ON THE PERTURBATION THEORY FOR THE MULTIDIMENSIONAL SCHRÖDINGER OPERATOR

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İZMİR

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ABSTRACT

The Schrödinger operator is one of the fundamental operators in quantum theory.

In this thesis, we obtain the asymptotic formulas for the eigenvalues of the self-adjoint Schrödinger operator defined by the differential expression

$$Lu = -\Delta u + q(x)u$$

in d-dimensional paralellepiped F, with the Neumann boundary conditions

$$\frac{\partial u}{\partial n}|_{\partial F} = 0.$$

ÖZET

Schrödinger operatörü kuantum fiziĝinin temel operatörlerinden biridir. Bu çalışmada, d-boyutlu bir pirizma F uzerinde

$$Lu = -\Delta u + q(x)u$$

diferansiyel ifadesi ve Neumann sınır koşulları

$$\frac{\partial u}{\partial n}|_{\partial F} = 0$$

ile tanımlanan kendine eş Schrödinger operatörünün özdeĝerleri için asimptotik formüller elde edilmiştir.

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CHAPTER ONE
INTRODUCTION

It is well known that the time independent Schrödinger operator,

$$Lu = -\Delta u + q(x)u,$$

is fundamental in quantum mechanics. The experimental experience and Schrödinger's work have shown that this operator is particularly useful in the following sense: It explains and improves Bohr's theory. It also gives a theoretical foundation for a number of basic physical effects which were observed experimentally but could not be explained sufficiently well by older theories.

The Schrödinger operator has different areas of application according to the properties of the potential $q(x)$. When the potential $q(x)$ is a periodic function, the Schrödinger operator arises in the description of periodic structures of various kind. This happens in the most natural way in the quantum theory of solids. Namely, the ions of a metal forming a crystal lattice give rise to a periodic field, in which a free electron can be considered. Then, in accordance with the fundamental principles of quantum mechanics, the possible values of the energy of a free electron belong to the spectrum of the Schrödinger operator with a periodic potential and the eigenfunctions describe the state of the particle. Therefore, it is important to have a detailed analysis of the spectral properties of this operator.

Both physicists and mathematicians have been studying the periodic Schrödinger operator for a long time . The most significant progress has been achieved in one dimensional case. The crucial property in analysis of the problem in one dimensional case is that the distance between the consecutive eigenvalues becomes larger and larger, so that the perturbation theory can be applied and asymptotic formulas for sufficiently large eigenvalues can be easily obtained, see

(Marchenko,1986), (Naimark,1968), (Birkhoff,1908), (Tamarkin,1917). The two and three dimensional cases are still of great challenge.

For physical applications, it is important to have a perturbation theory of the Schrödinger operator in many dimensional cases. In this case, to construct a perturbation theory turns out to be rather difficult, because the denseness of the eigenvalues of the free operator increases infinitely with increasing energy. The eigenvalues of the free operator are situated very close to each other in a high energy region. Therefore, when perturbation disturbs them, they strongly influence each other. This presents considerable difficulties as the arbitrarily small differences become small divisors in an asymptotic expansion, in particular, "the small denominators problem". Thus, to describe the perturbation of one of the eigenvalues, we must also study all the other surrounding eigenvalues.

In this thesis, we consider the d-dimensional Schrödinger operator $L_N(q(x))$, defined by the differential expression

$$Lu = -\Delta u + q(x)u \tag{1.1}$$

in F, with the Neumann boundary condition

$$\frac{\partial u}{\partial n}|_{\partial F} = 0, \tag{1.2}$$

where $F = [0, a_1] \times [0, a_2] \times ... \times [0, a_d]$, $a_1, a_2, ..., a_d \in R$, ∂F denotes the boundary of the domain F, $x = (x_1, x_2, ..., x_d) \in R^d$, $d \geq 2$, $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + ... + \frac{\partial^2}{\partial x_d^2}$ is the Laplace operator in R^d, and $q(x)$ is a real-valued function in $L_2(F)$.

Our aim is to find the asymptotic formulas for the eigenvalues of this Schrödinger operator $L_N(q(x))$ in an arbitrary dimension. For this, we use the method of papers (Veliev,1987), (Veliev,1988).

For the first time, the asymptotic formulas for the eigenvalues of the Schrödinger operator in parallelepiped with quasiperiodic boundary conditions were obtained in papers (Veliev,1987), (Veliev,1988) and the eigenvalues of the unperturbed operator were divided into two groups: Resonance and non-resonance ones.

The method introduced in these papers is general and can be used to solve all other boundary value problems. By some other methods, the asymptotic formulas for quasiperiodic boundary conditions in two and three dimensional cases were also obtained in (Feldman,1990), (Feldman, 1991), (Karpeshina,1992), (Karpeshina,1996) and (Friedlanger,1990).

The asymptotic formulas for the eigenvalues of the Schrödinger operator with Dirichlet boundary conditions in two-dimensional case were obtained in (Hald & McLaughlin,1996).

In Chapter One, some properties of multidimensional periodic functions are studied and the well known relation between the boundary value problems is given.

In Chapter Two, the eigenvalue problem for the unperturbed operator $L_N(0)$ is considered. In order to give the definitions of resonance and non-resonance eigenvalues, we divide R^d into two domains; resonance and non-resonance domains. We obtain the asymptotic formulas for the eigenvalues of the operator $L_N(q(x))$ in non-resonance domain. That is we prove that there is an eigenvalue of the operator $L_N(q(x))$ which is close to the non-resonance eigenvalue of the unperturbed operator $L_N(0)$.

In Chapter Three, we consider the problem in resonance domains. We find the relation between the eigenvalues of the operator $L_N(q(x))$ and the eigenvalues of the infinite matrix $C(\gamma, \gamma_1, ..., \gamma_k)$. Finally restricting the problem in a single resonance domain, we obtain the main result of the chapter: The eigenvalues of $L_N(q(x))$ are close to the eigenvalues of the corresponding one-dimensional Sturm-Liouville operators $T(Q(s), \beta)$.

In Chapter Four, the problem is considered for special case of single resonance domains and the relation between the eigenvalues of $L_N(q(x))$ and the eigenvalues of the one-dimensional Sturm-Liouville operators $T_{e_i}(Q(s))$ is given by a different approach.

1.1 Properties of the Periodic Functions on $\mathbf{R}^d$

Definition 1.1.1. A function $q(x)$, where $x \in R^d$, is said to be periodic if there are d linearly independent vectors $w_1, w_2, ..., w_d$ such that

$$q(x + w_k) = q(x), \qquad k = 1, 2, ..., d.$$

Note that, this definition is equivalent to

$$q(x + w) = q(x), \ \ \forall w \in \Omega, \tag{1.3}$$

where

$$\Omega = \{w : w = \sum_{i=1}^{d} m_i w_i; m_i \in Z, i = 1, 2, ..., d\}$$

is the lattice generated by the vectors $w_1, w_2, ..., w_d$.

Hence the function $q(x)$ satisfying (1.3) is said to be periodic with respect to Ω and related with this lattice, there is a d-dimensional parallelepiped

$$F = \{\sum_{i=1}^{d} t_i w_i; 0 \le t_i < 1, i = 1, 2, ..., d\},$$

called the fundamental domain of Ω, which is the period parallelepiped of $q(x)$.

We define the dual lattice Γ of Ω, as

$$\Gamma = 2\pi\Theta,$$

where the lattice

$$\Theta = \{\sum_{i=1}^{d} n_i \gamma_i, n_i \in Z, i = 1, 2, ..., d\}$$

is called the reciprocal lattice of Ω, and the vectors $\gamma_1, \gamma_2, ..., \gamma_d$ are linearly independent vectors satisfying

$$\langle w_j, \gamma_k \rangle = \delta_{jk} = \begin{cases} 1 \ , \ j = k \\ 0 \ , \ j \neq k \end{cases}$$

where $\langle .,. \rangle$ is the standard inner product in R^d.
Clearly, for any $w \in \Omega, \gamma \in \Gamma$

$$\langle w, \gamma \rangle = \langle \sum_{i=1}^{d} m_i w_i, \sum_{i=1}^{d} n_i \gamma_i \rangle = \sum_{i=1}^{d} m_i n_i w_i \gamma_i = 2\pi k,$$

where $k \in \mathbf{Z}$.

The functions $e^{i\langle \gamma, x \rangle}$ for $\gamma \in \Gamma$ are periodic with respect to Ω. Really,

$$e^{i\langle \gamma, x+w \rangle} = e^{i\langle \gamma, x \rangle} e^{i\langle \gamma, w \rangle} = e^{i\langle \gamma, x \rangle} e^{i2\pi k} = e^{i\langle \gamma, x \rangle}.$$

Let $q(x)$ be a real-valued and periodic (with respect to Ω) function of the space

$$W_2^l(F) = \{f : D^\alpha f \in L_2(F), \forall |\alpha| \leq l\},$$

where $\alpha = (\alpha_1, \alpha_2, ..., \alpha_d) \in Z^d, |\alpha| = |\alpha_1| + |\alpha_2| + ... + |\alpha_d|$, $D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} ... \partial x_d^{\alpha_d}}$, $l \in \mathbb{N}$ and $l \geq \frac{(d+20)(d-1)}{2} + d + 3$.

Since $\{e^{i\langle \gamma, x \rangle}\}_{\gamma \in \Gamma}$ forms an orthonormal basis in $L_2(F)$, we write

$$q(x) = \sum_{\gamma \in \Gamma} q_\gamma e^{i\langle \gamma, x \rangle},$$

where $q_\gamma = (q(x), e^{i\langle \gamma, x \rangle}) = \int_F q(x) \overline{e^{i\langle \gamma, x \rangle}} dx$ are the Fourier coefficients of $q(x)$. For simplicity, we can assume that $q_0 = \int_F q(x) dx = 0$ and F is a d-dimensional rectangle, say $F = [0, a_1] \times [0, a_2] \times ... \times [0, a_d]$.

Property 1: Let $q(x)$ be a real-valued function which is periodic with respect to Ω. Then $q(x)$ is a function of $W_2^l(F)$ if and only if the fourier coefficients q_γ of $q(x)$ satisfy the following relation

$$\sum_{\gamma \in \Gamma} |q_\gamma|^2 (1 + |\gamma|^{2l}) < \infty. \tag{1.4}$$

Proof. By definition of q_γ, we have

$$q_\gamma = \int_0^{a_1} \int_0^{a_2} ... \int_0^{a_d} q(x) e^{-i(\gamma_1 x_1 - i\gamma_2 x_2 - ... - i\gamma_d x_d)} dx_1 dx_2 ... dx_d,$$

where $\gamma = (\gamma_1, \gamma_2, ..., \gamma_d) \in \Gamma$. In the last expression, integrating by parts α_i times with respect to x_i, for each $i = 1, 2, ..., d$ and using the fact that $q(x)$ and $e^{i\langle\gamma,x\rangle}$ are periodic functions with respect to Ω, we get

$$q_\gamma = \frac{(-1)^{|\alpha|}}{(i\gamma_1)^{\alpha_1}.(i\gamma_2)^{\alpha_2}...(i\gamma_d)^{\alpha_d}} \int_F \frac{\partial^{|\alpha|} q(x)}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} ... \partial x_d^{\alpha_d}} \overline{e^{i\langle\gamma,x\rangle}} dx, \tag{1.5}$$

where

$$a_\gamma^\alpha = \int_F \frac{\partial^{|\alpha|} q(x)}{\partial x_1^{\alpha_1} x_2^{\alpha_2} ... \partial x_d^{\alpha_d}} \overline{e^{i\langle\gamma,x\rangle}} dx, \tag{1.6}$$

for $|\alpha| \leq l$, are the Fourier coefficients of the function $D^\alpha q(x) \in L_2(F)$ with respect to the basis $\{e^{i\langle\gamma,x\rangle}\}_{\gamma\in\Gamma}$. Then, by (1.5) and (1.6), we obtain

$$|q_\gamma|^2 \leq \frac{1}{|\gamma|^{2l}} |a_\gamma^\alpha|^2 \tag{1.7}$$

from which the relation (1.4) follows. Conversely, if $q(x)$ is periodic and (1.4) holds, then by (1.5) and (1.6)

$$\sum_{\gamma\in\Gamma} |a_\gamma^\alpha|^2 \leq \sum_{\gamma\in\Gamma} |\gamma|^{2l} |q_\gamma|^2 < \infty,$$

which implies that $q(x) \in W_2^l(F)$.

Property 2: For a large parameter ρ we can write a periodic function $q(x) \in W_2^l(F)$ as

$$q(x) = \sum_{\gamma\in\Gamma(\rho^{\alpha'})} q_\gamma e^{i\langle\gamma,x\rangle} + O(\rho^{-p\alpha'}), \tag{1.8}$$

where

$$\Gamma(\rho^{\alpha'}) = \{\gamma \in \Gamma : 0 < |\gamma| < \rho^{\alpha'}\},$$

$\alpha' > 0$, $p = l - d$, and $O(\rho^{-p\alpha'})$ is a function in $L_2(F)$, with norm of order $\rho^{-p\alpha'}$. That is, $f(\xi) = O(g(\xi))$, if there exists a constant c, such that $|\frac{f(\xi)}{g(\xi)}| < c$ at some neighborhood of ∞.

Proof. By (1.7) we have

$$\| \sum_{|\gamma|\geq\rho^{\alpha'}} q_\gamma e^{i\langle\gamma,x\rangle} \| \leq (\sum_{|\gamma|\geq\rho^{\alpha'}} |q_\gamma|^2)^{\frac{1}{2}}$$

$$\leq (\sum_{|\gamma|\geq\rho^{\alpha'}} \frac{|a_\gamma|^2}{|\gamma|^{2l}})^{\frac{1}{2}} \leq (\sum_{|\gamma|\geq\rho^{\alpha'}} \frac{1}{|\gamma|^{2l}})^{\frac{1}{2}} (\sum_{|\gamma|\geq\rho^{\alpha'}} |a_\gamma|^2)^{\frac{1}{2}}$$

$$= O(\rho^{-(l-d)\alpha'}) = O(\rho^{-p\alpha'}),$$

where $\sum_{\gamma\in\Gamma}|a_\gamma|^2<\infty$, since $a_\gamma=(D^\alpha q(x),e^{i\langle\gamma,x\rangle})$ are the Fourier coefficients of $D^\alpha q(x)\in L_2(F)$, $|\alpha|\le l$ and $(\sum_{|\gamma|\ge\rho^{\alpha'}}\frac{1}{|\gamma|^{2l}})^{\frac{1}{2}}=O(\rho^{-(l-d)\alpha'})$.

Property 3: For a periodic function $q(x)\in W_2^l(F)$, we have

$$\sum_{\gamma\in\Gamma}|q_\gamma|<\infty. \tag{1.9}$$

Proof. Using the relation (1.7) and the Cauchy-Schwarz inequality we have

$$\sum_{\gamma\in\Gamma}|q_\gamma|\le\sum_{\gamma\in\Gamma}\frac{|a_\gamma|}{|\gamma|^l}\le(\sum_{\gamma\in\Gamma}\frac{1}{|\gamma|^{2l}})^{\frac{1}{2}}(\sum_{\gamma\in\Gamma}|a_\gamma|^2)^{\frac{1}{2}}<\infty$$

where $(\sum_{\gamma\in\Gamma}\frac{1}{|\gamma|^{2l}})^{\frac{1}{2}}$ converges for $l>\frac{d}{2}$. That is, for $l>\frac{d}{2}$ we obtain (1.9).

1.2 The Relation between the Boundary Value Problems in d-Dimensional Parallelepiped

Associated with the Schrődinger operator in $L_2(F)$ we consider the following boundary value problems:

i) The Dirichlet problem is defined as :

$$-\Delta u+q(x)u=\lambda u$$

$$u|_{\partial F}=0,$$

where we denote the eigenvalues and the eigenfunctions of the Dirichlet problem by Λ_N and Ψ_N respectively.

ii) The Neumann problem is defined as :

$$-\Delta u+q(x)u=\lambda u$$

$$\frac{\partial u}{\partial n}|_{\partial F}=0, \tag{1.10}$$

where $\frac{\partial}{\partial n}$ denotes differentiation along the outward normal of the boundary ∂F. The eigenvalues and the eigenfunctions of the Neumann problem are denoted by Υ_N and Φ_N, respectively.

iii) The quasiperiodic or t-periodic problem is:

$$-\Delta u + q(x)u = \lambda u$$

$$u(x+w) = u(x)e^{i(w,t)}, \tag{1.11}$$

where $w \in \Omega$, $t = \sum_{i=1}^{d} \gamma_i t_i$, $\{\gamma_i\}_{i=1}^{d}$ is the basis of the dual lattice Γ, and $t_1, t_2, ..., t_d$ are real parameters in $[0, 1]$. We denote the eigenvalues and the eigenfunctions of this problem by λ_N and φ_N respectively.

To see the relation between the eigenvalues of the Dirichlet, Neumann and quasiperiodic problems, see (Eastham, 1973), first we give some definitions.

Let A denote the set of all complex-valued functions $f(x)$ which are continuous in F and have piecewise continuous first-order partial derivatives in F. Then the Dirichlet integral $J(f,g)$ in d-dimension is defined by

$$J(f,g) = \int_F \{\text{grad}f(x).\text{grad}\overline{g(x)} + q(x)f(x)\overline{g(x)}\}dx, \tag{1.12}$$

for $f(x)$ and $g(x)$ in A, where

$$\text{grad}f(x) = \frac{\partial f}{\partial x_1}e_1 + \frac{\partial f}{\partial x_2}e_2 + ... + \frac{\partial f}{\partial x_d}e_d.$$

If in (1.12), $g(x)$ has also piecewise continuous second-order partial derivatives in F, then a Green's theorem gives

$$J(f,g) = -\int_F f(x)\{\Delta\overline{g(x)} - q(x)\overline{g(x)}\}dx + \int_{\partial F} f\frac{\partial \overline{g}}{\partial n}dS, \tag{1.13}$$

where dS denotes an element of surface area of ∂F. We consider $J(f,g)$ as applied first to the quasiperiodic problem. If $f(x)$ and $g(x)$ satisfy the boundary conditions (1.11), the integral over ∂F in (1.13) is zero because the integrals over opposite faces of ∂F cancel out. In particular, when $g(x) = \varphi_N(x)$, (1.13) gives

$$J(f,g) = \lambda_N f_N, \tag{1.14}$$

where $f_N = \int_F f(x)\overline{\varphi_N(x)}dx$ is the Fourier coefficient, and we have used the fact that $\varphi_N(x)$ is the eigenfunction of the quasiperiodic problem with the corresponding eigenvalue λ_N. A particular case of (1.14) is

$$J(\varphi_N, \varphi_M) = \begin{cases} \lambda_N \ , \ M = N \\ 0 \ , \ M \neq N \end{cases}$$

It follows that

$$\sum_{N=0}^{\infty} \lambda_N |f_N|^2 \leq J(f, f) \tag{1.15}$$

for all $f(x) \in A$ which satisfy (1.11).
From (1.15) we obtain the following

$$\lambda_0 = \min\left(\frac{J(f,f)}{\int_F |f(x)|^2 dx}\right), \tag{1.16}$$

the minimum being taken over all $f(x)$ ($\neq 0$) in A which satisfy (1.11). Furthermore, the minimum in (1.16) is attained only when $f_N = 0$ for all N such that $\lambda_N > \lambda_0$, i.e., only when $f(x)$ is an eigenfunction corresponding to λ_0. In the case of the quasiperiodic problems, the eigenfunctions are real valued and therefore, for these problems , $f(x)$ can be confined to being real valued in (1.16). The results (1.14) and (1.16) follow from (1.13) because of the vanishing of the integral over ∂F when g(x) is an eigenfunction. Corresponding results hold for the Dirichlet and Neumann problems. In the first problem, the integral over ∂F in (1.13) vanishes if $f(x) = 0$ on ∂F. In the second problem, the integral vanishes without any boundary condition on $f(x)$ since, by (1.10), $\frac{\partial g}{\partial n} = 0$ on ∂F when $g(x)$ is an eigenfunction. Thus, for the Neumann problem , we have

$$\sum_{N=0}^{\infty} \Upsilon_N |f_N|^2 \leq J(f, f) \tag{1.17}$$

for all $f(x) \in A$, where $f_N = \int_F f(x)\Phi_N(x)dx$, while the corresponding result holds for the Dirichlet problem if $f(x)$ is in A and $f(x) = 0$ on ∂F. All the above results are used in the proof of the following theorem, see (Estham,1973).

Theorem 1.2.1. *For* $N \geq 0$,

$$\Upsilon_N \leq \lambda_N \leq \Lambda_N. \tag{1.18}$$

Proof. To prove the left-hand inequality, we first take $f(x) = \varphi_0$ in (1.17). Since $J(f, f) = \lambda_0$ by the particular case of (1.14), we obtain

$$\lambda_0 \geq \sum_{N=0}^{\infty} \Upsilon_N |f_N|^2 \geq \Upsilon_0 \sum_{N=0}^{\infty} |f_N|^2.$$

By the Parseval's formula, one has

$$\sum_{N=0}^{\infty} |f_N|^2 = \int_F |f(x)|^2 dx = 1.$$

Hence

$$\lambda_0 \geq \Upsilon_0.$$

Next we take

$$f(x) = c_0\varphi_0(x) + c_1\varphi_1(x),$$

where c_0 and c_1 are constants such that $|c_0|^2 + |c_1|^2 = 1$ and

$$c_0 \int_F \varphi_0(x)\Phi_0(x)dx + c_1 \int_F \varphi_1(x)\Phi_0(x)dx = 0,$$

Such a choice of c_0 and c_1 is always possible. The first condition makes $\int_F |f(x)|^2 dx = 1$ and the second makes $f_0 = 0$. By a particular case of (1.14)

$$J(f, f) = \lambda_0|c_0|^2 + \lambda_1|c_1|^2 \leq \lambda_1(|c_0|^2 + |c_1|^2) = \lambda_1.$$

Also, by (1.17) and the fact that $f_0 = 0$, we have

$$J(f, f) \geq \sum_{i=1}^{\infty} \Upsilon_N |f_N|^2 \geq \Upsilon_1 \sum_{i=1}^{\infty} |f_N|^2 = \Upsilon_1 \int_F |f_N|^2 dx = \Upsilon_1.$$

Hence

$$\lambda_1 \geq \Upsilon_1.$$

The argument can be extended to the general case N. We consider

$$f(x) = c_0\varphi_0(x) + c_1\varphi_1(x) + ... + c_N\varphi_N(x),$$

where c_i are real constants such that $|c_0|^2 + |c_1|^2 + ... + |c_N|^2 = 1$ and $f_i = 0$ for $0 \leq i \leq N - 1$. These conditions are N linear algebraic equations to be satisfied by $N + 1$ numbers $c_0, c_1, ..., c_N$ and such numbers do always exist. The proof of the theorem for general N is same as the proof for $N = 1$. The proof of the right-hand side of the inequality in Theorem 1.2.1 is similar and we use (1.15), instead of (1.17). □

CHAPTER TWO

SEPARATION OF EIGENVALUES FOR THE UNPERTURBED OPERATOR

In this chapter, we consider the eigenvalue problem for the unperturbed operator $L_N(0)$. As in papers (Veliev,1987) and (Veliev,1988), we divide the eigenvalues of the unperturbed operator into two groups: Resonance and non-resonance eigenvalues. In this chapter, these groups will be introduced and some estimations including eigenfunctions of the unperturbed operator will be obtained. These estimations will be used in the next chapters. Also the asymptotic formulas for the eigenvalues of the operator $L_N(q(x))$ are obtained in the non-resonance domain.

The non-resonance case was considered in (Atılgan et al., 2002). Here, we write the non-resonance case in an improved and enlarged form which can easily be used in the later chapters. Moreover it helps very much when reading the resonance case.

2.1 The Eigenvalues and the Eigenfunctions of the Unperturbed Operator $L_N(0)$

To begin with, we consider the operator $L_N(0)$ which is defined by the expressions (1.1) and (1.2) in the domain F, when $q(x) = 0$, where

$$F \equiv [0, a_1] \times [0, a_2] \times ... \times [0, a_d]. \tag{2.19}$$

Let

$$\Omega = \{\sum_{i=1}^{d} m_i w_i; m_i \in Z, i = 1, 2, ..., d\} \tag{2.20}$$

be a lattice in R^d with the reduced basis

$$w_1 = (a_1, 0, ..., 0), w_2 = (0, a_2, 0, ..., 0), ..., w_d = (0, ..., 0, a_d)$$

and

$$\Gamma = \{\sum_{i=1}^{d} n_i \gamma_i : n_i \in Z, i = 1, 2, ...d\} \tag{2.21}$$

be the dual lattice of Ω, where $\langle w_i, \gamma_j \rangle = 2\pi\delta_{ij}$.

Lemma 2.1.1. *The eigenvalues and the corresponding eigenfunctions of the operator $L_N(0)$ are $|\gamma|^2$ and $v_\gamma(x) = \cos\gamma_1 x_1 \cos\gamma_2 x_2 ... \cos\gamma_d x_d$ respectively, where $\gamma = (\gamma_1, \gamma_2, ..., \gamma_d) \in \frac{\Gamma}{2}$.*

Proof. First we show that

$$-\Delta v_\gamma = |\gamma|^2 v_\gamma.$$

Indeed,

$$\frac{\partial^2}{\partial x_k^2}(\prod_{i=1}^{d} \cos\gamma_i x_i) = -\gamma_k^2(\prod_{i=1}^{d} \cos\gamma_i x_i)$$

for all $k = 1, 2, ..., d$, which implies

$$-\Delta v_\gamma = -\sum_{k=1}^{d} \frac{\partial^2}{\partial x_k^2} v_\gamma = (\gamma_1^2 + \gamma_2^2 + ... + \gamma_d^2) v_\gamma = |\gamma|^2 v_\gamma.$$

Now, we show that the boundary conditions $\frac{\partial v_\gamma}{\partial n}(x) = 0$ for $x \in \partial F$ where

$$\partial F = \{x = (t_1 a_1, t_2 a_2, ..., t_d a_d) : t_i = 0 \text{ or } 1 \text{ at least for some } i, i = 1, 2, ..., d\}$$

hold. We know that $\frac{\partial v_\gamma}{\partial n}$ is the derivative of the function v_γ in the direction of the vector n, which is the normal to the boundary ∂F of F. By definition (2.19) of F, the boundary ∂F lies in the hyperplanes $\Pi_k = \{x \in R^d : \langle x, e_k \rangle = 0$ or on its shifts $a_k e_k + \Pi_k$, where $k = 1, 2, ..., d$; $e_1 = (1, 0, ..., 0), e_2 = (0, 1, 0, ..., 0), ..., e_d = (0, 0, , ..., 1)$. So, the normal vector to the hyperplanes Π_k and $a_k e_k + \Pi_k$ are e_k or $-e_k$, respectively. Therefore, $\frac{\partial v_\gamma}{\partial n}$ coincides with the first partial derivatives $\frac{\partial v_\gamma}{\partial x_j}$ or $-\frac{\partial v_\gamma}{\partial x_j}$ of u_γ. Thus,

$$\frac{\partial v_\gamma}{\partial x_j}|_{x\in\Pi_j} = \cos\gamma_1 x_1 ... \sin\gamma_j x_j ... \cos\gamma_d x_d|_{x_j=0} = 0,$$

and

$$\frac{\partial v_\gamma}{\partial x_j}|_{x\in a_j e_j+\Pi_j} = \cos\gamma_1 x_1 ... \sin\gamma_j x_j ... \cos\gamma_d x_d|_{x_j=a_j} = 0,$$

since $\gamma_j = \frac{m_j \pi}{a_j}, m \in Z$, by definition 2.21 of dual lattice Γ. □

Notice that, if for some k the component $\gamma_k = 0$, then the corresponding multiplier $\cos\gamma_k x_k$ does not take part in $v_\gamma(x)$.

Consider the norm of the function $v_\gamma(x)$ in $L_2(F)$

$$c_\gamma = \| v_\gamma(x) \| = \sqrt{\frac{a_1 a_2 ... a_d}{2^{d-k}}},$$

where $k, (0 \leq k \leq d)$, is the number of components γ_i of $\gamma = (\gamma_1, \gamma_2, ..., \gamma_d)$ such that $\gamma_i = 0$.

For the sake of simplicity in the calculations, we shall write $v_\gamma(x)$ in the form

$$v_\gamma(x) = (\prod_{i=1}^{d} \cos\gamma_i x_i) = \frac{1}{|A_\gamma|} \sum_{\alpha\in A_\gamma} e^{i\langle\alpha,x\rangle},$$

where $A_\gamma = \{\alpha = (\alpha_1, \alpha_2, ..., \alpha_d) \in R^d : |\alpha_i| = |\gamma_i|, i = 1, 2, ..., d\}$ and $|A_\gamma|$ is the number of vectors in A_γ.

Lemma 2.1.2.

$$(\sum_{\gamma'\in A_a} e^{i\langle\gamma',x\rangle})(\sum_{\alpha\in A_\gamma} e^{i\langle\alpha,x\rangle}) = \sum_{\gamma'\in A_a} \sum_{\alpha\in A_{\gamma+\gamma'}} e^{i\langle\alpha,x\rangle}, \tag{2.22}$$

for all $\gamma, \gamma' \in \frac{\Gamma}{2}$.

Proof. The expression (2.22) is equivalent to

$$\sum_{\{y:y=\alpha+\gamma',\gamma'\in A_a,\alpha\in A_\gamma\}} e^{i\langle y,x\rangle} = \sum_{\{y:y\in A_{\gamma+\gamma'},\gamma'\in A_a\}} e^{i\langle y,x\rangle}.$$

Therefore, we have only to show that the sets :

$$C = \{y : y = \alpha + \gamma', \gamma' \in A_a, \alpha \in A_\gamma\}$$

and

$$D = \{y : y \in A_{\gamma+\gamma'}, \gamma' \in A_a\}$$

are equal. For this, let $y = \alpha + \gamma' \in C$. Then, $\alpha \in A_\gamma$ and $\gamma' \in A_a$, that is, the components of the vectors are $\alpha_i = \pm\gamma_i$ and $\gamma_i' = \pm a_i$, by definition. Thus,

$$y_i = \pm(\gamma_i + a_i) \quad \text{or} \quad y_i = \pm(\gamma_i - a_i)\ ,\ \forall i = 1, 2, ..., d. \tag{2.23}$$

Now, let $z \in D$. Then, $z \in A_{\gamma+\gamma'}$, where $\gamma' \in A_a$. Again, by definition, $z_i = \pm(\gamma_i + \gamma_i')$, where $\gamma_i' = \pm a_i$. Hence,

$$z_i = \pm(\gamma_i + a_i) \text{ or } z_i = \pm(\gamma_i - a_i)\ ,\ \forall i = 1, 2, ..., d. \tag{2.24}$$

Then, by (2.23) and (2.24) we have $C = D$. □

2.2 Resonance and Non-Resonance Eigenvalues of $L_N(0)$

As it is mentioned before, the eigenvalues $|\gamma|^2$, for $|\gamma| \sim \rho$, of $L_N(0)$ are divided into two groups, where $|\gamma| \sim \rho$ means that $c_1\rho \leq |\gamma| \leq c_2\rho$ and by c_i, $i = 1, 2, ...$, we denote the positive real constants which do not depend on ρ. For this, first R^d is divided into two domains as follows:

Let $\alpha < \frac{1}{d+20}$, $\alpha_k = 3^k\alpha$, $k = 1, 2, ..., d-1$ and define the following sets

$$V_b(\rho^{\alpha_1}) \equiv \{x \in R^d : ||x|^2 - |x+b|^2| < \rho^{\alpha_1}\}$$

$$E_1(\rho^{\alpha_1}, p) \equiv \bigcup_{b\in\Gamma(p\rho^{\alpha})} V_b(\rho^{\alpha_1})$$

$$U(\rho^{\alpha_1}, p) \equiv R^d \backslash E_1(\rho^{\alpha_1}, p)$$

$$E_k(\rho^{\alpha_k}, p) \equiv \bigcup_{\gamma_1,\gamma_2,\ldots,\gamma_k \in \Gamma(p\rho^{\alpha})} (\bigcap_{i=1}^{k} V_{\gamma_i}(\rho^{\alpha_k})),$$

where the intersection $\bigcap_{i=1}^{k} V_{\gamma_i}(\rho^{\alpha_k})$ in E_k is taken over $\gamma_1, \gamma_2, ..., \gamma_k$ which are linearly independent vectors and the length of γ_i is not greater than the length of the other vectors in $\Gamma \bigcap \gamma_i R$, (see Remark 2.2.1). The set $U(\rho^{\alpha_1}, p)$ is said to be a non-resonance domain, and the eigenvalue $|\gamma|^2$ is called a non-resonance eigenvalue if $\gamma \in U(\rho^{\alpha_1}, p)$. The domains $V_b(\rho^{\alpha_1})$, for all $b \in \Gamma(p\rho^{\alpha})$ are called resonance domains and the eigenvalue $|\gamma|^2$ is a resonance eigenvalue if $\gamma \in V_b(\rho^{\alpha_1})$.

Remark 2.2.1. Note that, the elements of the single resonance domain

$$V_b(\rho^{\alpha_1}) = \{x \in R^d : ||x|^2 - |x+b|^2| < \rho^{\alpha_1}\}$$

are contained between the two hyperplanes

$$\Pi_1 = \{x : |x|^2 - |x+b|^2 = -\rho^{\alpha_1}\}$$

and

$$\Pi_2 = \{x : |x|^2 - |x+b|^2 = \rho^{\alpha_1}\}.$$

Since

$$|x|^2 - |x+b|^2 = \langle x, x\rangle - \langle x+b, x+b\rangle = -2\langle x, b\rangle - |b|^2 = \mp\rho^{\alpha_1},$$

$$\langle x, b\rangle + \frac{|b|^2}{2} \mp \frac{\rho^{\alpha_1}}{2} = 0,$$

we have

$$\Pi_1 = \{x : \langle x + \frac{b}{2} + \frac{\rho^{\alpha_1} b}{2|b|^2}, b\rangle = 0\} = \Pi_b + (\frac{b}{2} + \frac{\rho^{\alpha_1} b}{2|b|^2})$$

$$\Pi_2 = \{x : \langle x + \frac{b}{2} - \frac{\rho^{\alpha_1} b}{2|b|^2}, b\rangle = 0\} = \Pi_b + (\frac{b}{2} - \frac{\rho^{\alpha_1} b}{2|b|^2}),$$

where $\Pi_b \equiv \{x : \langle x, b\rangle = 0\}$ is the hyperplane passing through the origin. It is clear that the distance between the two parallel hyperplanes Π_1 and Π_2 is $\frac{\rho^{\alpha_1}}{|b|}$. So, without loss of generality, we can consider the resonance domains $V_b(\rho^{\alpha_1})$ for vectors $b \in \Gamma(p\rho^{\alpha})$ which are minimal in its direction, since $V_{sb}(\rho^{\alpha_1}) \subset V_b(\rho^{\alpha_1})$ for all integers $s > 1$.

Lemma 2.2.2. *The non-resonance domain $U(\rho^{\alpha_1}, p)$ has asymptotically full measure on R^d, i.e.,*

$$\frac{\mu(U(\rho^{\alpha_1}, p) \bigcap B(\rho))}{\mu(B(\rho))} \to 1 \quad \text{as} \rho \to \infty,$$

where $B(\rho) = \{x \in R^d : |x| \leq \rho\}$.

Proof. It is clear that $V_b(\rho^{\alpha_1}) \bigcap B(\rho)$ is the part of $B(\rho)$ which is contained between the two parallel hyperplanes Π_1 and Π_2. Since the distance between these hyperplanes is $\frac{\rho^{\alpha_1}}{|b|}$, we have

$$\mu(V_b(\rho^{\alpha_1}) \bigcap B(\rho)) = O(\rho^{d-1+\alpha_1}).$$

The number of the vectors γ in $\Gamma(p\rho^{\alpha})$ is $O(\rho^{d\alpha})$ and $\mu(B(\rho)) \sim \rho^d$, where $f(\rho) \sim g(\rho)$ means that there are positive independent on ρ constants c_1 and c_2 such that $c_1|g(\rho)| < |f(\rho)| < c_2|g(\rho)|$. Thus,

$$\begin{aligned} \mu(\bigcup_{b \in \Gamma(p\rho^{\alpha})} V_b(\rho^{\alpha_1}) \bigcap B(\rho)) &= O(\rho^{d-1+\alpha_1+d\alpha}) \\ &= \mu(B(\rho))O(\rho^{(d\alpha+\alpha_1-1)}). \end{aligned} \tag{2.25}$$

Using that, $R^d = U(\rho^{\alpha_1}, p) \bigcup E_1$, and

$$R^d \bigcap B(\rho) = (U(\rho^{\alpha_1}, p) \bigcap B(\rho)) \bigcup (E_1 \bigcap B(\rho))$$

we have,

$$\mu(B(\rho)) = \mu(U(\rho^{\alpha_1}, p) \bigcap B(\rho)) + \mu(E_1 \bigcap B(\rho)),$$

which together with (2.25) imply

$$\mu(U(\rho^{\alpha_1}, p) \bigcap B(\rho)) = \mu(B(\rho))(1 - O(\rho^{\alpha_1+d\alpha-1})). \tag{2.26}$$

Thus, from (2.26) the result follows, since $\alpha_1 + d\alpha < 1$. That is, the domain $U(\rho^{\alpha_1}, p)$ has asymptotically full measure on R^d. □

Note that Lemma2.2.2 implies that the number of non-resonance eigenvalues is essentially greater than the number of resonance eigenvalues. Namely, if

$N_n(\rho)$ denotes the number of $\gamma \in U(\rho^{\alpha_1}, p) \bigcap (B(2\rho) \backslash B(\rho))$ and $N_r(\rho)$ denotes the number of $\gamma \in \bigcup_{b \in \Gamma(p\rho^{\alpha})} V_b(\rho^{\alpha_1}) \bigcap (B(2\rho) \backslash B(\rho))$, then since $\alpha_1 + d\alpha < 1$,

$$\frac{N_r(\rho)}{N_n(\rho)} = O(\rho^{d\alpha + \alpha_1 - 1}) = o(1). \tag{2.27}$$

Remark 2.2.3. Consider

$$E \equiv (\bigcap_{i=1}^{k} V_{\gamma_i}(\rho^{\alpha_k})) \bigcap B(\rho).$$

We turn the coordinate axis so that $span\{\gamma_1, \gamma_2, ..., \gamma_k\}$ coincides with the span of the vectors $e_1 = (1, 0, 0, ..., 0), e_2 = (0, 1, 0, ..., 0), e_k = (0, 0, ..., 1)$. Clearly, $\gamma_s = \sum_{i=1}^{k} \gamma_{s,i} e_i$ for $s = 1, 2, ..., k$. Therefore, if $x \in \bigcap_{i=1}^{k} V_{\gamma_i}(\rho^{\alpha_k})$, then by definition, (see Remark 2.2.1) we have

$$\langle \gamma_s, x \rangle = \langle \sum_{i=1}^{k} \gamma_{s,i} e_i, x \rangle = \sum_{i=1}^{k} \gamma_{s,i} x_i = O(\rho^{\alpha_k}),$$

for $s = 1, 2, ..., k$ and $x = (x_1, x_2, ..., x_d)$.

Writing the last estimation for all $s = 1, 2, ..., k$, we get the system of linear algebraic equations

$$\begin{gathered} \gamma_{1,1}x_1 + \gamma_{1,2}x_2 + ... + \gamma_{1,k}x_k = O(\rho^{\alpha_k}) \\ \gamma_{2,1}x_1 + \gamma_{2,2}x_2 + ... + \gamma_{2,k}x_k = O(\rho^{\alpha_k}) \\ \vdots \\ \gamma_{k,1}x_1 + \gamma_{k,2}x_2 + ... + \gamma_{k,k}x_k = O(\rho^{\alpha_k}). \end{gathered}$$

By Crammer's rule, we obtain

$$x_n = \frac{\det(b_{j,i}^n)}{\det(\gamma_{j,i})}, \ n = 1, 2, ..., k, \tag{2.28}$$

where $\gamma_j = (\gamma_{j,1}, \gamma_{j,2}, ..., \gamma_{j,k}, 0, 0, ..., 0), b_{j,i}^n = \gamma_{j,i}$ for $n \neq i$ and $b_{j,i}^n = O(\rho^{\alpha_k})$ for $n = i$. Taking into account that the determinant $(\gamma_{j,i})$ of the system is the volume of the parallelepiped

$$\{\sum_{i=1}^{k} b_i \gamma_i : b_i \in [0, 1], i = 1, 2, ..., k\},$$

i.e, greater than some constant and using that $|\gamma_{j,i}| < p\rho^{\alpha}$, since $\gamma_j \in \Gamma(p\rho^{\alpha})$, we obtain the estimation for the components x_n of $x \in \bigcap_{i=1}^{k} V_{\gamma_i}(\rho^{\alpha_k})$,

$$x_n = O(\rho^{\alpha_k + (k-1)\alpha}), \ \forall n = 1, 2, ..., k. \tag{2.29}$$

Therefore, for the projection v of $x \in \bigcap_{i=1}^{k} V_{\gamma_i}(\rho^{\alpha_k})$ onto the $span\{\gamma_1, \gamma_2, ..., \gamma_k\}$, we obtain

$$|v| = O(\rho^{\alpha_k+(k-1)\alpha}). \tag{2.30}$$

Lemma 2.2.4. *The domain $V_b(\rho^{\alpha_1}) \setminus E_2$ has asymptotically full measure on $V_b(\rho^{\alpha_1})$, that is*

$$\frac{\mu((V_b(\rho^{\alpha_1}) \setminus E_2) \bigcap B(\rho))}{\mu(V_b(\rho^{\alpha_1}) \bigcap B(\rho))} \to 1,$$

as $\rho \to \infty$.

Proof. Let $k = 2$, then for a fixed $b \in \Gamma(p\rho^{\alpha})$, we consider

$$E_2(\rho^{\alpha_2}, p) = \bigcup_{\gamma_2 \in \Gamma(p\rho^{\alpha})} (V_b(\rho^{\alpha_2}) \bigcap V_{\gamma_2}(\rho^{\alpha_2}))$$

and

$$E = ((V_b(\rho^{\alpha_2}) \bigcap V_{\gamma_2}(\rho^{\alpha_2})) \bigcap B(\rho)).$$

If $x \in E$, then by (2.29), there exists an index $i > 2$ such that $|x_i| > \frac{\rho}{d}$. Define, $E(+i) = \{x \in E : x_i > \frac{\rho}{d}\}$ and $E(-i) = \{x \in E : x_i < -\frac{\rho}{d}\}$.
It is clear that,

$$E = (\bigcup_i E(+i)) \bigcup (\bigcup_i E(-i)).$$

Using the formulas

$$\mu(E(+i)) = \int_{Pr(E(+i))} \frac{|x|}{x_i} dx_1 dx_2 ... dx_{i-1} dx_{i+1} ... dx_d,$$

where $Pr(E(+i))$ is the projection of $E(+i)$ onto the hyperplane $x_i = 0$, and

$$\mu(Pr(E(+i))) = O(\rho^{2(\alpha_2+\alpha)+(d-3)}),$$

which follows from (2.29), doing the same estimation for $\mu(Pr(E(-i)))$, we get

$$\mu(E) = O(\rho^{2(\alpha_2+\alpha)+(d-3)}).$$

Then, using that the number of elements in $\Gamma(p\rho^{\alpha})$ is $O(\rho^{d\alpha})$, we have

$$\mu(E_2 \bigcap B(\rho)) = \mu(\bigcup_{\gamma_2 \in \Gamma(p\rho^{\alpha})} (V_b(\rho^{\alpha_2}) \bigcap V_{\gamma_2}(\rho^{\alpha_2})) \bigcap B(\rho))$$
$$= O(\rho^{2(\alpha_2+\alpha)+(d-3)+d\alpha}). \tag{2.31}$$

Also, we have

$$\mu(V_b(\rho^{\alpha_1})\bigcap B(\rho)) \sim \frac{\rho^{\alpha_1+d-2}}{|b|}. \tag{2.32}$$

Therefore, by (2.31) and (2.32), we get

$$\frac{\mu((V_b(\rho^{\alpha_1})\setminus E_2)\bigcap B(\rho))}{\mu(V_b(\rho^{\alpha_1})\bigcap B(\rho))} = \frac{\mu((V_b(\rho^{\alpha_1})\bigcap B(\rho)) - \mu(E_2\bigcap B(\rho)))}{\mu(V_b(\rho^{\alpha_1})\bigcap B(\rho))}$$
$$= 1 - \frac{\mu(E_2\bigcap B(\rho)))}{\mu(V_b(\rho^{\alpha_1})\bigcap B(\rho))} = 1 - \frac{O(\rho^{2(\alpha_2+\alpha)+(d-3)+d\alpha})}{O(\rho^{\alpha_1+d-2)})} \to 1$$

as $\rho \longrightarrow \infty$, since

$$2\alpha_2 - \alpha_1 + (d+3)\alpha < 1 \tag{2.33}$$

and

$$\alpha_2 > 2\alpha_1, \tag{2.34}$$

hold for $\alpha < \frac{1}{d+20}$. □

Notice that $\gamma \notin V_{e_k}(\rho^{\alpha_1})$ by definition means $||\gamma|^2 - |\gamma + e_k|^2| > \rho^{\alpha_1}$, which implies that

$$|\gamma_k| > \frac{1}{3}\rho^{\alpha_1}, \quad \forall k = 1, 2, ..., d. \tag{2.35}$$

Lemma 2.2.5. *Let* $v_\gamma(x)$ *be an eigenfunction of* $L_N(0)$ *. Then,*

$$v_a(x)v_\gamma(x) = \frac{1}{|A_a|}\sum_{\gamma'\in A_a} v_{\gamma+\gamma'},$$

for all $\gamma \in \frac{\Gamma}{2}$, $\gamma \notin V_{e_k}(\rho^{\alpha_1})$, $\forall k = 1, 2, ..., d$ *and for all* $a \in \Gamma(\rho^\alpha)$.

Proof. By (2.22), we have

$$v_a(x)v_\gamma(x) = \frac{1}{|A_a|}\frac{1}{|A_\gamma|}(\sum_{\gamma'\in A_a} e^{i\langle\gamma',x\rangle})(\sum_{\alpha\in A_\gamma} e^{i\langle\alpha,x\rangle})$$
$$= \frac{1}{|A_a|}\frac{1}{|A_\gamma|}\sum_{\gamma'\in A_a}\sum_{\alpha\in A_{\gamma+\gamma'}} e^{i\langle\alpha,x\rangle} = \frac{1}{|A_a|}\sum_{\gamma'\in A_a} v_{\gamma+\gamma'}, \tag{2.36}$$

since $|A_\gamma| = |A_{\gamma+\gamma'}| = 2^d$, because all components of γ and $\gamma+\gamma'$ are different from zero. Indeed, $\gamma \notin V_{e_k}(\rho^{\alpha_1})$ implies $|\gamma_k| > \frac{1}{3}\rho^{\alpha_1}$, $\forall k = 1, 2, ..., d$. Also, if $a \in \Gamma(\rho^\alpha)$, then $\gamma' \in \Gamma(\rho^\alpha)$ and thus $|\gamma'_k| < \rho^\alpha$, $\forall k = 1, 2, ..., d$. Therefore, $|\gamma_k + \gamma'_k| \geq ||\gamma_k| - |\gamma'_k|| > \frac{1}{4}\rho^{3\alpha}$. □

2.3 On the Potential of the Neumann Problem

It is known that the system $\{v_{\gamma'} = \frac{1}{|A_{\gamma'}|}\sum_{\alpha\in A_{\gamma'}} e^{i\langle\alpha,x\rangle}\}_{\gamma'\in\frac{\Gamma}{2}}$ is complete, i.e. form an orthogonal basis in $L_2(F)$. Hence, the potential $q(x)$ in the operator $L_N(q(x))$ can be written in the following form;

$$q(x) = \sum_{\gamma'\in\frac{\Gamma}{2}} q_{\gamma'} v_{\gamma'}, \tag{2.37}$$

where $q_{\gamma'} = (q(x), v_{\gamma'}(x))$ are the Fourier coefficients of the potential $q(x)$ with respect to the basis $\{v_{\gamma'}\}_{\gamma'\in\frac{\Gamma}{2}}$, $(.,.)$ is the inner product in $L_2(F)$. Without loss of generality we take $q_0 = 0$, otherwise we replace $q(x)$ by $q(x) - \frac{q_0}{\mu(F)}$.

In this work, we assume that the Fourier coefficients of the potential $q(x)$ satisfies the condition

$$\sum_{\gamma\in\frac{\Gamma}{2}} |q_{\gamma'}|^2(1 + |\gamma'|^{2l}) < \infty, \tag{2.38}$$

where $l > \frac{(d+20)(d-1)}{2} + d + 3$.

Therefore, we can write

$$q(x) = \sum_{\gamma'\in\Gamma(\rho^\alpha)} q_{\gamma'} v_{\gamma'} + O(\rho^{-p\alpha}), \tag{2.39}$$

where $p = l - d$, $\Gamma(\rho^\alpha) = \{\gamma' \in \frac{\Gamma}{2} : 0 < |\gamma'| < \rho^\alpha\}, \alpha < \frac{1}{(d+20)}$ and ρ is a large parameter.

Indeed, by (2.38) we have (see section 1.1)

$$\left\| \sum_{|\gamma'|>\rho^\alpha} q_{\gamma'} v_{\gamma'} \right\| \leq (\sum_{|\gamma'|>\rho^\alpha} |q_{\gamma'}|^2)^{\frac{1}{2}} = (\sum_{|\gamma'|>\rho^\alpha} \frac{|q_{\gamma'}|^2|\gamma'|^{2l}}{|\gamma'|^{2l}})^{\frac{1}{2}}$$
$$\leq (\sum_{|\gamma'|>\rho^\alpha} |q_{\gamma'}|^2|\gamma'|^{2l})^{\frac{1}{2}}(\sum_{|\gamma'|>\rho^\alpha} \frac{1}{|\gamma'|^{2l}})^{\frac{1}{2}} = O(\rho^{-(l-d)\alpha}) = O(\rho^{-p\alpha}).$$

Also, we have

$$\sum_{\gamma'\in\frac{\Gamma}{2}} |q_{\gamma'}| = \sum_{\gamma'\in\frac{\Gamma}{2}} \frac{|q_{\gamma'}||\gamma'|^l}{|\gamma'|^l} < (\sum_{\gamma'\in\frac{\Gamma}{2}} |q_{\gamma'}|^2|\gamma'|^{2l})^{\frac{1}{2}}(\sum_{\gamma'\in\frac{\Gamma}{2}} |\frac{1}{|\gamma'|^{2l}})^{\frac{1}{2}} < \infty,$$

and thus, one can define

$$M = \sum_{\gamma' \in \frac{\Gamma}{2}} |q_{\gamma'}|. \tag{2.40}$$

Remark 2.3.1. Notice that, if $q(x)$ is sufficiently smooth ($q(x) \in W_2^l(F)$) and the support of grad$q(x) = (\frac{\partial q}{\partial x_1}, \frac{\partial q}{\partial x_2}, ..., \frac{\partial q}{\partial x_d})$ is contained in the interior of the domain F, then $q(x)$ satisfies the condition (2.38).

There is also another class of functions $q(x)$, such that $q(x) \in W_2^l(F)$, and

$$q(x) = \sum_{\gamma' \in \Gamma} q_{\gamma'} v_{\gamma'},$$

which is periodic with respect to Ω and thus also satisfies the condition (2.38).

Lemma 2.3.2.

$$\sum_{\gamma' \in \Gamma(\rho^\alpha)} q_{\gamma'} v_{\gamma'}(x) v_\gamma(x) = \sum_{\gamma' \in \Gamma(\rho^\alpha)} q_{\gamma'} v_{\gamma+\gamma'}(x), \tag{2.41}$$

for all $\gamma \in \frac{\Gamma}{2}$, $\gamma \notin V_{e_k}(\rho^{\alpha_1})$.

Proof. From Lemma 2.2.5, for all $\gamma \in \frac{\Gamma}{2}$, $\gamma \notin V_{e_k}(\rho^{\alpha_1})$ and for all $a \in \Gamma(\rho^\alpha)$, we have

$$v_a(x) v_\gamma(x) = \frac{1}{|A_a|} \sum_{\gamma' \in A_a} v_{\gamma+\gamma'}.$$

The set A_a consists of the vectors $a^1, a^2, ..., a^s$, where $s = |A_a|$ and since

$$A_{a^1} = A_{a^2} = ... = A_{a^s} = A_a, \quad v_{a^1} = v_{a^2} = ... = v_{a^s} = v_a, \tag{2.42}$$

in the above expression, the vector a can be replaced by $a^1, a^2, ..., a^s$, that is

$$v_{a^k}(x) v_\gamma(x) = \frac{1}{|A_{a^k}|} \sum_{\gamma' \in A_{a^k}} v_{\gamma+\gamma'}(x) = \frac{1}{|A_a|} \sum_{\gamma' \in A_a} v_{\gamma+\gamma'}(x),$$

for all $k = 1, 2, ..., s$. Therefore, summing the obtained s equalities, we get

$$\sum_{k=1}^{s} v_{a^k}(x) v_\gamma(x) = \frac{s}{|A_a|} \sum_{\gamma' \in A_a} v_{\gamma+\gamma'}(x) = \sum_{\gamma' \in A_a} v_{\gamma+\gamma'}(x),$$

or equivalently

$$\sum_{\gamma' \in A_a} v_{\gamma'}(x) v_\gamma(x) = \sum_{\gamma' \in A_a} v_{\gamma+\gamma'}(x).$$

Thus,

$$\sum_{\gamma' \in A_a} q_{\gamma'} v_{\gamma'}(x) v_\gamma(x) = \sum_{\gamma' \in A_a} q_{\gamma'} v_{\gamma+\gamma'}(x), \tag{2.43}$$

since $q_{\gamma'} = q_a$ for all $\gamma' \in A_a$ (see (2.42)).

Clearly, there exist vectors $a_1, a_2, ..., a_n \in \frac{\Gamma}{2}$ such that

$$\Gamma(\rho^\alpha) = \bigcup_{j=1}^{n} A_{a_j}, \quad A_{a_j} \cap A_{a_k} = \emptyset, \quad \forall j \neq k. \tag{2.44}$$

In (2.43), replacing a by a_j for $j = 1, 2, ..., n$, summing all obtained n equality and using (2.44), we have

$$\sum_{j=1}^{n} \sum_{\gamma' \in A_{a_j}} q_{\gamma'} v_{\gamma'}(x) v_\gamma(x) = \sum_{j=1}^{n} \sum_{\gamma' \in A_{a_j}} q_{\gamma'} v_{\gamma+\gamma'}(x),$$

which is equivalent to

$$\sum_{\gamma' \in \Gamma(\rho^\alpha)} q_{\gamma'} v_{\gamma'}(x) v_\gamma(x) = \sum_{\gamma' \in \Gamma(\rho^\alpha)} q_{\gamma'} v_{\gamma+\gamma'}(x).$$

□

2.4 Asymptotic Formulas For the Eigenvalues In the Non-Resonance Domain

In this section , we consider the non-resonance eigenvalue $|\gamma|^2$ of the unperturbed operator $L_N(0)$, i.e., $\gamma \in U(\rho^{\alpha_1}, p)$ and prove that there exists an eigenvalue Υ_N of the operator $L_N(q)$ which is close to the non-resonance eigenvalue $|\gamma|^2$ of the operator $L_N(0)$.

Let the sets F, Ω and Γ be as defined in (2.20), (2.21), and (2.19) respectively. To obtain the asymptotic formulas for the eigenvalues of the operator

$L_N(q(x))$ in the non-resonance domain, we use the following formula for the operators $L_N(q(x))$ and $L_N(0)$, which we call the binding formula for these operators, that is;

$$(\Upsilon_N - |\gamma|^2)(\Phi_N, v_\gamma) = (\Phi_N, q(x)v_\gamma), \tag{2.45}$$

where $(.,.)$ is the inner product in $L_2(F)$.

The binding formula (2.45) can be obtained as follows: By multiplying both sides of the equation

$$-\Delta\Phi_N(x) + q(x)\Phi_N(x) = \Upsilon_N\Phi_N(x) \tag{2.46}$$

by $v_\gamma(x)$, we get

$$(-\Delta\Phi_N(x) + q(x)\Phi_N(x), v_\gamma(x)) = (\Upsilon_N\Phi_N(x), v_\gamma(x)),$$

using the properties of inner product, we obtain

$$(-\Delta\Phi_N(x), v_\gamma(x)) + (q(x)\Phi_N(x), v_\gamma(x)) = \Upsilon_N(\Phi_N(x), v_\gamma(x)),$$

since $L_N(0) = -\Delta$ is a self-adjoint operator, $v_\gamma(x)$ is the eigenfunction of the operator $L_N(0)$ corresponding to the eigenvalue $|\gamma|^2$ and $q(x)$ is a real valued function, we have

$$(\Phi_N(x), -\Delta v_\gamma(x)) + (\Phi_N(x), q(x)v_\gamma(x)) = \Upsilon_N(\Phi_N(x), v_\gamma(x)),$$

$$|\gamma|^2(\Phi_N(x), v_\gamma(x)) + (\Phi_N(x), q(x)v_\gamma(x)) = \Upsilon_N(\Phi_N(x), v_\gamma(x)),$$

which gives (2.45).

In order to start iteration, we substitute the decomposition (2.39) of $q(x)$ into the formula (2.45),i.e.,

$$(\Upsilon_N - |\gamma|^2)(\Phi_N, v_\gamma) = (\Phi_N, \sum_{\gamma_1 \in \Gamma(\rho^\alpha)} q_{\gamma_1} v_{\gamma_1} v_\gamma) + O(\rho^{-p\alpha}). \tag{2.47}$$

Since $\gamma \in U(\rho^{\alpha_1}, p)$ implies $\gamma \notin V_{e_k}(\rho^{\alpha_1})$, we can use Lemma 2.3.2 to obtain

$$\begin{aligned}(\Upsilon_N - |\gamma|^2)(\Phi_N, v_\gamma) &= (\Phi_N, \sum_{\gamma_1 \in \Gamma(\rho^\alpha)} q_{\gamma_1} v_{\gamma+\gamma_1}) + O(\rho^{-p\alpha}) \\ &= \sum_{\gamma_1 \in \Gamma(\rho^\alpha)} q_{\gamma_1} (\Phi_N, v_{\gamma+\gamma_1}) + O(\rho^{-p\alpha}). \end{aligned} \tag{2.48}$$

Similarly, the expressions (2.45) and (2.39) also imply that

$$(\Phi_N, v_{\gamma'}) = \frac{(\Phi_N, q(x)v_{\gamma'})}{\Upsilon_N - |\gamma'|^2} = \sum_{\gamma_1 \in \Gamma(\rho^\alpha)} q_{\gamma_1} \frac{(\Phi_N, v_{\gamma'+\gamma_1})}{\Upsilon_N - |\gamma'|^2} + O(\rho^{-p\alpha}), \tag{2.49}$$

for every vector $\gamma' \in \frac{\Gamma}{2}$, satisfying the condition

$$|\Upsilon_N - |\gamma'|^2| > \frac{1}{2}\rho^{\alpha_1}, \tag{2.50}$$

which is called the iterability condition.

Lemma 2.4.1. *Let $\gamma \in U(\rho^{\alpha_1}, p)$, i.e. $|\gamma|^2$ be a non-resonance eigenvalue of $L_N(0)$ and Υ_N be the eigenvalue of $L_N(q(x))$ satisfying the inequality*

$$|\Upsilon_N - |\gamma|^2| < \frac{1}{2}\rho^{\alpha_1}.$$

Then,

$$|\Upsilon_N - |\gamma + b|^2| > \frac{1}{2}\rho^{\alpha_1},$$

for all $b \in \Gamma(p\rho^\alpha)$.

Proof. If $\gamma \in U(\rho^{\alpha_1}, p)$, then for all $b \in \Gamma(p\rho^\alpha)$ we have the following inequality

$$||\gamma|^2 - |\gamma + b|^2| \geq \rho^{\alpha_1}$$

which, together with the inequality $|\Upsilon_N - |\gamma|^2| < \frac{1}{2}\rho^{\alpha_1}$, implies

$$\begin{aligned} |\Upsilon_N - |\gamma + b|^2| &= |\Upsilon_N - |\gamma + b|^2 - |\gamma|^2 + |\gamma|^2| \\ \geq |||\gamma|^2 - |\gamma + b|^2| - |\Upsilon_N - |\gamma|^2|| &\geq |\rho^{\alpha_1} - \frac{1}{2}\rho^{\alpha_1}|, \end{aligned}$$

the result follows. □

We say that $|\gamma|^2$ is of the order of ρ^2 if $|\gamma|^2 \sim \rho^2$.

Lemma 2.4.2. *Let $|\gamma|^2$ be an eigenvalue of the operator $L_N(0)$ of the order of ρ^2. Then, there is an integer N, such that $|\Upsilon_N - |\gamma|^2| < 2M$, and*

$$|(\Phi_N, v_\gamma)| > c_3 \rho^{\frac{-(d-1)}{2}}, \tag{2.51}$$

where M is the number defined in (2.40).

Proof. The set $\{\Phi_N\}_{N=1}^{\infty}$ of eigenfunctions of the self-adjoint operator $L_N(q(x))$ is an orthonormal basis for $L_2(F)$. So, we have

$$v_\gamma(x) = \sum_{N=1}^{\infty}(\Phi_N(x), v_\gamma(x))\Phi_N(x),$$

and the Parseval's identity (without loss of generality we assume $\|v_\gamma\| = 1$):

$$1 = \|v_\gamma\|^2 = \sum_{N=1}^{\infty}|(\Phi_N, v_\gamma)|^2 = \sum_{N:|\Upsilon_N - |\gamma|^2|>2M}|(\Phi_N, v_\gamma)|^2 + \sum_{N:|\Upsilon_N - |\gamma|^2|\leq 2M}|(\Phi_N, v_\gamma)|^2.$$

Using the binding formula (2.45), Bessel's inequality and (2.40) we have,

$$\sum_{N:|\Upsilon_N-|\gamma|^2|>2M}|(\Phi_N, v_\gamma)|^2 = \sum_{N:|\Upsilon_N-|\gamma|^2|>2M}\frac{|(\Phi_N, q(x)v_\gamma)|^2}{|\Upsilon_N - |\gamma|^2|^2}$$
$$\leq \frac{1}{4M^2}\sum_{N:|\Upsilon_N-|\gamma|^2|>2M}|(\Phi_N, q(x)v_\gamma)|^2 \leq \frac{1}{4M^2}\|q(x)\|^2\|v_\gamma\|^2 < \frac{1}{4}.$$

Therefore, by Parseval's identity,

$$\sum_{N:|\Upsilon_N-|\gamma|^2|\leq 2M}|(\Phi_N, v_\gamma)|^2 \geq \frac{3}{4}.$$

On the other hand, it is well known that if $a \sim \rho$ then the number of $\gamma \in \frac{\Gamma}{2}$ satisfying $||\gamma|^2 - a^2| < 1$ is less than $c_4\rho^{d-1}$. Therefore the number of eigenvalues of $L_N(0)$ lying in $(a^2 - 1, a^2 + 1)$ is less than $c_5\rho^{d-1}$. And since, by the general perturbation theory, the N-th eigenvalue of $L_N(q(x))$ lies in M-neighborhood of the N-th eigenvalue of $L_N(0)$, the number of the eigenvalues Υ_N of the operator $L_N(q(x))$ in the interval $I = [|\gamma|^2 - 2M, |\gamma|^2 + 2M]$ is less than $c_6\rho^{d-1}$. By this fact and the above inequality , there exists $N \in I$ such that

$$\frac{3}{4} < \sum_{N:|\Upsilon_N-|\gamma|^2|\leq 2M}|(\Phi_N, v_\gamma)|^2 < c_3\rho^{d-1}|(\Phi_N, v_\gamma)|^2.$$

That is,

$$|(\Phi_N(x), v_\gamma(x))| > c_3\rho^{-\frac{(d-1)}{2}}.$$

Hence, the lemma is proved. □

Theorem 2.4.3. *For every non-resonance eigenvalue,* $|\gamma|^2 \sim \rho^2$*, of the operator* $L_N(0)$*, there exists an eigenvalue* Υ_N *of the operator* $L_N(q(x))$ *satisfying the following formula:*

$$\Upsilon_N = |\gamma|^2 + O(\rho^{-\alpha_1}). \tag{2.52}$$

Proof. By Lemma 2.4.2, there is an index N, such that

$$|\Upsilon_N - |\gamma|^2| \leq 2M < \frac{1}{2}\rho^{\alpha_1}, \quad |(\Phi_N(x), v_\gamma(x))| > c_3\rho^{-\frac{(d-1)}{2}}. \tag{2.53}$$

We prove that this eigenvalue satisfies the formula (2.52). By the first inequality in (2.53) and Lemma2.4.1, the vector $\gamma+\gamma_1$, for $\gamma_1 \in \Gamma(\rho^\alpha)$, satisfies the iterability condition (2.50). Therefore in (2.49), one can replace γ' by $\gamma+\gamma_1$ and obtain

$$(\Phi_N, v_{\gamma+\gamma_1}) = \sum_{\gamma_2 \in \Gamma(\rho^\alpha)} q_{\gamma_2} \frac{(\Phi_N, v_{\gamma+\gamma_1+\gamma_2})}{\Upsilon_N - |\gamma+\gamma_1|^2} + O(\rho^{-p\alpha}).$$

Substituting this into the right-hand side of the equation (2.48), we get

$$\begin{aligned}(\Upsilon_N - |\gamma|^2)(\Phi_N, v_\gamma) &= \sum_{\gamma_1 \in \Gamma(\rho^\alpha)} q_{\gamma_1}(\Phi_N, v_{\gamma+\gamma_1}) + O(\rho^{-p\alpha}) \\ &= \sum_{\gamma_1,\gamma_2 \in \Gamma(\rho^\alpha)} q_{\gamma_1} q_{\gamma_2} \frac{(\Phi_N, v_{\gamma+\gamma_1+\gamma_2})}{\Upsilon_N - |\gamma+\gamma_1|^2} + O(\rho^{-p\alpha}).\end{aligned}$$

Isolating the terms with coefficient (Φ_N, v_γ) in the last expression, we obtain

$$\begin{aligned}&(\Upsilon_N - |\gamma|^2)(\Phi_N, v_\gamma) \\ &= \sum_{\substack{\gamma_1,\gamma_2 \in \Gamma(\rho^\alpha) \\ \gamma_1+\gamma_2=0}} q_{\gamma_1} q_{\gamma_2} \frac{(\Phi_N, v_\gamma)}{\Upsilon_N - |\gamma+\gamma_1|^2} \\ &+ \sum_{\substack{\gamma_1,\gamma_2 \in \Gamma(\rho^\alpha) \\ \gamma_1+\gamma_2\neq 0}} q_{\gamma_1} q_{\gamma_2} \frac{(\Phi_N, v_{\gamma+\gamma_1+\gamma_2})}{\Upsilon_N - |\gamma+\gamma_1|^2} + O(\rho^{-p\alpha}).\end{aligned}$$

Since $\gamma_1+\gamma_2 \in \Gamma(2\rho^\alpha)$, by (2.53) and Lemma2.4.1, the vector $\gamma+\gamma_1+\gamma_2$ satisfy the iterability condition, i.e.,

$$|\Upsilon_N - |\gamma+\gamma_1+\gamma_2|^2| > \frac{1}{2}\rho^{\alpha_1}.$$

Thus, again in the formula (2.49) replacing γ' by $\gamma+\gamma_1+\gamma_2$ and putting the obtained equation into the second sum of the last expression, we obtain

$$(\Upsilon_N - |\gamma|^2)(\Phi_N, v_\gamma)$$
$$= \sum_{\substack{\gamma_1,\gamma_2\in\Gamma(\rho^\alpha)\\ \gamma_1+\gamma_2=0}} q_{\gamma_1}q_{\gamma_2}\frac{(\Phi_N, v_\gamma)}{\Upsilon_N - |\gamma+\gamma_1|^2}$$
$$+ \sum_{\gamma_1,\gamma_2,\gamma_3\in\Gamma(\rho^\alpha)} q_{\gamma_1}q_{\gamma_2}q_{\gamma_3}\frac{(\Phi_N, v_{\gamma+\gamma_1+\gamma_2+\gamma_3})}{(\Upsilon_N - |\gamma+\gamma_1|^2)(\Upsilon_N - |\gamma+\gamma_1+\gamma_2|^2)} + O(\rho^{-p\alpha}).$$

Again, isolating the terms with coefficient (Φ_N, v_γ) in the above expression, we have

$$(\Upsilon_N - |\gamma|^2)(\Phi_N, v_\gamma)$$
$$= \sum_{\substack{\gamma_1,\gamma_2\in\Gamma(\rho^\alpha)\\ \gamma_1+\gamma_2=0}} q_{\gamma_1}q_{\gamma_2}\frac{(\Phi_N, v_\gamma)}{\Upsilon_N - |\gamma+\gamma_1|^2}$$
$$+ \sum_{\substack{\gamma_1,\gamma_2,\gamma_3\in\Gamma(\rho^\alpha)\\ \gamma_1+\gamma_2+\gamma_3=0}} q_{\gamma_1}q_{\gamma_2}q_{\gamma_3}\frac{(\Phi_N, v_\gamma)}{(\Upsilon_N - |\gamma+\gamma_1|^2)(\Upsilon_N - |\gamma+\gamma_1+\gamma_2|^2)}$$
$$+ \sum_{\substack{\gamma_1,\gamma_2,\gamma_3\in\Gamma(\rho^\alpha)\\ \gamma_1+\gamma_2+\gamma_3\neq 0}} q_{\gamma_1}q_{\gamma_2}q_{\gamma_3}\frac{(\Phi_N, v_{\gamma+\gamma_1+\gamma_2+\gamma_3})}{(\Upsilon_N - |\gamma+\gamma_1|^2)(\Upsilon_N - |\gamma+\gamma_1+\gamma_2|^2)} + O(\rho^{-p\alpha}).$$

By the same method, repeating the iteration $p_1 = [\frac{p+1}{2}]$ times and isolating each time the terms with multiplicand (Φ_N, v_γ), we get

$$(\Upsilon_N - |\gamma|^2)(\Phi_N, v_\gamma) = (\sum_{i=1}^{p_1} S_i)(\Phi_N, v_\gamma) + C_{p_1} + O(\rho^{-p\alpha}), \tag{2.54}$$

where

$$S_i(\Upsilon_N) = \sum_{\gamma_1,\dots\gamma_{i+1}\in\Gamma(\rho^\alpha)} \frac{q_{\gamma_1}\dots q_{\gamma_{i+1}}}{(\Upsilon_N - |\gamma+\gamma_1|^2)\dots(\Upsilon_N - |\gamma+\gamma_1+\dots+\gamma_i|^2)} \tag{2.55}$$

$$C_{p_1} = \sum_{\substack{\gamma_1,\dots\gamma_{p+1}\in\Gamma(\rho^\alpha)\\ \gamma_1+\gamma_2+\dots+\gamma_{p_1+1}\neq 0}} \frac{q_{\gamma_1}\dots q_{\gamma_{p_1+1}}(\Phi_N, v_{\gamma+\gamma_1+\dots+\gamma_{p_1+1}})}{(\Upsilon_N - |\gamma+\gamma_1|^2)\dots(\Upsilon_N - |\gamma+\gamma_1+\dots+\gamma_{p_1}|^2)}$$

$\gamma_i \in \Gamma(\rho^\alpha)$ and $|\gamma_1+\gamma_2+\dots+\gamma_i| < p_1\rho^\alpha$, for all $i = 1, 2, \dots, p_1$, Therefore, using Lemma 2.4.1 and the estimation (2.40), we have

$$|S_i(\Upsilon_N)| = \sum_{\gamma_1,\dots\gamma_{i+1}\in\Gamma(\rho^\alpha)} \frac{|q_{\gamma_1}|\dots|q_{\gamma_{i+1}}|}{|\Upsilon_N - |\gamma+\gamma_1|^2|\dots|\Upsilon_N - |\gamma+\gamma_1+\dots+\gamma_i|^2|}$$
$$\leq \frac{M^{i+1}}{(\frac{1}{2}\rho^{\alpha_1})^i}, \tag{2.56}$$

that is,

$$S_i(\Upsilon_N) = O(\rho^{-i\alpha_1}) \tag{2.57}$$

for $i = 1, 2, ..., p_1$, which implies

$$\sum_{i=1}^{p_1} S_i(\Upsilon_N) = O(\rho^{-\alpha_1}). \tag{2.58}$$

Similarly,

$$|C_{p_1}| = \sum_{\gamma_1,...\gamma_{p+1}\in\Gamma(\rho^\alpha)} \frac{|q_{\gamma_1}|...|q_{\gamma_{p_1+1}}||(\Phi_N, v_{\gamma+\gamma_1+...+\gamma_{p_1+1}})|}{|\Upsilon_N - |\gamma+\gamma_1|^2|...|\Upsilon_N - |\gamma+\gamma_1+...+\gamma_{p_1}|^2|}$$
$$\leq \frac{M^{p_1+1}}{(\frac{1}{2})^{p_1}\rho^{p_1\alpha_1}}$$

and thus,

$$C_{p_1} = O(\rho^{-p_1\alpha_1}). \tag{2.59}$$

Note that to obtain(2.58), we have only used the condition $\Upsilon_N \in I$, hence we may write

$$\sum_{i=1}^{p_1} S_i(a) = O(\rho^{-\alpha_1}) \ , \qquad \forall a \in I. \tag{2.60}$$

If we substitute (2.58) and (2.59) into (2.54), we get

$$(\Upsilon_N - |\gamma|^2)(\Phi_N, v_\gamma(x))$$
$$= O(\rho^{-\alpha_1})(\Phi_N, v_\gamma(x)) + O(\rho^{-p_1\alpha_1}) + O(\rho^{-p\alpha}). \tag{2.61}$$

Dividing both sides of the equation (2.61) by $(\Phi_N, v_\gamma(x))$, using (2.51) and using that $p_1 = [\frac{p+1}{2}]$, $p_1 \geq \frac{p}{2}$, $\alpha_1 = 3\alpha > 2\alpha$, $p_1\alpha_1 > p\alpha$ we obtain

$$|\Upsilon_N - |\gamma|^2| = O(\rho^{-\alpha_1}) + \frac{O(\rho^{-p\alpha})}{O(\rho^{-\frac{(d-1)}{2}})}.$$

Choosing p such that $p > \frac{d-1}{2\alpha} + 1$, the result follows. □

Let us define $c = [\frac{d-1}{2\alpha}] + 1$, where $[\frac{d-1}{2\alpha}]$ is the integer part of $\frac{d-1}{2\alpha}$ then in Lemma2.4.2, instead of (2.51), we have

$$|(\Phi_N, v_\gamma)| > \rho^{(-\frac{d-1}{2})} > \rho^{-c\alpha}. \tag{2.62}$$

Theorem 2.4.4. *Let $\gamma \in U(\rho^{\alpha_1}, p)$, $|\gamma| \sim \rho$. Then, there is an eigenvalue Υ_N of the operator $L_N(q(x))$ satisfying the formulas*

$$\Upsilon_N = |\gamma|^2 + F_{k-1} + O(\rho^{-k\alpha_1}), \tag{2.63}$$

for all $k = 1, 2, ...p - c$, where

$$F_0 = 0, F_1 = \sum_{\gamma_1 \in \Gamma(\rho^\alpha)} \frac{|q_{\gamma_1}|^2}{|\gamma|^2 - |\gamma - \gamma_1|^2},$$

$$F_j = \sum_{i=1}^{j} S_i(|\gamma|^2 + F_{j-1}),\ j = 2, 3, ..., p - c$$

Proof. The proof is done by mathematical induction on k.
For $k = 1$; by Theorem 2.4.3, Υ_N satisfies the equation

$$\Upsilon_N = |\gamma|^2 + F_0 + O(\rho^{-\alpha_1}),$$

where $F_0 = 0$.
For $k = j$; assume that

$$\Upsilon_N = |\gamma|^2 + F_{j-1} + O(\rho^{-j\alpha_1}). \tag{2.64}$$

Now, we prove that, for $k = j + 1$, (2.63) holds, i.e.,

$$\Upsilon_N = |\gamma|^2 + F_j + O(\rho^{-(j+1)\alpha_1}). \tag{2.65}$$

For this, we put the expression (2.64) into $S_i(\Upsilon_N)$ in formula (2.54)

$$(\Upsilon_N - |\gamma|^2)(\Phi_N, v_\gamma) = (\sum_{i=1}^{p_1} S_i(|\gamma|^2 + F_{j-1} + O(\rho^{-j\alpha_1})))(\Phi_N, v_\gamma) + C_{p_1} + O(\rho^{-p\alpha})$$

and divide both sides of the above equation by $(\Phi_N(x), v_\gamma)$. Using (2.62), we get

$$\Upsilon_N = |\gamma|^2 + \sum_{i=1}^{p_1} S_i(|\gamma|^2 + F_{j-1} + O(\rho^{-j\alpha_1})) + O(\rho^{-(p-c)\alpha}). \tag{2.66}$$

Adding and subtracting the term $\sum_{i=1}^{j} S_i(|\gamma|^2 + F_{j-1})$ in (2.66) , we have

$$\Upsilon_N = |\gamma|^2 + [\sum_{i=1}^{j} S_i(|\gamma|^2 + F_{j-1} + O(\rho^{-j\alpha_1})) - S_i(|\gamma|^2 + F_{j-1})]$$

$$+ \sum_{i=j+1}^{p_1} S_i(|\gamma|^2 + F_{j-1} + O(\rho^{-j\alpha_1})) + \sum_{i=1}^{j} S_i(|\gamma|^2 + F_{j-1}) + O(\rho^{-(p-n)\alpha}) \quad (2.67)$$

Notice that, (2.57) implies

$$\sum_{i=j+1}^{p_1} S_i(|\gamma|^2 + F_{j-1} + O(\rho^{-j\alpha_1})) = O(\rho^{-(j+1)\alpha_1}).$$

So, we need only to show that the expression in the square brackets in (2.67) is equal to $O(\rho^{-(j+1)\alpha_1})$. First we prove by induction that

$$F_j = O(\rho^{-\alpha}), \quad \forall j = 0, 1, 2, ..., p_1. \quad (2.68)$$

By Theorem 2.4.3, $F_0 = 0$. Suppose $F_{j-1} = O(\rho^{-\alpha})$, then by (2.60) we have $F_j = S_i(|\gamma|^2 + F_{j-1}) = O(\rho^{-\alpha_1})$, since $a = |\gamma|^2 + F_{j-1} = |\gamma|^2 + O(\rho^{-\alpha_1}) \in I$.

By (2.68) and Lemma2.4.1, we obtain

$$||\gamma|^2 + F_{j-1} + O(\rho^{-j\alpha}) - |\gamma + \gamma_1 + ... + \gamma_i|^2| > \frac{1}{3}\rho^{\alpha_1}$$

and

$$||\gamma|^2 + F_{j-1} - |\gamma + \gamma_1 + ... + \gamma_i|^2| > \frac{1}{3}\rho^{\alpha_1},$$

for all $i = 1, 2, ..., p_1$.

Hence, by direct calculations and using the above inequalities:

$$|S_1(|\gamma|^2 + F_{j-1} + O(\rho^{-j\alpha_1})) - S_1(|\gamma|^2 + F_{j-1})|$$

$$= \left| \sum_{\gamma_1,\gamma_2 \in \Gamma(\rho^\alpha)} \left(\frac{|q_{\gamma_1}||q_{\gamma_2}|}{(|\gamma|^2 + F_{j-1} + O(\rho^{-j\alpha_1}) - |\gamma + \gamma_1|^2)} - \frac{|q_{\gamma_1}||q_{\gamma_2}|}{(|\gamma|^2 + F_{j-1} - |\gamma + \gamma_1|^2)} \right) \right|$$

$$= \left| \sum_{\gamma_1,\gamma_2 \in \Gamma(\rho^\alpha)} \frac{-O(\rho^{-j\alpha})|q_{\gamma_1}||q_{\gamma_2}|}{(|\gamma|^2 + F_{j-1} + O(\rho^{-j\alpha_1}) - |\gamma + \gamma_1|^2)(|\gamma|^2 + F_{j-1} - |\gamma + \gamma_1|^2)} \right|$$

$$\leq \left| \sum_{\gamma_1,\gamma_2 \in \Gamma(\rho^\alpha)} \frac{-O(\rho^{-j\alpha_1})|q_{\gamma_1}||q_{\gamma_2}|}{(\frac{1}{3}\rho^{\alpha_1})^2} \right| = O(\rho^{(-j+2)\alpha_1}),$$

and

$$|S_2(|\gamma|^2 + F_{j-1} + O(\rho^{-j\alpha_1})) - S_2(|\gamma|^2 + F_{j-1})| = O(\rho^{-(j+3)\alpha_1}),$$

since

$$|\frac{|q_{\gamma_1}||q_{\gamma_2}||q_{\gamma_3}|}{(|\gamma|^2+F_{j-1}+O(\rho^{-j\alpha_1})-|\gamma+\gamma_1|^2)(|\gamma|^2+F_{j-1}+O(\rho^{-j\alpha_1})-|\gamma+\gamma_1+\gamma_2|^2)}-$$

$$\frac{|q_{\gamma_1}||q_{\gamma_2}||q_{\gamma_3}|}{(|\gamma|^2+F_{j-1}-|\gamma+\gamma_1|^2)(|\gamma|^2+F_{j-1}-|\gamma+\gamma_1+\gamma_2|^2)}|=O(\rho^{-(j+3)\alpha_1})$$

Similarly, we can calculate

$$|S_i(|\gamma|^2+F_{j-1}+O(\rho^{-j\alpha_1}))-S_i(|\gamma|^2+F_{j-1})|=O(\rho^{-(j+i+1)\alpha_1})$$

for all $i=1,2,...,j$.

Therefore, the expression in the square brackets in (2.67) is equal to $O(\rho^{-(j+1)\alpha_1})$, where $1\leq j+i\leq p-c$, from which the theorem follows. □

CHAPTER THREE
ASYMPTOTIC FORMULAS FOR THE EIGENVALUES IN THE RESONANCE DOMAIN

In this chapter, without loss of generality we assume that $\gamma \notin V_{e_k}(\rho^{\alpha_1})$, for $k = 1, 2, ..., d$, where $e_1 = (\frac{\pi}{a_1}, 0, ..., 0), e_2 = (0, \frac{\pi}{a_2}, 0, ..., 0), ..., e_d = (0, ..., 0, \frac{\pi}{a_d})$.

3.1 Estimations for the Eigenvalues in the Resonance Domain

Let $|\gamma|^2$ be a resonance eigenvalue of the operator $L_N(0)$, i.e. $\gamma \in (\bigcap_{i=1}^{k} V_{\gamma_i}(\rho^{\alpha_k}))\setminus E_{k+1}$, $k \geq 1$.

Define the following sets :

$$B_k(\gamma_1, \gamma_2, ..., \gamma_k) = \{b : b = \sum_{i=1}^{k} n_i\gamma_i, n_i \in Z, |b| < \frac{1}{2}\rho^{\frac{1}{2}\alpha_{k+1}}\}$$

$$B_k(\gamma) = \gamma + B_k(\gamma_1, \gamma_2, ..., \gamma_k) = \{\gamma + b : b \in B_k(\gamma_1, \gamma_2, ..., \gamma_k)\}$$

$$B_k(\gamma, p_1) = B_k(\gamma) + \Gamma(p_1\rho^{\alpha}).$$

First, we shall prove that, if $|\gamma|^2$ is a resonance eigenvalue of the operator $L_N(0)$, then the corresponding eigenvalue of the operator $L_N(q)$ is close to the eigenvalue of the matrix $C(\gamma,\gamma_1,...,\gamma_k)$ defined as :

$$C(\gamma,\gamma_1,...,\gamma_k)=(c_{ij})=\begin{cases} q_{h_i-h_j}\,,\ i\neq j\\ |h_i|^2\,,\ i=j\,,\end{cases}$$

for $i,j=1,2,...,b_k$, where by h_i we denote the vectors of $B_k(\gamma,p_1)$ and b_k is the number of vectors in $B_k(\gamma,p_1)$.

Theorem 3.1.1. *Let* $\gamma\in(\bigcap_{i=1}^{k}V_{\gamma_i}(\rho^{\alpha_k}))\setminus E_{k+1}\,,\ k=1,2,...,d-1,\ |\gamma|\sim\rho.$ *Then, for any eigenvalue* Υ_N *of* $L_N(q)$ *satisfying*

$$|\Upsilon_N-|\gamma|^2|<\frac{1}{2}\rho^{\alpha_1}\quad \textit{and}\quad |(\Phi_N,v_\gamma)|>c_{13}\rho^{-c\alpha},$$

there exists an eigenvalue $\lambda_i(\gamma)$ *of the matrix* $C(\gamma,\gamma_1,...,\gamma_k)$ *such that*

$$\Upsilon_N=\lambda_i(\gamma)+O(\rho^{-(p-c-\frac{1}{4}d3^d)\alpha}).$$

Proof. The binding formula for $L_N(q)$ and $L_N(0)$, (see 2.45) and (2.48) gives:

$$(\Upsilon_N-|\gamma|^2)c(N,\gamma)=\sum_{\gamma'\in\Gamma(\rho^\alpha)}q_{\gamma'}c(N,\gamma-\gamma')+O(\rho^{-p\alpha}),$$

where we use the notation $(\Phi_N,v_\gamma)=c(N,\gamma)$. Then, writing this equation for all $h_i\in B_k(\gamma,p_1)$ we have :

$$(\Upsilon_N-|h_i|^2)c(N,h_i)=\sum_{\gamma'\in\Gamma(\rho^\alpha)}q_{\gamma'}c(N,h_i-\gamma')+O(\rho^{-p\alpha}). \tag{3.69}$$

First, we will show that for $h_i-\gamma'\notin B_k(\gamma,p_1)$, we have

$$c(N,h_i-\gamma')=O(\rho^{-p\alpha}) \tag{3.70}$$

and thus,

$$\sum_{\substack{\gamma'\in\Gamma(\rho^\alpha)\\ h_i-\gamma'\notin B_k(\gamma,p_1)}}q_{\gamma'}c(N,h_i-\gamma')=O(\rho^{-p\alpha}). \tag{3.71}$$

To prove this, we use the inequality

$$|\Upsilon_N-|h_i-\gamma'-\gamma_1-\gamma_2-...-\gamma_s|^2|>\frac{1}{6}\rho^{\alpha_{k+1}}, \tag{3.72}$$

where $h_i \in B_k(\gamma, p_1)$, $h_i - \gamma' \notin B_k(\gamma, p_1)$, and $\gamma_i \in \Gamma(\rho^\alpha)$; $i = 1, 2, ..., s \quad ; s = 0, 1, 2, ..., p_1 - 1$. It is clear that, the relations $h_i \in B_k(\gamma, p_1)$, $h_i - \gamma' \notin B_k(\gamma, p_1)$, $p > 2p_1$ and $|\gamma'|, |\gamma_1|, ..., |\gamma_{p_1-1}| < \rho^\alpha$ imply that

$$a_s = h_i - \gamma' - \gamma_1 - \gamma_2 - ... - \gamma_s \in B_k(\gamma, p) \setminus B_k(\gamma)$$

for $s = 0, 1, ..., p_1 - 1$. To prove (3.72) we use the decomposition

$$a_s = \gamma + b + a,$$

where $b \in B_k$ and $a \in \Gamma(p\rho^\alpha)$. Clearly, $|b| < \frac{1}{2}\rho^{\frac{1}{2}\alpha_{k+1}}$ and $|a| < p_1\rho^\alpha$. First let us show that

$$||\gamma + b + a|^2 - |\gamma|^2| > \frac{1}{5}\rho^{\alpha_{k+1}}. \tag{3.73}$$

To prove (3.73), we consider two cases :

Case 1 : If $a \in P = span\{\gamma_1, \gamma_2, ..., \gamma_k\}$, then $a + b \in P$ and $\gamma + b + a \notin B_k(\gamma) = \gamma + B_k$ imply that $a + b \in P \setminus B_k$, i.e.

$$|a + b| \geq \frac{1}{2}\rho^{\frac{1}{2}\alpha_{k+1}}.$$

Now, consider the orthogonal decomposition of γ as $\gamma = x + v$, where $v \in P$ and $x \perp v$. Using that, $\langle x, a\rangle = \langle x, b\rangle = \langle x, v\rangle = 0$, $|a + b| \geq \frac{1}{2}\rho^{\frac{1}{2}\alpha_{k+1}}$, and $|v| < \rho^{\alpha_1}$, (see 2.2.2), we get :

$$\begin{aligned} ||\gamma + b + a|^2 - |\gamma|^2| &= ||x + v + b + a|^2 - |x + v|^2| \\ &= ||v + b + a|^2 - |v|^2| > \frac{1}{5}\rho^{\frac{1}{2}\alpha_{k+1}} \end{aligned}$$

Thus, for Case 1, (3.73) is proved.

Case 2: If $a \notin P$, then by definition of $\gamma \in (\bigcap_{i=1}^{k} V_{\gamma_i}(\rho^{\alpha_k})) \setminus E_{k+1}$, we have

$$||\gamma + a|^2 - |\gamma|^2| > \rho^{\alpha_{k+1}}. \tag{3.74}$$

Consider the difference

$$||\gamma + b + a|^2 - |\gamma|^2| = ||\gamma + b + a|^2 - |\gamma + b|^2 + |\gamma + b|^2 - |\gamma|^2|,$$

where

$$d_1 = |\gamma + b + a|^2 - |\gamma + b|^2 = |\gamma + a|^2 - |\gamma|^2 + 2\langle a, b\rangle,$$

and thus by (3.74) and $|2\langle a, b\rangle| \leq 2|a||b| < p\rho^\alpha\rho^{\frac{1}{2}\alpha_{k+1}} < \frac{1}{3}\rho^{\alpha_{k+1}}$ it is clear that

$$|d_1| > \frac{2}{3}\rho^{\alpha_{k+1}}.$$

Using that, $|\gamma+b+a|^2-|\gamma|^2=|v+b+a|^2-|v|^2$ and taking $a=0$, we get

$$d_2=|\gamma+b|^2-|\gamma|^2=|v+b|^2-|v|^2=(|v+b|-|v|)(|v+b|+|v|),$$

and thus

$$|d_2|<\frac{1}{3}\rho^{\alpha_{k+1}}.$$

Then,

$$||d_1|-|d_2||>\frac{1}{5}\rho^{\alpha_{k+1}}.$$

So, in any case (3.73) is true.

Therefore , $|\Upsilon_N-|\gamma|^2|<\frac{1}{2}\rho^{\alpha_1}$ and (3.73) imply that

$$|\Upsilon_N-|\gamma+b+a|^2|=|\Upsilon_N-|\gamma|^2|-||\gamma+b+a|^2-|\gamma|^2|>\frac{1}{6}\rho^{\alpha_{k+1}},$$

which completes the proof of (3.72).

Now, applying the formula (3.69) p_1 times and using the inequality (3.72) we have

$$|c(N,h_i-\gamma')|$$

$$=\left|\sum_{\gamma_1,\gamma_2,\ldots,\gamma_{p_1-1}\in\Gamma(\rho^\alpha)} q_{\gamma_1}q_{\gamma_2}\ldots q_{\gamma_{p_1-1}}\frac{c(N,h_i-\gamma'-\gamma_1-\ldots-\gamma_{p_1-1})}{\prod_{j=0}^{p_1-1}(\Upsilon_N-|h_i-\gamma'-\sum_{k=1}^{j}\gamma_k|^2)}\right|$$

$$\leq(\frac{1}{6}\rho^{\alpha_{k+1}})^{-p_1}\sum_{\gamma_1,\gamma_2,\ldots,\gamma_{p_1-1}\in\Gamma(\rho^\alpha)}|q_{\gamma_1}||q_{\gamma_2}|\ldots|q_{\gamma_{p_1-1}}||c(N,h_i-\gamma'-\gamma_1-\ldots-\gamma_{p_1-1})|$$

$$\leq M^{p_1-1}(\frac{1}{6}\rho^{\alpha_{k+1}})^{-p_1}=O(\rho^{-p\alpha})$$

since $p_1\alpha_{k+1}\geq p_1\alpha_2>p\alpha$.

Thus, (3.71) is true.

Therefore, the formula (3.69) becomes :

$$(\Upsilon_N-|h_i|^2)c(N,h_i)-\sum_{\gamma'\in\Gamma(\rho^\alpha)}q_{\gamma'}c(N,h_i-\gamma')=O(\rho^{-p\alpha}),\tag{3.75}$$

where the sum is taken under the condition $h_i-\gamma'\in B_k(\gamma,p_1)$. Writing the equation (3.75) for all h_i , $i=1,2,\ldots,b_k$ and using the notation $h_j=h_i-\gamma'$, we obtain the following system of equations :

$$(\Upsilon_N-|h_1|^2)c(N,h_1)-\sum_{h_1-h_j\in\Gamma(\rho^\alpha)}q_{h_1-h_j}c(N,h_j)=O(\rho^{-p\alpha}),$$

$$(\Upsilon_N - |h_2|^2)c(N,h_2) - \sum_{h_2-h_j\in\Gamma(\rho^\alpha)} q_{h_2-h_j}c(N,h_j) = O(\rho^{-p\alpha}),$$

$$\vdots$$

$$(\Upsilon_N - |h_{b_k}|^2)c(N,h_{b_k}) - \sum_{h_{b_k}-h_j\in\Gamma(\rho^\alpha)} q_{h_{b_k}-h_j}c(N,h_j) = O(\rho^{-p\alpha})$$

or briefly :

$$(C-\Upsilon_N I)[c(N,h_1),c(N,h_2),...,c(N,h_{b_k})]$$
$$= [O(\rho^{-p\alpha}),O(\rho^{-p\alpha}),...,O(\rho^{-p\alpha})]. \tag{3.76}$$

Then, one has

$$[c(N,h_1),c(N,h_2),...,c(N,h_{b_k})] = (C-\Upsilon_N I)^{-1}[O(\rho^{-p\alpha}),O(\rho^{-p\alpha}),...,O(\rho^{-p\alpha})].$$

Taking the norm and using the Cauchy-Schwarz inequality in the last expression we get :

$$(\sum_{i=1}^{b_k}|c(N,h_i)|^2)^{\frac{1}{2}} \leq \|(C-\Upsilon_N I)^{-1}\|(b_k O(\rho^{-p\alpha})^2)^{\frac{1}{2}} \leq \|(C-\Upsilon_N I)^{-1}\|\sqrt{b_k}\rho^{-p\alpha}.$$

By, 2.51 we also have,

$$(\sum_{i=1}^{b_k}|c(N,h_i)|^2)^{\frac{1}{2}} > \rho^{-c\alpha},$$

and thus

$$\|(C-\Upsilon_N I)^{-1}\| = \max|\Upsilon_N - \lambda_i|^{-1} > c_{14}\frac{\rho^{-c\alpha+p\alpha}}{\sqrt{b_k}},$$

where maximum is taken over all $\lambda_i(\gamma)\,, i=1,2,...,b_k$. Using that the number of vectors in B_k is $|B_k| = O(\rho^{\frac{k}{2}\alpha_{k+1}})$, the number of vectors in $\Gamma(p_1\rho^\alpha)$ is $|\Gamma(p_1\rho^\alpha)| = O(\rho^{d\alpha})$ and $d\alpha < \frac{1}{2}\alpha_d$, we get

$$b_k = O(\rho^{d\alpha+\frac{k}{2}\alpha_{k+1}}) = O(\rho^{\frac{d}{2}\alpha_d}) = O(\rho^{\frac{d}{2}3^d\alpha})$$

for all $k=1,2,...,d-1$. Therefore,

$$\min|\Upsilon_N - \lambda_i| < \rho^{-(p-c-\frac{1}{4}d3^d)\alpha},$$

where minimum is taken over all $\lambda_i(\gamma)\,, i=1,2,...,b_k$, and the result

$$\Upsilon_N = \lambda_i + O(\rho^{-(p-c-\frac{1}{4}d3^d)\alpha})$$

follows. □

Corollary 3.1.2. *Every eigenvalue $\Upsilon_N \sim \rho^2$ of the operator $L(q(x))$ satisfies either*

$$\Upsilon_N = |\gamma|^2 + O(\rho^{-\alpha_1}) \ , \ \textit{for } \gamma \in U(\rho^{\alpha_1}, p)$$

or

$$\Upsilon_N = \lambda_i(\gamma) + O(\rho^{-(p-c-\frac{1}{4}d3^d)\alpha}),$$

for $\gamma \in (\bigcap_{i=1}^{k} V_{\gamma_i}(\rho^{\alpha_k})) \setminus E_{k+1}$, $k = 1, 2, ..., d-1.$

Proof. Let Υ_N be any eigenvalue of $L(q(x))$ of the order of ρ^2. Since $\{v_\gamma(x)\}_{\gamma \in \frac{\Gamma}{2}}$ forms a complete system in $L_2(F)$, then $\Phi_N = \sum_{\gamma \in \frac{\Gamma}{2}} (\Phi_N, v_\gamma) v_\gamma$, and

$$\sum_{\gamma : |\Upsilon_N - |\gamma|^2| > \frac{1}{2}\rho^{\alpha_1}} |(\Phi_N, v_\gamma)|^2 = \sum_{\gamma : |\Upsilon_N - |\gamma|^2| > \frac{1}{2}\rho^{\alpha_1}} \left| \frac{(\Phi_N, q(x) v_\gamma)}{\Upsilon_N - |\gamma|^2} \right|^2$$
$$\leq \frac{1}{4}\rho^{-2\alpha_1} \sum_{\gamma : |\Upsilon_N - |\gamma|^2| > \frac{1}{2}\rho^{\alpha_1}} |(q(x)\Phi_N, v_\gamma)|^2$$
$$\leq \frac{1}{4}\rho^{-2\alpha_1} \|q(x)\|^2 \|\Phi_N\|^2 = O(\rho^{-2\alpha})$$

Then, by Parseval's identity

$$\sum_{\gamma : |\Upsilon_N - |\gamma|^2| < \frac{1}{2}\rho^{\alpha_1}} |(\Phi_N, v_\gamma)|^2 = 1 - O(\rho^{-2\alpha_1})$$

Since the number of γ in $|\Upsilon_N - |\gamma|^2| < \frac{1}{2}\rho^{\alpha_1}$ is $O(\rho^{d-1})$, we have

$$\max_{\gamma : |\Upsilon_N - |\gamma|^2| < \frac{1}{2}\rho^{\alpha_1}} |(\Phi_N, v_\gamma)| > \sqrt{\frac{1 - O(\rho^{-2\alpha})}{\rho^{d-1}}} > c_{15}\rho^{\frac{1-d}{2}}.$$

Thus the proof follows from Theorem 2.4.3 and Theorem 3.1.1. □

Theorem 3.1.3. *Let $\gamma \in (\bigcap_{i=1}^{k} V_{\gamma_i}(\rho^{\alpha_k})) \setminus E_{k+1}$, $k = 1, 2, ..., d-1$, $|\gamma| \sim \rho$. Then, for every eigenvalue $\lambda_i(\gamma)$ of the matrix $C(\gamma, \gamma_1, ..., \gamma_k)$ satisfying $|\lambda_i - |\gamma|^2| < \frac{3}{8}\rho^{\alpha_1}$, there is an eigenvalue Υ_N of the operator $L_N(q)$ such that*

$$\Upsilon_N = \lambda_i(\gamma) + O(\rho^{-(p-\frac{d}{4}3^d)\alpha + \frac{d-1}{2}}).$$

Proof. By the general perturbation theory, there is Υ_N such that $|\Upsilon_N - |\gamma|^2| < \frac{1}{2}\rho^{2\alpha_1}$. Therefore, we can use the system from Theorem 3.1.1,

$$(C - \Upsilon_N I)[c(N, h_1), c(N, h_2), ..., c(N, h_{b_k})] = [O(\rho^{-p\alpha}), O(\rho^{-p\alpha}), ..., O(\rho^{-p\alpha})]. \tag{3.77}$$

Let λ_i be any eigenvalue of the matrix C and $v_i = (v_{i1}, v_{i2}, ..., v_{ib_k})$ be the corresponding normalized eigenvector ($\|v_i\| = 1$). Multiplying (3.77) by v_i we have,

$$\langle (C - \Upsilon_N I)[c(N, h_1), c(N, h_2), ..., c(N, h_{b_k})], v_i \rangle = \langle [O(\rho^{-p\alpha}), O(\rho^{-p\alpha}), ..., O(\rho^{-p\alpha})], v_i \rangle,$$

and by the Cauchy-Schwarz inequality one has

$$|\langle [O(\rho^{-p\alpha}), O(\rho^{-p\alpha}), ..., O(\rho^{-p\alpha})], v_i \rangle | \leq c_{16}\sqrt{b_k(\rho^{-p\alpha})^2}\|v_i\| = O(\rho^{-(p-\frac{d}{4}3^d)\alpha}).$$

Using that C is a symmetric matrix and that $Cv_i = \lambda_i v_i$ we have,

$$\langle [c(N, h_1), c(N, h_2), ..., c(N, h_{b_k})], (C - \Upsilon_N I)v_i \rangle = O(\rho^{-(p-\frac{1}{4}3^d)\alpha}),$$

that is

$$(\Upsilon_N - \lambda_i)\sum_{j=1}^{b_k} c(N, h_j)v_{ij} = O(\rho^{-(p-\frac{d}{4}3^d)\alpha}),$$

or explicitly

$$(\Upsilon_N - \lambda_i)\left(\sum_{j=1}^{b_k} v_{ij}v_{h_j}, \Phi_N\right) = O(\rho^{-(p-\frac{d}{4}3^d)\alpha}). \tag{3.78}$$

So, we need to prove that there exists an integer N such that

$$\left|\left(\sum_{j=1}^{b_k} v_{ij}v_{h_j}, \Phi_N\right)\right| > c_{17}\rho^{-\frac{d-1}{2}}, \tag{3.79}$$

from which the theorem follows. For this, we first consider the decomposition of the matrix $C(\gamma, \gamma_1, ..., \gamma_k)$ as $C = A + B$, where the matrix A is defined as:

$$A = (a_{ij}) = \begin{cases} 0, & i \neq j \\ |h_j|^2, & i = j \end{cases}$$

for $i, j = 1, 2, ..., b_k$, and clearly, $B = C - A$. Here $|h_j|^2$ is an eigenvalue of the diagonal matrix A and $e_j = (0, 0, ..., 1, ...0)$ is the corresponding eigenvector, that is, $Ae_j = |h_j|^2 e_j$. We denote by $v_i(h_j) \equiv \langle v_i, e_j\rangle \equiv v_{ij}$, the j-th component of the vector v_i.

Remark 3.1.4. Note that, B is a symmetric operator for which the diagonal elements are zero, and the sum of the elements in each row of B is less than $M = \sum_{\gamma \in \frac{\Gamma}{2}} |q_\gamma|$. Therefore, the eigenvalues of B are less than M and the norm of B satisfies the inequality $\|B\| \leq M$.

Now, multiplying the equation $Cv_i = \lambda_i v_i$ by e_j and using that A and B are symmetric matrices we have

$$\begin{aligned}\langle Cv_i, e_j\rangle &= \langle (A+B)v_i, e_j\rangle = \langle v_i, Ae_j\rangle + \langle v_i, Be_j\rangle \\ &= |h_j|^2 v_i(h_j) + \langle v_i, Be_j\rangle = \lambda_i \langle v_i, e_j\rangle,\end{aligned}$$

which gives the binding formula for the matrices C and A, that is

$$(\lambda_i - |h_j|^2) v_i(h_j) = \langle v_i, Be_j\rangle. \tag{3.80}$$

Using the formula (3.80) and the Bessel's inequality we obtain :

$$\begin{aligned}\sum_{j:|\lambda_i - |h_j|^2| > \frac{1}{8}\rho^{\alpha_1}} |v_i(h_j)|^2 &= \sum_{j:|\lambda_i - |h_j|^2| > \frac{1}{8}\rho^{\alpha_1}} \frac{|\langle v_i, Be_j\rangle|^2}{|\lambda_i - |h_j|^2|^2} \\ &= \sum_{j:|\lambda_i - |h_j|^2| > \frac{1}{8}\rho^{\alpha_1}} \frac{|\langle Bv_i, e_j\rangle|^2}{|\lambda_i - |h_j|^2|^2} \leq \frac{1}{64}\rho^{-2\alpha_1} \|B\|^2 \|v_i\|^2 \\ &\leq \frac{M^2}{64}\rho^{-2\alpha_1} = O(\rho^{-2\alpha_1})\end{aligned} \tag{3.81}$$

To prove (3.79) , we consider the Parseval's identity

$$\begin{aligned}1 &= \left\| \sum_{j=1}^{b_k} v_i(h_j) v_{h_j} \right\|^2 \\ &= \sum_{N=1}^{\infty} |(\sum_{j=1}^{b_k} v_i(h_j) v_{h_j}, \Phi_N)|^2 \\ &= \sum_{N:|\Upsilon_N - |\gamma|^2| < \frac{1}{2}\rho^{2\alpha_1}} |(\sum_{j=1}^{b_k} v_i(h_j) v_{h_j}, \Phi_N)|^2 \\ &+ \sum_{N:|\Upsilon_N - |\gamma|^2| > \frac{1}{2}\rho^{2\alpha_1}} |(\sum_{j=1}^{b_k} v_i(h_j) v_{h_j}, \Phi_N)|^2\end{aligned} \tag{3.82}$$

First, we give an estimate for the second sum in the last expression, that is

$$\sum_{N:|\Upsilon_N-|\gamma|^2|>\frac{1}{2}\rho^{2\alpha_1}} |(\sum_{j=1}^{b_k} v_i(h_j)v_{h_j},\Phi_N)|^2$$
$$= \sum_{N:|\Upsilon_N-|\gamma|^2|>\frac{1}{2}\rho^{2\alpha_1}} |(\sum_{j:|\lambda_i-|h_j|^2|<\frac{1}{8}\rho^{\alpha_1}} v_i(h_j)v_{h_j},\Phi_N)$$
$$+(\sum_{j:|\lambda_i-|h_j|^2|>\frac{1}{8}\rho^{\alpha_1}} v_i(h_j)v_{h_j},\Phi_N)|^2$$
$$\leq 2 \sum_{N:|\Upsilon_N-|\gamma|^2|>\frac{1}{2}\rho^{2\alpha_1}} |(\sum_{j:|\lambda_i-|h_j|^2|<\frac{1}{8}\rho^{\alpha_1}} v_i(h_j)v_{h_j},\Phi_N)|^2$$
$$+2 \sum_{N:|\Upsilon_N-|\gamma|^2|>\frac{1}{2}\rho^{2\alpha_1}} |(\sum_{j:|\lambda_i-|h_j|^2|>\frac{1}{8}\rho^{\alpha_1}} v_i(h_j)v_{h_j},\Phi_N)|^2 \tag{3.83}$$

Now, notice that $|\lambda_i - |h_j|^2| < \frac{1}{8}\rho^{\alpha_1}$ and the condition of the theorem $|\lambda_i - |\gamma|^2| < \frac{3}{8}\rho^{\alpha_1}$ together imply that $||\gamma|^2 - |h_j|^2| < \frac{1}{2}\rho^{\alpha_1}$. Therefore, using the binding formula (3.80) and the well-known formula

$$\frac{1}{\Upsilon_N - |h_j|^2} = \frac{1}{(\Upsilon_N - |\gamma|^2)(1 - \frac{|h_j|^2-|\gamma|^2}{\Upsilon_N-|\gamma|^2})}$$
$$= \frac{1}{\Upsilon_N - |\gamma|^2} + \frac{|h_j|^2 - |\gamma|^2}{(\Upsilon_N - |\gamma|^2)^2} + \dots + \frac{(|h_j|^2 - |\gamma|^2)^{k-1}}{(\Upsilon_N - |\gamma|^2)^k} + O(\rho^{-(k+1)\alpha_1}) \tag{3.84}$$

for $|\Upsilon_N - |\gamma|^2| > \frac{1}{2}\rho^{2\alpha_1}$ and $||\gamma|^2 - |h_j|^2| < \frac{1}{2}\rho^{\alpha_1}$, we have

$$\sum_{N:|\Upsilon_N-|\gamma|^2|>\frac{1}{2}\rho^{2\alpha_1}} | \sum_{j:|\lambda_i-|h_j|^2|<\frac{1}{8}\rho^{\alpha_1}} v_i(h_j)(v_{h_j},\Phi_N)|^2$$
$$= \sum_{N:|\Upsilon_N-|\gamma|^2|>\frac{1}{2}\rho^{2\alpha_1}} | \sum_{j:|\lambda_i-|h_j|^2|<\frac{1}{8}\rho^{\alpha_1}} F(j)\frac{1}{\Upsilon_N - |h_j|^2}|^2$$
$$\leq (k+1) \sum_{N:|\Upsilon_N-|\gamma|^2|>\frac{1}{2}\rho^{2\alpha_1}} | \sum_{j:|\lambda_i-|h_j|^2|<\frac{1}{8}\rho^{\alpha_1}} \frac{F(j)}{\Upsilon_N - |\gamma|^2}|^2$$
$$+(k+1) \sum_{N:|\Upsilon_N-|\gamma|^2|>\frac{1}{2}\rho^{2\alpha_1}} | \sum_{j:|\lambda_i-|h_j|^2|<\frac{1}{8}\rho^{\alpha_1}} \frac{F(j)}{(\Upsilon_N - |\gamma|^2)^2}(|\gamma|^2 - |h_j|^2)|^2$$
$$\vdots$$
$$+(k+1) \sum_{N:|\Upsilon_N-|\gamma|^2|>\frac{1}{2}\rho^{2\alpha_1}} | \sum_{j:|\lambda_i-|h_j|^2|<\frac{1}{8}\rho^{\alpha_1}} \frac{F(j)}{(\Upsilon_N - |\gamma|^2)^k}(|\gamma|^2 - |h_j|^2)^{k-1}|^2$$
$$+(k+1) \sum_{N:|\Upsilon_N-|\gamma|^2|>\frac{1}{2}\rho^{2\alpha_1}} | \sum_{j:|\lambda_i-|h_j|^2|<\frac{1}{8}\rho^{\alpha_1}} O(\rho^{-(k+1)\alpha_1})F(j)|^2, \tag{3.85}$$

where

$$F(j) = (v_i(h_j)q(x)v_{h_j}, \Phi_N).$$

To calculate the order of each term in (3.85)we use the Bessel's inequality with respect to the basis $\{\Phi_N\}_{N=1}^{\infty}$, and the orthogonality of the eigenfunctions $v_{h_j}(x)$. Thus, we obtain:

$$\sum_{N:|\Upsilon_N-|\gamma|^2|>\frac{1}{2}\rho^{2\alpha_1}} |\sum_{j:|\lambda_i-|h_j|^2|<\frac{1}{8}\rho^{\alpha_1}} \frac{F(j)}{(\Upsilon_N-|\gamma|^2)^s}(|\gamma|^2-|h_j|^2)^{s-1}|^2$$

$$\sum_{N:|\Upsilon_N-|\gamma|^2|>\frac{1}{2}\rho^{2\alpha_1}} |\sum_{j:|\lambda_i-|h_j|^2|<\frac{1}{8}\rho^{\alpha_1}} \frac{(v_i(h_j)q(x)v_{h_j}(x),\Phi_N)}{(\Upsilon_N-|\gamma|^2)^s}(|\gamma|^2-|h_j|^2)^{s-1}|^2$$

$$\sum_{N:|\Upsilon_N-|\gamma|^2|>\frac{1}{2}\rho^{2\alpha_1}} |(q(x) \sum_{j:|\lambda_i-|h_j|^2|<\frac{1}{8}\rho^{\alpha_1}} v_i(h_j)v_{h_j}(x)(|\gamma|^2-|h_j|^2)^{s-1}, \frac{\Phi_N}{(\Upsilon_N-|\gamma|^2)^s})|^2$$

$$\leq c_{18}\rho^{-4s\alpha_1} \sum_{N:|\Upsilon_N-|\gamma|^2|>\frac{1}{2}\rho^{2\alpha_1}} |(q(x) \sum_{j:|\lambda_i-|h_j|^2|<\frac{1}{8}\rho^{\alpha_1}} v_i(h_j)v_{h_j}(x)(|\gamma|^2-|h_j|^2)^{s-1}, \Phi_N)|^2$$

$$\leq c_{18}\rho^{-4s\alpha_1}\|(q(x) \sum_{j:|\lambda_i-|h_j|^2|<\frac{1}{8}\rho^{\alpha_1}} v_i(h_j)v_{h_j}(x)(|\gamma|^2-|h_j|^2)^{s-1}\|^2$$

$$\leq c_{18}\rho^{-4s\alpha_1}M^2\| \sum_{j:|\lambda_i-|h_j|^2|<\frac{1}{8}\rho^{\alpha_1}} |v_i(h_j)|^2||\gamma|^2-|h_j|^2|^{2(s-1)}\|$$

$$\leq c_{18}\rho^{-4s\alpha_1}M^2\rho^{2(s-1)\alpha_1} \sum_{j:|\lambda_i-|h_j|^2|<\frac{1}{8}\rho^{\alpha_1}} |v_i(h_j)|^2 = O(\rho^{-2(s+1)\alpha_1})$$

for $s = 0, 1, ..., k$.

Now, let N_1 be the number of h_j, satisfying $|\lambda_i - |h_j|^2| < \frac{1}{8}\rho^{\alpha_1}$, then the order of the last term is:

$$\sum_{N:|\Upsilon_N-|\gamma|^2|>\frac{1}{2}\rho^{2\alpha_1}} |\sum_{j:|\lambda_i-|h_j|^2|<\frac{1}{8}\rho^{\alpha_1}} O(\rho^{-(k+1)\alpha_1})(v_i(h_j)q(x)v_{h_j}(x),\Phi_N)|^2$$
$$\leq N_1 \sum_{N:|\Upsilon_N-|\gamma|^2|>\frac{1}{2}\rho^{2\alpha_1}} \sum_{j:|\lambda_i-|h_j|^2|<\frac{1}{8}\rho^{\alpha_1}} |O(\rho^{-(k+1)\alpha_1})|^2|(v_i(h_j)|^2|(q(x)v_{h_j}(x),\Phi_N)|^2$$
$$\leq c_{19}N_1\rho^{-2(k+1)\alpha_1} \sum_{j:|\lambda_i-|h_j|^2|<\frac{1}{8}\rho^{\alpha_1}} |(v_i(h_j)|^2 \sum_{N:|\Upsilon_N-|\gamma|^2|>\frac{1}{2}\rho^{2\alpha_1}}^{\infty} |(q(x)v_{h_j}(x),\Phi_N)|^2$$
$$\leq c_{19}N_1\rho^{-2(k+1)\alpha_1} \sum_{j:|\lambda_i-|h_j|^2|<\frac{1}{8}\rho^{\alpha_1}} \|(q(x)v_{h_j}(x)\|^2$$
$$\leq c_{19}N_1^2M^2\rho^{-2(k+1)\alpha_1} = N_1^2O(\rho^{-2(k+1)\alpha_1}) = O(\rho^{-2\alpha_1}),$$

since $N_1 = O(\rho^{\frac{d}{2}\alpha_d})$, and we can always choose k in $O(\rho^{-2(k+1)\alpha_1})$ such that

$$N_1^2O(\rho^{-2(k+1)\alpha_1}) = O(\rho^{-2\alpha_1}).$$

Also, using the Bessel's inequality, the orthogonality of the eigenfunctions and (3.81) we have

$$\sum_{N:|\Upsilon_N-|\gamma|^2|>\frac{1}{2}\rho^{2\alpha_1}} |(\sum_{j:|\lambda_i-|h_j|^2|>\frac{1}{8}\rho^{\alpha_1}} v_i(h_j)v_{h_j},\Phi_N)|^2$$
$$\leq \|\sum_{j:|\lambda_i-|h_j|^2|>\frac{1}{8}\rho^{\alpha_1}} v_i(h_j)v_{h_j}\|^2$$
$$= \sum_{j:|\lambda_i-|h_j|^2|>\frac{1}{8}\rho^{\alpha_1}} |v_i(h_j|^2 = O(\rho^{-2\alpha_1}) \tag{3.86}$$

Therefore, by (3.82) it is clear that

$$\sum_{N:|\Upsilon_N-|\gamma|^2|<\frac{1}{2}\rho^{2\alpha_1}} |(\sum_{j=1}^{b_k} v_i(h_j)v_{h_j},\Phi_N)|^2 = 1 - O(\rho^{-2\alpha_1}) \tag{3.87}$$

Now, taking into account that the number of indexes N satisfying $|\Upsilon_N - |\gamma|^2| < \frac{1}{2}\rho^{2\alpha_1}$ is less than $c_{20}\rho^{d-1}$, we get (3.79)

$$|(\sum_{j=1}^{b_k} v_i(h_j)v_{h_j},\Phi_N)| > \sqrt{\frac{(1-O(\rho^{-2\alpha_1}))}{\rho^{d-1}}} > c_{17}\rho^{-\frac{d-1}{2}}.$$

So, dividing both sides of (3.78) by (3.79) we get the result

$$\Upsilon_N = \lambda_i + \frac{O(\rho^{-(p-\frac{d}{4}3^d)\alpha)}}{O(\rho^{-\frac{d-1}{2}})} = O(\rho^{-(p-\frac{d}{4}3^d)\alpha+\frac{d-1}{2}}).$$

□

3.2 Asymptotic Formulas for the Eigenvalues in a Single Resonance Domain

Now, we investigate in detail the eigenvalues of $L_N(q(x))$ in a single resonance domain. Namely, we find the relation between the eigenvalues of $L_N(q(x))$ in a single resonance domain and the eigenvalues of the Sturm-Liouville operators. In order the inequalities (2.33) and (2.34) to be satisfied, we can choose α, α_1 and α_2 as follows

$$\alpha = \frac{1}{d+p}, \quad \alpha_1 = \frac{p_2}{d+p}, \quad \alpha_2 = \frac{2p_2+1}{d+p},$$

where $p_2 = [\frac{p-5}{3}] - 1$ and $[\frac{p-5}{3}]$ is the integer part of the number $\frac{p-5}{3}$.
Let $\gamma \in V_\delta(\rho^{\alpha_1}) \setminus E_2$, $\delta \in \frac{\Gamma}{2}$, where δ is minimal in its direction. Consider the following sets :

$$B_1(\delta) = \{b : b = n\delta \, , n \in Z \, , \, |b| < \frac{1}{2}\rho^{\frac{1}{2}\alpha_2}\},$$

$$B_1(\gamma) = \gamma + B_1(\delta) = \{\gamma + b : b \in B_1(\delta)\},$$

$$B_1(\gamma, p_1) = B_1(\gamma) + \Gamma(p_1\rho^{\alpha}).$$

As before, denote by h_i, $i = 1, 2, ..., b_1$ the vectors of $B_1(\gamma, p_1)$, where b_1 is the number of vectors in $B_1(\gamma, p_1)$. Then the matrix $C(\gamma, \delta)$ is defined as :

$$C(\gamma, \delta) = (c_{ij}) = \begin{cases} q_{h_i - h_j}, i \neq j \\ |h_i|^2, i = j \end{cases}$$

where $i, j = 1, 2, ..., b_1$.
Also we define the matrix $D(\gamma, \delta) = (c_{ij})$ for $i, j = 1, 2, ..., a_1$, where $h_1, h_2, ..., h_{a_1}$ are the vectors of $B_1(\gamma, p_1) \bigcap \{\gamma + n\delta : n \in Z\}$, and a_1 is the number of vectors in $B_1(\gamma, p_1) \bigcap \{\gamma + n\delta : n \in Z\}$. Clearly $a_1 = O(\rho^{\frac{1}{2}\alpha_2})$.

Lemma 3.2.1. *a) If λ_i is an eigenvalue of the matrix $C(\gamma, \delta)$ such that $|\lambda_i - |h_i|^2| < M$ for $i = 1, 2, ..., a_1$, then*

$$|\lambda_i - |h_j|^2| > \frac{1}{4}\rho^{\alpha_2} \ \text{for} \ \forall j = a_1 + 1, a_1 + 2, ..., b_1.$$

b) If λ_i is an eigenvalue of the matrix $C(\gamma, \delta)$ such that $|\lambda_i - |h_i|^2| < M$ for $i = a_1 + 1, a_1 + 2, ..., b_1$, then

$$|\lambda_i - |h_j|^2| > \frac{1}{4}\rho^{\alpha_2} \ \text{for} \ \forall j = 1, 2, ..., a_1.$$

Proof. First we prove

$$||h_j|^2 - |h_i|^2| \geq \frac{1}{3}\rho^{\alpha_2}, \quad \forall i \leq a_1, \quad \forall j > a_1. \tag{3.88}$$

By definition, if $i \leq a_1$ then $h_i = \gamma + n\delta$, where $|n\delta| < \frac{1}{2}\rho^{\frac{1}{2}\alpha_2} + p_1\rho^{\alpha}$. If $j > a_1$ then $h_j = \gamma + s\delta + a$, where $|s\delta| < \frac{1}{2}\rho^{\frac{1}{2}\alpha_2}$, $a \in \Gamma(p_1\rho^{\alpha}) \setminus \delta R$. Therefore

$$|h_j|^2 - |h_i|^2 = 2\langle\gamma, a\rangle + 2s\langle\delta, a\rangle + 2s\langle\gamma, \delta\rangle + |s\delta|^2 + |a|^2 - 2n\langle\gamma, \delta\rangle - |n\delta|^2.$$

Since $\gamma \notin V_a(\rho^{\alpha_2})$, $|a| < p_1\rho^{\alpha}$, we have

$$|2\langle\gamma, a\rangle| > \rho^{\alpha_2} - c_9\rho^{2\alpha}.$$

The relation $\gamma \in V_\delta(\rho^{\alpha_1})$ and the inequalities for s and n imply that

$$2s\langle\gamma, \delta\rangle + 2s\langle\gamma, a\rangle + |a|^2 - 2n\langle\gamma, \delta\rangle = O(\rho^{\frac{1}{2}\alpha_2 + \alpha_1}),$$

$$||s\delta|^2 - |n\delta|^2| < \frac{1}{4}\rho^{\alpha_2} + c_{10}\rho^{\frac{1}{2}\alpha_2 + \alpha}.$$

Thus (3.88) follows from these relations, since $\frac{1}{2}\alpha_2 + \alpha_1 < \alpha_2$ and $\frac{1}{2}\alpha_2 + \alpha < \alpha_2$.

The eigenvalues of $D(\gamma, \delta)$ and $C(\gamma, \delta)$ lay in M-neighborhood of the numbers $|h_k|^2$ for $k = 1, 2, ..., a_1$ and for $k = 1, 2, ..., b_1$, respectively. The inequality (3.88) shows that one can enumerate the eigenvalues λ_i $(i = 1, 2, ..., b_1)$ of $C(\gamma, \delta)$ such that λ_i for $i \leq a_1$ lay in M-neighborhood of the numbers $|h_k|^2$ for $k \leq a_1$ and λ_i for $i > a_1$ lay in M-neighborhood of the numbers $|h_k|^2$ for $k > a_1$.Then by (3.88), we get

$$|\lambda_i - |h_j|^2| > \frac{1}{4}\rho^{\alpha_2}, \tag{3.89}$$

for $i \leq a_1$, $j > a_1$ and $i > a_1$, $j \leq a_1$. □

Theorem 3.2.2. *Let $\gamma \in V_\delta(\rho^{\alpha_1}) \setminus E_2$ and $|\gamma| \sim \rho$. Then, for any eigenvalue $\lambda_i(\gamma)$ of the matrix $C(\gamma, \delta)$ satisfying $|\lambda_i - |h_i|^2| < M, i = 1, 2, ..., a_1$, there exists an eigenvalue $\widetilde{\lambda}_{k(i)}$ of the matrix $D(\gamma, \delta)$ such that*

$$\lambda_i = \widetilde{\lambda}_{k(i)} + O(\rho^{-\frac{3}{4}\alpha_2}),$$

Proof. Let λ_i be an eigenvalue of $C(\gamma, \delta)$, and $v_i = (v_{i1}, v_{i2}, ..., v_{ib_1})$ be the corresponding normalized eigenvector. Denote by $e_j = (0, 0, ..., 1, 0, ..., 0)$ the eigenvector of $A(\gamma, \delta)$ corresponding to the j-th eigenvalue $|h_j|^2$, where

$$A(\gamma, \delta) = (a_{ij}) = \begin{cases} 0\ , & i \neq j \\ |h_i|^2\ , & i = j\ , \end{cases}$$

for $i,\ j = 1, 2, ..., b_1$.

The binding formula for $C(\gamma, \delta)$ and $A(\gamma, \delta)$, (see 3.80) is :

$$(\lambda_i - |h_j|^2)v_{ij} = \langle v_i, Be_j \rangle. \tag{3.90}$$

Substituting the orthogonal decomposition $Be_j = \sum_{k_1}^{b_1} \langle Be_j, e_{k_1}\rangle e_{k_1}$ in equation 3.90, we get

$$(\lambda_i - |h_j|^2)v_{ij} = \langle v_i, \sum_{k_1}^{b_1} \langle Be_j, e_{k_1}\rangle e_{k_1}\rangle = \sum_{k_1=1}^{b_1} \langle Be_j, e_{k_1}\rangle v_{ik_1}$$
$$= \sum_{k_1=1}^{a_1} \langle Be_j, e_{k_1}\rangle v_{ik_1} + \sum_{k_1=a_1+1}^{b_1} \langle Be_j, e_{k_1}\rangle v_{ik_1}.$$

It is clear that, $\langle Be_j, e_{k_1}\rangle = q_{h_{k_1}-h_j}$, and thus one has

$$(\lambda_i - |h_j|^2)v_{ij} - \sum_{k_1=1}^{a_1} q_{h_{k_1}-h_j} v_{ik_1} = \sum_{k_1=a_1+1}^{b_1} q_{h_{k_1}-h_j} v_{ik_1}. \tag{3.91}$$

Now, taking any $\lambda_i \in [|h_i|^2 - M, |h_i|^2 + M|]$, $i = 1, 2, ..., a_1$ and writing the equation (3.91) for all h_j , $j = 1, 2, ..., a_1$, we get the system of linear algebraic equations:

$$(\lambda_i - |h_1|^2)v_{i1} - q_{h_2-h_1}v_{i2} - ... - q_{h_{a_1}-h_1}v_{ia_1} = \sum_{k_1=a_1+1}^{b_1} q_{h_{k_1}-h_1}v_{ik_1}$$
$$-q_{h_1-h_2}v_{i1} + (\lambda_i - |h_2|^2)v_{i2} - ... - q_{h_{a_1}-h_2}v_{ia_1} = \sum_{k_1=a_1+1}^{b_1} q_{h_{k_1}-h_2}v_{ik_1}$$
$$\vdots$$
$$-q_{h_1-h_{a_1}}v_{i1} - q_{h_2-h_{a_1}}v_{i2} - ... + (\lambda_i - |h_{a_1}|^2)v_{ia_1} = \sum_{k_1=a_1+1}^{b_1} q_{h_{k_1}-h_{a_1}}v_{ik_1} \tag{3.92}$$

Using the binding formula (3.90) and the Lemma 3.89, we find an estimation for the right hand side of the above system. That is,

$$|\sum_{k_1=a_1+1}^{b_1} q_{h_{k_1}-h_s}v_{ik_1}| = |\sum_{k_1=a_1+1}^{b_1} q_{h_{k_1}-h_s}\frac{\langle v_i, Be_{k_1}\rangle}{\lambda_i - |h_{k_1}|^2}|$$
$$\leq \sum_{k_1=a_1+1}^{b_1} |q_{h_{k_1}-h_s}|\frac{\|v_i\|\|B\|\|e_{k_1}\|}{|\lambda_i - |h_{k_1}|^2|} \leq \frac{1}{4}\rho^{-\alpha_2}M \sum_{k_1=a_1+1}^{b_1} |q_{h_{k_1}-h_s}|$$
$$\leq \frac{1}{4}\rho^{-\alpha_2}M^2 = O(\rho^{-\alpha_2})$$

for $\forall s = 1, 2, ..., a_1$. Then, the system (3.92) becomes

$$(D(\gamma,\delta) - \lambda_i I)[v_{i1}, v_{i2}, ..., v_{ia_1}]^t = [O(\rho^{-\alpha_2}), O(\rho^{-\alpha_2}), ..., O(\rho^{-\alpha_2})]^t,$$

or

$$[v_{i1}, v_{i2}, ..., v_{ia_1}]^t = (D(\gamma,\delta) - \lambda_i I)^{-1}[O(\rho^{-\alpha_2}), O(\rho^{-\alpha_2}), ..., O(\rho^{-\alpha_2})]^t \qquad (3.93)$$

Using the binding formula (3.90) and Lemma 3.89 we have :

$$\sum_{k_1=a_1+1}^{b_1} |v_{ij}|^2 = \sum_{k_1=a_1+1}^{b_1} \frac{|\langle v_i, Be_{k_1}\rangle|^2}{|\lambda_i - |h_{k_1}|^2|^2} = \sum_{k_1=a_1+1}^{b_1} \frac{|\langle Bv_i, e_{k_1}\rangle|^2}{|\lambda_i - |h_{k_1}|^2|^2}$$
$$\leq \frac{1}{16}M^2\rho^{-2\alpha_2} = O(\rho^{-2\alpha_2}).$$

and thus, by Parseval's identity we get :

$$\sum_{k_1=1}^{a_1} |v_{ij}|^2 > 1 - O(\rho^{-2\alpha_2}). \qquad (3.94)$$

Now, taking the norm of (3.93) and using the above inequality we have :

$$\sqrt{1 - O(\rho^{-2\alpha_2})} < (\sum_{k_1=a_1+1}^{b_1} |v_{ij}|^2)^{\frac{1}{2}} \leq \|(D(\gamma,\delta) - \lambda_i I)^{-1}\| O(\sqrt{a_1}\rho^{-\alpha_2})$$

Thus,

$$max|\lambda_i - \widetilde{\lambda}_{k(i)}|^{-1} > \frac{\sqrt{1 - O(\rho^{-2\alpha_2})}}{\sqrt{a_1}\rho^{-\alpha_2}},$$

or

$$min|\lambda_i - \widetilde{\lambda}_{k(i)}| = O(\sqrt{a_1}\rho^{-\alpha_2}) = O(\rho^{-\frac{3}{4}\alpha_2})$$

where the maximum (minimum) is taken over all $\widetilde{\lambda}_{k(i)}$, $i = 1, 2, ..., a_1$. So, the result follows. □

Theorem 3.2.3. *For any eigenvalue $\widetilde{\lambda}_j$ of the matrix $D(\gamma,\delta)$, there exists an eigenvalue $\lambda_{i(j)}$ of the matrix $C(\gamma,\delta)$ such that*

$$\lambda_{i(j)} = \widetilde{\lambda}_j + O(\rho^{-\frac{1}{2}\alpha_2})$$

Proof. Define the matrix $D'(\gamma,\delta) = (d'_{ij})$ as; $d'_{ij} = |h_i|^2, i = j, i = 1, 2, ..., b_1$, $d'_{ij} = q_{h_j - h_i}, i \neq j,\ \ i, j = 1, 2, ..., a_1$ and $d'_{ij} = 0$, otherwise. Then the spectrum of the matrix $D'(\gamma,\delta)$ is :

$$spec(D') = spec(D)\bigcup\{|h_{a_1+1}|^2, |h_{a_1+2}|^2, ..., |h_{b_1}|^2\}$$
$$\equiv \{\widetilde{\lambda}_1, \widetilde{\lambda}_2, ..., \widetilde{\lambda}_{a_1}, |h_{a_1+1}|^2, |h_{a_1+2}|^2, ..., |h_{b_1}|^2\}.$$

Let us denote by $w_j = (w_{j1}, w_{j2}, ..., w_{ja_1}, 0, ..., 0)$ the normalized eigenvector corresponding to the j-th eigenvalue of the matrix D', when $j = 1, 2, ..., a_1$ and by $w_k = (0, 0, ..., 1, 0, ...0)$ the eigenvector corresponding to the k-th eigenvalue of D', when $k = a_1 + 1, a_1 + 2, ..., b_1$.

Now, using the system from the previous theorem, we have

$$(D' - \lambda_i I)[v_{i1}, v_{i2}, ..., v_{ib_1}]$$
$$= [(D - \lambda_i I)[v_{i1}, v_{i2}, ..., v_{ia_1}], (|h_{a_1+1}|^2 - \lambda_i)v_{ia_1+1}, ..., (|h_{b_1}|^2 - \lambda_i)v_{ib_1}]$$
$$= [O(\rho^{-\alpha_2}), ..., O(\rho^{-\alpha_2}), (|h_{a_1+1}|^2 - \lambda_i)v_{ia_1+1}, ..., (|h_{b_1}|^2 - \lambda_i)v_{ib_1}].$$

Taking the inner product of the last system by w_j , $j = 1, 2, ..., a_1$, using that D' is symmetric and $D'w_j = \widetilde{\lambda}_j w_j$ we have :

$$(\lambda_{i(j)} - \widetilde{\lambda}_j)\sum_{k=1}^{a_1} v_{ik}w_{jk} = \sum_{k=1}^{a_1} O(\rho^{-\alpha_2})w_{jk}, \tag{3.95}$$

where by the Cauchy-Schwarz inequality one has

$$|\sum_{k=1}^{a_1} O(\rho^{-\alpha_2})w_{jk}| \leq \sqrt{\sum_{k=1}^{a_1} O(\rho^{-\alpha_2})^2}\sqrt{\sum_{k=1}^{a_1} w_{jk}^2} \leq \sqrt{a_1(\rho^{-\alpha_2})^2} = O(\sqrt{a_1}\rho^{-\alpha_2}).$$

Thus, we get

$$(\lambda_{i(j)} - \widetilde{\lambda}_j)\sum_{k=1}^{a_1} v_{ik}w_{jk} = O(\rho^{-\frac{3}{4}\alpha_2}). \tag{3.96}$$

So, we need to show that for any $j = 1, 2, ..., a_1$ there exists $v_{i(j)}$, such that

$$|\sum_{k=1}^{a_1} v_{ik}w_{jk}| = |\langle v_i, w_j\rangle| > \sqrt{\frac{1 - O(\rho^{-\frac{3}{2}\alpha_2})}{a_1}} > c_{23}\rho^{-\frac{1}{4}\alpha_2}. \tag{3.97}$$

For this, we consider the decomposition $w_j = \sum_{i=1}^{b_1} \langle w_j, v_i \rangle v_i$ and the Parseval's identity

$$1 = \sum_{i=1}^{b_1} |\langle w_j, v_i \rangle|^2 = \sum_{i=1}^{a_1} |\langle w_j, v_i \rangle|^2 + \sum_{i=a_1+1}^{b_1} |\langle w_j, v_i \rangle|^2$$

First, let us show that

$$\sum_{i=a_1+1}^{b_1} |\langle w_j, v_i \rangle|^2 = O(\rho^{-\frac{3}{2}\alpha_2}). \tag{3.98}$$

Using that, $w_j = \sum_{k=1}^{a_1} \langle w_j, e_k \rangle e_k$, the binding formula for $C(\gamma, \delta)$ and $A(\gamma, \delta)$, the Lemma 3.89, part b) and the Bessel's inequality we obtain the estimation:

$$\sum_{i=a_1+1}^{b_1} |\langle w_j, v_i \rangle|^2 = \sum_{i=a_1+1}^{b_1} |\langle \sum_{k=1}^{a_1} w_{jk} e_k, v_i \rangle|^2$$

$$= \sum_{i=a_1+1}^{b_1} |\sum_{k=1}^{a_1} w_{jk} \langle e_k, v_i \rangle|^2 = \sum_{i=a_1+1}^{b_1} |\sum_{k=1}^{a_1} w_{jk} \frac{\langle Be_k, v_i \rangle}{|\lambda_i - |h_k|^2|}|^2$$

$$\leq \sum_{i=a_1+1}^{b_1} \rho^{-2\alpha_2} (\sum_{k=1}^{a_1} |w_{jk}||\langle Be_k, v_i \rangle|)^2 \leq \sum_{i=a_1+1}^{b_1} \rho^{-2\alpha_2} |a_1| \sum_{k=1}^{a_1} |w_{jk}|^2 |\langle Be_k, v_i \rangle|^2$$

$$\leq |a_1| \rho^{-2\alpha_2} \sum_{k=1}^{a_1} |w_{jk}|^2 \sum_{k=1}^{a_1} |\langle Be_k, v_i \rangle|^2 \leq |a_1| \rho^{-2\alpha_2} \sum_{k=1}^{a_1} |w_{jk}|^2 \|Be_k\|^2$$

$$\leq M^2 |a_1| \rho^{-2\alpha_2} \sum_{k=1}^{a_1} |w_{jk}|^2 = O(\rho^{-\frac{3}{2}\alpha_2}).$$

Therefore one has

$$\sum_{i=1}^{a_1} |\langle w_j, v_i \rangle|^2 = 1 - O(\rho^{-\frac{3}{2}\alpha_2})$$

from which it follows that there exists $v_{i(j)}$ such that (3.97) holds. Dividing both sides of (3.96) by (3.97) we get the result

$$\lambda_{i(j)} = \widetilde{\lambda_j} + O(\rho^{-\frac{1}{2}\alpha_2}).$$

□

Now, using the notation $h_i = \gamma - (\frac{i}{2})\delta$ if i is even, $h_i = \gamma + (\frac{i-1}{2})\delta$ if i is odd, for $i = 1, 2, ..., a_1$, (without loss of generality assume that a_1 is even) and using the orthogonal decomposition for $\gamma \in \frac{\Gamma}{2}$ as $\gamma = \beta + (l + v(\beta))\delta$, where $\beta \in H_\delta \equiv \{x \in R^d : \langle x, \delta \rangle = 0\}$, $l \in Z$, $v \in [0,1)$ we can write the matrix $D(\gamma, \delta)$ as :

$$D(\gamma, \delta) = |\beta|^2 I + E(\gamma, \delta),$$

where $E(\gamma, \delta)$ is:

$$E(\gamma,\delta) = \begin{bmatrix} (l+v)^2|\delta|^2 & q_\delta & q_{-\delta} & \cdot\ \cdot & q_{\frac{a_1}{2}\delta} \\ q_{-\delta} & (l-1+v)^2|\delta|^2 & q_{-2\delta} & \cdot\ \cdot & \cdot \\ q_\delta & q_{-2\delta} & (l+1+v)^2|\delta|^2 & \cdot\ \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot\ \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot\ \cdot & \cdot \\ q_{\frac{-a_1}{2}\delta} & \cdot & \cdot & \cdot\ \cdot & (l-\frac{a_1}{2}+v)^2|\delta|^2 \end{bmatrix}$$

Denote $n_k = -\frac{k}{2}$ if k is even, $n_k = \frac{k-1}{2}$ if k is odd. The system $\{e^{i(n_k+v)s} : k = 1, 2, ...\}$ is a basis in $L_2[0, 2\pi]$. Let $T(\gamma, \delta) \equiv T(Q(s), \beta)$ be the operator in ℓ_2 corresponding to the Sturm-Liouville operator T, generated by

$$-|\delta|^2 y''(s) + Q(s)y(s) = \mu y(s) \tag{3.99}$$

$$y(s + 2\pi) = e^{i2\pi v(\beta)} y(s),$$

where $Q(s) = \sum_{k=1}^{\infty} q_{n_k\delta} e^{in_k s}$, and $q_{n_k\delta} = (q(x), \sum_{\alpha \in A_{n_k\delta}} e^{i\langle\alpha,x\rangle})$, $s = \langle x, \delta \rangle$. It means that $T(\gamma, \delta)$ is the infinite matrix $(Te^{i(l+n_k+v)s}, e^{i(l+n_m+v)s})$, $k, m = 1, 2, ...$

Theorem 3.2.4. *For every eigenvalue μ_i of the Sturm-Liouville operator $T(\gamma, \delta)$, there exists an eigenvalue $\widetilde{\mu}_i$ of the matrix $E(\gamma, \delta)$ such that*

$$\mu_i = \widetilde{\mu}_i + O(\rho^{-\frac{3}{4}\alpha_2}).$$

proof: Decompose the infinite matrix $T(\gamma, \delta)$ as $T(\gamma, \delta) = \widetilde{A} + \widetilde{B}$, where the matrix $\widetilde{A} = (\widetilde{a}_{ij})$ is defined by $\widetilde{a}_{ij} = 0$ if $i \neq j$, $\widetilde{a}_{ii} = |(l - \frac{i}{2} + v)\delta|^2$ if i is even and $\widetilde{a}_{ii} = |(l + \frac{i-1}{2} + v)\delta|^2$ if i is odd, for $i, j = 1, 2, ...$ and $\widetilde{B} = T(\gamma, \delta) - \widetilde{A}$
Let μ_i be an eigenvalue of $T(\gamma, \delta)$, and $u_i = (u_{i1}, u_{i2}, u_{i3}, ...)$ be the corresponding normalized eigenvector, that is $Tu_i = \mu_i u_i$. Denote by $e_j = (0, 0, ..., 1, 0, ...)$ the

j-th eigenvector of the matrix $\widetilde{A}$. Then, the corresponding j-th eigenvalue of $\widetilde{A}$ is $|(j'+v)\delta|^2$, that is $\widetilde{A}e_j = |(j'+v)\delta|^2 e_j$, where $j' = l - \frac{j}{2}$ if j is even, $j' = l + \frac{j-1}{2}$ if j is odd, for $j = 1, 2, ...$

One can easily verify that

$$(\mu_i - |(j'+v)\delta|^2)u_{ij} = \langle u_i, \widetilde{B}e_j\rangle \tag{3.100}$$

and using the orthogonal decomposition $\widetilde{B}e_j = \sum_{k=1}^{\infty}\langle \widetilde{B}e_j, e_k\rangle e_k$, we get

$$(\mu_i - |(j'+v)\delta|^2)u_{ij} = \sum_{k=1}^{\infty}\langle \widetilde{B}e_j, e_k\rangle u_{ik}$$

and since $\langle \widetilde{B}e_j, e_k\rangle = q_{(n_k-n_j)\delta}$,

$$(\mu_i - |(j'+v)\delta|^2)u_{ij} - \sum_{k=1}^{a_1} q_{(n_k-n_j)\delta}u_{ik} = \sum_{k=a_1+1}^{\infty} q_{(n_k-n_j)\delta}u_{ik}. \tag{3.101}$$

Now take any eigenvalue μ_i of $T(\gamma, \delta)$, satisfying $|\mu_i - |(i'+v)\delta|^2| < sup|Q(s)|$ for $i = 1, 2, ..., \frac{a_1}{2}$, where $i' = l - \frac{i}{2}$ if i is even, $i' = l + \frac{i-1}{2}$ if i is odd. The relations $\gamma \in V_\delta(\rho^{\alpha_1})$ and $\gamma = \beta + (l+v)\delta$, $\langle \beta, \delta\rangle = 0$ imply

$$|2\langle\gamma, \delta\rangle + |\delta|^2| = |(l+v)|\delta|^2 + |\delta|^2| < \rho^{\alpha_1}, \quad |l| < c_{12}\rho^{\alpha_1}.$$

Therefore using the definition of i' and j', we have

$$|(i'+v)\delta| < \frac{|a_1\delta|}{4} + c_{23}\rho^{\alpha_1},$$

for $i = 1, 2, ...\frac{a_1}{2}$ and

$$|(j'+v)\delta| > \frac{|a_1\delta|}{2} - c_{24}\rho^{\alpha_1},$$

for $j > a_1$. Since $|a_1| > c_{25}\rho^{\frac{\alpha_2}{2}}$ and $\alpha_2 > 2\alpha_1$, we have

$$||(i'+v)\delta|^2 - |(j'+v)\delta|^2| > c_{26}\rho^{\alpha_2}, \tag{3.102}$$

for $i \leq \frac{a_1}{2}$, $j > a_1$, which implies

$$\begin{aligned} &|\mu_i - |(j'+v)\delta|^2| \\ &= ||\mu_i - |(i'+v)\delta|^2| - ||(j'+v)\delta|^2| - |(i'+v)\delta|^2|| > c_{17}\rho^{\alpha_2}, \end{aligned} \tag{3.103}$$

for $i = 1, 2, \ldots \frac{a_1}{2}$, $j > a_1$.

Thus,

$$|\sum_{k=a_1+1}^{\infty} q_{(n_k-n_j)\delta} u_{ik}| = \sum_{k=a_1+1}^{\infty} |q_{(n_k-n_j)\delta}||\frac{\langle u_i, \widetilde{B}e_k\rangle}{\mu_i - |(k+v)\delta|^2}|$$
$$\leq \sum_{k=a_1+1}^{\infty} |q_{(n_k-n_j)\delta}|\frac{\|u_i\|\|\widetilde{B}\|\|e_k\|}{|\mu_i - |(k+v)\delta|^2|} \leq M\rho^{-\alpha_2} \sum_{k=a_1+1}^{\infty} |q_{(n_k-n_j)\delta}| \quad (3.104)$$
$$\leq c_{18}\rho^{-\alpha_2},$$

since $\|\widetilde{B}\| \leq M$. Indeed, $\widetilde{B}$ corresponds to the operator $Q : y \to Q(s)y$ in $L_2[0, 2\pi]$, which has norm $sup|Q(s)| \leq M$. Therefore writing the equation (3.101) for all $j = 1, 2, \ldots, a_1$, and using (4.167) we get the following system

$$(E(\gamma,\delta) - \mu_i I)[u_{i1}, u_{i2}, \ldots, u_{ia_1}] = [O(\rho^{-\alpha_2}), O(\rho^{-\alpha_2}), \ldots, O(\rho^{-\alpha_2})]. \quad (3.105)$$

Using that, $u_i = \sum_{j=1}^{\infty} u_{ij}e_j$, the formula (3.100) and (3.103), we have

$$\sum_{j=a_1+1}^{\infty} |u_{ij}|^2 = \sum_{j=a_1+1}^{\infty} |\frac{\langle u_i, \widetilde{B}e_j\rangle}{\mu_i - |(j'+v)\delta|^2}|^2 = O(\rho^{-2\alpha_2})$$

and thus ,

$$\sum_{j=1}^{a_1} |u_{ij}|^2 = 1 - O(\rho^{-2\alpha_2}).$$

Taking the norm of the vector (see (4.168))

$$[u_{i1}, u_{i2}, \ldots, u_{ia_1}] = (E(\gamma,\delta) - \mu_i I)^{-1}[O(\rho^{-\alpha_2}), \ldots, O(\rho^{-\alpha_2})],$$

we get

$$\sqrt{1 - O(\rho^{-2\alpha_2})} = \|(E(\gamma,\delta) - \mu_i I)^{-1}\| O(\sqrt{a_1}\rho^{-\alpha_2})$$

or

$$\min_j |\mu_i - \widetilde{\mu}_j| = \frac{O(\sqrt{a_1}\rho^{-\alpha_2})}{\sqrt{1 - O(\rho^{-2\alpha_2})}} = O(\rho^{-\frac{3}{4}\alpha_2}),$$

where the minimum is taken over all eigenvalues $\widetilde{\mu}_j$ of the matrix $E(\gamma,\delta)$. Thus, the result follows. □

Theorem 3.2.5. *(Main result) For every $\beta \in H_\delta$, $|\beta| \sim \rho$ and for every eigenvalue $\mu_i(v(\beta))$ of the Sturm-Liouville operator $T(\gamma,\delta)$, there is an eigenvalue Υ_N of the operator $L_N(q(x))$ satisfying*

$$\Upsilon_N = |\beta|^2 + \mu_i + O(\rho^{-\frac{1}{2}\alpha_2}).$$

proof: From Theorem 3.2.4 and the definition of $E(\gamma, \delta)$, there exists an eigenvalue $\widetilde{\lambda}_{j(i)}$ of the matrix $D(\gamma, \delta)$, where γ has a decomposition $\gamma = \beta + (j + v(\beta))\delta$, satisfying $\widetilde{\lambda}_{j(i)} = |\beta|^2 + \mu_i + O(\rho^{-\frac{3}{4}\alpha_2})$. Therefore, the result follows from Theorem 3.2.3 and Theorem 3.1.3. □

CHAPTER FOUR
EIGENVALUES IN A SPECIAL SINGLE RESONANCE DOMAIN

In chapter three, we obtained the asymptotic formulas for the resonance eigenvalues $|\gamma|^2$, when $\gamma \in V_b(\rho^{\alpha_1})$, under the condition
$\gamma \notin V_{e_k}(\rho^{\alpha_1})$, for all $k = 1, 2, ..., d$, where $e_1 = (\frac{\pi}{a_1}, 0, ..., 0), e_2 = (0, \frac{\pi}{a_2}, 0, ..., 0), ...,$
$e_d = (0, ..., 0, \frac{\pi}{a_d})$.

In this chapter, we investigate the perturbation of the eigenvalue $|\gamma|^2$, when $\gamma \in V_\delta(\rho^{\alpha_1}) \setminus E_2 \equiv V'_\delta(\rho^{\alpha_1})$, where $\delta = e_i$, $i = 1, 2, ..., d$.

4.1 On the Unperturbed Operator $L_N(q^{e_i}(x))$

Now let

$$e_i = (0, ..., 0, \frac{\pi}{a_i}, 0, ..., 0) \in \frac{\Gamma}{2},$$

for $i = 1, 2, ..., d$ and

$$H_{e_i} = \{x \in \mathbb{R} : \langle x, e_i \rangle = 0\}$$
$$= \{(x_1, ..., x_{i-1}, 0, x_{i+1}, ..., x_d) : x_k \in \mathbb{R}, k = 1, 2, ..., d\}$$

be the hyperplane which is orthogonal to e_i. Then, we define the following sets:

$$\Omega_{e_i} = \Omega \cap H_{e_i}$$
$$= \{(m_1a_1, ..., m_{i-1}a_{i-1}, 0, m_{i+1}a_{i+1}, ..., m_da_d) : m_k \in \mathbb{Z}, \forall k = 1, 2, ..., d\},$$
$$\Gamma_{e_i} = \{(\frac{n_1\pi}{a_1}, ..., \frac{n_{i-1}\pi}{a_{i-1}}, 0, \frac{n_{i+1}\pi}{a_{i+1}}, ..., \frac{n_d\pi}{a_d}) : n_k \in \mathbb{Z}, \forall k = 1, 2, ..., d\}.$$

Clearly, for all $\gamma = (\frac{n_1\pi}{a_1}, ..., \frac{n_d\pi}{a_d}) \in \frac{\Gamma}{2}$, we have the following decompositions

$$\gamma = n_ie_i + \beta, \quad \beta \in \Gamma_{e_i}, \quad n_i \in \mathbb{Z},$$

or equivalently, redenoting n_i by j, we have

$$\gamma = je_i + \beta, \quad \beta \in \Gamma_{e_i}, \quad j \in \mathbb{Z} \tag{4.106}$$

and

$$v_\gamma = \cos\frac{n_1\pi}{a_1}x_1 ... \cos\frac{n_i\pi}{a_i}x_i ... \cos\frac{n_d\pi}{a_d}x_d$$
$$= \cos\frac{j\pi}{a_i}x_iv_\beta.$$

We write the decomposition (2.37) of $q(x)$ as follows

$$q(x) = \sum_{\gamma' \in \frac{\Gamma}{2}} q_{\gamma'}v_{\gamma'}(x) = q^{e_i}(x) + \sum_{\gamma' \in \frac{\Gamma}{2} \backslash e_iR} q_{\gamma'}v_{\gamma'}(x), \tag{4.107}$$

where $q^{e_i}(x) = \sum_{n\in\mathbb{Z}} q_{ne_i} \cos n\frac{\pi}{a_i}x_i \equiv P(s)$, $s = \langle e_i, x\rangle = \frac{\pi}{a_i}x_i$.

We consider the operator $L_N(q^{e_i}(x))$, as the unperturbed operator, defined by the differential expression

$$Lu = -\Delta u + q^{e_i}(x)u \tag{4.108}$$

with the Neumann boundary condition

$$\frac{\partial u}{\partial n}|_{\partial F} = 0. \tag{4.109}$$

Lemma 4.1.1. *The eigenvalues and the corresponding eigenfunctions of $L_N(q^{e_i}(x))$ are*

$$\lambda_{j_{e_i},\beta} \equiv \mu_{j_{e_i}} + |\beta|^2$$

and

$$\Theta_{j_{e_i},\beta} \equiv \varphi_{j_{e_i}}(s)\prod_{k\neq i}\cos\beta^k x_k = \varphi_{j_{e_i}}(s)v_\beta(x),$$

respectively, where $\beta = (\beta^1,...,\beta^{i-1},0,\beta^{i+1}...,\beta^d) \in \Gamma_\delta$, $\mu_{j_{e_i}}$ *and* $\varphi_{j_{e_i}}(s)$ *are the eigenvalues and the corresponding eigenfunctions of the operator* $T_{e_i}(P(s))$ *defined by the differential expression*

$$Ty(s) = -|\frac{\pi}{a_i}|^2 y''(s) + P(s)y(s) \tag{4.110}$$

in $[0,\pi]$, *with the Neumann boundary conditions*

$$y'(0) = y'(\pi) = 0. \tag{4.111}$$

Proof. Using the separation of variables method, we seek the solution $u(x)$ of

$$-(\frac{\partial^2 u(x)}{\partial x_1^2} + ... + \frac{\partial^2 u(x)}{\partial x_i^2} + ... + \frac{\partial^2 u(x)}{\partial x_d^2}) + q^{e_i}(x)u(x) = \lambda u(x), \tag{4.112}$$

satisfying the boundary condition (4.109) which coincides with the following boundary conditions

$$\frac{\partial u(x)}{\partial x_k}|_{x_k=0} = \frac{\partial u(x)}{\partial x_k}|_{x_k=a_k} = 0, \quad \forall k = 1,2,...,d, \tag{4.113}$$

(see the proof of Lemma2.1.1) in the form $u(x) = u_1(x_1)...u_i(x_i)...u_d(x_d)$. Substituting this into (4.112) and dividing both sides of the obtained equation by $u_1(x_1)...u_i(x_i)...u_d(x_d)$, we have

$$-\frac{u_1''(x_1)}{u_1(x_1)} - ... + [-\frac{u_i''(x_i)}{u_i(x_i)} + P(s)u_i(x_i)] - ... - \frac{u_d''(x_d)}{u_d(x_d)} = \lambda$$

this together with (4.113), give the following eigenvalue problems

$$-u_k''(x_k) = \lambda_k u_k(x_k), \quad u_k'(0) = u_k'(a_k) = 0, \quad k \neq i \tag{4.114}$$

and

$$-u_i''(x_i) + P(s)u_i(x_i) = \lambda_i u_i(x_i), \quad u_i'(0) = u_i'(a_i) = 0,$$

or equivalently

$$-|\frac{\pi}{a_i}|^2 u_i''(s) + P(s)u_i(s) = \lambda_i u_i(s), \quad u_i'(0) = u_i'(\pi) = 0, \tag{4.115}$$

where $s = \langle x, e_i\rangle = \frac{\pi}{a_i}x_i$.

Thus, the eigenvalues and the corresponding eigenfunctions of the problems (4.114) are

$$\lambda_{k,n_k} = |\frac{n_k \pi}{a_i}|^2, \quad u_{k,n_k}(x_k) = \cos\frac{n_k \pi}{a_k}(x_k),$$

for $k \neq i$, respectively. Denoting the eigenvalues and the corresponding eigenfunctions of the problem (4.115), i.e. of the operator $T_{e_i}(P(s))$, by μ_{je_i} and $\varphi_{je_i}(s)$, we get the eigenvalues and the corresponding eigenfunctions of $L_N(q^{e_i})$ as

$$\lambda_{je_i,\beta} \equiv |\frac{n_1\pi}{a_1}|^2 + ... + \mu_{je_i} + ... + |\frac{n_d\pi}{a_d}|^2 = \mu_{je_i} + |\beta|^2,$$

and

$$\Theta_{je_i,\beta} \equiv \cos\frac{n_1\pi}{a_1}(x_k) + ... + \varphi_{je_i}(s) + ... + \cos\frac{n_d\pi}{a_d}(x_k) = \varphi_{je_i}(s) + v_\beta(x),$$

respectively, where $\beta = (\frac{n_1\pi}{a_1}, ..., \frac{n_{i-1}\pi}{a_{i-1}}, 0, \frac{n_{i+1}\pi}{a_{i+1}}, ..., \frac{n_d\pi}{a_d}) \in \Gamma_{e_i}$. □

For the sake of simplicity, let us redenote μ_{je_i} by μ_j and $\varphi_{je_i}(s)$ by $\varphi_j(s)$, i.e.,

$$\mu_j \equiv \mu_{je_i}, \quad \varphi_j(s) \equiv \varphi_{je_i}(s).$$

The eigenvalues and the corresponding eigenfunctions of $T_{e_i}(0)$ are $|je_i|^2$ and $\cos js$, $j = 0, 1, 2, ...$ respectively. It is well known that the eigenvalue μ_j of $T_{e_i}(P(s))$ satisfying $|\mu_j - |je_i|^2| < \sup P(s)$, together with the corresponding eigenfunction $\varphi_j(s)$ satisfy the following relations

$$\mu_j = |je_i|^2 + O(\frac{1}{|je_i|}), \quad \varphi_j(s) = \cos js + O(\frac{1}{|je_i|}). \tag{4.116}$$

The main result of this chapter is that we find connection between the eigenvalues of the Schrödinger operator $L_N(q)$ corresponding to the single resonance domain $V_{e_i}(\rho^{\alpha_1}) \setminus E_2$ and the eigenvalues of the Sturm-Liouville operator $T_{e_i}(P(s))$.

Lemma 4.1.2. *Let* $\gamma = je_i + \beta \in V_{e_i}(\rho^{\alpha_1}) \setminus E_2$ *then*

$$|j| < r_1, \quad r_1 \equiv \rho^{\alpha_1}|e_i|^{-2} + 1. \tag{4.117}$$

Proof. $\gamma = je_i + \beta \in V_{e_i}(\rho^{\alpha_1}) \setminus E_2$, definition of $V_{e_i}(\rho^{\alpha_1})$ and Pythagorean relation ($e_i \perp \beta$) imply

$$||\gamma|^2 - |\gamma + e_i|^2| = ||je_i + \beta|^2 - |(j+1)e_i + \beta|^2|$$
$$= ||je_i|^2 + |\beta|^2 - |(j+1)e_i|^2 - |\beta|^2| = |2j+1||e_i|^2 < \rho^{\alpha_1},$$

which implies

$$|j| < \frac{\rho^{\alpha_1}}{|e_i|^2} + 1.$$

□

Note that, since $\gamma = je_i + \beta \in V_{e_i}(\rho^{\alpha_1}) \setminus E_2$, we have $\beta \notin V_{e_k}(\rho^{\alpha_1})$, for all $k \neq i$. Thus this relation implies that

$$|\beta^k| > \frac{1}{3}\rho^{\alpha_1}, \quad \forall k \neq i. \tag{4.118}$$

By (4.116), to the eigenvalue $|\gamma|^2 = |\beta|^2 + |je_i|^2$ of $L_N(0)$, corresponds the eigenvalue $|\beta|^2 + \mu_j$ of $L_N(q^{e_i})$. Now we prove that there is an eigenvalue Υ_N of $L_N(q)$ which is closed to the eigenvalue $|\beta|^2 + \mu_j$ of $L_N(q^{e_i})$. For this we use the following binding formula for $L_N(q)$ and $L_N(q^{e_i})$

$$(\Upsilon_N - \lambda_{j,\beta})(\Phi_N, \Theta_{j,\beta}) = (\Phi_N, (q(x) - q^{e_i}(x))\Theta_{j,\beta}). \tag{4.119}$$

Now as in the non-resonance case, we decompose $(q(x) - q^{e_i}(x))\Theta_{j,\beta}$ by $\Theta_{j',\beta'}$ and put this decomposition into (4.119).

Now, we find this decomposition. Writing (4.106) for every $\gamma_1 \in \Gamma(\rho^\alpha)$ and using (2.37), we have

$$\gamma_1 = n_1 e_i + \beta_1, \quad v_{\gamma_1}(x) = \prod_{i=1}^{d} \cos n_i \frac{\pi}{a_i} x_i = \cos n_1 s v_{\beta_1}(x),$$

$$q(x) - P(s) = \sum_{(\beta_1, n_1) \in \Gamma'(\rho^\alpha)} d(\beta_1, n_1) \cos n_1 s v_{\beta_1}(x) + O(\rho^{-p\alpha}), \tag{4.120}$$

where $\beta_1 = (\beta_1^1, ..., \beta_1^{i-1}, 0, \beta_1^{i+1}, ..., \beta_1^d) \in \Gamma_{e_i}$,
$d(\beta_1, n_1) = \int_F q(x) \cos n_1 s v_{\beta_1}(x)$ and

$\Gamma'(\rho^\alpha) = \{(\beta_1, n_1) : \beta_1 \in \Gamma_{e_i} \setminus \{0\}, n_1 \in \mathbb{Z}, n_1 e_i + \beta_1 \in \Gamma(\rho^\alpha)\}$. Clearly for $(\beta_1, n_1) \in \Gamma'(p\rho^\alpha)$, we have $|n_1 e_i + \beta_1| < p\rho^\alpha$ and since β_1 is orthogonal to e_i,

$$|\beta_1| < p\rho^\alpha, |n_1| < p\rho^\alpha, |n_1| < \frac{1}{2}r_1, \tag{4.121}$$

(see 4.117).

Lemma 4.1.3.

$$\sum_{(\beta_1,n_1)\in\Gamma'(\rho^\alpha)} d(\beta_1, n_1)\cos n_1 s v_{\beta_1}(x) v_{\beta'}(x) = \sum_{(\beta_1,n_1)\in\Gamma'(\rho^\alpha)} d(\beta_1, n_1)\cos n_1 s v_{\beta_1+\beta'}(x),$$

for all $\beta' \in \Gamma_{e_i}$, $\beta' \notin V_{e_k}(\rho^{\alpha_1})$, $k \neq i$.

Proof. We may consider the vectors β_1, β' of Γ_{e_i} as the vectors of $\mathbb{R}^{d-1}$, since the ith components of β_1 and β' are zero.Also note that Lemma2.2.5 is proved for arbitrary dimension d, hence it is true for the $(d-1)$-dimensional case under the condition $\beta' \notin V_{e_k}(\rho^{\alpha_1})$, $k \neq i$. Thus

$$v_\beta(x) v_{\beta'}(x) = \frac{1}{A_\beta} \sum_{\alpha' \in A_\beta} v_{\beta'+\alpha'}(x).$$

Arguing as the proof of Lemma2.3.2, instead of (2.43), we have

$$\sum_{\alpha' \in A_\beta} d(\alpha', n_1)\cos n_1 s v_{\alpha'}(x) v_{\beta'}(x) = \sum_{\alpha' \in A_\beta} d(\alpha', n_1)\cos n_1 s v_{\alpha'+\beta'}(x), \tag{4.122}$$

for all $n_1 \in \mathbb{Z}$, since $d(\alpha', n_1)\cos n_1 s = d(\beta, n_1)\cos n_1 s$, for all $\alpha' \in A_\beta$, $n_1 \in \mathbb{Z}$.

Clearly, there exist vectors $\beta_1, \beta_2, ..., \beta_m \in \Gamma_{e_i}$ such that

$$\Gamma'(\rho^\alpha) \subseteq (\bigcup_{j=1}^m A_{\beta_j}) \times \{n_1 \in \mathbb{Z} : |n_1| < \frac{1}{2}r_1\}. \tag{4.123}$$

In (4.122), replacing β by β_j, for $j = 1, 2, ..., m$, summing all obtained m equations, we get

$$\sum_{j=1}^m \sum_{\alpha' \in A_\beta} d(\alpha', n_1)\cos n_1 s v_{\alpha'}(x) v_{\beta'}(x) = \sum_{j=1}^m \sum_{\alpha' \in A_\beta} d(\alpha', n_1)\cos n_1 s v_{\alpha'+\beta'}(x),$$

by (4.123), which is equivalent to

$$\sum_{(\beta_1,n_1)\in\Gamma'(\rho^\alpha)} d(\beta_1, n_1)\cos n_1 s v_{\beta_1}(x) v_{\beta'}(x) = \sum_{(\beta_1,n_1)\in\Gamma'(\rho^\alpha)} d(\beta_1, n_1)\cos n_1 s v_{\beta_1+\beta'}(x).$$

□

Now multiplying the both sides of the equation (4.120) by $\Theta_{j',\beta'}$, where β' satisfies (4.118) and using Lemma4.1.3 , we get

$$(q(x)-P(s))\Theta_{j',\beta'} = \sum_{(\beta_1,n_1)\in\Gamma'(\rho^{\alpha})} d(\beta_1,n_1)\cos n_1 s v_{\beta_1}(x)\Theta_{j',\beta'} + O(\rho^{-p\alpha})$$
$$= \sum_{(\beta_1,n_1)\in\Gamma'(\rho^{\alpha})} d(\beta_1,n_1)\cos n_1 s v_{\beta_1}(x) v_{\beta'}(x)\varphi_{j'}(s) + O(\rho^{-p\alpha})$$
$$= \sum_{(\beta_1,n_1)\in\Gamma'(\rho^{\alpha})} d(\beta_1,n_1)\cos n_1 s \varphi_{j'}(s) v_{\beta_1+\beta'}(x) + O(\rho^{-p\alpha}) \quad (4.124)$$

To decompose the right hand side of (4.119) by $\Theta_{j',\beta'}$, we use the following lemmas:

Lemma 4.1.4. *Let r be a number no less than r_1, i.e. $r \geq r_1$, and j, m be integers satisfying $|j|+1<r, |m|\geq 2r$. Then*

$$(\varphi_j(s), \cos ms) = O(\frac{1}{|me_i|^{l-1}}), \quad (4.125)$$

$$\varphi_j(s) = \sum_{|m|<2r} (\varphi_j(s), \cos ms)\cos ms + O(\frac{1}{\rho^{(l-2)\alpha}}). \quad (4.126)$$

proof. We use the following binding formula for $T_{e_i}(P(s))$ and $T_{e_i}(0)$

$$(\mu_j - |me_i|^2)(\varphi_j(s), \cos ms) = (\varphi_j(s)P(s), \cos ms) \quad (4.127)$$

and the obvious decomposition (see (2.38))

$$P(s) = \sum_{|l_1 e_i| < \frac{|me_i|}{2l}} q_{l_1 e_i}\cos l_1 s + O(|me_i|^{-(l-1)}). \quad (4.128)$$

Putting (4.128) into (4.127), we get

$$(\mu_j - |me_i|^2)(\varphi_j(s), \cos ms) = (\varphi_j(s) \sum_{|l_1 e_i| < \frac{|me_i|}{2l}} q_{l_1 e_i}\cos l_1 s, \cos ms) + O(|me_i|^{-(l-1)})$$
$$= \sum_{|l_1 e_i| < \frac{|me_i|}{2l}} q_{l_1 e_i}(\varphi_j(s), \cos l_1 s . \cos ms) + O(|me_i|^{-(l-1)})$$
$$= \sum_{|l_1 e_i| < \frac{|me_i|}{2l}} q_{l_1 e_i}(\varphi_j(s), \frac{1}{2}[\cos(m+l_1)s + \cos(m-l_1)s]) + O(|me_i|^{-(l-1)})$$
$$= \sum_{|l_1 e_i| < \frac{|me_i|}{2l}} q_{l_1 e_i}(\varphi_j(s), \cos(m-l_1)s) + O(|me_i|^{-(l-1)},$$

again using (4.127), we get

$$(\mu_j - |me_i|^2)(\varphi_j(s), \cos ms) = \sum_{|l_1 e_i| < \frac{|me_i|}{2l}} q_{l_1 e_i} \frac{(\varphi_j(s)P(s), \cos(m-l_1)s)}{\mu_j - |(m-l_1)e_i|^2} + O(|me_i|^{-(l-1)}).$$

Putting (4.128) into the last equation, we obtain

$$(\mu_j - |me_i|^2)(\varphi_j(s), \cos ms) = \sum_{\substack{|l_1 e_i| < \frac{|me_i|}{2l}, \\ |l_2 e_i| < \frac{|me_i|}{2l}}} q_{l_1 e_i} q_{l_2 e_i} \frac{(\varphi_j(s), \cos(m-l_1-l_2)s)}{\mu_j - |(m-l_1)e_i|^2}$$

$$+O(|me_i|^{-(l-1)}).$$

In this way, iterating $k = [\frac{l}{2}]$ times and dividing both sides of the obtained equation by $\mu_j - |me_i|^2$, we have

$$(\varphi_j(s), \cos ms) = \sum_{\substack{|l_1 e_i| < \frac{|me_i|}{2l}, \dots, \\ |l_k e_i| < \frac{|me_i|}{2l}}} q_{l_1 e_i} \dots q_{l_k e_i} \frac{(\varphi_j(s), \cos(m-l_1-\dots-l_k)s)}{\prod_{t=0}^{k-1}(\mu_j - |(m-l_1-\dots-l_t)e_i|^2}$$

$$+O(|me_i|^{-(l-1)}) \tag{4.129}$$

where the integers $m, l_1, \dots, l_k$ satisfy the conditions
$|l_i| < \frac{|m|}{2l}, i = 1, 2, \dots, k, \quad |j| + 1 < \frac{|m|}{2}$ (see assumption of the lemma). These conditions imply that $||m - l_1 - \dots - l_t| - |j|| > \frac{|m|}{5}$. This together with (4.116) give

$$\frac{1}{|\mu_j - |(m-l_1-\dots-l_t)e_i|^2|} = \frac{1}{||je_i|^2 + O(\frac{1}{|je_i|}) - |(m-l_1-\dots-l_t)e_i|^2|} = O(|me_i|^{-2}), \tag{4.130}$$

for $t = 0, 1, \dots, k-1$. Hence by (4.129),(4.130) and (2.40), we have

$$|(\varphi_j(s), \cos ms)| \leq O(|me_i|^{-(l-1)})$$

$$+ \sum_{\substack{|l_1 e_i| < \frac{|me_i|}{2l}, \dots, \\ |l_k e_i| < \frac{|me_i|}{2l}}} \frac{|q_{l_1 e_i} \dots q_{l_k e_i}| ||\varphi_j(s)|| || \cos(m-l_1-\dots-l_k)s||}{\prod_{t=0}^{k-1} |\mu_j - |(m-l_1-\dots-l_t)e_i|^2|}$$

$$= O(|me_i|^{-(l-1)}).$$

(4.125) is proved. To prove (4.126), for j satisfying $|j| + 1 < r$, we write the fourier series of $\varphi_j(s)$ with respect to the basis $\{\cos ms : m \in \mathbb{Z}\}$, i.e.,

$$\varphi_j(s) = \sum_{m \in \mathbb{Z}} (\varphi_j(s), \cos ms) \cos ms$$

$$= \sum_{|m| < 2r} (\varphi_j(s), \cos ms) \cos ms + \sum_{m \geq 2r} (\varphi_j(s), \cos ms) \cos ms.$$

By (4.125), for $|m| \geq 2r$ and $|j|+1<r$, $(\varphi_j(s), \cos ms) = O(|me_i|^{-(l-1)})$. Using this relation, we get

$$\varphi_j(s) = \sum_{|m|<2r} (\varphi_j(s), \cos ms) \cos ms + O(|me_i|^{-(l-2)}),$$

since $|me_i| > \rho^\alpha$, (4.126) is proved.□

Lemma 4.1.5. *Let r be a number no less than r_1, i.e. $r \geq r_1$, and j be integer satisfying $|j|+1<r$ Then*

$$\cos n_1 s \varphi_j(s) = \sum_{|j_1|<6r} a(n_1, j, j+j_1) \varphi_{j+j_1}(s) + O(r^{-(l-3)}), \tag{4.131}$$

for $(n_1, \beta_1) \in \Gamma'(p_1\rho^\alpha)$,
where $a(n_1, j, j+j_1) = (\cos n_1 s \varphi_j(s), \varphi_{j+j_1}(s))$.

proof. Consider the fourier series of $\cos n_1 s \varphi_j(s)$ with respect to the basis $\{\varphi_{j+j_1}(s) : j_1 \in \mathbb{Z}\}$

$$\cos n_1 s \varphi_j(s) = \sum_{j_1 \in \mathbb{Z}} (\cos n_1 s \varphi_j(s), \varphi_{j+j_1}(s)) \varphi_{j+j_1}(s)$$
$$= \sum_{|j_1|<6r} a(n_1, j, j+j_1) \varphi_{j+j_1}(s) + \sum_{|j_1|\geq 6r} a(n_1, j, j+j_1) \varphi_{j+j_1}(s).$$

To prove (4.131), we must prove

$$\sum_{|j_1|\geq 6r} |a(n_1, j, j+j_1)| = O(r^{-(l-3)}) \tag{4.132}$$

or equivalently

$$|a(n_1, j, j+j_1)| = O(r^{-(l-2)}), \quad \forall j_1 : |j_1| \geq 6r. \tag{4.133}$$

Decomposing $\varphi_j(s)$ by $\cos ms$, we have

$$\varphi_j(s) = \sum_{m \in \mathbb{Z}} (\varphi_j(s), \cos ms) \cos ms,$$

multiplying this decomposition by $\cos n_1 s$, we obtain

$$\cos n_1 s \varphi_j(s) = \sum_{m \in \mathbb{Z}} (\varphi_j(s), \cos ms) \cos ms \cos n_1 s,$$
$$= \sum_{m \in \mathbb{Z}} (\varphi_j(s), \cos ms) \frac{1}{2} [\cos(n_1+m)s + \cos(n_1-m)s]$$
$$= \sum_{m \in \mathbb{Z}} (\varphi_j(s), \cos ms) \cos(n_1+m)s. \tag{4.134}$$

Using (4.134) and the decomposition

$$\varphi_{j+j_1}(s) = \sum_{k\in\mathbb{Z}} (\varphi_{j+j_1}(s), \cos ks) \cos ks,$$

we get

$$\begin{aligned} a(n_1, j, j+j_1) &= (\cos n_1 s \varphi_j(s), \varphi_{j+j_1}(s)) \\ = (\sum_{m\in\mathbb{Z}} (\varphi_j(s), \cos ms) \cos(n_1+m)s, &\sum_{k\in\mathbb{Z}} (\varphi_{j+j_1}(s), \cos ks) \cos ks) \\ = \sum_{m,k\in\mathbb{Z}} (\varphi_j(s), \cos ms) \overline{(\varphi_{j+j_1}(s), \cos ks)} &(\cos(n_1+m)s, \cos ks) \\ = \sum_{k\in\mathbb{Z}} (\varphi_j(s), \cos(k-n_1)s) &\overline{(\varphi_{j+j_1}(s), \cos ks)}. \end{aligned} \tag{4.135}$$

Consider the two cases:

Case 1: $|k| > \frac{1}{2}|j_1| \geq 3r$. Since $|n_1| + 1 < r$ (see 4.121), $|k - n_1| > 2r$. Hence by (4.126)

$$\sum_{|k|>\frac{1}{2}|j_1|} |(\varphi_j(s), \cos(k-n_1)s)| = \sum_{|k-n_1|>2r} O(\frac{1}{|(k-n_1)e_i|^{l-1}}) = O(r^{-(l-2)}). \tag{4.136}$$

Case 2: $|k| \leq \frac{1}{2}|j_1|$. By assumptions $|j| < r$ and $|j_1| \geq 6r$, we have $|j_1+j| > 5r$. For any integers $l_1, ..., l_t$ satisfying $|l_i| < \frac{|j_1|}{3l}, i = 1, 2, ..., t$, where $t = [\frac{l}{2}]$, we have $|j_1 + j| - |k - l_1 - ... - l_t| > \frac{1}{6}|j_1|$.This together with (4.116) give

$$\frac{1}{|\mu_j - |(k - l_1 - ... - l_i)e_i|^2|} = O(|j_1 e_i|^{-2}), \tag{4.137}$$

for $i = 0, 1, ..., t$. Arguing as the proof of (4.126), we get

$$\sum_{|k|\leq\frac{1}{2}|j_1|} |(\varphi_{j_1+j}(s), \cos ks)| = O(r^{-(l-2)}). \tag{4.138}$$

Using (4.136) and (4.138), we have

$$\begin{aligned} |a(n_1, j, j+j_1)| \leq &\sum_{|k|\leq\frac{1}{2}|j_1|} |(\varphi_j(s), \cos(k-n_1)s)||\overline{(\varphi_{j+j_1}(s), \cos ks)}| \\ + &\sum_{|k|>\frac{1}{2}|j_1|} |(\varphi_j(s), \cos(k-n_1)s)||\overline{(\varphi_{j+j_1}(s), \cos ks)}| = O(r^{-(l-2)}). \end{aligned}$$

Lemma is proved.□

Now using (4.124) and (4.131), we get

$$(q(x)-P(s))\Theta_{j',\beta'} = O(\rho^{-p\alpha})$$
$$+\sum_{(\beta_1,n_1)\in\Gamma'(\rho^\alpha),|j_1|\le 6r} d(\beta_1,n_1)a(n_1,j',j'+j_1)\varphi_{j'+j_1}(s)v_{\beta_1+\beta'}(x),$$

for every j' satisfying $|j'|+1<r$, i.e.,

$$(q(x)-P(s))\Theta_{j',\beta'} = \sum_{(\beta_1,j_1)\in Q(\rho^\alpha,6r)} A(j',\beta',j'+j_1,\beta'+\beta_1)\Theta_{j'+j_1,\beta'+\beta_1}+O(\rho^{-p\alpha}), \tag{4.139}$$

where

$$Q(\rho^\alpha,6r)=\{(j,\beta):|je_i|<6r, 0<|\beta|<\rho^\alpha\},$$
$$A(j',\beta',j'+j_1,\beta'+\beta_1)=\sum_{n_1:(\beta_1,n_1)\in\Gamma'(\rho^\alpha)} d(\beta_1,n_1)a(n_1,j',j'+j_1).$$

We need to prove the following lemma

Lemma 4.1.6.

$$\sum_{(\beta_1,j_1)\in Q(\rho^\alpha,6r)} |A(j',\beta',j'+j_1,\beta\prime+\beta_1)|<c_{13}. \tag{4.140}$$

Proof. By definition of $A(j',\beta',j'+j_1,\beta\prime+\beta_1)$, $d(\beta_1,n_1)$, (2.40) and (4.135), we have

$$\sum_{(\beta_1,j_1)\in Q(\rho^\alpha,6r)} |A(j',\beta',j'+j_1,\beta\prime+\beta_1)| \le \sum_{(\beta_1,n_1)\in\Gamma'(\rho^\alpha)} |d(\beta_1,n_1)| \sum_{|j_1|\le 6r} |a(n_1,j',j'+j_1)|$$
$$\le M\sum_{k\in Z}|(\varphi_j(s),\cos(k-n_1)s)| \sum_{|j_1|\le 6r}|(\varphi_{j+j_1}(s),\cos ks)|.$$

Hence (4.140) follows from the inequalities

$$\sum_{k\in Z}|(\varphi_j(s),\cos(k-n_1)s)|<c_{14}, \quad \sum_{|j_1|\le 6r}|(\varphi_{j+j_1}(s),\cos ks)|<c_{14},$$

which can be obtained by (4.127). □

4.2 On the Iterability Condition for the Triples $(N, j', \beta'))$

The decomposition (4.139) together with the binding formula (4.119) for $L_N(q)$ and $L_N(q^{e_i})$ give

$$(\Upsilon_N - \lambda_{j',\beta'})(\Phi_N, \Theta_{j',\beta'}) = (\Phi_N, (q(x) - P(s))\Theta_{j',\beta'})$$
$$= \sum_{(\beta_1,j_1)\in Q(\rho^\alpha,6r)} A(j', \beta', j' + j_1, \beta\prime + \beta_1)(\Phi_N, \Theta_{j'+j_1,\beta'+\beta_1}) + O(\rho^{-p\alpha}). \quad (4.141)$$

If the condition (iterability condition for the triple (N, j', β'))

$$|\Upsilon_N - \lambda_{j',\beta'}| > c_{15} \quad (4.142)$$

holds then the formula (4.141) can be written in the following form

$$(\Phi_N, \Theta_{j',\beta'}) = \frac{(\Phi_N, (q(x) - P(s))\Theta_{j',\beta'})}{\Upsilon_N - \lambda_{j',\beta'}}$$
$$= \sum_{(\beta_1,j_1)\in Q(\rho^\alpha,6r)} \frac{A(j', \beta', j' + j_1, \beta' + \beta_1)(\Phi_N, \Theta_{j'+j_1,\beta'+\beta_1})}{\Upsilon_N - \lambda_{j',\beta'}} + O(\rho^{-p\alpha}). \quad (4.143)$$

Using (4.141) and (4.143), we are going to find Υ_N which is close to $\lambda_{j,\beta}$, where $|j| + 1 < r_1$. For this, first in (4.141) instead of j', β', taking j and β, hence instead of r taking r_1, we get

$$(\Upsilon_N - \lambda_{j,\beta})(\Phi_N, \Theta_{j,\beta}) = (\Phi_N, (q(x) - P(s))\Theta_{j,\beta})$$
$$= \sum_{(\beta_1,j_1)\in Q(\rho^\alpha,6r_1)} A(j, \beta, j + j_1, \beta + \beta_1)(\Phi_N, \Theta_{j+j_1,\beta+\beta_1}) + O(\rho^{-p\alpha}). \quad (4.144)$$

To iterate it by using (4.143) for $j' = j + j_1$ and $\beta' = \beta + \beta_1$, we will prove that there is a number N such that

$$|\Upsilon_N - \lambda_{j+j_1,\beta+\beta_1}| > \frac{1}{2}\rho^{\alpha_2}, \quad (4.145)$$

where $|j+j_1|+1 < 7r_1 \equiv r_2$, since $|j|+1 < r_1$ and $|j_1| < 6r_1$. Then $(j+j_1, \beta+\beta_1)$ satisfies (4.142). This means that, in formula (4.143), the pair (j', β') can be replaced by the pair $(j + j_1, \beta + \beta_1)$. Then, in (4.143) instead of r, taking r_2, we get

$$(\Phi_N, \Theta_{j+j_1,\beta+\beta_1}) = O(\rho^{-p\alpha})$$

$$+ \sum_{(\beta_2,j_2)\in Q(\rho^\alpha,6r_2)} \frac{A(j+j_1,\beta+\beta_1,j+j_1+j_2,\beta+\beta_1+\beta_2)(\Phi_N,\Theta_{j+j_1+j_2,\beta+\beta_1+\beta_2})}{\Upsilon_N - \lambda_{j+j_1,\beta+\beta_1}}.$$

Putting the above formula into (4.144), we obtain

$$(\Upsilon_N - \lambda_{j,\beta})c(N,j,\beta) = O(\rho^{-p\alpha}) + \sum_{\substack{(\beta_1,j_1)\in Q(\rho^\alpha,6r_1),\\ (\beta_2,j_2)\in Q(\rho^\alpha,6r_2)}} \frac{A(j,\beta,j^1,\beta^1)A(j^1,\beta^1,j^2,\beta^2)c(N,j^2,\beta^2)}{\Upsilon_N - \lambda_{j^1,\beta^1}}, \tag{4.146}$$

where $c(N,j,\beta) = (\Phi_N, \Theta_{j,\beta})$, $j^k = j+j_1+j_2+...+j_k$ and $\beta^k = \beta+\beta_1+\beta_2+...+\beta_k$. Thus we are going to find a number N such that $c(N,j,\beta)$ is not too small and the condition (4.145) is satisfied.

Lemma 4.2.1. *(a) Suppose $h_1(x), h_2(x), ..., h_m(x) \in L_2(F)$, where $m = [\frac{d}{2\alpha_2}]+1$. Then for every eigenvalue $\lambda_{j,\beta}$ of the operator $L_N(q^{e_i})$, there exists an eigenvalue Υ_N of $L_N(q)$ satisfying*

(i) $|\Upsilon_N - \lambda_{j,\beta}| < 2M$, where $M = \sup|q(x)|$,

(ii) $|c(N,j,\beta)| > \rho^{-q\alpha}$, where $q\alpha = [\frac{d}{2\alpha}+2]\alpha$,

(iii) $|c(N,j,\beta)|^2 > \frac{1}{2m}\sum_{i=1}^m |(\Phi_N, \frac{h_i}{\|h_i\|})|^2 > \frac{1}{2m}|(\Phi_N, \frac{h_i}{\|h_i\|})|^2, \quad \forall i = 1,2,...,m.$

(b) Let $\gamma = \beta + je_i \in V'_{e_i}(\alpha)$ and $(\beta_1, j_1) \in Q(\rho^\alpha, 6r_1), (\beta_k, j_k) \in Q(\rho^\alpha, 6r_k)$, where $r_k = 7r_{k-1}$ for $k = 2,3,...,p$. Then for $k = 1,2,3,...,p_1$, we have

$$|\lambda_{j,\beta} - \lambda_{j^k,\beta^k}| > \frac{3}{5}\rho^{\alpha_2}, \quad \forall \beta^k \neq \beta. \tag{4.147}$$

proof. (a) Let A, B, C be the set of indexes N satisfying (i), (ii), (iii), respectively. Using the binding formula (??) for $L_N(q)$ and $L_N(q^{e_i})$ and the Bessel's inequality, we get

$$\sum_{N\notin A} |c(N,j,\beta)|^2 = \sum_{N\notin A} |\frac{(\Phi_N, (q(x)-P(s))\Theta_{j,\beta})}{\Upsilon_N - \lambda_{j,\beta}}|^2$$
$$\le \frac{1}{4M^2}\|(q(x)-P(s))\Theta_{j,\beta}\|^2 \le \frac{1}{4}.$$

Hence by Parseval's relation, we obtain

$$\sum_{N\in A}|c(N,j,\beta)|^2>\frac{3}{4}.$$

Using the fact that the number of indexes N in A is less than $\rho^{d\alpha}$ and by the relation $N\notin B \quad \Rightarrow \quad |c(N,j,\beta)|<\rho^{-q\alpha}$, we have

$$\sum_{N\in A\setminus B}|c(N,j,\beta)|^2<\rho^{d\alpha}\rho^{-q\alpha}<\rho^{-\alpha}.$$

Since $A=(A\setminus B)\bigcup(A\bigcap B)$, by above inequalities, we get

$$\frac{3}{4}<\sum_{N\in A}|c(N,j,\beta)|^2=\sum_{N\in A\setminus B}|c(N,j,\beta)|^2+\sum_{N\in A\bigcap B}|c(N,j,\beta)|^2,$$

which implies

$$\sum_{N\in A\bigcap B}|c(N,j,\beta)|^2>\frac{3}{4}-\rho^{-\alpha}>\frac{1}{2}. \tag{4.148}$$

Now, suppose that $A\bigcap B\bigcap C=\emptyset$, i.e., for all $N\in A\bigcap B$, the condition (iii) does not hold. Then by (4.148) and Bessel's inequality, we have

$$\frac{1}{2}<\sum_{N\in A\bigcap B}|c(N,j,\beta)|^2\leq\sum_{N\in A\bigcap B}\frac{1}{2m}\sum_{i=1}^{m}|(\Phi_N,\frac{h_i}{\|h_i\|})|^2$$
$$=\frac{1}{2m}\sum_{i=1}^{m}\sum_{N\in A\bigcap B}|(\Phi_N,\frac{h_i}{\|h_i\|})|^2<\frac{1}{2m}\sum_{i=1}^{m}\|\frac{h_i}{\|h_i\|}\|^2=\frac{1}{2},$$

which is a contradiction.

(b) The definition of $\lambda_{j,\beta}$ gives

$$|\lambda_{j,\beta}-\lambda_{j^k,\beta^k}|=||\beta|^2+\mu_j-|\beta+\beta_1+...+\beta_k|^2-\mu_{j^k}|$$
$$\geq|||\beta|^2-|\beta+\beta_1+...+\beta_k|^2|-|\mu_j-\mu_{j^k}||. \tag{4.149}$$

The condition of the lemma $\gamma_i=\beta_i+n_ie_i\in\Gamma(\rho^\alpha)$ and the relation $\gamma=\beta+je_i\in V_{e_i}(\rho^{\alpha_1})\setminus E_2$ together with $|je_i|<c_{16}\rho^{\alpha_1}$ (see (4.117)) and $|n_ie_i|<c_{16}\rho^{\alpha_1}$ (see (4.121)) imply that

$$\rho^{\alpha_2}<||\gamma|^2-|\gamma+\gamma^k|^2|=||\beta|^2+|je_i|^2-|\beta_k|^2-|n^ke_i|^2|$$
$$<||\beta|^2-|\beta_k|^2|+c_{17}\rho^{\alpha_1},\qquad \beta_1+...+\beta_k\neq 0,$$

since $\beta,\beta_1,...,\beta_k$ are orthogonal to e_i, that is, we have

$$||\beta|^2-|\beta_k|^2|>c_{18}\rho^{\alpha_2}.$$

This last inequality together with (4.149) and the asymptotic formula (4.116) give

$$|\lambda_{j,\beta} - \lambda_{j^k,\beta^k}| > c_{19}\rho^{\alpha_2}.$$

Lemma is proved. □

4.3 Asymptotic Formulas

Now we consider the following function

$$h_i(x) = \sum_{\substack{(j_1,\beta_1)\\(j_2,\beta_2)}} \frac{A(j,\beta,j+j_1,\beta+\beta_1)A(j+j_1,\beta+\beta_1,j^2,\beta^2)\Theta_{j^2,\beta^2}(x)}{(\lambda_{j,\beta} - \lambda_{j+j_1,\beta+\beta_1})^i}, \quad (4.150)$$

where $(j_1,\beta_1) \in Q(\rho^\alpha, 6r_1)$ and $(j_2,\beta_2) \in Q(\rho^\alpha, 6r_2)$. Since $\{\Theta_{j^2,\beta^2}(x)\}$ is a total system and $\beta_1 \neq 0$, by (4.140) and (4.147), we have

$$\sum_{(j',\beta')} |(h_i(x), \Theta_{j',\beta'})|^2$$

$$= \sum_{\substack{(j_1,\beta_1)\\(j_2,\beta_2)}} \frac{|A(j,\beta,j+j_1,\beta+\beta_1)A(j+j_1,\beta+\beta_1,j^2,\beta^2)\Theta_{j^2,\beta^2}(x)|^2}{|(\lambda_{j,\beta} - \lambda_{j+j_1,\beta+\beta_1})^i|^2}$$

$$\leq O(\rho^{-2i\alpha_2})(\sum_{\substack{(j_1,\beta_1)\\(j_2,\beta_2)}} |A(j,\beta,j+j_1,\beta+\beta_1)||A(j+j_1,\beta+\beta_1,j^2,\beta^2)|)^2$$

$$\leq c_{20}\rho^{-2i\alpha_2}, \quad (4.151)$$

i.e., $h_i(x) \in L_2(F)$ and $\|h_i(x)\| = O(\rho^{-i\alpha_2})$.

Theorem 4.3.1. *For every eigenvalue $\lambda_{j,\beta}$ of the operator $L_N(q^{e_i})$ with $\beta + je_i \in V'_{e_i}(\rho^{\alpha_1})$, there exists an eigenvalue Υ_N of the operator $L_N(q)$ satisfying*

$$\Upsilon_N = \lambda_{j,\beta} + O(\rho^{-\alpha_2}). \quad (4.152)$$

proof. By Lemma4.2.1, for the chosen $h_i(x), i = 1,2,...,m$ in (4.150), there exists a number N, satisfying (i), (ii), (iii). Since $\beta_1 \neq 0$, by (4.149), we have

$$|\lambda_{j,\beta} - \lambda_{j^1,\beta^1}| > c_{19}\rho^{\alpha_2}.$$

The above inequality together with (i) imply

$$|\Upsilon_N - \lambda_{j^1,\beta^1}| > c_{21}\rho^{\alpha_2}.$$

Using the following well known decomposition

$$\frac{1}{|\Upsilon_N - \lambda_{j^1,\beta^1}|} = \sum_{i=1}^{m} \frac{|\Upsilon_N - \lambda_{j,\beta}|^{i-1}}{|\lambda_{j,\beta} - \lambda_{j^1,\beta^1}|^i} + O(\rho^{-(m+1)\alpha_2}),$$

we see that the formula (4.146) can be written as

$$(\Upsilon_N - \lambda_{j,\beta})c(N,j,\beta) = O(\rho^{-p\alpha})$$

$$+ \sum_{\substack{(\beta_1,j_1)\in Q(\rho^\alpha,6r_1),\\ (\beta_2,j_2)\in Q(\rho^\alpha,6r_2)}} \frac{A(j,\beta,j+j_1,\beta+\beta_1)A(j+j_1,\beta+\beta_1,j^2,\beta^2)c(N,j^2,\beta^2)}{\Upsilon_N - \lambda_{j+j_1,\beta+\beta_1}}$$

$$= \sum_{i=1}^{m} |\Upsilon_N - \lambda_{j,\beta}|^{i-1}(\Phi_N, \frac{h_i}{\|h_i\|})\|h_i\| + O(\rho^{-(m+1)\alpha_2}).$$

Now dividing both sides of the last equation by $c(N,j,\beta)$ and using (ii), (iii), we have

$$|\Upsilon_N - \lambda_{j,\beta}| \le \frac{|(\Phi_N, \frac{h_1}{\|h_1\|})|}{|c(N,j,\beta)|}\|h_1\| + \frac{|\Upsilon_N - \lambda_{j,\beta}||(\Phi_N, \frac{h_2}{\|h_2\|})|}{|c(N,j,\beta)|}\|h_2\|$$

$$+ ... + \frac{|\Upsilon_N - \lambda_{j,\beta}|^{(m-1)}|(\Phi_N, \frac{h_m}{\|h_m\|})|}{|c(N,j,\beta)|}\|h_m\| + O(\rho^{-(m+1)\alpha_2+q\alpha})$$

$$\le \|h_1\| + 2M\|h_2\| + ... + (2M)^{m-1}\|h_m\| + O(\rho^{-(m+1)\alpha_2+q\alpha}).$$

Hence by (4.151), we obtain

$$\Upsilon_N = \lambda_{j,\beta} + O(\rho^{-\alpha_2}),$$

since $(m+1)\alpha_2 - q\alpha > \alpha_2$. Theorem is proved.□

It follows from (4.147) and (4.152) that the triples (N, j^k, β^k) for $k = 1, 2, ..., p_1$, satisfy the iterability condition (4.142). By (4.143) instead of j', β' and r taking j^2, β^2 and r_3 , we have

$$c(N,j^2,\beta^2) = \sum_{(\beta_3,j_3)\in Q(\rho^\alpha,6r_3)} \frac{A(j^2,\beta^2,j^3,\beta^3)(\Phi_N, \Theta_{j^3,\beta^3})}{\Upsilon_N - \lambda_{j^2,\beta^2}} + O(\rho^{-p\alpha}). \quad (4.153)$$

To obtain the other terms of the asymptotic formula of Υ_N, we iterate the formula (4.141). Now we isolate the terms with multiplicand $c(N,j,\beta)$ in the right hand side of (4.155)

$$(\Upsilon_N - \lambda_{j,\beta})c(N,j,\beta) = O(\rho^{-p\alpha})$$
$$+ \sum_{\substack{(\beta_1,j_1)\in Q(\rho^\alpha,6r_1)\\(\beta_2,j_2)\in Q(\rho^\alpha,6r_2)\\(j+j_1+j_2,\beta+\beta_1+\beta_2)=(j,\beta)}} \frac{A(j,\beta,j^1,\beta^1)A(j^1,\beta^1,j,\beta)}{\Upsilon_N - \lambda_{j^1,\beta^1}} c(N,j,\beta)$$
$$+ \sum_{\substack{(\beta_1,j_1)\in Q(\rho^\alpha,6r_1)\\(\beta_2,j_2)\in Q(\rho^\alpha,6r_2)\\(j+j_1+j_2,\beta+\beta_1+\beta_2)\neq(j,\beta)}} \frac{A(j,\beta,j^1,\beta^1)A(j^1,\beta^1,j^2,\beta^2)}{\Upsilon_N - \lambda_{j^1,\beta^1}} c(N,j^2,\beta^2). \quad (4.154)$$

Substituting the equation (4.153) into the second sum of the equation (4.154), we get

$$(\Upsilon_N - \lambda_{j,\beta})c(N,j,\beta) = O(\rho^{-p\alpha})$$
$$+ \sum_{\substack{(\beta_1,j_1)\in Q(\rho^\alpha,6r_1)\\(\beta_2,j_2)\in Q(\rho^\alpha,6r_2)\\(j^2,\beta^2)=(j,\beta)}} \frac{A(j,\beta,j^1,\beta^1)A(j^1,\beta^1,j,\beta)}{\Upsilon_N - \lambda_{j^1,\beta^1}} c(N,j,\beta) +$$
$$\sum_{\substack{(\beta_1,j_1)\in Q(\rho^\alpha,6r_1)\\(\beta_2,j_2)\in Q(\rho^\alpha,6r_2)\\(j^2,\beta^2)\neq(j,\beta)\\(j_3,\beta_3)\in Q(\rho^\alpha,6r_3)}} \frac{A(j,\beta,j^1,\beta^1)A(j^1,\beta^1,j^2,\beta^2)A(j^2,\beta^2,j^3,\beta^3)}{(\Upsilon_N - \lambda_{j^1,\beta^1})(\Upsilon_N - \lambda_{j^2,\beta^2})} c(N,j^3,\beta^3). (4.155)$$

Again isolating the terms $c(N,j,\beta)$ in the last sum of the equation (4.155), we obtain

$$(\Upsilon_N - \lambda_{j,\beta})c(N,j,\beta) = [\sum_{\substack{(\beta_1,j_1)\in Q(\rho^\alpha,6r_1)\\(\beta_2,j_2)\in Q(\rho^\alpha,6r_2)\\(j^2,\beta^2)=(j,\beta)}} \frac{A(j,\beta,j^1,\beta^1)A(j^1,\beta^1,j,\beta)}{\Upsilon_N - \lambda_{j^1,\beta^1}}$$
$$+ \sum_{\substack{(\beta_1,j_1)\in Q(\rho^\alpha,6r_1)\\(\beta_2,j_2)\in Q(\rho^\alpha,6r_2)\\(\beta_3,j_3)\in Q(\rho^\alpha,6r_3)\\(j^2,\beta^2)\neq(j,\beta)\\(j^3,\beta^3)=(j,\beta)}} \frac{A(j,\beta,j^1,\beta^1)A(j^1,\beta^1,j^2,\beta^2)A(j^2,\beta^2,j,\beta)}{\Upsilon_N - \lambda_{j^1,\beta^1}}]c(N,j,\beta)$$
$$\sum_{\substack{(\beta_1,j_1)\in Q(\rho^\alpha,6r_1)\\(\beta_2,j_2)\in Q(\rho^\alpha,6r_2)\\(j_3,\beta_3)\in Q(\rho^\alpha,6r_3)\\(j^2,\beta^2)\neq(j,\beta)\\(j^3,\beta^3)\neq(j,\beta)}} \frac{A(j,\beta,j^1,\beta^1)A(j^1,\beta^1,j^2,\beta^2)A(j^2,\beta^2,j^3,\beta^3)}{(\Upsilon_N - \lambda_{j^1,\beta^1})(\Upsilon_N - \lambda_{j^2,\beta^2})} c(N,j^3,\beta^3)$$
$$+O(\rho^{-p\alpha}). (4.156)$$

In this way, iterating $2p_2$ times, we get

$$(\Upsilon_N - \lambda_{j,\beta})c(N,j,\beta) = [\sum_{k=1}^{2p_2} S'_k]c(N,j,\beta) + C'_{2p_2} + O(\rho^{-p\alpha}), \quad (4.157)$$

where

$$S'_k(\Upsilon_N, \lambda_{j,\beta}) = \sum_{\substack{(\beta_1,j_1)\in Q(\rho^\alpha,6r_1),\ldots,\\(j_{k+1},\beta_{k+1})\in Q(\rho^\alpha,6r_{k+1})\\(j^{k+1},\beta^{k+1})=(j,\beta)\\(j^s,\beta^s)\neq(j,\beta),s=2,\ldots,k}} \left(\prod_{i=1}^{k} \frac{A(j^{i-1},\beta^{i-1},j^i,\beta^i)}{(\Upsilon_N - \lambda_{j^i,\beta^i})}\right) A(j^k,\beta^k,j,\beta) \tag{4.158}$$

and

$$C'_k = \sum_{\substack{(\beta_1,j_1)\in Q(\rho^\alpha,6r_1),\ldots,\\(j_{k+1},\beta_{k+1})\in Q(\rho^\alpha,6r_{k+1})\\(j^s,\beta^s)\neq(j,\beta),s=2,\ldots,k+1}} \left(\prod_{i=1}^{k} \frac{A(j^{i-1},\beta^{i-1},j^i,\beta^i)}{(\Upsilon_N - \lambda_{j^i,\beta^i})}\right) A(j^k,\beta^k,j^{k+1},\beta^{k+1}) c(N,j^{k+1},\beta^{k+1}). \tag{4.159}$$

Now, we estimate S'_k and C'_k. For this, we consider the terms which appear in the denominators of (4.158) and (4.159). By the conditions under the summations in (4.158) and (4.159), we have $j_1 + j_2 + ... + j_i \neq 0$ or $\beta_1 + \beta_2 + ... + \beta_i \neq 0$, for $i = 2, 3, ..., k$.

If $\beta_1 + \beta_2 + ... + \beta_i \neq 0$, then by (4.149) and (4.152), we have

$$|\Upsilon_N - \lambda_{j^i,\beta^i}| > \frac{1}{2}\rho^{\alpha_2}. \tag{4.160}$$

If $\beta_1 + \beta_2 + ... + \beta_i = 0$, i.e., $j_1 + j_2 + ... + j_i \neq 0$, then by well-known theorem

$$|\lambda_{j,\beta} - \lambda_{j^i,\beta^i}| = |\mu_j - \mu_{j^i}| > c,$$

hence by (4.152), we obtain

$$|\Upsilon_N - \lambda_{j^i,\beta^i}| > \frac{1}{2}c. \tag{4.161}$$

Since $\beta_k \neq 0$ for all $k \leq 2p_2$, the relation $\beta_1 + \beta_2 + ... + \beta_i = 0$ implies $\beta_1 + \beta_2 + ... + \beta_{i\pm 1} \neq 0$. Therefore the number of multiplicands $\Upsilon_N - \lambda_{j^i,\beta^i}$ in (4.160) is no less than p_2. Thus by (4.140), (4.160) and (4.161), we get

$$S'_1 = O(\rho^{-\alpha_2}), \quad C'_{2p_2} = O(\rho^{-p\alpha_2}). \tag{4.162}$$

Theorem 4.3.2. *(a) For every eigenvalue $\lambda_{j,\beta}$ of $L_N(q^{e_i})$ such that $\beta + je_i \in V'_{e_i}(\rho^{\alpha_1})$, there exists an eigenvalue Υ_N of the operator $L_N(q)$ satisfying*

$$\Upsilon_N = \lambda_{j,\beta} + E_{k-1} + O(\rho^{-k\alpha_2}), \tag{4.163}$$

where $E_0 = 0$, $E_s = \sum_{k=1}^{2p_2} S'_k(E_{s-1} + \lambda_{j,\beta}, \lambda_{j,\beta})$, $s = 1, 2, ...$

(b) If

$$|\Upsilon_N - \lambda_{j,\beta}| < c_{14} \tag{4.164}$$

and

$$|c(N, j, \beta)| > \rho^{-n\alpha} \tag{4.165}$$

hold then Υ_N satisfies (4.163).

proof. By Lemma4.2.1 (a)-(b), there exists N satisfying the conditions (4.164) and (4.165) in part (b). Hence it suffices to prove part (b). By (4.147) and (4.164), the triples (N, j^k, β^k) satisfy the iterability condition in (4.142). Hence we can use (4.157) and (4.162) . Now, we prove the theorem by induction:

For $k = 1$, to prove (4.163), we divide both sides of the equation (4.157) by $c(N, j, \beta)$ and use the estimations (4.162).

Suppose that (4.163) holds for $k = s$, i.e.,

$$\Upsilon_N = \lambda_{j,\beta} + E_{s-1} + O(\rho^{-s\alpha_2}). \tag{4.166}$$

To prove that (4.163) is true for $k = s+1$, in (4.157) we substitute the expression (4.166) for Υ_N into $\sum_{k=1}^{2p_2} S'_k(\Upsilon_N, \lambda_{j,\beta})$, then we get

$$(\Upsilon_N - \lambda_{j,\beta})c(N, j, \beta) = (\sum_{k=1}^{2p_2} S'_k(\lambda_{j,\beta} + E_{s-1} + O(\rho^{-s\alpha_2}), \lambda_{j,\beta}))c(N, j, \beta) + C'_{2p_2} + O(\rho^{-p\alpha})$$

dividing the both sides of the last equality by $c(N, j, \beta)$ and using Lemma4.2.1-(ii), (4.162), we obtain

$$\Upsilon_N = \lambda_{j,\beta} + \sum_{k=1}^{2p_2} S'_k(\lambda_{j,\beta} + E_{s-1} + O(\rho^{-s\alpha_2}), \lambda_{j,\beta}) + O(\rho^{-(p-q)\alpha}). \tag{4.167}$$

Now we add and subtract the term $\sum_{k=1}^{2p_2} S'_k(E_{s-1}+\lambda_{j,\beta},\lambda_{j,\beta})$ in (4.167), then we have

$$\Upsilon_N = \lambda_{j,\beta} + E_s + O(\rho^{-(p-q)\alpha})$$
$$+[\sum_{k=1}^{2p_2} S'_k(\lambda_{j,\beta}+E_{s-1}+O(\rho^{-s\alpha_2}),\lambda_{j,\beta}) - \sum_{k=1}^{2p_2} S'_k(E_{s-1}+\lambda_{j,\beta},\lambda_{j,\beta})]. \quad (4.168)$$

Now, we first prove that $E_j = O(\rho^{-\alpha_2})$ by induction. $E_0 = 0$. Suppose that $E_{j-1} = O(\rho^{-\alpha_2})$, then $a = \lambda_{j,\beta} + E_{j-1}$ satisfies (4.160) and (4.161). Hence we get

$$S'_1(a,\lambda_{j,\beta}) = O(\rho^{-1\alpha_2}) \Rightarrow E_j = O(\rho^{-\alpha_2}). \quad (4.169)$$

So to prove the theorem, we need to show that the expression in the square brackets in (4.168) is equal to $O(\rho^{-(s+1)\alpha_2})$. This can be easily checked by (4.169) and the obvious relation

$$\frac{1}{\lambda_{j,\beta}+E_{s-1}+O(\rho^{-s\alpha_2})-\lambda_{j^k,\beta^k}} - \frac{1}{\lambda_{j,\beta}+E_{s-1}-\lambda_{j^k,\beta^k}} = O(\rho^{-(s+1)\alpha_2}),$$

for $\beta^k \neq \beta$. The theorem is proved.□

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