# ON DEHN AND WIRTINGER PRESENTATION

**Dehn ve Wirtinger Temsilleri Üzerine**

**Yrd.Doç.Dr. Arif DANE**
**Cumhuriyet University**
**Faculty of Science**
**SİVAS**

**ABSTRACT**

In this study ,Dehn [7] and Wirtinger [1] presentation were used to express knot group relators in any crossing point. After the relators were transformed into a linear homogenous equation .Solition of the equation system was inwestigated for $\forall t \in Z$ according to mod$\Delta(t)$,Where $Z$ is integers and $\Delta(t)$ Alexander polynomial of knot. It is found out that the equation system has zero solition for $t=1$ and infinite solition for $t \neq 1$.

**Key Words:** Knot Theory

**OZET**

Bu çalışmada, herhangibir geçit noktasında düğüm grubunun bağıntılarını ifade elmek için Dehn [7] veWirtinger [1] temsilleri kullanıldı. Daha sonra bu bağıntılar bir lineer homogen denkleme dönüştürüldü . Lineer homogen denklem sisteminin $\forall t \in Z$ için mod$\Delta(t)$ ye göre çözümleri araştırıldı. Burada $Z$ tamsayılar $\Delta(t)$ düğümün Alexander polinomudur. $t=1$ için sıfır çozüm $t \neq 1$ için sonsuz çozümlere sahıp oldukları saptandı.

## 1.INTRODUCTION

If $K$ is any knot in $R^3$ such that its projection on to the plane $z=0$ is regular and p is any point in $R^3$-K (base point) Then, The fundamentul group $\Pi_1(R^3\text{-}K,p)$ is called the knot group and is show with

$$G=\Pi_1(R^3\text{-}K,p)=(x_1,x_2,\ldots,x_n;r_1,r_2,\ldots,r_m).$$

Where $x_1,x_2,\ldots,x_n$ are the generators of G and $r_1,r_2,\ldots,r_m$ are the relations of it.There are two fundamental methods to find the generators and relations of knot group.

i) Wirtinger presentation
ii) Dehn presentation

Now ,we define these two knot presentations.

**Definition 1.1 (Wirtinger Presentation):** If we consider a knot K with the directed ,the n double points of the regular projection.In this diagram ,there are n numbered pieces of curve counterparting the overpasses.

The homolopy clases of closed curves ,starting at $p \in R^3$-K point and ending at p and encirling the overpasses simply are the generators of the group K .These generators are illustrated by the little arrows put on the pieces of curve.We fix a relation at every $c_i$ double point in the following way .

Representative curves of $x_i,x_j,x_k$ generators belonging to the overpass at $c_i$ are directed so as to form the left hand system according to the direction of the knot .Around $c_i$ we choise a reading direction starting with the one of $x_i,x_j,x_k$ genarators, if the direction of each generator is the same as the chosen direction ,we take generators itself ,if it is the inverse of it ,we take the inverse of it and equate this multiply with one .

As is seen in the figure 1,the relation at $c_i$ double point is illustrated in the following way

$$x_i x_j x_i^{-1} x_k^{-1} = 1$$

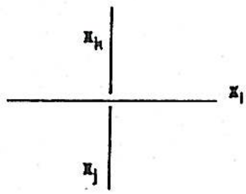


Figure 1.

**Definition 1.2 (Dehn Presentation) :** Let be a knot in regular projection in $R^3$ and let the regions of the regular diagram be $b_0, b_1, b_2, \dots, b_n$.

The homotopy classes of the curves,starting at $p \in R^3$-K a base point just above the regular projection plane, passing from $b_n$ n=1,2,3,...,n district region to the back region of the regular projection plane and return to p through $b_0$,are taken as the generators of knot group.

While moving in the direction of the knot, if underpasses crossing is formed,two spots are placed on the two left regions,as one spot on each of them the diagram formed in this way is also called "pointed diagram".

As is seen in the figure 2 ,let pointed regions be $b_i$ ,$b_j$, let the other two regions be $b_k$,$b_l$. Then, if the generators corresponding to these regions are respectively $x_i$ ,$x_j$,$x_k$,$x_l$ the relation attached to that crossing point are written as $x_i x_j^{-1} x_k x_l^{-1} = 1$.

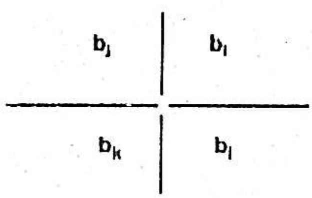


Figure 2

The relations at crossing points neigboring to $b_0$ region involve only three elements. Because an closed curve,going to $b_0$ district and return to p again through $b_0$, represents the element 1.

**Definition 1.3 (Free Derivation):** A mapping D:ZG → ZG is said to be a derivative if and only if

**(1) $D(f+g)=D(f)+D(g)$**
**(2) $D(f.g)=Df\varepsilon(g)+fD(g)$ for $f,g \in G$**

where $\varepsilon$ is the augmentation homomorphism for all f and g in ZG as given in following

$\varepsilon$:ZG → Z
$\varepsilon(\sum n_g g)=\sum n_g$
$\varepsilon$:G → Z , $\varepsilon(g)=1$.

Note that if g belongs to G, then (2) reads

$D(f.g)=D f + f D(g)$

It is obtained the following statements which are the result of (1),(2) and (3) statements.

1) $D(n)=0 \quad n\in Z$

2) $D(\sum n_i g_i)=\sum n_i D(g_i)$

3) $D(g^{-1})=-(g^{-1})D(g)$

4) $D(g^n)=(1+g+g^2+,\dots+g^{n-1})D(g)$

5) $D(g^{-n})=-(g^{-n})(1+g+g^2+,\dots+g^{n-1})D(g)=-(g^{-n})D(g^n)$

**Definition 1.4 (Jacobien matrix):** If $(x_1,x_2,x_3,\dots x_n;r_1,r_2,r_3,\dots r_m)$ is any presentation of G the matrix

$$\left(\frac{d(r_1,r_2,\dots r_m)}{d(x_1,x_2,\dots x_n)}\right)^{\phi}=\begin{bmatrix} \left(\frac{dr_1}{dx_1}\right)^{\phi} & \left(\frac{dr_1}{dx_2}\right)^{\phi} & \left(\frac{dr_1}{dx_n}\right)^{\phi} \\ \dots\dots & \dots\dots & \dots\dots \\ \left(\frac{dr_m}{dx_1}\right)^{\phi} & \left(\frac{dr_m}{dx_2}\right)^{\phi} & \left(\frac{dr_m}{dx_n}\right)^{\phi} \end{bmatrix}$$

that we call the Jacabien of presentation.Actually it is not quite unique because the rows and/or the columns could agrear in any order.The entries in the Jacobien matrix are elements of the integral group ring ZG of G.
The canonical homomorphism $\phi$ of F onto G extends in an obvious way to a homomrphism of ZF onto ZG which we perversely continue to denote by $\phi$.

**Definition1.5 (Alexander matrix) :** If H is any group upon which G can be mapped by a homomorphism $\varphi$ ,we can similary extend $\varphi$ to a homomorphism $\varphi$ of ZG upon ZH and thus define the matrix

$$\left(\frac{d(r_1,r_2,\dots r_m)}{d(x_1,x_2,\dots x_n)}\right)^{\varphi\phi}=\begin{bmatrix} \frac{dr_1}{dx_1} & \frac{dr_1}{dx_2} & \frac{dr_1}{dx_n} \\ \dots\dots & \dots\dots & \dots\dots \\ \frac{dr_m}{dr_1} & \frac{dr_m}{dx_2} & \frac{dr_m}{dx_n} \end{bmatrix}^{\varphi\phi}$$

that we call Alexander matrix at $\varphi$ .

where

$$F \xrightarrow{\phi} G \xrightarrow{\varphi} H$$

$$ZF \xrightarrow{\phi} ZG \xrightarrow{\varphi} ZH$$

The choice of $\varphi$ ranges from $\varphi$ being identity mapping of G onto itself to $\varphi$ being the map of into the trivial group 1.We are going to be most interested in choosing H to be the commutator quation group G/G' and $\varphi$ the abelianizer /2.3/.

**Example 1.1 :** According to Wirtinger presentation the relation, attached to any crossing point of a K knot is known as $r_i = x_i x_j x_i^{-1} x_k^{-1}$

According to the definition.1.4

$$\frac{dr_i}{dx_i} = 1 - x_i x_j x_i^{-1}, \quad \frac{dr_i}{dx_j} = x_i, \quad \frac{dr_i}{dx_k} = -x_i x_j x_i^{-1} x_k^{-1}$$ from here

$$\left(\frac{dr_i}{dx_i}\right)^{\varphi\phi} = 1-t, \qquad \left(\frac{dr_i}{dx_j}\right)^{\varphi\phi} = t, \qquad \left(\frac{dr_i}{dx_j}\right)^{\varphi\phi} = -1$$

That is, $\varphi$ and $\phi$ $x_i \to t, x_j \to t, x_k \to t$ is an tranformation taken to t.

## THE DEFINITION OF THE KNOT RELATIONS THROUGH LINEAR HOMOGENOUS EQATIONS

At this stage of our study,We will transform the knot group relations obtained through Dehn and Wirtinger presentations through a special transformation to according to mod $\Delta$ (t) a linear homogenous equation .

Then ,will survey according to $\Delta$ (t) the solitions of the equation system formed in this way.For that let us prove the following lemmas.

**Lemma 2.1:** A relation $r_i = x_i x_j x_i^{-1} x_k^{-1}$ at any $c_i$ passing point obtained through Wirtinger presentation, can be define through linear homogenous equation as $\alpha_i(1-t)+\alpha_j t-\alpha_k = 0$ by means of a special transformation.

**Proof:** As is seen the figure 1, the relation attached to $c_i$ passing point was $r_i = x_i x_j x_i^{-1} x_k^{-1}$ $\alpha_i, \alpha_j, \alpha k$ are respectively the unknown ones representing overpass and underpass and $\phi : ZG \to Z | x:r |$, $x_i \to t$, $x_j \to t$, $x_k \to t$ are transformations belonging to the same subject. In this case, it may be defined through a linear homogenous equation as $\alpha_i (1-t) + \alpha_j t - \alpha k = 0$ .

These equations are called the diagram equation. Here, the statements 1-t, t, -1 are the value of relation $r_i$ found in example 1.1. This transformation is one to one. The equation $\alpha_i (1-t) + \alpha_j t - \alpha_k = 0$ is written in the following way $\alpha_i + \alpha_j t - \alpha_i t - \alpha_k = 0$ . Here, if $\alpha_i$ , $\alpha_j$, $\alpha_k$ representing overpass and underpass is subsituted for generators, if t=1 and + is subsituted for. (multiplication), again the relation $x_i x_j x_i^{-1} x_k^{-1} = 1$ is obtained. Then the diagrams equations of a knot, when its relations is given, can be written and if the diagrams equations of a knot is given, its relations can be written.

**Lemma 2.2.** The diagram equations of a knot, involving n numbered crossing points, obtained by Wirtinger presentation while $t \in Z$ and $\Delta$ (t) are Alexander polinomial of the knot form a homogenous equation, involving n equations with n unknows.

**Proof:** For a K knot involving n numbered crossing points, n numbered linear homogenous equations may be written in return for involving n equations and these form a equation system

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$$\alpha_i (1-t) + \alpha_j t - \alpha_k = 0 . \quad \text{mod } \Delta(t) \quad \forall t \in Z$$

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The coefficient matrix of this system given in the definition 1.5. is the Alexander matrix. Continualy,as $a_{ij}$ =0 /1/. There are infinite solitions of the above system. Specially, for t =1 $\Delta(t)=1$, / 4/ for that reason, the solition of the system according to mod 1 is zero solition.

**Lemma 2.3:** The relation $r_l = x_i x_j^{-1} x_k x_l^{-1}$ at any crossing point obtained Dehn presentation, through a special transformation , can be defined a linear homegenous equation shown as

$$\alpha_i t - \alpha_j t + \alpha_k - \alpha_l = 0$$

**Proof:** As is seen the figure 2, the relation entagled in $c_i$ crossing point was $r_l = x_i x_j^{-1} x_k x_l^{-1}$. In case $\alpha_i, \alpha_j, \alpha_k, \alpha_l$ are the unknows ones symbolizing the diagram regions and $\phi$ ;$x_i \to t$, $x_j \to t$, $x_k \to 1$ ,$x_l \to 1$ the transformation taking to, we may define through linear homogenous equation. Agein, this transformation is one to one. If $\alpha_i, \alpha_j, \alpha_k, \alpha_l$ symbolizing diagram regions, is substituted for $x_i, x_j, x_k, x_l$ generators corresponding to the regions, t = 1 is taken and instead of +, . is used, $x_i x_j^{-1} x_k x_l^{-1} = 1$ is obtained as a relation of the group knot.

**Lemma 2.4:** The diagram equations of a knot involving n crossing point, obtained according to Dehn presentation , in case $t \in Z$ and $\Delta(t)$ is the Alexander polinomial of the knot, according to mod $\Delta(t)$ form a system of linear homogenous equation involving n equations with (n+2) unknows.

**Proof:** In the diagram of a knot involving n crossing point, according to Euler theorem there are (n+2) regions. In this case, for each equation , involving (n+2) unknows n linear homogenous equations can be written as follows. The equations formed in this way, according to mod $\Delta(t)$ form a system of equation.

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$$\alpha_i t - \alpha_j t + \alpha_k - \alpha_l = 0 \quad \text{mod } \Delta(t) , \ \forall t \in Z$$

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Again , the rank of the system of equation formed above is n-2 . Consequently , for t=1, there is zero solition, for $t \neq 1$ there are infinite solition.

Now, we must survey the solition of the equation system to which we give some examples.

**Example 2.1:** The relations of the trefoil knot,of which diagram and diagram regions are given,according to Wirtinger and Dehn presentation ,are

$r_1 = xyx^{-1}z^{-1}$ , $r_2 = zxz^{-1}y^{-1}$ , $r_3 = yzy^{-1}x^{-1}$ (Wirtinger presentation )

and $r_1 = x^{-1}zy^{-1}$ ,$r_2 = y^{-1}zt^{-1}$ ,$r_3 = t^{-1}zx^{-1}$ (Dehn presentation )

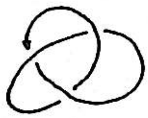

figure 3

Here ,according to Wirtinger presentation ,the generators belonging overpass and underpass are x,y,z, according to Dehn presentetion,the generators,correspondin to $b_0,b_1,b_2,b_3,b_4$ regions,are shown through 1,x,y,z,t.

Firstly,let us write the diagram according to the Wirtinger presentetion .The Alexander polinomial of the trefoil knot $\Delta(t)=t^2-t+1$. for $t=2, \Delta(2)=3$. Let $\alpha_1,\alpha_2,\alpha_3$ be the unknows symbolizing overpass and underpass, the equations, corresponding to $r_1,r_2,r_3$ relations, according to mod 3 ,form a system of equation as follows

$$\begin{aligned} &\alpha_1(1-t)+\alpha_2 t-\alpha_3=0 \\ &\alpha_1 t-\alpha_2+(1-t)\alpha_3=0 \bmod 3 \\ &-\alpha_1+\alpha_2(1-t)+\alpha_3 t=0 \end{aligned}$$

the coeffecient matrix given the Alexander matrix of the knot $a_{ij}=0$ , We must see to the rank of the system. Let us write

$t=2$ ,

$$\begin{vmatrix} -1 & 2 & -1 \\ 2 & -1 & -1 \\ -1 & -1 & 2 \end{vmatrix} \sim \begin{vmatrix} -1 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}$$

In this case,n-r=3-1=2 . It has a solition dependent one parameter.

$$\alpha_1=2\alpha_2-\alpha_3 \quad , \quad \alpha_2=\alpha_2 \quad , \quad \alpha_3=\alpha_3$$

For example , $\alpha_2=1$, $\alpha_3=0$ are chosen , the set of solition {2,1,0} become a solution of the system above.

Now ,let us write diagram equation according to mod 3 ,We are may form a system of equation as follows

$$\begin{aligned} &\alpha_0 t-\alpha_1 t+\alpha_3-\alpha_2=0 \\ &\alpha_0 t-\alpha_4 t+\alpha_3-\alpha_1=0 \bmod 3 \\ &\alpha_0 t-\alpha_2 t+\alpha_3-\alpha_4=0 \end{aligned}$$

Again, We must see to the rank of the system, let us write t=2

$$\begin{vmatrix} 2 & -2 & -1 & 1 & 0 \\ 2 & -1 & 0 & 1 & -2 \\ 2 & 0 & -1 & 0 & -2 \end{vmatrix} \sim \begin{vmatrix} 2 & -2 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & -2 \\ 0 & 2 & -1 & 0 & -1 \end{vmatrix} \sim \begin{vmatrix} 2 & -2 & -2 & 1 & 0 \\ 0 & 1 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{vmatrix}$$

In this case , n-r =5-2=3 It has a solition depandent tree parameters.

$\alpha_0=\alpha_2+2\alpha_4-2\alpha_3$ , $\alpha_1=-\alpha_2+2\alpha_4$ , $\alpha_2=\alpha_2$ , $\alpha_3=\alpha_3$ , $\alpha_4=\alpha_4$,

For example, for $\alpha_2=1$ , $\alpha_3=1$ , $\alpha_4=0$

the set of solition of system is **{-1,-1,1,1,0}** .

**Example 2.2:** As is seen the figure 4 for $5_2$ knot ,We make the some operation .

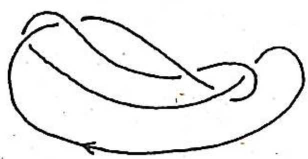

**figure 4**

the Alexander polinomial of $5_2$ knot is $\Delta(t)=2t^2-3t+2$ for $t=-1$ , $\Delta(-1)=7$

According to Wirtinger presentation the relations of $5_2$ knot group are $xux^{-1}t^{-1}=1$ , $yty^{-1}z^{-1}=1$ , $zxz^{-1}u^{-1}=1$ , $tyt^{-1}x^{-1}=1$ , $uzu^{-1}y^{-1}=1$.

Let $\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5$ be the generators x,y,z,t,u symbolizing .According to mod 7 ,we may form a system of equation as follow .

$$\begin{aligned}
&\alpha_1(1-t)+\alpha_5 t-\alpha_4=0\\
&\alpha_2(1-t)+\alpha_4 t-\alpha_3=0\\
&\alpha_3(1-t)+\alpha_1 t-\alpha_5=0 \quad \text{mod } 7\\
&\alpha_4(1-t)+\alpha_2 t-\alpha_1=0\\
&\alpha_5(1-t)+\alpha_3 t-\alpha_2=0
\end{aligned}$$

For t=-1 , the coeffecient matrix

$$\begin{vmatrix} 2 & 0 & 0 & -1 & -1\\ 0 & 2 & -1 & -1 & 0\\ -1 & 0 & 2 & 0 & -1\\ -1 & -1 & 0 & 2 & 0\\ 0 & -1 & -1 & 0 & 2 \end{vmatrix} \sim \begin{vmatrix} 0 & -2 & 0 & 3 & -1\\ 0 & 2 & -1 & -1 & 0\\ 0 & 1 & 2 & -2 & -1\\ -1 & -1 & 0 & 2 & 0\\ 0 & -1 & -1 & 0 & 2 \end{vmatrix} \sim \begin{vmatrix} 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 2 & -2 & -1\\ -1 & -1 & 0 & 2 & 0\\ 0 & 0 & 1 & -2 & 1 \end{vmatrix}$$

n-r=5-3=2 It has a solition dependant two paremeters.it is obtained $\alpha_0=4\alpha_4-3\alpha_5$ , $\alpha_1=-2\alpha_4+3\alpha_5$ , $\alpha_2=2\alpha_4-\alpha_5$ , $\alpha_4=\alpha_4$ , $\alpha_5=\alpha_5$

If the generators corresponding to $b_0,b_1,b_2,b_3,b_4,b_5,b_6$ regions are 1,x,y,z,t,u,v, according to Dehn presentation the relations of $5_2$ knot are $y^{-1}zx^{-1}=1$, $zy^{-1}u^{-1}=1$, $zt^{-1}x^{-1}=1$, $t^{-1}uv^{-1}=1$, $uz^{-1}zv^{-1}=1$.

Let $\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6$ be the unknowns symbolizing $b_0,b_1,b_2,b_3,b_4,b_5,b_6$ regions.Then ,

$$\alpha_0 t-\alpha_2 t+\alpha_3-\alpha_1=0$$
$$\alpha_3 t-\alpha_2 t+\alpha_0-\alpha_6=0$$
$$\alpha_3 t-\alpha_4 t+\alpha_0-\alpha_1=0 \mod 7$$
$$\alpha_0 t-\alpha_4 t+\alpha_5-\alpha_6=0$$
$$\alpha_6 t-\alpha_4 t+\alpha_3-\alpha_6=0$$

For t=1 ,the coeffecient matrix of this system is ,

$$\begin{vmatrix} -1 & -1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & -1 & 0 & 0 & -1 \\ 1 & -1 & 0 & -1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 & 1 & -1 & -1 \end{vmatrix} \sim \begin{vmatrix} -1 & -1 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 & -2 \\ 0 & 0 & 0 & 1 & 1 & -1 & -1 \end{vmatrix}$$

The rank of system is n-r=7-4=3. It has a solition depandent three parameters .The set of solution is $\alpha_0=-\alpha_4+\alpha_5+\alpha_6$ , $\alpha_1=-4\alpha_4+5\alpha_6$ , $\alpha_2=-2\alpha_4+3\alpha_6$ , $\alpha_3=-\alpha_4+\alpha_5+\alpha_6$ , $\alpha_4=\alpha_4$ , $\alpha_5=\alpha_5$ , $\alpha_6=\alpha_6$.

**Result:** Can all knots be classifed in solition space as above ?

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# ON RACKS IN FUZZY ALGEBRAC TOPOLOGY

Arif DANE

C.Ü. Department of Mathematics, Sivas, TURKEY.

## ABSTRACT

In this study, given a path and homotopy concepts in a topological space are investigated for fuzzy topological spaces. We defined fuzzy sets in $I_1^X$ and also fuzzy paths families in $X^I$.. We formed a rack struckture on $I_1^X$ and $X^I$. Then, some theorems and results about these concepts are proved.

**Keywords:** Path, Homotpy, Fuzzy set, Fuzzycontinuous ,Rack.

## ÖZET

Bu çalışmada, bir topolojik uzayda verilen yol ve homotopi kavramları fuzzy topolojik uzaylarda incelendi. Sonra bu kavramlarla ilgili bazı teoremler ve sonuçlar ispatlandı. $I_1^X$ de fuzzy kümeleri ve $X^I$ da da fuzzy yol ailelerini tanımladık $I_1^X$ ve $X^I$ üzerinde bir rack yapısı oluşturduk.Sonra bu kavramlarla ilgili ba teoremler ve sonuçları ispatladık.

**Anahtar Kelimeler:** Yol, Homotopi, Fuzzy küme, Fuzzy süreklilik, Rack.

## INTRODUCTION

In this section, we have defined relation on $R^+\cup\{0\}$ such that if $a,b\in R\cup\{0\}$ then, for $a\in I_n$ $b\in I_m$, there is a unique $x\in I_1$ such that

$a= x+(n-1)\, k_1$

$b= x+(m-1)k_2$

where $k_1,k_2\in N$.

After than we hawe proveu that $I_1\equiv I_n$ mod (n-1). Here, we have defined a modulo operation on $R^+\cup\{0\}$ using the above relation.

First, with the following theorem, we will show that there is an equivalent relation on $R^+\cup\{0\}$ with above relation.

**Theorem 1:** Above relation is an equivalent relation on $R^+\cup\{0\}$.

**Proof:** (i). $a \equiv a$ if $a \in I_n$ then thare is a unique $x \in I_1$ such that $a = x+ (n-1)\, k$ thus $a \equiv a$ is obvious.

(ii). $a \equiv b \Rightarrow b \equiv a$ ; if $a \equiv b$, then there is a unique $x \in I_1$ such that

$a = x+ (n-1)\, k_1$

$b = x + (m-1)\, k_2$

$k_1, k_2 \in N$ and $a \in I_n$, $b \in I_m$. It is clear that b a.

(iii). $a \equiv b$ and $b \equiv c \Rightarrow a \equiv c$;

if $a \equiv b$, then $a = x + (n-1)\, k_1$

$b \equiv c$, then $b = x + (m-1) k_2$

$c = x + (p-1)\, k_3$

$k_1, k_2, k_3 \in N$ and $a \in I_n$, $b \in I_m$ and $c \in I_p$ thus $a \equiv c$.

For $x \in I_1 = [\,0,1\,]$, [x] will be show an equivalent class. All the equivalent classes of $R^{+} \cup \{0\}$ is $I_1 = [\,0,1\,]$.

Now, we want to show that $I_1 \equiv I_n\ \forall\, n \in N$. To see this first we define a modulo relation on $R^{+} \cup \{0\}$ with using the equivalent relation . Let $a \in I_n = [\,n-1, n\,]$ and $b \in I_1$, then $a \equiv b \bmod (n-1) \leftrightarrow a = b + (n-1) \cdot k,\ k \in N$.

**Theorem 2:** For every $n \in N$, $I_n \equiv I_1 \bmod (n-1)$.

**Proof:** To prove this theorem we shall use the induction principle.

(1). For n=1 , $I_1 \equiv I_1 \bmod (0)$. It is obvious.

(2). For n=k , we suppose that the theorem is true.

(3). For n=k+1, let $I_{k+1} = [\,k, k+1\,]$ and $k, k+1 \in I_{k+1}$

$k \equiv 0 \bmod k \leftrightarrow k = 0 + k.n,\ n \in N$, if we take n=1, it is true.

$k+1 \equiv 1 \bmod k \leftrightarrow k+1 = 1 + k.n,\ n \in N$, if we take n=1, it is true.

Let $x \in (k, k+1)$, suppose that y=x-k, then $y \in I_1$ and $x \equiv y \bmod k$, that is x=y+n.k for n=1. Thus for k+1 the theorem is true. Finally for all $n \in N$, the theorem is true.

**Definition 1:** Let X be a non-empty set. A fuzzy set in X is an element in $I_1^{X}$ i.e.,a function from X into $I_1$ / 1 /.

We will denote fuzzy sets with $\alpha, \beta, \gamma \ldots$,and $\alpha(x)$ the image of x with a fuzzy set $\alpha$ in X. Where, 0(x)=0, 1(x)=1 for all x in X.Contain,Unions and Intersections of fuzzy set are denoted by $\leq$ , V and $\Lambda$ respectively.

Let $\alpha$ and $\beta$ be two fuzzy sets in X we have

$\alpha \leq \beta \leftrightarrow \alpha(x) \leq \beta(x)$ for all x in X.

$\alpha = \beta \leftrightarrow \alpha(x) = \beta(x)$ for all x in X

$\mu = \alpha V \beta \leftrightarrow \mu(x) = \max\{\alpha(x), \beta(x)\}$ for all x in X

$\mu = \alpha \Lambda \beta \leftrightarrow \mu(x) = \min\{\alpha(x), \beta(x)\}$ for all x in X

**Definition 2:** The complement of $\alpha$ denoted by $\alpha'$ is defined by the formula $\alpha' = 1 - \alpha(x)$ , for all x in X.

**Definition 3:** A fuzzy set in X is called a fuzzy point iff it takes the value 0 for all $y \in X$ except one,say $x \in X$. If it is value at x is $\alpha$ $(0 < \alpha < 1)$, we denote this fuzzy point by $x_\alpha$, where the point x is called its support. we can write the fuzzy point $x_\alpha$, with.

$$x_\alpha(y) = \begin{cases} \alpha & y = x \\ 0 & y \neq x \end{cases}$$

and we can denote the support of $x_\alpha$ with supp$x_\alpha$=x.

**Definition 4:** A fuzzy topology is a familly $\tau$ of fuzzy sets in X which satisfies the following conditions.

T.1) $0, 1 \in \tau$

T.2) if $\alpha, \beta \in \tau$ ,then $\alpha \Lambda \beta \in \tau$

T.3) if $\mu_i \in \tau$, for each $i \in I$, then $V_i \mu_i \in \tau$

$\tau$ is called a fuzzy topology for X , and thepair $(X, \tau)$ is a fuzzy topological space. Every member of $\tau$ is called an open fuzzy set. A fuzzy set is closed fuzzy set iff its complement is open / 2 /.

**Definition 5:** Let X and Y be fuzzy topological space and let f be in Y. Then, the inverse of $\beta$ , written as $f^{-1}(\beta)=\beta(f(x))$ for all x in X / 1 /.

**Definition 6:** A rack is a non-empty set X with a binary operation the following two axioms.

**Axiom 1.** Given $a,b \in X$ there is a unique $c \in X$ such that $a=c^b$.

**Axiom 2.** Given $a,b,c \in X$ the formula

$a^{bc} = a^{cb^c}$ holds. Where $c^b$ operates like $c^{-1}bc$, i.e, $c^b = c^{-1}bc$.

Now, we want to define a rack on $I_1^X$. For this purpose we must define an operation on $I_1^X$ that must satisfy the rack axioms / 3 /.

We consider fuzzy sets in $I_1^X$ with a binary operation which we shall write exponentialy. For $\alpha,\beta \in I_1^X \Rightarrow (\alpha,\beta) \rightarrow \alpha^\beta=\beta^{-1}\alpha\beta$.

Here, for $\beta \in I_1^X$,

$$\beta^{-1}(x)=\begin{cases} \dfrac{1}{\beta(x)} & \beta(x)\neq 0 \\ 1 & \beta(x)=0 \end{cases}$$

and $(\alpha^\beta)(x)= \beta^{-1}(x)\ \alpha(x)\ \beta(x)$. We will suppose that if $\beta^{-1}(x) \in I_n$ , then since $I_n \equiv I_1 \mathrm{mod}\,(n-1)$ $\beta^{-1}(x) \in I_1^X$.

**Theorem** . With above binary operation on $I_1^X$, $I_1^X$ is arack.

**Proof:** (i). For all $\alpha,\beta \in I_1^X$,there is a unique $\gamma \in I_1^X$ such that $\alpha=\gamma^\beta$.

Indeed; $\alpha=\gamma^\beta= \beta^{-1}\gamma\beta \Rightarrow \gamma=\beta\alpha\beta^{-1}$ , $\gamma \in I_1^X$.

(ii). For all $\alpha,\beta,\gamma \in I_1^X$,the formula

$\alpha^{\beta\gamma} = \alpha^{\gamma\beta^\gamma}$ holds. Indeed;

$$\alpha^{\gamma.\beta^\gamma} = \alpha^{\gamma.\gamma^{-1}.\beta.\gamma} = \alpha^{\beta.\gamma} = \left(\alpha^\beta\right)^\gamma.$$

**Example 1:** Let X={a,b}. We have a rack structure on $I_1^X$.

(i). Let $\alpha,\beta \in I_1^X$, where $\alpha(a)= \alpha(b)= \frac{1}{2}$ and $\beta(a)= \beta(b)= \frac{1}{3}$. If we chose $\gamma \in I_1^X$ with $\gamma(a)=\gamma(b)= \frac{1}{2}$ then , $\alpha=\gamma^\beta$. Indeed;

$$\alpha(a)=(\beta^{-1}\gamma\beta)(a)=\beta^{-1}(a).\gamma(a).\beta(a)=3.\frac{1}{2}.\frac{1}{3}=\frac{1}{2}$$

$$\alpha(b)=(\beta^{-1}\gamma\beta)(b)=\beta^{-1}(b).\gamma(b).\beta(b)=3.\frac{1}{2}\frac{1}{3}=\frac{1}{2}$$

(ii). If we choose $\gamma\in I_1^X$ with $\gamma(a)=\gamma(b)=\frac{1}{9}$, then

$$(\alpha^{\beta\gamma})(a)=(\beta\gamma)^{-1}(a).\alpha(a).(\beta\gamma)(a)=\frac{1}{\beta(a)\gamma(a)}.\alpha(a).\beta(a)\gamma(a)=\alpha(a)=\frac{1}{2}$$

and

$$\left(\alpha^{\gamma\beta^{\gamma}}\right)(a)=\left(\gamma\beta^{\gamma}\right)^{-1}(a)\alpha(a)\left(\gamma\beta^{\gamma}\right)(a)$$

$$=\left(\gamma(a)\gamma^{-1}(a)\beta(a)\gamma(a)\right)^{-1}\alpha(a)\gamma(a)\gamma^{-1}(a)=$$

$$\beta^{-1}(a)\gamma^{-1}(a)\alpha(a)\beta(a)\gamma(a)=\alpha(a)=\frac{1}{2}$$

Similar way

$$\left(\alpha^{\beta\gamma}\right)(b)=\left(\alpha^{\gamma\beta^{\gamma}}\right)(b)=\alpha(b)=\frac{1}{2}$$

Now, if we choose binary operation on $I_1^X$ with for $\alpha,\beta\in I_1^X$ $(\alpha+\beta)(x)=\alpha(x)+\beta(x)$ and $\alpha^{-1}(x)=1-\alpha(x)=\alpha'(x)$. Then we have a rack structure on $I_1^X$ with this operation.

**Theorem 5:** If we consider fuzzy sets in $I_1^X$ with a binary operation which we shall write exponentially for $I_1^X$, $(\alpha,\beta)\rightarrow\alpha^{\beta}=\beta^{-1}+\alpha+\beta$, then $I_1^X$ is a rack with this operation.

**Proof:** (i). For all $\alpha,\beta\in I_1^X$, if we choose $\gamma=\alpha$, then $\alpha=\gamma^{\beta}$. Indeed;

$\alpha=\gamma^{\beta}=\beta^{-1}+\gamma+\beta \Rightarrow \beta+\alpha+\beta^{-1}=1+\alpha$

we know that $I_2\equiv I_1 \text{mod}(1)$. Thus $(1+\alpha)(x)=1+\alpha(x)\equiv\alpha(x)\text{mod}(1)$.

Hence we can take $\alpha$ instead of $1+\alpha$.

(ii). For all $\alpha,\beta,\gamma\in I_1^X$, the formula

$$\alpha^{\beta+\gamma}=\alpha^{\gamma+\beta^{\gamma}}$$ holds.

Indeed; $\alpha^{\gamma+\beta^{\gamma}}=\alpha^{\gamma+\gamma^{-1}+\beta+\gamma}=\alpha^{1+\beta+\gamma}$ since, $1+\beta+\gamma\equiv(\beta+\gamma)(\text{mod } 2)$,

$$\alpha^{\gamma+\beta^{\gamma}}=\alpha^{\beta+\gamma}$$

**Example 2:** Let X={a,b } and let $\alpha,\beta\in I_1^X$ where $\alpha(a)=\alpha(b)=\frac{1}{2}$ and $\beta(a)=\beta(b)=\frac{1}{3}$.

(i). If we choose $\gamma\equiv\alpha$ mod2 then $\alpha=\gamma^{\beta}=\beta^{-1}+\gamma+\beta$ and

$$\alpha(a)=\frac{1}{2}=(\gamma^{\beta})(a)=\beta^{-1}(a)+\gamma(a)+\beta(a)=1+\gamma(a)=\alpha(a)=\frac{1}{2}.$$

(ii). If we choose $\gamma\equiv\alpha$ mod2 then

$$\left(\alpha^{\beta+\gamma}\right)(a)=\left(\alpha^{\gamma+\beta^{\gamma}}\right)(a)=2+\alpha(a)\equiv\alpha(a)\bmod 2=\left(\alpha^{\beta+\gamma}\right)(b)$$

$$\left(\alpha^{\gamma+\beta^{\gamma}}\right)(a)=2+\alpha(a)\equiv\alpha(a)\bmod 2=\frac{1}{2}=\left(\alpha^{\gamma+\beta^{\gamma}}\right)(b)$$

In this section we have defined a rack on the familly all paths in X.

**Definition 7:** A path in a topological space X is a continuous mapping $p: I\rightarrow(X,\tau)$, where $I=\left[0,\|a\|\right]$ The points p(0) and $p(\|a\|)$ are called the initial point and the end point of the path p respectively / 4 /.

A topological space X is said to be pathwise connected if and only if for every pair of points $x_0$ and $x_{\|a\|}$ of X there exists a path p in X such that,

$$p(x_0)=x_0 \text{ and } p(\|a\|)=x_a.$$

Let $p_0$ and $p_1$ be two paths in a topological space X, which the same initial point and have the same end point. It say that $p_0$ is homotopic to $p_1$ if and only if there exists a continuous mapping.

$$H: IxI\rightarrow X$$

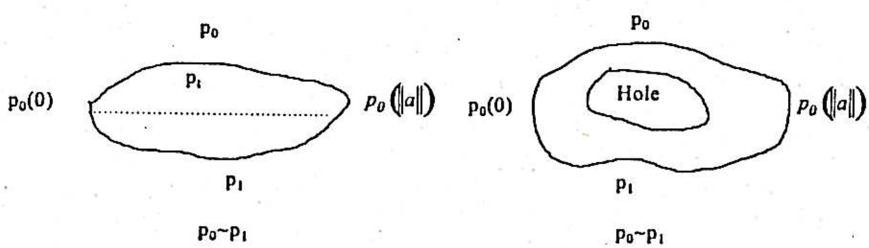
such that

$$H(s,0)=p_0(s),\quad H(s,1)=p_1(s) \quad \text{for all s in I}$$

and

$$H(s,t)=p_0(0)=p_1(0) \quad \text{for all s in I}$$

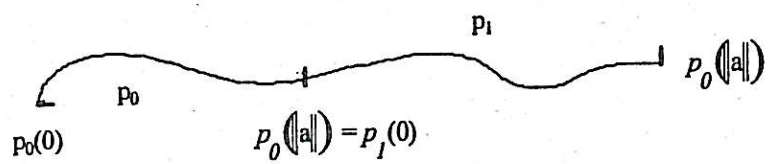
$$H(\|a\|,t)=p_0(\|a\|)=p_1(\|a\|) \quad \text{for all s in I.}$$



Instead of $p_0$ is homotopic to $p_1$ one also says that $p_0$ can be continuously deformed into $p_1$ and writes $p_0 \sim p_1$.

Suppoze that $p_0$ and $p_1$ are two paths in a topological space X such that $p_0(\|a\|)=p_1(0)$. Then, the product $p_1.p_0$ is defined to be

$$(p_1.p_0)(s)=\begin{cases} p_0(2s) & \text{for } 0\le s\le \frac{1}{2} \\ p_1(2s-|a|) & \text{for } \frac{1}{2}\le s\le |a| \end{cases}$$




**Definition 8:** A path in a topological space X whose end point and initial point coincide is called a loop in X . The loop $l$ is said to be based at $l(0)=l(\|a\|)$.

**Definition 9:** Suppose that e is the trivial loop in a topological space X at $x_0$, that is $e(x)=x_0$ for all x in X. Then, $e.l\sim l$ .

**Lemma 1:** Suppose that $l$ is a loop in a topological space X at $x_0$ and $l^1$ is the loop in X at $x_0$ defined by $l^1(s)=l(\|a\| -s)$ for all x in I. Then, $l^1.l\sim e$.

**Proof:** / /.

Now, we will show with $X^I$ the familly of allpaths in X.

$X^I = \{ p|\ p: I \to X \text{ continuous} \}$

Using the production of two paths in X, we arrive that a rack structure on $X^I$.

Indeed; (i). For all $p,q \in X^I$, if we take $r=q.p.q^{-1}\in X^I$ then $p=rq=q^{-1}rq=q^{-1}qpq^{-1}q=p$ .

(ii). For all $p,q,r\in X^I$, the formula ,

$$p^{r.q^r} = p^{r.r^{-1}.q.r} = p^{q.r} \text{ holds.}$$

The following theorem gives a semi-group structure on $X^I$ with the operation $\wedge$. For $p,q \in X^I$, $p\wedge q = \begin{cases} p & p\approx q \\ e & p\approx q \end{cases}$ .

**Theorem 6:** $(X^I,\wedge)$ is a semi group.

**Proof:** G.1: For $p,q \in X^I$

$p\wedge q = \begin{cases} p & p\approx q \\ e & p\approx q \end{cases}$ . Hence $p\wedge q\in X^I$

G.2: For $p,q,r\in X^I$, $p\wedge( q\wedge r) =p\wedge \begin{cases} q & q\approx r \\ e & q\approx r \end{cases} =$

$$\begin{cases} p\wedge q & q\approx r \\ p\wedge e & q\approx r \end{cases} = \begin{cases} p & p\approx q \quad and\, q\approx r \\ e & p\approx q \quad and\, q\approx r \end{cases}$$

$$\left(p\wedge q\right)\wedge r = \begin{cases} p & p\approx q \\ e & p\approx q \end{cases} \wedge r = \begin{cases} p\wedge r & p\approx q \\ e\wedge r & p\approx q \end{cases} = \begin{cases} p & p\approx q \quad and\, p\approx r \\ e & p\approx q \quad and\, p\approx r \end{cases}$$

Thus, $p\wedge(q\wedge r)=(p\wedge q)\wedge r$

G.3: For $p\in X^I$, since $p\wedge e=e$ and $e\wedge p=e$ there is not an unit element.

G.4:For $p\in X^I$, since $p\wedge p^{-1}=\begin{cases} p & p\approx p^{-1} \\ e & p\approx p^{-1} \end{cases}=p$ there is not an inverse element of p.

G.5: For $p,q\in X^I$, since $p\wedge q\neq q\wedge p$ $(X^I,\wedge)$ is not abelian.

**Theorem 7:** For $p,q\in X^I$ if we define $p\vee q=p.q$, then $(X^I,\vee)$ is a group except for abelian.

**Proof:** It is clear from the definition $p\vee q$.

**Corollary 1 :** We have the following proporties with "$\wedge$" and "$\vee$" operations.

(i). $(p\vee q)\wedge r=(p\wedge q)\vee(q\wedge r)$

(ii). $(p\wedge q)\vee r=(p\vee q)\wedge(q\vee r)$

**Proof :** It is obvious.

**Definition10:** A fuzzy path in a fuzzy topological space X is a fuzzy continuous mapping $p: I\rightarrow(X,\tau)$, where $I=\left[0,\|a\|\right]$ The points $p(0)$ and $p(\|a\|)$ are called the initial point and the end point of the path p respectively.

A fuzzy topological space X is said to be pathwise connected if and only if for every pair of points $x_0$ and x of X there exists a path p in X such that

$$p(x_0)=x_0 \text{ and } p(\|a\|)=x_a.$$

Let $p_0$ and $p_1$ be two fuzzy paths in a fuzzy topological space X, which the same initial point and have the same end point. It say that $p_0$ is homotopic to $p_1$ if and only if there exists a fuzzy continuous mapping.

$$H: IxI\rightarrow X$$

such that

$$H(s,0)=p_0(s),\quad H(s,1)=p_1(s) \qquad \text{for all s in I}$$

and

$$H(0,t)=p_0(0),\quad H(1,t)=p_0(1)=p_1(1) \quad \text{for all t in I}$$

Instead of $p_0$ is fuzzy homotopic to $p_1$ one also says that $p_0$ can be fuzzy continuously deformed into $p_1$ and writes $p_0\sim p_1$.

Suppoze that $p_0$ and $p_1$ are two fuzzy paths in a fuzzy topological space X such that $p_0(\|a\|)=p_1(0)$. Then, the product $p_1.p_0$ is as adapted in definition 7.

Now, we will show with $X^I$ the familly of all fuzzy paths in X.

$X^I=\{p \mid p : I \rightarrow X$ fuzzy continuous $\}$.

Using the production of two fuzzy paths in X, we arrive that a rack a structure on $X^I$.

Indeed;

(i). For all $p,q \in X^I$, if we take $r=q.p.q^{-1} \in X^I$, then

$$p=r.q=q^{-1}.r.q=q^{-1}.q.p.q^{-1}.q=p.$$

(ii). For every $p,q,r \in X^I$, the formula

$$p^{q.r}=p^{r.q^r}=p^{r.r^{-1}.q.r}$$ holds.

Then, $X^I$ is a rack.

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