HOMOLOGICAL APPROACH TO COMPLEMENTS AND SUPPLEMENTS

A Thesis Submitted to the
Graduate School of Natural and Applied Sciences of
Dokuz Eylül University
In Partial Fulfilment of the Requirements for
the Degree of Doctor of Philosophy in Mathematics

by
Engin MERMUT

November, 2004
İZMİR

Ph. D. THESIS EXAMINATION RESULT FORM

We certify that we have read the thesis, entitled **"HOMOLOGICAL APPROACH TO COMPLEMENTS AND SUPPLEMENTS"** completed by **ENGİN MERMUT** under supervision of **PROF. DR. GÜZİN GÖKMEN** and that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Doctor of Philosophy.

Prof. Dr. Güzin Gökmen

Supervisor

Prof. Dr. Rafail Alizade

Jury Member

Prof. Dr. Halil İbrahim Karakaş

Jury Member

YRD DOÇ.DR. KAREN Ö AKINCI

Jury Member

Yard.Doç.Dr. Halil Oruç

Jury Member

Approved by the

Graduate School of Natural and Applied Sciences

Prof. Dr. Cahit HELVACI

Director

ACKNOWLEDGEMENTS

I would like to express my deepest gratitude to Rafail Alizade* for many reasons. Firstly, when I started the Ph.D. programme of mathematics, in my first term, his contract with Dokuz Eylül University have been ended although our developing small mathematics department needed such researchers to educate young researchers. Even in this case, when he started with the support of TÜBİTAK in İzmir Institute of Technology (İYTE, İzmir Yüksek Teknoloji Enstitüsü), he continued to be my advisor, officially as second advisor, but he was my thesis advisor. My academic advisor Güzin Gökmen†, officially my supervisor, supported in many respects the continuation of our research with Rafail Alizade since she, as a founder of our department, sees the importance of education of young researchers for the sake of the future of the department. Rafail Alizade continued to work in İYTE and using the advantage of his being in İzmir, we have worked together with also his other students every week. In these weekly meetings, we have been able to keep our interest in our research subject, improving our background. These lectures formed a good basis for our preparation as researchers. As an advisor, he has done much for his students on introducing them to the creative part of mathematical research, guiding them on this research area and giving the courage to attack the problems he set forth. Because of all these reasons, I have always found myself lucky to do my Ph. D. thesis under the supervision of my thesis advisor Rafail Alizade and support of my academic advisor Güzin Gökmen to whom I wish to express my thanks once more.

I would like to express my gratitude to TÜBİTAK (The Scientific and Technical Research Council of Turkey) for its support during my Ph. D. research. Besides, TÜBİTAK has given services like in obtaining articles easily from various libraries (in and outside Turkey) via ULAKBİM which has been so useful.

Engin MERMUT‡

---

*Prof. Dr. Rafail Alizade, İzmir Yüksek Teknoloji Enstitüsü, Fen Fakültesi, Matematik Bölümü, Gülbahçeköyü, 35437, Urla/İZMİR, TURKEY. e-mail: `rafailalizade@iyte.edu.tr`

†Prof. Dr. Güzin Gökmen, Dokuz Eylül Üniversitesi, Kaynaklar Yerleşkesi, Fen-Edebiyat Fak., Matematik Böl., 35160, Buca/İZMİR, TURKEY. e-mail: `guzin.gokmen@deu.edu.tr`

‡Engin Mermut, Dokuz Eylül Üniversitesi, Kaynaklar Yerleşkesi, Fen-Edebiyat Fakültesi, Matematik Bölümü, 35160, Buca/İZMİR, TURKEY. e-mail: `engin.mermut@deu.edu.tr`

ABSTRACT

This Ph. D. thesis deals with the proper classes $\mathcal{C}ompl_{R\text{-}\mathcal{M}od}$ and $\mathcal{S}uppl_{R\text{-}\mathcal{M}od}$ of R-modules determined by complement (closed) and supplement submodules, and related proper classes like $\mathcal{N}eat_{R\text{-}\mathcal{M}od}$ and $\mathcal{C}o\text{-}\mathcal{N}eat_{R\text{-}\mathcal{M}od}$, determined by neat and coneat submodules for a ring R. $\mathcal{C}o\text{-}\mathcal{N}eat_{R\text{-}\mathcal{M}od}$ is injectively generated by modules with zero radical and contains $\mathcal{S}uppl_{R\text{-}\mathcal{M}od}$. A submodule A of a module B is coneat in B if and only if there exists a submodule $K \leq B$ such that $A + K = B$ and $A \cap K \leq \operatorname{Rad} A$. For a semilocal ring R, $\mathcal{C}o\text{-}\mathcal{N}eat_{R\text{-}\mathcal{M}od}$ is injectively generated by all (semi-)simple R-modules and equals $\mathcal{S}uppl_{R\text{-}\mathcal{M}od}$ if R is left perfect. $\mathcal{S}uppl_{R\text{-}\mathcal{M}od}$-coprojectives are only projectives if $\operatorname{Rad} R = 0$. For a Dedekind domain W, $\mathcal{C}ompl_{W\text{-}\mathcal{M}od}$-coprojectives are only torsion-free W-modules. The inductive closure of the proper class $\mathcal{S}uppl_{\mathbb{Z}\text{-}\mathcal{M}od}$ is $\mathcal{C}ompl_{\mathbb{Z}\text{-}\mathcal{M}od}$ for the ring $\mathbb{Z}$ of integers. For a Dedekind domain W, $\mathcal{C}ompl_{W\text{-}\mathcal{M}od} = \mathcal{N}eat_{W\text{-}\mathcal{M}od}$ is projectively, injectively and flatly generated by all simple W-modules. c-injective modules over a Dedekind domain are direct summands of a direct product of homogeneous semisimple modules and of injective envelopes of cyclic modules. For a Dedekind domain W, every supplement in a W-module is a complement; if W is not a field, $\mathcal{S}uppl_{W\text{-}\mathcal{M}od} \subsetneq \mathcal{C}o\text{-}\mathcal{N}eat_{W\text{-}\mathcal{M}od} \subseteq \mathcal{C}ompl_{W\text{-}\mathcal{M}od}$, where the second inclusion is an equality if and only if $\operatorname{Rad} W \neq 0$. A finitely generated torsion submodule of a module over a Dedekind domain is a complement if and only if it is a supplement. In a torsion module over a Dedekind domain, neat submodules and coneat submodules coincide. For a Dedekind domain W, if $\operatorname{Rad} W = 0$ and W is not a field, then the functors $\operatorname{Ext}_{\mathcal{S}uppl_{W\text{-}\mathcal{M}od}}$ and $\operatorname{Ext}_{\mathcal{C}o\text{-}\mathcal{N}eat_{W\text{-}\mathcal{M}od}}$ are not factorizable as $W\text{-}\mathcal{M}od \times W\text{-}\mathcal{M}od \xrightarrow{\operatorname{Ext}_W} W\text{-}\mathcal{M}od \xrightarrow{H} W\text{-}\mathcal{M}od$ for any functor H.
Key words: Complement, supplement, closed submodule, neat, coneat, c-injective, proper class, coprojective, coinjective, factorizable Ext, inductive closure, flatly generated, injectively generated, projectively generated.

ÖZET

Bu doktora tezinde, bir R halkası için R-modüllerde, tamamlayan (kapalı) ve tümleyen alt modüller aracılığıyla tanımlanan $\mathcal{C}ompl_{R\text{-}\mathcal{M}od}$ ve $\mathcal{S}uppl_{R\text{-}\mathcal{M}od}$ öz sınıfları ile bunlarla ilgili düzenli ve kodüzenli alt modüllerle tanımlanan $\mathcal{N}eat_{R\text{-}\mathcal{M}od}$ ve $\mathcal{C}o\text{-}\mathcal{N}eat_{R\text{-}\mathcal{M}od}$ gibi öz sınıfları incelenmektedir. $\mathcal{C}o\text{-}\mathcal{N}eat_{R\text{-}\mathcal{M}od}$ radikali sıfır olan tüm modüller tarafından injektif olarak üretilir ve $\mathcal{S}uppl_{R\text{-}\mathcal{M}od}$'u içerir. Bir B modülünün A alt modülü, B'de kodüzenlidir ancak ve ancak B'nin bir K alt modülü için, $A + K = B$ ve $A \cap K \leq \operatorname{Rad} A$ ise. Yarı-yerel bir R halkası için, $\mathcal{C}o\text{-}\mathcal{N}eat_{R\text{-}\mathcal{M}od}$ tüm (yarı-)basit modüller tarafından injektif olarak üretilir ve eğer R sol mükemmel bir halka ise $\mathcal{S}uppl_{R\text{-}\mathcal{M}od}$'a eşittir. Eğer $\operatorname{Rad} R = 0$ ise $\mathcal{S}uppl_{R\text{-}\mathcal{M}od}$-koprojektifler sadece projektiflerdir. Bir Dedekind tamlık bölgesi W için, $\mathcal{C}ompl_{W\text{-}\mathcal{M}od}$-koprojektifler sadece burulmasız W-modüllerdir. Tamsayılar halkası $\mathbb{Z}$ için, $\mathcal{S}uppl_{\mathbb{Z}\text{-}\mathcal{M}od}$'un direkt limite göre kapanışı $\mathcal{C}ompl_{\mathbb{Z}\text{-}\mathcal{M}od}$'dur. Bir Dedekind tamlık bölgesi W için, $\mathcal{C}ompl_{W\text{-}\mathcal{M}od} = \mathcal{N}eat_{W\text{-}\mathcal{M}od}$ tüm basit W-modüller tarafından projektif, injektif ve düz olarak üretilir. Bir Dedekind tamlık bölgesi üzerindeki c-injektif modüller, homojen yarı-basit modüllerin ve devirli modüllerin injektif bürümlerinin direkt çarpımının direkt toplam terimidir. Bir Dedekind tamlık bölgesi W için, bir W-modüldeki her tümleyen tamamlayandır; eğer W bir cisim değilse, $\mathcal{S}uppl_{W\text{-}\mathcal{M}od} \subsetneqq \mathcal{C}o\text{-}\mathcal{N}eat_{W\text{-}\mathcal{M}od} \subseteq \mathcal{C}ompl_{W\text{-}\mathcal{M}od}$ sağlanır ve ikinci içerme eşitliktir ancak ve ancak $\operatorname{Rad} W \neq 0$ ise. Bir Dedekind tamlık bölgesi üzerindeki bir modülün sonlu üretilmiş burulmalı bir alt modülü tamamlayandır ancak ve ancak tümleyen ise. Bir Dedekind tamlık bölgesi üzerindeki burulmalı bir modülde, düzenli alt modüller ve kodüzenli alt modüller aynıdır. Bir Dedekind tamlık bölgesi W için, $\operatorname{Ext}_{\mathcal{S}uppl_{W\text{-}\mathcal{M}od}}$ ve $\operatorname{Ext}_{\mathcal{C}o\text{-}\mathcal{N}eat_{W\text{-}\mathcal{M}od}}$ funktorları, hiçbir H funktoru için, $W\text{-}\mathcal{M}od \times W\text{-}\mathcal{M}od \xrightarrow{\operatorname{Ext}_W} W\text{-}\mathcal{M}od \xrightarrow{H} W\text{-}\mathcal{M}od$ şeklinde parçalanamaz, eğer $\operatorname{Rad} W = 0$ ve W bir cisim değil ise.

Anahtar Sözcükler: tamamlayan, tümleyen, kapalı alt modül, düzenli, kodüzenli, c-injektif, öz sınıf, koprojektif, koinjektif, parçalanabilir Ext, direkt limite göre kapanış, düz olarak üretilmiş, injektif olarak üretilmiş, projektif olarak üretilmiş.

CONTENTS

Chapter Two
PROPER CLASSES

Chapter Three
THE PROPER CLASSES RELATED TO
COMPLEMENTS AND SUPPLEMENTS

Chapter Four

COMPLEMENTS AND SUPPLEMENTS IN ABELIAN GROUPS

Chapter Five

COMPLEMENTS AND SUPPLEMENTS IN MODULES OVER DEDEKIND DOMAINS

NOTATION

R	an associative ring with unit unless otherwise stated
$\mathbb{Z}$, $\mathbb{Z}^+$	the ring of integers, the set of all positive integers
$\mathbb{Q}$	the field of rational numbers
$\mathbb{Z}_{p^\infty}$	the Prüfer (divisible) group for the prime p
W	a (commutative) Dedekind domain
R-module	*left* R-module
R-$\mathcal{M}od$, $\mathcal{M}od$-R	the categories of *left* R-modules, *right* R-modules
$\mathcal{A}b = \mathbb{Z}$-$\mathcal{M}od$	the categeory of abelian groups ($\mathbb{Z}$-modules)
σ[M]	the category of all M-*sub*generated modules (i.e. all submodules of all M-generated modules) for a module M
$\cong$	isomorphic
$\mathrm{Hom}_R(M,N)$	all R-module homomorphisms from M to N
$\mathrm{Hom}_R^{\mathrm{Im}\ll}(M,N)$	all $f \in \mathrm{Hom}_R(M,N)$ such that $\mathrm{Im}(f)$ is small in N
$\mathrm{Hom}_R^{\mathrm{Im}\leq\mathrm{Rad}}(M,N)$	all $f \in \mathrm{Hom}_R(M,N)$ such that $\mathrm{Im}(f)$ is in $\mathrm{Rad}(N)$
$M \otimes_R N$	the tensor product of the *right* R-module M and the *left* R-module N
$\mathrm{Ker}(f)$	the kernel of the map f
$\mathrm{Im}(f)$	the image of the map f
$\mathrm{E}(M)$	the injective envelope (hull) of a module M
$\mathrm{Soc}(M)$	the socle of the R-module M
$\mathrm{Rad}(M)$	the radical of the R-module M
$\mathrm{Rad}(R)$	the Jacobson radical of the ring R
$\mathrm{u.dim}(M)$	the uniform dimension (=Goldie dimension) of M
$\mathrm{h.dim}(M)$	the hollow dimension (=dual Goldie dimension) of M
$\mathrm{Ann}_R^l(X)$	$= \{r \in R \mid rX = 0\}$ = the *left* annihilator of a subset X of a *left* R-module M
$\mathrm{Ann}_R^r(X)$	$= \{r \in R \mid Xr = 0\}$ = the *right* annihilator of a subset X of a *right* R-module M
$\varinjlim M_i$	direct limit of the direct system $\{M_i(i \in I); \pi_i^j(i \leq j)\}$
$\lvert X\rvert$	Cardinality of a set X

$\mathcal{P}$	a proper class of R-modules
$A \leq_{\mathcal{P}} B$	A is a $\mathcal{P}$-submodule of B, i.e. the inclusion monomorphism $A \hookrightarrow B$ is a $\mathcal{P}$-monomorphism
$\pi(\mathcal{P})$	all $\mathcal{P}$-projective modules
$\pi^{-1}(\mathcal{M})$	the proper class of R-modules projectively generated by a class $\mathcal{M}$ of R-modules
$\iota(\mathcal{P})$	all $\mathcal{P}$-injective modules
$\iota^{-1}(\mathcal{M})$	the proper class of R-modules injectively generated by a class $\mathcal{M}$ of R-modules
$\tau(\mathcal{P})$	all $\mathcal{P}$-flat *right* R-modules
$\tau^{-1}(\mathcal{M})$	the proper class of R-modules flatly generated by a class $\mathcal{M}$ of *right* R-modules
$\widetilde{\mathcal{P}}$	the inductive closure of a proper class $\mathcal{P}$
$\text{Ext}_{\mathcal{P}} = \text{Ext}^1_{\mathcal{P}}$	$\text{Ext}_{\mathcal{P}}(C, A) = \text{Ext}^1_{\mathcal{P}}(C, A)$ is the subgroup of $\text{Ext}_R(C, A)$ consisting of equivalence classes of short exact sequences in $\mathcal{P}$ starting with the R-module A and ending with the R-module C
$\mathcal{S}plit_{R\text{-}\mathcal{M}od}$	the smallest proper class of R-modules consisting of *only splitting* short exact sequences of R-modules
$\mathcal{A}bs_{R\text{-}\mathcal{M}od}$	the largest proper class of R-modules consisting of *all* short exact sequences of R-modules (absolute purity)
$\mathcal{P}ure_{\mathbb{Z}\text{-}\mathcal{M}od}$	the proper class of pure-exact sequences of abelian groups
$\mathcal{N}eat_{\mathbb{Z}\text{-}\mathcal{M}od}$	the proper class of neat-exact sequences of abelian groups
$\mathcal{A}$	an abelian category (like R-$\mathcal{M}od$, $\mathbb{Z}$-$\mathcal{M}od = \mathcal{A}b$ or σ[M]) *For a suitable abelian category $\mathcal{A}$ like R-$\mathcal{M}od$, $\mathbb{Z}$-$\mathcal{M}od = \mathcal{A}b$ or σ[M], the following proper classes are defined:*
$\mathcal{C}ompl_{\mathcal{A}}$, $\mathcal{C}$	the proper class of complements in the abelian category $\mathcal{A}$
$\mathcal{S}uppl_{\mathcal{A}}$, $\mathcal{S}$	the proper class of supplements in the abelian category $\mathcal{A}$
$\mathcal{N}eat_{\mathcal{A}}$, $\mathcal{N}$	the proper class of neats in the abelian category $\mathcal{A}$
$\mathcal{C}o\text{-}\mathcal{N}eat_{\mathcal{A}}$, $c\mathcal{N}$	the proper class of coneats in the abelian category $\mathcal{A}$

$\leq$	submodule
$\ll$	small (=superfluous) submodule
$\trianglelefteq$	essential submodule
$\leq_c$	complement submodule (=closed submodule)
$\leq_s$	supplement submodule
$\leq_{\mathcal{N}}$	neat submodule
$\leq_{c\mathcal{N}}$	coneat submodule

I^{-1}	Inverse of an invertible fractional ideal of a commutative domain R with field of fractions K, $I^{-1} = (R : I) = \{q \in K \mid qI \subseteq R\}$

$\in\in$	$\mathbb{E} \in\in \mathrm{Ext}_R(C, A)$ means that $\mathbb{E}$ is an element of an element of the group $\mathrm{Ext}_R(C, A)$ whose elements are equivalence classes of short exact sequences of R-modules starting with A and ending with C, so $[\mathbb{E}] \in \mathrm{Ext}_R(C, A)$
$[\mathbb{E}]$	the equivalence class of a short exact sequence $\mathbb{E}$ of R-modules
$\alpha\mathbb{E}$	pushout of a short exact sequence $\mathbb{E} \in\in \mathrm{Ext}_R(C, A)$ along $\alpha : A \longrightarrow M$; $\alpha\mathbb{E} \in\in \mathrm{Ext}_R(C, M)$
$\triangle_{\mathbb{E}}$	connecting homomorphism $\triangle_{\mathbb{E}} : \mathrm{Hom}_R(H, A) \longrightarrow \mathrm{Ext}_R(C, A)$, $\triangle_{\mathbb{E}}(\alpha) = [\alpha\mathbb{E}]$, sending each $\alpha \in \mathrm{Hom}_R(H, A)$ to the equivalence class of the pushout $\alpha\mathbb{E}$ of $\mathbb{E}$ along α

CHAPTER ONE
INTRODUCTION

In this introductory chapter, we will give the motivating ideas for our thesis problems and summarize what we have done. To explain these problems and results, we will summarize what proper classes are in Section 1.3; for some more details, see Chapter 2. What is assumed as preliminary notions is sketched in Section 1.2. See Section 1.1 for the definition of *complement* and *supplement*. In Section 1.9, we will summarize the main results of this thesis. The other sections of this chapter summarize the motivating ideas and related concepts some of which are explained in more detail in the following chapters: Neat subgroups (Section 1.4), C-rings of Renault (Section 1.5), complements and supplements in modules over Dedekind domains (Section 1.6), c-injective modules (Section 1.7), extending(CS) modules and lifting modules (Section 1.8).

We deal with complements (closed submodules) and supplements in unital R-modules for an associative ring R with unity using relative homological algebra via the known two dual proper classes of short exact sequences of R-modules and R-module homomorphisms, $\mathcal{C}ompl_{R\text{-}\mathcal{M}od}$ and $\mathcal{S}uppl_{R\text{-}\mathcal{M}od}$, and related other proper classes like $\mathcal{N}eat_{R\text{-}\mathcal{M}od}$ and $\mathcal{C}o\text{-}\mathcal{N}eat_{R\text{-}\mathcal{M}od}$. $\mathcal{C}ompl_{R\text{-}\mathcal{M}od}$ [$\mathcal{S}uppl_{R\text{-}\mathcal{M}od}$] consists of all short exact sequences

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

of R-modules and R-module homomorphisms such that $\mathrm{Im}(f)$ is a complement [resp. supplement] in B. $\mathcal{N}eat_{R\text{-}\mathcal{M}od}$ [$\mathcal{C}o\text{-}\mathcal{N}eat_{R\text{-}\mathcal{M}od}$] consists of all short exact sequences of R-modules and R-module homomorphisms with respect to which every simple module is projective [resp. every module with zero radical is injective].

We seek for the projectives, injectives, coprojectives, coinjectives with respect to these proper classes. When does these proper classes have enough projectives? Enough injectives? In Chapter 3, we deal with these proper classes for an arbitrary ring R. Firstly, we search for the results when R is the ring of integers, i.e. for abelian groups, in Chapter 4, since these form the motivating ideas for our research. The next step is to generalize these results for the case $R = W$, a Dedekind domain in Chapter 5. The results for some other classes of rings have been collected in Section 3.8. Some problems for complements and supplements can be interpreted in terms of homological algebra using these proper classes of complements and supplements and their relations with other proper classes.

1.1 Complements and supplements

We try to understand a module through its submodules, or better to say through its relation with its submodules. More precisely, let R denote an associative ring with unity, B be an R-module and let K be a submodule of B. It would be best if K is a *direct summand* of B, that is if there exists another submodule A of B such that $B = K \oplus A$; that means,

$$B = K + A \quad \text{and} \quad K \cap A = 0.$$

When K is not a direct summand, we hope at least to retain one of these conditions. These give rise to two concepts: *complement* and *supplement*.

If A is a submodule of B such that $B = K + A$ (that is the above first condition for direct sum holds) and A is *minimal* with respect to this property (that is there is *no* submodule $\tilde{A}$ of B such that $\tilde{A} \subsetneq A$ but still $B = K + \tilde{A}$), then A is called a *supplement of* K *in* B and K is said to *have a supplement in* B. Equivalently, $K + A = B$ and $K \cap A$ is *small* (=*superfluous*) in A (which is denoted by $K \cap A \ll A$, meaning that for *no* proper submodule X of A, $K \cap A + X = A$). K need *not* have a supplement. If a module B is such that every submodule of it has a supplement, then it is called a *supplemented module*. For the definitions and related properties see Wisbauer (1991, §41). In a series of papers from 1974, H. Zöschinger considered the class of supplemented modules (Zöschinger, 1974a,b,c, 1976, 1978, 1979a,b, 1980, 1981, 1982a,b, 1986, 1994). In recent years, the research on related concepts has regained interest; see Talebi & Vanaja (2004), Koşan & Harmanci (2004), Nebiyev & Pancar (2003), Orhan (2003), Kuratomi (2003), Keskin Tütüncü & Orhan (2003), Idelhadj & Tribak (2003b,a), Güngöroglu & Keskin Tütüncü (2003), Alizade & Büyükaşık (2003), Tuganbaev (2002), Talebi & Vanaja (2002), Özcan (2002), Keskin (2002b,a), Ganesan & Vanaja (2002), Oshiro (2001), Keskin & Xue (2001), Alizade et al. (2001), Smith (2000b), Keskin (2000b,a), Lomp (1999), Keskin et al. (1999), Harmancı et al. (1999), Oshiro & Wisbauer (1995), Xin (1994), Vanaja (1993), Liu (1992), Al-Khazzi & Smith (1991), Baba & Harada (1990), Fieldhouse (1985), Oshiro (1984b,a), Inoue (1983), Hausen & Johnson (1983b,a), Hausen (1982).

If A is a submodule of B such that $K \cap A = 0$ (that is the above second condition for direct sum holds) and A is *maximal* with respect to this property (that is there is *no* submodule $\tilde{A}$ of B such that $\tilde{A} \supsetneq A$ but still $K \cap \tilde{A} = 0$), then A is called a *complement of* K *in* B and K is said to *have a complement in* B. By Zorn's Lemma, it is seen that K always has a complement in B (unlike the case for supplements). In fact, by Zorn's Lemma, we know that if we have a submodule A' of B such that $A' \cap K = 0$, then there exists a complement A of K in B such that $A \supseteq A'$. See the monograph Dung et al. (1994) for a survey of results in the related concepts.

We deal with the collection of submodules each of which is a complement of some submodule or supplement of some submodule. A submodule A of a module B is said to be a *complement in* B if A is a complement of some submodule of B; shortly, we also say that A is a *complement submodule of* B in this case and denote this by $A \leq_c B$. It is said that A is *closed in* B if A has *no* proper essential extension in B, that is, there exists *no* submodule $\tilde{A}$ of B such that $A \subsetneqq \tilde{A}$ and A is *essential* in $\tilde{A}$ (which is denoted by $A \trianglelefteq \tilde{A}$ and meaning that for every *nonzero* submodule X of $\tilde{A}$, we have $A \cap X \neq 0$). We also say in this case that A is a *closed submodule* and it is known that closed submodules and complement submodules in a module coincide (see Dung et al. (1994, §1)). So the "*c*" in the notation $A \leq_c B$ can be interpreted as complement or closed. Dually, a submodule A of a module B is said to be a *supplement in* B if A is a supplement of some submodule of B; shortly, we also say that A is a *supplement submodule of* B in this case and denote this by $A \leq_s B$.

1.2 Preliminaries, terminology and notation

Throughout this thesis, by a *ring* we mean an associative ring with unity; R will denote such a general ring, unless otherwise stated. So, *if nothing is said about* R *in the statement of a theorem, proposition, etc., then that means* R *is just an arbitrary ring.* We consider unital left R-modules; R-module will mean *left* R-module. R-$\mathcal{M}od$ denotes the category of all *left* R-modules. $\mathcal{M}od$-R denotes the category of *right* R-modules. $\mathbb{Z}$ denotes the ring of integers. $\mathcal{A}b$, or $\mathbb{Z}$-$\mathcal{M}od$, denotes the category of abelian groups ($\mathbb{Z}$-modules). *Group* will mean *abelian group* only. We denote a (commutative) *Dedekind domain* by W; see Section 5.1. *Integral domain*, or shortly *domain*, will mean a nonzero ring without zero divisors, *not* necessarily commutative. But following the general convention a *principal ideal domain* (shortly *PID*) will mean a *commutative* domain in which every ideal is principal, i.e. generated by one element. Also a Dedekind domain is assumed to be commutative as usual.

All definitions not given here can be found in Anderson & Fuller (1992), Wisbauer (1991), Dung et al. (1994) and Fuchs (1970).

The ***notation*** *we use have been given on pages (xi-xiii) just before this chapter. There is also an* ***index*** *in the end and the page number of the definition of a term in the index, if it exists, has been written in boldface.*

We do not delve into the details of definitions of every term in modules, rings and homological algebra. Essentially, we accept fundamentals of module theory, categories, pullback and pushout, the Hom and tensor ($\otimes$) functors, projective modules, injective modules, flat modules, homology functor, projective and injective resolutions, derived functors, the functor $\mathrm{Ext}_R = \mathrm{Ext}_R^1 : R\text{-}\mathcal{M}od \times R\text{-}\mathcal{M}od \longrightarrow \mathcal{A}b$, the functors $\mathrm{Ext}_R^n : R\text{-}\mathcal{M}od \times R\text{-}\mathcal{M}od \longrightarrow \mathcal{A}b$ ($n \in \mathbb{Z}^+$), projective dimension of a module, injective dimension of a module, Goldie dimension (uniform dimension), dual Goldie dimension (hollow dimension) are known.

For more details in homological algebra see the books Alizade & Pancar (1999), Rotman (1979), Cartan & Eilenberg (1956) and Maclane (1963). For modules and rings see the books Anderson & Fuller (1992), Lam (2001, 1999) and Facchini (1998). For abelian groups, see Fuchs (1970). For relative homological algebra, our main references are the books Maclane (1963) and Enochs & Jenda (2000) and the article Sklyarenko (1978). We will explain most of the terms and summarize the necessary concepts.

The book Wisbauer (1991) gives the concepts in module theory relative to the category σ[M] for a module M. This category σ[M] is the full subcategory of R-$\mathcal{M}od$ consisting of all M-subgenerated modules, that is all *submodules* of M-generated modules, where a module N is said to be an M-generated module if there exists an R-module epimorhism $f : \bigoplus_{\lambda\in\Lambda} M \longrightarrow N$ for some index set Λ. This category reflects the properties of the module M. For example, $\sigma[R] = R\text{-}\mathcal{M}od$, where the ring R is considered as a left R-module.

1.3 Proper classes of R-modules for a ring R

Let $\mathcal{P}$ be a class of short exact sequences of R-modules and R-module homomorphisms. If a short exact sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0 \qquad (1.3.1)$$

belongs to $\mathcal{P}$, then f is said to be a *$\mathcal{P}$-monomorphism* and g is said to be a *$\mathcal{P}$-epimorphism* (both are said to be *$\mathcal{P}$-proper* and the short exact sequence is said to be a *$\mathcal{P}$-proper* short exact sequence.).

The class $\mathcal{P}$ is said to be *proper* (in the sense of Buchsbaum) if it satisfies the following conditions (see Buschbaum (1959), Maclane (1963, Ch. 12, §4), Stenström (1967a, §2) and Sklyarenko (1978, Introduction)):

1. If a short exact sequence $\mathbb{E}$ is in $\mathcal{P}$, then $\mathcal{P}$ contains every short exact sequence isomorphic to $\mathbb{E}$.
2. $\mathcal{P}$ contains all splitting short exact sequences.
3. The composite of two $\mathcal{P}$-monomorphisms is a $\mathcal{P}$-monomorphism if this composite is defined. The composite of two $\mathcal{P}$-epimorphisms is a $\mathcal{P}$-epimorphism if this composite is defined.
4. If g and f are monomorphisms, and $g \circ f$ is a $\mathcal{P}$-monomorphism, then f is a $\mathcal{P}$-monomorphism. If g and f are epimorphisms, and $g \circ f$ is a $\mathcal{P}$-epimorphism, then g is a $\mathcal{P}$-epimorphism.

An important example for proper classes in abelian groups is $\mathcal{P}ure_{\mathbb{Z}\text{-}\mathcal{M}od}$: The proper class of all short exact sequences (1.3.1) of abelian groups and abelian group homomorphisms such that $\mathrm{Im}(f)$ is a pure subgroup of B, where a subgroup A of a group B is *pure* in B if $A \cap nB = nA$ for all integers n (see Fuchs (1970, §26-30) for the important notion of purity in abelian groups). The short exact sequences in $\mathcal{P}ure_{\mathbb{Z}\text{-}\mathcal{M}od}$ are called *pure-exact sequences* of abelian groups. The proper class $\mathcal{P}ure_{\mathbb{Z}\text{-}\mathcal{M}od}$ forms one of the origins of *relative* homological algebra; it is the reason why a proper class is also called *purity* (as in Mishina & Skornyakov (1976), Generalov (1972, 1978, 1983)).

The smallest proper class of R-modules consists of only *splitting* short exact sequences of R-modules which we denote by $\mathcal{S}plit_{R\text{-}\mathcal{M}od}$.The largest proper class of R-modules consists of *all* short exact sequences of R-modules which we denote by $\mathcal{A}bs_{R\text{-}\mathcal{M}od}$ (*absolute purity*).

For a proper class $\mathcal{P}$ of R-modules, call a submodule A of a module B a $\mathcal{P}$-submodule of B, if the inclusion monomorphism $i_A : A \to B$, $i_A(a) = a$, $a \in A$, is a $\mathcal{P}$-monomorphism. We denote this by $A \leq_{\mathcal{P}} B$.

Denote by $\mathcal{P}$ a proper class of R-modules. An R-module M is said to be *$\mathcal{P}$-projective* [*$\mathcal{P}$-injective*] if it is projective [injective] with respect to all short exact sequences in $\mathcal{P}$, that is, $\mathrm{Hom}(M, \mathbb{E})$ [$\mathrm{Hom}(\mathbb{E}, M)$] is exact for every $\mathbb{E}$ in $\mathcal{P}$. Denote all $\mathcal{P}$-projective [$\mathcal{P}$-injective] modules by $\pi(\mathcal{P})$ [$\iota(\mathcal{P})$]. For a given class $\mathcal{M}$ of modules, denote by $\pi^{-1}(\mathcal{M})$ [$\iota^{-1}(\mathcal{M})$] the largest proper class $\mathcal{P}$ for which each $M \in \mathcal{M}$ is $\mathcal{P}$-projective [$\mathcal{P}$-injective]; it is called the proper class *projectively generated* [*injectively generated*] by $\mathcal{M}$. When the ring R is *not* commutative, we must be careful with the sides for the tensor product analogues of above. Remember that by an R-module, we mean a *left* R-module. A *right* R-module M is said to be *$\mathcal{P}$-flat* if $M \otimes \mathbb{E}$ is exact for every $\mathbb{E}$ in $\mathcal{P}$. Denote all $\mathcal{P}$-flat *right* R-modules by $\tau(\mathcal{P})$. For a given class $\mathcal{M}$ of *right* R-modules, denote by

$\tau^{-1}(\mathcal{M})$ the largest proper class $\mathcal{P}$ of (left) R-modules for which each $M \in \mathcal{M}$ is $\mathcal{P}$-flat; it is called the proper class *flatly generated* by the class $\mathcal{M}$ of *right* R-modules. When the ring R is commutative, there is no need to mention the sides of the modules since a right R-module may also be considered as a left R-module and vice versa. An R-module C is said to be $\mathcal{P}$-*coprojective* if *every* short exact sequence of R-modules and R-module homomorphisms of the form (1.3.1) *ending* with C is in the proper class $\mathcal{P}$. An R-module A is said to be $\mathcal{P}$-*coinjective* if *every* short exact sequence of R-modules and R-module homomorphisms of the form (1.3.1) *starting* with A is in the proper class $\mathcal{P}$. See Sklyarenko (1978, §1-3,8-9) for these concepts in relative homological algebra in categories of modules.

A proper class $\mathcal{P}$ of R-modules is said to have *enough projectives* [*enough injectives*] if for every R-module M, there exists a $\mathcal{P}$-projective P [resp. a $\mathcal{P}$-injective I] with a $\mathcal{P}$-epimorphism $P \longrightarrow M$ [rep. a $\mathcal{P}$-monomorphism $M \longrightarrow I$]. A proper class $\mathcal{P}$ of R-modules with enough projectives [enough injectives] is also said to be a *projective proper class* [resp. *injective proper class*].

For a proper class $\mathcal{P}$ and R-modules A, C, denote by $\text{Ext}^1_{\mathcal{P}}(C, A)$ or just by $\text{Ext}_{\mathcal{P}}(C, A)$, the equivalence classes of all short exact sequences in $\mathcal{P}$ which start with A and end with C, i.e. a short exact sequence in $\mathcal{P}$ of the form (1.3.1). This turns out to be a subgroup of $\text{Ext}_R(C, A)$ and a bifunctor $\text{Ext}^1_{\mathcal{P}} : R\text{-}\mathcal{M}od \times R\text{-}\mathcal{M}od \longrightarrow \mathcal{A}b$ is obtained which is a subfunctor of Ext^1_R (see Maclane (1963, Ch. 12, §4-5)).

A proper class $\mathcal{P}$ is said to be *inductively closed* if for every direct system $\{\mathbb{E}_i (i \in I); \pi_i^j (i \leq j)\}$ in $\mathcal{P}$, the direct limit $\mathbb{E} = \varinjlim \mathbb{E}_i$ is also in $\mathcal{P}$ (see Fedin (1983) and Sklyarenko (1978, §8)). As in Fedin (1983), for a proper class $\mathcal{P}$, denote by $\widetilde{\mathcal{P}}$, the smallest inductively closed proper class containing $\mathcal{P}$; it is called the *inductive closure* of $\mathcal{P}$.

1.4 Motivating ideas from abelian groups

The classes $\mathcal{C}ompl_{R\text{-}\mathcal{M}od}$ and $\mathcal{S}uppl_{R\text{-}\mathcal{M}od}$ defined in the introduction to this chapter really form proper classes as has been shown more generally by Generalov (1978, Theorem 1), Generalov (1983, Theorem 1), Stenström (1967b, Proposition 4 and Remark after Proposition 6). In Stenström (1967b), following the terminology in abelian groups, the term 'high' is used instead of complements and 'low' for supplements. Generalov (1978, 1983) use the terminology 'high' and 'cohigh' for complements and supplements, and give more general definitions for proper classes of complements and supplements related to another given proper class (motivated by the considerations as pure-high extensions and neat-high extensions in Harrison et al. (1963)); 'weak purity' in Generalov (1978) is what we denote by $\mathcal{C}ompl_{R\text{-}\mathcal{M}od}$. See also Erdoğan (2004, Theorem 2.7.15 and Theorem 3.1.2) for the proofs of $\mathcal{C}ompl_{R\text{-}\mathcal{M}od}$ and $\mathcal{S}uppl_{R\text{-}\mathcal{M}od}$ being proper classes.

A subgroup A of a group B is said to be *neat* in B if $A \cap pB = pA$ for all prime numbers p (see Fuchs (1970, §31); the notion of neat subgroup has been introduced in Honda (1956, pp. 42-43)). This is a weaker condition than being a pure subgroup. What is important for us is that neat subgroups of an abelian group coincide with complements in that group. Denote by $\mathcal{N}eat_{\mathbb{Z}\text{-}\mathcal{M}od}$, the proper class of all short exact sequences

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

of abelian groups and abelian group homomorphisms where $\text{Im}(f)$ is a neat subgroup of B; call such short exact sequences *neat-exact sequences* of abelian groups (like the terminology for pure-exact sequences). The following result is one of the motivations for us to deal with complements and its dual supplements: The proper class $\mathcal{C}ompl_{\mathbb{Z}\text{-}\mathcal{M}od} = \mathcal{N}eat_{\mathbb{Z}\text{-}\mathcal{M}od}$ is projectively generated, flatly generated and injectively generated by simple groups $\mathbb{Z}/p\mathbb{Z}$, p prime number:

$$\begin{aligned} \mathcal{C}ompl_{\mathbb{Z}\text{-}\mathcal{M}od} &= \mathcal{N}eat_{\mathbb{Z}\text{-}\mathcal{M}od} = \pi^{-1}(\{\mathbb{Z}/p\mathbb{Z} | p \text{ prime}\}) \\ &= \tau^{-1}(\{\mathbb{Z}/p\mathbb{Z} | p \text{ prime}\}) = \iota^{-1}(\{\mathbb{Z}/p\mathbb{Z} | p \text{ prime}\}). \end{aligned}$$

(see Theorem 4.1.1)

The second equality $\mathcal{N}eat_{\mathbb{Z}\text{-}\mathcal{M}od} = \pi^{-1}(\{\mathbb{Z}/p\mathbb{Z} | p \text{ prime}\})$ was the motivation to define for any ring R, as said in the introduction to this chapter,

$$\begin{aligned} \mathcal{N}eat_{R\text{-}\mathcal{M}od} &\overset{def.}{=} \pi^{-1}(\{\text{all simple } R\text{-modules}\}) \\ &= \pi^{-1}(\{R/P | P \text{ maximal left ideal of } R\}), \end{aligned}$$

following Stenström (1967a, 9.6) (and Stenström (1967b, §3)). For a submodule A of an R-module B, say that A is a *neat submodule* of B, or say that A is *neat in* B, if A is a $\mathcal{N}eat_{R\text{-}\mathcal{M}od}$-submodule, and denote this shortly by $A \leq_{\mathcal{N}} B$. We always have $\mathcal{C}ompl_{R\text{-}\mathcal{M}od} \subseteq \mathcal{N}eat_{R\text{-}\mathcal{M}od}$ for any ring R (by Stenström (1967b, Proposition 5)) (see Theorem 3.1.1, Corollary 3.2.7).

1.5 C-rings of Renault and torsion-free covering modules of Enochs

If R is a commutative Noetherian ring in which every *nonzero* prime ideal is maximal, then

$$\mathcal{C}ompl_{R\text{-}\mathcal{M}od} = \mathcal{N}eat_{R\text{-}\mathcal{M}od},$$

by Stenström (1967b, Corollary to Proposition 8) (see Proposition 3.3.1).

Generalov (1978, Theorem 5) gives a characterization of this equality in terms of the ring R (see Theorem 3.3.2):

$$\mathcal{C}ompl_{R\text{-}\mathcal{M}od} = \mathcal{N}eat_{R\text{-}\mathcal{M}od} \qquad \textit{if and only if} \qquad R \text{ is a left C-ring.}$$

The notion of C-ring has been introduced by Renault (1964): A ring R is said to be a *left* C-ring if for every (left) R-module B and for every essential proper submodule A of B, $\mathrm{Soc}(B/A) \neq 0$, that is B/A has a simple submodule. Equivalently, for every essential *left* ideal I of R, there exist $r \in R$ such that the left annihilator $\mathrm{Ann}^l_R(r+I) = \{s \in R | s(r+I) = 0\} = \{s \in R | sr \in I\}$ of the element $r+I$ in the left R-module R/I is a maximal left ideal of R. Similarly *right* C-rings are defined. For example, a commutative Noetherian ring in which every *nonzero* prime ideal is maximal is a C-ring. So, of course, in particular a Dedekind domain and therefore a PID is also a C-ring. See Section 3.3.

A commutative domain R is a C-ring if and only if every nonzero torsion module has a simple submodule (see Proposition 3.3.9) and such rings have been considered in Enochs & Jenda (2000, §4.4). See Enochs & Jenda (2000, Theorem 4.4.1) related to torsion-free covering modules.

1.6 Complements and supplements in modules over Dedekind domains

Generalov (1983, Corollary 1 and 6) gives the following interesting result (the equality from Generalov (1978, Theorem 5) as a Dedekind domain is a C-ring): For a Dedekind domain W,

$$\mathcal{S}uppl_{W\text{-}\mathcal{M}od} \subseteq \mathcal{C}ompl_{W\text{-}\mathcal{M}od} = \mathcal{N}eat_{W\text{-}\mathcal{M}od},$$

where the inclusion is strict if W is *not* a field. So if A is a *supplement* in an W-module B where W is a Dedekind domain, then A is a *complement* (see Theorem 5.2.1). As in abelian groups (Theorem 4.1.1), $\mathcal{C}ompl_{W\text{-}\mathcal{M}od}$ is both projectively generated, injectively generated and flatly generated (see Theorem 5.2.2): The following five proper classes of W-modules are *equal* for a Dedekind domain W:

1. $\mathcal{C}ompl_{W\text{-}\mathcal{M}od}$,

2. $\mathcal{N}eat_{W\text{-}\mathcal{M}od} \stackrel{def.}{=} \pi^{-1}(\{W/P | P \text{ maximal ideal of } W\})$,

3. $\iota^{-1}(\{M | M \in W\text{-}\mathcal{M}od \text{ and } PM = 0 \text{ for some maximal ideal } P \text{ of } W\})$,

4. $\tau^{-1}(\{W/P | P \text{ maximal ideal of } W\})$

5. The proper class of all short exact sequences

$$\mathbb{E}: \qquad 0 \longrightarrow A \stackrel{f}{\longrightarrow} B \stackrel{g}{\longrightarrow} C \longrightarrow 0$$

   of W-modules and W-module homomorphisms such that for every maximal ideal P of W,

$$A' \cap PB = PA', \qquad \text{where } A' = \operatorname{Im}(f)$$

   (or $A \cap PB = PA$ when A is identified with its image and f is taken as the inclusion homomorphism).

One of the main steps in the proof is the equality of the second, third and fifth proper classes and these follow from Nunke (1959, Lemmas 4.4, 5.2 and Theorem 5.1).

Another consequence of Nunke (1959, Theorem 5.1) is that for a Dedekind domain W, and W-modules A, C,

$$\operatorname{Ext}_{\mathcal{C}ompl_{W\text{-}\mathcal{M}od}}(C, A) = \operatorname{Ext}_{\mathcal{N}eat_{W\text{-}\mathcal{M}od}}(C, A) = \operatorname{Rad}(\operatorname{Ext}_W(C, A)).$$

(see Theorem 5.2.3). So in abelian groups, for the proper class $\mathcal{C} = \mathcal{C}ompl_{\mathbb{Z}\text{-}\mathcal{M}od} = \mathcal{N}eat_{\mathbb{Z}\text{-}\mathcal{M}od}$,

$$\operatorname{Ext}_{\mathcal{C}}(C, A) = \bigcap_{p \text{ prime}} p \operatorname{Ext}(C, A).$$

(Fuchs (1970, Exercise 53.4)).

1.7 c-injectivity

Let X and M be R-modules. The module X is called *M-c-injective* if, for every *closed* submodule A of M, every homomorphism $f : A \longrightarrow X$ can be lifted to M, i.e. there exists a homomorphism $\tilde{f} : M \longrightarrow X$ such that $\tilde{f}|_A = f$:

$$\begin{array}{ccc} A & \leq_c & M \\ {\scriptstyle f}\downarrow & \swarrow {\scriptstyle \tilde{f}} & \\ X & & \end{array}$$

A module M is called *self-c-injective* if M is M-c-injective. For a discussion of c-injectivity and related problems see Santa-Clara & Smith (2000), Smith (2000a) and Santa-Clara & Smith (2004).

We say that an R-module X is *c-injective* if it is M-c-injective for every R-module M. This is just $Compl_{R\text{-}\mathcal{M}od}$-injectives since closed submodules and complement submodules of a module coincide. Santa-Clara & Smith (2004, Theorem 6) shows that for a Dedekind domain W, every direct product of simple W-modules is self-c-injective.

An R-module M is said to be a *homogenous* (*isotypic*) *semisimple* R-module if M is a semisimple R-module whose simple submodules are all isomorphic, that is, $M = \bigoplus_{\lambda \in \Lambda} S_\lambda$ for some index set Λ and simple submodules S_λ of M such that for some maximal left ideal P of R, $S_\lambda \cong R/P$ for every $\lambda \in \Lambda$.

Santa-Clara & Smith (2004, after Theorem 6) has also noted that: for a Dedekind domain W, if M is a direct product of homogeneous semisimple W-modules, then M is self-c-injective and any simple W-module is M-c-injective. We will see that all these mentioned self-c-injective modules over a Dedekind domain are c-injective and we are able to describe c-injective modules by the general theorems for injectively generated proper classes since for a Dedekind domain W,

$Compl_{W\text{-}\mathcal{M}od}$ is injectively generated by homogenous semisimple W-modules.

1.8 Extending(CS) modules and lifting modules

A module M is said to be an *extending module* or CS-module if every closed (=complement) submodule is a direct summand. The "CS" here is for "complements are summands". See the monograph Dung et al. (1994) on extending modules.

A module M is *amply supplemented* if for all submodules U and V of M such that $U + V = M$, there is a supplement V' of U in M such that $V' \leq V$. See (Wisbauer, 1991, §41, before 41.7).

Dualizing extending modules, a module M is said to be a *lifting module* if M is amply supplemented and every supplement in M is a direct summand. See Lomp (1996, Ch. 4) for lifting modules, its equivalent definitions. For the property in the definition we gave, see Wisbauer (1991, 41.12).

For which rings R, all R-modules are extending (CS)? Similarly, for which rings R, all R-modules are lifting? The answer to these extreme cases is known and will be summarized in the theorem below. Since these conditions are for all R-modules, these extreme cases are in fact extreme cases for the proper classes $Compl_{R\text{-}\mathcal{M}od}$ and $Suppl_{R\text{-}\mathcal{M}od}$. All R-modules are extending (CS) if and only if $Compl_{R\text{-}\mathcal{M}od} = Split_{R\text{-}\mathcal{M}od}$. All R-modules are lifting if and only if all R-modules are amply supplemented and $Suppl_{R\text{-}\mathcal{M}od} = Split_{R\text{-}\mathcal{M}od}$. All R-modules are (amply) supplemented if and only if R is a left perfect ring by characterization of left perfect rings in Wisbauer (1991, 43.9). A ring R is said to be *left perfect* if every left R-module M has a *projective cover*, that is, an epimorphism $f : P \longrightarrow M$

from a projective module P onto M with Ker(f) small in P. So, all R-modules are lifting if and only if R is left perfect and $\mathcal{S}uppl_{R\text{-}\mathcal{M}od} = \mathcal{S}plit_{R\text{-}\mathcal{M}od}$. In fact, these extreme ends have been considered more generally in the category σ[M] for a module M. As is seen from the theorem below, we have the equivalence of left and right conditions. A module M is said to be *uniserial* if its submodules are linearly ordered by inclusion. A module is said to be *serial* if it is a direct sum of uniserial modules. A ring R is said to be a *left serial* ring if the left R-module ${}_RR$ is a serial module. Similarly, *right* serial rings are defined. See Wisbauer (1991, §55) for serial modules and rings.

Theorem 1.8.1. *(by Dung et al. (1994, 13.5) and Oshiro & Wisbauer (1995, Corollary 2.5)) For a ring R, the following are equivalent:*

(i) $\mathcal{C}ompl_{R\text{-}\mathcal{M}od} = \mathcal{S}plit_{R\text{-}\mathcal{M}od}$ *(all* left *R-modules are extending (CS)),*

(ii) $\mathcal{C}ompl_{\mathcal{M}od\text{-}R} = \mathcal{S}plit_{\mathcal{M}od\text{-}R}$ *(all* right *R-modules are extending (CS)),*

(iii) R is left *perfect and* $\mathcal{S}uppl_{R\text{-}\mathcal{M}od} = \mathcal{S}plit_{R\text{-}\mathcal{M}od}$ *(all* left *R-modules are lifting),*

(iv) R is right *perfect and* $\mathcal{S}uppl_{\mathcal{M}od\text{-}R} = \mathcal{S}plit_{\mathcal{M}od\text{-}R}$ *(all* right *R-modules are lifting),*

(v) Every (cyclic) left *R-module is the direct sum of an injective module and a semisimple module,*

(vi) Every left *R-module is the direct sum of a projective module and a semisimple module,*

(vii) Every left *R-module is a direct sum of modules of length* ≤ 2*,*

(viii) The right handed versions of (v)-(vii),

*(ix) R is (*left *and* right*) artinian serial and* $J^2 = 0$ *for the Jacobson radical J of R.*

1.9 Main results of this thesis

1.9.1 The proper class $\mathcal{C}o\text{-}\mathcal{N}eat_{R\text{-}\mathcal{M}od}$

We have,

$$\begin{aligned}\mathcal{N}eat_{R\text{-}\mathcal{M}od} &= \pi^{-1}(\{\text{all semisimple } R\text{-modules}\}) \\ &= \pi^{-1}(\{M \,|\, \operatorname{Soc} M = M,\ M \text{ an } R\text{-module}\}),\end{aligned}$$

where $\operatorname{Soc} M$ is the socle of M, that is the sum of all simple submodules of M. Dualizing this, we have defined the proper class $\mathcal{C}o\text{-}\mathcal{N}eat_{R\text{-}\mathcal{M}od}$ as said in the introduction to this chapter by

$$\begin{aligned}\mathcal{C}o\text{-}\mathcal{N}eat_{R\text{-}\mathcal{M}od} &= \iota^{-1}(\{\text{all } R\text{-modules with zero radical}\}) \\ &= \iota^{-1}(\{M \,|\, \operatorname{Rad} M = 0,\ M \text{ an } R\text{-module}\}).\end{aligned}$$

If A is a $\mathcal{C}o\text{-}\mathcal{N}eat_{R\text{-}\mathcal{M}od}$-submodule of an R-module B, denote this by $A \leq_{c\mathcal{N}} B$ and say that A is a *coneat submodule* of B, or that the submodule A of the module B is *coneat in* B.

For any ring R (see Proposition 3.4.1),

$$\mathcal{S}uppl_{R\text{-}\mathcal{M}od} \subseteq \mathcal{C}o\text{-}\mathcal{N}eat_{R\text{-}\mathcal{M}od} \subseteq \iota^{-1}(\{\text{ all (semi-)simple } R\text{-modules}\}).$$

Being a coneat submodule is like being a supplement: For a submodule A of a module B, A is coneat in B if and only if there exists a submodule $K \leq B$ such that

$$A + K = B \quad \text{and} \quad A \cap K \leq \operatorname{Rad} A$$

(Proposition 3.4.2). So, when $\operatorname{Rad} A \ll A$, A is coneat in B if and only if it is a supplement in B. See Theorem 3.5.5 for a description of $\operatorname{Ext}_{\mathcal{C}o\text{-}\mathcal{N}eat_{R\text{-}\mathcal{M}od}}(C, A)$ for given R-modules A, C, like the description for $\mathrm{Ext}_{\mathcal{S}uppl_{R\text{-}\mathcal{M}od}}(C, A)$ in Generalov (1983, Corollary 3).

For a semilocal ring R,

$$\mathcal{C}o\text{-}\mathcal{N}eat_{R\text{-}\mathcal{M}od} = \iota^{-1}(\{\text{all (semi-)simple } R\text{-modules}\}),$$

and for a left perfect ring R,

$$\mathcal{S}uppl_{R\text{-}\mathcal{M}od} = \mathcal{C}o\text{-}\mathcal{N}eat_{R\text{-}\mathcal{M}od} = \iota^{-1}(\{\text{all (semi-)simple } R\text{-modules}\})$$

(Theorem 3.8.7 and Corollary 3.8.8).

1.9.2 Coinjective and coprojective modules with respect to $\mathcal{C}ompl_{R\text{-}\mathcal{M}od}$ and $\mathcal{S}uppl_{R\text{-}\mathcal{M}od}$

The injective envelope of a module is constructed by embedding that module into an injective module and taking a maximal essential extension in that injective module. It is already known from this construction that $\mathcal{C}ompl_{R\text{-}\mathcal{M}od}$-coinjective modules are only injective modules. Dually, if the ring R has zero Jacobson radical, then $\mathcal{S}uppl_{R\text{-}\mathcal{M}od}$-coprojective modules are only projective modules (Theorem 3.7.2). Similarly, $\mathcal{C}o\text{-}\mathcal{N}eat_{R\text{-}\mathcal{M}od}$-coprojective modules are also only projective modules if $\mathrm{Rad}\, R = 0$ (Theorem 3.7.3).

For a left C-ring R, an R module M is injective if and only if $\mathrm{Ext}^1_R(S, M) = 0$ for all simple R-modules S (Proposition 3.7.4).

1.9.3 Results in abelian groups

A supplement in an abelian group is a complement (Theorem 4.1.4). For a *finite* subgroup A of a group B, A is a complement in B if and only if it is a supplement in B (Theorem 4.3.1).

To every proper class $\mathcal{P}$, we have a relative $\text{Ext}_{\mathcal{P}}$ functor and for the proper class $\mathcal{S}uppl_{\mathbb{Z}\text{-}\mathcal{M}od}$, this functor behaves badly unlike $\mathcal{C}ompl_{\mathbb{Z}\text{-}\mathcal{M}od}$: The functor $\text{Ext}_{\mathcal{S}uppl_{\mathbb{Z}\text{-}\mathcal{M}od}}$ is *not* factorizable as $\mathbb{Z}\text{-}\mathcal{M}od \times \mathbb{Z}\text{-}\mathcal{M}od \xrightarrow{Ext_{\mathbb{Z}}} \mathcal{A}b \xrightarrow{H} \mathcal{A}b$ for any functor $H : \mathcal{A}b \longrightarrow \mathcal{A}b$ on the category of abelian groups (Theorem 4.5.3).

The inductive closure of the proper class $\mathcal{S}uppl_{\mathbb{Z}\text{-}\mathcal{M}od}$ is flatly generated by all simple abelian groups ($\mathbb{Z}/p\mathbb{Z}$, p prime), so it is equal to $\mathcal{C}ompl_{\mathbb{Z}\text{-}\mathcal{M}od} = \mathcal{N}eat_{\mathbb{Z}\text{-}\mathcal{M}od}$ (Theorem 4.4.4).

The proper class $\mathcal{C}o\text{-}\mathcal{N}eat_{\mathbb{Z}\text{-}\mathcal{M}od}$ is *strictly* between $\mathcal{S}uppl_{\mathbb{Z}\text{-}\mathcal{M}od}$ and $\mathcal{C}ompl_{\mathbb{Z}\text{-}\mathcal{M}od}$ (Theorem 4.6.5). Like $\text{Ext}_{\mathcal{S}uppl_{\mathbb{Z}\text{-}\mathcal{M}od}}$, the functor $\text{Ext}_{\mathcal{C}o\text{-}\mathcal{N}eat_{\mathbb{Z}\text{-}\mathcal{M}od}}$ is *not* factorizable as $\mathbb{Z}\text{-}\mathcal{M}od \times \mathbb{Z}\text{-}\mathcal{M}od \xrightarrow{Ext_{\mathbb{Z}}} \mathcal{A}b \xrightarrow{H} \mathcal{A}b$ for any functor $H : \mathcal{A}b \longrightarrow \mathcal{A}b$ on the category of abelian groups (Theorem 4.6.7). For a *finite* subgroup A of a group B, A is neat in B if and only if it is coneat in B (Theorem 4.6.6). For a *torsion* group B, neat subgroups and coneat subgroups coincide (Theorem 4.6.8).

$\mathcal{C}ompl_{\mathbb{Z}\text{-}\mathcal{M}od}$-coinjectives are only injective (divisible) abelian groups.
$\mathcal{S}uppl_{\mathbb{Z}\text{-}\mathcal{M}od}$-coinjectives are also only injective (divisible) abelian groups.
$\mathcal{S}uppl_{\mathbb{Z}\text{-}\mathcal{M}od}$-coprojectives are only free abelian groups. $\mathcal{C}ompl_{\mathbb{Z}\text{-}\mathcal{M}od}$-coprojectives are only torsion-free abelian groups (Theorem 4.7.1).

1.9.4 Results in modules over Dedekind domains

Let W be a Dedekind domain.

In Section 1.6, it has been noted that, the proper class $\mathcal{C}ompl_{W\text{-}\mathcal{M}od}$ is both projectively generated, injectively generated and flatly generated (see Theorem 5.2.2). In particular, we can describe $\mathcal{C}ompl_{W\text{-}\mathcal{M}od}$-injective modules by Proposition 2.4.4 for proper classes injectively generated by a class of modules closed under taking submodules, since for a Dedekind domain W, $\mathcal{C}ompl_{W\text{-}\mathcal{M}od}$ equals the injectively generated proper class

$$\iota^{-1}(\{M|M \in W\text{-}\mathcal{M}od \text{ and } PM = 0 \text{ for some maximal ideal } P \text{ of } W\})$$

If for a module M, $PM = 0$ and P is a maximal ideal in W (so W/P is a field), then M may be considered as a W/P-vector space. If its dimension is α, then it is isomorphic to a direct sum of α copies of W/P. Hence it is a homogenous (isotypic) semisimple W-module. Thus,

$$\mathcal{C}ompl_{W\text{-}\mathcal{M}od} = \iota^{-1}(\{M|M \text{ is a homogeneous semisimple } W\text{-module }\}).$$

Hence, for a Dedekind domain W, every direct summand of a direct product of homogeneous semisimple W-modules and of injective envelopes of cyclic W-modules are c-injective (i.e. $\mathcal{C}ompl_{W\text{-}\mathcal{M}od}$-injective) and these are the only c-injective W-modules (Theorem 5.2.4).

Like in abelian groups, $\mathcal{C}ompl_{W\text{-}\mathcal{M}od} = \mathcal{N}eat_{W\text{-}\mathcal{M}od}$ is also injectively generated by all simple W-modules (Proposition 5.2.5).

As has been noted in Section 1.6, $\mathcal{S}uppl_{W\text{-}\mathcal{M}od} \subseteq \mathcal{C}ompl_{W\text{-}\mathcal{M}od}$. A partial converse is the following: A *finitely generated torsion* submodule of a W-module is a complement if and only if it is a supplement (Theorem 5.3.1).

For a Dedekind domain W which is not a field,

(i) If $\operatorname{Rad} W = 0$, then

$$\mathcal{S}uppl_{W\text{-}\mathcal{M}od} \subsetneqq \mathcal{C}o\text{-}\mathcal{N}eat_{W\text{-}\mathcal{M}od} \subsetneqq \mathcal{N}eat_{W\text{-}\mathcal{M}od} = \mathcal{C}ompl_{W\text{-}\mathcal{M}od},$$

(ii) If $\operatorname{Rad} W \neq 0$, then

$$\mathcal{S}uppl_{W\text{-}\mathcal{M}od} \subsetneqq \mathcal{C}o\text{-}\mathcal{N}eat_{W\text{-}\mathcal{M}od} = \mathcal{N}eat_{W\text{-}\mathcal{M}od} = \mathcal{C}ompl_{W\text{-}\mathcal{M}od},$$

(Theorem 5.4.6).

For a Dedekind domain W, if $\operatorname{Rad} W = 0$ and W is not a field, then the functors $\operatorname{Ext}_{\mathcal{S}uppl_{W\text{-}\mathcal{M}od}}$ and $\operatorname{Ext}_{\mathcal{C}o\text{-}\mathcal{N}eat_{W\text{-}\mathcal{M}od}}$ are *not* factorizable as

$$W\text{-}\mathcal{M}od \times W\text{-}\mathcal{M}od \xrightarrow{Ext_W} W\text{-}\mathcal{M}od \xrightarrow{H} W\text{-}\mathcal{M}od$$

for any functor $H : W\text{-}\mathcal{M}od \longrightarrow W\text{-}\mathcal{M}od$ (Theorem 5.4.8).

Like in abelian groups, for a *torsion* W-module B, neat submodules and coneat submodules coincide (Theorem 5.4.9).

$\mathcal{C}ompl_{W\text{-}\mathcal{M}od}$-coinjectives are only injective W-modules. $\mathcal{S}uppl_{W\text{-}\mathcal{M}od}$-coinjectives are also only injective W-modules. If $\operatorname{Rad} W = 0$, then $\mathcal{S}uppl_{W\text{-}\mathcal{M}od}$-coprojectives are only projective W-modules. $\mathcal{C}ompl_{W\text{-}\mathcal{M}od}$-coprojectives are only torsion-free W-modules (Theorem 5.5.1).

CHAPTER TWO
PROPER CLASSES

We will not see the general definition of proper classes in an abelian category as in Maclane (1963, Ch. 12) since our main investigations are in the proper classes of modules. The definition of proper classes and related terminology have been summarized in Section 1.3 in the first chapter. In Section 2.2, we review the definitions, which have been given in Section 1.3, for projectives, injectives, coprojectives, coinjectives with respect to a proper class, using diagrams and $\text{Ext}_{\mathcal{P}}$ with respect to a proper class $\mathcal{P}$ mentioned in Section 2.1. In the other sections of this chapter, we have summarized the results that we refer frequently for proper classes of R-modules which are projectively generated or injectively generated or flatly generated. In the last Section 2.6, coinjective and coprojective modules with respect to a projectively or injectively generated proper class is described. Our summary is from the survey Sklyarenko (1978).

2.1 $\text{Ext}_{\mathcal{P}}$ with respect to a proper class $\mathcal{P}$

The functor Ext^n_R, $n \in \mathbb{Z}^+ \cup \{0\}$: In the proper class $\mathcal{A}bs_{R\text{-}\mathcal{M}od}$, there are enough injectives and enough projectives. So every module has a projective resolution and an injective resolution. Thus for given R-modules A, C we can take an injective resolution

$$0 \longrightarrow A \xrightarrow{\delta} E_0 \xrightarrow{d_0} E_1 \xrightarrow{d_1} E_2 \longrightarrow \cdots$$

which is an exact sequence with all $E_0, E_1, E_2, \ldots$ injective and define for each $n \in \mathbb{Z}^+ \cup \{0\}$, $\mathrm{Ext}^n(C,A) = \mathrm{Ker}(\mathrm{Hom}(C,d_n))/\,\mathrm{Im}(\mathrm{Hom}(C,d_{n-1}))$, that is $\mathrm{Ext}^n(C,-)$ is the n^{th}-right derived functor of the functor $\mathrm{Hom}(C,-) : R\text{-}\mathcal{M}od \longrightarrow \mathcal{A}b$ (we set $d_{-1} = 0$, so that $\mathrm{Ext}^0(C,A) \cong \mathrm{Hom}(C,A)$). This group $\mathrm{Ext}^n(C,A)$ is well-defined, it is up to isomorphism independent of the choice of the injective resolution and in fact can also be defined using projective resolutions. The functor Ext remedies the inexactness of the functor Hom. See for example Alizade & Pancar (1999), Rotman (1979), Maclane (1963) and Cartan & Eilenberg (1956).

The functor Ext^1_R: There is an alternative definition of Ext^1_R using the so called Baer sum. Let A and C be R-modules. Two short exact sequences

$$\mathbb{E} : 0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0 \quad \text{and} \quad \mathbb{E}' : 0 \longrightarrow A \xrightarrow{f'} B' \xrightarrow{g'} C \longrightarrow 0$$

of R-modules and R-module homomorphisms starting with A and ending with C are said to be *equivalent* if we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 1_A} & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle 1_C} & & \\ 0 & \longrightarrow & A & \xrightarrow{f'} & B' & \xrightarrow{g'} & C & \longrightarrow & 0 \end{array}$$

with some R-module homomorphism $\psi : B \longrightarrow B'$, where $1_A : A \longrightarrow A$ and $1_C : C \longrightarrow C$ are identity maps. Denote by $[\mathbb{E}]$ the equivalence class of the short exact sequence $\mathbb{E}$. $\mathrm{Ext}^1_R(C,A)$ consists of all equivalence classes of short exact sequences of R-modules and R-module homomorphisms starting with A and ending with C. The addition in $\mathrm{Ext}^1_R(C,A)$ is given by Baer sum. A bifunctor $Ext^1_R : R\text{-}\mathcal{M}od \times R\text{-}\mathcal{M}od \longrightarrow \mathcal{A}b$ is obtained along these lines. Denote Ext^1_R shortly by Ext_R. See Maclane (1963, Ch. III).

Let A, C be R-modules. $\mathbb{E} \in\in \mathrm{Ext}_R(C,A)$ means that $\mathbb{E}$ is an element of an element of the group $\mathrm{Ext}_R(C,A)$, that is the equivalence class $[\mathbb{E}] \in \mathrm{Ext}_R(C,A)$, so it just means that $\mathbb{E}$ is a short exact sequence of R-modules starting with A

and ending with C. If the underlying ring R is fixed, we just write $\text{Ext}(C, A)$ instead of $\text{Ext}_R(C, A)$ when there is no ambiguity.

Note that when the ring R is *commutative*, $\text{Ext}_R(C, A)$ has a natural R-module structure for R-modules A, C. So, we have in this case a bifunctor $Ext^1_R : R\text{-}\mathcal{M}od \times R\text{-}\mathcal{M}od \longrightarrow R\text{-}\mathcal{M}od$.

The functor $\text{Ext}^1_{\mathcal{P}}$**:** In a proper class $\mathcal{P}$, we may not have enough injectives and enough projectives, so it is not possible in this case to use derived functors to give relative versions of Ext. But the alternative definition of Ext^1_R may be used in this case.

For a proper class $\mathcal{P}$ and R-modules A, C, denote by $\text{Ext}^1_{\mathcal{P}}(C, A)$ or shortly by $\text{Ext}_{\mathcal{P}}(C, A)$, the equivalence classes of all short exact sequences *in* $\mathcal{P}$ which start with A and end with C. This turns out to be a subgroup of $\text{Ext}_R(C, A)$ and a bifunctor $\text{Ext}^1_{\mathcal{P}} : R\text{-}\mathcal{M}od \times R\text{-}\mathcal{M}od \longrightarrow \mathcal{A}b$ is obtained which is a subfunctor of Ext^1_R. See Maclane (1963, Ch. 12, §4-5). Alternatively, using such a subfunctor will help to define a proper class.

The functor $\text{Ext}^n_{\mathcal{P}}$, $n \in \mathbb{Z}^+ \cup \{0\}$**:** Similar to the construction for $\text{Ext}^1_{\mathcal{P}}$, by considering long extensions

$$0 \longrightarrow A \longrightarrow B_1 \longrightarrow B_2 \longrightarrow \cdots \longrightarrow B_n \longrightarrow C \longrightarrow 0$$

with a suitable equivalence relation and addition gives us a bifunctor $\text{Ext}^n_{\mathcal{P}} : R\text{-}\mathcal{M}od \times R\text{-}\mathcal{M}od \longrightarrow \mathcal{A}b$. See Maclane (1963, Ch. 12, §4-5).

Using these functors $\text{Ext}^n_{\mathcal{P}}$, the *global dimension* of a proper class $\mathcal{P}$ is defined as the smallest nonnegative integer n such that $\text{Ext}^{n+1}_{\mathcal{P}}(C, A) = 0$ for all modules A, C, but $\text{Ext}^n_{\mathcal{P}}(C, A) \neq 0$ for some modules A, C, if of course such an n exits; otherwise it is set ∞.

2.2 Projectives, injectives, coprojectives and coinjectives with respect to a proper class

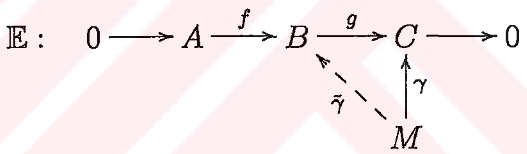
Take a short exact sequence

$$\mathbb{E}: \qquad 0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

of R-modules and R-module homomorphisms.

An R-module M is said to be *projective with respect to the short exact sequence* $\mathbb{E}$, or *with respect to the epimorphism* g if any of the following equivalent conditions holds:

1. every diagram



   where the first row is $\mathbb{E}$ and $\gamma : M \longrightarrow C$ is an R-module homomorphism can be embedded in a commutative diagram by choosing an R-module homomorphism $\tilde{\gamma} : M \longrightarrow B$; that is, for every homomorphism $\gamma : M \longrightarrow C$, there exits a homomorphism $\tilde{\gamma} : M \longrightarrow B$ such that $g \circ \tilde{\gamma} = \gamma$.

2. The sequence

$$\mathrm{Hom}(M, \mathbb{E}): \quad 0 \longrightarrow \mathrm{Hom}(M, A) \xrightarrow{f_*} \mathrm{Hom}(M, B) \xrightarrow{g_*} \mathrm{Hom}(M, C) \longrightarrow 0$$

   is exact.

Dually, an R-module M is said to be *injective with respect to the short exact sequence* $\mathbb{E}$, or *with respect to the monomorphism* g if any of the following equivalent conditions holds:

1. every diagram

$$\begin{array}{llllllllll} \mathbb{E}: & 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & & {\scriptstyle\alpha}\big\downarrow & \swarrow_{\tilde{\alpha}} & & & & & \\ & & & M & & & & & & \end{array}$$

where the first row is $\mathbb{E}$ and $\alpha : A \longrightarrow M$ is an R-module homomorphism can be embedded in a commutative diagram by choosing an R-module homomorphism $\tilde{\alpha} : B \longrightarrow M$; that is, for every homomorphism $\alpha : A \longrightarrow M$, there exists a homomorphism $\tilde{\alpha} : B \longrightarrow M$ such that $\tilde{\alpha} \circ f = \alpha$.

2. The sequence

$$\mathrm{Hom}(\mathbb{E}, M): \quad 0 \longrightarrow \mathrm{Hom}(C, M) \xrightarrow{g^*} \mathrm{Hom}(B, M) \xrightarrow{f^*} \mathrm{Hom}(A, M) \longrightarrow 0$$

is exact.

Denote by $\mathcal{P}$ a proper class of R-modules.

The following definitions have been given in Section 1.3. An R-module M is said to be *$\mathcal{P}$-projective* [*$\mathcal{P}$-injective*] if it is projective [injective] with respect to all short exact sequences in $\mathcal{P}$. Denote all $\mathcal{P}$-projective [$\mathcal{P}$-injective] modules by $\pi(\mathcal{P})$ [$\iota(\mathcal{P})$]. An R-module C is said to be *$\mathcal{P}$-coprojective* if *every* short exact sequence of R-modules and R-module homomorphisms of the form

$$\mathbb{E}: \quad 0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

ending with C is in the proper class $\mathcal{P}$. An R-module A is said to be *$\mathcal{P}$-coinjective* if *every* short exact sequence of R-modules and R-module homomorphisms of the form

$$\mathbb{E}: \quad 0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

starting with A is in the proper class $\mathcal{P}$.

Using the functor $\mathrm{Ext}_{\mathcal{P}}$, the $\mathcal{P}$-projectives, $\mathcal{P}$-injectives, $\mathcal{P}$-coprojectives, $\mathcal{P}$-coinjectives are simply described as extreme ends for the subgroup $\mathrm{Ext}_{\mathcal{P}}(C, A) \leq \mathrm{Ext}_R(C, A)$ being 0 or the whole of $\mathrm{Ext}_R(C, A)$:

1. An R -module C is *$\mathcal{P}$-projective* if and only if
$$\mathrm{Ext}_{\mathcal{P}}(C,A)=0 \text{ for all } R\text{-modules } A.$$

2. An R -module C is *$\mathcal{P}$-coprojective* if and only if
$$\mathrm{Ext}_{\mathcal{P}}(C,A)=\mathrm{Ext}_{R}(C,A) \text{ for all } R\text{-modules } A.$$

3. An R -module A is *$\mathcal{P}$-injective* if and only if
$$\mathrm{Ext}_{\mathcal{P}}(C,A)=0 \text{ for all } R\text{-modules } C.$$

4. An R -module A is *$\mathcal{P}$-coinjective* if and only if
$$\mathrm{Ext}_{\mathcal{P}}(C,A)=\mathrm{Ext}_{R}(C,A) \text{ for all } R\text{-modules } C.$$

2.3 Projectively generated proper classes

For a given class $\mathcal{M}$ of modules, denote by $\pi^{-1}(\mathcal{M})$ the class of all short exact sequences $\mathbb{E}$ of R-modules and R-module homomorphisms such that $\mathrm{Hom}(M,\mathbb{E})$ is exact for all $M\in\mathcal{M}$, that is,

$$\pi^{-1}(\mathcal{M})=\{\mathbb{E}\in \mathcal{A}bs_{R\text{-}\mathcal{M}od}|\,\mathrm{Hom}(M,\mathbb{E}) \text{ is exact for all } M\in\mathcal{M}\}.$$

$\pi^{-1}(\mathcal{M})$ is the largest proper class $\mathcal{P}$ for which each $M\in\mathcal{M}$ is $\mathcal{P}$-projective. It is called the proper class *projectively generated* by $\mathcal{M}$.

For a proper class $\mathcal{P}$, the *projective closure* of $\mathcal{P}$ is the proper class $\pi^{-1}(\pi(\mathcal{P}))$ which contains $\mathcal{P}$. If the projective closure of $\mathcal{P}$ is equal to itself, then it is said to be *projectively closed*, and that occurs if and only if it is projectively generated.

A proper class $\mathcal{P}$ of R-modules is said to have *enough projectives* if for every R-module M, there exists a $\mathcal{P}$-projective P with a $\mathcal{P}$-epimorphism $P\longrightarrow M$. A proper class $\mathcal{P}$ of R-modules with enough projectives is also said to be a *projective proper class.*

Proposition 2.3.1. *(Sklyarenko, 1978, Proposition 1.1) Every projective proper class is projectively generated.*

Let $\mathcal{P}$ be a proper class of R-modules. Direct sums of $\mathcal{P}$-projective modules is $\mathcal{P}$-projective. Direct summand of an $\mathcal{P}$-projective module is $\mathcal{P}$-projective.

A proper class $\mathcal{P}$ is called $\prod$-*closed* if for *every* collection $\{\mathbb{E}_\lambda\}_{\lambda\in\Lambda}$ in $\mathcal{P}$, the product $\mathbb{E} = \prod_{\lambda\in\Lambda} \mathbb{E}_\lambda$ is in $\mathcal{P}$, too.

Proposition 2.3.2. *(Sklyarenko, 1978, Proposition 1.2) Every projectively generated proper class is* $\prod$*-closed.*

A subclass $\mathcal{M}$ of a class $\overline{\mathcal{M}}$ of modules is called a *projective basis* for $\overline{\mathcal{M}}$ if every module in $\overline{\mathcal{M}}$ is a direct summand of a direct sum of modules in $\mathcal{M}$ and of free modules.

Proposition 2.3.3. *(Sklyarenko, 1978, Proposition 2.1) If $\mathcal{M}$ is a set, then the proper class $\pi^{-1}(\mathcal{M})$ is projective, and $\mathcal{M}$ is a projective basis for the class of all $\mathcal{P}$-projective modules.*

Even when $\mathcal{M}$ is not a set but:

Proposition 2.3.4. *(Sklyarenko, 1978, Proposition 2.3) If $\mathcal{M}$ is a class of modules closed under passage to factor modules, then the proper class $\pi^{-1}(\mathcal{M})$ is projective, and $\mathcal{M}$ is a projective basis for the class of all $\mathcal{P}$-projective modules.*

2.4 Injectively generated proper classes

For a given class $\mathcal{M}$ of modules, denote by $\iota^{-1}(\mathcal{M})$ the class of all short exact sequences $\mathbb{E}$ of R-modules and R-module homomorphisms such that $\mathrm{Hom}(\mathbb{E}, M)$ is exact for all $M \in \mathcal{M}$, that is,

$$\iota^{-1}(\mathcal{M}) = \{\mathbb{E} \in \mathcal{A}bs_{R\text{-}\mathcal{M}od} | \, \mathrm{Hom}(\mathbb{E}, M) \text{ is exact for all } M \in \mathcal{M}\}.$$

$\iota^{-1}(\mathcal{M})$ is the largest proper class $\mathcal{P}$ for which each $M \in \mathcal{M}$ is $\mathcal{P}$-injective. It is called the proper class *injectively generated* by $\mathcal{M}$.

For a proper class $\mathcal{P}$, the *injective closure* of $\mathcal{P}$ is the proper class $\iota^{-1}(\iota(\mathcal{P}))$ which contains $\mathcal{P}$. If the injective closure of $\mathcal{P}$ is equal to itself, then it is said to be *injectively closed*, and that occurs if and only if it is injectively generated.

A proper class $\mathcal{P}$ of R-modules is said to have *enough injectives* if for every R-module M, there exists a $\mathcal{P}$-injective I with a $\mathcal{P}$-monomorphism $M \longrightarrow I$. A proper class $\mathcal{P}$ of R-modules with enough injectives is also said to be an *injective proper class.*

Proposition 2.4.1. *(Sklyarenko, 1978, Proposition 3.1) Every injective proper class is injectively generated.*

Let $\mathcal{P}$ be a proper class of R-modules. Direct product of $\mathcal{P}$-injective modules is $\mathcal{P}$-injective. Direct summand of an $\mathcal{P}$-injective module is $\mathcal{P}$-injective.

A proper class $\mathcal{P}$ is called $\oplus$-*closed* if for *every* collection $\{\mathbb{E}_\lambda\}_{\lambda\in\Lambda}$ in $\mathcal{P}$, the direct sum $\mathbb{E} = \bigoplus_{\lambda\in\Lambda} \mathbb{E}_\lambda$ is in $\mathcal{P}$, too.

Proposition 2.4.2. *(Sklyarenko, 1978, Proposition 1.2) Every injectively generated proper class is $\oplus$-closed.*

An injective module is called *elementary* if it coincides with the injective envelope of some *cyclic* submodule. Such modules form a set and every injective module can be embedded in a direct product of elementary injective modules (Sklyarenko, 1978, Lemma 3.1).

A subclass $\mathcal{M}$ of a class $\overline{\mathcal{M}}$ of modules is called an *injective basis* for $\overline{\mathcal{M}}$ if every module in $\overline{\mathcal{M}}$ is a direct summand of a direct product of modules in $\mathcal{M}$ and of certain elementary injective modules.

Proposition 2.4.3. *(Sklyarenko, 1978, Proposition 3.3) If $\mathcal{M}$ is a set, then the proper class $\iota^{-1}(\mathcal{M})$ is injective, and $\mathcal{M}$ is an injective basis for the class of all $\mathcal{P}$-injective modules.*

Even when $\mathcal{M}$ is not a set but:

Proposition 2.4.4. *(Sklyarenko, 1978, Proposition 3.4) If $\mathcal{M}$ is a class of modules closed under taking submodules, then the proper class $\iota^{-1}(\mathcal{M})$ is injective, and $\mathcal{M}$ is an injective basis for the class of all $\mathcal{P}$-injective modules.*

2.5 Flatly generated proper classes

When the ring R is *not* commutative, we must be careful with the sides for the tensor product analogues of projectives and injectives with respect to a proper class. Remember that by an R-module, we mean a *left* R-module.

Take a short exact sequence

$$\mathbb{E}: \qquad 0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

of R-modules and R-module homomorphisms. We say that a *right* R-module M

is *flat with respect to the short exact sequence* $\mathbb{E}$, or *with respect to the monomorphism* g if

$$M \otimes \mathbb{E}: \quad 0 \longrightarrow M \otimes A \xrightarrow{1_m \otimes f} M \otimes B \xrightarrow{1_m \otimes g} M \otimes C \longrightarrow 0$$

is exact.

Denote by $\mathcal{P}$ a proper class of R-modules.

A *right* R-module M is said to be $\mathcal{P}$-*flat* if M is flat with respect to every short exact sequence $\mathbb{E} \in \mathcal{P}$, that is, $M \otimes \mathbb{E}$ is exact for every $\mathbb{E}$ in $\mathcal{P}$. Denote all $\mathcal{P}$-flat *right* R-modules by $\tau(\mathcal{P})$. The class of all such modules is closed under direct sums, direct limits, and direct summands. Hence it is natural to define a basis for $\mathcal{P}$-flat modules as a family from which each such module can be obtained by these operations.

For a given class $\mathcal{M}$ of *right* R-modules, denote by $\tau^{-1}(\mathcal{M})$ the class of all short exact sequences $\mathbb{E}$ of R-modules and R-module homomorphisms such that $M \otimes \mathbb{E}$ is exact for all $M \in \mathcal{M}$:

$$\tau^{-1}(\mathcal{M}) = \{\mathbb{E} \in \mathcal{A}bs_{R\text{-}\mathcal{M}od} | M \otimes \mathbb{E} \text{ is exact for all } M \in \mathcal{M}\}.$$

$\tau^{-1}(\mathcal{M})$ is the largest proper class $\mathcal{P}$ of (left) R-modules for which each $M \in \mathcal{M}$ is $\mathcal{P}$-flat. It is called the proper class *flatly generated* by the class $\mathcal{M}$ of *right* R-modules.

When the ring R is commutative, there is no need to mention the sides of the modules since a right R-module may also be considered as a left R-module and vice versa.

Let M be a *finitely presented* R-module, that is, $M \cong F/G$ for some *finitely generated free* R-module F and some *finitely generated* submodule G of F. So, we have a short exact sequence

$$0 \longrightarrow G \longrightarrow F \longrightarrow M \longrightarrow 0$$

Any short exact sequence

$$0 \longrightarrow H \longrightarrow F' \longrightarrow M \longrightarrow 0$$

where F' is a *finitely generated free* module and H is a *finitely generated* module is called a *free presentation* of M. An exact sequence

$$F'' \longrightarrow F' \longrightarrow M \longrightarrow 0$$

where F' and F'' are *finitely generated free* modules is also called a *free presentation* of M. More generally, an exact sequence

$$P_1 \xrightarrow{f} P_0 \xrightarrow{g} M \longrightarrow 0$$

where P_0 and P_1 are *finitely generated projective* modules will be called a *presentation* of M. Apply $\mathrm{Hom}(-, R)$ to this presentation:

$$0 \longrightarrow \mathrm{Hom}(M, R) \xrightarrow{g^*} \mathrm{Hom}(P_0, R) \xrightarrow{f^*} \mathrm{Hom}(P_1, R)$$

Fill the right side of this sequence of *right* R-modules by the module $F^0 := \mathrm{Hom}(P_1, R)/\operatorname{Im}(f^*)$ to obtain the exact sequence

$$\mathrm{Hom}(P_0, R) \xrightarrow{f^*} \mathrm{Hom}(P_1, R) \xrightarrow{\sigma} F^0 = \mathrm{Hom}(P_1, R)/\operatorname{Im}(f^*) \longrightarrow 0, \qquad (2.5.1)$$

where σ is the canonical epimorphism. For a finitely generated projective R-module P, $\mathrm{Hom}(P, R)$ is a finitely generated projective *right* R-module. So $\mathrm{Hom}(P_0, R)$ and $\mathrm{Hom}(P_1, R)$ are finitely generated projective modules, hence the exact sequence (2.5.1) is a presentation for the *finitely presented right* module F^0. Note that the correspondence $F \mapsto F^0$ is not one-to-one, for it depends on the presentation of F. Also F can be interpreted as $(F^0)^0$ (by taking a free presentation of F). See Sklyarenko (1978, §5).

Proposition 2.5.1. *(Sklyarenko, 1978, Corollary 5.1) For any finitely presented module F and any short exact sequence $\mathbb{E}$ of R-modules, the sequence* $\mathrm{Hom}(F, \mathbb{E})$ *is exact if and only if the sequence $F^0 \otimes \mathbb{E}$ is exact.*

Using this:

Theorem 2.5.2. *(Sklyarenko, 1978, Theorem 8.3) Let $\mathcal{M}$ be a set of finitely presented R-modules. Associate with each $F \in \mathcal{M}$, the right R-module F^0 and let $\mathcal{M}^0$ be the set of all these F^0. We may assume that $(\mathcal{M}^0)^0 = \mathcal{M}$. Then*

$$\pi^{-1}(\mathcal{M}) = \tau^{-1}(\mathcal{M}^0) \quad \textit{and} \quad \tau^{-1}(\mathcal{M}) = \pi^{-1}(\mathcal{M}^0)$$

Proposition 2.5.3. *(Sklyarenko, 1978, Lemma 5.1) For any short exact sequence $\mathbb{E}$ of R-modules, any* right *R-module M, the sequence $M \otimes \mathbb{E}$ is exact if and only if the sequence* $\mathrm{Hom}(M, \mathbb{E}^*)$ *is exact, where $\mathbb{E}^* := \mathrm{Hom}_{\mathbb{Z}}(\mathbb{E}, \mathbb{Q}/\mathbb{Z})$.*

A proper class $\mathcal{P}$ is said to be *inductively closed* if for every direct system $\{\mathbb{E}_i (i \in I); \pi_i^j (i \leq j)\}$ in $\mathcal{P}$, the direct limit $\mathbb{E} = \varinjlim \mathbb{E}_i$ is also in $\mathcal{P}$ (see Fedin (1983) and Sklyarenko (1978, §8)). As in Fedin (1983), for a proper class $\mathcal{P}$, denote by $\widetilde{\mathcal{P}}$, the smallest inductively closed proper class containing $\mathcal{P}$; it is called the *inductive closure* of $\mathcal{P}$.

Since tensor product and direct limit commutes, a flatly generated proper class is inductively closed; moreover:

Theorem 2.5.4. *(Sklyarenko, 1978, Theorem 8.1) For a given class $\mathcal{M}$ of* right *R-modules, the proper class $\tau^{-1}(\mathcal{M})$ is inductively closed. It is injectively generated, and if $\mathcal{M}$ is a set, then it is an injective proper class. A short exact sequence $\mathbb{E}$ belongs to $\tau^{-1}(\mathcal{M})$ if and only if $\mathbb{E}^* \in \pi^{-1}(\mathcal{M})$, where $\mathbb{E}^* := \mathrm{Hom}_{\mathbb{Z}}(\mathbb{E}, \mathbb{Q}/\mathbb{Z})$.*

2.6 Coinjective and coprojective modules with respect to a projectively or injectively generated proper class

Throughout this section let $\mathcal{P}$ be a proper class of R-modules.

Proposition 2.6.1. *(Sklyarenko, 1978, Proposition 9.1) The intersection of the classes of all $\mathcal{P}$-projective modules and $\mathcal{P}$-coprojective modules coincides with the class of all projective R-modules.*

Proposition 2.6.2. *(Sklyarenko, 1978, Proposition 9.2) The intersection of the classes of all $\mathcal{P}$-injective modules and $\mathcal{P}$-coinjective modules is the class of all injective R-modules.*

Proposition 2.6.3. *(Sklyarenko, 1978, Proposition 9.3)*

(i) If $\mathcal{P}$ is injectively closed, then every direct sum of $\mathcal{P}$-coinjective modules is $\mathcal{P}$-coinjective.

(ii) If $\mathcal{P}$ is $\prod$-closed, then every product of $\mathcal{P}$-coinjective modules is $\mathcal{P}$-coinjective.

(iii) If $\mathcal{P}$ is $\oplus$-closed, then every direct sum of $\mathcal{P}$-coprojective modules is $\mathcal{P}$-coprojective.

Proposition 2.6.4. *(Sklyarenko, 1978, Proposition 9.4) If $\mathcal{P}$ is injectively generated, then for an R-module C, the condition $\mathrm{Ext}^1_R(C, J) = 0$ for all $\mathcal{P}$-injective J is equivalent to C being $\mathcal{P}$-coprojective.*

More directly:

Proposition 2.6.5. *If $\mathcal{P} = \iota^{-1}(\mathcal{M})$ for a class $\mathcal{M}$ of modules, then for an R-module C, the condition $\mathrm{Ext}^1_R(C, M) = 0$ for all $M \in \mathcal{M}$ is equivalent to C being $\mathcal{P}$-coprojective.*

Proof. Suppose C is a $\mathcal{P}$-coprojective module. Let $M \in \mathcal{M}$. Take an element $[\mathbb{E}] \in \operatorname{Ext}_R^1(C, M)$:

$$\mathbb{E}: \qquad 0 \longrightarrow M \longrightarrow B \longrightarrow C \longrightarrow 0$$

Since C is $\mathcal{P}$-coprojective, $\mathbb{E} \in \mathcal{P}$. Then $\mathbb{E}$ splits because M, being an element of $\mathcal{M}$, is $\mathcal{P}$-injective as $\mathcal{P} = \iota^{-1}(\mathcal{M})$. Hence $[\mathbb{E}] = 0$ as required. Thus $\operatorname{Ext}_R^1(C, M) = 0$.

Conversely, suppose for an R-module C, $\operatorname{Ext}_R^1(C, M) = 0$ for all $M \in \mathcal{M}$. Take any short exact sequence $\mathbb{E}$ of R-modules ending with C:

$$\mathbb{E}: \qquad 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

Applying $\operatorname{Hom}(-, M)$, we obtain the following exact sequence by the long exact sequence connecting Hom and Ext:

$$0 \longrightarrow \operatorname{Hom}(C, M) \longrightarrow \operatorname{Hom}(B, M) \longrightarrow \operatorname{Hom}(A, M) \longrightarrow \operatorname{Ext}_R^1(C, M) = 0$$

So $\operatorname{Hom}(\mathbb{E}, M)$ is exact for every $M \in \mathcal{M}$. This means $\mathbb{E} \in \iota^{-1}(\mathcal{M}) = \mathcal{P}$. $\square$

Proposition 2.6.6. *(Sklyarenko, 1978, Proposition 9.5) If $\mathcal{P}$ is projectively generated, then for an R-module A, the condition $\operatorname{Ext}_R^1(P, A) = 0$ for all $\mathcal{P}$-projective P is equivalent to A being $\mathcal{P}$-coinjective.*

More directly:

Proposition 2.6.7. *If $\mathcal{P} = \pi^{-1}(\mathcal{M})$ for a class $\mathcal{M}$ of modules, then for an R-module A, the condition $\operatorname{Ext}_R^1(M, A) = 0$ for all $M \in \mathcal{M}$ is equivalent to A being $\mathcal{P}$-coinjective.*

Proof. Suppose A is a $\mathcal{P}$-coinjective module. Let $M \in \mathcal{M}$. Take an element $[\mathbb{E}] \in \operatorname{Ext}_R^1(M, A)$:

$$\mathbb{E}: \qquad 0 \longrightarrow A \longrightarrow B \longrightarrow M \longrightarrow 0$$

Since A is $\mathcal{P}$-coinjective, $\mathbb{E} \in \mathcal{P}$. Then $\mathbb{E}$ splits because M, being an element of $\mathcal{M}$, is $\mathcal{P}$-projective as $\mathcal{P} = \pi^{-1}(\mathcal{M})$. Hence $[\mathbb{E}] = 0$ as required. Thus $\mathrm{Ext}^1_R(M, A) = 0$.

Conversely, suppose for an R-module A, $\mathrm{Ext}^1_R(M, A) = 0$ for all $M \in \mathcal{M}$. Take any short exact sequence $\mathbb{E}$ of R-modules starting with A:

$$\mathbb{E}: \qquad 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

Applying $\mathrm{Hom}(M, -)$, we obtain the following exact sequence by the long exact sequence connecting Hom and Ext:

$$0 \longrightarrow \mathrm{Hom}(M, A) \longrightarrow \mathrm{Hom}(M, B) \longrightarrow \mathrm{Hom}(M, C) \longrightarrow \mathrm{Ext}^1_R(M, A) = 0$$

So $\mathrm{Hom}(M, \mathbb{E})$ is exact for every $M \in \mathcal{M}$. This means $\mathbb{E} \in \pi^{-1}(\mathcal{M}) = \mathcal{P}$. □

CHAPTER THREE

THE PROPER CLASSES RELATED TO COMPLEMENTS AND SUPPLEMENTS

The proper classes $\mathcal{C}ompl_{R\text{-}\mathcal{M}od}$, $\mathcal{S}uppl_{R\text{-}\mathcal{M}od}$, $\mathcal{N}eat_{R\text{-}\mathcal{M}od}$ and $\mathcal{C}o\text{-}\mathcal{N}eat_{R\text{-}\mathcal{M}od}$ of R-modules can be defined more generally in suitable abelian categories (Section 3.1). In fact, we will not deal with general abelian categories; our main focus is on the category $R\text{-}\mathcal{M}od$ for an arbitrary ring R. The proper class $\mathcal{N}eat_{R\text{-}\mathcal{M}od}$ contains the proper class $\mathcal{C}ompl_{R\text{-}\mathcal{M}od}$ (Section 3.2). Equality holds if and only if R is a left C-ring (Section 3.3). Examples for C-rings and some properties of them are given in Section 3.3. The injectively generated proper class $\mathcal{C}o\text{-}\mathcal{N}eat_{R\text{-}\mathcal{M}od}$ defined dually to $\mathcal{N}eat_{R\text{-}\mathcal{M}od}$ contains $\mathcal{S}uppl_{R\text{-}\mathcal{M}od}$; a submodule A of a module B is coneat in B if and only if there exists a submodule $K \leq B$ such that $A+K=B$ and $A \cap K \leq \operatorname{Rad} A$, which is a weakened form of being a supplement (Section 3.4). For R-modules A and C, $\operatorname{Ext}_{\mathcal{C}o\text{-}\mathcal{N}eat_{R\text{-}\mathcal{M}od}}(C,A)$ has been described in Section 3.5 like the description of $\operatorname{Ext}_{\mathcal{S}uppl_{R\text{-}\mathcal{M}od}}(C,A)$ in Generalov (1983). In a module with finite uniform dimension (=Goldie dimension), there is a criterion for a submodule to be a complement via its uniform dimension, and in a module with finite hollow dimension (=dual Goldie dimension), there is a criterion for a submodule to be a supplement via its hollow dimension (Section 3.6). It is well known that $\mathcal{C}ompl_{R\text{-}\mathcal{M}od}$-coinjective modules are only injective modules; dually, if $\operatorname{Rad} R = 0$, $\mathcal{S}uppl_{R\text{-}\mathcal{M}od}$-coprojective modules are only projective modules (Section 3.7). Properties of $\mathcal{S}uppl_{R\text{-}\mathcal{M}od}$ and $\mathcal{C}o\text{-}\mathcal{N}eat_{R\text{-}\mathcal{M}od}$ over some rings like left quasi-duo rings, left max rings, semilocal rings and left perfect rings are dealt with in

Section 3.8.

3.1 $\mathcal{C}ompl_{\mathcal{A}}$, $\mathcal{S}uppl_{\mathcal{A}}$, $\mathcal{N}eat_{\mathcal{A}}$ and $\mathcal{C}o\text{-}\mathcal{N}eat_{\mathcal{A}}$ for a suitable abelian category $\mathcal{A}$

Let $\mathcal{A}$ be an abelian category (see for example Maclane (1963, Ch. 9, §1-2), or Enochs & Jenda (2000, §1.3) for the definition of abelian categories).

The class $\mathcal{C}ompl_{\mathcal{A}}$ consists of all short exact sequences

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0 \qquad (3.1.1)$$

in $\mathcal{A}$ such that A is a complement of some subobject K of B, that is $A \cap K = 0$ and K is maximal with respect to this.

The class $\mathcal{N}eat_{\mathcal{A}}$ consists of all short exact sequences (3.1.1) in $\mathcal{A}$ such that every simple object is a relative projective for it, where an object S is called *simple* if it has no subobjects except 0 and S, denoted by,

$$\mathcal{N}eat_{\mathcal{A}} = \pi_{\mathcal{A}}^{-1}(\{S \in \mathcal{A} | S \text{ simple}\}).$$

Theorem 3.1.1. *(Stenström, 1967b, Propositons 4-6) Let $\mathcal{A}$ be an abelian category in which every object M has an injective envelope $E(M)$ (for any subobject L of M, $E(L)$ is considered as a well-defined subobject of $E(M)$). Then:*

(i) $\mathcal{C}ompl_{\mathcal{A}}$ and $\mathcal{N}eat_{\mathcal{A}}$ form proper classes.

(ii) $\mathcal{C}ompl_{\mathcal{A}} \subseteq \mathcal{N}eat_{\mathcal{A}}$.

(iii) If $\mathcal{M}$ is a class of objects in $\mathcal{A}$ such that for every $A \neq 0$ in $\mathcal{A}$, there exists a monomorphism $M \longrightarrow A$ for some $M \neq 0$ in $\mathcal{M}$, then

$$\pi_{\mathcal{A}}^{-1}(\mathcal{M}) \subseteq \mathcal{C}ompl_{\mathcal{A}}.$$

Stenström (1967b, Remark after Proposition 6) points out that the proper class $\mathcal{S}uppl_{\mathcal{A}}$ can also be defined. Similarly $\mathcal{C}o\text{-}\mathcal{N}eat_{\mathcal{A}}$ may be defined. But as we said in the introduction, we will not deal with these generalizations.

3.2 The proper class $\mathcal{N}eat_{R\text{-}\mathcal{M}od}$

We will firstly see an element wise criterion for being a $\mathcal{N}eat_{R\text{-}\mathcal{M}od}$-submodule like in abelian groups. In abelian groups, for a subgroup A of the group B, the inclusion $A \hookrightarrow B$ is a $\pi^{-1}(\{\mathbb{Z}/p\mathbb{Z} | p \text{ prime}\})$-monomorphism if and only if $A \cap pB = pA$ for all prime numbers p, i.e. A is a neat subgroup of B. This result holds because:

Proposition 3.2.1. *(by Warfield (1969, Proposition 2)) For an element r in a ring R, the following are equivalent for a short exact sequence*

$$\mathbb{E}: \qquad 0 \longrightarrow A \xrightarrow{i_A} B \xrightarrow{g} C \longrightarrow 0$$

of R-modules and R-module homomorphisms where A is a submodule of B and i_A is the inclusion map:

(i) $\mathrm{Hom}(R/Rr, B) \xrightarrow{g_*} \mathrm{Hom}(R/Rr, C)$ *is epic (i.e. R/Rr is projective relative to the short exact sequence $\mathbb{E}$)*

(ii) $R/rR \otimes A \xrightarrow{1_{R/rR} \otimes i_A} R/rR \otimes B$ *is monic (i.e. R/rR is flat relative to the short exact sequence $\mathbb{E}$)*

(iii) $A \cap rB = rA$.

The equivalence of the last two assertions for the principal *right* ideal rR holds for any *right* ideal I:

Proposition 3.2.2. *(Sklyarenko, 1978, Lemma 6.1) Let A be a submodule of an R-module B and $i_A : A \hookrightarrow B$ be the inclusion map. For a* right *ideal I of R, $A \cap IB = IA$ if and only if*

$$R/I \otimes A \xrightarrow{1_{R/I} \otimes i_A} R/I \otimes B$$

is monic.

For a not necessarily principal ideal I of R, a criterion for R/I to be projective with respect to a short exact sequence is obtained using the following elementary lemma:

Lemma 3.2.3. *(see for example Fuchs & Salce (2001, Lemma I.8.4) or Sklyarenko (1978, Lemma 1.2, without proof)) Suppose*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle \alpha} & & \uparrow & & \uparrow{\scriptstyle \beta} & & \\ 0 & \longrightarrow & A_1 & \xrightarrow[f_1]{} & B_1 & \longrightarrow & C_1 & \longrightarrow & 0 \end{array}$$

is a commutative diagram of modules and module homomorphisms with exact rows. Then, β can be lifted to a homomorphism $C_1 \longrightarrow B$ if and only if α can be extended to a map $B_1 \longrightarrow A$, that is, there exists $\tilde{\beta} : C_1 \longrightarrow B$ such that $g \circ \tilde{\beta} = \beta$ if and only if there exists $\tilde{\alpha} : B_1 \longrightarrow A$ such that $\tilde{\alpha} \circ f_1 = \alpha$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle \alpha} & \overset{\tilde{\alpha}}{\nwarrow} \circlearrowleft & \uparrow & \overset{\tilde{\beta}}{\nwarrow} \circlearrowleft & \uparrow{\scriptstyle \beta} & & \\ 0 & \longrightarrow & A_1 & \xrightarrow[f_1]{} & B_1 & \longrightarrow & C_1 & \longrightarrow & 0 \end{array}$$

Proposition 3.2.4. *For a left ideal I in a ring R, the following are equivalent for a short exact sequence*

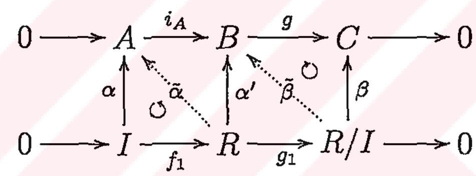
$$\mathbb{E}: \qquad 0 \longrightarrow A \xrightarrow{i_A} B \xrightarrow{g} C \longrightarrow 0$$

of R-modules and R-module homomorphisms where A is a submodule of B and i_A is the inclusion map:

(i) $\operatorname{Hom}(R/I, B) \xrightarrow{g_*} \operatorname{Hom}(R/I, C)$ *is epic (i.e. R/I is projective relative to the short exact sequence $\mathbb{E}$)*

(ii) For every $b \in B$, if $Ib \leqslant A$, then there exists $a \in A$ such that $I(b-a) = 0$.

Proof. (i)⇒(ii): Let $b \in B$ be such that $Ib \leq A$. Then we can define $\alpha : I \longrightarrow A$ by $\alpha(r) = rb$ for each $r \in I$, as $rb \in A$ for $r \in I$. Also define $\alpha' : R \longrightarrow B$ by $\alpha'(r) = rb$ for each $r \in R$. Then $\alpha(r) = \alpha'(r)$ for all $r \in I$. Let $f_1 : I \longrightarrow R$ be the inclusion map and $g_1 : R \longrightarrow R/I$ be the canonical epimorphism. Define $\beta : R/I \longrightarrow C$ by $\beta(r + I) = g(\alpha'(r))$. Since, by our hypothesis (i), R/I is projective with respect to $\mathbb{E}$, there exists $\tilde{\beta} : R/I \longrightarrow B$ such that $g \circ \tilde{\beta} = \beta$. Then by Lemma 3.2.3, there exists $\tilde{\alpha} : R \longrightarrow A$ such that $\tilde{\alpha} \circ f_1 = \alpha$:



Let $a = \tilde{\alpha}(1) \in A$. Then, for each $r \in I$,

$$rb = \alpha(r) = \tilde{\alpha}(f_1(r)) = \tilde{\alpha}(r) = r\tilde{\alpha}(1) = ra.$$

So, $r(b-a) = 0$ for all $r \in I$, i.e. $I(b-a) = 0$ as required.
(ii)⇒(i): Let $\beta : R/I \longrightarrow C$ be a given homomorphism. Let $f_1 : I \longrightarrow R$ be the inclusion map and $g_1 : R \longrightarrow R/I$ be the canonical epimorphism. Since R is projective, there exists $\alpha' : R \longrightarrow B$ such that $g \circ \alpha' = \beta \circ g_1$. So $\alpha'(I) \leq \operatorname{Ker} g = \operatorname{Im}(i_A) = A$, hence we can define $\alpha : I \longrightarrow A$ by $\alpha(r) = \alpha'(r)$ for each $r \in I$. Let $b = \alpha'(1)$. Then for each $r \in R$, $\alpha'(r) = rb$. So, $\alpha(I) = Ib \leq A$ and hence, by our hypothesis (ii), there exists $a \in A$ such that $I(b-a) = 0$, that is, for each $r \in I$, $rb = ra$. Define $\tilde{\alpha} : R \longrightarrow A$ by $\tilde{\alpha}(r) = ra$ for each $r \in R$. Then $\tilde{\alpha} \circ f_1 = \alpha$, and by Lemma 3.2.3, there exists a homomorphism $\tilde{\beta} : R/I \longrightarrow B$ such that $g \circ \tilde{\beta} = \beta$; see just the above diagram. This proves that $\operatorname{Hom}(R/I, B) \xrightarrow{g_*} \operatorname{Hom}(R/I, C)$ is epic. □

With this criterion we observe that:

Corollary 3.2.5. *For a short exact sequence*

$$\mathbb{E}: \qquad 0 \longrightarrow A \xrightarrow{i_A} B \xrightarrow{g} C \longrightarrow 0$$

of R-modules where A is a submodule of B and i_A is the inclusion map, the following are equivalent:

(i) $\mathbb{E} \in \mathcal{N}eat_{R\text{-}\mathcal{M}od} = \pi^{-1}(\{R/P | P \text{ maximal left ideal of } R\})$,

(ii) For every maximal left ideal P of R, for every $b \in B$, if $Pb \leqslant A$, then there exists $a \in A$ such that $P(b-a) = 0$.

Proposition 3.2.6. *Let A be a* complement *of a submodule K in the R-module B and P be a* maximal *left ideal of R. If $Pb \leqslant A$ for some $b \in B$, then $b \in A \oplus K$, so $b = a + k$ for some $a \in A$, $k \in K$ and*

$$P(b-a) = 0.$$

Proof. If $b \in A$, we are done. Assume $b \notin A$. As $b \notin A$, $A + Rb \gneqq A$, so since A is a complement of K in B,

$$0 \neq (A + Rb) \cap K.$$

Hence there exist some $a' \in A, u \in R, k' \in K$ such that

$$0 \neq a' + ub = k'.$$

Here u can not be in P because otherwise $ub \in A$ which would imply $0 \neq k' = a' + ub \in A \cap K = 0$, a contradiction. So necessarily $u \notin P$. Then, since P is a maximal ideal of R,

$$P + Ru = R.$$

So $p + su = 1$ for some $p \in P$, $s \in R$. Hence:

$$b = 1b = (p + su)b = pb + sub \quad \Rightarrow \quad sub = b - pb.$$

Since $a' + ub = k'$, multiplying by s, we obtain

$$sa' + \underbrace{sub}_{=b-pb} = sk' \quad \Rightarrow \quad b = \underbrace{-sa'}_{\in A} + \underbrace{pb}_{\in A} + \underbrace{sk'}_{\in K} \in A \oplus K.$$

So $b = a + k$ for some $a \in A$, $k \in K$. For each $q \in P$,

$$\underbrace{qb}_{\in A} = \underbrace{qa}_{\in A} + \underbrace{qk}_{\in K} \quad \Rightarrow \quad qb = qa \text{ and } 0 = qk \text{ as we have direct sum } A \oplus K$$

So we obtain $qk = 0$ for every $q \in P$, hence $Pk = 0$, where $b - a = k \in K$ and $a \in A$. □

This proposition also gives a proof of Theorem 3.1.1-(ii) for R-modules:

Corollary 3.2.7. *(Stenström, 1967b, by Proposition 5) For any ring R,*

$$\mathcal{C}ompl_{R\text{-}\mathcal{M}od} \subseteq \mathcal{N}eat_{R\text{-}\mathcal{M}od}.$$

Proof. Another proof may be given just by showing that every short exact sequence

$$\mathbb{E}: \qquad 0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} S \longrightarrow 0$$

in $\mathcal{C}ompl_{R\text{-}\mathcal{M}od}$ splits if S is a simple module. Without loss of generality, assume A is a submodule of B and f is the inclusion homomorphism. Since A is a complement in B, it is closed there and since $B/A \cong S \neq 0$, A is a proper closed submodule of B. So A is not essential in B, hence there exists a nonzero module K in B such that $A \cap K = 0$. Since $K \neq 0$, there exists $0 \neq k \in K$ which is not in A as $A \cap K = 0$. Since $B/A \cong S$ is simple, it is generated by any nonzero element. Hence $B/A = R(k + A)$, which implies $B = A + Rk$. Since $A \cap K = 0$, $A \cap Rk = 0$, too. So, $B = A \oplus Rk$, and the sequence $\mathbb{E}$ splits. □

3.3 C-rings

Proposition 3.3.1. *(Stenström, 1967b, Corollary to Proposition 8) If R is a* *commutative Noetherian ring in which every* nonzero *prime ideal is maximal, then*

$$\mathcal{C}ompl_{R\text{-}\mathcal{M}od} = \mathcal{N}eat_{R\text{-}\mathcal{M}od}.$$

Generalov (1978) gives a characterization of this equality in terms of the ring R: Equality holds if and only if R is a *left* C-ring. The notion of C-ring has been introduced by Renault (1964): A ring R is said to be a *left* C-ring if for every (left) R-module B and for every essential proper submodule A of B, $\mathrm{Soc}(B/A) \neq 0$, that is B/A has a simple submodule. Similarly *right* C-rings are defined.

Theorem 3.3.2. *(by Generalov (1978, Theorem 5)) For a ring R,*

$$\mathcal{C}ompl_{R\text{-}\mathcal{M}od} = \mathcal{N}eat_{R\text{-}\mathcal{M}od} \quad \text{if and only if} \quad R \text{ is a left } C\text{-ring.}$$

Proof. ($\Rightarrow$): Let B be an R-module and A an essential proper submodule of B. So A is not closed in B as it has a proper essential extension, so not a complement in B (as complement submodules and closed submodules are the same). Since $\mathcal{C}ompl_{R\text{-}\mathcal{M}od} = \mathcal{N}eat_{R\text{-}\mathcal{M}od}$, this shows that A is not a $\mathcal{N}eat_{R\text{-}\mathcal{M}od}$-submodule of B. Then by Corollary 3.2.5, there exists a maximal left ideal P of R and an element $b \in B$ such that $Pb \leq A$ but $P(b-a) \neq 0$ for every $a \in A$. Then the cyclic submodule $R(b+A)$ of B/A is simple as $R(b+A) \cong R/P$ since $Pb \leq A$, $b \notin A$ (because if b were in A, then $P(b-b) = 0$ would hold, contradicting $P(b-a) \neq 0$ for every $a \in A$) and P is a maximal left ideal of R.

($\Leftarrow$): Let B be an R-module and A a $\mathcal{N}eat_{R\text{-}\mathcal{M}od}$-submodule of B. Let K be a complement of A in B and let A' be a complement of K in B containing A. Then it is easily seen that $A \trianglelefteq A'$. Suppose for the contrary that $A \neq A'$. Since R is a C-ring, there exists a simple submodule S of A'/A. Say $S = R(a'+A)$ for some $a' \in A'$. As S is simple, $S \cong R/P$ for some maximal left ideal P of R. Then $Pa' \leq A$. Since A is a $\mathcal{N}eat_{R\text{-}\mathcal{M}od}$-submodule of B, there exists $a \in A$ such that $P(a'-a) = 0$ by Corollary 3.2.5. Consider the cyclic submodule $R(a'-a)$ of A'. Since $S = R(a'+A) = R((a'-a)+A)$ is simple (so nonzero), we cannot have

$R(a'-a)=0$. As $A \trianglelefteq A'$, $R(a'-a)\cap A \neq 0$. Let $I=\{r\in R | r(a'-a)\in A\}$. Then $R(a'-a)\cap A = I(a'-a)$. Since $P(a'-a)=0\le A$, $P\le I\le R$. So $I=P$ or $I=R$ as P is a maximal left ideal of R. But both cases leads to a contradiction: If $I=P$, then $0\neq R(a'-a)\cap A = P(a'-a)=0$. If $I=R$, then $1\in I$ implies $a'-a=1(a'-a)\in A$, so $0\neq S=R((a'-a)+A)=0$. This contradiction shows that we must have $A=A'$, hence A is a complement in B. □

So for a left C-ring R, we know $\mathcal{C}ompl_{R\text{-}\mathcal{M}od}$-projectives by Proposition 2.3.4 for projectively generated classes (by classes of modules closed under passage to factor modules):

Corollary 3.3.3. *If R is a C-ring, then a module A is in $\pi(\mathcal{C}ompl_{R\text{-}\mathcal{M}od}) = \pi(\mathcal{N}eat_{R\text{-}\mathcal{M}od})$ if and only if A is a direct summand of a direct sum of free modules and simple modules.*

Before giving examples of C-rings, let's firstly give equivalent conditions for being a C-ring:

Proposition 3.3.4. *(by Renault (1964, Proposition 1.2)) For a ring R, the following are equivalent:*

(i) R is a left C-ring,

(ii) For every essential proper left ideal I of R, $\mathrm{Soc}(R/I)\neq 0$,

(iii) For every essential proper left ideal I of R, there exist $r\in R$ such that the left annihilator $\mathrm{Ann}^l_R(r+I)=\{s\in R | s(r+I)=0\}=\{s\in R | sr\in I\}$ of the element $r+I$ in the left R-module R/I is a maximal left ideal of R.

Proof. (i)⇔(ii) is Renault (1964, Proposition 1.2).
(ii)⇒(iii): Since $\mathrm{Soc}(R/I)\neq 0$, there exists a simple (cyclic) submodule $S=R(r+I)$ of R/I for some $r\in R$. Since $S=R(r+I)\cong R/\mathrm{Ann}^l_R(r+I)$ is simple,

$\mathrm{Ann}^l_R(r+I)$ is a maximal left ideal of R.
(iii)⇒(ii): $R(r+I) \cong R/\mathrm{Ann}^l_R(r+I)$ is a simple submodule of R/I since $\mathrm{Ann}^l_R(r+I)$ is a maximal left ideal of R. So $\mathrm{Soc}(R/I) \neq 0$. □

A ring R is said to be *left semi-artinian* if $\mathrm{Soc}(R/I) \neq 0$ for every proper left ideal I of R. So a left semi-artinian ring is clearly a left C-ring by Proposition 3.3.4.

Left Noetherian C-rings are:

Proposition 3.3.5. *(Renault, 1964, Corollary to Theorem 1.2) For a left* Noetherian *ring R, the following are equivalent:*

(i) R is a left C*-ring,*

(ii) For every essential (proper) left ideal I of R, the left R-module R/I has finite length.

A Dedekind domain is a C-ring (see Proposition 5.1.6). So a PID is also a C-ring. More generally:

Proposition 3.3.6. *A commutative Noetherian ring in which every* nonzero *prime ideal is maximal is a* C*-ring.*

Proof. Let R be such a ring. By Proposition 3.3.5, we need to show that for every essential proper ideal I of R, the R-module R/I has finite length. A module has finite length if and only if it is Noetherian and Artinian. So, R/I has finite length if and only if it is Noetherian and Artinian. R/I is Noetherian as R is so. Note that we have assumed that R is commutative, so R/I is also a commutative ring. Clearly, R/I is a Noetherian ring. Similarly, R/I is Artinian as a left R-module if and only R/I is Artinian as a commutative ring. By, for example Atiyah & Macdonald (1969, Theorem 8.5), a commutative ring is Artinian if and only if it is Noetherian and its Krull dimension is 0 (i.e. every prime ideal is maximal). $I \neq 0$ since I is an essential ideal of R. Every prime ideal of R/I is of the form P/I where P is a prime ideal of R such that $0 \neq I \leq P$. Since we assume that every nonzero prime ideal is maximal, P must be a maximal ideal of R. This shows that R/I has Krull dimension 0. Since it is also Noetherian, it is Artinian as required. □

A Dedekind domain is a commutative hereditary (and Noetherian) domain. If we take a hereditary Noetherian ring which is not necessarily commutative or a domain but the hereditary and Noetherian conditions are left and right symmetric, then such a ring is also a left (and right) C-ring:

Proposition 3.3.7. *(McConnell & Robson, 2001, 5.4.5 Proposition) Let R be a left and right Noetherian, and, left and right hereditary ring. Then for every essential (proper) left ideal I of R, the left R-module R/I has finite length.*

So, by Proposition 3.3.5,

Corollary 3.3.8. *A left and right Noetherian, and, left and right hereditary ring is a left (and right)* C*-ring.*

For commutative domains:

Proposition 3.3.9. *A commutative domain R is a* C*-ring if and only if every nonzero torsion R-module has a simple submodule.*

Proof. Firstly, note that in a commutative domain R, every *nonzero* ideal is essential.
($\Rightarrow$): Let M be a nonzero torsion module. Since $M \neq 0$, there exists $0 \neq m \in M$. Since M is torsion, there exits $0 \neq r \in R$ such that $rm = 0$. So, $Rm \cong R/I$ for some nonzero proper ideal I of R, namely $I = \{s \in R | sm = 0\}$. Hence Rm has

a simple submodule since $\mathrm{Soc}(R/I) \neq 0$ as R is a C-ring (by Proposition 3.3.4).
($\Leftarrow$): By Proposition 3.3.4, it suffices to show that $\mathrm{Soc}(R/I) \neq 0$ for every nonzero proper ideal I of R. This follows since R/I is a torsion module, so has a simple submodule. □

Commutative domains R such that every nonzero torsion R-module has a simple submodule have been considered in Enochs & Jenda (2000, §4.4). See page 58 for Enochs & Jenda (2000, (1)⇔(3) in Theorem 4.4.1).

Proposition 3.3.10. *(by Enochs & Jenda (2000, Remark 4.4.2))*

(i) If R is a commutative Noetherian domain which is not a field, then R is a C*-ring if and only if its Krull dimension is 1, that is every nonzero prime ideal is maximal.*

(ii) If R is a commutative local domain with maximal ideal M, then R is a C*-ring if and only if for every sequence $(a_i)_{i=1}^{\infty}$ of elements of M and every ideal $I \leq R$, $a_1 a_2 \ldots a_n \in I$ for some $n \geq 1$.*

3.4 The proper class $\mathcal{C}o\text{-}\mathcal{N}eat_{R\text{-}\mathcal{M}od}$

Generalov (1983) gives some results for the proper class $\mathcal{S}uppl_{R\text{-}\mathcal{M}od}$ to be injective and not to be injective. For a *left max ring* R (that is a ring in which *every* (left) R-module has small radical, equivalently *every* R-module has a maximal submodule), $\mathcal{S}uppl_{R\text{-}\mathcal{M}od}$ is injective by Generalov (1983, Theorem 4). A ring R is said to be a *left B-domain* if it is a *left duo ring* (that is, all of its *left* ideals are two-sided ideals) which is a left hereditary domain. For example a PID or a Dedekind domain is a B-domain. For a B-domain R which is not a division ring, $\mathcal{S}uppl_{R\text{-}\mathcal{M}od}$ is *not* an injective proper class by Generalov (1983, Proposition 5), that is does *not* have enough $\mathcal{S}uppl_{R\text{-}\mathcal{M}od}$-injectives. Generalov (1983, Corollary 6) uses this to prove that for a Dedekind domain W which is not a field, the inclusion $\mathcal{S}uppl_{W\text{-}\mathcal{M}od} \subsetneqq \mathcal{C}ompl_{W\text{-}\mathcal{M}od}$ is strict. Over a left Noetherian left hereditary ring R, Generalov (1983, Theorems 5-6) describes $\mathcal{S}uppl_{R\text{-}\mathcal{M}od}$-injective modules. The proper class $\mathcal{C}o\text{-}\mathcal{N}eat_{R\text{-}\mathcal{M}od}$, defined in Section 1.9.1, will help in dealing with these results also:

$$\mathcal{C}o\text{-}\mathcal{N}eat_{R\text{-}\mathcal{M}od} = \iota^{-1}(\{M \mid \operatorname{Rad} M = 0,\ M \text{ an } R\text{-module}\}).$$

Firstly, the proper class $\mathcal{C}o\text{-}\mathcal{N}eat_{R\text{-}\mathcal{M}od}$ is an injectively generated proper class containing $\mathcal{S}uppl_{R\text{-}\mathcal{M}od}$:

Proposition 3.4.1. *For any ring R,*

$$\mathcal{S}uppl_{R\text{-}\mathcal{M}od} \subseteq \mathcal{C}o\text{-}\mathcal{N}eat_{R\text{-}\mathcal{M}od} \subseteq \iota^{-1}(\{\textit{all (semi-)simple } R\textit{-modules}\})$$

Proof. Let M be an R-module such that $\operatorname{Rad} M = 0$. To prove the first inclusion it suffices to show that any short exact sequence

$$\mathbb{E}: \qquad 0 \longrightarrow M \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

in $\mathcal{S}uppl_{R\text{-}\mathcal{M}od}$ splits. Since $\mathbb{E} \in \mathcal{S}uppl_{R\text{-}\mathcal{M}od}$, $\operatorname{Im}(f)$, call it M', is a supplement in B; so there exists a submodule K of B such that M' is a supplement of K in B, hence

$$M' + K = B \text{ and } M' \cap K \ll M'$$

As $M' \cap K \ll M'$, $M' \cap K \subseteq \operatorname{Rad} M' \cong \operatorname{Rad} M = 0$, so $M' \cap K = 0$. Thus $B = M' \oplus K$, hence the sequence $\mathbb{E}$ splits.

The last inclusion holds since for a semisimple module M, $\operatorname{Rad} M = 0$. □

The criterion for being a coneat submodule is like being a supplement in the following weaker sense:

Proposition 3.4.2. *For a submodule A of a module B, the following are equivalent:*

(i) A is coneat in B,

(ii) There exists a submodule $K \leq B$ such that ($K \geq \operatorname{Rad} A$ and,)

$$A + K = B \quad \textit{and} \quad A \cap K = \operatorname{Rad} A.$$

(iii) There exists a submodule $K \leq B$ such that

$$A + K = B \quad \textit{and} \quad A \cap K \leq \operatorname{Rad} A.$$

Proof. Let $\mathbb{E}$ be the following short exact sequence of R-modules:

$$\mathbb{E}: \qquad 0 \longrightarrow A \xrightarrow{i_A} B \xrightarrow{g} B/A \longrightarrow 0,$$

where i_A is the inclusion homomorphism and g is the natural epimorphism.

(i)⇒(ii): Since $A/\operatorname{Rad} A$ has zero radical, it is injective with respect to the short exact sequence $\mathbb{E} \in \mathcal{C}o\text{-}\mathcal{N}eat_{R\text{-}\mathcal{M}od}$, so the natural epimorphism $\sigma : A \longrightarrow A/\operatorname{Rad} A$ can be extended to a map $h : B \longrightarrow A/\operatorname{Rad} A$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{i_A} & B & \longrightarrow & B/A & \longrightarrow & 0 \\ & & \sigma\downarrow & \swarrow_{h} & & & & & \\ & & A/\operatorname{Rad} A & & & & & & \end{array}$$

So this h induces a homomorphism $h' : B/\operatorname{Rad} A \longrightarrow A/\operatorname{Rad} A$ such that

$$h'(a + \operatorname{Rad} A) = a + \operatorname{Rad} A.$$

Hence the short exact sequence

$$0 \longrightarrow A/\operatorname{Rad} A \xrightarrow{i'_A} B/\operatorname{Rad} A \longrightarrow B/A \longrightarrow 0$$

splits, where i'_A is the inclusion homomorphism, and we obtain for some submodule K of B containing $\operatorname{Rad} A$

$$B/\operatorname{Rad} A = A/\operatorname{Rad} A \oplus K/\operatorname{Rad} A.$$

So $A + K = B$ and $A \cap K = \operatorname{Rad} A$.

(ii)⇒(iii) is trivial.

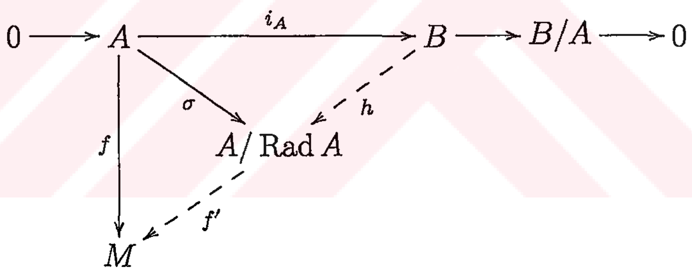
(iii)⇒(ii) follows by taking $K + \operatorname{Rad} A$ instead of K.

(ii)⇒(i): We have,

$$B/\operatorname{Rad} A = A/\operatorname{Rad} A \oplus K/\operatorname{Rad} A.$$

This gives us a homomorphism $h : B \longrightarrow A/\operatorname{Rad} A$ such that $h(a) = a + \operatorname{Rad} A$ for every $a \in A$.

To show that $\mathbb{E}$ is in $\mathcal{C}o\text{-}\mathcal{N}eat_{R\text{-}\mathcal{M}od}$, we must show that every module M with $\operatorname{Rad} M = 0$ is injective with respect to the short exact sequence $\mathbb{E}$. So let $f : A \longrightarrow M$ be a given homomorphism, where $\operatorname{Rad} M = 0$. Then $f(\operatorname{Rad} A) \leq \operatorname{Rad}(f(A)) \leq \operatorname{Rad} M = 0$, hence $\operatorname{Rad} A \leq \operatorname{Ker}(f)$, so the map f factors through the canonical epimorphism $\sigma : A \longrightarrow A/\operatorname{Rad} A$: $f = f' \circ \sigma$ for some homomorphism $f' : A/\operatorname{Rad} A \longrightarrow M$. But then we can use the above homomorphism $h : B \longrightarrow A/\operatorname{Rad} A$ to define the map $h' : B \longrightarrow M$ as $h' = f' \circ h$ which satisfies $h'|_A = f$: Follow the commutative diagram:



□

$\mathcal{C}o\text{-}\mathcal{N}eat_{R\text{-}\mathcal{M}od}$ is an injective proper class (i.e. it has enough injectives) by Proposition 2.4.4 since $\{M \mid \operatorname{Rad} M = 0,\ M \text{ an } R\text{-module}\}$ is a class of modules closed under taking submodules, or more directly:

Proposition 3.4.3. *Given an R-module A, denote by $\operatorname{E}(A)$ the injective envelope of A. Then the monomorphism*

$$\begin{aligned} f : A &\longrightarrow \operatorname{E}(A) \oplus (A/\operatorname{Rad} A) \\ x &\longmapsto (x, x + \operatorname{Rad} A) \end{aligned}$$

is a $\mathcal{C}o\text{-}\mathcal{N}eat_{R\text{-}\mathcal{M}od}$*-monomorphism and* $\mathrm{E}(A) \oplus (A/\operatorname{Rad} A)$ *is* $\mathcal{C}o\text{-}\mathcal{N}eat_{R\text{-}\mathcal{M}od}$*-injective.*

Proof. Like in the proof of Proposition 3.4.2, (iii)⇒(i), since from the module $B := \mathrm{E}(A) \oplus (A/\operatorname{Rad} A)$, we clearly have a projection $B \longrightarrow A/\operatorname{Rad} A$ and any map $A \longrightarrow M$, with $\operatorname{Rad} M = 0$, factors through $A \longrightarrow A/\operatorname{Rad} A$. □

Corollary 3.4.4. *An R-module M is* $\mathcal{C}o\text{-}\mathcal{N}eat_{R\text{-}\mathcal{M}od}$*-injective if and only if it is a direct summand of a module of the form* $E \oplus A$*, where E is an injective R-module and A is an R-module with* $\operatorname{Rad} A = 0$.

Proof. (⇐) is clear since a module with zero radical is $\mathcal{C}o\text{-}\mathcal{N}eat_{R\text{-}\mathcal{M}od}$-injective, and injective modules are of course $\mathcal{C}o\text{-}\mathcal{N}eat_{R\text{-}\mathcal{M}od}$-injective.
(⇒): By Proposition 3.4.3, we can embed any R-module M as a $\mathcal{C}o\text{-}\mathcal{N}eat_{R\text{-}\mathcal{M}od}$-submodule into a $\mathcal{C}o\text{-}\mathcal{N}eat_{R\text{-}\mathcal{M}od}$-injective module of the form $E \oplus A$, where E is an injective R-module and A is an R-module with $\operatorname{Rad} A = 0$:

$$M \leq_{c\mathcal{N}} E \oplus A \quad \text{and} \quad E \oplus A \text{ is } \mathcal{C}o\text{-}\mathcal{N}eat_{R\text{-}\mathcal{M}od}\text{-injective.}$$

If M is a $\mathcal{C}o\text{-}\mathcal{N}eat_{R\text{-}\mathcal{M}od}$-injective R-module, then M is a direct summand of $E \oplus A$. □

Proposition 3.4.5. *Let A be a submodule of a module B and suppose*

$$\operatorname{Rad} A \ll A.$$

Then, A is coneat in B if and only if it is a supplement in B.

Proof. (⇐) always holds by Proposition 3.4.1. Conversely, suppose A is coneat in B. Then by Proposition 3.4.2, there exist a submodule K of B such that

$$A + K = B \quad \text{and} \quad A \cap K = \operatorname{Rad} A.$$

Since $\operatorname{Rad} A \ll A$, $A \cap K \ll A$, so that A is a supplement (of K) in B. □

Corollary 3.4.6. *Let A be a submodule of a module B.*
Suppose A is finitely generated or $\operatorname{Rad} A = 0$.
Then, A is coneat in B if and only if it is a supplement in B.

Proof. Follows from Proposition 3.4.5, since when A is finitely generated, $\operatorname{Rad} A \ll A$ (and when $\operatorname{Rad} A = 0$, $\operatorname{Rad} A \ll A$ clearly). $\square$

3.5 $\mathrm{Ext}_{\mathcal{C}o\text{-}\mathcal{N}eat_{R\text{-}\mathcal{M}od}}$

Generalov (1983, Proposition 2, Lemma 4, Theorem 2, Corollaries 2-4) has described the subgroup $\mathrm{Ext}_{\mathcal{S}uppl_{R\text{-}\mathcal{M}od}}(C, A)$ of $\mathrm{Ext}_R(C, A)$ for R-modules A, C as follows:

Theorem 3.5.1. *(Generalov, 1983, Corollary 3) Given R-modules A and C, take a short exact sequence $\mathbb{E}$ of R-modules and R-module homomorphisms such that $\mathbb{E}$ ends with C and its middle term is projective, that is, $\mathbb{E} \in\in \mathrm{Ext}_R(C, H)$ for some R-module H such that*

$$\mathbb{E}: \qquad 0 \longrightarrow H \xrightarrow{f} P \xrightarrow{g} C \longrightarrow 0,$$

where P is a projective *module. Then,*

$$\mathrm{Ext}_{\mathcal{S}uppl_{R\text{-}\mathcal{M}od}}(C, A) = \Delta_{\mathbb{E}}\left(\mathrm{Hom}_R^{\mathrm{Im}\ll}(H, A)\right),$$

where

$$\mathrm{Hom}_R^{\mathrm{Im}\ll}(H, A) = \{\alpha \in \mathrm{Hom}_R(H, A) \mid \mathrm{Im}(\alpha) \ll A\},$$

and

$$\begin{aligned} \Delta_{\mathbb{E}} : \mathrm{Hom}_R(H, A) &\longrightarrow \mathrm{Ext}_R(C, A) \\ \alpha &\longmapsto [\alpha\mathbb{E}] \end{aligned}$$

is the connecting homomorphism sending each $\alpha \in \mathrm{Hom}_R(H, A)$ to the equivalence class of the pushout $\alpha\mathbb{E}$ of $\mathbb{E}$ along α:

$$\begin{array}{lccccccccc} \mathbb{E}: & 0 & \longrightarrow & H & \xrightarrow{f} & P & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & & \downarrow{\scriptstyle\alpha} & & \downarrow & & \| & & \\ \alpha\mathbb{E}: & 0 & \longrightarrow & A & \dashrightarrow & \bullet & \dashrightarrow & C & \longrightarrow & 0 \end{array}$$

Since being a coneat submodule criterion is like being a supplement (Proposition 3.4.2), following similar steps as in Generalov (1983), we will describe $\mathrm{Ext}_{\mathcal{C}o\text{-}\mathcal{N}eat_{R\text{-}\mathcal{M}od}}(C, A)$. Instead of $\mathrm{Hom}_R^{\mathrm{Im}\ll}(H, A)$, the following will appear for any R-modules H and A:

$$\mathrm{Hom}_R^{\mathrm{Im}\leq\mathrm{Rad}}(H, A) = \{\alpha \in \mathrm{Hom}_R(H, A) \,|\, \mathrm{Im}(\alpha) \leq \mathrm{Rad}\, A\}.$$

Proposition 3.5.2. *Given R-modules A and C, and any R-module H, take a short exact sequence $\mathbb{E} \in\in \mathrm{Ext}_R(C, H)$, say,*

$$\mathbb{E}: \qquad 0 \longrightarrow H \xrightarrow{f} G \xrightarrow{g} C \longrightarrow 0.$$

Consider the connecting homomorphism $\triangle_{\mathbb{E}} : \mathrm{Hom}_R(H, A) \longrightarrow \mathrm{Ext}_R(C, A)$ corresponding to this short exact sequence $\mathbb{E}$. Then,

$$\triangle_{\mathbb{E}}\left(\mathrm{Hom}_R^{\mathrm{Im}\leq\mathrm{Rad}}(H, A)\right) \leq \mathrm{Ext}_{\mathcal{C}o\text{-}\mathcal{N}eat_{R\text{-}\mathcal{M}od}}(C, A).$$

Proof. Let $\alpha \in \mathrm{Hom}_R^{\mathrm{Im}\leq\mathrm{Rad}}(H, A)$ and consider the pushout $\alpha\mathbb{E}$ of $\mathbb{E}$ along α:

$$\begin{array}{lccccccccc} \mathbb{E}: & 0 & \longrightarrow & H & \xrightarrow{f} & G & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \| & & \\ \alpha\mathbb{E}: & 0 & \longrightarrow & A & \xrightarrow[f']{} & B & \xrightarrow[g']{} & C & \longrightarrow & 0 \end{array}$$

For simplicity, let us suppose that H and A are submodules of G and B, and f, f' are inclusion homomorphisms. By properties of the pushout, we see that

$$A + \beta(G) = B \quad \text{and} \quad A \cap \beta(G) = \alpha(H).$$

Since $\mathrm{Im}(\alpha) \leq \mathrm{Rad}\, A$, we have for $K := \beta(G) \leq B$,

$$A + K = B \quad \text{and} \quad A \cap K = \alpha(H) \leq \mathrm{Rad}\, A.$$

Then, by Proposition 3.4.2, A is coneat in B. $\square$

Proposition 3.5.3. *Given R-modules A and C, and any R-modules H and H', let $\mathbb{E} \in\in \mathrm{Ext}_R(C,H)$ and $\mathbb{E}' \in\in \mathrm{Ext}_R(C,H')$ be such that*

$$\begin{array}{ll} \mathbb{E}: & 0 \longrightarrow H \xrightarrow{f} G \xrightarrow{g} C \longrightarrow 0, \\ \mathbb{E}': & 0 \longrightarrow H' \xrightarrow{f} G' \xrightarrow{g'} C \longrightarrow 0, \end{array}$$

and G' is projective with respect to *the short exact sequence $\mathbb{E}$.*
Consider the connecting homomorphisms $\triangle_{\mathbb{E}} : \mathrm{Hom}_R(H,A) \longrightarrow \mathrm{Ext}_R(C,A)$ and $\triangle_{\mathbb{E}'} : \mathrm{Hom}_R(H',A) \longrightarrow \mathrm{Ext}_R(C,A)$ corresponding to these short exact sequences $\mathbb{E}$ and $\mathbb{E}'$. Then:

$$\triangle_{\mathbb{E}}\left(\mathrm{Hom}_R^{\mathrm{Im}\leq\mathrm{Rad}}(H,A)\right) \leq \triangle_{\mathbb{E}'}\left(\mathrm{Hom}_R^{\mathrm{Im}\leq\mathrm{Rad}}(H',A)\right).$$

If G is also projective with respect to *the short exact sequence $\mathbb{E}'$, then*

$$\triangle_{\mathbb{E}}\left(\mathrm{Hom}_R^{\mathrm{Im}\leq\mathrm{Rad}}(H,A)\right) = \triangle_{\mathbb{E}'}\left(\mathrm{Hom}_R^{\mathrm{Im}\leq\mathrm{Rad}}(H',A)\right).$$

Proof. Take $\alpha \in \mathrm{Hom}_R^{\mathrm{Im}\leq\mathrm{Rad}}(H,A)$ and consider the pushout $\alpha\mathbb{E}$ of $\mathbb{E}$ along α. As G' is projective with respect to $\mathbb{E}$, the first two rows in the following diagram have been filled commutatively by the maps β and γ:

$$\begin{array}{llcccccccc} \mathbb{E}': & 0 & \longrightarrow & H' & \xrightarrow{f'} & G' & \xrightarrow{g'} & C & \longrightarrow & 0 \\ & & & \downarrow \beta & & \downarrow \gamma & & \| & & \\ \mathbb{E}: & 0 & \longrightarrow & H & \xrightarrow{f} & G & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & & \downarrow \alpha & & \downarrow & & \| & & \\ \alpha\mathbb{E}: & 0 & \longrightarrow & A & \xrightarrow[f']{} & \bullet & \xrightarrow[g']{} & C & \longrightarrow & 0 \end{array}$$

Then for $\alpha' = \alpha \circ \beta$, $\alpha' \in \mathrm{Hom}_R^{\mathrm{Im}\leq\mathrm{Rad}}(H',A)$. By commutativity of the diagram, $\alpha\mathbb{E}$ is a pushout of $\mathbb{E}'$ along α'; so $\alpha\mathbb{E}$ and $\alpha'\mathbb{E}'$ are equivalent short exact sequences, hence

$$\triangle_{\mathbb{E}}(\alpha) = [\alpha\mathbb{E}] = [\alpha'\mathbb{E}'] = \triangle_{\mathbb{E}'}(\alpha').$$

The last claim of the lemma follows by changing the roles of $\mathbb{E}$ and $\mathbb{E}'$. $\square$

Proposition 3.5.4. *For R-modules A and C,*

$$\mathrm{Ext}_{\mathcal{C}o\text{-}\mathcal{N}eat_{R\text{-}\mathcal{M}od}}(C,A) = \bigcup_{\substack{H \leq \mathrm{Rad}\, A,\\ [\mathbb{E}] \in \mathrm{Ext}_R(C,H)}} \Delta_{\mathbb{E}}\left(\mathrm{Hom}_R^{\mathrm{Im}\leq\mathrm{Rad}}(H,A)\right),$$

where the union is over all submodules H of $\mathrm{Rad}\, A$ and all representative short exact sequences $\mathbb{E}$ of elements of $\mathrm{Ext}_R(C,H)$, and for each such short exact sequence $\mathbb{E}$, $\Delta_{\mathbb{E}} : \mathrm{Hom}_R(H,A) \longrightarrow \mathrm{Ext}_R(C,A)$ is the connecting homomorphism corresponding to $\mathbb{E}$.

Proof. '$\supseteq$' follows from Proposition 3.5.2. To prove the converse, let $\mathbb{E}_1 \in\in \mathrm{Ext}_{\mathcal{C}o\text{-}\mathcal{N}eat_{R\text{-}\mathcal{M}od}}(C,A)$:

$$\mathbb{E}_1 : \qquad 0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

For simplicity, suppose A is a submodule of B and f is the inclusion homomorphism. As A is coneat in B, by Proposition 3.4.2, there exits a submodule K of B such that

$$A + K = B \quad \text{and} \quad A \cap K \leq \mathrm{Rad}\, A.$$

Let $H = A \cap K$. Then $H \leq \mathrm{Rad}\, A$ and since

$$K/H = K/A \cap K \cong (A+K)/A = B/A \cong C,$$

we obtain a short exact sequence $\mathbb{E} \in\in \mathrm{Ext}_R(C,H)$, and the following commutative diagram:

$$\begin{array}{lccccccccc}
\mathbb{E}: & 0 & \longrightarrow & H & \xrightarrow{f'} & K & \xrightarrow{g'} & C & \longrightarrow & 0 \\
 & & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \| & & \\
\mathbb{E}_1 = \alpha\mathbb{E}: & 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0
\end{array}$$

where the maps $\alpha : H \to A$ and $\beta : K \to B$ are inclusion homomorphisms. This means $\mathbb{E}_1$ is a pushout of $\mathbb{E}$ along α, so $[\mathbb{E}_1] = \Delta_{\mathbb{E}}(\alpha)$; since $\alpha(H) = H \leq \mathrm{Rad}\, A$, $\alpha \in \mathrm{Hom}_R^{\mathrm{Im}\leq\mathrm{Rad}}(H,A)$ as required. $\square$

From the last two propositions, we obtain:

Theorem 3.5.5. *Given R-modules A and C, take a short exact sequence* $\mathbb{E}$ *of R-modules and R-module homomorphisms such that* $\mathbb{E}$ *ends with C and its middle term is projective, that is,* $\mathbb{E} \in\in \mathrm{Ext}_R(C, H)$ *for some R-module H such that*

$$\mathbb{E}: \qquad 0 \longrightarrow H \xrightarrow{f} P \xrightarrow{g} C \longrightarrow 0,$$

where P is a projective *module. Then,*

$$\mathrm{Ext}_{\mathcal{C}o\text{-}\mathcal{N}eat_{R\text{-}\mathcal{M}od}}(C, A) = \Delta_{\mathbb{E}}\left(\mathrm{Hom}_R^{\mathrm{Im}\leq\mathrm{Rad}}(H, A)\right),$$

where $\Delta_{\mathbb{E}} : \mathrm{Hom}_R(H, A) \longrightarrow \mathrm{Ext}_R(C, A)$ *is the connecting homomorphism corresponding to* $\mathbb{E}$ *(as for example described in Theorem 3.5.1).*

3.6 Criterion for complements and supplements via Goldie dimension (uniform dimension) and dual Goldie dimension (hollow dimension)

For Goldie dimension (uniform dimension) and dual Goldie dimension (hollow dimension), see for example (Facchini, 1998, §2.6-2.8) or Lomp (1996, Chapters 2-3). We denote the uniform dimension and hollow dimension of a module M by $\mathrm{u.dim}\, M$ and $\mathrm{h.dim}\, M$ respectively.

When considering complements in modules with finite uniform dimension (Goldie dimension), we have the following criterion:

Theorem 3.6.1. *(Dung et al., 1994, 5.10 (1)) Let B be a module with finite uniform dimension (Goldie dimension) (denoted by* $\mathrm{u.dim}\, B < \infty$*). Then a* *submodule A of B is a complement in B (closed in B) if and only if A and B/A have finite uniform dimension and*

$$\mathrm{u.}\dim B = \mathrm{u.}\dim A + \mathrm{u.}\dim(B/A).$$

Dually for supplements in modules with finite hollow dimension, we have:

Theorem 3.6.2. *(Lomp, 1996, Corollary 3.2.3) Let B be a module with finite hollow dimension (dual Goldie dimension) (denoted by $\mathrm{h.}\dim B < \infty$). Then a submodule A of B is a supplement in B if and only if A and B/A have finite hollow dimension and*

$$\mathrm{h.}\dim B = \mathrm{h.}\dim A + \mathrm{h.}\dim(B/A).$$

3.7 Coinjectives and coprojectives with respect to $\mathcal{C}ompl_{R\text{-}\mathcal{M}od}$ and $\mathcal{S}uppl_{R\text{-}\mathcal{M}od}$

In Erdoğan (2004), $\mathcal{C}ompl_{R\text{-}\mathcal{M}od}$-coinjective modules have been called *absolutely complement* modules and $\mathcal{C}ompl_{R\text{-}\mathcal{M}od}$-coprojective modules have been called *absolutely co-complement* modules. Similarly, $\mathcal{S}uppl_{R\text{-}\mathcal{M}od}$-coinjective modules have been called *absolutely supplement* modules and $\mathcal{S}uppl_{R\text{-}\mathcal{M}od}$-coprojective modules have been called *absolutely co-supplement* modules in Erdoğan (2004). For some properties of these modules, see Erdoğan (2004, Chapters 3-4).

Remember the construction of an injective envelope of a module. It is seen from this construction that a module is injective if and only if it has no proper essential extension, that is, it is a closed submodule of every module containing it (see for example Maclane (1963, Proposition III.11.2)). Since closed submodules and complement submodules of a module coincide, that means the following:

Theorem 3.7.1. *(by Erdoğan (2004, Proposition 4.1.4))* $\mathcal{C}ompl_{R\text{-}\mathcal{M}od}$*-coinjective modules are only injective modules.*

Dually, $\mathcal{S}uppl_{R\text{-}\mathcal{M}od}$-coprojectives are only projectives if the ring R has zero Jacobson radical:

Theorem 3.7.2. *If* $\operatorname{Rad} R = 0$*, then* $\mathcal{S}uppl_{R\text{-}\mathcal{M}od}$*-coprojective modules are only projective modules.*

Proof. Suppose M is a $\mathcal{S}uppl_{R\text{-}\mathcal{M}od}$-coprojective module. There exists an epimorphism $g : F \longrightarrow M$ from a free module F. So, for $H := \operatorname{Ker} g$ and f the inclusion homomorphism, we obtain the following short exact sequence

$$\mathbb{E}: \qquad 0 \longrightarrow H \xrightarrow{f} F \xrightarrow{g} M \longrightarrow 0$$

Since M is $\mathcal{S}uppl_{R\text{-}\mathcal{M}od}$-coprojective, $\mathbb{E}$ is in $\mathcal{S}uppl_{R\text{-}\mathcal{M}od}$. So, H is a supplement in F. Clearly, $\operatorname{Rad} H \leq \operatorname{Rad} F$. Since $\operatorname{Rad} F = JF$ for $J := \operatorname{Rad} R$, the Jacobson radical of R (by for example Lam (2001, Proposition 24.6-(3))), we obtain that $\operatorname{Rad} F = 0$ as $J = 0$ by our assumption. Hence $\operatorname{Rad} H = 0$. Then the short exact sequence $\mathbb{E} \in \mathcal{S}uppl_{R\text{-}\mathcal{M}od}$ splits since modules with zero radical are $\mathcal{S}uppl_{R\text{-}\mathcal{M}od}$-injective by Proposition 3.4.1. Then, $F \cong H \oplus M$, and so M is also a projective module. □

This proof, in fact, gives the following:

Theorem 3.7.3. *If* $\operatorname{Rad} R = 0$*, then* $\mathcal{C}o\text{-}\mathcal{N}eat_{R\text{-}\mathcal{M}od}$*-coprojective modules are only projective modules.*

In Enochs & Jenda (2000, (1)⇔(3) in Theorem 4.4.1), it is shown that for a commutative domain R, the following are equivalent:

(i) Every torsion R-module has a simple submodule,

(ii) A module E is injective if and only if $\mathrm{Ext}^1_R(S, E) = 0$ for all simple modules S.

By Proposition 3.3.9, the first condition is equivalent to R being C-ring. The following result generalizes (i)⇒(ii) above:

Proposition 3.7.4. *For a left* C*-ring R, an R module M is injective if and only if* $\mathrm{Ext}^1_R(S, M) = 0$ *for all simple R-modules S.*

Proof. Since R is a C-ring, by Theorem 3.3.2, the proper class $\mathcal{C}ompl_{R\text{-}\mathcal{M}od}$ is projectively generated by all simple modules:

$$\mathcal{C}ompl_{R\text{-}\mathcal{M}od} = \mathcal{N}eat_{R\text{-}\mathcal{M}od} = \pi^{-1}(\{R/P | P \text{ maximal left ideal of } R\})$$

Let A be an R-module. By Proposition 2.6.7, A is $\mathcal{C}ompl_{R\text{-}\mathcal{M}od}$-coinjective if and only if $\mathrm{Ext}^1_R(S, A) = 0$ for all simple modules S. By Theorem 3.7.1, A is $\mathcal{C}ompl_{R\text{-}\mathcal{M}od}$-coinjective if and only if A is injective. □

3.8 $\mathcal{S}uppl_{R\text{-}\mathcal{M}od}$ and $\mathcal{C}o\text{-}\mathcal{N}eat_{R\text{-}\mathcal{M}od}$ over some classes of rings

A ring R is said to be a *left quasi-duo ring* if each *maximal* left ideal is a two-sided ideal.

Lemma 3.8.1. *(Generalov, 1983, Lemma 3) Let R be a left quasi-duo ring. Then for each module M,*

$$\mathrm{Rad}\, M = \bigcap_{P \underset{max.}{\leq} {}_R R} PM,$$

where the intersection is over all maximal left *ideals of R.*

Proposition 3.8.2. *Let R be a left quasi-duo ring. Then,*

$$\mathcal{C}o\text{-}\mathcal{N}eat_{R\text{-}\mathcal{M}od} \subseteq \tau^{-1}(\{R/P | P \textit{ maximal left ideal of } R\})$$

Proof. The proof is the proof in Generalov (1983, Proposition 1) where it has been shown that

$$\mathcal{S}uppl_{R\text{-}\mathcal{M}od} \subseteq \tau^{-1}(\{R/P | P \text{ maximal left ideal of } R\}).$$

Take a short exact sequence $\mathbb{E} \in \mathcal{C}o\text{-}\mathcal{N}eat_{R\text{-}\mathcal{M}od}$:

$$\mathbb{E}: \qquad 0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

Without loss of generality, suppose that A is a submodule of B and f is the inclusion homomorphism. So A is a coneat submodule of the module B. By Proposition 3.2.2, to end the proof it suffices to show that $A \cap PB = PA$ for each maximal left ideal P of R.

As $A \leq_{c\mathcal{N}} B$, by Proposition 3.4.2, there exists a submodule K of B such that $A + K = B$ and $A \cap K \leq \operatorname{Rad} A$. Then,

$$\begin{aligned} A \cap PB &= A \cap P(A+K) \leq A \cap (PA + PK) = PA + A \cap PK \\ &\leq PA + A \cap K \leq PA + \operatorname{Rad} A = PA, \end{aligned}$$

where the last equality follows from Lemma 3.8.1, since each maximal left ideal is assumed to be a two-sided ideal. So $A \cap PB \leq PA$, and since the converse is clear, we obtain $A \cap PB = PA$ as required. □

A ring R is a *left max ring* R if $\operatorname{Rad} M \ll M$ for all (left) R-modules M, equivalently every R-module M has a maximal submodule.

Proposition 3.8.3. *For a left max ring R, $\mathcal{S}uppl_{R\text{-}\mathcal{M}od} = \mathcal{C}o\text{-}\mathcal{N}eat_{R\text{-}\mathcal{M}od}$.*

Proof. Follows from Proposition 3.4.5. □

Proposition 3.8.4. *(Generalov, 1983, Proposition 4) Let R be a ring that can be embedded in an R-module S such that* $\operatorname{Rad} S = R$. *Then:*

(i) *For each module M, there exists a module H such that* $\operatorname{Rad} H = M$.

(ii) *If, in addition, the R-module S/R is* semisimple, *then an* essential *extension H of the module M such that H/M is a semisimple module can be taken such that* $\operatorname{Rad} H = M$.

A module M is said to be a *small module* if it is a small submodule of a module containing it, equivalently if it is a small submodule of its injective envelope. See Leonard (1966) for *small modules.*

A ring R is said to be a *left small ring* if R, considered as a (left) R-module, is a small R-module, equivalently R is small in its injective envelope $\mathrm{E}(R)$. It is noted in Lomp (2000, Proposition 3.3) that a ring R is left small, if and only if, $\operatorname{Rad} E = E$ for every injective R-module E, if and only if, $\operatorname{Rad} \mathrm{E}(R) = \mathrm{E}(R)$.

Proposition 3.8.5. *(Generalov, 1983, Corollary 5) If R is a ring that can be embedded in an R-module S such that* $\operatorname{Rad} S = R$ *and S/R is a semisimple R-module (and R is essential in S), then R is a left small ring, so* $\operatorname{Rad} E = E$ *for every injective R-module E and in particular no injective R-module is finitely generated.*

Proposition 3.8.6. *A left quasi-duo domain which is not a division ring is a left small ring.*

Proof. Let R be left quasi-duo domain which is not a division ring and E be an injective R-module. Since E is injective, it is also a divisible R-module (by for example Cohn (2002, Proposition 4.7.8)). Since R is not a division ring, any maximal left ideal P of R is nonzero and so $PE = E$ as E is divisible. By Lemma 3.8.1,

$$\operatorname{Rad} E = \bigcap_{P \underset{max.}{\leq} {}_R R} PE = \bigcap_{P \underset{max.}{\leq} {}_R R} E = E.$$

□

A ring R is said to be *semilocal* if $R/\operatorname{Rad} R$ is a semisimple ring, that is a left (and right) semisimple R-module. See for example (Lam, 2001, §20). Such rings are also called as rings semisimple modulo its radical as in Anderson & Fuller (1992, in §15, pp. 170-172).

Theorem 3.8.7. *If R is a semilocal ring, then*

$$\mathcal{C}o\text{-}\mathcal{N}eat_{R\text{-}\mathcal{M}od} = \iota^{-1}(\{\textit{all (semi-)simple } R\textit{-modules}\}).$$

Proof. For any ring R, the left side is contained in the right side by Proposition 3.4.1. We prove equality for a semilocal ring R. By Anderson & Fuller (1992, Corollary 15.18), for every R-module A, $A/\operatorname{Rad} A$ is semisimple. So every R-module M with $\operatorname{Rad} M = 0$ is semisimple. Conversely, every semisimple R-module has zero radical (for any ring R). Hence,

$$\{M | \operatorname{Rad} M = 0,\ M \text{ an } R\text{-module}\} = \{\text{ all semisimple } R\text{-modules }\}.$$

So,

$$\begin{aligned} \mathcal{C}o\text{-}\mathcal{N}eat_{R\text{-}\mathcal{M}od} &= \iota^{-1}(\{M | \operatorname{Rad} M = 0,\ M \text{ an } R\text{-module}\}) \\ &= \iota^{-1}(\{\text{ all semisimple } R\text{-modules }\}). \end{aligned}$$

The reason for the equality

$$\iota^{-1}(\{\text{all semisimple } R\text{-modules}\}) = \iota^{-1}(\{\text{all simple } R\text{-modules}\})$$

comes from the characterization of semilocal rings in Anderson & Fuller (1992, Proposition 15.17): every product of simple left R-modules is semisimple. Denote $\iota^{-1}(\{\text{all semisimple } R\text{-modules}\})$ shortly by $\mathcal{P}$ and $\iota^{-1}(\{\text{all simple } R\text{-modules}\})$ shortly by $\mathcal{P}'$. Clearly $\mathcal{P} \subseteq \mathcal{P}'$. Conversely, it suffices to show that every semisimple R-module M is injective with respect to the proper class $\mathcal{P}'$. Since M is a semisimple R-module, $M = \bigoplus_{\lambda\in\Lambda} S_\lambda$ for some index set Λ and simple submodules S_λ of M. Then $M \leq N := \prod_{\lambda\in\Lambda} S_\lambda$. The right side N is also a semisimple R-module (by Anderson & Fuller (1992, Proposition 15.17)). So its submodule M

is a direct summand of N. But N, being a product of simple modules which are injective with respect to the proper class $\mathcal{P}'$, is injective with respect to proper class $\mathcal{P}'$. Then so is its direct summand M as required. □

By characterization of left perfect rings as in for example Anderson & Fuller (1992, Theorem 28.4), a ring R is *left perfect* if and only if R is a semilocal left max ring. So:

Corollary 3.8.8. *For a left perfect ring R,*

$$\mathcal{S}uppl_{R\text{-}\mathcal{M}od} = \mathcal{C}o\text{-}\mathcal{N}eat_{R\text{-}\mathcal{M}od} = \iota^{-1}(\{\textit{all (semi-)simple R-modules}\}).$$

Proof. Since a left perfect ring R is a semilocal left max ring, the result follows from Theorem 3.8.7 and Proposition 3.8.3. □

CHAPTER FOUR

COMPLEMENTS AND SUPPLEMENTS IN ABELIAN GROUPS

The proper class $\mathcal{N}eat_{\mathbb{Z}\text{-}\mathcal{M}od}$ is one of the motivating ideas for the proper classes we are dealing with (Section 4.1). We will see in Section 4.1 that a supplement in an abelian group is a complement (neat) in that group. In a finitely generated abelian group, there is a criterion for a subgroup to be a complement via uniform dimension, and in a finite abelian group, there is a criterion for a subgroup to be a supplement via hollow dimension (Section 4.2). In Section 4.3, we will see that finite subgroups which are complements are supplements. The inductive closure of the proper class $\mathcal{S}uppl_{\mathbb{Z}\text{-}\mathcal{M}od}$ is $\mathcal{C}ompl_{\mathbb{Z}\text{-}\mathcal{M}od} = \mathcal{N}eat_{\mathbb{Z}\text{-}\mathcal{M}od}$ (Section 4.4). The functor $\text{Ext}_{\mathcal{S}uppl_{\mathbb{Z}\text{-}\mathcal{M}od}}$ is *not* factorizable (Section 4.5). The proper class $\mathcal{C}o\text{-}\mathcal{N}eat_{\mathbb{Z}\text{-}\mathcal{M}od}$ is *strictly* between $\mathcal{S}uppl_{\mathbb{Z}\text{-}\mathcal{M}od}$ and $\mathcal{C}ompl_{\mathbb{Z}\text{-}\mathcal{M}od} = \mathcal{N}eat_{\mathbb{Z}\text{-}\mathcal{M}od}$ (Sections 4.3 and 4.6). We will also see in Section 4.6 that for a torsion group, neat subgroups and coneat subgroups coincide. $\mathcal{C}ompl_{\mathbb{Z}\text{-}\mathcal{M}od}$-coprojectives are only torsion-free abelian groups (Section 4.7).

4.1 The proper class $\mathcal{N}eat_{\mathbb{Z}\text{-}\mathcal{M}od}$

As mentioned in the first chapter, the following result is one of the motivations for us to deal with complements and its dual supplements:

Theorem 4.1.1. *The proper class $Compl_{\mathbb{Z}\text{-}\mathcal{M}od} = \mathcal{N}eat_{\mathbb{Z}\text{-}\mathcal{M}od}$ is projectively generated, flatly generated and injectively generated by simple groups $\mathbb{Z}/p\mathbb{Z}$, p prime number:*

$$\begin{aligned} Compl_{\mathbb{Z}\text{-}\mathcal{M}od} &= \mathcal{N}eat_{\mathbb{Z}\text{-}\mathcal{M}od} = \pi^{-1}(\{\mathbb{Z}/p\mathbb{Z} | p \text{ prime}\}) \\ &= \tau^{-1}(\{\mathbb{Z}/p\mathbb{Z} | p \text{ prime}\}) = \iota^{-1}(\{\mathbb{Z}/p\mathbb{Z} | p \text{ prime}\}). \end{aligned}$$

Proof. The first equality has been pointed out in Harrison et al. (1963, after Definition 3 in §4); more generally, the equality of the first and third proper classes above will follow from Stenström (1967b, Corollary to Proposition 8) or Generalov (1978, Theorem 5). The second and third equalities follow from Proposition 3.2.1, since this proposition gives for the ring $\mathbb{Z}$ and prime number p that for a short exact sequence

$$\mathbb{E}: \qquad 0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0 \tag{4.1.1}$$

of abelian groups and homomorphisms, $\mathrm{Hom}(\mathbb{Z}/p\mathbb{Z}, \mathbb{E})$ is exact if and only if $\mathbb{Z}/p\mathbb{Z} \otimes \mathbb{E}$ is exact if and only if $\mathrm{Im}(f) \cap pB = p\,\mathrm{Im}(f)$, i.e. $\mathrm{Im}(f)$ is a neat subgroup of B.

So, it remains only to prove $\mathcal{N}eat_{\mathbb{Z}\text{-}\mathcal{M}od} = \iota^{-1}(\{\mathbb{Z}/p\mathbb{Z} | p \text{ prime}\})$. ($\subseteq$) part follows from Harrison et al. (1963, Lemma 4 in §4) (as it implies in particular that every simple group $\mathbb{Z}/p\mathbb{Z}$ (p prime), is $\mathcal{N}eat_{\mathbb{Z}\text{-}\mathcal{M}od}$-injective); as proved there, this follows because if a simple group $\mathbb{Z}/p\mathbb{Z}$ is neat in a group, then it is pure there, so being of bounded order, it is a direct summand. The proof of ($\supseteq$) part (and also ($\subseteq$) part) is done in Bilhan (1995, Theorem 7). To prove the ($\supseteq$) (as in Bilhan (1995, Theorem 7)), take a short exact sequence $\mathbb{E}$ of the form (4.1.1) in $\iota^{-1}(\{\mathbb{Z}/p\mathbb{Z} | p \text{ prime}\})$, where we can suppose without loss of generality that A

is a subgroup of B and f is the inclusion map. Suppose for the contrary $\mathbb{E}$ is not in $\mathcal{N}eat_{\mathbb{Z}\text{-}\mathcal{M}od}$, i.e. A is not a neat subgroup of B. Hence, there exists a prime number p such that $A \cap pB \neq pA$, so there exists an element $a \in (A \cap pB) \setminus pA$. By Zorn's lemma, the collection of subgroups $\{D \leq A | pA \leq D \text{ and } a \notin D\}$ ($\neq \emptyset$ as it contains $A \cap pB$) has a maximal element with respect to inclusion, say M. So, $pA \leq M$, $a \notin M$ and M is maximal with respect to this property. Since, $p(A/M) = 0$, A/M is a direct sum of simple groups each isomorphic to $\mathbb{Z}/p\mathbb{Z}$. By maximality of M, this sum must consist of just one term, i.e. $A/M \cong \mathbb{Z}/p\mathbb{Z}$. Since $\mathbb{E} \in \iota^{-1}(\{\mathbb{Z}/q\mathbb{Z} | q \text{ prime}\})$, A/M is injective with respect to the short exact sequence $\mathbb{E}$, hence for the canonical epimorphism $\sigma : A \longrightarrow A/M$, there exists a homomorphism $\psi : B \longrightarrow A/M$ such that $\psi \circ f = \sigma$, so $\psi(x) = x + M$ for all $x \in A$. But then we have,

$$0 \neq a + M = \psi(a) \in \psi(A \cap pB) \leq \psi(pB) = p\psi(B) \leq p(A/M) = 0,$$

which gives the required contradiction. □

So in this case we know $\mathcal{C}ompl_{\mathbb{Z}\text{-}\mathcal{M}od}$-projectives and $\mathcal{C}ompl_{\mathbb{Z}\text{-}\mathcal{M}od}$-injectives as it is given generally for projectively generated and injectively generated proper classes:

Corollary 4.1.2. *A group A is in $\pi(\mathcal{C}ompl_{\mathbb{Z}\text{-}\mathcal{M}od}) = \pi(\mathcal{N}eat_{\mathbb{Z}\text{-}\mathcal{M}od})$ if and only if A is a (direct summand of a) direct sum of cyclic groups of infinite order or prime order.*

Proof. For such a projectively generated class, we know by Proposition 2.3.3 or 2.3.4 that a group is a relative projective for the proper class $\mathcal{C}ompl_{\mathbb{Z}\text{-}\mathcal{M}od} = \mathcal{N}eat_{\mathbb{Z}\text{-}\mathcal{M}od}$ if it is a direct summand of a direct sum of cyclic groups of infinite order or prime order. By a special theorem for abelian groups, Fuchs (1970, Theorem 18.1), a direct summand of a direct sum of cyclic groups (of prime order or infinite order) must be again a direct sum of cyclic groups. We can assume that in this direct sum, finite cyclic groups have also been expanded as a direct sum of cyclic groups of prime power order. But in our case these cyclic groups must again have prime order or must be of infinite order. $\square$

For a torsion group A,

$$A = \bigoplus_{p \text{ prime}} A_p,$$

where for each prime p, A_p is the *p-primary component* of A consisting of all elements of A whose order is a power of the prime p (see for example Fuchs (1970, Theorem 8.4)).

A *cocyclic* group is a group isomorphic to $\mathbb{Z}/p^k\mathbb{Z}$, p prime number and $k \in \mathbb{Z}^+$ or the Prüfer (divisible) group $\mathbb{Z}_{p^\infty}$, p prime (see Fuchs (1970, §3) for this notion dual to cyclic). The *Prüfer group* $\mathbb{Z}_{p^\infty}$ is the abelian group generated by a sequence of elements $c_0, c_1, c_2, \ldots$ such that $pc_n = c_{n-1}$ for every $n \in \mathbb{Z}^+$ and $pc_0 = 0$. Its subgroups form a chain:

$$0 < \mathbb{Z}c_0 < \mathbb{Z}c_1 < \mathbb{Z}c_2 < \cdots < \mathbb{Z}c_{n-1} < \mathbb{Z}c_n < \cdots \mathbb{Z}_{p^\infty},$$

where each subgroup $\mathbb{Z}c_n \cong \mathbb{Z}/p^{n+1}\mathbb{Z}$ is a cyclic group of order p^{n+1}. The Prüfer group $\mathbb{Z}_{p^\infty}$ is isomorphic to the p-primary component of the torsion group $\mathbb{Q}/\mathbb{Z}$; this p-primary component is generated by all $\frac{1}{p^k} + \mathbb{Z}$, $k \in \mathbb{Z}^+$:

$$\mathbb{Q}/\mathbb{Z} = \bigoplus_{p \text{ prime}} \underbrace{(\mathbb{Q}/\mathbb{Z})_p}_{p\text{-primary component}} = \bigoplus_{p \text{ prime}} \left(\bigcup_{k=1}^{\infty} \mathbb{Z}(\tfrac{1}{p^k} + \mathbb{Z}) \right) \cong \bigoplus_{p \text{ prime}} \mathbb{Z}_{p^\infty}$$

(see for example Fuchs (1970, Example 2 after Theorem 8.4 at p. 43)). The injective envelope of $\mathbb{Z}/p^k\mathbb{Z}$ is $\mathbb{Z}_{p^\infty}$ for any prime p and $k \in \mathbb{Z}^+$.

Proposition 4.1.3. *(Harrison et al., 1963, Lemma 4 in §4) A group A is in $\iota(Compl_{\mathbb{Z}\text{-}\mathcal{M}od}) = \iota(\mathcal{N}eat_{\mathbb{Z}\text{-}\mathcal{M}od})$, if and only if, A is a direct summand of direct product of cyclic groups of prime order ($\cong \mathbb{Z}/p\mathbb{Z}$, p prime) and of infinite cocyclic groups ($\cong \mathbb{Z}_{p^\infty}$, p prime), and of groups isomorphic to $\mathbb{Q}$, if and only if,*

$$A = D \oplus \prod_{p \text{ prime}} T_p,$$

where D is a divisible subgroup of A and for each prime p, the subgroups T_p of A satisfies $pT_p = 0$ (so each T_p is a direct sum of simple groups isomorphic to $\mathbb{Z}/p\mathbb{Z}$).

Proof. Proposition 2.4.3 gives the general description of relative injectives for a proper class injectively generated by a set of modules (or by Proposition 2.4.4). In our case, an elementary injective $\mathbb{Z}$-module is the injective envelope (=divisible hull) of a cyclic group. The divisible hull of the infinite cyclic group $\mathbb{Z}$ is $\mathbb{Q}$. A finite cyclic group is a finite direct sum of cyclic groups of prime power order, so its divisible hull is the finite direct sum of the injective hulls of those cyclic groups of prime power order, which are infinite cocyclic groups. This proves the first 'if and only if'.

The last statement gives the characterization of neat-injective abelian groups (i.e. $\mathcal{N}eat_{\mathbb{Z}\text{-}\mathcal{M}od}$-injective abelian groups) given by Harrison et al. (1963, Lemma 4 in §4). Note, of course that, it gives much more information about the structure of neat-injective abelian groups. □

An interesting result for abelian groups with a one line proof using proper classes, but a direct proof of which seems not available, is that:

Theorem 4.1.4. *A supplement in an abelian group is a complement:*

$$\mathcal{S}uppl_{\mathbb{Z}\text{-}\mathcal{M}od} \subseteq \mathcal{C}ompl_{\mathbb{Z}\text{-}\mathcal{M}od}.$$

Proof. $\mathcal{S}uppl_{\mathbb{Z}\text{-}\mathcal{M}od} \subseteq \iota^{-1}(\{\mathbb{Z}/p\mathbb{Z} | p \text{ prime}\}) = \mathcal{N}eat_{\mathbb{Z}\text{-}\mathcal{M}od} = \mathcal{C}ompl_{\mathbb{Z}\text{-}\mathcal{M}od}$
Here the first inclusion follows from Proposition 3.4.1. □

Proposition 4.1.5. *(Fuchs, 1970, Exercise 53.4) (by Nunke (1959, Theorem 5.1), see Theorem 5.2.3) For the proper class $\mathcal{C} = \mathcal{C}ompl_{\mathbb{Z}\text{-}\mathcal{M}od} = \mathcal{N}eat_{\mathbb{Z}\text{-}\mathcal{M}od}$,*

$$\operatorname{Ext}_{\mathcal{C}}(C, A) = \bigcap_{p \text{ prime}} p \operatorname{Ext}(C, A).$$

4.2 Complements and supplements in abelian groups via uniform dimension and hollow dimension

In a finitely generated abelian group, to determine whether a (finitely generated) subgroup is a complement or not, we can use the criterion Theorem 3.6.1 for complements in modules with finite uniform dimension. A finitely generated abelian group, being a direct sum of cyclic groups of infinite or prime power order, has *finite* uniform dimension which is the number of summands in such a direct sum decomposition because $\mathrm{u.dim}\,\mathbb{Z} = 1 = \mathrm{u.dim}\,\mathbb{Z}/p^k\mathbb{Z}$ for a prime number p and $k \in \mathbb{Z}^+$, and u. dim is additive on finite direct sums. So a subgroup A of a finitely generated abelian group is a complement if and only if $\mathrm{u.dim}\,B = \mathrm{u.dim}\,A + \mathrm{u.dim}\,B/A$, or in terms of short exact sequences:

Proposition 4.2.1. *For a* finitely generated *abelian group B, a short exact sequence*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

of abelian groups and homomorphisms is in $\mathcal{C}ompl_{\mathbb{Z}\text{-}\mathcal{M}od} = \mathcal{N}eat_{\mathbb{Z}\text{-}\mathcal{M}od}$ if and only if

$$\mathrm{u.dim}\,B = \mathrm{u.dim}\,A + \mathrm{u.dim}\,C.$$

For cyclic groups of prime power order, $\mathrm{h.dim}\,\mathbb{Z}/p^k\mathbb{Z} = 1$ for p prime and $k \in \mathbb{Z}^+$, but $\mathrm{h.dim}\,\mathbb{Z} = \infty$. $\mathbb{Z}/p^k\mathbb{Z}$, being a uniserial $\mathbb{Z}$-module, is both uniform and hollow but $\mathbb{Z}$, although uniform, does *not* have finite hollow dimension. Among finitely generated abelian groups only the torsion ones (i.e. finite groups) has finite hollow dimension which is the number of summands in its decomposition as a direct sum of cyclic groups of prime power order. So, by Theorem 3.6.2:

Corollary 4.2.2. *For a* finite *abelian group B, a short exact sequence*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

of abelian groups and homomorphisms is in $Suppl_{\mathbb{Z}\text{-}\mathcal{M}od}$ if and only if

$$\text{h.}\dim B = \text{h.}\dim A + \text{h.}\dim C.$$

4.3 Finite subgroups which are complements are supplements

We look for some converse results to Theorem 4.1.4, that is when is a complement also a supplement? We will give an example of a complement in a finitely generated abelian group which is *not* a supplement. We will see that finite subgroups of a group which are complements in that group are also supplements.

Firstly, let us take a *finite* nonzero p-group A, where p is a prime number. By the fundamental theorem for the structure of the finitely generated abelian groups, we know that A is isomorphic to a finite direct sum of cyclic groups of order a power of p: For some positive integers $n, k_1, k_2, \dots, k_n$,

$$A \cong \bigoplus_{i=1}^{n} \mathbb{Z}/p^{k_i}\mathbb{Z}$$

Then considered as a $\mathbb{Z}$-module, $\operatorname{Rad} A$ is the Frattini subgroup of A and as A is a p-group, for a prime $q \neq p$, $qA = A$, so

$$\operatorname{Rad} A = \bigcap_{q \text{ prime}} qA = pA.$$

Hence,

$$A/\operatorname{Rad} A \cong \bigoplus_{i=1}^{n} (\mathbb{Z}/p^{k_i}\mathbb{Z})/(p\mathbb{Z}/p^{k_i}\mathbb{Z}) \cong \bigoplus_{i=1}^{n} \mathbb{Z}/p\mathbb{Z}$$

is a finite direct sum of simple modules. Suppose A is a neat subgroup of a group B, so we have a neat monomorphism (inclusion map) $0 \longrightarrow A \longrightarrow B$, i.e. the short exact sequence

$$\mathbb{E}: \qquad 0 \longrightarrow A \longrightarrow B \longrightarrow B/A \longrightarrow 0$$

is in $\mathcal{C}ompl_{\mathbb{Z}\text{-}\mathcal{M}od} = \mathcal{N}eat_{\mathbb{Z}\text{-}\mathcal{M}od}$. Since $A/\operatorname{Rad} A$ is a finite direct sum of simple groups, it is injective with respect to the short exact sequence $\mathbb{E}$ by Theorem 4.1.1. Thus there exists a homomorphism $h : B \longrightarrow A/\operatorname{Rad} A$ such that $h \circ i = \sigma$, that is, $h(a) = a + \operatorname{Rad} A$ for all $a \in A$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{\ i\ } & B & \longrightarrow & B/A & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \sigma} & \swarrow{\scriptstyle h} & & & & & \\ & & A/\operatorname{Rad} A & & & & & & \end{array}$$

Here i is the inclusion map and σ is the canonical epimorphism. So this h induces a homomorphism $h' : B/\operatorname{Rad} A \longrightarrow A/\operatorname{Rad} A$ such that $h'(a+\operatorname{Rad} A) = a+\operatorname{Rad} A$, hence the short exact sequence

$$0 \longrightarrow A/\operatorname{Rad} A \xrightarrow{inc.} B/\operatorname{Rad} A \longrightarrow B/A \longrightarrow 0$$

splits and we obtain for some subgroup A' of B containing $\operatorname{Rad} A$

$$B/\operatorname{Rad} A = A/\operatorname{Rad} A \oplus A'/\operatorname{Rad} A.$$

So $A + A' = B$ and $A \cap A' = \operatorname{Rad} A$. As $A \cong \bigoplus_{i=1}^{n} \mathbb{Z}/p^{k_i}\mathbb{Z}$, $\operatorname{Rad} A = pA \cong \bigoplus_{i=1}^{n} p\mathbb{Z}/p^{k_i}\mathbb{Z}$ and as for $i = 1, 2, \ldots, n$, $p\mathbb{Z}/p^{k_i}\mathbb{Z} \ll \mathbb{Z}/p^{k_i}\mathbb{Z}$, we obtain $\operatorname{Rad} A \ll A$. Hence $A \cap A' \ll A$, so A is a supplement of A' in B. The above argument gives that if a complement in an abelian group is a *finite* p-group, then it is a supplement. Of course, the same argument works for any *finite* group.

Theorem 4.3.1. *Let A be a* finite *subgroup of an abelian group B. Then A is a complement in B if and only if it is a supplement in B.*

Proof. ($\Leftarrow$) always holds (for any subgroup A) by Theorem 4.1.4. Conversely suppose $A \neq 0$ is a complement in B. The proof goes as the same with the arguments given for a p-group before the theorem. We only need to show that $A/\operatorname{Rad} A$ is a direct sum of simple groups and $\operatorname{Rad} A \ll A$. But these are clear, since A, being a finite group, is a *finite* direct sum of cyclic groups of prime power order. □

In the proof of this theorem, note that as A, being finite, is finitely generated, we clearly have $\operatorname{Rad} A \ll A$. But to have that $A/\operatorname{Rad} A$ is a direct sum of simple groups, being finitely generated is *not* enough; for example, the group $\mathbb{Z}$ is finitely generated, $\operatorname{Rad}\mathbb{Z} = 0 \ll \mathbb{Z}$, but $\mathbb{Z}/\operatorname{Rad}\mathbb{Z} \cong \mathbb{Z}$ is not a direct sum of simple abelian groups. Really, we can *not* generalize this theorem to include finitely generated abelian groups as the following example shows.

Example 4.3.2. The following short exact sequence of abelian groups is in $\mathcal{C}ompl_{\mathbb{Z}\text{-}\mathcal{M}od} = \mathcal{N}eat_{\mathbb{Z}\text{-}\mathcal{M}od}$ but it is *not* in $\mathcal{C}o\text{-}\mathcal{N}eat_{\mathbb{Z}\text{-}\mathcal{M}od}$, so also *not* in $\mathcal{S}uppl_{\mathbb{Z}\text{-}\mathcal{M}od}$:

$$\mathbb{E}: \qquad 0 \longrightarrow \mathbb{Z} \xrightarrow{f} (\mathbb{Z}/p\mathbb{Z}) \oplus \mathbb{Z} \xrightarrow{g} \mathbb{Z}/p^2\mathbb{Z} \longrightarrow 0$$

where $f(k) = k \cdot (-1 + p\mathbb{Z}, p)$, $k \in \mathbb{Z}$, and $g(a + p\mathbb{Z}, b) = (pa + b) + p^2\mathbb{Z}$, $a, b \in \mathbb{Z}$.

Firstly, $\mathbb{E}$ is in $\mathcal{C}ompl_{\mathbb{Z}\text{-}\mathcal{M}od}$ by Corollary 4.2.1, as

$$\operatorname{u.dim}((\mathbb{Z}/p\mathbb{Z}) \oplus \mathbb{Z}) = 2 = 1 + 1 = \operatorname{u.dim}(\mathbb{Z}) + \operatorname{u.dim}(\mathbb{Z}/p^2\mathbb{Z}).$$

It is *not* in $\mathcal{C}o\text{-}\mathcal{N}eat_{\mathbb{Z}\text{-}\mathcal{M}od}$ because if it were, then it would split as $\operatorname{Rad}(\mathbb{Z}) = 0$. But we cannot have this since it would imply $(\mathbb{Z}/p\mathbb{Z}) \oplus \mathbb{Z} \cong \mathbb{Z} \oplus (\mathbb{Z}/p^2\mathbb{Z})$ which cannot hold by the uniqueness of the cyclic factors of infinite or prime power order in a finitely generated abelian group up to isomorphism.

In fact, this example is obtained through the following considerations which make it clear. Denote $\mathcal{C}ompl_{\mathbb{Z}\text{-}\mathcal{M}od}$, $\mathcal{S}uppl_{\mathbb{Z}\text{-}\mathcal{M}od}$ and $\mathcal{C}o\text{-}\mathcal{N}eat_{\mathbb{Z}\text{-}\mathcal{M}od}$ shortly by $\mathcal{C}$, $\mathcal{S}$ and $c\mathcal{N}$ respectively. $\operatorname{Ext}(\mathbb{Z}/p^2\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z}/p^2\mathbb{Z}$ (see for example Fuchs (1970, §52) for details on elementary properties of $\operatorname{Ext}_{\mathbb{Z}}$). By Proposition 4.1.5,

$$\operatorname{Ext}_{\mathcal{C}}(\mathbb{Z}/p^2\mathbb{Z}, \mathbb{Z}) = \operatorname{Rad}(\operatorname{Ext}(\mathbb{Z}/p^2\mathbb{Z}, \mathbb{Z})) \cong \operatorname{Rad}(\mathbb{Z}/p^2\mathbb{Z}) = p\mathbb{Z}/p^2\mathbb{Z} \neq 0.$$

But $\mathrm{Ext}_{c\mathcal{N}}(\mathbb{Z}/p^2\mathbb{Z}, \mathbb{Z}) = 0$ because any short exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow B \longrightarrow \mathbb{Z}/p^2\mathbb{Z} \longrightarrow 0$$

in $c\mathcal{N}$ splits since $\mathrm{Rad}\,\mathbb{Z} = 0$. Note that, since $\mathcal{S}uppl_{\mathbb{Z}\text{-}\mathcal{M}od} \subseteq \mathcal{C}o\text{-}\mathcal{N}eat_{\mathbb{Z}\text{-}\mathcal{M}od}$ by Proposition 3.4.1, $\mathrm{Ext}_{\mathcal{S}}(\mathbb{Z}/p^2\mathbb{Z}, \mathbb{Z}) = 0$, too. So a short exact sequence

$$\mathbb{E} \in\in \mathrm{Ext}_{\mathcal{C}}(\mathbb{Z}/p^2\mathbb{Z}, \mathbb{Z}) \cong p\mathbb{Z}/p^2\mathbb{Z} \neq 0$$

which is *not* splitting (i.e. the equivalence class of which is a nonzero element in $\mathrm{Ext}_{\mathcal{C}}(\mathbb{Z}/p^2\mathbb{Z}, \mathbb{Z})$) will give the example we look for. In fact, in the isomorphism $\mathrm{Ext}(\mathbb{Z}/p^2\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z}/p^2\mathbb{Z}$, a generator for the cyclic group $\mathrm{Ext}(\mathbb{Z}/p^2\mathbb{Z}, \mathbb{Z})$ of order p^2 can be taken as the equivalence class $[\mathbb{F}]$ of the following short exact sequence

$$\mathbb{F}: \qquad 0 \longrightarrow \mathbb{Z} \xrightarrow{\;u\;} \mathbb{Z} \xrightarrow{\;v\;} \mathbb{Z}/p^2\mathbb{Z} \longrightarrow 0$$

where u is multiplication by p^2 and v is the canonical epimorphism. The example $\mathbb{E}$ above is in the equivalence class $p[\mathbb{F}]$ whose representative can be taken as the short exact sequence denoted by $p\mathbb{F}$ which is obtained by pushout:

$$\begin{array}{lccccccccc}
\mathbb{F}: & 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\quad u \quad} & \mathbb{Z} & \xrightarrow{\quad v \quad} & \mathbb{Z}/p^2\mathbb{Z} & \longrightarrow & 0 \\
 & & & \big\downarrow h & & \big\downarrow & & \big\downarrow & & \\
\mathbb{E} = p\mathbb{F}: & 0 & \dashrightarrow & \mathbb{Z} & \overset{f}{\dashrightarrow} & (\mathbb{Z}/p\mathbb{Z}) \oplus \mathbb{Z} & \overset{g}{\dashrightarrow} & \mathbb{Z}/p^2\mathbb{Z} & \dashrightarrow & 0
\end{array}$$

where $h : \mathbb{Z} \longrightarrow \mathbb{Z}$ is multiplication by p, i.e. $h(k) = pk$, $k \in \mathbb{Z}$, and f, g are found to be given as in the beginning of the example by pushout computations.

Corollary 4.3.3. *$\mathcal{C}o\text{-}\mathcal{N}eat_{\mathbb{Z}\text{-}\mathcal{M}od} \subsetneqq \mathcal{C}ompl_{\mathbb{Z}\text{-}\mathcal{M}od}$ and $\mathcal{S}uppl_{\mathbb{Z}\text{-}\mathcal{M}od} \subsetneqq \mathcal{C}ompl_{\mathbb{Z}\text{-}\mathcal{M}od}$.*

Proof. $\mathcal{S}uppl_{\mathbb{Z}\text{-}\mathcal{M}od} \subseteq \mathcal{C}o\text{-}\mathcal{N}eat_{\mathbb{Z}\text{-}\mathcal{M}od} \subseteq \iota^{-1}(\{\mathbb{Z}/p\mathbb{Z} | p \text{ prime}\}) = \mathcal{C}ompl_{\mathbb{Z}\text{-}\mathcal{M}od}$.
The first two inclusions follow from Proposition 3.4.1 and the last equality follows from Theorem 4.1.1. The second inclusion is strict by the previous Example 4.3.2. □

4.4 The inductive closure of the proper class $\mathcal{S}uppl_{\mathbb{Z}\text{-}\mathcal{M}od}$ is $\mathcal{C}ompl_{\mathbb{Z}\text{-}\mathcal{M}od} = \mathcal{N}eat_{\mathbb{Z}\text{-}\mathcal{M}od}$

We will use the following two lemmas in proving this claim; these lemmas say that each $\mathbb{Z}/p^n\mathbb{Z}$, p prime and integer $n \geq 2$, and the Prüfer group $\mathbb{Z}_{p^\infty}$ are *not* flat with respect to the proper class $\mathcal{S}uppl_{\mathbb{Z}\text{-}\mathcal{M}od}$, i.e. *not* in $\tau(\mathcal{S}uppl_{\mathbb{Z}\text{-}\mathcal{M}od})$. In the proofs of these lemmas, we use elementary properties of tensor product of abelian groups; for these properties, see for example Fuchs (1970, §59-60). We use the following natural isomorphism a few times:

Proposition 4.4.1. *(Fuchs, 1970, §59, property H) For every positive integer m and abelian group C, we have a natural isomorphism*

$$\psi : C \otimes (\mathbb{Z}/m\mathbb{Z}) \longrightarrow C/mC$$

such that for $k \in \mathbb{Z}$ and $c \in C$,

$$\psi(c \otimes (k + m\mathbb{Z})) = kc + mC.$$

Lemma 4.4.2. *Let p be a prime number and $n \geq 2$ an integer. $\mathbb{Z}/p^n\mathbb{Z}$ is* not *flat with respect to following short exact sequence in $\mathcal{S}uppl_{\mathbb{Z}\text{-}\mathcal{M}od}$:*

$$\mathbb{E}: \qquad 0 \longrightarrow \mathbb{Z}/p^n\mathbb{Z} \xrightarrow{f} (\mathbb{Z}/p\mathbb{Z}) \oplus (\mathbb{Z}/p^{2n-1}\mathbb{Z}) \xrightarrow{g} \mathbb{Z}/p^n\mathbb{Z} \longrightarrow 0,$$

where

$$\begin{aligned} f(k + p^n\mathbb{Z}) &= (k + p\mathbb{Z}, kp^{n-1} + p^{2n-1}\mathbb{Z}), \quad k \in \mathbb{Z}, \\ g(a + p\mathbb{Z}, b + p^{2n-1}\mathbb{Z}) &= (b - ap^{n-1}) + p^n\mathbb{Z}, \quad a, b \in \mathbb{Z}. \end{aligned}$$

Proof. Firstly, $\mathbb{E}$ is in $\mathcal{C}ompl_{\mathbb{Z}\text{-}\mathcal{M}od}$ by Proposition 4.2.1, as

$$\text{u.}\dim((\mathbb{Z}/p\mathbb{Z}) \oplus (\mathbb{Z}/p^{2n-1}\mathbb{Z})) = 2 = \text{u.}\dim(\mathbb{Z}/p^n\mathbb{Z}) + \text{u.}\dim(\mathbb{Z}/p^n\mathbb{Z}).$$

As $\mathbb{Z}/p^n\mathbb{Z}$ is finite, this sequence $\mathbb{E}$ is in $\mathcal{S}uppl_{\mathbb{Z}\text{-}\mathcal{M}od}$ by Theorem 4.3.1. Tensoring $\mathbb{E}$ with $\mathbb{Z}/p^n\mathbb{Z}$, we do *not* obtain a monomorphism $f \otimes 1_{\mathbb{Z}/p^n\mathbb{Z}}$ (where $1_{\mathbb{Z}/p^n\mathbb{Z}} : \mathbb{Z}/p^n\mathbb{Z} \longrightarrow \mathbb{Z}/p^n\mathbb{Z}$ is the identity map). Because, the element

$$(1 + p^n\mathbb{Z}) \otimes (p + p^n\mathbb{Z}) \neq 0 \text{ in } (\mathbb{Z}/p^n\mathbb{Z}) \otimes (\mathbb{Z}/p^n\mathbb{Z})$$

as it corresponds, under the natural isomorphism

$$(\mathbb{Z}/p^n\mathbb{Z}) \otimes (\mathbb{Z}/p^n\mathbb{Z}) \cong \mathbb{Z}/p^n\mathbb{Z},$$

to the element $p + p^n\mathbb{Z} \neq 0$ in $\mathbb{Z}/p^n\mathbb{Z}$ as $n \geq 2$ (see Proposition 4.4.1). But

$$\begin{aligned}(f \otimes 1_{\mathbb{Z}/p^n\mathbb{Z}})((1 + p^n\mathbb{Z}) \otimes (p + p^n\mathbb{Z})) &= (1 + p\mathbb{Z}, p^{n-1} + p^{2n-1}\mathbb{Z}) \otimes (p + p^n\mathbb{Z}) \\ &= (1 + p\mathbb{Z}, 0) \otimes (p + p^n\mathbb{Z}) + (0, p^{n-1} + p^{2n-1}\mathbb{Z}) \otimes (p + p^n\mathbb{Z}) \\ &= (p + p\mathbb{Z}, 0) \otimes (1 + p^n\mathbb{Z}) + (0, 1 + p^{2n-1}\mathbb{Z}) \otimes (p^n + p^n\mathbb{Z}) \\ &= 0 \otimes (1 + p^n\mathbb{Z}) + (0, 1 + p^{2n-1}\mathbb{Z}) \otimes 0 = 0 + 0 = 0\end{aligned}$$

□

We want to find a $\mathcal{S}uppl_{\mathbb{Z}\text{-}\mathcal{M}od}$-monomorphism with respect to which $\mathbb{Z}_{p^\infty}$ is *not* flat. Since $\mathcal{S}uppl_{\mathbb{Z}\text{-}\mathcal{M}od} \subseteq \mathcal{C}ompl_{\mathbb{Z}\text{-}\mathcal{M}od} = \mathcal{N}eat_{\mathbb{Z}\text{-}\mathcal{M}od}$, we must search it through $\mathcal{N}eat_{\mathbb{Z}\text{-}\mathcal{M}od}$-monomorphisms. It is known that there are enough neat-injectives; more precisely it is shown in Harrison et al. (1963, Lemma 5) that for a group X, if we embed X into a divisible group D, then the homomorphism

$$\begin{array}{ccc} X & \longrightarrow & D \oplus \prod\limits_{p \text{ prime}} (X/pX) \\ x & \mapsto & (x, (x + pX)_{p \text{ prime}}) \end{array}$$

is a $\mathcal{N}eat_{\mathbb{Z}\text{-}\mathcal{M}od}$-monomorphism and $D \oplus \prod_{p \text{ prime}}(X/pX)$ is $\mathcal{N}eat_{\mathbb{Z}\text{-}\mathcal{M}od}$-injective. Motivated by such considerations the following example is found:

Lemma 4.4.3. *Let p be a prime. Denote by $\mathbb{Q}$ all rational numbers and by $\mathbb{Q}_p$ the additive subgroup of $\mathbb{Q}$ consisting all rational numbers whose denominators* *are relatively prime to p. Then the monomorphism*

$$f \ : \ \mathbb{Q}_p \longrightarrow \mathbb{Q} \oplus (\mathbb{Q}_p/p\mathbb{Q}_p)$$
$$f(x) = (x, x + p\mathbb{Q}_p), \quad x \in \mathbb{Q}_p$$

is a $Suppl_{\mathbb{Z}\text{-}\mathcal{M}od}$-monomorphism and the Prüfer group $\mathbb{Z}_{p^\infty}$ is not *flat with respect to this monomorphism.*

Proof. Denote by $\mathbb{Q}^{(p)}$, the subgroup of $\mathbb{Q}$ consisting of all rational numbers whose denominators are powers of the prime p. Observe that $\mathbb{Q}_p + \mathbb{Q}^{(p)} = \mathbb{Q}$ and $\mathbb{Q}_p \cap \mathbb{Q}^{(p)} = \mathbb{Z}$.

Also note that $p\mathbb{Q}_p + \mathbb{Z} = \mathbb{Q}_p$: For $a/b \in \mathbb{Q}_p$, $a, b \in \mathbb{Z}$, $b \neq 0$, $p \nmid b$, as the greatest common divisor of p and b is 1, there exits $u, v \in \mathbb{Z}$ such that $pu + bv = 1$, so $a/b = p(ua/b) + va \in p\mathbb{Q}_p + \mathbb{Z}$.

Our claim is that $\text{Im}(f)$ is a supplement of $\mathbb{Q}^{(p)} \oplus 0$ in $\mathbb{Q} \oplus (\mathbb{Q}_p/p\mathbb{Q}_p)$.

1. $\text{Im}(f) + (\mathbb{Q}^{(p)} \oplus 0) = \mathbb{Q} \oplus (\mathbb{Q}_p/p\mathbb{Q}_p)$.

   Really for an element $(y, z + p\mathbb{Q}_p) \in \mathbb{Q} \oplus (\mathbb{Q}_p/p\mathbb{Q}_p)$, where $y \in \mathbb{Q}$, $z \in \mathbb{Q}_p$, as $\mathbb{Q}_p + \mathbb{Q}^{(p)} = \mathbb{Q}$, $y = u + v$ for some $u \in \mathbb{Q}_p$ and $v \in \mathbb{Q}^{(p)}$. Since $\mathbb{Q}_p = p\mathbb{Q}_p + \mathbb{Z}$, $u = pu' + k$ and $z = pz' + n$ for some $u', z' \in \mathbb{Q}_p$ and $k, n \in \mathbb{Z}$. So, $z + p\mathbb{Q}_p = n + p\mathbb{Q}_p = (pu' + n) + p\mathbb{Q}_p$, hence,

   $$(y, z + p\mathbb{Q}_p) = \underbrace{(pu' + n, (pu' + n) + p\mathbb{Q}_p)}_{= f(pu'+n)} + (v + k - n, 0 + p\mathbb{Q}_p)$$

   is in $\text{Im}(f) + (\mathbb{Q}^{(p)} \oplus 0)$ as $v + k - n \in \mathbb{Q}^{(p)}$ since $\mathbb{Z} \subseteq \mathbb{Q}^{(p)}$ and $v \in \mathbb{Q}^{(p)}$.

2. $\text{Im}(f) \cap (\mathbb{Q}^{(p)} \oplus 0) = [(p\mathbb{Q}_p) \cap \mathbb{Q}^{(p)}] \oplus 0 = (p\mathbb{Z}) \oplus 0 \ll \text{Im}(f)$

   where the smallness in the end is obtained as follows:

   $$\text{Rad}(\mathbb{Q}_p) = \bigcap_{q \text{ prime}} q\mathbb{Q}_p = p\mathbb{Q}_p$$

   as $\mathbb{Q}_p$ is q-divisible for each prime $q \neq p$ and $p\mathbb{Z} \leq p\mathbb{Q}_p$, being a cyclic subgroup of the radical, is small in $\mathbb{Q}_p$; hence, $f(p\mathbb{Z}) \ll f(\mathbb{Q}_p) = \text{Im}(f)$. But $f(p\mathbb{Z}) = p\mathbb{Z} \oplus 0$.

To prove that $\mathbb{Z}_{p^\infty}$ is *not* flat with respect to this $\mathcal{S}uppl_{\mathbb{Z}\text{-}\mathcal{M}od}$-monomorphism, we will show that $\mathbb{Q}_p \otimes \mathbb{Z}_{p^\infty}$ is *not* 0, while $(\mathbb{Q} \oplus (\mathbb{Q}_p/p\mathbb{Q}_p)) \otimes \mathbb{Z}_{p^\infty} = 0$.

$$[\mathbb{Q} \oplus (\mathbb{Q}_p/p\mathbb{Q}_p)] \otimes \mathbb{Z}_{p^\infty} \cong (\mathbb{Q} \otimes \mathbb{Z}_{p^\infty}) \oplus [(\mathbb{Q}_p/p\mathbb{Q}_p) \otimes \mathbb{Z}_{p^\infty}] = 0$$

because $\mathbb{Q}\otimes\mathbb{Z}_{p^\infty} = 0$ as $\mathbb{Q}$ is divisible and $\mathbb{Z}_{p^\infty}$ is torsion, and $(\mathbb{Q}_p/p\mathbb{Q}_p)\otimes\mathbb{Z}_{p^\infty} = 0$ as $\mathbb{Q}_p/p\mathbb{Q}_p$ is torsion and $\mathbb{Z}_{p^\infty}$ is divisible.

To prove that $\mathbb{Q}_p \otimes \mathbb{Z}_{p^\infty} \neq 0$, we will show that the element $1 \otimes c_0 \neq 0$ in $\mathbb{Q}_p \otimes \mathbb{Z}_{p^\infty}$, where we consider $\mathbb{Z}_{p^\infty}$ to be generated by a sequence of elements $c_0, c_1, c_2, \ldots$ such that $pc_0 = 0$ and $pc_n = c_{n-1}$ for each integer $n \geq 1$ (look at page 67 for the Prüfer (divisible) group $\mathbb{Z}_{p^\infty}$). Suppose for the contrary that $1 \otimes c_0 = 0$ in $\mathbb{Q}_p \otimes \mathbb{Z}_{p^\infty}$. Then by properties of tensor product (see for example Fuchs (1970, Exercise 59.7), or Atiyah & Macdonald (1969, Corollary 2.13)), we know that there exists a finitely generated subgroup $A \leq \mathbb{Q}_p$ and a finitely generated subgroup $B \leq \mathbb{Z}_{p^\infty}$ such that $1 \in A$, $c_0 \in B$ and $1 \otimes c_0 = 0$ in $A \otimes B$, so also $1 \otimes c_0 = 0$ in $\mathbb{Q}_p \otimes B$. Since $B \neq 0$ is finitely generated, $B = \mathbb{Z}c_n$ for some integer $n \geq 0$, so $B \cong \mathbb{Z}/p^{n+1}\mathbb{Z}$. Note that under this isomorphism c_n corresponds to $1 + p^{n+1}\mathbb{Z}$ and $c_0 = p^n c_n$ corresponds to $p^n + p^{n+1}\mathbb{Z}$. We have then,

$$\mathbb{Q}_p \otimes B \cong \mathbb{Q}_p \otimes (\mathbb{Z}/p^{n+1}\mathbb{Z}) \cong \mathbb{Q}_p/p^{n+1}\mathbb{Q}_p,$$

where the second isomorphism is the natural isomorphism as described in Proposition 4.4.1. Under these isomorphisms $1 \otimes c_0 = 0$ in $\mathbb{Q}_p \otimes B$ corresponds to $p^n + p^{n+1}\mathbb{Q}_p$ in $\mathbb{Q}_p/p^{n+1}\mathbb{Q}_p$ which is *not* zero and this gives the required contradiction. □

Theorem 4.4.4. *The inductive closure of the proper class $\mathcal{S}uppl_{\mathbb{Z}\text{-}\mathcal{M}od}$ is $\mathcal{C}ompl_{\mathbb{Z}\text{-}\mathcal{M}od} = \mathcal{N}eat_{\mathbb{Z}\text{-}\mathcal{M}od}$.*

Proof. Denote $\mathcal{C}ompl_{\mathbb{Z}\text{-}\mathcal{M}od}$ shortly by $\mathcal{C}$ and $\mathcal{S}uppl_{\mathbb{Z}\text{-}\mathcal{M}od}$ shortly by $\mathcal{S}$. Let $\widetilde{\mathcal{S}}$ be the inductive closure of $\mathcal{S}$. By a remarkable theorem for abelian groups (Manovcev (1975), Kuz'minov (1976), Sklyarenko (1981), Sklyarenko (1978, Theorem 8.2

(without proof))), every inductively closed proper class of abelian groups is flatly generated by some subcollection Ω of

$$\{\mathbb{Z}/p^k\mathbb{Z} | p \text{ prime and } k \in \mathbb{Z}^+\} \cup \{\mathbb{Z}_{p^\infty} | p \text{ prime}\}$$

So the inductively closed class $\widetilde{\mathcal{S}} = \tau^{-1}(\Omega)$ for some such collection Ω. To show that $\widetilde{\mathcal{S}} = \mathcal{C}$, it will suffice to show that in the collection Ω, we can have only $\mathbb{Z}/p\mathbb{Z}$, p prime. But that follows from the previous lemmas as each $\mathbb{Z}/p^n\mathbb{Z}$, p prime and integer $n \geq 2$, and $\mathbb{Z}_{p^\infty}$ are *not* in $\tau(\mathcal{S})$, so also are not in $\tau(\widetilde{\mathcal{S}})$. Since $\mathcal{C}$ is inductively closed (as it is flatly generated by $\{\mathbb{Z}/p\mathbb{Z} | p \text{ prime}\}$, by Theorem 4.1.1) and $\mathcal{S} \subseteq \mathcal{C}$ by Theorem 5.2.1, we have $\widetilde{\mathcal{S}} \subseteq \mathcal{C}$, so, of course, all $\mathbb{Z}/p\mathbb{Z}$ for each prime p is in Ω and we obtain that $\Omega = \{\mathbb{Z}/p\mathbb{Z} | p \text{ prime}\}$. Hence $\widetilde{\mathcal{S}} = \tau^{-1}(\Omega) = \mathcal{C}$. □

4.5 The functor $\mathrm{Ext}_{\mathcal{S}uppl_{\mathbb{Z}\text{-}\mathcal{M}od}}$ is *not* factorizable

For a proper class $\mathcal{P}$ of R-modules, let us say that $\mathrm{Ext}_{\mathcal{P}}$ is *factorizable* as

$$R\text{-}\mathcal{M}od \times R\text{-}\mathcal{M}od \xrightarrow{Ext_R} \mathcal{A}b \longrightarrow \mathcal{A}b\,,$$

if it is a composition $H \circ \mathrm{Ext}_R$ for some functor $H : \mathcal{A}b \longrightarrow \mathcal{A}b$: the diagram

$$\begin{array}{ccc} R\text{-}\mathcal{M}od \times R\text{-}\mathcal{M}od & \xrightarrow{\quad Ext_{\mathcal{P}} \quad} & \mathcal{A}b \\ & {\scriptstyle Ext_R}\searrow \quad \nearrow{\scriptstyle H} & \\ & \mathcal{A}b & \end{array}$$

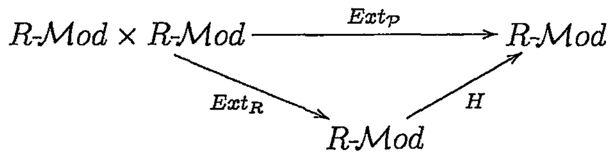
is commutative, that is, for all R-modules A, C,

$$\mathrm{Ext}_{\mathcal{P}}(C, A) = H(\mathrm{Ext}_R(C, A)).$$

When the ring R is *commutative*, since the functor Ext_R can be considered to have range $R\text{-}\mathcal{M}od$, we say that $\mathrm{Ext}_{\mathcal{P}}$ is *factorizable* as

$$R\text{-}\mathcal{M}od \times R\text{-}\mathcal{M}od \xrightarrow{Ext_R} R\text{-}\mathcal{M}od \longrightarrow R\text{-}\mathcal{M}od\,,$$

if it is a composition $H \circ \mathrm{Ext}_R$ for some functor $H : R\text{-}\mathcal{M}od \longrightarrow R\text{-}\mathcal{M}od$: the diagram



is commutative, that is, for all R-modules A, C,

$$\mathrm{Ext}_{\mathcal{P}}(C, A) = H(\mathrm{Ext}_R(C, A)).$$

For the ring $R = \mathbb{Z}$, since we identify the categories $\mathcal{A}b$ and $\mathbb{Z}\text{-}\mathcal{M}od$, both the above two definitions coincide so that for a proper class $\mathcal{P}$ of $\mathbb{Z}$-modules (abelian groups), we just say that $\mathrm{Ext}_{\mathcal{P}}$ is *factorizable* in the above cases.

Example 4.5.1. Let $U : \mathcal{A}b \longrightarrow \mathcal{A}b$ be the Ulm functor which associates to each abelian group A, its Ulm subgroup $U(A) = \bigcap_{n=1}^{\infty} nA$. $\mathrm{Ext}_{\mathcal{P}ure_{\mathbb{Z}\text{-}\mathcal{M}od}}$ is denoted by Pext in Fuchs (1970, §53) and by Nunke (1959, Theorem 5.1), it is shown that for abelian groups A, C,

$$\mathrm{Ext}_{\mathcal{P}ure_{\mathbb{Z}\text{-}\mathcal{M}od}}(C, A) = \mathrm{Pext}(C, A) = \bigcap_{n=1}^{\infty} n\,\mathrm{Ext}_{\mathbb{Z}}(C, A) = U(\mathrm{Ext}_{\mathbb{Z}}(C; A)).$$

Thus, $\mathrm{Ext}_{\mathcal{P}ure_{\mathbb{Z}\text{-}\mathcal{M}od}}$ is factorizable.

Example 4.5.2. Denote $\mathcal{C}ompl_{\mathbb{Z}\text{-}\mathcal{M}od} = \mathcal{N}eat_{\mathbb{Z}\text{-}\mathcal{M}od}$ shortly by $\mathcal{C}$. Consider the functor $\mathrm{Rad} : \mathcal{A}b \longrightarrow \mathcal{A}b$ which associates with each abelian group A, its Frattini subgroup (which is its radical as $\mathbb{Z}$-module and which equals) $\mathrm{Rad}(A) = \bigcap_{p \text{ prime}} pA$. Then by Proposition 4.1.5, since

$$\mathrm{Ext}_{\mathcal{C}}(C, A) = \bigcap_{p \text{ prime}} p\,\mathrm{Ext}_{\mathbb{Z}}(C, A) = \mathrm{Rad}(\mathrm{Ext}_{\mathbb{Z}}(C; A)),$$

$\mathrm{Ext}_{\mathcal{C}}$ is factorizable.

But the proper classes $\mathcal{S}uppl_{\mathbb{Z}\text{-}\mathcal{M}od}$ behaves badly in this sense:

Theorem 4.5.3. *The functor* $\mathrm{Ext}_{\mathcal{S}uppl_{\mathbb{Z}\text{-}\mathcal{M}od}}$ *is* not *factorizable as*

$$\mathbb{Z}\text{-}\mathcal{M}od \times \mathbb{Z}\text{-}\mathcal{M}od \xrightarrow{Ext_{\mathbb{Z}}} \mathcal{A}b \xrightarrow{H} \mathcal{A}b$$

for any functor $H : \mathcal{A}b \longrightarrow \mathcal{A}b$ *on the category of abelian groups.*

Proof. Denote $\mathcal{C}ompl_{\mathbb{Z}\text{-}\mathcal{M}od}$ and $\mathcal{S}uppl_{\mathbb{Z}\text{-}\mathcal{M}od}$ shortly by $\mathcal{C}$ and $\mathcal{S}$ respectively. Suppose for the contrary that $\mathrm{Ext}_{\mathcal{S}}$ is factorizable, so that there exits a functor $H : \mathcal{A}b \longrightarrow \mathcal{A}b$ such that for all abelian groups A, C,

$$\mathrm{Ext}_{\mathcal{S}}(C, A) = H(\mathrm{Ext}_{\mathbb{Z}}(C, A)).$$

In Example 4.3.2, we have found that $\mathrm{Ext}_{\mathcal{S}}(\mathbb{Z}/p^2\mathbb{Z}, \mathbb{Z}) = 0$. As $\mathrm{Ext}(\mathbb{Z}/p^2\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z}/p^2\mathbb{Z}$, this implies that $H(\mathbb{Z}/p^2\mathbb{Z}) = 0$. But also $\mathrm{Ext}(\mathbb{Z}/p^2\mathbb{Z}, \mathbb{Z}/p^2\mathbb{Z}) \cong \mathbb{Z}/p^2\mathbb{Z}$ and as $\mathbb{Z}/p^2\mathbb{Z}$ is a finite group, by Theorem 4.3.1,

$$\begin{aligned}\mathrm{Ext}_{\mathcal{S}}(\mathbb{Z}/p^2\mathbb{Z}, \mathbb{Z}/p^2\mathbb{Z}) &= \mathrm{Ext}_{\mathcal{C}}(\mathbb{Z}/p^2\mathbb{Z}, \mathbb{Z}/p^2\mathbb{Z}) \\ &= \mathrm{Rad}(\mathrm{Ext}_{\mathbb{Z}}(\mathbb{Z}/p^2\mathbb{Z}, \mathbb{Z}/p^2\mathbb{Z})) \cong p(\mathbb{Z}/p^2\mathbb{Z}) \neq 0.\end{aligned}$$

So in this case $H(\mathbb{Z}/p^2\mathbb{Z})$ must be nonzero. This contradiction ends the proof. □

4.6 The proper class $\mathcal{C}o\text{-}\mathcal{N}eat_{\mathbb{Z}\text{-}\mathcal{M}od}$

The proper class $\mathcal{C}o\text{-}\mathcal{N}eat_{\mathbb{Z}\text{-}\mathcal{M}od}$ is between $\mathcal{S}uppl_{\mathbb{Z}\text{-}\mathcal{M}od}$ and $\mathcal{C}ompl_{\mathbb{Z}\text{-}\mathcal{M}od}$:

Proposition 4.6.1.

$$\begin{aligned}\mathcal{S}uppl_{\mathbb{Z}\text{-}\mathcal{M}od} \subseteq \mathcal{C}o\text{-}\mathcal{N}eat_{\mathbb{Z}\text{-}\mathcal{M}od} &\subseteq \iota^{-1}(\{\mathbb{Z}/p\mathbb{Z} | p \text{ prime}\}) \\ &= \mathcal{N}eat_{\mathbb{Z}\text{-}\mathcal{M}od} = \mathcal{C}ompl_{\mathbb{Z}\text{-}\mathcal{M}od}\end{aligned}$$

Proof. The first two inclusions follow from Proposition 3.4.1. The last two equalities follow from Theorem 4.1.1. □

The above inclusions are *strict.* To prove this, we will follow mainly the proofs in Generalov (1983, Theorems 6-7, Propositions 4-5) for the particular ring $\mathbb{Z}$, which simplifies some steps and for which some missing steps in Generalov (1983, proofs of Theorem 6 and Proposition 5) are easily done. After two lemmas, we will give an example of a $\mathcal{C}o\text{-}\mathcal{N}eat_{\mathbb{Z}\text{-}\mathcal{M}od}$-monomorphism which is *not* a $\mathcal{S}uppl_{\mathbb{Z}\text{-}\mathcal{M}od}$-monomorphism.

Lemma 4.6.2. *(by Generalov (1983, Theorem 7, Proposition 4, Corollary 5)) For the submodule* $S := \sum_{p \text{ prime}} \mathbb{Z}\frac{1}{p} \leq \mathbb{Q}$ *of the* $\mathbb{Z}$*-module* $\mathbb{Q}$ *of rational numbers, we have:*

(i) $S/\mathbb{Z} = \operatorname{Soc}(\mathbb{Q}/\mathbb{Z})$ *is a semisimple* $\mathbb{Z}$*-module.*

(ii) $\operatorname{Rad} S = \mathbb{Z}$.

(iii) For the countably generated free $\mathbb{Z}$*-module* $F := \bigoplus_{i \in \mathbb{Z}^+} \mathbb{Z}$*, take the* $\mathbb{Z}$*-module* $A := \bigoplus_{i \in \mathbb{Z}^+} S$*. Then* $\operatorname{Rad} A = F$ *and* $A/\operatorname{Rad} A$ *is a semisimple* $\mathbb{Z}$*-module.*

Proof. (i) See page 67 for the decomposition of $\mathbb{Q}/\mathbb{Z}$; from this decomposition, it is seen that $\operatorname{Soc}(\mathbb{Q}/\mathbb{Z}) = S/\mathbb{Z}$.

(ii) Since $S/\mathbb{Z}$ is semisimple, $\operatorname{Rad}(S/\mathbb{Z}) = 0$. Hence $\operatorname{Rad} S \leq \mathbb{Z}$ clearly. For any prime q, $qS = q\left(\sum_{p \text{ prime}} \mathbb{Z}\frac{1}{p}\right) \geq q\mathbb{Z}\frac{1}{q} = \mathbb{Z}$. So $\operatorname{Rad} S = \bigcap_{q \text{ prime}} qS \geq \mathbb{Z}$. Thus, $\operatorname{Rad} S = \mathbb{Z}$.

(iii) $\operatorname{Rad} A = \bigoplus_{i \in \mathbb{Z}^+} \operatorname{Rad} S = \bigoplus_{i \in \mathbb{Z}^+} \mathbb{Z} = F$ and $A/\operatorname{Rad} A = \bigoplus_{i \in \mathbb{Z}^+} (S/\mathbb{Z})$ is semisimple. □

Lemma 4.6.3. *(Leonard, 1966, after Corollary to Theorem 2, p. 528) The countably generated free* $\mathbb{Z}$*-module* $F := \bigoplus_{i \in \mathbb{Z}^+} \mathbb{Z}$ *is* not *a small* $\mathbb{Z}$*-module.*

Proof. Since the $\mathbb{Z}$-module $\mathbb{Q}$ is countably generated, there exists an epimorphism $g : F \longrightarrow \mathbb{Q}$ from the free $\mathbb{Z}$-module $F := \bigoplus_{i \in \mathbb{Z}^+} \mathbb{Z}$. Let $H := \operatorname{Ker} g$. Then $F/H \cong \mathbb{Q}$.

By Leonard (1966, Lemma 6), F is *not* a small $\mathbb{Z}$-module since $F/H \cong \mathbb{Q}$ is a nonzero injective module (also pointed out in Leonard (1966, after Corollary to Theorem 2, p. 528)). In fact, this is simply because if F is a small module, then F is small in its injective envelope $\mathrm{E}(F)$ by Leonard (1966, Theorem 1). So, the quotient module F/H is small in $\mathrm{E}(F)/H$. But since $F/H \cong \mathbb{Q}$ is injective, F/H is a direct summand of $\mathrm{E}(F)/H$ which contradicts with F/H being small in $\mathrm{E}(F)/H$. □

Example 4.6.4. (by Generalov (1983, Proposition 5)) Consider the $\mathbb{Z}$-modules

$$F := \bigoplus_{i \in \mathbb{Z}^+} \mathbb{Z} \leq A := \bigoplus_{i \in \mathbb{Z}^+} S \leq \bigoplus_{i \in \mathbb{Z}^+} \mathbb{Q} = \mathrm{E}(A),$$

where $S := \sum_{p \text{ prime}} \mathbb{Z}\frac{1}{p} \leq \mathbb{Q}$, $\operatorname{Rad} A = F$ and $\mathrm{E}(A)$ denotes the injective envelope of A. Then the monomorphism

$$\begin{aligned} f : A &\longrightarrow \mathrm{E}(A) \oplus (A/\operatorname{Rad} A) \\ x &\longmapsto (x, x + \operatorname{Rad} A) \end{aligned}$$

is a $\mathcal{C}o\text{-}\mathcal{N}eat_{\mathbb{Z}\text{-}\mathcal{M}od}$-monomorphism but *not* a $\mathcal{S}uppl_{\mathbb{Z}\text{-}\mathcal{M}od}$-monomorphism. So

$$\mathcal{S}uppl_{\mathbb{Z}\text{-}\mathcal{M}od} \neq \mathcal{C}o\text{-}\mathcal{N}eat_{\mathbb{Z}\text{-}\mathcal{M}od}.$$

Proof. By Lemma 4.6.2, $\operatorname{Rad} A = F$. By Proposition 3.4.3, f is a $\mathcal{C}o\text{-}\mathcal{N}eat_{\mathbb{Z}\text{-}\mathcal{M}od}$-monomorphism and $\mathrm{E}(A) \oplus (A/\operatorname{Rad} A)$ is $\mathcal{C}o\text{-}\mathcal{N}eat_{\mathbb{Z}\text{-}\mathcal{M}od}$-injective.

Suppose for the contrary that f is a $\mathcal{S}uppl_{\mathbb{Z}\text{-}\mathcal{M}od}$-monomorphism.

Let $M := f(A)$ and $N := \mathrm{E}(A) \oplus (A/\operatorname{Rad} A)$. Then M is a supplement in N. That means there exits a submodule $K \leq N$ such that

$$M + K = N \quad \text{and} \quad M \cap K \ll M.$$

Let $C := M \cap K$. Since $C \ll M$, $C \leq \operatorname{Rad} M = \operatorname{Rad} f(A) \cong \operatorname{Rad} A = F$, so C is also a free $\mathbb{Z}$-module and rank of the free $\mathbb{Z}$-module C is at most $|\mathbb{Z}^+|$. Since C is a small module, C cannot have rank $|\mathbb{Z}^+|$ because otherwise $C \cong \bigoplus_{i\in\mathbb{Z}^+} \mathbb{Z} = F$, but F is not a small module by Lemma 4.6.3. Hence C is necessarily a free module of finite rank.

As $C \ll M$,

$$C \leq \operatorname{Rad} M \leq \operatorname{Rad} N = \operatorname{Rad}(\mathrm{E}(A) \oplus (A/\operatorname{Rad} A)) = \operatorname{Rad} \mathrm{E}(A) \leq \mathrm{E}(A).$$

So,

$$\begin{aligned}(\mathrm{E}(A)/C) \oplus (A/\operatorname{Rad} A) &\cong (\mathrm{E}(A) \oplus (A/\operatorname{Rad} A))/C = N/C = (M+K)/C \\ &\cong (M/C) \oplus (K/C)\end{aligned}$$

Since the left side is $\mathcal{C}o\text{-}\mathcal{N}eat_{\mathbb{Z}\text{-}\mathcal{M}od}$-injective, so is the direct summand M/C of the right side. Hence by Corollary 3.4.4, M/C is a direct summand of a module of the form $E_1 \oplus A_1$, where E_1 is an injective $\mathbb{Z}$-module and A_1 is a $\mathbb{Z}$-module such that $\operatorname{Rad} A_1 = 0$. So there exists a submodule X of $E_1 \oplus A_1$ such that $(M/C) \oplus X = E_1 \oplus A_1$. Then, since radical of an injective (divisible) $\mathbb{Z}$-module is itself, we obtain that

$$((\operatorname{Rad} M)/C) \oplus \operatorname{Rad} X = (\operatorname{Rad}(M/C)) \oplus \operatorname{Rad} X = \operatorname{Rad} E_1 \oplus \operatorname{Rad} A_1 = E_1 \oplus 0 = E_1.$$

So $\operatorname{Rad} M/C$ is an injective module as it is a direct summand of an injective module.

But $\operatorname{Rad} M \cong F$ is a free $\mathbb{Z}$-module of rank $|\mathbb{Z}^+|$ and C is a finitely generated free submodule of $\operatorname{Rad} M$. Let $\{x_k | k \in \mathbb{Z}^+\}$ be a basis for the free $\mathbb{Z}$-module $\operatorname{Rad} M$ and $\{y_1, y_2, \ldots, y_n\}$ a basis for C (where n is the rank of free module C). Express each y_i in terms of the basis elements x_k, $k \in \mathbb{Z}^+$, for $\operatorname{Rad} M$. Let F_1 be the submodule of the free $\mathbb{Z}$-module $\operatorname{Rad} M$ spanned by the finitely many basis elements x_k which occur with a nonzero coefficient in the expansion of at least one y_i, $i = 1, 2, \ldots, n$. Then F_1 is finitely generated. Let F_2 be the submodule of the free $\mathbb{Z}$-module $\operatorname{Rad} M$ spanned by the remaining x_k's. Then $\operatorname{Rad} M = F_1 \oplus F_2$ and $F_2 \neq 0$ as $\operatorname{Rad} M$ is not finitely generated. Since $C \leq F_1$,

$$\operatorname{Rad} M/C \cong (F_1/C) \oplus F_2.$$

This implies that F_2 is also an injective $\mathbb{Z}$-module since $\operatorname{Rad} M/C$ is so. But a nonzero free $\mathbb{Z}$-module is not injective. This contradiction ends the proof. □

Theorem 4.6.5. $\mathcal{S}uppl_{\mathbb{Z}\text{-}\mathcal{M}od} \subsetneq \mathcal{C}o\text{-}\mathcal{N}eat_{\mathbb{Z}\text{-}\mathcal{M}od} \subsetneq \mathcal{C}ompl_{\mathbb{Z}\text{-}\mathcal{M}od}$

Proof. The inclusions follow from Proposition 4.6.1. Example 4.6.4 shows that the first inclusion is strict and Example 4.3.2 shows that the second inclusion is strict. □

Theorem 4.6.6. *A* finite *subgroup A of a group B is coneat in B if and only if it is neat in B.*

Proof. As A is a finite group, it has small radical. So it is coneat in B if and only if it is a supplement in B if and only if it is a complement in B if and only if it is neat in B, by Corollary 3.4.6, Theorem 4.3.1 and Theorem 4.1.1. □

Theorem 4.6.7. $\operatorname{Ext}_{\mathcal{C}o\text{-}\mathcal{N}eat_{\mathbb{Z}\text{-}\mathcal{M}od}}$ *is* not *factorizable as*

$$\mathbb{Z}\text{-}\mathcal{M}od \times \mathbb{Z}\text{-}\mathcal{M}od \xrightarrow{Ext_{\mathbb{Z}}} \mathcal{A}b \xrightarrow{H} \mathcal{A}b$$

for any functor $H : \mathcal{A}b \longrightarrow \mathcal{A}b$ on the category of abelian groups.

Proof. We use Example 4.3.2 as in the proof of Theorem 4.5.3. The proof goes as the same lines in that proof. Denote $\mathcal{C}o\text{-}\mathcal{N}eat_{\mathbb{Z}\text{-}\mathcal{M}od}$ shortly by $c\mathcal{N}$. Only note that, with the notation in that proof, $\operatorname{Ext}_{c\mathcal{N}}(\mathbb{Z}/p^2\mathbb{Z}, \mathbb{Z}) = 0$ since $\operatorname{Rad} \mathbb{Z} = 0$, and

$$\operatorname{Ext}_{c\mathcal{N}}(\mathbb{Z}/p^2\mathbb{Z}, \mathbb{Z}/p^2\mathbb{Z}) = \operatorname{Ext}_{\mathcal{C}}(\mathbb{Z}/p^2\mathbb{Z}, \mathbb{Z}/p^2\mathbb{Z}),$$

by Theorem 4.6.6, as $\mathbb{Z}/p^2\mathbb{Z}$ is a finite group and neat subgroups are complement subgroups (by Theorem 4.1.1). □

For a *torsion* group B, neat subgroups and coneat subgroups coincide:

Theorem 4.6.8. *Let B be a* torsion *group, and A any subgroup of B. Then A is neat in B if and only if A is coneat in B.*

Proof. ($\Leftarrow$) always holds (for any W-module B) by Corollary 4.3.3. Conversely, suppose A is neat in B. To show that A is coneat in B, we must show that for every module M with $\operatorname{Rad} M = 0$, any homomorphism $f : A \longrightarrow M$ can be extended to B. Since B is a torsion group, so is its subgroup A, hence $f(A)$ is also a torsion group. So, without loss of generality, we may suppose that M is also a torsion group. Decompose A, B and M into their p-primary components: $A = \bigoplus_p A_p$, $B = \bigoplus_p B_p$ and $M = \bigoplus_p M_p$, where the index p runs through all prime numbers (see page 67 for p-primary components of a torsion group). For each prime p, let $f_p : A_p \longrightarrow M_p$ be the restriction of f to A_p, with range restricted to M_p also (note that $f(A_p) \leq M_p$). Since $0 = \operatorname{Rad} M = \bigoplus_p \operatorname{Rad} M_p = \bigoplus_p pM_p$, we have $pM_p = 0$ for each prime p. So, each M_p is a neat-injective abelian group by Proposition 4.1.3. Suppose each A_p is neat in B_p. Then there exists $\tilde{f}_p : B_p \longrightarrow M_p$ extending $f_p : A_p \longrightarrow M_p$. Define $\tilde{f} : B \longrightarrow M$, by $\tilde{f}(\sum_p b_p) = \sum_p \tilde{f}_p(b_p)$ for each $\sum_p b_p \in \bigoplus_p B_p = B$ where $b_p \in B_p$ for every prime p. Then $\tilde{f} : B \longrightarrow M$ is the required homomorphism extending $f : A \longrightarrow M$:

$$\begin{array}{ccc} A = \bigoplus_p A_p & \leq_c & \bigoplus_p B_p = B \\ {\scriptstyle f=\bigoplus_p f_p}\downarrow & \swarrow {\scriptstyle \tilde{f}=\bigoplus_p \tilde{f}_p} & \\ M = \bigoplus_p M_p & & \end{array}$$

Thus, it only remains to show that each A_p is neat in B_p. One can show this either directly, or it follows since $\mathcal{N}eat_{\mathbb{Z}\text{-}\mathcal{M}od}$ is a proper class: A_p is neat in A as it is a direct summand of A, and A is neat in B. So, A_p is neat in B as the composition of two $\mathcal{N}eat_{\mathbb{Z}\text{-}\mathcal{M}od}$-monomorphisms is a $\mathcal{N}eat_{\mathbb{Z}\text{-}\mathcal{M}od}$-monomorphism by proper class axioms. Since $A_p \leq B_p \leq B$, we have that the composition $A_p \hookrightarrow B_p \hookrightarrow B$ of inclusion monomorphisms is a $\mathcal{N}eat_{\mathbb{Z}\text{-}\mathcal{M}od}$-monomorphism, so the first inclusion monomorphism $A_p \hookrightarrow B_p$ must also be a $\mathcal{N}eat_{\mathbb{Z}\text{-}\mathcal{M}od}$-monomorphism by proper class axioms. $\square$

4.7 Coinjectives and coprojectives with respect to $\mathcal{C}ompl_{\mathbb{Z}\text{-}\mathcal{M}od}$ and $\mathcal{S}uppl_{\mathbb{Z}\text{-}\mathcal{M}od}$

By Theorem 3.7.1, we already know that $\mathcal{C}ompl_{\mathbb{Z}\text{-}\mathcal{M}od}$-coinjectives are only injective (divisible) abelian groups. By Theorem 3.7.2, as $\operatorname{Rad}\mathbb{Z} = 0$, $\mathcal{S}uppl_{\mathbb{Z}\text{-}\mathcal{M}od}$-coprojectives are only projective abelian groups, so free abelian groups. Since $\mathcal{S}uppl_{\mathbb{Z}\text{-}\mathcal{M}od} \subseteq \mathcal{C}ompl_{\mathbb{Z}\text{-}\mathcal{M}od}$, $\mathcal{S}uppl_{\mathbb{Z}\text{-}\mathcal{M}od}$-coinjectives are also only injective (divisible) abelian groups.

Theorem 4.7.1. *$\mathcal{C}ompl_{\mathbb{Z}\text{-}\mathcal{M}od}$-coprojectives are only torsion-free abelian groups.*

Proof. Firstly, each torsion-free abelian group C is $\mathcal{C}ompl_{\mathbb{Z}\text{-}\mathcal{M}od}$-coprojective because every short exact sequence

$$\mathbb{E}: \qquad 0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

of abelian groups is in $\mathcal{P}ure_{\mathbb{Z}\text{-}\mathcal{M}od}$ (see for example Fuchs (1970, §26, property (d), p. 114)). Since pure subgroups are neat, we obtain that the short exact sequence $\mathbb{E}$ is in $\mathcal{N}eat_{\mathbb{Z}\text{-}\mathcal{M}od} = \mathcal{C}ompl_{\mathbb{Z}\text{-}\mathcal{M}od}$ (equality by Theorem 4.1.1).
Conversely suppose C is a $\mathcal{C}ompl_{\mathbb{Z}\text{-}\mathcal{M}od}$-coprojective abelian group. Since the proper class

$$\mathcal{C}ompl_{\mathbb{Z}\text{-}\mathcal{M}od} = \iota^{-1}(\{\mathbb{Z}/p\mathbb{Z} | p \text{ prime}\})$$

is injectively generated by simple abelian groups (by Theorem 4.1.1), we know that an abelian group C is $\mathcal{C}ompl_{\mathbb{Z}\text{-}\mathcal{M}od}$-coprojective if and only if $\operatorname{Ext}^1_{\mathbb{Z}}(C,T) = 0$ for all simple abelian groups T by Proposition 2.6.5. Suppose for the contrary that C is not torsion-free. Hence there exists $0 \neq c \in C$ which has a finite order. Without loss of generality we can assume that c has prime order, say for a prime p, we have $pc = 0$ and $c \neq 0$. Then $S = \mathbb{Z}c \cong \mathbb{Z}/p\mathbb{Z}$ is a simple abelian group. Consider the short exact sequence

$$0 \longrightarrow S \xrightarrow{f} C \xrightarrow{g} C/S \longrightarrow 0$$

where f is the inclusion homomorphism and g is the natural epimorphism. By the long exact sequence connecting Hom and Ext, we have the following exact sequence:

$$\dots \longrightarrow \mathrm{Ext}^1_{\mathbb{Z}}(C,S)=0 \longrightarrow \mathrm{Ext}^1_{\mathbb{Z}}(S,S)=0 \longrightarrow \mathrm{Ext}^2_{\mathbb{Z}}(C/S,S)=0 \longrightarrow \dots$$

Here $\mathrm{Ext}^1_{\mathbb{Z}}(C,S)=0$ because C is $\mathcal{C}ompl_{\mathbb{Z}\text{-}\mathcal{M}od}$-coprojective and $\mathrm{Ext}^2_{\mathbb{Z}}(C/S,S)=0$ since $\mathrm{Ext}^2_{\mathbb{Z}}=0$ as $\mathbb{Z}$ is a hereditary ring. Thus the above exact sequence implies that $\mathrm{Ext}^1_{\mathbb{Z}}(S,S)=0$ which is the required contradiction since $\mathrm{Ext}^1_{\mathbb{Z}}(S,S)\cong\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z},\mathbb{Z}/p\mathbb{Z})\cong\mathbb{Z}/p\mathbb{Z}$ by, for example, Fuchs (1970, §52, property (D)). □

CHAPTER FIVE

COMPLEMENTS AND SUPPLEMENTS IN MODULES OVER DEDEKIND DOMAINS

Throughout this chapter W denotes a (commutative) Dedekind domain.

In Section 5.1, we summarize the main properties of Dedekind domains we will use. c-injective modules over Dedekind domains are described in Section 5.2. We will see that finitely generated torsion complement submodules are supplements in modules over Dedekind domains in Section 5.3. For a Dedekind domain W which is not a field, we will see in Section 5.4 that

(i) If $\operatorname{Rad} W = 0$, then

$$\mathcal{S}uppl_{W\text{-}\mathcal{M}od} \subsetneqq \mathcal{C}o\text{-}\mathcal{N}eat_{W\text{-}\mathcal{M}od} \subsetneqq \mathcal{N}eat_{W\text{-}\mathcal{M}od} = \mathcal{C}ompl_{W\text{-}\mathcal{M}od}.$$

(ii) If $\operatorname{Rad} W \neq 0$, then

$$\mathcal{S}uppl_{W\text{-}\mathcal{M}od} \subsetneqq \mathcal{C}o\text{-}\mathcal{N}eat_{W\text{-}\mathcal{M}od} = \mathcal{N}eat_{W\text{-}\mathcal{M}od} = \mathcal{C}ompl_{W\text{-}\mathcal{M}od}.$$

In this Section 5.4, we will also see that like in abelian groups, if $\operatorname{Rad} W = 0$ and W is not a field, then $\operatorname{Ext}_{\mathcal{S}uppl_{W\text{-}\mathcal{M}od}}$ and $\operatorname{Ext}_{\mathcal{C}o\text{-}\mathcal{N}eat_{W\text{-}\mathcal{M}od}}$ are *not* factorizable as

$$W\text{-}\mathcal{M}od \times W\text{-}\mathcal{M}od \xrightarrow{Ext_W} W\text{-}\mathcal{M}od \xrightarrow{H} W\text{-}\mathcal{M}od$$

for any functor $H : W\text{-}\mathcal{M}od \longrightarrow W\text{-}\mathcal{M}od$. Like in abelian groups, in a torsion W-module, neat submodules and coneat submodules coincide. In Section 5.5, it is shown that $Compl_{W\text{-}\mathcal{M}od}$-coprojectives are only torsion-free W-modules.

5.1 Dedekind domains

We summarize the main facts that we use for Dedekind domains from Cohn (2002, §10.5-6) and Berrick & Keating (2000, Ch. 5-6).

Let R be a commutative domain and K be its field of fractions. By a *fractional ideal* I of R, we understand an R-submodule of the R-module K such that

$$Rz \leq I \leq Ru \quad \text{for some } z, u \in K \setminus \{0\}.$$

An ordinary ideal I of R is a fractional ideal if and only if it is nonzero. The nonzero ideals of R are called *integral ideals* of R. Multiplication of fractional ideals can be defined as for usual ideals. With this multiplication, the set of fractional ideals turns out to be a monoid with identity element R. For each fractional ideal I, we can define an "*inverse*":

$$(R : I) = \{q \in K | qI \subseteq R\}.$$

$(R : I)$ is also a fractional ideal and satisfies

$$I(R : I) \subseteq R.$$

But here equality need not hold. If it does hold, then I is said to be *invertible* and we also write I^{-1} instead of $(R : I)$. An ideal in R is invertible if and only if it is a nonzero projective R-module. An element of K is said to be *integral* over R if it is a root of a monic polynomial in $R[X]$. A commutative domain R is *integrally closed* if the elements of K which are integral over R are just the elements of R.

A commutative domain satisfying any of the equivalent conditions in the following theorem is said to be a *Dedekind domain*:

Theorem 5.1.1. *(by Cohn (2002, Propositions 10.5.1,4,6)) For a commutative domain R, the following are equivalent:*

(i) The set of fractional ideals of R is a group under multiplication,

(ii) Every fractional ideal of R is invertible,

(iii) Every integral ideal of R is invertible,

(iv) Every ideal of R is projective,

(v) R is Noetherian, integrally closed and every nonzero prime ideal of R is maximal,

(vi) Every nonzero prime ideal of R is invertible,

(vii) Every integral ideal of R can be expressed uniquely as a finite product of maximal ideals,

(viii) Every integral ideal of R can be expressed as a finite product of prime ideals,

Note that by the above theorem, a Dedekind domain is always Noetherian.

Theorem 5.1.2. *(Cohn, 2002, Theorem 10.6.8) Any finitely generated torsion module M over a Dedekind domain W is a direct sum of cyclic W-modules: for some $n \in \mathbb{Z}^+$ and nonzero ideals $I_1, \ldots, I_n$ in W,*

$$A \cong (W/I_1) \oplus \ldots \oplus (W/I_n), \qquad I_1 \geq I_2 \geq \ldots \geq I_n,$$

and this decomposition is unique up to isomorphism.

Using this theorem, we give a proof of the following result to review primary decomposition of finitely generated torsion modules over Dedekind domains like in abelian groups. For the complete description of finitely generated modules over Dedekind domains and their invariants see for example Berrick & Keating (2000, Theorem 6.3.23).

Proposition 5.1.3. *Let A be a finitely generated torsion W-module, where W is a Dedekind domain. Then $A/\operatorname{Rad} A$ is a finitely generated* semisimple *W-module and* $\operatorname{Rad} A \ll A$

Proof. Let A be a finitely generated torsion W-module.

As A is finitely generated, we have $\operatorname{Rad} A \ll A$. We also show this more directly below.

Consider any nonzero ideal I in W. Since W is a Dedekind domain,

$$I = P_1^{r_1} \cdot \ldots \cdot P_k^{r_k}$$

for *distinct* maximal ideals $P_1, \ldots, P_k$ in W and $r_1, \ldots, r_k \in \mathbb{Z}^+$. Then

$$W/I \cong \bigoplus_{i=1}^{k} (W/P_i^{r_i})$$

as $P_1^{r_1}, \ldots, P_k^{r_k}$ are pairwise comaximal ideals. So

$$\operatorname{Rad}(W/I) \cong \bigoplus_{i=1}^{k} \operatorname{Rad}(W/P_i^{r_i}) = \bigoplus_{i=1}^{k} P_i(W/P_i^{r_i}) \ll \bigoplus_{i=1}^{k} (W/P_i^{r_i}) \cong (W/I)$$

since $W/P_i^{r_i}$ is a uniserial W-module because for a maximal ideal P and $r \in \mathbb{Z}^+$, the W-submodules of W/P^r form the following chain:

$$0 < P^{r-1}/P^r < \ldots < P^2/P^r < P/P^r < W/P^r,$$

(so $\operatorname{Rad}(W/P^r) = P/P^r \ll W/P^r$). Since

$$(W/P^r)/\operatorname{Rad}(W/P^r) = (W/P^r)/(P/P^r) \cong W/P$$

is simple,

$$(W/I)/\operatorname{Rad}(W/I) \cong \bigoplus_{i=1}^{k} (W/P_i^{r_i})/\operatorname{Rad}(W/P_i^{r_i})$$

is also a finitely generated semisimple W-module.

Turning back to A, as A is a finitely generated torsion W-module, it is a direct sum of finitely many cyclic W-modules by Theorem 5.1.2: for some $n \in \mathbb{Z}^+$ and nonzero ideals $I_1, \ldots, I_n$ in W,

$$A \cong (W/I_1) \oplus \ldots \oplus (W/I_n).$$

So,

$$\operatorname{Rad} A \cong \bigoplus_{i=1}^{n} \operatorname{Rad}(W/I_i) \ll \bigoplus_{i=1}^{n} W/I_i \cong A \quad \text{and}$$

$$A/\operatorname{Rad} A \cong \bigoplus_{i=1}^{n} (W/I_i)/(\operatorname{Rad}(W/I_i)).$$

Hence $A/\operatorname{Rad} A$ is also a finitely generated semisimple W-module. □

Proposition 5.1.4. *(by Cohn (2002, Proposition 10.6.9)) Any torsion W-module M over a Dedekind domain W is a direct sum of its primary parts in a unique way:*

$$M = \bigoplus_{0 \neq P \underset{max.}{\leq} W} M_P,$$

where for each nonzero prime ideal P of W (so P is a maximal ideal of W),

$$M_P = \{x \in M | P^n x = 0 \text{ for some } n \in \mathbb{Z}^+\}$$

*is the P-*primary part *of the W-module M.*

For a nonzero prime ideal P of a Dedekind domain W, we say that a W-module M is *P-primary* if $M = M_P$.

Proposition 5.1.5. *Let W be a Dedekind domain, P be a nonzero prime ideal of W and M be a P-primary W-module. Then* $\operatorname{Rad} M = PM$.

Proof. Let Q be a maximal ideal of W and suppose $Q \neq P$. Let $m \in M$. Since M is P-primary, $P^n m = 0$ for some $n \in \mathbb{Z}^+$. Since $P^n \nsubseteq Q$ and Q is a maximal ideal,

$P^n + Q = W$. So, $Wm = P^n m + Qm = 0 + Qm = Qm$, hence $m \in Wm = Qm$. Thus $M \leq QM$, so $QM = M$ for every maximal ideal $Q \neq P$. Hence, By Lemma 3.8.1,

$$\operatorname{Rad} M = \bigcap_{Q \underset{max.}{\leq} {}_W W} QM = PM.$$

□

Proposition 5.1.6. *Any nonzero torsion module over a Dedekind domain has a simple submodule, so any Dedekind domain is a* C*-ring.*

Proof. Let M be a nonzero torsion module over a Dedekind domain W. Then by Proposition 5.1.4,

$$M = \bigoplus_{0 \neq P \underset{max.}{\leq} W} M_P,$$

where for each nonzero maximal ideal P of W, M_P is the *P-primary part* of M. Since $M \neq 0$, there exists a nonzero maximal ideal P such that $M_P \neq 0$. So there exits $0 \neq m \in M_P$. Since $m \in M_p$, there exits a smallest $n \in \mathbb{Z}^+ \cup \{0\}$ such that $P^n m = 0$ but $P^{n-1} m \neq 0$ (where we take $P^0 = W$). Then for the submodule $A := P^{n-1} m \neq 0$ of M, $PA = 0$. Hence A is a nonzero (homogenous) semisimple W-module (see page 19), so has a simple submodule. This also shows that W is a C-ring by Proposition 3.3.9. □

Proposition 5.1.7. *Let W be a Dedekind domain which is not a field. For an injective W-module E,* $\operatorname{Rad} E = E$.

Proof. Since E is injective, it is also a divisible W-module (by for example Cohn (2002, Proposition 4.7.8)). Since W is not a field, any maximal ideal P of W is nonzero and so $PE = E$ as E is divisible. By Lemma 3.8.1,

$$\operatorname{Rad} E = \bigcap_{P \underset{max.}{\leq} {}_W W} PE = \bigcap_{P \underset{max.}{\leq} {}_W W} E = E.$$

□

Theorem 5.1.8. *(by Kaplansky (1958, Theorem 3) and Kaplansky (1952, Theorem 2-(b)), or by Fuchs & Salce (2001, Theorem VI.1.14))* Projective *modules over Dedekind domains which are* not *finitely generated are* free.

Proposition 5.1.9. *For a Dedekind domain W which is not a field, the following are equivalent:*

(i) $\operatorname{Rad} W \neq 0$,

(ii) W is semilocal,

(iii) W has only finitely many maximal ideals,

(iv) W is a PID with only finitely many maximal ideals.

Proof. By Lam (2001, Proposition 20.2), a commutative ring is semilocal if and only if it has finitely many maximal ideals. This shows (ii)⇔(iii). By Cohn (2002, Proposition 10.6.2), a Dedekind domain with finitely many prime ideals is a PID. This proves (iii)⇔(iv). It remains to show (i)⇔(iii).
(i)⇒(iii): Since W is a Dedekind domain, the nonzero ideal $\operatorname{Rad} W$ can be expressed uniquely as a finite product of maximal ideals by Theorem 5.1.1-(vii): For some $n \in \mathbb{Z}^+$ and maximal ideals $P_1, P_2, \ldots, P_n$ of W,

$$\operatorname{Rad} W = P_1 P_2 \ldots P_n.$$

For any maximal ideal P of W, since $\operatorname{Rad} W \leq P$, we have

$$P \supseteq P_1 P_2 \ldots P_n.$$

Then, for example by Sharp (2000, Lemma 3.55), $P \supseteq P_j$ for some $j \in \{1, 2, \ldots, n\}$. Since P and P_j are maximal ideals, we obtain $P = P_j$. Hence $\{P_1, P_2, \ldots, P_n\}$ is the set of all maximal ideals of W. So, W has finitely many maximal ideals.
(iii)⇒(i): If W has finitely many maximal ideals $P_1, P_2, \ldots, P_n$ for some $n \in \mathbb{Z}^+$, then

$$\operatorname{Rad} W = P_1 \cap P_2 \cap \ldots \cap P_n \supseteq P_1 P_2 \ldots P_n \neq 0,$$

as each P_i is nonzero since W is not a field. □

Proposition 5.1.10. *(Fuchs & Salce, 2001, Exercise I.5.5-(c)) For a commutative domain R, an ideal J of R and any R-module M,*

$$\mathrm{Ext}_R(J^{-1}/R, M) \cong M/JM,$$

if J is an invertible ideal.

Corollary 5.1.11. *For a Dedekind domain W, a* nonzero *ideal J of W and any W-module M,*

$$\mathrm{Ext}_W(W/J, M) \cong M/JM$$

Proof. Since W is a Dedekind domain, the nonzero ideal J of W is invertible. So, the result follows from Proposition 5.1.10 since $J^{-1}/W \cong W/J$ by Nunke (1959, Lemma 4.4). □

5.2 c-injective modules over Dedekind domains

Generalov (1983) gives the following interesting result (the equality from Generalov (1978, Theorem 5)):

Theorem 5.2.1. *(Generalov, 1983, Corollaries 1 and 6) For a Dedekind domain W,*

$$\mathcal{S}uppl_{W\text{-}\mathcal{M}od} \subseteq \mathcal{C}ompl_{W\text{-}\mathcal{M}od} = \mathcal{N}eat_{W\text{-}\mathcal{M}od}.$$

where the inclusion is strict if W is not *a field. So if A is a* supplement *in an W-module B where W is a Dedekind domain, then A is a* complement.

As in abelian groups (Theorem 4.1.1), $\mathcal{C}ompl_{W\text{-}\mathcal{M}od}$ is both projectively generated, injectively generated and flatly generated:

Theorem 5.2.2. *The following five proper classes of W-modules are* equal *for a Dedekind domain W:*

(i) $\mathit{Compl}_{W\text{-}\mathcal{M}od}$,

(ii) $\mathcal{N}eat_{W\text{-}\mathcal{M}od} \overset{def.}{=} \pi^{-1}(\{W/P \mid P \text{ maximal ideal of } W\})$,

(iii) $\iota^{-1}(\{M \mid M \in W\text{-}\mathcal{M}od \text{ and } PM = 0 \text{ for some maximal ideal } P \text{ of } W\})$,

(iv) $\tau^{-1}(\{W/P \mid P \text{ maximal ideal of } W\})$

(v) *The proper class of all short exact sequences*

$$\mathbb{E}: \qquad 0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0 \tag{5.2.1}$$

of W-modules and W-module homomorphisms such that for every maximal ideal P of W,

$$A' \cap PB = PA', \qquad \text{where } A' = \mathrm{Im}(f)$$

(or $A \cap PB = PA$ when A is identified with its image and f is taken as the inclusion homomorphism).

Proof. The equality of the first two classes follows by Proposition 3.3.1 or Theorem 3.3.2.

The equality of the last two proper classes follows from, for example, Sklyarenko (1978, Lemma 6.1) (see Proposition 3.2.2).

The equality of the second and fifth proper classes follows from Nunke (1959, Lemmas 4.4 and 5.2).

The equality of the third and fifth proper classes follows from Nunke (1959, Theorem 5.1) as pointed out in Generalov (1983, Corollary 1): Take a short exact sequence $\mathbb{E}$ in the third proper class above, of the form (5.2.1) where without loss of generality A is identified with its image in B and f is taken as the inclusion homomorphism. For a maximal ideal P in W, since A/PA is a module such that $P(A/PA) = 0$, it is injective with respect to $\mathbb{E}$, so it is trivial over A/PA in the sense defined in Nunke (1959, before Theorem 5.1); hence by that theorem, $A \cap PB = PA$. So $\mathbb{E}$ is in the last proper class. Conversely, to show that the last proper class is contained in the third proper class, it suffices to show that any short exact sequence $\mathbb{E}$ in the last proper class above, of the form (5.2.1), where without loss of generality A is identified with its image in B and f is taken as the inclusion homomorphism, is splitting if A is a module such that $PA = 0$ for some maximal ideal P of W. By Nunke (1959, Theorem 5.1), as $A \cap PB = PA$, $\mathbb{E}$ is trivial over $A/PA = A/0 \cong A$, which means the existence of a map $\psi : B \longrightarrow A$ such that $\psi(a) = a$ for all $a \in A$, hence $\mathbb{E}$ is splitting as required. □

Another consequence of Nunke (1959, Theorem 5.1) is:

Theorem 5.2.3. *For a Dedekind domain W, and W-modules A, C,*

$$\mathrm{Ext}_{\mathcal{C}ompl_{W\text{-}\mathcal{M}od}}(C, A) = \mathrm{Ext}_{\mathcal{N}eat_{W\text{-}\mathcal{M}od}}(C, A) = \mathrm{Rad}(\mathrm{Ext}_W(C, A)).$$

Proof. $\mathcal{C}ompl_{W\text{-}\mathcal{M}od} = \mathcal{N}eat_{W\text{-}\mathcal{M}od}$ by Theorem 5.2.2. Take a short exact sequence $\mathbb{E}$ in $\mathcal{C}ompl_{W\text{-}\mathcal{M}od}$, of the form (5.2.1) where without loss of generality A is identified with its image in B and f is taken as the inclusion homomorphism. For each maximal ideal P in W, by the previous Theorem 5.2.2, $A \cap PB = PA$, hence by Nunke (1959, Theorem 5.1), the equivalence class $[\mathbb{E}]$ of the short exact sequence $\mathbb{E}$ is in $P\,\mathrm{Ext}(C, A)$. Thus $[\mathbb{E}] \in \bigcap_{P \underset{\text{maximal}}{\leq} W} P\,\mathrm{Ext}(C, A) = \mathrm{Rad}(\mathrm{Ext}(C, A))$, where the last equality follows from, for example, Generalov (1983, Lemma 3) (see Lemma 3.8.1). □

We can describe c-injective modules over Dedekind domains (which are then also self-c-injective):

Theorem 5.2.4. *For a Dedekind domain W, since $\mathcal{C}ompl_{W\text{-}\mathcal{M}od}$ is injectively generated by homogenous semisimple W-modules, every* c*-injective-module is a direct summand of a direct product of homogeneous semisimple W-modules and of injective envelopes of cyclic W-modules.*

Proof. By Proposition 2.4.4, $\mathcal{C}ompl_{W\text{-}\mathcal{M}od}$-injectives are given as above since by Theorem 5.2.2, $\mathcal{C}ompl_{W\text{-}\mathcal{M}od}$ equals the injectively generated proper class

$$\iota^{-1}(\{M|M \text{ is a homogenous semisimple } W\text{-module}\}).$$

□

Like in abelian groups, the proper class $\mathcal{C}ompl_{W\text{-}\mathcal{M}od}$ is injectively generated by simple W-modules:

Proposition 5.2.5. *For a Dedekind domain W,*

$$\mathcal{C}ompl_{W\text{-}\mathcal{M}od} = \iota^{-1}(\{W/P|P \textit{ maximal ideal of } W\}).$$

Proof. Denote $\mathcal{C}ompl_{W\text{-}\mathcal{M}od}$ shortly by $\mathcal{C}$:

$$\mathcal{C} = \iota^{-1}(\{M|M \text{ is a homogenous semisimple } W\text{-module}\}).$$

Let $\mathcal{C}'$ be the proper class

$$\mathcal{C}' = \iota^{-1}(\{W/P|P \text{ maximal ideal of } W\}).$$

Clearly $\mathcal{C} \subseteq \mathcal{C}'$. Conversely, it suffices to show that every homogenous semisimple W-module M is injective with respect to the proper class $\mathcal{C}'$. Since M is a homogenous semisimple W-module, $M = \bigoplus_{\lambda\in\Lambda} S_\lambda$ for some index set Λ and simple submodules S_λ of M such that for some maximal left ideal P of R, $S_\lambda \cong R/P$ for every $\lambda \in \Lambda$. Then $M \leq N := \prod_{\lambda\in\Lambda} S_\lambda$. Since $PN = 0$, N may be considered as a vector space over the field W/P. If α is the dimension of the W/P-vector space N, then N is isomorphic to a direct sum of α copies of W/P. So N is a homogenous semisimple W-module. Since N is semisimple, its submodule M is a direct summand of N. But $N = \prod_{\lambda\in\Lambda} S_\lambda$, being a product of simple modules which are injective with respect to the proper class $\mathcal{C}'$, is injective with respect to proper class $\mathcal{C}'$. Then so is its direct summand M as required. □

Proposition 5.2.6. *For a Dedekind domain W,*

$$\mathcal{S}uppl_{W\text{-}\mathcal{M}od} \subseteq \mathcal{C}o\text{-}\mathcal{N}eat_{W\text{-}\mathcal{M}od} \subseteq \mathcal{N}eat_{W\text{-}\mathcal{M}od} = \mathcal{C}ompl_{W\text{-}\mathcal{M}od}.$$

Proof. By Proposition 3.4.1, $\mathcal{S}uppl_{W\text{-}\mathcal{M}od} \subseteq \mathcal{C}o\text{-}\mathcal{N}eat_{W\text{-}\mathcal{M}od}$. By Proposition 3.8.2,

$$\mathcal{C}o\text{-}\mathcal{N}eat_{R\text{-}\mathcal{M}od} \subseteq \tau^{-1}(\{W/P | P \text{ maximal ideal of } W\}).$$

By Theorem 5.2.2, the right side equals $\mathcal{N}eat_{W\text{-}\mathcal{M}od} = \mathcal{C}ompl_{W\text{-}\mathcal{M}od}$. □

Like in abelian groups, we have:

Theorem 5.2.7. *If R is a PID, then*

$$\begin{aligned} \mathcal{C}ompl_{R\text{-}\mathcal{M}od} &= \mathcal{N}eat_{R\text{-}\mathcal{M}od} = \pi^{-1}(\{R/Rp | p \textit{ prime element of } R\}) \\ &= \tau^{-1}(\{R/Rp | p \textit{ prime element of } R\}) \\ &= \iota^{-1}(\{R/Rp | p \textit{ prime element of } R\}). \end{aligned}$$

Proof. By Theorem 5.2.2 and Proposition 5.2.5. □

By Proposition 3.2.1 for principal *left* ideals, we obtain:

Theorem 5.2.8. *If R is a ring in which each maximal left ideal is a principal left ideal (that is a left ideal generated by one element), then $\mathcal{N}eat_{R\text{-}\mathcal{M}od}$ is flatly generated also:*

$$\begin{aligned} \mathcal{N}eat_{R\text{-}\mathcal{M}od} &= \pi^{-1}(\{R/Rp | p \in R \textit{ such that } Rp \textit{ is a maximal left ideal of } R\}) \\ &= \tau^{-1}(\{R/pR | p \in R \textit{ such that } Rp \textit{ is a maximal left ideal of } R\}) \end{aligned}$$

5.3 Finitely generated torsion complement submodules are supplements in modules over Dedekind domains

We look for some converse results to Theorem 5.2.1, that is, when is a complement also a supplement? For finitely generated torsion modules over a Dedekind domain, we obtain by generalizing Theorem 4.3.1 in abelian groups :

Theorem 5.3.1. *Let W be a Dedekind domain. Take a W-module B and a submodule $A \leq B$. Suppose A is a* finitely generated torsion *W-module. Then A is a complement in B if and only if A is a supplement in B.*

Proof. ($\Leftarrow$) always holds (for any submodule A) by Theorem 5.2.1. Conversely suppose $A \neq 0$ is a complement in B. The proof goes as the same with the arguments given for a p-group before Theorem 4.3.1 for abelian groups. We only need to show the following:

(i) $A/\operatorname{Rad} A$ is a finitely generated semisimple W-module, so by Theorem 5.2.2, it is injective with respect to the inclusion map $A \longrightarrow B$ which is a $\mathcal{C}ompl_{W\text{-}\mathcal{M}od}$-monomorphism.

(ii) $\operatorname{Rad} A \ll A$.

These follow from Proposition 5.1.3. □

Generalizing Proposition 4.2.1 and Proposition 4.2.2 for modules over Dedekind domains using uniform dimension and hollow dimension, we obtain a weaker form of the previous theorem. However its proof might suggest another way for a generalization over some other class of rings:

Theorem 5.3.2. *Let W be a Dedekind domain, B be a* finitely generated torsion *W-module and A a submodule of B. Then A is a complement in B if and only if A is a supplement in B.*

Proof. From the proof of Proposition 5.1.3, it is seen that a finitely generated torsion W-module is a direct sum of finitely many W-modules of the form W/P^r (P a maximal ideal) which are both hollow and uniform, so have uniform dimension 1 and hollow dimension 1. As these dimensions are additive on finite direct sums (see Dung et al. (1994, 5.8 (2)) and Lomp (1996, 3.1.10 (1))), we see that a finitely generated torsion W-module has the same finite hollow dimension and uniform dimension. So the result follows from Theorems 3.6.1 and 3.6.2 as submodules and quotient modules of a finitely generated torsion W-module are also finitely generated torsion W-modules. □

We can *not* generalize this result to include finitely generated modules which are *not* torsion even in abelian groups as has been shown by Example 4.3.2. Also this gives a proof of the strict inclusion in Theorem 5.2.1 for the case $W = \mathbb{Z}$ by an example.

5.4 The proper class $\mathcal{C}o\text{-}\mathcal{N}eat_{W\text{-}\mathcal{M}od}$ for a Dedekind domain W

Throughout this section, let W be a Dedekind domain and suppose it is *not* a field to exclude the trivial cases.

We will show that if $\operatorname{Rad} W = 0$, then the proper class $\mathcal{C}o\text{-}\mathcal{N}eat_{W\text{-}\mathcal{M}od}$ is *strictly* between $\mathcal{S}uppl_{W\text{-}\mathcal{M}od}$ and $\mathcal{C}ompl_{W\text{-}\mathcal{M}od}$. When $\operatorname{Rad} W \neq 0$, still $\mathcal{S}uppl_{W\text{-}\mathcal{M}od} \neq$

$Co\text{-}\mathcal{N}eat_{W\text{-}\mathcal{M}od}$, but $Co\text{-}\mathcal{N}eat_{W\text{-}\mathcal{M}od} = \mathcal{N}eat_{W\text{-}\mathcal{M}od} = Compl_{W\text{-}\mathcal{M}od}$. To prove that $Suppl_{W\text{-}\mathcal{M}od} \subsetneq Co\text{-}\mathcal{N}eat_{W\text{-}\mathcal{M}od}$, we will follow mainly the proofs in Generalov (1983, Theorems 6-7, Propositions 4-5) for the Dedekind domain W, which simplifies some steps and for which some missing steps in Generalov (1983, proofs of Theorem 6 and Proposition 5) can be done. After two lemmas, we give an example of a $Co\text{-}\mathcal{N}eat_{W\text{-}\mathcal{M}od}$-monomorphism which is *not* a $Suppl_{W\text{-}\mathcal{M}od}$-monomorphism.

Lemma 5.4.1. *(by Generalov (1983, Theorem 7, Proposition 4, Corollary 5)) Let W be a Dedekind domain which is not a field and Q the field of fractions of W. Let $S \leq Q$ be the submodule of the W-module Q such that $S/W = \operatorname{Soc}(Q/W)$. Then:*

(i) $\operatorname{Rad} S = W$ *and* S/W *is a semisimple* W*-module,*

(ii) For a free W-module $F := \bigoplus_{\lambda\in\Lambda} W$ for some index set Λ, take the W-module $A := \bigoplus_{\lambda\in\Lambda} S$. Then $\operatorname{Rad} A = F$ *and* $A/\operatorname{Rad} A$ *is a semisimple* W*-module.*

Proof. (i) Since $S/W = \operatorname{Soc}(Q/W)$, it is clearly semisimple. So $\operatorname{Rad}(S/W) = 0$. Hence $\operatorname{Rad} S \leq W$.

Let P be a maximal ideal of W. Since W is not a field, $P \neq 0$. So P is an invertible ideal, that is, for the submodule $P^{-1} \leq Q$, $PP^{-1} = W$. Hence P^{-1} is a homogenous semisimple W-module with each simple submodule isomorphic to W/P (see page 19). So, the quotient P^{-1}/W is also semisimple. Hence $P^{-1}/W \leq \operatorname{Soc}(Q/W) = S/W$, which implies that $P^{-1} \leq S$. So

$$W = PP^{-1} \leq PS.$$

Then, by Lemma 3.8.1,

$$\operatorname{Rad} S = \bigcap_{P \underset{max.}{\leq}_R R} PS \geq W.$$

Thus, $\operatorname{Rad} S = W$.

(ii) $\operatorname{Rad} A = \bigoplus_{\lambda\in\Lambda} \operatorname{Rad} S = \bigoplus_{\lambda\in\Lambda} W = F$ and $A/\operatorname{Rad} A = \bigoplus_{\lambda\in\Lambda}(S/W)$ is semisimple. $\square$

Lemma 5.4.2. *(by Leonard (1966, Lemma 6)) Let W be a Dedekind domain which is not a field and Q the field of fractions of W. There exists an epimorphism $g : F \longrightarrow Q$ from a free W-module $F := \bigoplus_{\lambda\in\Lambda} W$ for some index set Λ. The free W-module $F := \bigoplus_{\lambda\in\Lambda} W$ is* not *a small W-module, and so the index set Λ is necessarily* infinite.

Proof. Let $H := \operatorname{Ker} g$. Then $F/H \cong Q$. By Leonard (1966, Lemma 6), F is *not* a small W-module since $F/H \cong Q$ is a nonzero injective module. In fact, this is simply because if F is a small module, then F is small in its injective envelope $\mathrm{E}(F)$ by Leonard (1966, Theorem 1). So, the quotient module F/H is small in $\mathrm{E}(F)/H$. But since $F/H \cong Q$ is injective, F/H is a direct summand of $\mathrm{E}(F)/H$ which contradicts with F/H being small in $\mathrm{E}(F)/H$.
Since Q is injective, $\operatorname{Rad} Q = Q$ by Proposition 5.1.7. So the finitely generated submodule W of $\operatorname{Rad} Q = Q$ is small in Q. If the index set Λ were finite, then $W \ll Q$ would imply $F = \bigoplus_{\lambda\in\Lambda} W \ll \bigoplus_{\lambda\in\Lambda} Q$ so that F would be a small module. $\square$

Example 5.4.3. (by Generalov (1983, Proposition 5)) Let W be a Dedekind domain which is not a field and Q the field of fractions of W. Consider the W-modules

$$F := \bigoplus_{\lambda\in\Lambda} W = \operatorname{Rad} A \leq A := \bigoplus_{\lambda\in\Lambda} S \leq \bigoplus_{\lambda\in\Lambda} Q = \mathrm{E}(A),$$

where,

(i) $S \leq Q$ is the W-module given as in Lemma 5.4.1 such that $S/W = \operatorname{Soc}(Q/W)$,

(ii) the free W-module $F := \bigoplus_{\lambda\in\Lambda} W$ is as in Lemma 5.4.2 for some *infinite* index set Λ such that there exists an epimorphism $g : F \longrightarrow Q$,

(iii) $\mathrm{E}(A)$ denotes the injective envelope of A.

Then the monomorphism

$$\begin{aligned} f : A &\longrightarrow \mathrm{E}(A) \oplus (A/\operatorname{Rad} A) \\ x &\longmapsto (x, x + \operatorname{Rad} A) \end{aligned}$$

is a $\mathcal{C}o\text{-}\mathcal{N}eat_{W\text{-}\mathcal{M}od}$-monomorphism but *not* a $\mathcal{S}uppl_{W\text{-}\mathcal{M}od}$-monomorphism. So $\mathcal{S}uppl_{W\text{-}\mathcal{M}od} \neq \mathcal{C}o\text{-}\mathcal{N}eat_{W\text{-}\mathcal{M}od}$.

Proof. By Lemma 5.4.1, $\operatorname{Rad} A = F$. By Proposition 3.4.3, f is a $\mathcal{C}o\text{-}\mathcal{N}eat_{W\text{-}\mathcal{M}od}$-monomorphism and $\mathrm{E}(A) \oplus (A/\operatorname{Rad} A)$ is $\mathcal{C}o\text{-}\mathcal{N}eat_{W\text{-}\mathcal{M}od}$-injective.

Suppose for the contrary that f is a $\mathcal{S}uppl_{W\text{-}\mathcal{M}od}$-monomorphism.

Let $M := f(A)$ and $N := \mathrm{E}(A) \oplus (A/\operatorname{Rad} A)$. Then M is a supplement in N. That means there exits a submodule $K \leq N$ such that

$$M + K = N \quad \text{and} \quad M \cap K \ll M.$$

Let $C := M \cap K$. Since $C \ll M$, $C \leq \operatorname{Rad} M = \operatorname{Rad} f(A) \cong \operatorname{Rad} A = F$, so C is also a projective W-module. Suppose C is not finitely generated. Then by Theorem 5.1.8, C is free. So, rank of C is at most $|\Lambda|$, the rank of F. But, rank of C cannot be $|\Lambda|$ because then $C \cong F$ would be a small module, contradicting that F is not a small module by Lemma 5.4.2. Since rank of C is strictly less than Λ, C has a basis whose cardinality is strictly less than Λ. Thus C has a *generating set* whose whose cardinality is strictly less than Λ, if C is not finitely generated. But that is also true if C is finitely generated since Λ is an infinite set. So, in any case, C has a generating set $Y = \{y_\gamma | \gamma \in \Gamma\}$ for some index set Γ such that $|\Gamma| < |\Lambda|$.

As $C \ll M$,

$$C \leq \operatorname{Rad} M \leq \operatorname{Rad} N = \operatorname{Rad}(\mathrm{E}(A) \oplus (A/\operatorname{Rad} A)) = \operatorname{Rad} \mathrm{E}(A) \leq \mathrm{E}(A).$$

So,

$$\begin{aligned} (\mathrm{E}(A)/C) \oplus (A/\operatorname{Rad} A) &\cong (\mathrm{E}(A) \oplus (A/\operatorname{Rad} A))/C = N/C = (M + K)/C \\ &\cong (M/C) \oplus (K/C) \end{aligned}$$

Since the left side is $\mathcal{C}o\text{-}\mathcal{N}eat_{W\text{-}\mathcal{M}od}$-injective, so is the direct summand M/C of the right side. Hence by Corollary 3.4.4, M/C is a direct summand of a module of the form $E_1 \oplus A_1$, where E_1 is an injective W-module and A_1 is a W-module such that $\operatorname{Rad} A_1 = 0$. So there exists a submodule X of $E_1 \oplus A_1$ such that $(M/C) \oplus X = E_1 \oplus A_1$. Then, since radical of an injective W-module is equal to itself (by Proposition 5.1.7), we obtain that

$$((\operatorname{Rad} M)/C)\oplus \operatorname{Rad} X = (\operatorname{Rad}(M/C))\oplus \operatorname{Rad} X = \operatorname{Rad} E_1 \oplus \operatorname{Rad} A_1 = E_1 \oplus 0 = E_1.$$

So $\operatorname{Rad} M/C$ is an injective module as it is a direct summand of an injective module.

But $\operatorname{Rad} M \cong F$ is a free W-module of rank $|\Lambda|$ and C has a generating set $Y = \{y_\gamma | \gamma \in \Gamma\}$ with $|\Gamma| < |\Lambda|$. Let $\{x_\lambda | \lambda \in \Lambda\}$ be a basis for the free W-module $\operatorname{Rad} M$. Express each y_γ in terms of the basis elements x_λ, $\lambda \in \Lambda$, for $\operatorname{Rad} M$. Let F_1 be the submodule of the free W-module $\operatorname{Rad} M$ spanned by the basis elements x_λ which occur with a nonzero coefficient in the expansion of at least one y_γ, $\gamma \in \Gamma$. Then F_1 has rank $\leq |\Gamma|$. Let F_2 be the submodule of the free W-module $\operatorname{Rad} M$ spanned by the remaining x_λ's. Then $\operatorname{Rad} M = F_1 \oplus F_2$ and $F_2 \neq 0$ as we have strict inequality for the cardinalities: $|\Gamma| < |\Lambda|$. Since $C \leq F_1$,

$$\operatorname{Rad} M/C \cong (F_1/C) \oplus F_2.$$

This implies that F_2 is also an injective W-module since $\operatorname{Rad} M/C$ is so. But a nonzero free W-module is *not* injective, because radical of an injective W-module is equal to itself (by Proposition 5.1.7) but a nonzero free module has proper radical (more generally any nonzero projective module has proper radical, see for example Anderson & Fuller (1992, Proposition 17.14)). This contradiction ends the proof. □

For a Dedekind domain W which is not a field, $\mathcal{C}o\text{-}\mathcal{N}eat_{W\text{-}\mathcal{M}od} = \mathcal{C}ompl_{W\text{-}\mathcal{M}od}$ only when $\operatorname{Rad} W \neq 0$:

Lemma 5.4.4. *Let W be a Dedekind domain such that* $\operatorname{Rad} W \neq 0$. *Then*

$$\mathcal{C}o\text{-}\mathcal{N}eat_{W\text{-}\mathcal{M}od} = \mathcal{N}eat_{W\text{-}\mathcal{M}od} = \mathcal{C}ompl_{W\text{-}\mathcal{M}od}.$$

Proof. The second equality holds for any Dedekind domain W by Theorem 5.2.2. Suppose $\operatorname{Rad} W \neq 0$. Then by Proposition 5.1.9, W is a semilocal ring. So by Theorem 3.8.7,

$$\mathcal{C}o\text{-}\mathcal{N}eat_{W\text{-}\mathcal{M}od} = \iota^{-1}(\{\text{all simple } W\text{-modules}\}).$$

By Proposition 5.2.5,

$$\iota^{-1}(\{\text{all simple } W\text{-modules}\}) = \mathcal{C}ompl_{W\text{-}\mathcal{M}od}.$$

□

Lemma 5.4.5. *Let W be a Dedekind domain which is not a field such that* $\operatorname{Rad} W = 0$. *For any maximal ideal P in W, there exits a short exact sequence $\mathbb{E} \in\in \operatorname{Ext}_W(W/P^2, W)$ which is in $\mathcal{N}eat_{W\text{-}\mathcal{M}od} = \mathcal{C}ompl_{W\text{-}\mathcal{M}od}$ but* not *in $\mathcal{C}o\text{-}\mathcal{N}eat_{W\text{-}\mathcal{M}od}$, and hence* not *in $\mathcal{S}uppl_{W\text{-}\mathcal{M}od}$.*

Proof. By Corollary 5.1.11, for the ideal $J = P^2$ we obtain

$$\operatorname{Ext}_W(W/P^2, W) = \operatorname{Ext}_W(W/J, W) \cong \operatorname{Ext}_W(J^{-1}/W, W) \cong W/JW = W/P^2$$

Denote $\mathcal{C}ompl_{W\text{-}\mathcal{M}od}$, $\mathcal{S}uppl_{W\text{-}\mathcal{M}od}$ and $\mathcal{C}o\text{-}\mathcal{N}eat_{W\text{-}\mathcal{M}od}$ by $\mathcal{C}$, $\mathcal{S}$ and $c\mathcal{N}$ respectively. By Theorem 5.2.3,

$$\operatorname{Ext}_{\mathcal{C}}(W/P^2, W) = \operatorname{Rad}(\operatorname{Ext}_W(W/P^2, W)) \cong \operatorname{Rad}(W/P^2) = P/P^2 \neq 0.$$

But $\operatorname{Ext}_{\mathcal{S}}(W/P^2, W) \leq \operatorname{Ext}_{c\mathcal{N}}(W/P^2, W) = 0$ since $\operatorname{Rad} W = 0$ by our assumption (the $\leq$ follows from Proposition 3.4.1). Take a *nonzero* element $[\mathbb{E}] \in \operatorname{Ext}_{\mathcal{C}}(W/P^2, W)$. Then $\mathbb{E}$ is in $\mathcal{C}ompl_{W\text{-}\mathcal{M}od}$ but *not* in $\mathcal{C}o\text{-}\mathcal{N}eat_{W\text{-}\mathcal{M}od}$. □

Theorem 5.4.6. *Let W be a Dedekind domain which is not a field.*

(i) If $\operatorname{Rad} W = 0$, *then*

$$\mathcal{S}uppl_{W\text{-}\mathcal{M}od} \subsetneq \mathcal{C}o\text{-}\mathcal{N}eat_{W\text{-}\mathcal{M}od} \subsetneq \mathcal{N}eat_{W\text{-}\mathcal{M}od} = \mathcal{C}ompl_{W\text{-}\mathcal{M}od}.$$

(ii) If $\operatorname{Rad} W \neq 0$*, then*

$$\mathcal{S}uppl_{W\text{-}\mathcal{M}od} \subsetneq \mathcal{C}o\text{-}\mathcal{N}eat_{W\text{-}\mathcal{M}od} = \mathcal{N}eat_{W\text{-}\mathcal{M}od} = \mathcal{C}ompl_{W\text{-}\mathcal{M}od}.$$

Proof. By Proposition 5.2.6,

$$\mathcal{S}uppl_{W\text{-}\mathcal{M}od} \subseteq \mathcal{C}o\text{-}\mathcal{N}eat_{W\text{-}\mathcal{M}od} \subseteq \mathcal{N}eat_{W\text{-}\mathcal{M}od} = \mathcal{C}ompl_{W\text{-}\mathcal{M}od}.$$

By Example 5.4.3, $\mathcal{S}uppl_{W\text{-}\mathcal{M}od} \neq \mathcal{C}o\text{-}\mathcal{N}eat_{W\text{-}\mathcal{M}od}$.

(i) If $\operatorname{Rad} W = 0$, then $\mathcal{C}o\text{-}\mathcal{N}eat_{W\text{-}\mathcal{M}od} \neq \mathcal{N}eat_{W\text{-}\mathcal{M}od} = \mathcal{C}ompl_{W\text{-}\mathcal{M}od}$ by Lemma 5.4.5.

(ii) If $\operatorname{Rad} W \neq 0$, then by Lemma 5.4.4, $\mathcal{C}o\text{-}\mathcal{N}eat_{W\text{-}\mathcal{M}od} = \mathcal{N}eat_{W\text{-}\mathcal{M}od} = \mathcal{C}ompl_{W\text{-}\mathcal{M}od}$.

□

Theorem 5.4.7. *Let W be a Dedekind domain. Take a W-module B and a submodule $A \leq B$. Suppose A is a* finitely generated torsion *W-module. Then A is neat in B if and only if A is coneat in B.*

Proof. By Theorem 5.4.6, we already have $\mathcal{C}o\text{-}\mathcal{N}eat_{W\text{-}\mathcal{M}od} \subseteq \mathcal{C}ompl_{W\text{-}\mathcal{M}od}$. So ($\Leftarrow$) holds for any W-module A. Conversely, if A is a *finitely generated torsion* W-module and A is neat in B, then by Theorem 5.3.1, A is a supplement in B, hence A is coneat in B since $\mathcal{S}uppl_{W\text{-}\mathcal{M}od} \subseteq \mathcal{C}o\text{-}\mathcal{N}eat_{W\text{-}\mathcal{M}od}$ by Proposition 3.4.1. □

Like in abelian groups, $\operatorname{Ext}_{\mathcal{C}ompl_{W\text{-}\mathcal{M}od}}$ is factorizable as

$$W\text{-}\mathcal{M}od \times W\text{-}\mathcal{M}od \xrightarrow{Ext_W} W\text{-}\mathcal{M}od \xrightarrow{\operatorname{Rad}} W\text{-}\mathcal{M}od$$

by Theorem 5.2.3, but:

Theorem 5.4.8. *Let W be a Dedekind domain which is not a field such that* $\operatorname{Rad} W = 0$. *Then the functors* $\operatorname{Ext}_{\mathcal{S}uppl_{W\text{-}\mathcal{M}od}}$ *and* $\operatorname{Ext}_{\mathcal{C}o\text{-}\mathcal{N}eat_{W\text{-}\mathcal{M}od}}$ *are* not *factorizable as*

$$W\text{-}\mathcal{M}od \times W\text{-}\mathcal{M}od \xrightarrow{Ext_W} W\text{-}\mathcal{M}od \xrightarrow{H} W\text{-}\mathcal{M}od$$

for any functor $H : W\text{-}\mathcal{M}od \longrightarrow W\text{-}\mathcal{M}od$.

Proof. Denote $\mathcal{C}ompl_{W\text{-}\mathcal{M}od}$, $\mathcal{S}uppl_{W\text{-}\mathcal{M}od}$ and $\mathcal{C}o\text{-}\mathcal{N}eat_{W\text{-}\mathcal{M}od}$ by $\mathcal{C}$, $\mathcal{S}$ and $c\mathcal{N}$ respectively.

Suppose for the contrary that $\operatorname{Ext}_{\mathcal{S}}$ is factorizable as

$$W\text{-}\mathcal{M}od \times W\text{-}\mathcal{M}od \xrightarrow{Ext_W} W\text{-}\mathcal{M}od \xrightarrow{H} W\text{-}\mathcal{M}od$$

for some functor $H : W\text{-}\mathcal{M}od \longrightarrow W\text{-}\mathcal{M}od$. So for all W-modules A and C, $\operatorname{Ext}_{\mathcal{S}}(C, A) = H(\operatorname{Ext}_W(C, A))$. Let P be a maximal ideal of W. In the proof of Lemma 5.4.5, we have found that

$$\operatorname{Ext}_W(W/P^2, W) \cong W/P^2 \quad \text{and} \quad \operatorname{Ext}_{\mathcal{S}}(W/P^2, W) = 0.$$

This implies that $H(W/P^2) \cong H(\operatorname{Ext}_W(W/P^2, W)) = \operatorname{Ext}_{\mathcal{S}}(W/P^2, W) = 0$, hence $H(W/P^2) = 0$. But also $\operatorname{Ext}_W(W/P^2, W/P^2) \cong W/P^2$ by Corollary 5.1.11. By Theorem 5.3.1, since W/P^2 is a finitely generated torsion W-module, we obtain

$$\begin{aligned}\operatorname{Ext}_{\mathcal{S}}(W/P^2, W/P^2) &= \operatorname{Ext}_{\mathcal{C}}(W/P^2, W/P^2)\\ &= \operatorname{Rad}(\operatorname{Ext}_W(W/P^2, W/P^2)) \cong P(W/P^2) = P/P^2 \neq 0.\end{aligned}$$

So in this case $H(W/P^2) \cong H(\operatorname{Ext}_W(W/P^2, W/P^2)) = \operatorname{Ext}_{\mathcal{S}}(W/P^2, W/P^2) \cong P/P^2 \neq 0$. This contradiction shows that $\operatorname{Ext}_{\mathcal{S}uppl_{W\text{-}\mathcal{M}od}}$ is *not* factorizable.

Similarly, $\operatorname{Ext}_{c\mathcal{N}}$ is *not* factorizable. In the above proof, just replace $\mathcal{S}$ by $c\mathcal{N}$. Note that $\operatorname{Ext}_{c\mathcal{N}}(W/P^2, W) = 0$ since $\operatorname{Rad} W = 0$, and

$$\operatorname{Ext}_{c\mathcal{N}}(W/P^2, W/P^2) = \operatorname{Ext}_{\mathcal{C}}(W/P^2, W/P^2)$$

by Theorem 5.4.7 as W/P^2 is a finitely generated torsion W-module and $\mathcal{N}eat_{W\text{-}\mathcal{M}od} = \mathcal{C}ompl_{W\text{-}\mathcal{M}od}$ by Theorem 5.2.2. □

Like in abelian groups, for a *torsion* W-module B, neat submodules and coneat submodules coincide:

Theorem 5.4.9. *Let W be a Dedekind domain. Let B be a* torsion *W-module, and A any submodule of B. Then A is neat in B if and only if A is coneat in B.*

Proof. ($\Leftarrow$) always holds (for any module B) by Proposition 5.2.6. Conversely, suppose A is neat in B. To exclude the trivial cases suppose that W is not a field, so its maximal ideals are nonzero. To show that A is coneat in B, we must show that for every W-module M with $\operatorname{Rad} M = 0$, any homomorphism $f : A \longrightarrow M$ can be extended to B. Since B is a torsion W-module, so is its submodule A, hence $f(A)$ is also a torsion W-module. So, without loss of generality, we may suppose that M is also a torsion W-module. Decompose A, B and M into their P-primary parts by Proposition 5.1.4: $A = \bigoplus_P A_P$, $B = \bigoplus_P B_P$ and $M = \bigoplus_P M_P$, where the index P runs through all *nonzero* prime ideals of W, hence P runs through all maximal ideals of W. For each maximal ideal P of W, let $f_P : A_P \longrightarrow M_P$ be the restriction of f to A_P, with range restricted to M_P also (note that $f(A_P) \leq M_P$). Since $0 = \operatorname{Rad} M = \bigoplus_P \operatorname{Rad} M_P = \bigoplus_P PM_P$ by Proposition 5.1.5, we have $PM_P = 0$ for each maximal ideal P. So, each M_P is a $\mathcal{N}eat_{W\text{-}\mathcal{M}od}$-injective module by Theorem 5.2.2. Suppose each A_P is neat in B_P. Then there exists $\tilde{f}_P : B_P \longrightarrow M_P$ extending $f_P : A_P \longrightarrow M_P$. Define $\tilde{f} : B \longrightarrow M$, by $\tilde{f}(\sum_P b_P) = \sum_P \tilde{f}_P(b_P)$ for each $\sum_P b_P \in \bigoplus_P B_P = B$ where $b_P \in B_P$ for every maximal ideal P. Then $\tilde{f} : B \longrightarrow M$ is the required homomorphism extending $f : A \longrightarrow M$:

$$\begin{array}{ccc} A = \bigoplus_P A_P & \leq_c & \bigoplus_P B_P = B \\ {\scriptstyle f=\bigoplus_P f_P}\big\downarrow & \swarrow {\scriptstyle \tilde{f}=\bigoplus_P \tilde{f}_P} & \\ M = \bigoplus_P M_P & & \end{array}$$

Thus, it only remains to show that each A_P is neat in B_P which follows since $\mathcal{N}eat_{W\text{-}\mathcal{M}od}$ is a proper class: A_P is neat in A as it is a direct summand of A, and A is neat in B. So, A_P is neat in B as the composition of two $\mathcal{N}eat_{W\text{-}\mathcal{M}od}$-monomorphisms is a $\mathcal{N}eat_{W\text{-}\mathcal{M}od}$-monomorphism by proper class axioms. Since

$A_P \leq B_P \leq B$, we have that the composition $A_P \hookrightarrow B_P \hookrightarrow B$ of inclusion monomorphisms is a $\mathcal{N}eat_{W\text{-}\mathcal{M}od}$-monomorphism, so the first inclusion monomorphism $A_P \hookrightarrow B_P$ must also be a $\mathcal{N}eat_{W\text{-}\mathcal{M}od}$-monomorphism by proper class axioms. □

5.5 Coinjectives and coprojectives with respect to $\mathcal{C}ompl_{W\text{-}\mathcal{M}od}$ and $\mathcal{S}uppl_{W\text{-}\mathcal{M}od}$

By Theorem 3.7.1, we already know that $\mathcal{C}ompl_{W\text{-}\mathcal{M}od}$-coinjectives are only injective W-modules and by Theorem 3.7.2, if $\operatorname{Rad} W = 0$, then $\mathcal{S}uppl_{W\text{-}\mathcal{M}od}$-coprojectives are only projective W-modules. Since $\mathcal{S}uppl_{W\text{-}\mathcal{M}od} \subseteq \mathcal{C}ompl_{W\text{-}\mathcal{M}od}$, $\mathcal{S}uppl_{W\text{-}\mathcal{M}od}$-coinjectives are also only injective W-modules.

Theorem 5.5.1. *For a Dedekind domain W, $\mathcal{C}ompl_{W\text{-}\mathcal{M}od}$-coprojectives are only torsion-free W-modules.*

Proof. Firstly, each torsion-free W-module C is $\mathcal{C}ompl_{W\text{-}\mathcal{M}od}$-coprojective because every short exact sequence

$$\mathbb{E}: \qquad 0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

of W-modules is in $\mathcal{C}ompl_{W\text{-}\mathcal{M}od}$: By Theorem 5.2.2,

$$\mathcal{C}ompl_{W\text{-}\mathcal{M}od} = \mathcal{N}eat_{W\text{-}\mathcal{M}od} = \pi^{-1}(\{W/P | P \text{ maximal ideal of } W\})$$

So, it suffices to show that every simple module W/P, where P is a maximal ideal of W, is projective with respect to $\mathbb{E}$. But that is clear since the image of a homomorphism $\alpha : W/P \longrightarrow C$ is torsion as W/P is torsion, so there is no homomorphism $W/P \longrightarrow C$ except the zero homomorphism which of course extends to $W/P \longrightarrow B$ as the zero homomorphism.

Conversely suppose C is a $\mathcal{C}ompl_{W\text{-}\mathcal{M}od}$-coprojective W-module. Since the proper class

$$\mathcal{C}ompl_{W\text{-}\mathcal{M}od} = \iota^{-1}(\{W/P|P \text{ maximal ideal of } W\})$$

is injectively generated by all simple W-modules (by Proposition 5.2.5), we know that a W-module C is $\mathcal{C}ompl_{W\text{-}\mathcal{M}od}$-coprojective if and only if $\mathrm{Ext}^1_W(C, S) = 0$ for all simple W-modules S by Proposition 2.6.5. Suppose for the contrary that C is not torsion-free. Hence there exists $0 \neq c \in C$ such that $Ic = 0$ for some nonzero ideal I of W. Consider the submodule Wc of C. Since Wc is a torsion module, it has a simple submodule S by Proposition 5.1.6. Say $S \cong W/P$ for some maximal ideal P of W. Consider the short exact sequence

$$0 \longrightarrow S \xrightarrow{f} C \xrightarrow{g} C/S \longrightarrow 0$$

where f is the inclusion homomorphism and g is the natural epimorphism. By the long exact sequence connecting Hom and Ext, we have the following exact sequence:

$$\ldots \longrightarrow \mathrm{Ext}^1_W(C, S) = 0 \longrightarrow \mathrm{Ext}^1_W(S, S) = 0 \longrightarrow \mathrm{Ext}^2_W(C/S, S) = 0 \longrightarrow \ldots$$

Here $\mathrm{Ext}^1_W(C, S) = 0$ because C is $\mathcal{C}ompl_{W\text{-}\mathcal{M}od}$-coprojective and $\mathrm{Ext}^2_W(C/S, S) = 0$ since $\mathrm{Ext}^2_W = 0$ as W is a hereditary ring. Thus the above exact sequence implies that $\mathrm{Ext}^1_W(S, S) = 0$ which is the required contradiction since $\mathrm{Ext}^1_W(S, S) \cong \mathrm{Ext}^1_W(W/P, W/P) \cong W/P \neq 0$ by Corollary 5.1.11. □

REFERENCES

Al-Khazzi, I., & Smith, P. F. (1991). Modules with chain conditions on superfluous submodules. Comm. Algebra, 19(8), 2331–2351.

Alizade, R., Bilhan, G., & Smith, P. F. (2001). Modules whose maximal submodules have supplements. Comm. Algebra, 29(6), 2389–2405.

Alizade, R., & Büyükaşık, E. (2003). Cofinitely weak supplemented modules. Comm. Algebra, 31(11), 5377–5390.

Alizade, R., & Pancar, A. (1999). Homoloji Cebire Giriş. Samsun: 19 Mayıs Üniversitesi.

Anderson, F. W., & Fuller, K. R. (1992). Rings and Categories of Modules. New-York: Springer.

Atiyah, M. F., & Macdonald, I. G. (1969). Introduction to commutative algebra. Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont.

Baba, Y., & Harada, M. (1990). On almost M-projectives and almost M-injectives. Tsukuba J. Math., 14(1), 53–69.

Berrick, A. J., & Keating, M. E. (2000). An introduction to rings and modules with K-theory in view, vol. 65 of Cambridge Studies in Advanced Mathematics. Cambridge: Cambridge University Press.

Bilhan, G. (1995). Neat exact sequences of abelian groups. M. Sc. Thesis, Dokuz Eylül University, The Graduate School of Natural and Applied Sciences, İzmir.

Buschbaum, D. (1959). A note on homology in categories. Ann. of Math., 69(1), 66–74.

Cartan, H., & Eilenberg, S. (1956). Homological Algebra. Princeton Landmarks in Mathematics and Physics series, New Jersey: Princeton Univesity, 13th ed.

Cohn, P. M. (2002). Basic Algebra: Groups, Rings and Fields. London: Springer-Verlag.

Dung, N. V., Huynh, D., Smith, P. F., & Wisbauer, R. (1994). Extending Modules. No. 313 in Putman Research Notes in Mathematics Series, Harlow: Longman.

Enochs, E., & Jenda, O. M. G. (2000). Relative Homological Algebra. No. 30 in de Gruyter Expositions in Mathematics, Berlin: de Gruyter.

Erdoğan, E. (2004). Absolutely supplement and absolutely complement modules. M. Sc. Thesis, İzmir Institute of Technology, İzmir.

Facchini, A. (1998). Module theory, vol. 167 of Progress in Mathematics. Basel: Birkhäuser Verlag, ISBN 3-7643-5908-0. Endomorphism rings and direct sum decompositions in some classes of modules.

Fedin, S. N. (1983). The concept of inductive closure of a proper class. Math. Notes., 33(3–4), 227–233. Translated from Russian from *Mat. Zametki* 33(3), 445-457 (1983).

Fieldhouse, D. J. (1985). Semiperfect and F-semiperfect modules. Internat. J. Math. Math. Sci., 8(3), 545–548.

Fuchs, L. (1970). Infinite Abelian Groups, vol. 1. New York: Academic Press.

Fuchs, L., & Salce, L. (2001). Modules over non-Noetherian domains, vol. 84 of Mathematical Surveys and Monographs. Providence, RI: American Mathematical Society, ISBN 0-8218-1963-1.

Ganesan, L., & Vanaja, N. (2002). Modules for which every submodule has a unique coclosure. Comm. Algebra, 30(5), 2355–2377.

Generalov, A. I. (1972). On the definition of purity of modules. Math. Notes, 11, 232–235. Translated from Russian from *Mat. Zametki* 11(4), 375-380 (1972).

Generalov, A. I. (1978). On weak and w-high purity in the category of modules. Math. USSR, Sb., 34, 345–356. Translated from Russian from *Mat. Sb., N. Ser.* 105(147), 389-402 (1978).

Generalov, A. I. (1983). The w-cohigh purity in a category of modules. Math. Notes, 33(5-6), 402–408. Translated from Russian from *Mat. Zametki* 33(5), 785-796 (1983).

Güngöroglu, G., & Keskin Tütüncü, D. (2003). Copolyform and lifting modules. Far East J. Math. Sci. (FJMS), 9(2), 159–165.

Harmancı, A., Keskin, D., & Smith, P. F. (1999). On $\oplus$-supplemented modules. Acta Math. Hungar., 83(1-2), 161–169.

Harrison, D., Irwin, J. M., Peercy, C. L., & Walker, E. A. (1963). High extensions of abelian groups. Acta Math. Acad. Sci. Hunger., 14, 319–330.

Hausen, J. (1982). Supplemented modules over Dedekind domains. Pacific J. Math., 100(2), 387–402.

Hausen, J., & Johnson, J. A. (1983a). A characterization of two classes of dedekind domains by properties of their modules. Publ. Math. Debrecen, 30(1-2), 53–55.

Hausen, J., & Johnson, J. A. (1983b). A new characterization of perfect and semiperfect rings. Bull. Calcutta Math. Soc., 75(1), 57–58.

Honda, K. (1956). Realism in the theory of abelian groups I. Comment. Math. Univ. St. Paul, 5, 37–75.

Idelhadj, A., & Tribak, R. (2003a). On injective ⊕-supplemented modules. In Comptes rendus d. l. p. r. m.-a. s. l. alg. et l. appl. (Tétouan, 2001), pp. 166–180, Univ. Abdelmalek Essaâdi. Fac. Sci. Tétouan, Tétouan.

Idelhadj, A., & Tribak, R. (2003b). On some properties of ⊕-supplemented modules. Int. J. Math. Math. Sci., (69), 4373–4387.

Inoue, T. (1983). Sum of hollow modules. Osaka J. Math., 20(2), 331–336.

Kaplansky, I. (1952). Modules over Dedekind rings and valuation rings. Trans. Amer. Math. Soc., 72, 327–340.

Kaplansky, I. (1958). Projective modules. Ann. of Math (2), 68, 372–377.

Keskin, D. (2000a). Characterizations of right perfect rings by ⊕-supplemented modules. In Algebra and its applications (Athens, OH, 1999), vol. 259 of Contemp. Math., pp. 313–318, Providence, RI: Amer. Math. Soc.

Keskin, D. (2000b). On lifting modules. Comm. Alg., 28(7), 3427–3440.

Keskin, D. (2002a). An approach to extending and lifting modules by modular lattices. Indian J. Pure Appl. Math., 33(1), 81–86.

Keskin, D. (2002b). Discrete and quasi-discrete modules. Comm. Algebra, 30(11), 5273–5282.

Keskin, D., Smith, P. F., & Xue, W. (1999). Rings whose modules are ⊕-supplemented. J. Algebra, 218(2), 470–487.

Keskin, D., & Xue, W. (2001). Generalizations of lifting modules. Acta Math. Hungar., 91(3), 253–261.

Keskin Tütüncü, D., & Orhan, N. (2003). CCSR-modules and weak lifting modules. East-West J. Math., 5(1), 89–96.

Koşan, T., & Harmanci, A. (2004). Modules supplemented relative to a torsion theory. Turkish J. Math., 28(2), 177–184.

Kuratomi, Y. (2003). Direct sums of lifting modules. In Proc. of the 35th Symp. on Ring Th. and Represent. Th. (Okayama, 2002), pp. 165–169, Symp. Ring Theory Represent Theory Organ. Comm., Okayama.

Kuz'minov, V. I. (1976). Groups of pure extensions of abelian groups. Siberian Math. J., 17(6), 959–968. Translated from Russian from *Sibirsk. Mat. Ž.* 17(6), 1308–1320, 1438 (1976).

Lam, T. Y. (1999). Lectures on modules and rings, vol. 189 of Graduate Texts in Mathematics. New York: Springer-Verlag, ISBN 0-387-98428-3.

Lam, T. Y. (2001). A first course in noncommutative rings, vol. 131 of Graduate Texts in Mathematics. New York: Springer-Verlag, 2nd ed., ISBN 0-387-95183-0.

Leonard, W. W. (1966). Small modules. Proc. Amer. Math. Soc., 17, 527–531.

Liu, Y. H. (1992). On small projective modules. J. Math. Res. Exposition, 12(2), 231–234.

Lomp, C. (1996). On dual Goldie dimension. Diplomarbeit (M. Sc. Thesis), Mathematischen Institut der Heinrich-Heine Universität, Düsseldorf. Revised version (2000).

Lomp, C. (1999). On semilocal modules and rings. Comm. Algebra, 27(4), 1921–1935.

Lomp, C. (2000). On the splitting of the dual Goldie torsion theory. In Algebra and its applications (Athens, OH, 1999), vol. 259 of Contemp. Math., pp. 377–386, Providence, RI: Amer. Math. Soc.

Maclane, S. (1963). Homology. Berlin-Göttingen-Heidelberg: Springer-Verlag.

Manovcev, A. A. (1975). Inductive purities in abelian groups. Math. USSR, Sb., 25(3), 389–418. Translated from Russian from *Mat. Sb., N. Ser.* 96(138), 414–446, 503 (1975).

McConnell, J. C., & Robson, J. C. (2001). Noncommutative Noetherian rings, vol. 30 of Graduate Studies in Mathematics. Providence, RI: American Mathematical Society, revised ed., ISBN 0-8218-2169-5. With the cooperation of L. W. Small.

Mishina, A. P., & Skornyakov, L. A. (1976). Abelian groups and modules, vol. 107 of American Mathematical Society Translations. Ser. 2. Providence, R. I.: American Mathematical Society. Translated from Russian from *Abelevy gruppy i moduli*, Izdat. Nauka, Moscow (1969).

Nebiyev, C., & Pancar, A. (2003). On amply supplemented modules. Int. J. Appl. Math., 12(3), 213–220.

Nunke, R. J. (1959). Modules of extensions over dedekind rings. Illunois J. of Math., 3, 222–241.

Orhan, N. (2003). Some characterizations of lifting modules in terms of preradicals. Hacet. J. Math. Stat., 32, 13–15.

Oshiro, K. (1984a). Lifting modules, extending modules and their applications to generalized uniserial rings. Hokkaido Math. J., 13(3), 339–346.

Oshiro, K. (1984b). Lifting modules, extending modules and their applications to QF-rings. Hokkaido Math. J., 13(3), 310–338.

Oshiro, K. (2001). Theories of Harada in Artinian rings and applications to classical Artinian rings. In International Symposium on Ring Theory (Kyongju, 1999), Trends Math., pp. 279–301, Boston, MA: Birkhäuser Boston.

Oshiro, K., & Wisbauer, R. (1995). Modules with every subgenerated module lifting. Osaka J. Math., 32(2), 513–519.

Özcan, A. Ç. (2002). Modules with small cyclic submodules in their injective hulls. Comm. Algebra, 30(4), 1575–1589.

Renault, G. (1964). Étude de certains anneaux a liés aux sous-modules compléments dun a-module. C. R. Acad. Sci. Paris, 259, 4203–4205.

Rotman, J. (1979). An Introduction to Homological Algebra. New York: Academic Press.

Santa-Clara, C., & Smith, P. F. (2000). Modules which are self-injective relative to closed submodules. In Algebra and its applications (Athens, OH, 1999), vol. 259 of Contemp. Math., pp. 487–499, Providence, RI: Amer. Math. Soc.

Santa-Clara, C., & Smith, P. F. (2004). Direct products of simple modules over Dedekind domains. Arch. Math. (Basel), 82(1), 8–12.

Sharp, R. Y. (2000). Steps in commutative algebra, vol. 51 of London Mathematical Society Student Texts. Cambridge: Cambridge University Press, 2nd ed.

Sklyarenko, E. G. (1978). Relative homological algebra in categories of modules. Russian Math. Surveys, 33(3), 97–137. Traslated from Russian from *Uspehi Mat. Nauk* 33, no. 3(201), 85-120 (1978).

Sklyarenko, E. G. (1981). On the Manovcev-Kuz'minov theorem. Siberian Math. J., 22(1), 106–110. Translated from Russian from *Sibirsk. Mat. Zh.* 22(1), 144–150, 230 (1981).

Smith, P. F. (2000a). Commutative domains whose finitely generated projective modules have an injectivity property. In Algebra and its applications (Athens, OH, 1999), vol. 259 of Contemp. Math., pp. 529–546, Providence, RI: Amer. Math. Soc.

Smith, P. F. (2000b). Finitely generated supplemented modules are amply supplemented. Arab. J. Sci. Eng. Sect. C Theme Issues, 25(2), 69–79.

Stenström, B. T. (1967a). Pure submodules. Arkiv för Matematik, 7(10), 159–171.

Stenström, B. T. (1967b). High submodules and purity. Arkiv för Matematik, 7(11), 173–176.

Talebi, Y., & Vanaja, N. (2002). The torsion theory cogenerated by M-small modules. Comm. Algebra, 30(3), 1449–1460.

Talebi, Y., & Vanaja, N. (2004). Copolyform Σ-lifting modules. Vietnam J. Math., 32(1), 49–64.

Tuganbaev, A. A. (2002). Semiregular, weakly regular, and π-regular rings. J. Math. Sci. (New York), 109(3), 1509–1588. Algebra, 16.

Vanaja, N. (1993). Characterizations of rings using extending and lifting modules. In Ring theory (Granville, OH, 1992), pp. 329–342, River Edge, NJ: World Sci. Publishing.

Warfield, R. B. (1969). Purity and algebraic compactness for modules. Pac. J. Math., 28, 699–719.

Wisbauer, R. (1991). Foundations of Module and Ring Theory. Reading: Gordon and Breach.

Xin, L. (1994). A note on dual Goldie dimension over perfect rings. Kobe J. Math., 11(1), 21–24.

Zöschinger, H. (1974a). Komplementierte Moduln über Dedekindringen. J. Algebra, 29, 42–56.

Zöschinger, H. (1974b). Komplemente als direkte Summanden. Arch. Math. (Basel), 25, 241–253.

Zöschinger, H. (1974c). Moduln, die in jeder Erweiterung ein Komplement haben. Math. Scand., 35, 267–287.

Zöschinger, H. (1976). Basis-Untermoduln und Quasi-kotorsions-Moduln über diskreten Bewertungsringen. Bayer. Akad. Wiss. Math-Nat. Kl. Sitzungsber., pp. 9–16 (1977).

Zöschinger, H. (1978). Über Torsions- und κ-Elemente von $\mathrm{Ext}(C, A)$. J. Algebra, 50(2), 299–336.

Zöschinger, H. (1979a). Quasiseparable und koseparable Moduln über diskreten Bewertungsringen. Math. Scand., 44(1), 17–36.

Zöschinger, H. (1979b). Komplemente für zyklische Moduln über Dedekindringen. Arch. Math. (Basel), 32(2), 143–148.

Zöschinger, H. (1980). Die κ-Elemente von ${\mathrm{Ext}^1}_R(C, \mathbf{R}^n)$. Hokkaido Math. J., 9(2), 155–174.

Zöschinger, H. (1981). Projektive Moduln mit endlich erzeugtem Radikalfaktormodul. Math. Ann., 255(2), 199–206.

Zöschinger, H. (1982a). Komplemente als direkte Summanden. II. Arch. Math. (Basel), 38(4), 324–334.

Zöschinger, H. (1982b). Gelfandringe und koabgeschlossene Untermoduln. Bayer. Akad. Wiss. Math.-Natur. Kl. Sitzungsber., pp. 43–70 (1983).

Zöschinger, H. (1986). Komplemente als direkte Summanden. III. Arch. Math. (Basel), 46(2), 125–135.

Zöschinger, H. (1994). Die globale Transformation eines lokalen Ringes. J. Algebra, 168(3), 877–902.

INDEX

*Numbers in **boldface** refer to pages on which the terms are defined.*