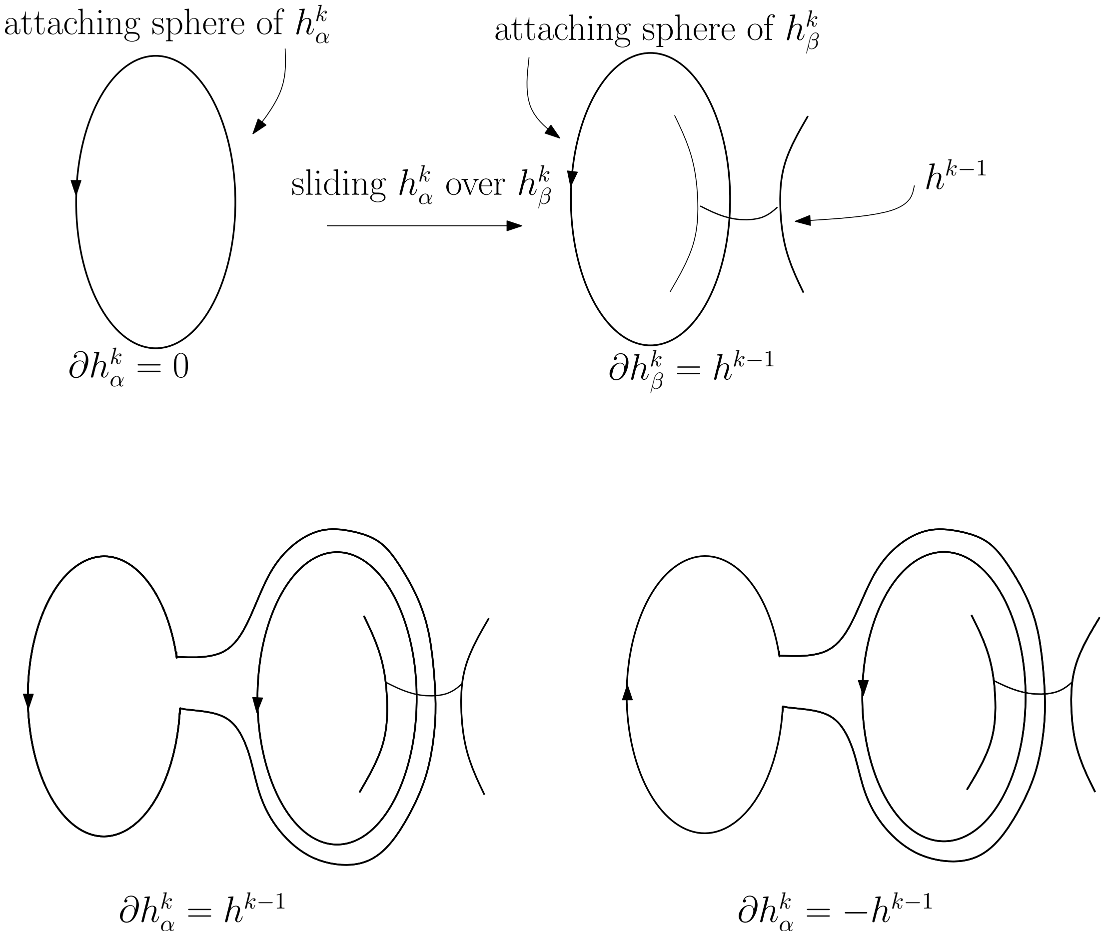


Figure 3.4 How sliding change the boundary operator

The change of boundary operation obviously change the basis element $h^2_\alpha \in H_2(M)$ to $h^2_\alpha + h^2_\beta \in H_2(M)$ or $h^2_\alpha \in H_2(M)$ to $h^2_\alpha - h^2_\beta \in H_2(M)$.

To avoid abuse of notation we use α and β instead of h^2_α and h^2_β. Thus it is easy to conclude that framing coefficient will be change from $\alpha.\alpha$ to

$$(\alpha \pm \beta).(\alpha \pm \beta) = \alpha^2 + \beta^2 \pm 2\alpha\beta = r_1 + r_2 \pm 2lk(K_1^{r_1}, K_2^{r_2})$$

since the intersection form is given with respect to the linking matrix of the framed link.

Therefore if one slides h_α over h_β then the sliding operation corresponds the below changes where the matrix represents the linking matrix.

$$\begin{array}{c c}h_\alpha & \multirow{2}{*}{$\begin{pmatrix}k & l\\ l & m\end{pmatrix}$}\\ h_\beta & \end{array}\longrightarrow\begin{array}{c c}h_\alpha \pm h_\beta & \multirow{2}{*}{$\begin{pmatrix}k \pm 2l + m & l + m\\ l + m & m\end{pmatrix}$}\\ h_\beta & \end{array}$$

This obviously represents the 2-handle slide over 2-handle according to above discussion. It works also for 2-handle slides over 1-handle since 1-handle can be seen zero framing 2-handle.

3.3 Handle Cancellation

If the attaching sphere of the k-handle intersect transversely ones with the belt sphere of the (k-1)-handle then this two handle create a cancelling pair. Or in other word we can describe this condition with the boundary map.

h^k_α and h^{k-1}_β create a cancelling pair iff $\partial h^k_\alpha = \pm h^{k-1}_\beta$

Proof of Cancellation theorem is explained in many book for example in Matsumoto (2002) or in Milnor et al. (1965). Here we avoid the proof of theorem but in the below picture the idea of cancelling can be seen.

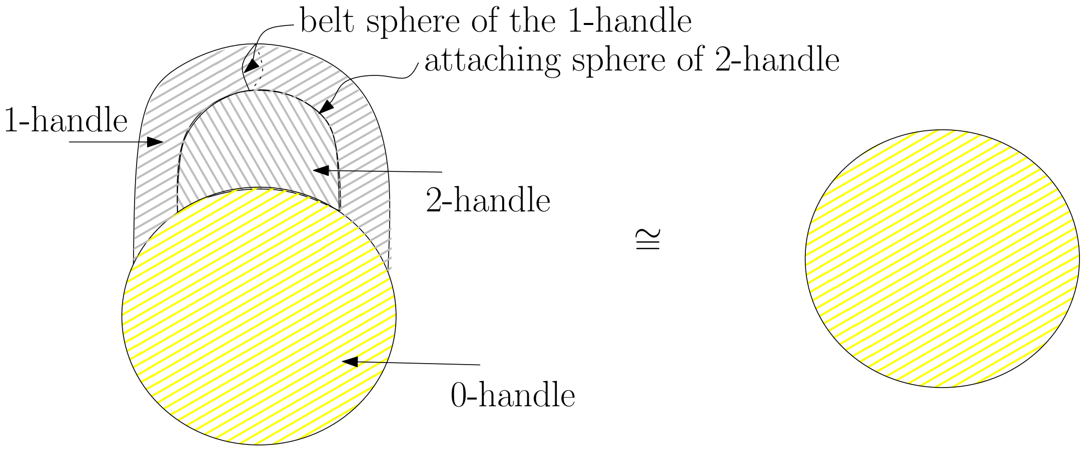


Figure 3.5 One dimensional cancelling pair

In 4-dimension it is visualised in same manner, but we need to first define dotted notation of one-handle.

3.4 Carving and Dotted Circle Notation

What we investigate here was introduced by Selman Akbulut. The dotted circle notation is one of the useful methods to draw one handle in the handle decomposition. Not all of them but some aspects of its power will be discuss in this section.

The dotted circle is used as alternative notation of one-handle attachment to the 0-handle. Let us start with an example which has an important place in the history of notation. The dotted circle notation used first to distinguish the one handle which is obtained in a way explained below example. This example is one of the exercises from Akbulut (2012).

Example 3.4.1. Let X be a manifold obtained D^4 attaching 2-handle with zero framing $X = D^4 \cup h^2$ and it is obvious that $X = S^2 \times D^2$ we draw this manifold by zero framing unknot. By surgery S^2 in X, obtain the manifold $Y = D^3 \times S^1$. The manifold Y is exactly $= D^4 \cup h^1$ and it is drawn by dotted circle. In same manner if we surger S^1 in Y we again obtain X. Notice that this two manifold has same boundary. Changing the notation from zero framing to dotted circle does not change the boundary of manifold.

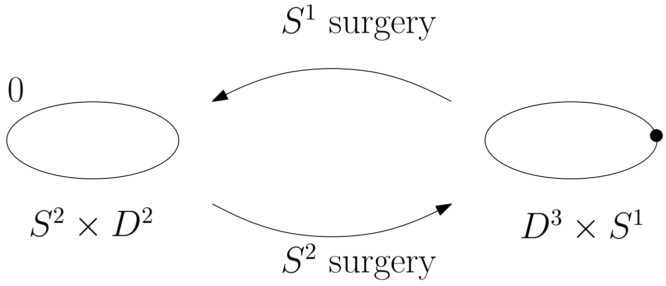


Figure 3.6 Relation between dotted circle and zero framing

Another helpful explanation is for this notation is *carving*. It is explained by pushing interior of the embedded D^2 in S^3 into D^4 and removing open tubular neighbourhood of D^2 from D^4.

Remember that if any two handle goes ones over 1-handle then they create 1-2 handle cancelling pair see picture below. So we can conclude that attaching one handle is equivalent to remove embedded 2-handle from the ∂N. The carving base on these explanations and the general definition is given in the lecturer note Akbulut (2012).

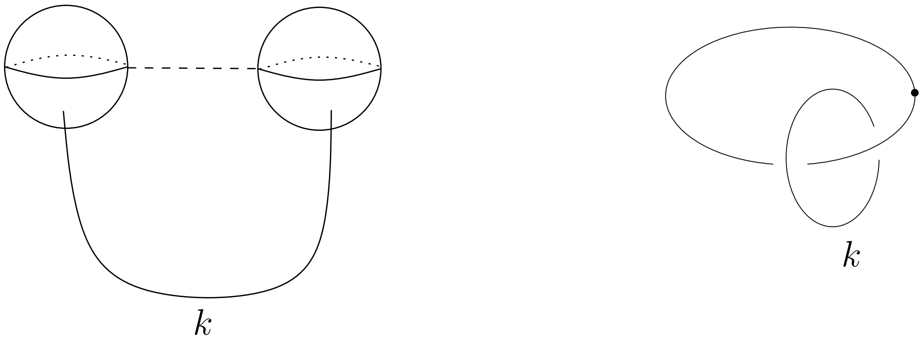


Figure 3.7 1-2 Cancelling pair

We assume that the attaching circle of the 2-handle goes over parallel to dashed line rather than over 1-handle. We can always remove the unlink cancelling pair from the diagram. If we add cancelling pair in the diagram we call it *1-2 birth*.

Definition 3.4.2. Let M^m be a connected manifold obtained from N^m by attaching k-handle $M = N \cup_{\phi} h^k$. If the attaching sphere of the k-handle $\phi(S^{k-1} \times 0)$ bounds a disk in the ∂N then M is obtained from N by drilling out an open tubular neighbourhood of the properly embedded disk D^{m-k-1}. This observation is called *carving*.

Example 3.4.3. We can always move in an un-knot ±1 framing to the rest of the link conversely we can always move out an un-knot ±1 framing from the rest of the link. Result of this movement is seen obviously in the below picture.

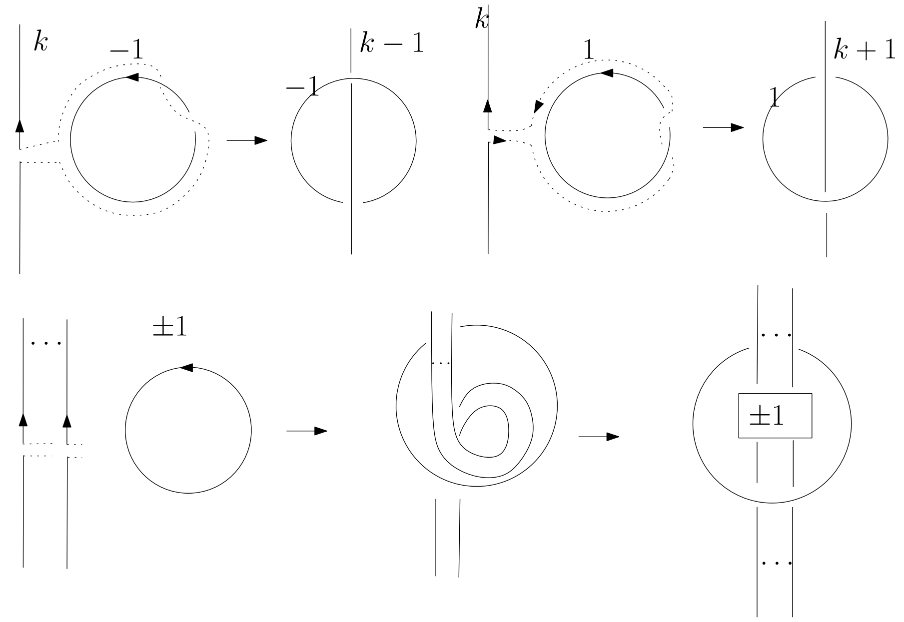


Figure 3.8 Two handles sliding over unlinked one handle

Above observations have some special place in Handle Calculus since it coincides with Blow-Up operation. This operation is reversible and its reverse is called as Blow-Down. Let us give definition of it.

Definition 3.4.4. The *Blowing up* operation is taking connected sum with $\mathbb{CP}^2$ or $\overline{\mathbb{CP}^2}$. In diagrammatic language adding ±1 framing unknot to the diagram without linking. Here we need to say that, clearly blow-up and blow-down operations does not effect the boundary of the manifold. Since $\mathbb{CP}^2$ is closed simply connected 4-manifold.

We learn many technique up to here so now time to give an concrete example. The below fact is firstly proved by Hirzebruch.

Example 3.4.5. We will show diffeomorphism between two manifold using sliding operation. $S^2 \times S^2 \# \mathbb{CP}^2 \cong \mathbb{CP}^2 \# \overline{\mathbb{CP}^2} \# \mathbb{CP}^2$

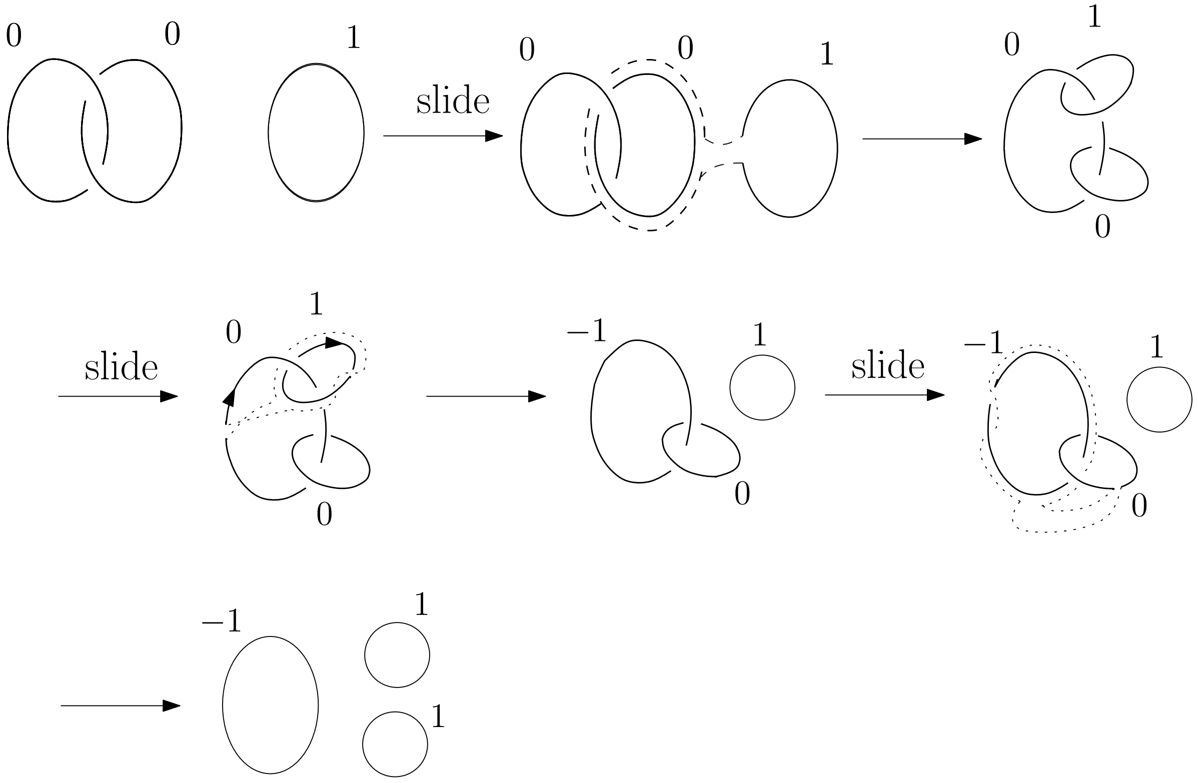


Figure 3.9 Diffeomorphism between 4-manifolds

CHAPTER FOUR

GLUCK TWIST

Firstly we discuss about the Gluck Twist operation which was introduced first by Herman Gluck in 1961 in Gluck (1962) and as a result of this operation some candidate of the exotic 4-spheres are obtained.

4.1 Definitions and Examples

Our main point is to understand description of the Gluck Twist operation. Then we will try to reinforce our description on some pictures.

Definition 4.1.1. Let $T : S^2 \times S^1 \rightarrow S^2 \times S^1$ be a self diffeomorphism defined by $T(x, y) = (\phi_y(x), y)$ where ϕ_y denote the rotation of S^2 about the diameter through the north and south poles trough an angle $2\pi y$ in some fixed direction. The *Gluck twist* operation is cutting out tubular neighbourhood of 2-sphere $\nu(S^2) = S^2 \times D^2$ and gluing it back by T. Here we need to remark that, the only non-trivial self diffeomorphism of $S^2 \times S^1$ is T.

$$X \mapsto X_s = (X - \nu(S)) \cup_T (S^2 \times D^2)$$

Assume X is simply connected as explained in Gluck (1962) this operation on homologically trivial S^2 always gives a homeomorphic copy of X. But the question is $X_s \approx X$ or not.

4.1.1 Handlebody Description of Gluck Twist

To see Gluck construction in the handle picture we will prove a theorem.

Theorem 4.1.2. *The operation below coincide with the Gluck Twist.*

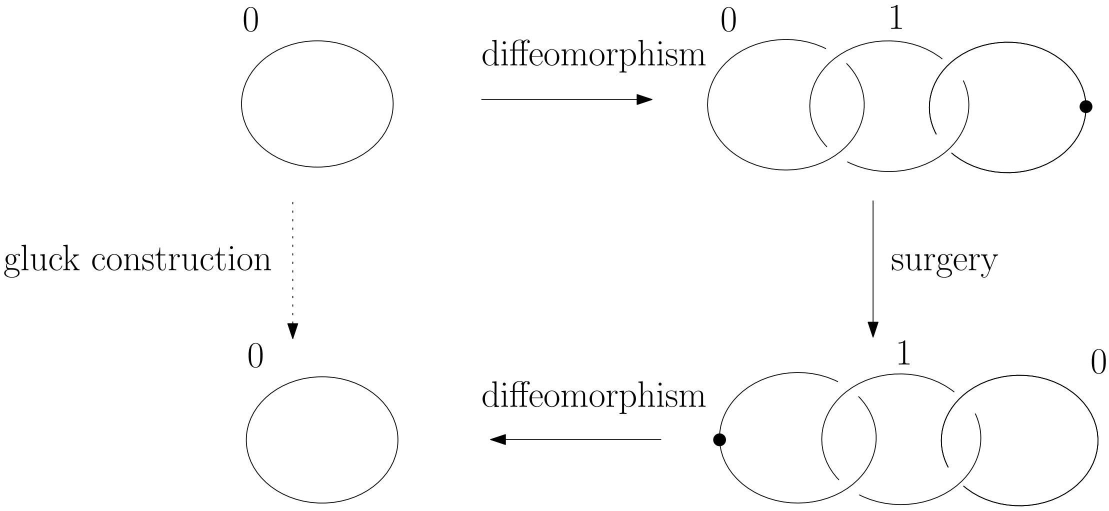


Figure 4.1 Construction of Gluck twist

Proof. It coincide with the cutting out $D^2 \times S^2$ and re-glue it back via self diffeomorphism of the boundary let illustrate it as ϕ . As we know there is only two self diffeomorphism of the $S^1 \times S^2$. Therefore, we only need to show that the self diffeomorphism of the boundary ϕ is not identity. To show it we apply the operation to $S^2 \times S^2$.

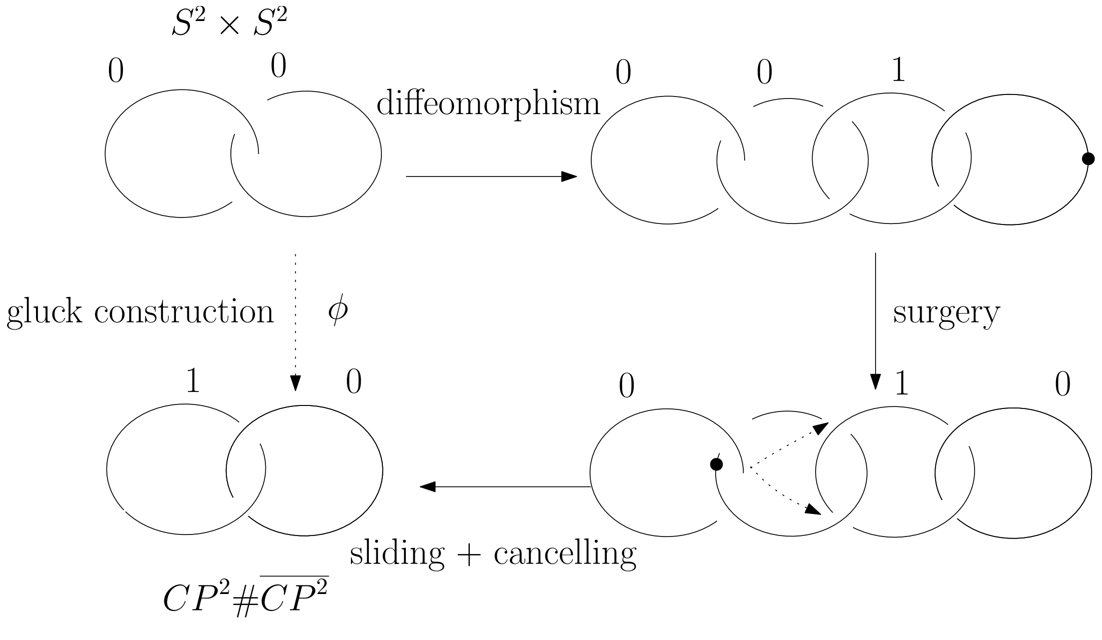


Figure 4.2 Proof of ϕ is not identity

Since $S^2 \times S^2$ is not diffeomorphic to the $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$ then we can conclude that, the operation ϕ is not identity. Therefore, the operation coincide with the Gluck construction.

□

Corollary 4.1.3. *The Gluck twist operation is pictured as below*

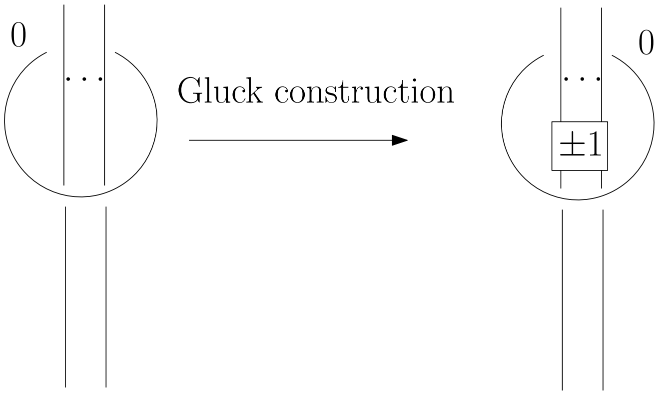


Figure 4.3 Picture of Gluck twist

Example 4.1.4. We can easily see that Gluck twist operation to unknotted S^2 in S^4 does not change diffeomorphism type of the S^4 by using standard handle decomposition of S^4. Indeed, We can image S^4 as one 2-handle and one 3-handle attached to 0-handle and capped with 4-handle. Using Theorem 2.5.12 we do not need to deal with 3-handles. Therefore its handlebody is shown as un-knotted circle with zero framing. It can be easily seen that Gluck twist does not change the diffeomorphism type of the S^4.

CHAPTER FIVE
CONCLUSION

In this thesis, we have researched into a handlebody decomposition of 4-dimensional closed connected smooth manifolds. Then we have introduce intersection theory, this study allows us to understand handlebody theory and calculation with handles deeply.

After this examination, we gave a brief introduction to Gluck construction in the general sense. Understanding handle decomposition and calculation methods with handles allows us to understand Gluck twist in terms of handlebody. Using this technique we easily show that Gluck twist to a trivial S^2 in S^4 does not change the diffeomorphism type of the S^4.

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