DOKUZ EYLÜL UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

MULTIPLE LIFE INSURANCE

by
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İZMİR

MULTIPLE LIFE INSURANCE

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by
Beste Hamiye SERTDEMİR

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M.Sc THESIS EXAMINATION RESULT FORM

We have read the thesis entitled **"MULTIPLE LIFE INSURANCE"** completed by **BESTE HAMİYE SERTDEMİR** under supervision of **DR. SEDAT ÇAPAR** and we certify that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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MULTIPLE LIFE INSURANCE

ABSTRACT

In this thesis, multiple life insurance products are investigated in the case of two lives. Initially two types of joint life policies, joint-life and last-survivor products, have been examined under the independence assumption of future lifetimes. In the case of married couples, the use of dependent mortality model have been studied the impact on pricing of last-survivor policies. The first purpose of the study is to compare the premium values of last - survivor products with the independence and dependence assumption of lifetimes of spouses. The second aim is to search whether using age difference factor in the model has the impact on dependence structure and premium valuation. Thus, Gumbel-Hougaard copula with Weibull marginal distribution function was chosen to generate the dependence structure because of its convenient functional form. Then, the parameters of Weibull survival distributions related to Turkey have been estimated for females and males according to three models: Independent, dependent and dependent with age difference variable. As a result, under the fixed interest rate assumption, the actuarial present values of joint last survivor insurances and annuities for all models have been calculated based on these parameter estimations related to Turkey, and the results have been compared as ratios together with three dimensional plots.

Keywords : Multiple life insurance, last-survivorship, joint life, actuarial present value, lifetime dependence, dependent mortality model, copula.

ÇOKLU YAŞAM SİGORTALARI

ÖZ

Bu tezde, iki yaşamın olduğu durumda çoklu hayat sigortası ürünleri incelenmiştir. İlk olarak bileşik hayat poliçelerinin iki türü, bileşik-yaşam ve son-hayatta kalan ürünleri, geriye kalan ömürlerin bağımsızlığı varsayımı altında incelenmiştir. Evli çiftlerin olduğu durumda, bağımlı ölüm modeli kullanımının son - hayatta kalan poliçelerinin fiyatlaması üzerindeki etkisine çalışılmıştır. Çalışmanın ilk amacı son-hayatta kalan ürünlerinin prim değerlerini eşlerin yaşamlarının bağımlılığı ve bağımsızlığı varsayımına göre karşılaştırmaktır. İkinci amaç modelde yaş farkı faktörü kullanımının bağımlılık yapısı ve prim değerlemesi üzerinde etkisi olup olmadığını araştırmaktır. Bu amaçla, fonksiyonel yapısının uygun olmasından dolayı bağımlılık yapısını oluşturmak için Gumbel-Hougaard copula ile Weibull marjinal dağılım fonksiyonu seçilmiştir. Daha sonra, Türkiye'ye ilişkin Weibull sağ kalım dağılımının parametreleri kadınlar ve erkekler için şu üç modele göre tahmin edilmiştir : Bağımsız, bağımlı ve yaş farkı değişkenin olduğu bağımlı model. Sonuç olarak, sabit faiz oranı varsayımı altında, bütün modeller için son - hayatta kalan sigortaları ve anüitelerinin aktüeryal peşin değerleri Türkiye' ye ilişkin parametre tahminlerine dayalı olarak hesaplanmış ve sonuçlar oran olarak üç boyutlu grafikleri ile birlikte karşılaştırılmıştır.

Anahtar sözcükler : Çoklu hayat sigortası, son-sağkalım, bileşik yaşam, aktüeryal peşin değer, yaşam bağımlılığı, bağımlı ölüm modeli, copula.

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CHAPTER ONE
INTRODUCTION

Life insurance is a contract between the insurance company (insurer) and the policy holder so that insurer promises to provide contingent payment contained in the policy and to help reduce financial adverse effects upon the death of the insured person. An important type of life insurance is multiple life insurance which is an extension of single life. These types of contracts cover two or more persons where the death benefit payable according to order of deaths. Multiple life insurances are usually preferred by couples to guarantee the future lifetime of the surviving spouse when one of the spouses dies. Also, these policies are used for family protection by parents. Moreover in business life, the companies prefer this policy type to guarantee their employers lifetime so-called group insurance. For these reasons, multiple life contracts are much more preferable by the policyholders. In this thesis, we will restrict our studies to situations covering two lives which are married.

Mainly, multi-life policies for two lives consist of two distinct statuses which give different benefits due to the order of deaths of insured and spouse. The first status exists if all members of group survive and fails upon the first death which is called as a joint-life status. The second status valids provided that at least one member is survive and fails upon the last death is known as a last-survivor status. In addition, various policies can be definable. These policy types are studied in detailed in Chapter 2. Understanding of multiple life insurance requires intensive mathematical background, and there are several beneficial references about multiple life insurances and annuities in actuarial literature. The good references for multi-life theory are Bowers et all (1986), Jordan (1991) and Dickson et all (2009). Chen (2010) is an excellent overview into analysis of joint life insurance with stochastic interest rate. Also Matvejevs and Matvejevs (2001), Das (2003), Bi (2008) and Hürlimann (2009) give good explanations for joint life insurance and its applications. Youn, Shemyakin and Herman (2002) presents a modified versions of some formulas involving basic relationships in multi-life functions related joint-life and last-survivor random variables.

Generally in insurance industry, organizations assume that mortalities of individuals are independent in case of more than one life, but the deaths of couples are not independent in real life. Some studies on this subject have showed dependency between age at deaths of individuals especially for married couples due to some factors such as common lifestyle, common disaster and broken-heart factor. Common life style is related to the partners' physical age, and it has a direct effect to the correlation between ages at death of spouses. The other two factors represent incidents occurring simultaneously such as traffic accident and catastrophe (common disaster) or close in calendar time (broken-heart factor). Particularly the third factor increases the mortality rate after the mortality of one's spouse. Such effects may have significant influence on present values related to multiple life actuarial functions (Dhaene, Vanneste and Wolthuis, 2000). Hürlimann (2009) showed that the models based on independence assumption overestimates the joint life net single and level premiums and underestimates the last survivor net single and level premiums. The dependency structure of future lifetimes of couples and the premium computations provided with the help of copula functions and common shock models. In this thesis, we focused on only copula functions to modeling the dependency of age at deaths, and it examined in detail in Chapter 3.

In recent years, the copula models have became an important part of actuarial science to construct dependencies between random variables and to calculate premium computations. The thing that makes it so appealing is its simplicity. Also, using copulas, it is possible to construct various dependence structures by using parametric or non-parametric models of the marginal distributions of lifetimes. The most useful reference using copulas in actuarial sciences and finance is Frees and Valdez (1998). They introduced actuaries to the concept of copulas, a tool for understanding relationships among multivariate outcomes. Also, Frees, Carriere and Valdez (1995) and Shemyakin and Youn (2006) give an overview to construction of copula models to investigate dependence effects on joint last survivor annuity values using parametric copulas. They fitted several copulas to the same data for actuarial calculations. Purwono (2005) briefly summarizes the foundations of construction of

Gaussian, Student and Archimedean models from parametric families of bivariate copulas, and have achieved joint last-survivor probabilities using copula and conditional copula models of joint survival assuming that the marginal survival functions are distributed two-parameter Weibull. Also, Shemyakin and Youn (2000) and Youn and Shemyakin (2001) examined the joint last survivor insurance valuation with dependent mortality models adding age difference between the spouses. Dhaene, Vanneste and Wolthuis (2000) handle the net single premiums of insurances and annuties based on more than one life statuses using Frechet lower and upper bounds. Denuit and Cornet (1999) deal with dependent future lifetimes on the reversionary annuity values taking into account Frechet-Hoeffding bounds and Norberg's Markov model. Luciano, Spreeuw and Vigna (2010) compared the values of reversionary annuities of three different generations under dependent mortality for couples, and achieved the result that dependence parameters located in copula of these generations should be distinct.

In this thesis, we examined both single and joint life insurances and annuities with its products, and these are presented detailed in Chapter 2. Chapter 3 includes exhaustive information about parametric copula families, and how the copulas using to construct dependency of couples lifetimes. In Chapter 4, we estimate the parameters of bivariate Gumbel-Hougaard copula functions of males and females for three models: Independent, dependent and dependent with age differences by using mortality table constructed for Turkey. We also calculate the actuarial present values of premiums of joint last-survivor insurances and annuities. The results are presented and compared as ratios of benefits for these models. The results given in Chapter 4 reveal that according to measures of association, the future lifetimes of spouses are actually dependent. In addition, age difference captures an extra dependency between survival functions of partners which may have a considerable impact of last survivor pricing. Calculations were performed in EXCEL.

CHAPTER TWO
MULTIPLE LIFE INSURANCES AND ANNUITIES

2.1 The Notion of Status

In applications of the life insurance and annuity, survival characteristics of several lives may be required for calculations of survival and failure probabilities and the payments. Thus, the concept of status has a great importance especially for life insurance products involving more than one life. Status is an artificially established life form for which there is a definition of survival and failure. The examples to understand concept of status would be a single life aged (*x*), which defines a status, fails when (*x*) dies exactly. Another example is a "life" $\overline{n|}$ that defines a status so-called term certain status. This status survives for exactly *n* unit times and then dies at the end of *n* unit times. The random variable of remaining lifetime of (*x*) denoted by *T*(*x*) will be considered as the period of survival of status, also can be considered as the time-until-failure of the status. Definition of status which depends on the type of insurance, brings about change of notations and formulations. For instance, the subscript $x:\overset{1}{\overline{n|}}$ shows that the payments are made if the first failure belongs to life $\overline{n|}$. The other subscript $\overset{1}{x}:\overline{n|}$ indicates us that the payments are made if the life (*x*) dies as first. The first of given examples is used to representation of the actuarial present value of pure endowment life insurance while the second is used to denote actuarial present value of term life insurance. If we consider from this perspective, these examples involve two lives, and we can perceive them as different types of multiple life statuses. When there are several lives, more complicated statuses are definable in various forms. Thus, firstly the status and its survival and failure should be well defined, and then we can apply based on definition in order to improve products of life annuities and insurances.

2.2 Multiple Life Statuses

As mentioned above, one of the main and well-known statuses is the single life status. In insurance market, the applications of multiple life are widespread as well as single life. We are familiar to the joint-life status which is one of the typical multiple life status, and the simplest form of this status is $x:\overline{n|}$. In this form, $\overline{n|}$ is the artificial life form, but we examine the statuses as combination of two or more individual lives. Such models are referred as multi-life models in actuarial science. Two well-known specific types of multiple life statuses are the joint-life and the last-survivor statuses. In this thesis, we restrict our interest to survival and failure of the status for two lives since we focus on some types of life insurances and annuities in which the time of the benefit payment based on two lives. But it can be extended for more than two lives if desired. In applications of life insurance, the future lifetime of two lives are assumed to be independent unless otherwise stated. In this case, the probabilistic expressions for single life can easily be expanded for multiple life. The same is true also for the formulas of the benefit payments. Briefly, under the independence assumption, actuarial functions of multiple-life can be expressed by means of single-life functions. On the other hand, the recent studies about future lifetime of two or more lives indicate that their future lifetimes are dependent because of some factors which are explained in detailed in Chapter 3. In dependent case, the future lifetimes are modelled by some copula functions which are also explained in Chapter 3.

2.2.1 The Joint-Life Status

The joint–life status is one of the common types of multiple life statuses. A joint life status involves several individual lives, and it requires the survival of all of these individual lives. In this case, it fails upon occurrence of the first death of one of its component lives. Joint-life status is denoted by $\left(x_1 x_2 ... x_m\right)$ for m lives with ages $x_1, x_2, ..., x_m$. The notation of this status for two lives with currently ages x and y is denoted by (xy) or ($x{:}y$). In this chapter, we examined the joint-life functions which are used in calculations of the actuarial present values of benefit payments of various life insurance policies. Also, we investigate the joint–life functions separately as continuous and discrete (or curtate) functions of joint–life status.

2.2.1.1 Continuous Joint-Life Functions

In continuous joint–life functions, the functions are obtained in terms of continuous future lifetime (or complete future lifetime) variable. In actuarial science, it is usually neccessary making probabilistic expression about this random variable. Therefore, we give essential formulas and notations based on this variable.

Let X represents a newborn's age-at-death random variable, assumed to be non-negatively continuous, and $T(x)$ represents the future lifetime random variable of an individual aged x, given that a newborn has survived to age x, is denoted as following:

$$T(x) = X - x \mid X > x \tag{2.1}$$

which is defined on the interval $[0,w_x]$ where w_x states the difference between the ultimate age of the lifetable and x. In the joint-life status, the two lives are regarded as a single entity which exists as long as both of them are alive, and fails upon the first death. In this case, time-to-failure random variable states the waiting time from now until either (x) or (y) dies and it is denoted as $T(xy)$. It equals the smaller of individuals's future lifetimes, $T(x)$ and $T(y)$, that are non-negative continuous random variables. The time-until-failure of the joint-life status is mathematically defined as:

$$T(xy) = \min\{T(x), T(y)\} = \begin{cases} T(x) & T(x) \le T(y) \\ T(y) & T(x) > T(y) \end{cases} \tag{2.2}$$

In most of the life insurance applications, survival probabilities are one of the necessary elements. At this stage, the survival function describes the probability of survival of a newborn until the age of x. The survival distribution function (sdf) of X, denoted by $s_X(x)$ is defined in equation (2.3).

$$s_X(x) = 1 - F_X(x) = P(X > x) = {}_x p_0 \qquad x \ge 0 \tag{2.3}$$

where $F_X(x)$ is the probability of death of a newborn at or prior to age x also denoted by ${}_x q_0$. The survival function of future lifetime random variable also denoted as ${}_t p_x$ gives the probability of attaining age $x+t$ of (x) is given as:

$$
\begin{aligned}
s_{T(x)}(t) = P(T(x) > t) = P(X > x+t \mid X > x) & \\
&= \frac{P\left(\{X > x+t\} \cap \{X > x\}\right)}{P(X > x)} = \frac{P\left(X > x+t\right)}{P(X > x)} \\
&= \frac{s_X(x+t)}{s_X(x)} = {}_t p_x
\end{aligned}
\tag{2.4}
$$

The existence of joint-life status requires the survival of all component lives in t years so that the event, $\{T(xy) > t\}$, and it is the intersection of two independent events, $\{T(x) > t\}$ and $\{T(y) > t\}$.

$$
\{T(xy) > t\} = \{T(x) > t\} \cap \{T(y) > t\} \tag{2.5}
$$

Then, the joint sdf of $T(xy)$ also denoted by ${}_t p_{xy}$ is obtained as:

$$
\begin{aligned}
s_{T(xy)}(t) &= P\left(T(xy) > t\right) = P\left(\min\{T(x), T(y)\} > t\right) \\
&= P\left(\{T(x) > t\} \ and \ \{T(y) > t\} \middle| X > x, Y > y\right) \\
&= P\left(T(x) > t \middle| X > x\right) P\left(T(y) > t \middle| Y > y\right)
\end{aligned}
\tag{2.6}
$$

which equals to product of the marginal survival probabilities ${}_t p_{xy} = {}_t p_x \, {}_t p_y$.

The probability of death of life-aged-x within t years, which is expressed by the cumulative distribution function (cdf) of the future lifetime variable, is obtained as:

$$
\begin{aligned}
F_{T(x)}(t) = P\left(T(x) \le t\right) = P\left(X \le x+t \mid X > x\right) & \\
&= 1 - P\left(X > x+t \mid X > x\right) = {}_t q_x
\end{aligned}
\tag{2.7}
$$

To obtain the cdf of the time–until–failure random variable of the joint–life status, the event, $\{T(xy) \le t\}$, should be examined. This event is equal the union of the two events as $\{T(x) \le t\}$ and $\{T(y) \le t\}$ since these two events are not mutually exclusive events.

$$\{T(xy) \le t\} = \{T(x) \le t\} \cup \{T(y) \le t\} \tag{2.8}$$

The cdf of $T(xy)$ gives the probability of the failure of the joint–life status when the first death occurs or both lives fails upon within time t. Then, this function is reflected as

$$\begin{aligned} F_{T(xy)}(t) &= P\left(T(xy) \le t\right) = P\left(\min\{T(x), T(y)\} \le t\right) \\ &= P\left(T(x) \le t \text{ } or \text{ } T(y) \le t\right) \\ &= P\left(T(x) \le t\right) + P\left(T(y) \le t\right) - P\left(T(x) \le t \text{ } and \text{ } T(y) \le t\right) \end{aligned} \tag{2.9}$$

It can be expressed by using standard actuarial notations

$${}_tq_{xy} = {}_tq_x + {}_tq_y - {}_tq_x \; {}_tq_y \tag{2.10}$$

The probability density function (pdf) of a random lifetime variable X is an instantaneous measurement of death for a given age, and it relates to the any point in time when the sdf and the cdf give the probabilities on time intervals. When the derivative of the cdf or the survival function exists, the pdf of X is given by:

$$f_X(x) = \frac{dF_X(x)}{d_x} = -\frac{ds_X(x)}{d_x} \qquad x \ge 0 \tag{2.11}$$

Also, this function is obtained by

$$f_X(x) = s_X(x)\mu(x) = {}_0p_x \; \mu(x) \tag{2.12}$$

where $\mu(x)$ denotes the force of mortality at age *x*. The pdf for the future lifetime random variable represents the conditional density at age *x+t*, given survival to age *x*. We can get an expression for the probability density function of *T(x)* of a single life:

$$
\begin{aligned}
f_{T(x)}(t) &= \frac{dF_{T(x)}(t)}{dt} = -\frac{ds_{T(x)}(t)}{dt} = \frac{f_X(x+t)}{1-F_X(x)} = \frac{f_X(x+t)}{s_X(x)} = \frac{{}_{x+t}p_0\ \mu(x+t)}{{}_x p_0} \\
&= \frac{{}_x p_0\ {}_t p_x\ \mu(x+t)}{{}_x p_0} = \underbrace{{}_t p_x}_{survival\ function}\ \underbrace{\mu(x+t)}_{force\ of\ mortality}
\end{aligned}
\tag{2.13}
$$

In terms of joint-life status, the pdf of time-until-failure random variable is obtained by using survival distribution function ${}_t p_{xy} = {}_t p_x\ {}_t p_y$.

$$
\begin{aligned}
f_{T(xy)}(t) &= \frac{dF_{T(xy)}(t)}{dt} = -\frac{ds_{T(xy)}(t)}{dt} = -\frac{d_t p_{xy}}{dt} = -\frac{d_t p_x\ {}_t p_y}{dt} \\
&= -\left(\left(\frac{d_t p_x}{dt}\right)\ {}_t p_y + \left(\frac{d_t p_y}{dt}\right)\ {}_t p_x\right) \\
&= -\left(\left(-{}_t p_x \mu(x+t)\right)\ {}_t p_y + \left(-{}_t p_y \mu(y+t)\right)\ {}_t p_x\right) \\
&= {}_t p_x \mu(x+t)\, {}_t p_y + {}_t p_y \mu(y+t)\, {}_t p_x \\
&= {}_t p_x\ {}_t p_y \left(\mu(x+t) + \mu(y+t)\right) \\
&= \underbrace{{}_t p_{xy}}_{survival\ function}\ \underbrace{\left(\mu(x+t) + \mu(y+t)\right)}_{force\ function}
\end{aligned}
\tag{2.14}
$$

The force of mortality has an important role in mortality analysis. The force of mortality at age *x* defines the probability of death between the ages of x and $x+\Delta x$ for a newborn, and this probability is conditioned on the survival to age *x* where Δx represents the short time interval. This conditional instantaneous measure provides a distribution which specify the probability of death in a very short period of time for a life of attained age *x*. Briefly, it is the conditional death rate at age *x* given survival to age *x*. Also, the terms failure rate or hazard rate function are used in reliability theory. The hazard rate is obtained depending on the definition as follows;

$$P\left(x < X \le x+\Delta x \middle| X > x\right) = \frac{F_X\left(x+\Delta x\right) - F_X\left(x\right)}{1 - F_X\left(x\right)} = \frac{F_X\left(x+\Delta x\right) - F_X\left(x\right)}{s_X\left(x\right)} \tag{2.15}$$

Since $f_X\left(x\right) = \dfrac{dF_X\left(x\right)}{dx}$, we can write $\dfrac{F_X\left(x+\Delta x\right) - F_X\left(x\right)}{\Delta x} \approx f_X\left(x\right)$. Therefore,

$$\frac{P\left(x < X \le x+\Delta x \middle| X > x\right)}{\Delta x} \approx \frac{f_X\left(x\right)}{s_X\left(x\right)} \tag{2.16}$$

In this case, the force of mortality at age *x* which is non-negative and piece-wise continuous function is denoted by μ_x or $\mu\left(x\right)$ and it is obtained as follows:

$$\mu\left(x\right) = \lim_{\Delta x \to 0} \frac{P\left(x < X \le x+\Delta x \middle| X > x\right)}{\Delta x} = \frac{f_X\left(x\right)}{s_X\left(x\right)} = \frac{f_X\left(x\right)}{1 - F_X\left(x\right)} \tag{2.17}$$

If we examine the force function in terms of future lifetime variable *T(x)*, it has similar meaning with lifetime variable *X*. The force of mortality of *T(x)* which is denoted by $\mu_{T(x)}\left(t\right)$ states the probability of the death at age *x+t* given survival to age *x+t*.

$$\mu_{T(x)}\left(t\right) = \lim_{\Delta t \to 0} \frac{P\left(t < T\left(x\right) \le t+\Delta t \middle| T\left(x\right) > t\right)}{\Delta t} \tag{2.18}$$

Since $T\left(x\right) = X - x$ the equation 2.18 holds as

$$\begin{aligned}
\mu_{T(x)}\left(t\right) &= \lim_{\Delta t \to 0} \frac{P\left(t < X - x \le t+\Delta t \middle| X - x > t\right)}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{P\left(x+t < X \le x+t+\Delta t \middle| X > x+t\right)}{\Delta t} \\
&= \mu\left(x+t\right)
\end{aligned} \tag{2.19}$$

The force function of time until-failure-random variable *T(xy)* of joint-life status represents the probability of occurrence of the first death in the future instant under the condition of the survival of both lives *x* and *y* for *t* years. The force function of *T*(*xy*) denoted as $\mu_{T(xy)}(t)$ or $\mu_{xy}(t)$ can be derived in a similar way with single life.

$$\begin{aligned}\mu_{T(xy)}(t) = \mu_{xy}(t) = \frac{f_{T(xy)}(t)}{1-F_{T(xy)}(t)} = \frac{f_{T(xy)}(t)}{s_{T(xy)}(t)} \\ = \frac{{}_t p_{xy}\ \left(\mu(x+t)+\mu(y+t)\right)}{{}_t p_{xy}} = \mu(x+t)+\mu(y+t)\end{aligned} \tag{2.20}$$

Conditional probabilities are frequently used in the valuation of contingent payments. The notation ${}_{n|}q_x$ represents a probability that a life aged *x* is alive for *n* years and then dies within the next year, and it is obtained as;

$$\begin{aligned}{}_{n|}q_x &= P\left[n < T(x) \le n+1\right] = P\left(T(x) \le n+1\right) - P\left(T(x) \le n\right) \\ &= {}_n p_x - {}_{n+1} p_x = {}_n p_x \left(1 - p_{x+n}\right) \\ &= {}_n p_x\ q_{x+n}\end{aligned} \tag{2.21}$$

where ${}_n p_x$ is the probability of survival of a life aged *x* within *n* years, and q_{x+n} is the probability of death of a life aged $x+n$ within a year. In case of *n* = 1, the prefixes in the notations are omitted and they are shown as q_x and p_x. When there are two lives, the probability of joint-life status (*x+n:y+n*) failing within one year can be expressed via probabilities of failures of individual lives as follows:

$$\begin{aligned}q_{x+n:y+n} &= 1 - p_{x+n:y+n} \\ &= 1 - p_{x+n}\ p_{y+n} \\ &= 1 - \left(1-q_{x+n}\right)\left(1-q_{y+n}\right) \\ &= 1 - \left(1 - q_{x+n} - q_{y+n} - q_{x+n}q_{y+n}\right) \\ &= q_{x+n} + q_{y+n} - q_{x+n}q_{y+n}\end{aligned} \tag{2.22}$$

The conditional probability of joint-life status describes the probability that the first death occurs between the *n*th and *n*+1th years since the status fails upon the first death of component lives. The probability conditional upon the survival of the status for *n* years is expressed as follows:

$$
\begin{aligned}
{}_{n|}q_{xy} &= P\left[n < T\left(xy\right) \le n+1\right] = P\left(T\left(xy\right) \le n+1\right) - P\left(T\left(xy\right) \le n\right) \\
&= {}_n p_{xy} - {}_{n+1} p_{xy} = {}_n p_{xy}\left(1 - p_{x+n:y+n}\right) \\
&= {}_n p_{xy}\ q_{x+n:y+n}
\end{aligned}
\tag{2.23}
$$

2.2.1.2 Curtate Joint –Life Functions

In life insurance applications, we are interested in individual's curtate future lifetime like continuous future lifetime. The curtate future lifetime random variable is associated with the continuous future lifetime random variable. This random variable is the integer part of the future lifetime $T(x)$, and it gives the number of complete years lived in the future by (x) prior to death. This random variable is denoted by $K(x)$ for a life aged x, and it is equal to the greatest integer of $T(x)$; $K(x) = \lfloor T(x) \rfloor$. The probability function of $K(x)$ can be expressed as:

$$
\begin{aligned}
P[K(x) = k] &= P[k \le T(x) < k+1] \\
&= {}_k p_x - {}_{k+1} p_x \\
&= {}_k p_x\ q_{x+k} \\
&= {}_{k|}q_x \qquad\qquad k = 0,1,2,\ldots
\end{aligned}
\tag{2.24}
$$

As seen above, this probability function represents the probability that a life aged x will survive for k years and then die within the following year. Its distribution function is the step function that is defined by:

$$
F_{K(x)}(k) = P[K(x) \le k] = \sum_{h=0}^{k} {}_{h|}q_x = {}_{k+1}q_x \qquad \text{k=0,1,2,\ldots}
\tag{2.25}
$$

The discrete functions of joint-life status are based on the curtate future lifetime random variable. As in single life case, the curtate future lifetime of (xy) describes the number of whole years completed by (xy) prior to first death, and it is equal to $K(xy) = \lfloor T(xy) \rfloor$. The probability mass function of this random variable is

$$
\begin{aligned}
P[K(xy) = k] &= P[k \le T(xy) < k+1] \\
&= {}_k p_{xy} - {}_{k+1} p_{xy} \\
&= {}_k p_{xy}\ q_{x+k:y+k} \\
&= {}_{k|}q_{xy} \qquad\qquad k = 0,1,2,\ldots
\end{aligned}
\tag{2.26}
$$

The distribution function of the curtate future lifetime variable of the joint - life status is obtained as:

$$F_{K(xy)}(k) = P[K(xy) \le k] = \sum_{h=0}^{k} {}_{h|}q_{xy} = {}_{k+1}q_{xy} \qquad \text{k=0,1,2,…} \tag{2.27}$$

2.2.2 The Last –Survivor Status

A last survivor status terminates upon the last death of component members, and it survives so long as at least one member remains alive. The status does not exist if and only if its all components die. Last - survivor status is denoted by $\left(\overline{x_1 x_2 ... x_m}\right)$ which is involving *m* lives with ages $x_1, x_2, ..., x_m$. Similarly as above, we are interested a pair of lives currently ages *x* and *y* and in this case the status is represented by $\left(\overline{xy}\right)$ or $\left(\overline{x:y}\right)$.

2.2.2.1 Continuous Last - Survivor Functions

Continuous functions related to the last-survivor status are obtained by considering the distribution of the time-until-failure random variable of this status. The time to failure random variable of last-survivor status is the largest of individual's remaining lifetimes, *T*(*x*) and *T*(*y*) because of the status fails on the second death in two lives case. Accordingly, the time until failure of the last-survivor status is the time until second death in bivariate case or the last death in general case. The future lifetime random variable denoted as $T\left(\overline{xy}\right)$ is equal to:

$$T\left(\overline{xy}\right) = \max\left\{T\left(x\right), T\left(y\right)\right\} = \begin{cases} T(x) & T(x) > T(y) \\ T(y) & T(x) \le T(y) \end{cases} \tag{2.28}$$

Survival of this status requires that (*x*) or (*y*) have been alive for *t* years, or both of them have remained as alive during *t* years. $\left\{T(\overline{xy}) > t\right\}$ represents the second death occurs after time *t*. Thus, the survival of last - survivor status is explained by union of two independent events $\left\{T(x) > t\right\}$ and $\left\{T(y) > t\right\}$;

$\left\{T(\overline{xy})>t\right\}=\{T(x)>t\}\cup\{T(y)>t\}$. In this case, the sdf is expressed as a probability of occurence of this event.

$$
\begin{aligned}
s_{T\left(\overline{xy}\right)}(t) &= P\left(T\left(\overline{xy}\right)>t\right)=P\left(\max\{T(x),T(y)\}>t\right) \\
&= P\left(\{T(x)>t\} \text{ or } \{T(y)>t\} \middle| X>x, Y>y\right) \\
&= P\left(T(x)>t \middle| X>x\right)+P\left(T(y)>t \middle| Y>y\right)-P\left(\{T(x)>t\} \text{ and } \{T(y)>t\} \middle| X>x, Y>y\right)
\end{aligned}
\tag{2.29}
$$

which is also reflected in actuarial science as;

$$
\begin{aligned}
{}_t p_{\overline{xy}} &= {}_t p_x + {}_t p_y - {}_t p_x \, {}_t p_y \\
&= {}_t p_x + {}_t p_y - {}_t p_{xy}
\end{aligned}
\tag{2.30}
$$

As it is seen in equation (2.30), the survival probability of the last-survivor status can be obtained by using survival probabilities of joint-life status and single life. The cumulative distribution function of $T\left(\overline{xy}\right)$ equals the probability of intersection of two independent events $\{T(x)\le t\}$ and $\{T(y)\le t\}$ since the failure of this status requires the death both of lives (x) and (y); $\left\{T(\overline{xy})\le t\right\}=\{T(x)\le t\}\cap\{T(y)\le t\}$. The cumulative distribution function is denoted as;

$$
\begin{aligned}
F_{T(\overline{xy})}(t) &= P\left(T(\overline{xy})\le t\right)=P\left(\max\{T(x),T(y)\}\le t\right) \\
&= P\left(T(x)\le t \text{ and } T(y)\le t\right) \\
&= P\left(T(x)\le t\right)P\left(T(y)\le t\right)
\end{aligned}
\tag{2.31}
$$

also denoted as actuarial notation as follows;

$$
\begin{aligned}
{}_t q_{\overline{xy}} &= 1-{}_t p_{\overline{xy}} = 1-\left({}_t p_x+{}_t p_y-{}_t p_{xy}\right)=1-\left({}_t p_x+{}_t p_y-{}_t p_x \, {}_t p_y\right) \\
&= 1-\left(\left(1-{}_t q_x\right)+\left(1-{}_t q_y\right)-\left(1-{}_t q_x\right)\left(1-{}_t q_y\right)\right) \\
&= {}_t q_x \, {}_t q_y
\end{aligned}
\tag{2.32}
$$

As mentioned earlier, the probability density function is obtained by a product of the survival and the hazard rate functions. So the pdf of $T\left(\overline{xy}\right)$ is calculated as;

$$f_{T(\overline{xy})}(t)=s_{T(\overline{xy})}(t)\ \mu_{T(\overline{xy})}(t)={}_t\,p_{\overline{xy}}\ \mu_{T(\overline{xy})}(t) \tag{2.33}$$

Based on the theory of probability, the density function is also stated as the derivative of the distribution function.

$$\begin{aligned}
f_{T(\overline{xy})}(t)&=\frac{dF_{T(\overline{xy})}(t)}{dt}=\frac{d_t q_{\overline{xy}}}{dt}=\frac{d\left({}_t q_x\ {}_t q_y\right)}{dt}\\
&=\frac{d_t q_x}{dt}\ {}_t q_y+\frac{d_t q_y}{dt}\ {}_t q_x\\
&=f_{T(x)}(t)\ {}_t q_y+f_{T(y)}(t)\ {}_t q_x\\
&=\left({}_t p_x\ \mu(x+t)\right)\ {}_t q_y+\left({}_t p_y\ \mu(y+t)\right)\ {}_t q_x
\end{aligned}$$

$$\boxed{f_{T(\overline{xy})}(t)={}_t p_x\ {}_t q_y\ \mu(x+t)+{}_t p_y\ {}_t q_x\ \mu(y+t)} \tag{2.34}$$

$$\begin{aligned}
&={}_t p_x\left(1-{}_t p_y\right)\ \mu(x+t)+{}_t p_y\left(1-{}_t p_x\right)\ \mu(y+t)\\
&={}_t p_x\ \mu(x+t)+{}_t p_y\ \mu(y+t)-\left({}_t p_x\,{}_t p_y\ \mu(x+t)+{}_t p_x\,{}_t p_y\ \mu(y+t)\right)\\
&={}_t p_x\ \mu(x+t)+{}_t p_y\ \mu(y+t)-\left({}_t p_x\,{}_t p_y\ \left(\mu(x+t)+\mu(y+t)\right)\right)
\end{aligned}$$

$$\boxed{f_{T(\overline{xy})}(t)={}_t p_x\ \mu(x+t)+{}_t p_y\ \mu(y+t)-{}_t\,p_{xy}\ \mu_{xy}(t)}$$

The hazard rate function or also known as the force function equals to the ratio of pdf to sdf, and it is denoted by $\mu_{T(\overline{xy})}(t)$.

$$\mu_{T(\overline{xy})}(t)=\mu_{\overline{xy}}(t)=\frac{f_{T(\overline{xy})}(t)}{1-F_{T(\overline{xy})}(t)}=\frac{f_{T(\overline{xy})}(t)}{s_{T(\overline{xy})}(t)} \tag{2.35}$$

In contrast to joint-life status, the hazard rate of $T\left(\overline{xy}\right)$ represents the probability of occurence of the second death in the future instant under the condition of survival of the status for *t* years. If the necessary functions are positioned in equation (2.35), the hazard rate function is hold as follows;

$$\mu_{T(\overline{xy})}(t) = \frac{{}_tp_x \; {}_tq_y \; \mu(x+t) + {}_tp_y \; {}_tq_x \; \mu(y+t)}{{}_tp_x + {}_tp_y - {}_tp_{xy}}$$
$$= \frac{{}_tp_x \; \mu(x+t) + {}_tp_y \; \mu(y+t) - {}_tp_{xy} \; \mu_{xy}(t)}{{}_tp_x + {}_tp_y - {}_tp_{xy}} \tag{2.36}$$

The conditional probability in terms of last-survivor status means that the probability of second death occurs between *n*th and *n*+1th years conditioned on at least one member is alive. This probability is calculated as in equation (2.37).

$$\begin{aligned} {}_{n|}q_{\overline{xy}} &= P\left[n < T\left(\overline{xy}\right) \le n+1\right] = P\left(T\left(\overline{xy}\right) \le n+1\right) - P\left(T\left(\overline{xy}\right) \le n\right) \\ &= {}_np_{\overline{xy}} - {}_{n+1}p_{\overline{xy}} = \left({}_np_x + {}_np_y - {}_np_{xy}\right) - \left({}_{n+1}p_x + {}_{n+1}p_y - {}_{n+1}p_{xy}\right) \\ &= {}_{n|}q_x + {}_{n|}q_y - {}_{n|}q_{xy} \end{aligned} \tag{2.37}$$

2.2.2.2 Curtate Last - Survivor Functions

The curtate future lifetime random variable of last-survivor status is explained as the number of completed years before the status fails. It is denoted by $K(\overline{xy})$.

$$K(\overline{xy}) = \left\lfloor T(\overline{xy}) \right\rfloor \tag{2.38}$$

The probability function of $K(\overline{xy})$ is expressed as in equation (2.39).

$$\begin{aligned} P[K(\overline{xy}) = k] &= P[k \le T(\overline{xy}) < k+1] \\ &= {}_kp_{\overline{xy}} - {}_{k+1}p_{\overline{xy}} \\ &= {}_kp_{\overline{xy}} \; q_{\overline{x+k:y+k}} \\ &= \left({}_kp_x + {}_kp_y - {}_kp_x \; {}_kp_y\right)\left(q_{x+k} \; q_{y+k}\right) \\ &= {}_kp_x \; q_{x+k} + {}_kp_y \; q_{y+k} - {}_kp_x \; {}_kp_y \left(q_{x+k} + q_{y+k} - q_{x+k} \; q_{y+k}\right) \\ &= {}_{k|}q_{\overline{xy}} \end{aligned} \tag{2.39}$$

The cdf of curtate future lifetime variable is

$$F_{K(\overline{xy})}(k) = P[K(\overline{xy}) \le k] = \sum_{h=0}^{k} {}_{h|}q_{\overline{xy}} = {}_{k+1}q_{\overline{xy}} \tag{2.40}$$

2.3 Life Annuity and Insurance Models

The purpose of established insurance organization is to take measures against the adverse financial impacts of random events. The basic logic of life insurance is an exchange. The policyholder named insured pays a consideration, called the premium, in return the insurance organization (insurer) pay a predetermined lump sum which is called the sum insured or benefit if the certain event defined in contract occurs. The life annuity and insurance models deal with valuation of the payments contingent on the survival or death of the insured and general sense these models are referred to as the life contingency models due to this reason. Insureds can change the insurance period in their policies if they want to have higher or lower sum insured at the end of this period. In this case, benefits which are depend on the amounts paid by the insureds need to be recalculated. The life insurance organization can invest the premiums and then yields of assets provide to pay the benefits. In the field of life contingent payment model, analysis of the payments consists of two important parts. The first part is living or death contingency which is modeled by means of probability theory. The other is considered as the time value of money in life insurance theory. The benefit payments and premiums can take place in different ways at various points of time.

Net single premiums are used to calculate the present value of benefits. Before the defining of the formula of net single premium, the present value random variable which is changeable according to types of life insurances and annuities should be determined. Let b_t and v_t represent benefit and discount rate, respectively. The present value function can be defined as follows;

$$z_t = b_t v_t \tag{2.41}$$

The expected value of equation (2.41) is named as actuarial present value. Since the contingent future benefit depends on discount rate by implication timing and the death or survival of an insured, the present value of the benefit depends on these elements, and it is modeled as a random variable. It should be note that, while benefit payments are made at the failure of the status for insurance policy, an annuity is payable provided that the status survives. Furthermore, insurance payments can be modelled contingent on the specific order of the deaths of the individuals.

2.3.1 Life Insurances for Multiple Life Statuses

In life insurance, benefit payments are provided contingent upon the survival of the insured for a certain period, or upon the death of the insured in a certain period. Usually benefits are payable contingent upon the death of the policyholder in order to designated beneficiary receives the payments. If benefit is paid at the moment of death, then it is evaluated within continuous model. So present value random variable, which is referred to as random present value of benefit payable in some sources $\overline{Z}$ is a decreasing function of future lifetime $T(x)$. If benefit is payable at the end of year of death, it is taken into consideration within the framework of discrete models. Thus in such models, present value random variable, Z is a function of curtate-future-lifetime $K(x)$. In next subsections we described the features of commonly issued life insurance policies.

2.3.1.1 Whole Life Insurance

Whole life insurance is one of the traditional products which provides lump sum of death benefit to the policyholder or beneficiary when the insured dies at any time in the future. Since this type of the life insurance covers whole life, in any case benefit payments are made by the insurer when the death of the insured occurs. Whole life insurance is the limiting case of n-year term insurance as $n \to \infty$ (Bowers et al, 1986). There are two models of whole life insurance; discrete and continuous life insurance models.

In discrete model, curtate-future-lifetime of insured at the policy issue which is associated with present value of the benefit is used because of the assumption that benefits of life insurance are paid at the end of the year of the death. Also this assumption simplify the calculations since it enables the use of life tables. For discrete life insurance model, the present value random variable in the case of single life is

$$Z = v^{K(x)+1} \tag{2.42}$$

The net single premium is described by

$$A_x = E[Z] = E\left[v^{K(x)+1}\right] = \sum_{k=0}^{\infty} v^{k+1} {}_k p_x \; q_{x+k} = \sum_{k=0}^{\infty} v^{k+1} {}_{k|}q_x \tag{2.43}$$

When $K(xy)$ denotes the curtate-future-lifetime random variable of joint-life status, the present value function of the benefit of a unit payment is given by

$$Z_{xy} = v^{K(xy)+1} \tag{2.44}$$

For a joint life status in which case benefits are paid on the first death, the net single premium is obtained as

$$A_{xy} = E\left[Z_{xy}\right] = E\left[v^{K(xy)+1}\right] = \sum_{k=0}^{\infty} v^{k+1} {}_k p_{xy} \; q_{x+k:y+k} = \sum_{k=0}^{\infty} v^{k+1} {}_{k|}q_{xy} \tag{2.45}$$

If we consider the last – survivor status, the present value random variable is a function of curtate - future - lifetime variable $K(\overline{xy})$.

$$Z_{\overline{xy}} = v^{K(\overline{xy})+1} \tag{2.46}$$

The actuarial present value of a unit payment on the second death is obtained as follows;

$$A_{\overline{xy}} = E\left[Z_{\overline{xy}}\right] = E\left[v^{K(\overline{xy})+1}\right] = \sum_{k=0}^{\infty} v^{k+1} \left({}_k p_x \; q_{x+k} + {}_k p_y \; q_{y+k} - {}_k p_{xy} \; q_{x+k:y+k} \right) = \sum_{k=0}^{\infty} v^{k+1} {}_{k|}q_{\overline{xy}} \tag{2.47}$$

On the other hand, in continuous case, for a single life aged (x), the benefit payments are made at the moment of the death of the insured. The present value function is demonstrated as the function of continuous future lifetime random variable, $T(x)$, as follows:

$$\overline{Z} = v^{T(x)} \tag{2.48}$$

Then benefit of a unit payment at age of issue x for continuous whole life insurance model is denoted by $\bar{A}_x$, and it is calculated by given formula:

$$\bar{A}_x = E\left[\bar{Z}\right] = E\left[v^{T(x)}\right] = \int_0^\infty v^t f_{T(x)}(t)dt = \int_0^\infty v^t \,{}_t p_x \mu(x+t)dt \tag{2.49}$$

If a unit payment is provided immediately on the first death of the pair, random present value of the payment is related to $T(xy)$.

$$\bar{Z}_{xy} = v^{T(xy)} \tag{2.50}$$

Its first moment is equal to the actuarial present value of joint life insurance which is denoted $\bar{A}_{xy}$:

$$\bar{A}_{xy} = E\left[\bar{Z}_{xy}\right] = E\left[v^{T(xy)}\right] = \int_0^\infty v^t f_{T(xy)}(t)dt = \int_0^\infty v^t \,{}_t p_{xy} \;\mu_{T(xy)}(t)dt = \int_0^\infty v^t \,{}_t p_x \;{}_t p_y \;\left(\mu(x+t)+\mu(y+t)\right)dt \tag{2.51}$$

When we take into account an insurance failed upon the second death of a couple lives, the present value random variable depends on the $T(\overline{xy})$.

$$\bar{Z}_{\overline{xy}} = v^{T(\overline{xy})} \tag{2.52}$$

The actuarial present value of last survivor insurance is shown by $\bar{A}_{\overline{xy}}$ and its formulation is obtained as follows:

$$\begin{aligned} \bar{A}_{\overline{xy}} &= E\left[\bar{Z}_{\overline{xy}}\right] = E\left[v^{T(\overline{xy})}\right] = \int_0^\infty v^t f_{T(\overline{xy})}(t)dt = \int_0^\infty v^t \,{}_t p_{\overline{xy}} \;\mu_{T(\overline{xy})}(t)\,dt \\ &= \int_0^\infty v^t \left(\,{}_t p_x \;\mu(x+t) + \,{}_t p_y \;\mu(y+t) - \,{}_t p_{xy} \left(\mu(x+t)+\mu(y+t)\right)\right) dt \\ &= \bar{A}_x + \bar{A}_y - \bar{A}_{xy} \end{aligned} \tag{2.53}$$

2.3.1.2 Term Life Insurance

In term life insurance which is cheaper than whole life insurance, a lump sum benefit is paid only if the insured dies within term of policy's issue. Unless death of the policyholder occurs during the period based on the policy, the indemnity must not be paid by the insurer. In case of death of the insured, the term life insurance ensures the sum insured to the dependents of the policyholder.

If insured life-aged (x) dies within n-year period of time, the compensation is paid by the insurer and this term insurance is referred to as n-year term life insurance. Under the discrete model, benefits are payable at the end of the year of the death as a general rule. When we assumes a unit of benefit payment for single life, the present-value random variable is obtained as following:

$$Z = \begin{cases} v^{K(x)+1} & K = 0,1,...,n\text{-}1 \\ 0 & K \geq n \end{cases}$$

The actuarial present value is denoted by $A^{1}_{x:\overline{n}|}$

$$A^{1}_{x:\overline{n}|} = E\left[Z\right] = \sum_{k=0}^{n-1} v^{k+1} P(K(x) = k) = \sum_{k=0}^{n-1} v^{k+1} {}_{k|}q_x \tag{2.54}$$

In joint-life status, n-year term life insurance provides payment if first death occurs within n years. The present value random variable is obtained by substituting (xy) for (x);

$$Z_{xy} = \begin{cases} v^{K(xy)+1} & K = 0,1,...,n\text{-}1 \\ 0 & K \geq n \end{cases}$$

Its actuarial present value is

$$A^{\ 1}_{xy:\overline{n}|} = E\left[Z_{xy}\right] = \sum_{k=0}^{n-1} v^{k+1} P(K(xy) = k) = \sum_{k=0}^{n-1} v^{k+1} {}_{k|}q_{xy} \tag{2.55}$$

In case of last–survivor status, discrete n-year term life insurance pays 1 at the end of year of the second death within n years, and its random present value is given by;

$$Z_{\overline{xy}} = \begin{cases} v^{K(\overline{xy})+1} & K = 0,1,...,n\text{-}1 \\ 0 & K \geq n \end{cases}$$

When last-survivor status is failed in period of n years, the actuarial present value of benefit of a unit payment is given as follows;

$$A^{1}_{\overline{xy}:\overline{n}|} = E\left[Z_{\overline{xy}}\right] = \sum_{k=0}^{n-1} v^{k+1} P(K(\overline{xy}) = k) = \sum_{k=0}^{n-1} v^{k+1}\,{}_{k|}q_{\overline{xy}} \tag{2.56}$$

In continuous case, the benefit is payable at the time of the death. The present value for continuous n - year term insurance policy is a decreasing function of $T(x)$.

$$\bar{Z} = \begin{cases} v^T & T \le n \\ 0 & T > n \end{cases}$$

The net single premium of a unit of benefit payment is denoted as

$$\bar{A}^{1}_{x:\overline{n}|} = E\left[\bar{Z}\right] = \int_0^n v^t f_{T(x)}(t)\,dt \tag{2.57}$$

If the first death of lives (x) and (y) occurs within n years, payment of 1 is made at the moment of death. The present value is a function of $T(xy)$.

$$\bar{Z}_{xy} = \begin{cases} v^T & T \le n \\ 0 & T > n \end{cases}$$

The actuarial present value of the joint-life status is denoted by $\bar{A}^{1}_{xy:\overline{n}|}$, and the "cup" in the notation shows that the joint - life status must fail before the term certain status fails.

$$\bar{A}^{1}_{xy:\overline{n}|} = E\left[\bar{Z}_{xy}\right] = \int_0^n v^t f_{T(xy)}(t)\,dt = \int_0^n v^t\,{}_t p_{xy}\;\mu_{T(xy)}(t)\,dt \tag{2.58}$$

If the second death of lives (x) and (y) occurs within n years, benefit is payable immediately on the death. The random present value is;

$$\bar{Z}_{\overline{xy}} = \begin{cases} v^T & T \le n \\ 0 & T > n \end{cases}$$

Its actuarial present value is given by as follows;

$$\bar{A}^{\;1}_{\overline{xy}:\overline{n}|} = E\left[\bar{Z}_{\overline{xy}}\right] = \int_0^n v^t f_{T(\overline{xy})}(t)\,dt = \int_0^n v^t \,{}_t p_{\overline{xy}}\; \mu_{T(\overline{xy})}(t)\,dt \tag{2.59}$$

2.3.1.3 Pure Endowment

Insurance company made payments of benefits after *n* years from the starting date of the contract for which is called *n*-year pure endowment insurance if and only if the insureds are alive at the end of *n* years. The actuarial present value of an *n*-years pure endowment insurance for single life, joint-life and last-survivor statuses are given in equations (2.60) - (2.62), respectively. While in joint - life status the benefit is paid if the first death does not occur within *n* years, in last survivor status the benefit is paid if the second death does not occur within n years.

$$A_{x:\overline{n}|}^{\;\;\;1} = {}_nE_x = v^n\,{}_np_x \tag{2.60}$$

$$A_{xy:\overline{n}|}^{\;\;\;1} = {}_nE_{xy} = v^n\,{}_np_{xy} \tag{2.61}$$

$$A_{\overline{xy}:\overline{n}|}^{\;\;\;1} = {}_nE_{\overline{xy}} = v^n\,{}_np_{\overline{xy}} \tag{2.62}$$

2.3.1.4 Endowment Life Insurance

Endowment life insurance or mixed life insurance, which has characteristics of risk and savings, is one of the traditional insurance contracts. It is considered as a combination of the term insurance and pure endowment insurance. The premiums can be higher than other life insurance products since the survival and failure probabilities are used in this type of life insurance. In *n*-year endowment insurance policy, if the insured person dies within *n* years, he or she gets a benefit which is equal to the benefit of an *n*-year term insurance policy. On the other hand, if the insured survives at the end of *n* years, the insured gets a benefit which is equal to the benefit of an *n*-year pure endowment insurance.

In discrete model, the benefit is paid to the insured or beneficiary at the policy anniversary immediately following the death, and the random present value for this case is a decreasing function of $K(x)$, is denoted as;

$$Z = \begin{cases} v^{K+1} & K \leq n\text{-}1 \\ v^{n} & K \geq n \end{cases} \tag{2.63}$$

As mentioned above, an n-year endowment policy is the combination of an n-year term insurance and n-year pure endowment insurance policies. This is valid for all cases (joint -life and last - survivor statuses), and they are given in equations (2.64) - (2.66).

$$\begin{aligned} A_{x:\overline{n}|} &= E[Z] = \sum_{k=0}^{n-1} v^{k+1} P(K(x)=k) + \sum_{k=n}^{\infty} v^{n} P(K(x)=k) = \sum_{k=0}^{n-1} v^{k+1}\, {}_{k|}q_{x} + v^{n}\, {}_{n}p_{x} \\ &= A^{1}_{x:\overline{n}|} + A_{x:\overset{1}{\overline{n}|}} \end{aligned} \tag{2.64}$$

$$\begin{aligned} A_{xy:\overline{n}|} &= E[Z] = \sum_{k=0}^{n-1} v^{k+1} P(K(xy)=k) + \sum_{k=n}^{\infty} v^{n} P(K(xy)=k) = \sum_{k=0}^{n-1} v^{k+1}\, {}_{k|}q_{xy} + v^{n}\, {}_{n}p_{xy} \\ &= A^{\ 1}_{xy:\overline{n}|} + A_{xy:\overset{1}{\overline{n}|}} \end{aligned} \tag{2.65}$$

$$\begin{aligned} A_{\overline{xy}:\overline{n}|} &= E\left[Z_{\overline{xy}}\right] = \sum_{k=0}^{n-1} v^{k+1} P(K(\overline{xy})=k) + \sum_{k=n}^{\infty} v^{n} P(K(\overline{xy})=k) = \sum_{k=0}^{n-1} v^{k+1}\, {}_{k|}q_{\overline{xy}} + v^{n}\, {}_{n}p_{\overline{xy}} \\ &= A^{1}_{\overline{xy}:\overline{n}|} + A_{\overline{xy}:\overset{1}{\overline{n}|}} \end{aligned} \tag{2.66}$$

In continuous model, the death benefit is paid to the insured immediately on death. As in discrete model, the random present value is the function of $T(x)$, and it is denoted in equation (2.67). The actuarial present values for three statuses are given in equations (2.68) - (2.70).

$$\overline{Z} = \begin{cases} v^{T} & T \leq n \\ v^{n} & T > n \end{cases} \tag{2.67}$$

$$\bar{A}_{x:\overline{n}|} = E\left[\bar{Z}\right] = \int_0^n v^t f_{T(x)}\left(t\right)dt + \int_n^\infty v^n f_{T(x)}(t)dt = \int_0^n v^t f_{T(x)}\left(t\right)dt + v^n{}_n p_x$$
$$= \bar{A}^{1}_{x:\overline{n}|} + A_{x:\overline{n}|}{}^{1} \tag{2.68}$$

$$\bar{A}_{xy:\overline{n}|} = E\left[\bar{Z}_{xy}\right] = \int_0^n v^t f_{T(xy)}\left(t\right)dt + \int_n^\infty v^n f_{T(xy)}(t)dt = \int_0^n v^t f_{T(xy)}\left(t\right)dt + v^n{}_n p_{xy}$$
$$= \bar{A}^{1}_{xy:\overline{n}|} + A_{xy:\overline{n}|}{}^{1} \tag{2.69}$$

$$\bar{A}_{\overline{xy}:\overline{n}|} = E\left[\bar{Z}_{\overline{xy}}\right] = \int_0^n v^t f_{T(\overline{xy})}\left(t\right)dt + \int_n^\infty v^n f_{T(\overline{xy})}(t)dt = \int_0^n v^t f_{T(\overline{xy})}\left(t\right)dt + v^n{}_n p_{\overline{xy}}$$
$$= \bar{A}^{1}_{\overline{xy}:\overline{n}|} + A_{\overline{xy}:\overline{n}|}{}^{1} \tag{2.70}$$

2.3.2 Life Annuities for Multiple Life Statuses

In non-random setting a series of payments, which can be equal to each other, increasing or decreasing made at regular intervals in time are known as annuity certain. A sequence of payments are made (or received) at equal intervals over the future lifetime of a person in life annuity. The recipient is called an annuitant (Dickson et al, 2009). Since annuities are contingent upon the survival of the annuitant which are opposite of life insurances, they are called life annuities also known as contingent annuities. A series of benefit payments are contingent on survival of policyholder's life, and the future lifetime of the annuitant is unknown, so the present values of life annuity benefits are unknown. Generally, life annuities are preferred by older people to obtain additional income during retirement. Buying a whole life annuity guarantees that the income will not run out before the annuitant dies (Dickson et al, 2009). Life annuities can be classified in various ways according to the period covered by annuity. Also like in life insurances, these annuities are examined within discrete or continuous model whether payments are made continuously or at equal interval. The essential ingredients of the life annuities are considered as survival probability and present value of payments. In terms of the time of payments, life annuities can be examined basically in three different models. If payments are made at the beginnings of each period, then they are called as annuities due. If the payments are made at the end of each period, these are known as annuities immediate. The third model is continuous model. In this model, the payments are made continuously. Apart from these, payments can be made semi – annually, quarterly or monthly etc. but we will not be interested in annuities payable more frequently than once a year for calculation of benefits. We are interested in annually premium payments. Annuities due and immediate are evaluated as discrete models. We wil consider annuity due under the discrete and continuous models of life annuity. We examined them for a single life, then we extended them to the joint life and last survivor statuses of two lives. In case of two lives, the major products are considered as joint life annuity and last survivor annuity which are particularly preferred by married couples. A joint life annuity continues as long as two lives survive, and it stops in case of the occurrence of the first death. In a last survivor annuity, payments continue to be made provided that at least one of the lives remains alive. In discrete model, the random present value is a function of *K(x)* while in continuous model, it is the function of $T(x)$ We deal with the present values of commonly used types of life annutities as net single premiums.

2.3.2.1 Whole Life Annuity

A whole life annuity continues until the annuitant dies. When the premiums are paid annually, there are a total of $K(x)+1$ payments since discrete life annuities depend on curtate future lifetime of (x). The random present value of a whole life annuity - due is given as

$$Y = \ddot{a}_{\overline{K(x)+1}|} \tag{2.71}$$

The actuarial present value of whole life annuity - due for single life is

$$\ddot{a}_x = \sum_{k=0}^{\infty} v^k \,{}_k p_x = \frac{1-A_x}{d} \tag{2.72}$$

The present value function of whole life annuity due paid at the beginning of the each period as long as both (x) and (y) are alive. Present value function for joint life status is defined as follows;

$$Y_{xy} = \ddot{a}_{\overline{K(xy)+1}|} \tag{2.73}$$

The actuarial present value for the status is

$$\ddot{a}_{xy} = \sum_{k=0}^{\infty} v^k {}_k p_{xy} = \frac{1 - A_{xy}}{d} \tag{2.74}$$

The whole annuity due for the last survivor status provides payments at beginning of the year as long as at least one of (x) and (y) survive and for present value random variable of this status, we have;

$$Y_{\overline{xy}} = \ddot{a}_{\overline{K(\overline{xy})+1}|} \tag{2.75}$$

The expected present value of whole life annuity due can be obtained as follows;

$$\ddot{a}_{\overline{xy}} = \sum_{k=0}^{\infty} v^k {}_k p_{\overline{xy}} = \frac{1 - A_{\overline{xy}}}{d} \tag{2.76}$$

The continuous whole life annuity provides a continuous payment stream at the rate of 1 per year as long as (x) survives. In this case the random present value of the payments is the function of complete future lifetime $T(x)$ and it is stated as;

$$\bar{Y} = \bar{a}_{\overline{T(x)}|} \tag{2.77}$$

The actuarial present value of the annuity for single life can be obtained by using current payment technique and also using relationship between annuity and insurance.

$$\bar{a}_x = \int_0^{\infty} v^t {}_t p_x dt = \frac{1 - \bar{A}_x}{\delta} \tag{2.78}$$

In the joint-life status, the present value function of continuous whole life annuity becomes a function of time-until-failure $T(xy)$.

$$\bar{Y}_{xy} = \bar{a}_{\overline{T(xy)}|} \tag{2.79}$$

The actuarial present value of this joint-life annuity is

$$\bar{a}_{xy} = \int_0^\infty v^t {}_t p_{xy} dt = \frac{1-\bar{A}_{xy}}{\delta} \tag{2.80}$$

In continuous model of whole life annuity, the random present value for the last-survivor status is stated as a function of $T\left(\overline{xy}\right)$.

$$\bar{Y}_{\overline{xy}} = \bar{a}_{\overline{T(\overline{xy})}|} \tag{2.81}$$

The expected present value of the last - survivor annuity is given as follows;

$$\bar{a}_{\overline{xy}} = \int_0^\infty v^t {}_t p_{\overline{xy}} dt = \frac{1-\bar{A}_{\overline{xy}}}{\delta} \tag{2.82}$$

2.3.2.2 Temporary Life Annuity

An *n*–year temporary life annuity continues making payments while annuitant survives during the next *n* years, and temporary life annuities are also known as term life annuities. Payments cease on the death of the annuitant and the expiration of *n* years after the date of issue. The random present value of an *n*-year temporary life annuity due of 1 per annum for life (*x*) can be written as follows:

$$Y = \begin{cases} \ddot{a}_{\overline{K(x)+1}|} = \dfrac{1-v^{K(x)+1}}{d} & K(x) \le n\text{-}1 \\ \ddot{a}_{\overline{n}|} = \dfrac{1-v^n}{d} & K(x) \ge n \end{cases} \tag{2.83}$$

The actuarial present value of benefit of a unit payment of the annuity is denoted by $\ddot{a}_{x:\overline{n}|}$.

$$\ddot{a}_{x:\overline{n}|} = \sum_{k=0}^{n-1} v^k {}_k p_x = \frac{1-A_{x:\overline{n}|}}{d} \tag{2.84}$$

The subscript in equation (2.84) indicates that the annuity is payable for a life aged (*x*) and payments are not made more than *n* years. Also, the actuarial present value can be calculated by using net single premium related to endowment life insurance. The n–year joint-life annuity make payments at the beginning of the years while both spouses are alive in the case of a married couple; it is defined as;

$$\ddot{a}_{xy:\overline{n}|} = \sum_{k=0}^{n-1} v^k \, {}_k p_{xy} = \frac{1 - A_{xy:\overline{n}|}}{d} \tag{2.85}$$

The n – year last - survivor annuity pays benefit of a unit payment at the start of the years provided that either spouse is alive, and its actuarial present value is given as in equation (2.86);

$$\ddot{a}_{\overline{xy}:\overline{n}|} = \sum_{k=0}^{n-1} v^k \, {}_k p_{\overline{xy}} = \frac{1 - A_{\overline{xy}:\overline{n}|}}{d} \tag{2.86}$$

Under the continuous model, the random present value of an n–year temporary life annuity is stated as follows depending on $T(x)$.

$$\overline{Y} = \begin{cases} \overline{a}_{\overline{T(x)}|} = \dfrac{1 - v^{T(x)}}{\delta} & T(x) \le n \\ \overline{a}_{\overline{n}|} = \dfrac{1 - v^n}{\delta} & T(x) > n \end{cases} \tag{2.87}$$

The actuarial present value of benefit for a unit payment of the annuity is denoted by $\overline{a}_{x:\overline{n}|}$. As it can be seen in the notations, bar is used to denote continuous payment.

$$\overline{a}_{x:\overline{n}|} = \int_0^n v^t \, {}_t p_x \; dt = \frac{1 - \overline{A}_{x:\overline{n}|}}{\delta} \tag{2.88}$$

The expected present value of n–year joint life annuity under continuous model is calculated as follows;

$$\overline{a}_{xy:\overline{n}|} = \int_0^n v^t \, {}_t p_{xy} \; dt = \frac{1 - \overline{A}_{xy:\overline{n}|}}{\delta} \tag{2.89}$$

Similarly, the actuarial present value for n–year last survivor annuity of 1 per year is as in equation (2.90);

$$\bar{a}_{\overline{xy}:\overline{n}|} = \int_0^n v^t \, {}_t p_{\overline{xy}} \; dt = \frac{1 - \bar{A}_{\overline{xy}:\overline{n}|}}{\delta} \tag{2.90}$$

2.4 Reversionary Annuities

A reversionary annuity is a special type of annuities which is contingent on two lives. Usually couples are interested in this kind of annuity, and a pension package can cover the reversionary annuity benefit as a part of pension plan. In reversionary annuity, benefits are paid to a specified life as long as this person remains alive, but only after another specified life has been failure. In this case, the person who receives the payment is referred to as annuitant and the other one is called insured. So it can be stated that benefits are not payable so long as the insured survives. If we suppose that a unit benefit payment is made to life-aged (y) on the death of (x) so long as (y) survives, then the present value of continuously payable reversionary annuity is as follows:

$$\bar{Z} = \begin{cases} \bar{a}_{\overline{T(y)-T(x)}|} & T(x) \leq T(y) \\ 0 & T(x) > T(y) \end{cases} \tag{2.91}$$

Assuming independent future lifetimes, the expected present value of this reversionary annuity is

$$\bar{a}_{x|y} = E\left[\bar{Z}\right] = \int_0^\infty v^t \;\; {}_t p_{xy} \;\; \mu(x+t) \;\; \bar{a}_{y+t} \;\; dt \tag{2.92}$$

2.5 Contingent Insurances

In contingent insurances, the payments are made contingent upon the occurence of the deaths in a specific order. If the first died is x when y is alive, the actuarial present value of contingent insurance is calculated as in equation (2.93).

$$\bar{A}^{1}_{xy} = \int_0^\infty v^t \, {}_t p_y \;\; {}_t p_x \;\; \mu(x+t) dt \tag{2.93}$$

In addition, if x dies after the death of y, the actuarial present value of this contingent insurance as follows;

$$\bar{A}_{xy}^{\;2} = \int_0^\infty v^t \, {}_tq_y \; {}_tp_x \; \mu(x+t)\, dt$$

The relationships of contingent insurances according to order of death x and y are given in equation (2.95) - (2.97). That is, at first equality, the actuarial present value of continuous whole life insurance for life x is the summation of the actuarial present value for the case in which if x dies when y is alive and the actuarial present value for the case in which if x dies after the death of y.

$$\bar{A}_x = \bar{A}_{xy}^{\;1} + \bar{A}_{xy}^{\;2} \tag{2.95}$$

$$\bar{A}_{xy} = \bar{A}_{xy}^{\;1} + \bar{A}_{xy}^{\quad 1} \tag{2.96}$$

$$\bar{A}_{\overline{xy}} = \bar{A}_{xy}^{\;2} + \bar{A}_{xy}^{\quad 2} \tag{2.97}$$

CHAPTER THREE
COPULA

In Chapter 2, survival distributions and premium calculations related to various life insurance and annuity products for multiple life statuses were examined depending on the independence assumption. In pricing joint-life and last-survivor products, the remaining lifetimes of the individuals are assumed to be mutually independent since it provides convenience in terms of computation. But, this assumption can not be realistic if the policy is issued to workers in the same workplace, a married couple or twin. The workers, spouses and twins can be exposed to the same risks since they spend time together. There are three types of dependence for such situations. One of them is instantaneous dependence which implies occurence of two events at the same time such as car or plane crash. The long-term dependence structure may appear when the individuals have a common risk environment. For instance, two partners often come from the same neighbourhood which determines their common risks (Spreeuw and Wang, 2008). The death of one life changes the remaining lifetime of other individual but this effect decreases over time. This case is explained by short-term dependence. The "broken heart syndrome" (as researched in Parkes et al., 1969 and Jagger and Sutton, 1991) is the most well known example of short - term dependence (Spreeuw and Wang, 2008).

In this chapter, we deal with dependent future lifetime models for a pair of lives. Generally common shock models, multiple state models and some varieties of copulas depending of the nature of the data are used to model dependence in mortality. Hougaard (2000) proposes that short-term dependence structure are more appropriate than other dependencies for married couples. Marshall and Olkin (1988) suggests that common shock models are suitable to model instantaneous dependence. The study of Spreeuw (2006) has demonstrated that most common Archimedean copulas present long-term dependence. Youn and Shemyakin (2001) show that a copula model combined with physical age difference of two partners shows better performance than ignoring the age difference. Also, the same authors combined Bayesian approach and copula functions to model the dependence structure. In this thesis, we studied with some types of copulas for modelling of dependency structure of mortality.

3.1 Copula Models

In recent years, the copula models have became an important part of actuarial science to construct dependencies between random variables and to calculate premium computations. The thing that makes it so appealing is its simplicity. Also using copulas, it is possible to construct various dependence structures by using parametric or non-parametric models of the marginal distributions of lifetimes. A copula is a function that links univariate marginals to their multivariate distribution (Frees and Valdez, 1998). A copula function defined as $C:[0,1]^2 \to [0,1]$ satisfies the following properties.

1. $C(u,0) = C(0,u) = 0$ for any $u \in [0,1]$
2. $C(u,1) = C(1,u) = u$ for any $u \in [0,1]$
3. For all $0 \le u_1 \le u_2 \le 1$ and $0 \le v_1 \le v_2 \le 1$
4. $C([u_1,v_1]\times[u_2,v_2]) = C(u_2,v_2) - C(u_1,v_2) - C(u_2,v_1) + C(u_1,v_1) \ge 0$

The first of them implies the groundedness property, and the last one is 2-increasing property. The presence of both properties gives the non – decreasing property in each place. The arguments *u* and *v* are univariate distribution functions denoted as $F_1(x)$ and $F_2(y)$, then the copula function $C(F_1(x), F_2(y))$ produces a joint distribution function at (x,y), $F(x,y)$. More generally any joint distribution functions $H(x,y)$ with continuous marginal distribution functions introduces a general representation as a copula function by using Sklar's theorem.

Sklar's Theorem (Sklar, 1959) ;

Let $F_1(x)$ and $F_2(y)$ be marginal distribution functions for every $(x,y) \in \Re^2$. Then

i. If C' is any subcopula whose domain is $Range(F_1) \times Range(F_2)$, then $C'(F_1(x), F_2(y))$ is a bivariate distribution function whose marginals are $F_1(x)$ and $F_2(y)$.

ii. On the contrary, if $F(x,y)$ is a bivariate distribution function whose marginals are $F_1(x)$ and $F_2(y)$, a unique subcopula C' exists with domain $Range(F_1) \times Range(F_2)$;
$F(x,y) = C'(F_1(x), F_2(y))$

Based on these two definitions, if $F_1(x)$ and $F_2(y)$ are continuous then the subcopula is a copula. Otherwise, a copula C exists for every $(u,v) \in Range(F_1) \times Range(F_2)$ such that $C(u,v) = C'(u,v)$. Then a unique representation of a copula function as follows:

$$C(u,v) = H\left(F_1^{-1}(u), F_2^{-1}(v)\right) \tag{3.1}$$

where F_1^{-1} and F_2^{-1} are quasi - inverses of the marginals.

There are several copula families including Gaussian, Student-t, Archimedean etc. Archimedean copula famiy includes Frank, Gumbel-Hougaard and Clayton copulas. The simplest way to obtain Gaussian and Student-t copula families is to use the inverse function of distribution families given by equation (3.1). For bivariate version of the Gaussian copula can be represented by this way as follows:

$$\begin{aligned} C(u,v:\rho) &= \Phi_\rho^2\left(\Phi^{-1}(u)\Phi^{-1}(v)\right) \\ &= \int_{-\infty}^{\Phi^{-1}(u)} \int_{-\infty}^{\Phi^{-1}(v)} \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{-\frac{x^2 - \rho xy + y^2}{2\left(1-\rho^2\right)}\right\} dxdy \end{aligned} \tag{3.2}$$

where $\Phi(x)$ is the standart normal distribution function and ρ is the correlation coefficient between the marginals. Alike the bivariate version of Student-t copula is obtained as

$$C(u,v:\rho,\upsilon)=t^2_{\rho,\upsilon}\left(t^{-1}_\upsilon(u)t^{-1}_\upsilon(v)\right)$$
$$=\int_{-\infty}^{t^{-1}_\upsilon(u)}\int_{-\infty}^{t^{-1}_\upsilon(v)}\frac{1}{2\pi\sqrt{1-\rho^2}}\frac{1}{\left\{1+\dfrac{x^2-\rho xy+y^2}{\upsilon\left(1-\rho^2\right)}\right\}^{1+\upsilon/2}}dxdy \qquad (3.3)$$

where $t_\upsilon(x)$ is the function of t distribution with υ degrees of freedom and ρ is the correlation coefficient between the marginals. On the other hand, generator function ϕ which uniquely determines its copula is used to obtain Archimedean copulas. The function *H* is called as generator of copula functions. Suppose that $\phi:[0,\infty]\to[0,1]$ is a strictly decreasing convex function so that $\phi(0)=1$. Then the general way to obtain Archimedean copulas is as follows;

$$C(u,v:\alpha)=\phi\left(\phi^{-1}(u)+\phi^{-1}(v)\right),\qquad u,v\in[0,1] \qquad (3.4)$$

where α is the parameter of association. Table 3.1 represents the commonly used Archimedean copulas, their generators and bivariate versions.

Table 3.1 Archimedean copulas and their generators

Family	Generator $\phi(t)$	Dependence Parameter α	Bivariate Version
Independence	$-\ln t$	-	uv
Frank (1979)	$\ln\dfrac{e^{\alpha t}-1}{e^{\alpha}-1}$	$-\infty<\alpha<\infty$	$\dfrac{1}{\alpha}\ln\left(1+\dfrac{\left(e^{\alpha u}-1\right)\left(e^{\alpha v}-1\right)}{e^{\alpha}-1}\right)$
Gumbel (1960), Hougaard (1986)	$(-\ln t)^{\alpha}$	$\alpha\geq 1$	$\exp\left\{-\left[(-\ln u)^{\alpha}+(-\ln v)^{\alpha}\right]^{1/\alpha}\right\}$
Clayton (1978), Cooke - Johnson (1981), Oakes (1982)	$t^{-\alpha}-1$	$\alpha>1$	$\left(u^{-\alpha}+v^{-\alpha}-1\right)^{-1/\alpha}$

In the Archimedean copulas, the measure of association is calculated by two measures. These are Kendall's and Spearman's correlation coefficients. Let X_1 and X_2 be any two random variables, and g_1 and g_2 be strictly increasing over the range of X_1 and X_2. Then the study of Schweizer and Wolff (1981) show that the transformed random variables of X_1 and X_2 such that $g_1(X_1)$ and $g_2(X_2)$ have the same copula as X_1 and X_2. The authors also showed that Kendall's and Spearman's correlation coefficients could be stated in terms of copula function. Kendall's correlation coefficient is defined as

$$\begin{aligned}\tau(X_1, X_2) &= P\{(X_1 - X_1^*)(X_2 - X_2^*) > 0\} - P\{(X_1 - X_1^*)(X_2 - X_2^*) < 0\} \\ &= \iint C(u,v)dC(u,v) - 1\end{aligned} \tag{3.5}$$

where $\tau(.)$ is the laplace transformation and P is the probability function. Spearman's correlation coefficient is defined as

$$\begin{aligned}\rho(X_1, X_2) &= 12E\{(F_1(x_1) - 1/2)(F_2(x_2) - 1/2)\} \\ &= 12\iint \{C(u,v) - uv\}dudv\end{aligned} \tag{3.6}$$

This measure depends on both copula and marginal distributions. Table 3.2 summarizes the calculation of this two correlation measures for Archimedean copulas. It should be note that the correlation coefficient of Frank's copula depends on Deibye functions which is defined as

$$D_k(x) = \frac{k}{x^k}\int_0^x \frac{t^k}{e^t - 1}dt \qquad k = 1, 2 \tag{3.7}$$

There is one-to-one relationship between the correlation coefficients and association parameter α. As it is shown in Table 3.1, Clayton /Cook-Johnson/ Oakes and Gumbel-Hougaard copula families have the limited dependence parameter space. These families allow only non - negative correlations. On the other hand, Frank's family allows both positive and negative dependence.

Table 3.2 Measures of dependence of Archimedean Copulas

Family	Bivariate Copula	Kendall's τ	Spearman's ρ
Independence	uv	0	0
Frank (1979)	$\frac{1}{\alpha}\ln\left(1+\frac{\left(e^{\alpha u}-1\right)\left(e^{\alpha v}-1\right)}{e^{\alpha}-1}\right)$	$1-\frac{4}{\alpha}\left\{D_1\left(-\alpha\right)-1\right\}$	$1-\frac{12}{\alpha}\left\{D_2\left(-\alpha\right)-D_1\left(-\alpha\right)\right\}$
Gumbel (1960), Hougaard (1986)	$\exp\left\{-\left[\left(-\ln u\right)^{\alpha}+\left(-\ln v\right)^{\alpha}\right]^{1/\alpha}\right\}$	$1-\alpha^{-1}$	No closed form
Clayton (1978), Cooke - Johnson (1981), Oakes (1982)	$\left(u^{-\alpha}+v^{-\alpha}-1\right)^{-1/\alpha}$	$\frac{\alpha}{\alpha+2}$	Complicated form

Correlation measures give useful informations about dependence and association between the random variables in terms of copula functions. For this reason, it is possible to specify the copula forms by using these measures. Let $\left(X_{11},X_{21}\right),...,\left(X_{1n},X_{2n}\right)$ be a random sample of bivariate observations. Assume that F be the distribution function of Arcimedean copula C. We can identify the form of ϕ by using the procedure of Genest and Rivest (1993). This procedure works with unobserved random variable $Z_i=F\left(X_{1i},X_{2i}\right)$ has distribution function $K\left(z\right)=P\left(Z_i\le z\right)$. It has been shown that in the study of Genest and Revest (1993), this distribution function is related to the generator function of Archimedean copula.

$$K\left(z\right)=z-\frac{\phi\left(z\right)}{\phi'\left(z\right)} \tag{3.8}$$

The algorithm of identifying ϕ is as follows:

Step1. Kendall's τ is estimated by using nonparametric or distribution-free techniques

$$\tau_n = \binom{n}{2}^{-1} \sum_{i>j} sign\left[\left(X_{1i} - X_{1j}\right)\left(X_{2i} - X_{2j}\right)\right]$$

Step2. Construct a nonparametric estimate of *K*

- Let $Z_i = \left\{\#\text{ of } \left(X_{1i}, X_{2j}\right) : X_{1j} < X_{1i} \text{ and } X_{2j} < X_{2i}\right\} / \left(n-1\right) \quad i = 1,\ldots,n$ represents the pseudo - observations.
- The estimation of *K* is constructed as $K_n\left(z\right) =$ proportion of $z_i\text{'}s \le z$.

Step3. Construct a parametric estimate of *K* by using

$$K_\phi\left(z\right) = z - \frac{\phi\left(z\right)}{\phi'\left(z\right)}$$

In this thesis, we are interested in the future life lengths of several lives. Since our application is the modelling of dependence structure of married couples, we restrict our interest with two lives. Let L_1 and L_2 be the associated pairs of lives during a limited period of time T. Let a_1 and a_2 represent the entry ages for lives L_1 and L_2, respectively. In survival analysis, this situation is called as left-truncation. In addition, if the life L_j is terminated by before the age $a_j + T$ or by the end of the observation period, the death will not be observed which represents the right - censoring. The right-censoring is generally represented by an indicator function such that

$$c_{ij} = \begin{cases} 0,\ t_{ij} = T \text{ (censoring)} \\ 1,\ t_{ij} < T \text{ (no censoring)} \end{cases}$$

In terms of our application, it is important to estimate the future life length probabilities for given entry ages. For joint-life status, the probability of future life length for two lives

$$\begin{aligned} &p_{JL}\left(t; a_1, a_2\right) \\ &= p\left(\min\left\{X_1 - a_1, X_2 - a_2\right\} > t \mid \min\left\{X_1 - a_1, X_2 - a_2\right\} > 0\right) \end{aligned} \tag{3.9}$$

and this probability for last - survivor status is

$$p_{LS}(t;a_1,a_2) \\ = p\left(\max\{X_1-a_1, X_2-a_2\} > t \mid \min\{X_1-a_1, X_2-a_2\} > 0\right) \tag{3.10}$$

To estimate $p_{JL}(t;a_1,a_2)$ and $p_{LS}(t;a_1,a_2)$, it requires the estimation of bivariate survival function $S(t_1,t_2)$ which defined in Chapter 2. The estimations of $S(t_1,t_2)$ for $p_{JL}(t;a_1,a_2)$ and $p_{LS}(t;a_1,a_2)$ are given in equations (3.11) and (3.12), respectively.

$$p_{JL}(t;a_1,a_2) = \frac{S(a_1+t, a_2+t)}{S(a_1,a_2)} \tag{3.11}$$

$$p_{LS}(t;a_1,a_2) = \frac{S(a_1, a_2+t) + S(a_1+t, a_2) - S(a_1+t, a_2+t)}{S(a_1,a_2)} \tag{3.12}$$

Estimation of $S(t_1,t_2)$ is constructed in two approaches. In first approach, the marginal survival functions $S_1(t_1)$ and $S_2(t_2)$ are estimated by using some nonparametric methods such as Kaplan-Meier or by using some parametric models such as Gompertz or Weibull distribution. In second approach, the estimated survival functions are combined with some copula models. In this approach, the association parameter α is estimated by maximum likelihood method. In second approach, the bivariate survival function is estimated by using a copula.

$$S(t_1,t_2:\theta_1,\theta_2,\alpha) = C\left(S_1(t_1:\theta_1), S_2(t_2:\theta_2);\alpha\right) \tag{3.13}$$

where θ_j are parameters of the margins and α is the association parameter. Let $y_i = (a_{i1}, a_{i2}, t_{i1}, t_{i2}, c_{i1}, c_{i2})$ represents the vector of associated pair of lives (L_{i1}, L_{i2}). Then the likelihood function for parameter vector $\theta = (\theta_1, \theta_2, \alpha)$ in case of presence of right - censoring is as follows

$$l(\theta \mid y)=\prod_{i=1}^{n} f\left(x_{i1},x_{i2};\theta_1,\theta_2\right)^{c_{i1}c_{i2}} f_1\left(x_{i1},x_{i2};\theta_1,\theta_2\right)^{c_{i1}(1-c_{i2})}\times$$
$$f_2\left(x_{i1},x_{i2};\theta_1,\theta_2\right)^{(1-c_{i1})c_{i2}} S\left(x_{i1},x_{i2};\theta_1,\theta_2\right)^{(1-c_{i1})(1-c_{i2})} \tag{3.14}$$

where

$$f\left(x_1,x_2;\theta_1,\theta_2\right)=\frac{\partial^2}{\partial x_1 \partial x_2} S\left(x_1,x_2;\theta_1,\theta_2\right)$$
$$f_j\left(x_1,x_2;\theta_1,\theta_2\right)=\frac{\partial}{\partial x_j} S\left(x_1,x_2;\theta_1,\theta_2\right)$$

and $x_{ij}=a_{ij}+t_{ij}$.

One of the most commonly used model is two parameter Weibull distribution which survival function denoted as

$$S\left(t_j\right)=p\left(X_j>t_j\right)=\exp\left\{-\left(\frac{t}{\beta_j}\right)^{\gamma_j}\right\}, \qquad t \ge 0 \tag{3.15}$$

where β_j and γ_j represent scale and shape parameters, respectively. The Gumbel-Hougaard copula for this model is

$$S\left(t_1,t_2\right)=C\left(S\left(t_1;\beta_1,\gamma_1\right),S\left(t_2;\beta_2,\gamma_2\right);\alpha\right)=\exp\left\{-\left[\left(\frac{t}{\beta_1}\right)^{\alpha\gamma_1}+\left(\frac{t}{\beta_2}\right)^{\alpha\gamma_2}\right]^{1/\alpha}\right\} \tag{3.16}$$

In addition, the studies of Youn and Shemyakin (2001) showed that some additional dependence between the couples mortalities could be captured by using an additional factor d which describes the age difference between husband and wife. In this case, the association parameter α depends on d while in the original case α is not allowed to depend on d. The choice of Hougaard's copula versus Frank's copula is explained in Youn and Shemyakin (2001) as to get a convenient functional form of dependence $\alpha=\alpha\left(d\right)$. The Hougaard's copula with Weibull survival functions with $\alpha\left(d\right)>1$ is described as

$$S_d\left(t_1,t_2+d\right)=C\left(S\left(t_1;\beta_1,\gamma_1\right),S\left(t_2+d;\beta_2,\gamma_2\right);\alpha\left(d\right)\right)=\exp\left\{-\left[\left(\frac{t}{\beta_1}\right)^{\alpha(d)\gamma_1}+\left(\frac{t+d}{\beta_2}\right)^{\alpha(d)\gamma_2}\right]^{1/\alpha(d)}\right\} \qquad (3.17)$$

In 3.17, only the parameter of association changes with d where α is a Cauchy-type function $\alpha\left(d:\beta,\gamma\right)=1+\dfrac{\beta}{1+\gamma d^2}$ with hyper parameters $\left(\beta,\gamma\right)$.

CHAPTER FOUR
APPLICATION

In this chapter, we analyzed the premiums of last - survivor insurances and annuities for married couples under the independence and dependence assumption of future lifetimes. Our purpose is to examine the impacts of different mortality assumptions on the premiums of last - survivor insurance and annuity. We studied the modelling of dependency between married couples for Turkey by using the dependence structures of the studies suggested by Frees, Carriere and Valdez (1995), Youn and Shemyakin (2001). The calculations are made by using the average ages of females and males from the mortality table of Turkey obtained by Taylan (2012). To construct and analyze the dependence structure of the future lifetimes for married couples, we used Hougaard's copula. The reason of choosing of this copula model is; when the Weibull marginals are used, it provides a convenient functional form of dependence for $\alpha = \alpha(d)$. Hence, the association parameter α only depends on age difference factor "*d*", not marginal distributions. Also, we investigated the importance of the age difference factor between spouses in pricing of the last-survivor products for Turkey.

To illustrate the importances of dependence and age difference factor, we studied with three different models. For all models, we assumed that the marginal survival functions of females and males are defined by two - parameter Weibull distributions with scales β_i and shapes γ_i, where i=1 represents the male mortality and i=2 to female mortality. We calculated the net single premium of last - survivor insurance and annuity for whole life for all models. In Model I, it is assumed that the mortalities of the spouses are independent. Since there is no association in this situation, the value of α is 1. In model II, the mortalities of the spouses are dependent, and α is fixed. In Model III, the mortalities of the spouses are dependent as in Model II but α depends on age difference of spouses. The age difference is calculated as $E_M - E_F$, where E_M and E_F denote the entry ages of males and females, respectively. In this model, the values of parameter $\alpha(d)$ were calculated by help of the function $a(d;\lambda,\theta)=1+\frac{\theta}{1+\lambda d^2}$ where λ and θ are the hyper-parameters. In literature, such studies perform with the same data. It is not easy to find the data to carry out these analyses. To obtain the calculations, we used the same data with Frees, Carriere and Valdez (1995), Youn and Shemyakin (2001). The data set is obtained from a large Canadian insurance company which has 14,947 last - survivor annuity contracts. The time period of these contracts is December 29, 1988 through December 31, 1993. Since there are some repetitive and single sex contracts, 11,457 contracts are used to obtain results.

In parameter estimations, we assumed that the lifetimes of married couples live in Turkey has the same variance with this data set. Also, the values of association parameter are estimated by using the same data. To calculate the Weibull parameters for Turkey, we shifted the mean of the distributions of lifetimes for males and females according to the mortality table obtained by Taylan (2012). A random variable X has a two - parameter Weibull distribution with parameters β and γ if the pdf of X is

$$f\left(X;\gamma,\beta\right)=\begin{cases}\frac{\gamma}{\beta^{\gamma}}x^{\gamma-1}\exp\left[-\left(\frac{x}{\beta}\right)^{\gamma}\right] & x\geq 0\\ 0 & x<0\end{cases} \tag{4.1}$$

where $\beta>0,\gamma>0$. If the random variable X has a weibull distribution, the mean and variance is calculated as follows, respectively

$$E\left(X\right)=\beta\,\Gamma\left(1+\frac{1}{\gamma}\right) \tag{4.2}$$

$$V\left(X\right)=\beta^2\left[\Gamma\left(1+\frac{2}{\gamma}\right)-\left(\Gamma\left(1+\frac{1}{\gamma}\right)\right)^2\right]=\beta^2\,\Gamma\left(1+\frac{2}{\gamma}\right)-\left[E\left(X\right)\right]^2 \tag{4.3}$$

Then, to estimate the parameters we used the property of Gamma function as,

$$\Gamma\left(x+1\right)=x\,\Gamma\left(x\right) \tag{4.4}$$

Finally, to solve the equation $\beta^2\,\Gamma\left(1+\frac{2}{\gamma}\right)$ we used the iterative methods. The parameter estimates for all models are given in Table 4.1. As it is seen from this table, the association parameter α has the maximum value at Model III. In addition, the Model II has bigger α value than Model I. This means that, there is a dependent structure between the lifetimes of males and females so that we can not assume they are independent. Moreover, with the age difference factor some additional dependence can be captured. Since there is a direct relationship between α and Kendall's τ such that $\tau=\frac{\alpha-1}{\alpha}$. Since the value of τ increases with high α, the dependent measures of the models can be ordered as Model III > Model II > Model I.

To calculate the actuarial present values, firstly we calculated the survival probability of the last–survivor status corresponding to the curtate future lifetime of the status for each model. Then by using the products of the discount rates and the probabilities, we calculated the actuarial present value of last - survivor annuity.

$$\ddot{a}_{\overline{xy}}=\sum_{k=0}^{\infty}v^k\,{}_kp_{\overline{xy}} \tag{4.5}$$

where the present value factor $v=\frac{1}{1+i}$ and interest rate is i = 0.05. The actuarial present values for all models are given in Tables 4.2 - 4.4. Also, to compare the actuarial present values of last - survivor annuities for all three models, we calculated the ratios of dependent to independent models given in Tables 4.5 and 4.6. From Table 4.5, it is seen that, when dependent mortality model is used and age difference is negative, there is some reduction in annuity values until the female age 75. After the female age 75, the annuity values are increased. On the other hand, when age difference is positive, the annuity values generally tend to be increased. The same comments are valid for Table 4.6. Depending on these comments and Figures 4.1-4.4, we can conclude that, the ratios of actuarial present values are change with both time and age difference. So, the mortalities of women and men are dependent.

Table 4.1 Parameter estimations for all models

	Parameters	Model 1	Model 2	Model 3
Male	β_1	79.104	79.231	79.192
	γ_1	7.192	6.971	7.037
Female	β_2	83.773	83.644	83.604
	γ_2	8.924	9.236	9.335
Association	$\alpha(0)$	1	1.64	2.02
	$\alpha(5)$	1	1.64	1.76
	$\alpha(10)$	1	1.64	1.33
	λ	0.021	0	0
	θ	1.018	0.64	0

Table 4.2 Actuarial present values of last - survivor annuity for Model I

Female Age	Age Difference				
	-10	-5	0	5	10
50	17.5666	17.1081	16.6941	16.3492	16.0786
55	16.6611	16.0991	15.5977	15.1854	14.8659
60	15.5627	14.8912	14.3013	13.8247	13.4608
65	14.2707	13.4951	12.8265	12.2975	11.9002
70	12.8108	11.9509	11.2249	10.6649	10.2510
75	11.2375	10.3278	9.5749	9.0113	8.6006
80	9.6291	8.7123	7.9642	7.4271	7.0387
85	8.0709	7.1893	6.4642	5.9840	5.6343
90	6.6368	5.8181	5.0782	4.6958	4.4043

Table 4.3 Actuarial present values of last - survivor annuity for Model II

Female Age	Age Difference				
	-10	-5	0	5	10
50	17.0484	16.5675	16.2741	16.1730	16.2176
55	16.1203	15.5508	15.2169	15.1206	15.2002
60	15.0586	14.3930	14.0135	13.9195	14.0328
65	13.8809	13.1134	12.6801	12.5799	12.7188
70	12.6164	11.7454	11.2487	11.1290	11.2774
75	11.3037	10.3353	9.7677	9.6137	9.7495
80	9.9876	8.9379	8.2950	8.0971	8.1957
85	8.7117	7.6051	6.8772	6.6403	6.6797
90	7.5072	6.3635	5.4846	5.2475	5.2175

Table 4.4 Actuarial present values of last - survivor annuity for Model III

Female Age	Age Difference				
	-10	-5	0	5	10
50	17.2430	16.4983	16.1377	16.1498	16.1616
55	16.3312	15.4674	15.0671	15.1011	15.0688
60	15.2658	14.2971	13.8587	13.9072	13.8033
65	14.0549	13.0094	12.5310	12.5773	12.3780
70	12.7222	11.6398	11.1160	11.1356	10.8305
75	11.3090	10.2362	9.6592	9.6256	9.2245
80	9.8708	8.8530	8.2150	8.1078	7.6410
85	8.4684	7.5409	6.8283	6.6443	6.1576
90	7.1524	6.3247	5.4692	5.2435	4.8031

Table 4.5 Ratio of actuarial present values of last - survivor annuities (Model II / Model I)

Female Age	Age Difference				
	-10	-5	0	5	10
50	0.9705	0.9684	0.9748	0.9892	1.0086
55	0.9675	0.9659	0.9756	0.9957	1.0225
60	0.9676	0.9665	0.9799	1.0068	1.0425
65	0.9727	0.9717	0.9886	1.0230	1.0688
70	0.9848	0.9828	1.0021	1.0435	1.1001
75	1.0059	1.0007	1.0201	1.0668	1.1336
80	1.0372	1.0259	1.0415	1.0902	1.1644
85	1.0794	1.0578	1.0639	1.1097	1.1855
90	1.1311	1.0937	1.0800	1.1175	1.1846

Table 4.6 Ratio of actuarial present values of last - survivor annuities (Model III / Model I)

Female Age	Age Difference				
	-10	-5	0	5	10
50	0.9816	0.9643	0.9667	0.9878	1.0051
55	0.9802	0.9608	0.9660	0.9944	1.0136
60	0.9809	0.9601	0.9690	1.0060	1.0254
65	0.9849	0.9640	0.9770	1.0227	1.0401
70	0.9931	0.9740	0.9903	1.0441	1.0565
75	1.0064	0.9911	1.0088	1.0682	1.0725
80	1.0251	1.0161	1.0315	1.0916	1.0856
85	1.0492	1.0489	1.0563	1.1103	1.0929
90	1.0777	1.0870	1.0770	1.1166	1.0905

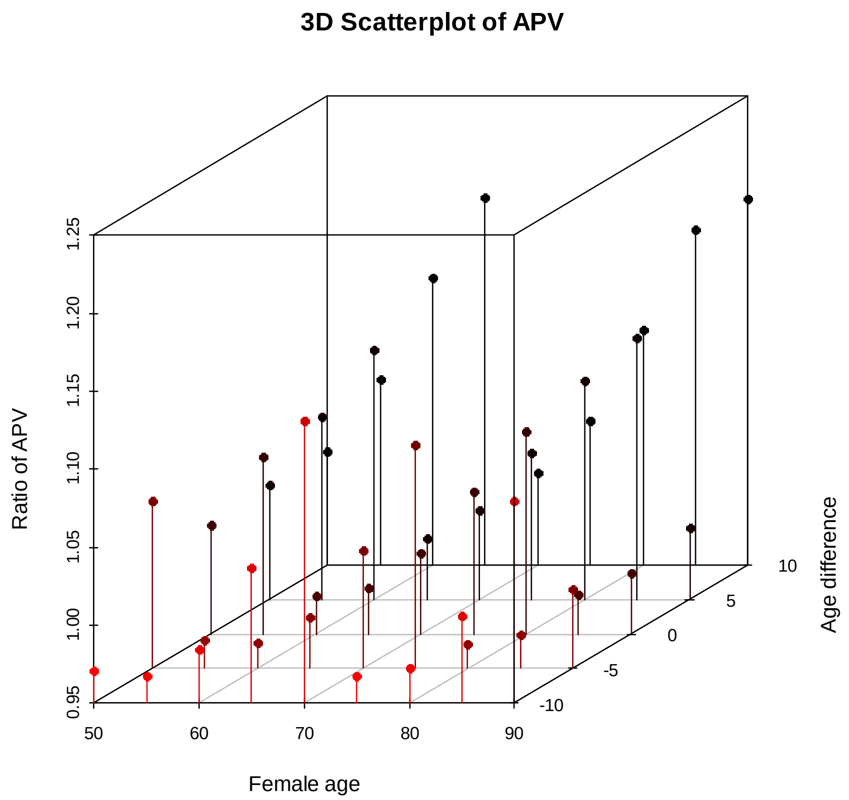

Figure 4.1 Scatterplot3D of female age, age difference and ratio of actuarial present values of annuities for Model II / Model I

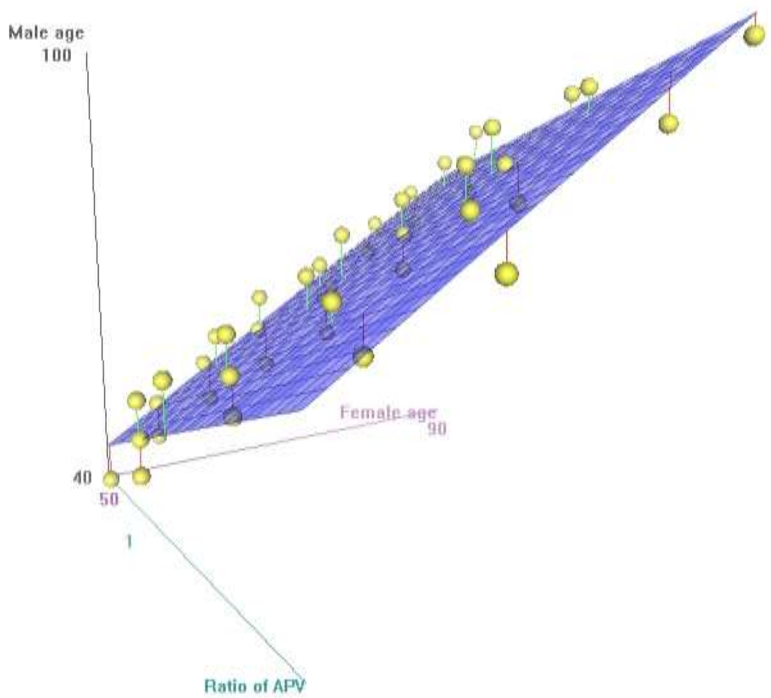

Figure 4.2 Scatter3D plot of female age, male age and ratio of actuarial present values of annuities for Model II / Model I

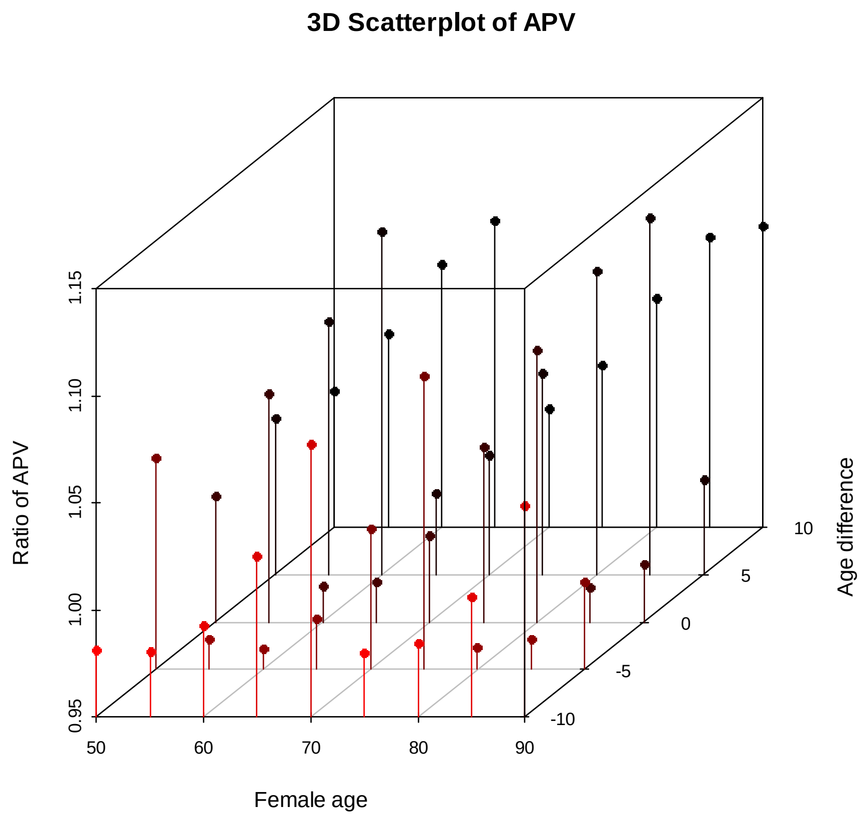

Figure 4.3 Scatterplot3D of female age, age difference and ratio of actuarial present values of annuities for ModelIII / Model I

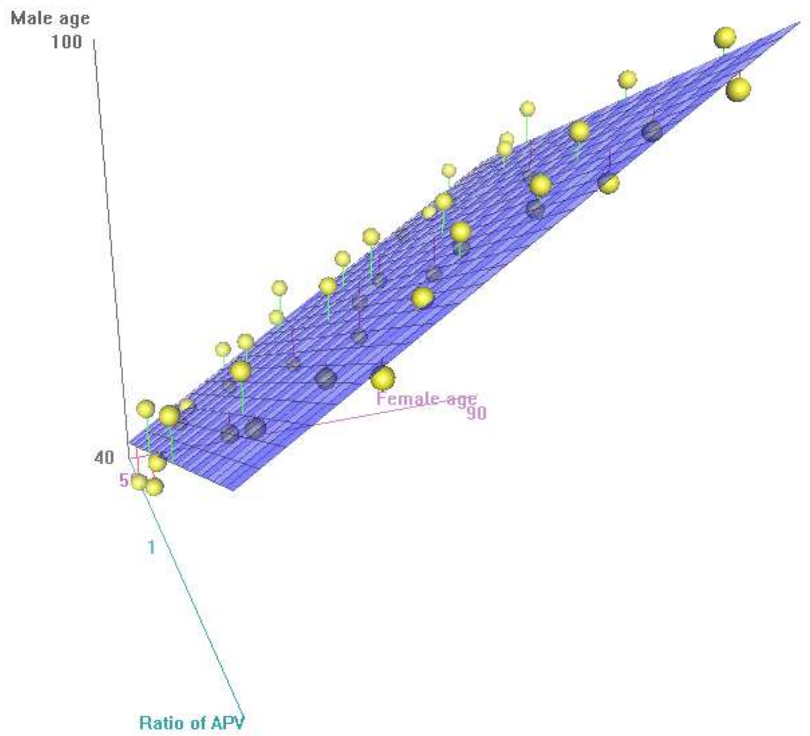

Figure 4.4 Scatter3D plot of female age, male age and ratio of actuarial present values of annuities for Model III / Model I

After the calculation of actuarial present values of last-survivor annuities $\ddot{a}_{\overline{xy}}$ for all three models, with the help of insurance - annuity relationship we calculate the actuarial present values of last - survivor insurances $A_{\overline{xy}}$.

$$A_{\overline{xy}} = 1 - d\ddot{a}_{\overline{xy}} \tag{4.6}$$

where $d = \dfrac{i}{1+i}$ and i = 0.05. The actuarial present values of last - survivor insurance for all models are given in Table 4.7 - 4.9. As in annuities, similarly we calculated the ratios of premium values of insurances. These ratios are given in Table 4.10 and Table 4.11. Since premium of insurance is a decreasing function of premium of annuity, the comments of insurance tables will be inverse of annuity's. From these tables, we can see that, the premium values under the dependence assumption are higher than independence assumption. That is, independence assumption underestimates the premium values compared to dependence case. We also give the 3-dimensional plots of ratios of premium values for insurance in Figures 4.5 - 4.8. The dependence structure also can be shown from these figures as in previous figures.

Table 4.7 Actuarial present values of last - survivor insurance for Model I

Female Age	Age Difference				
	-10	-5	0	5	10
50	0.1635	0.1853	0.2050	0.2215	0.2343
55	0.2066	0.2334	0.2572	0.2769	0.2921
60	0.2589	0.2909	0.3189	0.3417	0.3590
65	0.3204	0.3574	0.3892	0.4144	0.4333
70	0.3899	0.4309	0.4655	0.4921	0.5118
75	0.4648	0.5082	0.5440	0.5709	0.5904
80	0.5415	0.5851	0.6207	0.6463	0.6648
85	0.6156	0.6576	0.6922	0.7150	0.7317
90	0.6839	0.7229	0.7582	0.7764	0.7903

Table 4.8 Actuarial present values of last - survivor insurance for Model II

Female Age	Age Difference				
	-10	-5	0	5	10
50	0.1882	0.2111	0.2250	0.2298	0.2277
55	0.2324	0.2595	0.2754	0.2799	0.2762
60	0.2829	0.3146	0.3327	0.3372	0.3318
65	0.3390	0.3755	0.3962	0.4009	0.3943
70	0.3992	0.4407	0.4643	0.4700	0.4630
75	0.4617	0.5078	0.5348	0.5422	0.5357
80	0.5244	0.5744	0.6050	0.6144	0.6097
85	0.5851	0.6378	0.6725	0.6838	0.6819
90	0.6425	0.6969	0.7388	0.7501	0.7515

Table 4.9 Actuarial present values of last - survivor insurance for Model III

Female Age	Age Difference				
	-10	-5	0	5	10
50	0.1789	0.2143	0.2315	0.2309	0.2304
55	0.2223	0.2634	0.2825	0.2808	0.2824
60	0.2730	0.3192	0.3401	0.3377	0.3427
65	0.3307	0.3805	0.4033	0.4011	0.4105
70	0.3942	0.4457	0.4706	0.4697	0.4842
75	0.4615	0.5126	0.5400	0.5416	0.5607
80	0.5299	0.5784	0.6088	0.6139	0.6361
85	0.5967	0.6409	0.6748	0.6836	0.7068
90	0.6594	0.6988	0.7396	0.7503	0.7713

Table 4.10 Ratio of actuarial present values of last - survivor insurances (Model II / Model I)

Female Age	Age Difference				
	-10	-5	0	5	10
50	1.1511	1.1392	1.0975	1.0375	0.9718
55	1.1249	1.1118	1.0708	1.0108	0.9456
60	1.0927	1.0815	1.0433	0.9868	0.9242
65	1.0580	1.0506	1.0179	0.9674	0.9099
70	1.0238	1.0227	0.9974	0.9551	0.9046
75	0.9933	0.9992	0.9831	0.9497	0.9073
80	0.9684	0.9817	0.9747	0.9506	0.9171
85	0.9504	0.9699	0.9715	0.9564	0.9319
90	0.9395	0.9640	0.9744	0.9661	0.9509

Table 4.11 Ratio of actuarial present values of last - survivor insurances (Model III / Model I)

Female Age	Age Difference				
	-10	-5	0	5	10
50	1.0941	1.1565	1.1292	1.0424	0.9833
55	1.0759	1.1285	1.0983	1.0141	0.9668
60	1.0544	1.0973	1.0665	0.9883	0.9546
65	1.0321	1.0646	1.0362	0.9679	0.9474
70	1.0110	1.0343	1.0109	0.9545	0.9461
75	0.9929	1.0086	0.9926	0.9487	0.9497
80	0.9786	0.9885	0.9808	0.9499	0.9568
85	0.9693	0.9746	0.9748	0.9561	0.9659
90	0.9642	0.9667	0.9755	0.9664	0.9759

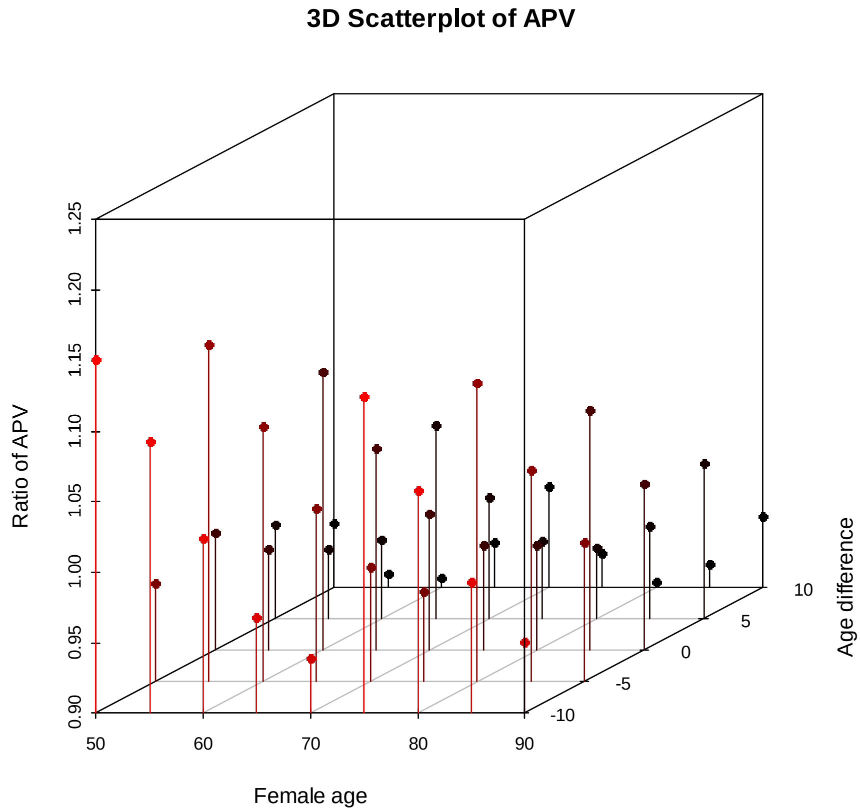

Figure 4.5 Scatterplot3D of female age, age difference and ratio of actuarial present values of insurances for Model II / Model I

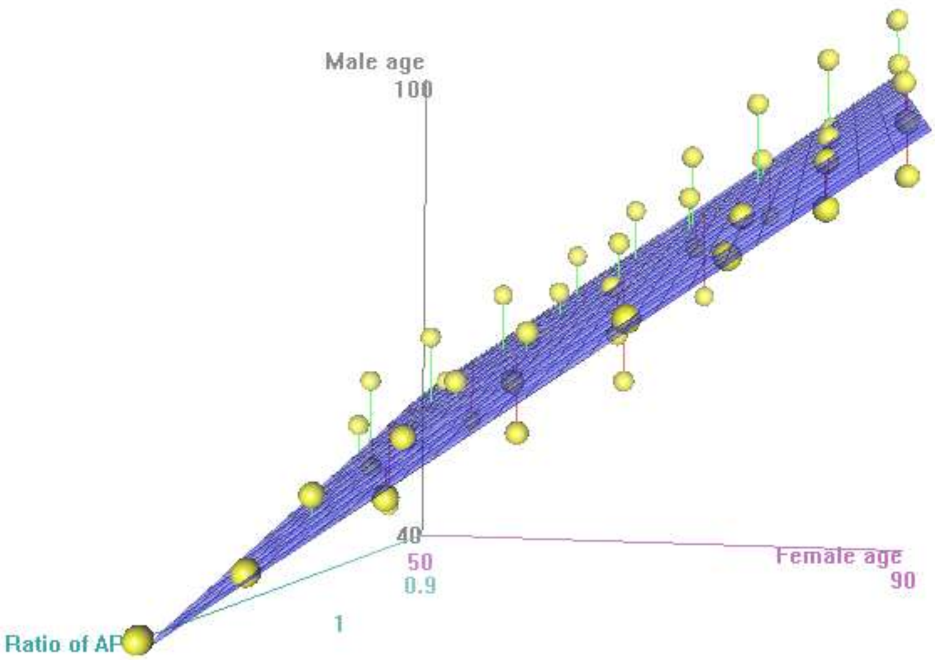

Figure 4.6 Scatter3D plot of female age, male age and ratio of actuarial present values of insurances for Model II / Model I

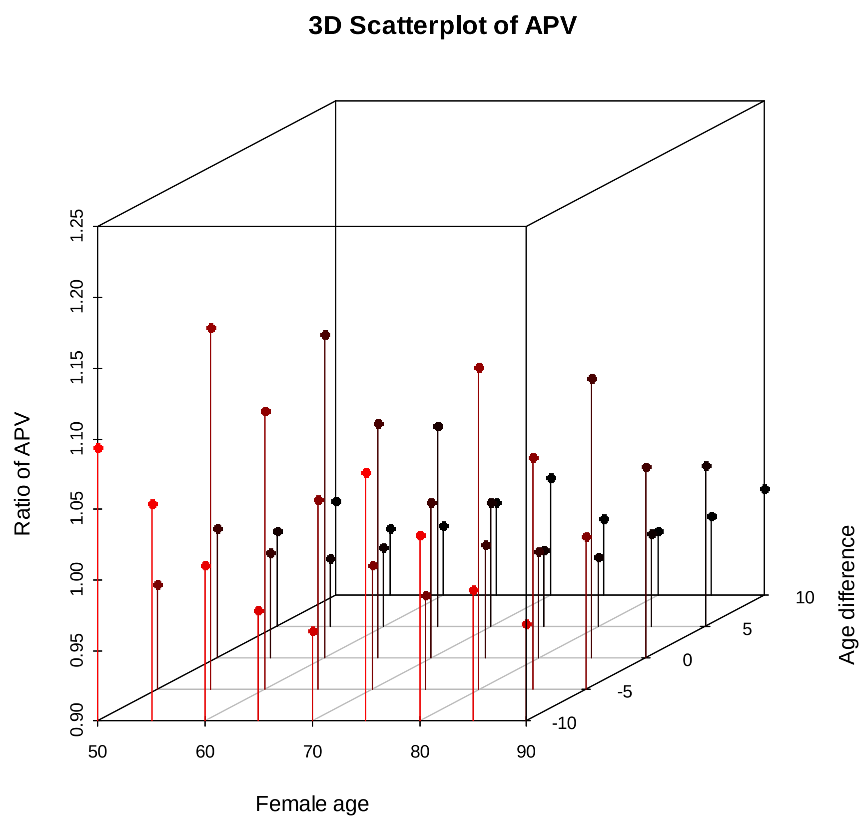

Figure 4.7 Scatterplot3D of female age, age difference and ratio of actuarial present values of insurances for Model III / Model I

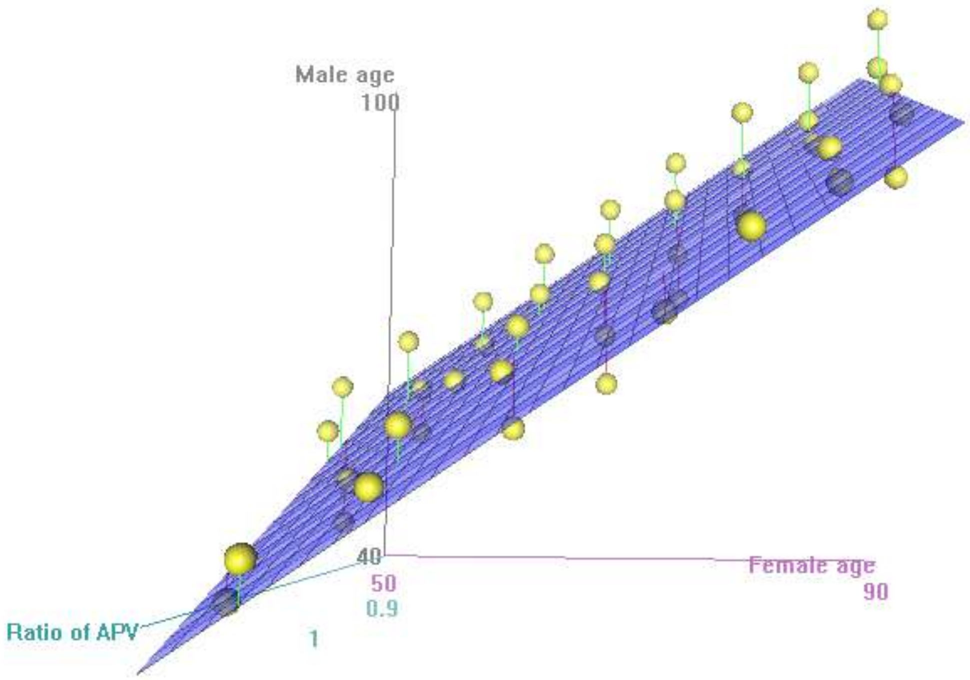

Figure 4.8 Scatter3D plot of female age, male age and ratio of actuarial present values of insurances for Model III / Model I

CHAPTER FIVE
CONCLUSION

Nowadays the multiple life insurance and annuity contracts have a wide range of applications. Such contracts are most commonly preferred by couples who jointly organize their financial status and by companies to protect their employers lives against to unexpected events, etc.. These types of policies cover two or more lives. In this thesis, since we are interested in married couples, we restrict ourselves by two lives. These contracts are called joint life contract, and they are based on the first or second death. Firstly, we are explored the fundamentals of single life insurances to better understand the general structure of multiple life insurances. Then, since the standard insurance industry assumes that the lives are independent, we studied the mathematical background of multiple life insurance and its various types depend on the order of death under independence assumption. But, recent studies about future lifetimes of multiple lives and pricing of their contracts showed that they are not independent. The dependence structure of multiple lives are commonly examined by using copula functions. Chapter 3 has detailed information about copulas and its usage for joint survivorship. To model the lifetimes, several families of distribution functions can be used. Frees, Carriere and Valdez (1995) who studied the same data with us showed that the Gompertz distribution provides satisfactory fit to the data. According to him, the main reason of this choosing is the data set has older people. On the other hand, Shemyakin and Youn (2000) proposed using Gumbel Hougaard's copula with Weibull distribution to capture some additional dependence by using age difference factor. We choose Gumbel-Hougaard's copula with Weibull marginal distribution function to generate the dependence structure because of its convenient functional form to associate the age difference factor and association parameter.

We used the same data with Frees, Carriere and Valdez (1995) and Shemyakin and Youn (2000, 2006). By using the variance and values of association parameters of this data, we shifted the mean of the lifetime distribution according to the expected lifetimes of males and females obtained by Taylan (2012). In Chapter 4, we estimated the parameters of Weibull survival distributions according to three models. Then we calculate the actuarial present values of last-survivor annuity and insurance for Turkey with fixed interest rate (0.05). To compare three models, we also calculated the ratios of premium values. Our results in Chapter 4 are consistent with the studies of Frees and et al. (1995) and Shemyakin and Youn (2000). According to our results, when dependent mortality model is used and age difference is negative, there is some reduction in annuity values until the female age 75. After the female age 75, the annuity values are increased. On the other hand, when age difference is positive, the annuity values generally tend to be increased. In addition, we get the similar results with dependent mortality model by using age difference factor. Additionally, we computed the actuarial present values of insurance, and we compared three models by using their proportions. Moreover, we plotted the three dimensional plots of ratio of actuarial present values, female and male ages (and age difference). These plots clearly show that the ratios of premium values change with the time and age difference. This means that the lifetimes of spouses are not independent. In brief, the premium values of annuities for the second death are overestimated when the independent joint survival model is used. On the other hand, the premium values of insurances are underestimated. So, in such studies using of dependent mortality models is better than independent model. Our findings are support this idea as previous studies.

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