DOKUZ EYLÜL UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

INVENTORY MANAGEMENT IN A TWO-ECHELON SUPPLY CHAIN

by
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İZMİR

INVENTORY MANAGEMENT IN A TWO-ECHELON SUPPLY CHAIN

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Ph.D. THESIS EXAMINATION RESULT FORM

We have read the thesis entitled **"INVENTORY MANAGEMENT IN A TWO ECHELON SUPPLY CHAIN"** completed by **UMAY UZUNOĞLU KOÇER** under supervision of **ASSOC. PROF. DR. C. CENGİZ ÇELİKOĞLU** and we certify that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Doctor of Philosophy.

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INVENTORY MANAGEMENT IN A TWO-ECHELON SUPPLY CHAIN

ABSTRACT

With the trend towards greater synergy between suppliers and customers, and since the customers become more sophisticated recently, supply chain inventory management has been gaining importance day by day.

In this study, the base stock policy for the firms, which form a two-echelon supply chain in the presence of independent but non-stationary (time dependent) demand will be discussed. A serial system in multi-period setting is studied. Each stage incurs certain holding cost and shortage cost at the end of each period. The system is examined under both centralized and decentralized control scheme. In decentralized frame, two different game formulations are considered. The first one is the repeated game. In each period a one-period game is played and for T-period finite horizon, these one-period games are played T times. Each one-period game has a unique Nash equilibrium, however when the one-period game is repeated, different equilibrium points may be obtained in each period. This fact derives the need for defining subgames. By working backward, subgame perfect equilibrium for the repeated game can be obtained.

First the case in which the decisions made independently in each period is studied, then the case in which the decisions made in the past has influence on current decisions is considered. This requires a stochastic game (Markovian game) formulation. The solution is found by Markov perfect equilibrium.

Keywords: base stock policy, multi-echelon inventory systems, repeated game, stochastic game, Markov perfect equilibrium

İKİ AŞAMALI TEDARİK ZİNCİRİNDE ENVANTER YÖNETİMİ

ÖZET

Tedarikçiler ve endüstriyel müşteriler arasında giderek artan sinerji ile birlikte ve müşteriler geçmiş zamana göre daha bilinçli olduklarından, tedarik zinciri envanter yönetimi günden güne önem kazanmaktadır.

Bu çalışmada, iki aşamalı tedarik zincirini oluşturan firmalar için bağımsız ancak durağan olmayan talep durumunda, eldeki envanteri belli bir düzeye çıkaracak envanter politikasının (base stock policy) incelenmesi hedeflenmiştir. Çok periyot durumunda seri bir sistem çalışılmıştır. Her periyot sonunda her aşama belli bir elde tutma ve stoksuzluk maliyetine katlanmaktadır. Sistem merkezi ve merkezi olmayan karar yapısı altında incelenmiştir. Merkezi olmayan karar yapısında iki tür oyun formüle edilmiştir. İlki tekrarlı oyunlardır. Her periyotta tek periyotluk oyun oynanır ve T periyotluk sınırlı planlama dönemi için bu tek periyotluk oyunlar T kez oynanır. Her tek periyotluk oyunun tek bir Nash dengesi vardır ancak tek periyotluk oyunlar tekrar edildiğinde, her periyotta farklı bir Nash dengesi bulunabilir. Bu da altoyun tanımını gerektirir. Geriye doğru çalışarak, tekrarlı oyun için altoyun mükemmel dengesi bulunabilir.

Her kararın her periyotta bağımsız olarak verildiği durumun yanı sıra, geçmişteki kararların verilecek kararları etkilediği durum da düşünülmüştür. Bu stokastik oyun formulasyonunu gerektirir. Stokastik oyun için çözüm Markov mükemmel dengesi ile bulunur.

Anahtar Sözcükler: Base stock politikası, çok aşamalı envanter sistemleri, tekrarlı oyunlar, stokastik oyunlar, Markov mükemmel dengesi

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CHAPTER ONE
INTRODUCTION

Organizations perform some certain activities that include purchasing and material releasing, inbound and outbound transportation, materials handling, warehousing and distribution, inventory control and management, demand and supply planning, order processing, production planning and scheduling, shipping, processing, and customer service. Although companies have been performing these activities for many years, they did not consider viewing these activities as interrelated activities that need to be coordinated, until recent years. Having considered these activities as interrelated, companies needed a concept that could increase operating and financial performance, provide new sources of competitive advantage, and lead to better managed business. This concept is supply chain management.

In addition to this fact, recently as customers become more sophisticated, they demand the right product at the right time, at the right price, and at the right place. The competitive weapon of the 1980s' was the quality, but nowadays since the differentiator is customer responsiveness; companies need to have some different structure to survive in the competitive market.

With the trend towards greater synergy between suppliers and industrial customers, most manufacturing enterprises are organized as networks of manufacturing and distribution sites that purchase raw materials, transform those materials into intermediate and finished products, and also distribute the finished goods to customers. Management of such networks (also referred to as "supply chains") has become as a major topic in operations research.

1.1 Supply Chain

Supply chain is a network of facilities and distributions options that performs the functions of procurement of materials, transformation of these materials into intermediate and finished products, and the distribution of these finished products to customers. Supply chain exists in both service and manufacturing organizations.

A supply chain typically consists of the geographically distributed facilities and transportation links connecting these facilities. In manufacturing industry this supply chain is the linkage, which defines the physical movement of raw materials (from suppliers), processing by the manufacturing units, their storage and final delivery as finished goods for the customers. In services, such as retail stores or a delivery service like UPS or Federal Express, the supply chain reduces the problem to distribution logistics where the start point is the finished product that has to be delivered to the client.

A typical supply chain consists of a *retail store*, which meets the customer demand directly, a *distribution center* or *warehouse* that supplies goods for retailers, a *manufacturer* that produces goods, and finally *suppliers* that provide manufacturers with raw materials and components for production. These are called elements of supply chain and they work with coordination for fulfilling a customer request. Figure 1.1 illustrates the elements of supply chain and interactions between them.

Supply chain is dynamic and involves the constant flow of information, product and funds between its elements or stages. Each stage of supply chain performs different processes and interacts with other stages of the chain. In reality, a manufacturer may receive material from several suppliers and then supply several distributors. Therefore most supply chains are actually networks, which may be called *supply networks*.

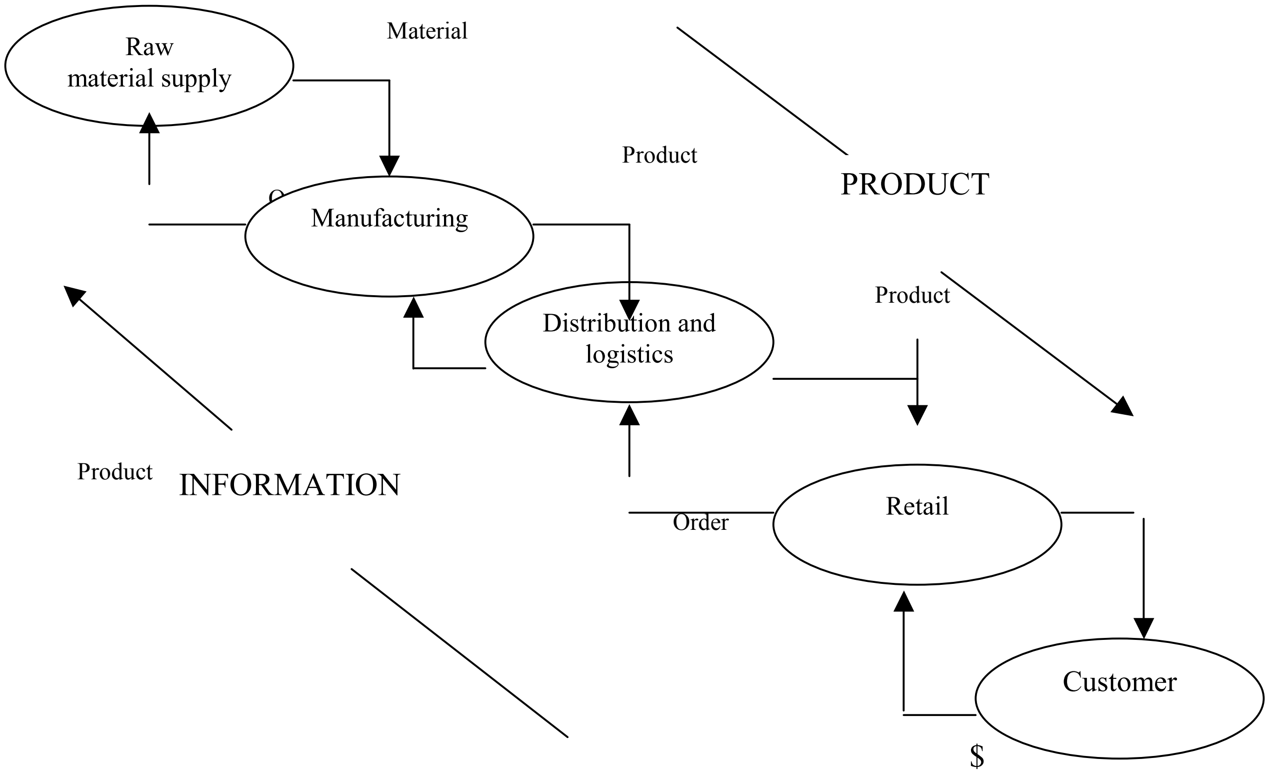


Figure 1.1 Elements of a Supply Chain.

1.2 Supply Chain Management

Managing the supply chain is coordinating all of the operations of a company with the operations of its suppliers and customers. Supply chain management is a set of approaches utilized to efficiently integrate suppliers, manufacturers, warehouses, and stores, so that merchandise is produced and distributed at the right quantities, to the right locations, and at the right time, in order to minimize systemwide cost while satisfying service level requirements.

The complexity of decision making across the supply chain network, makes supply chain management (planning) as difficult as it is important. It is reported in Supply Chain Management Review; May-June 2004 that; today, in United States, a typical company's supply chain related costs can represent more than 80% of revenue and 50% of assets. Because of that reason, beyond gaining a competitive advantage, companies need perfect management of supply chain to survive in a global marketplace. Mismanaging the supply chain may result in real financial or strategic damage for the company.

Since the well-managed supply chain works at the lowest cost, at the highest quality and the highest responsiveness, provides important advantages; in contemporary world, it is believed that competition will be not among firms, but between the supply chains, in which the firms are involved.

Besides these facts, it is hard to manage something that is not measured. In managing the supply chain, there may be several decision variables such as:

- Location- of facilities and sourcing points
- Production –what to produce in which facilities
- Inventory- how much to order, when to order, safety stocks
- Transportation –mode of transport, shipment size, routing and scheduling

Mathematical optimization models can be used to describe the relationships among decisions, constraints, and objectives for optimizing the supply chain. This relationship can be shown as in Figure 1.2.

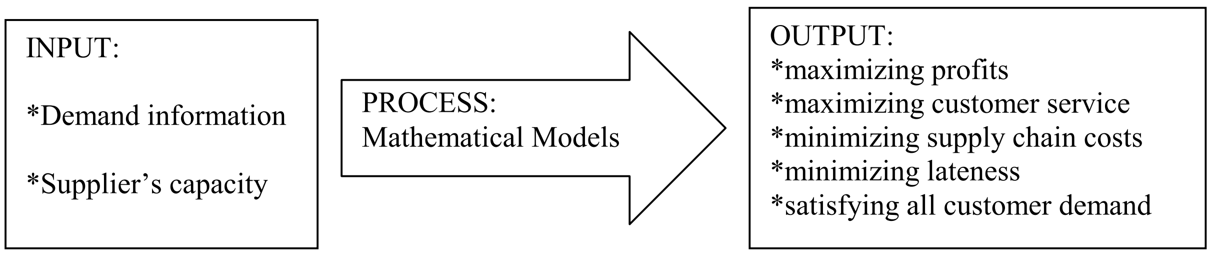


Figure 1.2 The mechanism which optimizes a supply chain.

1.3 Inventory Management in Supply Chain

Inventory can be simply defined as stock of items kept to meet internal or external customer demand. Usually people think inventory of a final product waiting to be sold to a customer. However in supply chain, inventory takes place in every stage. Besides finished goods, there may be inventories including raw materials and in-process products. Since inventory can be kept at every stage of a supply chain, management of inventory is quite important for supply chain success. Moreover, it is widely known that one of the largest costs come from the inventory or shortages (stockouts). For a manufacturer, retailer or distributor; the amount of inventory held directly impacts the costs and customer responsiveness.

There are essentially two motives for holding inventories. If the cost of obtaining or producing items is expected to rise in the near future, it is advantageous to hold inventories in anticipation of the price rise. Inventories may also be retained in advance of sales increases. If demand is expected to increase, it may be more economical to build up large inventories rather than to increase production capacity at a future time. However, it is often encountered that inventory build-ups occur as a result of poor sales. Variation in demand increases the challenge of maintaining inventory to avoid stockouts or to have inventory build-ups. In other words, the problem is complicated by the fact that demand is uncertain, and this uncertainty may cause stockouts in which inventory is depleted and orders cannot be filled.

To minimize supply and demand imbalances in the supply chain, there are certain methods of inventory management. In our study, we assume a model in which the inventory level is reviewed periodically, and orders are placed at regular intervals where order amount equal to one. This policy is known as a base stock policy.

1.4 Competitive World

There are two managerial insights managing the supply chain. Starting with Clark and Scarf in 1960, it is considered that there is a central decision maker who has ability to reach the inventory position information in different locations and makes the inventory decisions for the whole system. The central decision-maker has the information about the cost structure and demand pattern for the whole supply chain, and has the ability to create a global inventory policy. However, the centralized optimization point of view is just an approximation to the real supply chain because it is more realistic that a supply chain, works in decentralized mode where more than two decision makers optimizing different objectives. Wang et al. (2003) reported three important drawbacks of centralized supply chain models:

- Ignoring the independence of the supply chain members.
- The cost of information processing may be expensive. The central decision maker must gather all the information from every supply chain member and finally issue instructions to members.
- The huge capacity of the centralized optimization models. If the problem is fairly large and difficult, it may be impossible to model and solve.

Because of these constraints of centralized models and also need for a better reflection of real life, decentralized supply chain models have been studied recently.

In decentralized decision structure, independent managers (or decision makers) in every inventory location, observes the activities in that location and make the decisions. Since the effectiveness of the system decreases in competitive market, these decision maker's behaviors are rational locally, rather than being efficient globally. Different from the traditional supply chain structure, the firms in decentralized case may have different, even conflicting objectives. Moreover, each manager may have his own firm's cost structure and demand pattern, but have no idea about the whole chain.

When two or more decision maker's objectives are in conflict, there is a competitive world. If strategy determining measures of the decision maker are not certain and if we do not have any information about probabilities associated with these strategies, the problem is a special decision problem under uncertainty. In these cases, the decision maker makes the best decision not only according to the situations that he encountered, but also according to the strategies, which the opponents of the decision maker may choose. Each decision maker wants to optimize his objective and this case can be called as a "game".

Elements, which make up the supply chain, mostly have conflicting objectives because in decentralized control scheme, there are different decision-makers in different locations who want to optimize their objective. Game theory provides suitable methods and techniques for characterizing the behaviors of the firms, which have independent decision makers under competitiveness.

1.5 Scope of The Study

In this study, we aim to discuss the base stock policy for the firms, which consist of two-echelon supply chain in the presence of independent but non-stationary (time dependent) demand. We studied a serial system in multi-period setting. Each stage incurs certain holding cost and shortage cost at the end of each period. First, we investigate the system in centralized control scheme. In this case, the time horizon is divided into two parts where the demand is assumed non-stationary in the finite part and stationary in the infinite part. The base stock policies are determined by myopic policy. Second, we examine the system under decentralized control scheme. We considered that there is a repeated game for the finite horizon problem and it is formulated first. In each period there is a one-period game, which has a unique Nash equilibrium. In each one-period game there is only one Nash equilibrium but when we repeat the one-period game, we may obtain different equilibrium points. This fact derives the need for defining a subgame. Since we can find a Nash equilibrium for each subgame, we can obtain subgame perfect equilibrium for the whole (repeated) game. We assume the horizon is finite; hence it is possible to characterize the subgame perfect equilibrium using backward induction. The strategies in the last period (subgame) must be a Nash equilibrium of the one-period game played in that period. And then when we move to the backward, it is possible to find a strategy that chooses the action of the each subgame to minimize the expected value of the cost. When we reach the initial period, we have the equilibrium of the repeated game. Having studied the case in which the decisions are made independently in each period, in the next step the case in which the decisions made in the past has influence on current decisions is considered. This requires a Markovian game formulation.

In our study, we examined a multi-echelon inventory system, where the retailer faces a stochastic customer demand. For this serial system, which consists of one supplier and one retailer, firms are considered to follow the base stock policy and the optimal inventory policy is searched. The assumption that says demands are independent but nonstationary across periods is the one that makes our study different from the rest of the literature. In this context, the defined system is examined under centralized and decentralized control scheme. Decentralized control scheme of the system is examined under a repeated and then a Markovian game formulation. Our study differs from Cachon and Zipkin (1999) in that, they give a Stackelberg game formulation. Wang et al. (2004) may be considered very similar to ours. They study a distribution system with a game theoretical approach and consider a cooperative mechanism; in other words contract design to make the system more efficient.

Some general information about multi-echelon systems and dynamic games and the literature survey on related work will be given in chapter 2. In chapter 3, we declare the problem statement and define the model under the centralized control scheme. In chapter 4, the defined problem is discussed under the decentralized scheme and the game theoretic frame is defined. Finally in chapter 5 we summarize the findings and give directions for future research.

CHAPTER TWO
LITERATURE REVIEW

The functions of procuring the raw materials, transforming these raw materials into intermediate and finished products, and finally distributing the finished products to the customers constitute the supply chain. Supply chain is a dynamic structure and involves the constant flow of information, product and funds between stages. Each stage of supply chain performs different processes and interacts with other stages of the chain. In reality, a manufacturer may receive material from several suppliers and then supply several distributors. Therefore most supply chains are actually networks, which are also called supply networks. A schematic of a supply chain can be shown as in Figure 2.1.

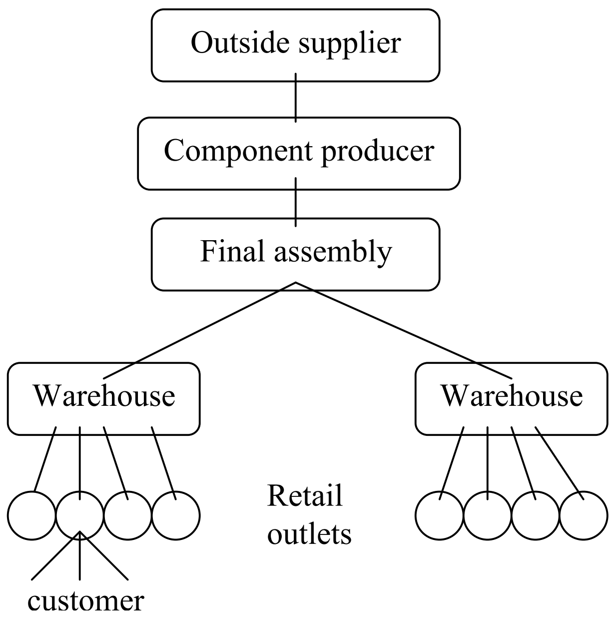


Figure 2.1 Supply chain network

In supply chain management concept, there are two main decision issues: structural and coordination. Structural or strategic issues are longer term decisions such as where to locate the factories, warehouses, and retail sites; how many facilities to have; what capacity should each of these faculties have; what modes of transportation should be used for which product. The result of these structural decisions is a network of facilities designed to produce and distribute the products under consideration.

Coordination decisions are usually taken after the structural decisions are made. Whereas structural decisions tend to be based on long-term, deterministic approaches, coordination decisions include short term issues such as; should inventory stocking and replenishment decisions be made centrally or in a decentralized fashion; should inventory be held at central warehouses; where should inventory be deployed, in other words should most inventory be held at a central location, or should it be pushed to the retail level; how should a limited or insufficient amount of stock be allocated to different locations that need it.

These decisions are quite complex. In this study, we are interested in coordination decisions especially about inventory management in the supply chain. Some models and insights will be presented to help decision-makers related with the inventory management decisions at different locations. Firstly it is convenient to give some information about how the decisions including inventory management are made in supply chain system. The following section explains the structure of supply chains briefly.

2.1 Centralized and Decentralized Systems

The simplest case, conceptually, is a centralized system. In a centralized system, all relevant information in the network flows to a single point, where all decisions are made. These decisions are then transmitted throughout the network to be implemented. From some point of view, this situation is ideal. It is perfect that there is a fully informed decision maker with full control over the whole system.

However in practice, centralization may be dysfunctional. A centralized system requires a fast, reliable and perfect communication system, and a powerful information-processing capability. And also it requires organizations to act in a synchronized fashion. For host of technical, economic and cultural reasons, these elements are often lacking. For this reason, decentralized systems are considered. Decentralized systems are systems where information and control are distributed throughout the network.

In decentralized decision structure, the decision–makers in each installation is independent each other. Each one observes the activities in his own location and make the decisions. Moreover, each manager may have his own firm's cost structure and demand pattern, but have no idea about the whole chain. Different from the centralized supply chain structure, the firms which are included in the chain may have different, even conflicting objectives.

Besides information flow throughout the network, the structure of the system is also important. When there is more than a single stocking point, there exists the possibility for many forms of interaction between stocking points. One of the simplest forms of interaction involves one stocking point which serves as a warehouse for one or more stocking points. This leads to what is referred to as a multiechelon inventory system.

2.2 Multi-echelon Inventory Systems

The multi-echelon inventory problem was first motivated by military logistics problems and has played a large role in the materials management of the armed forces. An item may be stocked in an inventory system at only a single physical location, or it may be stocked at many locations.. One possible multiechelon inventory system is illustrated in Figure 2.2. The arrows indicate the normal pattern for the flow of goods through the system.

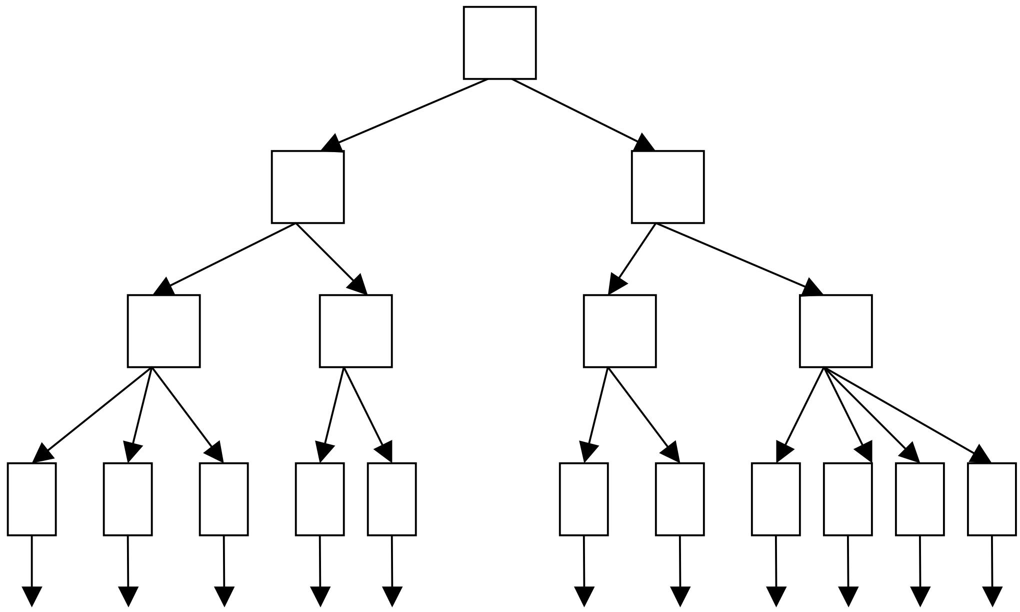


Figure 2.2 Multi-echelon inventory system

This might refferred to as a four echelon system since there are four levels. Each level is called as *echelon*. In the system shown, customer demands occur only at the stocking points in level 1. These stocking points have their stocks replenished by shipments from warehouses at level 2, which in turn receive replenishments for their stock from level 3, etc. Figure 2.2 represents only one type of multiechelon system. In other cases, customer demands might occur at all levels, or stocking points at any level might not only receive shipments from the next highest level but might also get replenishments from any higher level or from the source. Also, it might be allowable, on occasion to permit redisribution of stocks among various stocking points at a given level.

Most inventory systems encountered in the real world are multiechelon in nature. However, it is often true that one need not or cannot consider the multiechelon system in its entirety. The reason for this is that different organizations operate different parts of the system. For example, Figure 2.2 might refer to a production distribution system in which the source is a plant where the item is manufactured, level 4 is a factory warehouse, level 3 represents regional warehouses, level 2 represents warehouses in various cities, and level 1 represents the retail establishments which sell the item to the public. In such a system the manufacturer might control only the plant and factory warehouse, while different organizations operate the regional warehouses and still differet organizations operate the city warehouses and the retail establishments. Even at a given level many different organizations may be involved. For example, each of the warehouses in different cities may be under different ownership. In such a system each organization has the freedom to choose the operating doctrine for controlling the inventories under its jurisdiction. To sum up, it is tractable to handle the whole system. So, one could not attempt to analyse the system as a whole and one decision maker dictate what operating doctrine should be used by each stocking point at each level. This is called *centralized control*. Whereas one might be concerned with the best way for one of the warehouses at level 2 to control its inventories which is called *decentralized control*. In making the analysis, the customers would be the retailers at level 1 and the source from which replenishments are obtained would be the appropriate warehouse at level 3.

As explained before, in a supply chain structure, there may be many retail outlets, which are replenished from warehouses, and there may be many warehouses, which are supplied from a manufacturer. Since this structure is conceptually very similar to multi-echelon inventory systems, multi-echelon inventory systems can be used to optimize the deployment of inventory in a supply chain. Analyzing the multi-echelon system as a whole, may be intractable, there are lots of works in literature, which studied a two-echelon system.

Two echelon inventory systems are generally used to provide products and services for customers who are distributed over an extensive geographical region. Two echelon inventory structure consisting of a warehouse which supplies N retail stores is shown in Figure 2.3. The structure is common to many wholesale/retail systems today, although there are many variations in the method for their operation. The warehouse receives shipments from suppliers and manufacturers, and distributes them to the stores. In some cases, the warehouse itself also maintains inventory. Many retailers consider the role for the warehouse stock to be a strategic issue.

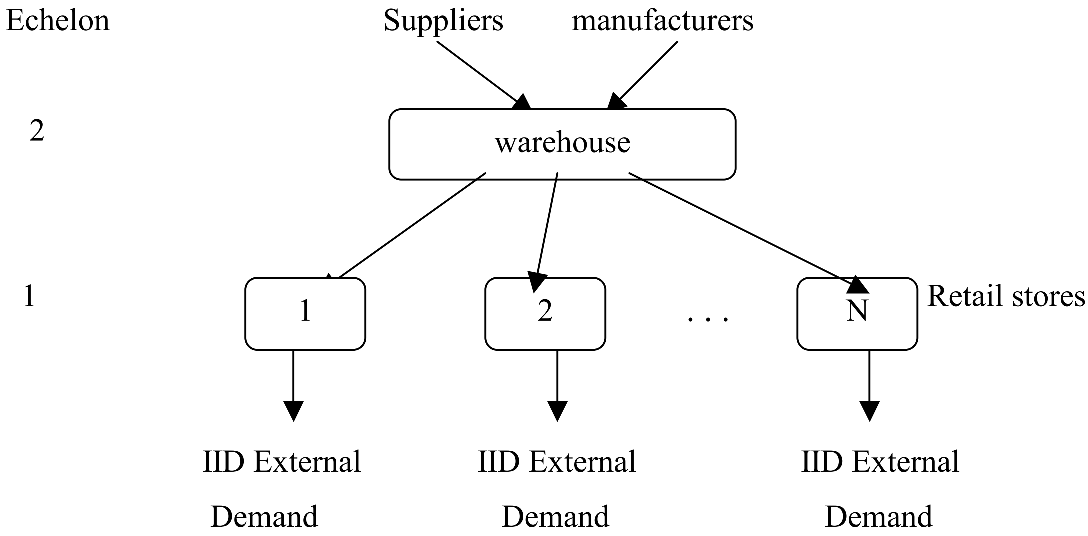


Figure 2.3 Two echelon inventory system

The defining feature of the multi-echelon structure is that lower-level locations are supplied by higher-level locations. However, in this framework, there may be many possible variations. Multi-echelon systems can be classified into three main groups. *Arborescent systems* have branches spreading apart, with the products flowing to different branches. *Coalescent systems* have materials coming together into one end item. *Series systems* have locations feeding each other in a direct path. These different kinds of systems are illustrated in Figure 2.4. Since the system, which will be analyzed in this study, is a series system, this kind of system will be explained in detail in the following section.

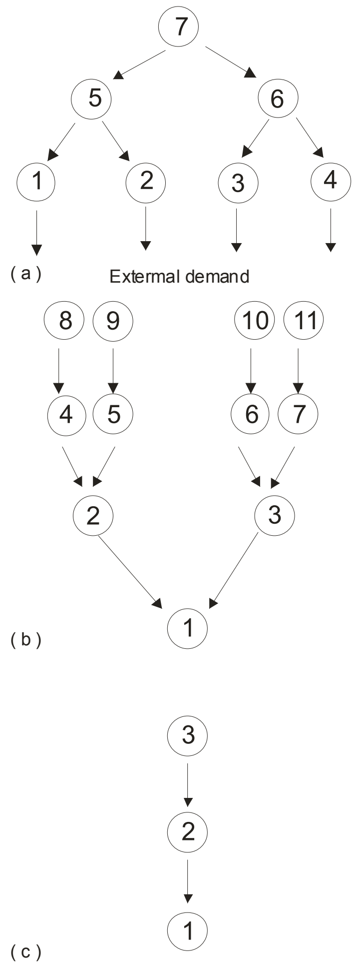


Figure 2.4 Multi-echelon systems: (a) Arborecent systems, (b) coalescent assembly systems, (c) series systems

2.2.1 Series Systems

When there are several locations and/or products, the items and relationships among them form a network; specifically a directed graph. The nodes represent the items and arcs depict the supply-demand relationships. It is important to distinguish several broad network structures.

The simplest structure is a series system, which is depicted in Figure 2.5. Here the items represent the outputs of successive production stages or stocking points along a supply chain. Each product is used as input to make the next one; or each location supplies the next one. Only the first item receives supplies from outside the system, and only the last one meets exogenous customer demands.

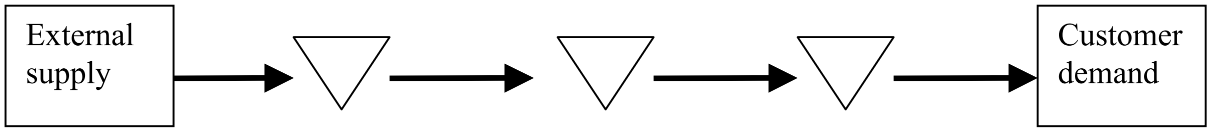


Figure 2.5 Series system

2.2.1.1 Description of Series Systems

Let us assume that there are *J* items numbered *j=1,2,…,J* from first to last, as in Figure 2.6. The items represent the outputs of successive production stages, or stocking points along a supply chain. Demand occurs only for item *J*, and an external source supplies item 1. All other supply links are internal; item 1 supplies item 2, item 2 supplies item 3, and so on. Another word for "item" is "stage". So, a series system is sometimes called as a *multistage system*.

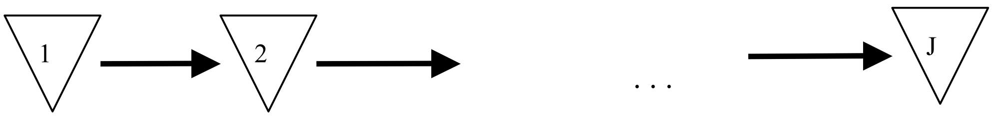


Figure 2.6. Series system

"Stock moves in discrete batches like in the EOQ model. An order is a decision to move a batch to any stage, whether the batch comes from the supplier or a prior stage. The stages do not make their own decisions. In other words, it is mostly assumed that information is fully centralized. However the system can be operated effectively in a decentralized manner. The order decisions must be coordinated; it makes no sense to order a batch to be sent to one stage, when the prior stage has insufficient inventory. The external supplier always has ample stock available. There are economies of scale in the form of fixed costs for all orders." (Zipkin P.,2000)

The problem is to find a good balance between fixed costs and inventory holding costs; like in most of the inventory problems.

2.2.1.2 Echelons and Echelon Inventories

The *echelon* of stage *j* (or *echelon j* shortly) comprises stage *j* itself and all downstream stages, i.e., all stages $i \geq j$. Echelon inventory at a stocking point includes all inventory that either is at that stocking point or has passed through that stocking point. The echelons of a four-stage system are indicated by rectangles in Figure 2.7, where *J=4*. This notion that is echelon concept, captures the supply-demand relationships in a useful manner: First stage *J* is its echelon, the external supplier and all the prior stages can be viewed as stage *J*'s supply process. Likewise, echelon *J-1* refers to the last two stages. This is another subsystem, whose supply process includes the earlier stages $i < J\text{-}1$. Continuing in this manner, the entire system can be viewed as a hierarchy of subsystems, the echelons, each with a clearly defined supply process.

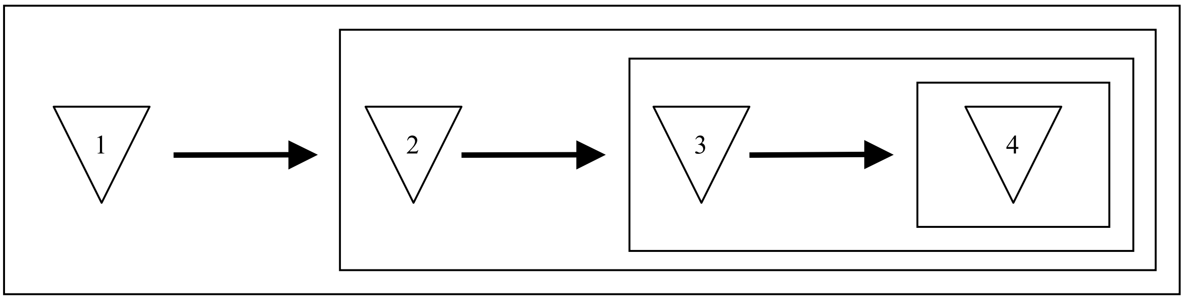


Figure 2.7 Echelon structure

Comparing echelon systems with conventional systems, Narasimhan et. al. found the following:

1. The same size and frequency of orders will be placed by the stocking point farthest from the customer, because the conventional and the echelon rates will be the same at this level.
2. The same total system inventory will be held. The larger lot sizes at the levels closer to the customer indicate that lots will not stay long at predecessor levels.
3. Inventories will be shifted toward the retail level.

In a multistage production process, suppose item 1 is a raw material. At each stage the material is transformed and another material is obtained. Numerous enhancements are added to the material at each stage, until final product *J* is obtained. This means a unit of item $i > j$ includes one of item *j*. The total system inventory of item *j* comprises not just local inventory of item *j*, but also the inventories downstream. In notation, the prime indicates the local quantity. When we define

$I_j^{'}(t)$: Local or installation inventory of item *j*.

$I_j(t)$: Echelon inventory of item *j* at time *t*.

The echelon inventory at time *t* for item *j* can be expressed by (2.1).

$$I_j(t) = \sum_{i \geq j} I_i'(t) \tag{2.1}$$

When it comes to costs; the echelon holding cost rate at a given inventory stocking point is the incremental cost of holding a unit of system inventory at that stocking point rather than at an earlier or predecessor point. For example, the echelon holding cost at a retailer would be the incremental cost of holding inventory at the retailer rather than at the regional warehouse. This relationship is expressed by Zipkin (2000) by the following equation:

$h_j^{'}$: Local inventory-cost rate for item *j*.

h_j : Echelon inventory-cost rate for item *j*.

$$h_j = h_j^{'} - h_{j-1}^{'} \tag{2.2}$$

where $h_0 = 0$. Assume that each $h_j > 0$.

The system wide inventory cost rate can be expressed as in Zipkin (2000), as follows:

$$\sum_j h_j^{'} I_j^{'}(t) = \sum_j h_j I_j(t) \qquad \text{for all } t \qquad (2.3)$$

Thus, the echelon inventories track stocks and their costs throughout the system just as well as the local inventories.

2.2.2 Base Stock Policy

The base stock system is a response to the difficulties of each echelon deciding when to reorder based only on demand from the next lower echelon.

The policy aims to keep the inventory position at the constant value *s* (base stock level). If the system starts with an inventory position (*IP*) less than *s*, or equivalently $IP(0) \leq s$, we immediately order the difference, so that $IP(0) = s$. Otherwise; or equivalently $IP(0) > s$, we order nothing until demand reduces $IP(t)$ to *s*. Once $IP(t)$ hits *s*, it remains there from then on. And this explains the name of *base-stock level* or *base-stock policy*.

Widely known *(r,q)* policy is continuous review inventory policy in which we order a quantity *q* whenever our inventory level reaches a reorder level *r*. *(r,q)* policy with batch size *q*=1 is called a *base-stock policy.* Such a policy makes sense when economies of scale in the supply system is negligible relative to the other factors. For example, when each individual unit is very valuable, holding and backorder costs clearly dominates any fixed order costs. Likewise, for a slow moving product (one with a low demand rate), the economy of the situation clearly rules out large batches. Also, there is a natural quantity unit for both demand and supply, and in terms of that unit it makes sense to set *q*=1.

Since *q* is fixed to 1, there is only one remaining policy variable, *r*. It is convenient to use the equivalent variable

s : base-stock level

$$s = r + 1$$

To sum up, a base stock policy is an inventory policy consisting of a reorder level *r*, and a base lot size equal to 1.

To find the optimal policy, the cost structure must be constructed first. The cost factors are defined as below:

k : fixed cost to place an order
h : cost to hold one unit in inventory for one unit of time
b : penalty cost for one unit backordered for one unit of time

All these factors are positive. For a general formulation, also define

$OF(r,q)$: order frequency
$I(r,q)$: average inventory on hand
$B(r,q)$: average backorder

Then total average cost will be

$$C(r,q) = kOF(r,q) + hI(r,q) + bB(r,q) \tag{2.4}$$

The goal is to determine the values of *r* and *q* that minimize (2.4). If we assume there is no fixed cost to place an order, this case is the one that base stock policy makes sense. As we defined, $s = r + 1$ is the base stock level. The average cost function becomes

$$C(s) = hI(s) + bB(s) \tag{2.5}$$

"Figure 2.8 illustrates this convex function. For small s, $I(s)$ *is negligible and* $B(s)$ *is nearly linear with slope -1; thus C(s) is nearly linear with slope –b in this range. For large s,* $B(s)$ *goes to zero, while* $I(s)$ *becomes linear with slope +1, so C(s) becomes linear with slope h."* (Zipkin P, 2000)

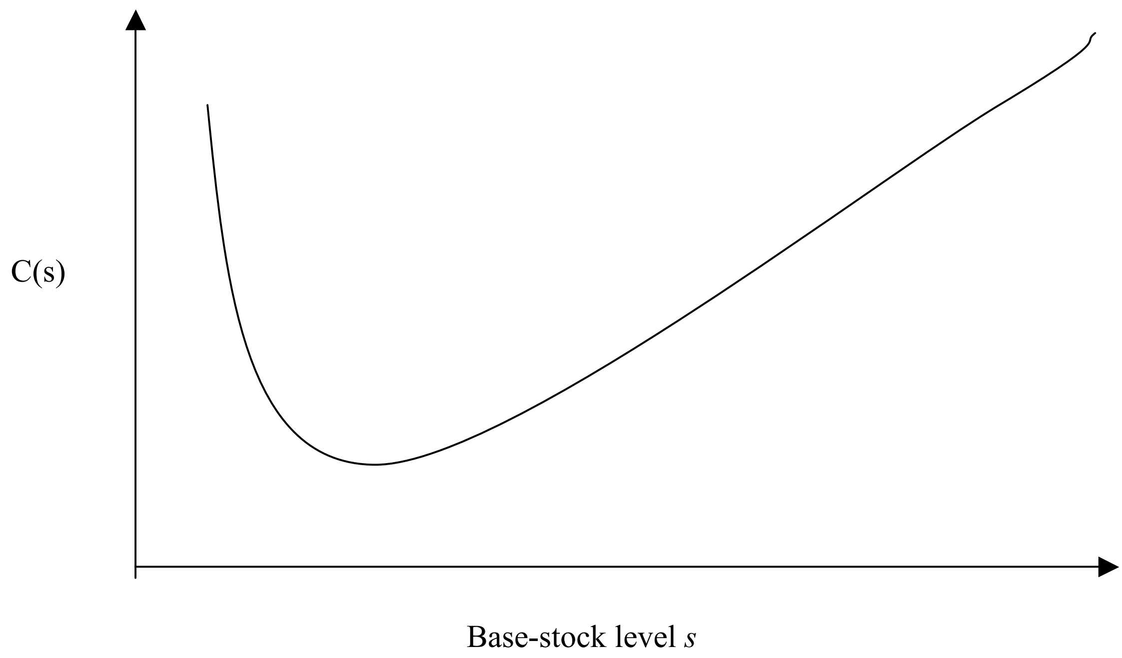


Figure 2.8. Average cost function

C(s) is expressed in another form by Zipkin (2000). The following function is defined.

$$\hat{C}(y) = h[y]^{+} + b[y]^{-} \quad (2.6)$$

for all real *y*. This is a nonnegative piecewise linear, convex function. Then if we denote the demand as D,

$$C(s) = E\left[\hat{C}(s - D)\right] \quad (2.7)$$

is defined again by Zipkin (2000). Since *C(s)* is a convex function, to minimize *C(s)*, all we have to do is to solve the equation $C'(s) = 0$.

Let us consider a series system with J stages. The numbering of stages follows the flow of goods. Stage 1 receives the supplies from an outside source. All other links are internal; each stage $j<J$ feeds its successor, stage $j+1$. Each stage has its own associated supply system. When stage $j-1$ sends a shipment toward stage j (or the source sends a shipment to stage $j=1$), the shipment must pass through stage j's supply system before arriving at j. Every stage can hold inventory. Demand is stochastic; it occurs only at the last stage, J. In a multistage system, there are two types of base-stock policy; local and echelon.

A *local base-stock policy* is a decentralized control scheme, where each stage monitors its own local inventory position, places orders with its predecessor, and responds to orders from its successor. Each stage j follows a standard, single-stage base-stock policy with parameter

$s_j^{'}$: local base-stock level for stage j

a nonnegative integer. The overall policy is described by the vector:

$$\mathbf{s}^{'} = (\mathbf{s}_j^{'})_{j=1}$$

The policy works as follows: Stage J monitors its own inventory position. It experiences demands and places orders with stage $J-1$ using a standard base-stock policy with base stock level $s_J^{'}$. Stage $J-1$ treats these incoming orders as its own demands, filling them when it has stock available and otherwise logging backorders to be filled later. It also follows a standard base-stock policy with parameter $s_{J-1}^{'}$ to determine the orders it places with stage $J-2$. This mechanism works like that until stage 1. Stage 1's orders go to external source, which fills them immediately.

An *echelon base-stock policy* is a centralized control scheme. We monitor the echelon inventory-order position. We determine the orders and interstage shipments so as to keep each inventory-order position constant. In other words, each stage j applies a base-stock policy. We can interpret a local base-stock policy in echelon terms. We need to define following:

B : system backorders

I_j : echelon inventory at stage j

IN_j : echelon net inventory at stage j

IOP_j : echelon inventory-order position at stage j

ITP_j : echelon inventory-transit position at stage j

h_j : echelon inventory holding cost rate at stage j

s_j : echelon base-stock level for stage j

The overall policy is described by the vector $\mathbf{s} = (\mathbf{s}_j)_{j=1}^{J}$.

An example can explain how the system works more precisely. Given **s**, if the s_j are nonincreasing, set $s_j' = s_j - s_{j+1}$ (where $s_{J+1} = 0$). In a two-stage system (*J=2)*, assume that $s_1 < s_2$. Suppose the system starts with no inventory at stage 1 and inventory s_1 at stage 2. Stage 2 immediately orders $s_2 - s_1$, but stage 1 has no inventory, so it backlog the orders. In fact, stage 1's echelon inventory position is already at its base-stock level s_1, so it orders only in response to subsequent demands. Thus the initial backlog at stage 1 remains there forever, that is, it remains at least $s_2 - s_1$. Stage 1 never hold inventory, and the inventory at stage 2 never exceeds s_1.

In conclusion every local-base stock policy is equivalent to some echelon base-stock policy. The expression for the average cost is defined by Zipkin (2000) as below:

$$C(s) = E[\sum_{j=1}^{J} h_j IN_j + (b + h_J')B] \tag{2.8}$$

An alternative way to organize these calculations is given by Zipkin (2000) as below:

$$\hat{C}_j(x|\ \mathbf{s}) = E[\sum_{i \geq j} h_i IN_i + (b + h_J')B | IN_j = x] \tag{2.9}$$

$$C_j(y|\ \mathbf{s}) = E[\sum_{i \geq j} h_i IN_i + (b + h_J')B | ITP_j = y] \tag{2.10}$$

$$\underline{C}_j(x|\ \mathbf{s}) = E[\sum_{i \geq j} h_i IN_i + (b + h_J^{'})B | IN_{j-1} = x] \quad (2.11)$$

These functions determine the best echelon base-stock policy: Set $\underline{C}_{J+1}(x) = (b + h_j^{'})[x]^{-}$.

For $j=J, J-1, ..., 1,$ given $\underline{C}_{j+1}$, compute

$$\hat{C}_j(x) = h_j x + \underline{C}_{j+1}(x)$$

$$C_j(y) = E[\hat{C}_j(y - D_j)]$$

$$s_j^* = arg\,min\{C_j(y)\}$$

$$\underline{C}_j(x) = C_j(\min\{s_j^*, x\}) \quad (2.12)$$

At termination, set $\mathbf{s}^* = (s_j^*)$ and $C^* = C_1(s_1^*)$. These quantities describe the optimal policy and the optimal cost. The recursion given (2.9) is called *the fundamental equation of supply chain theory*. It reflects the basic dynamics of the series systems." Zipkin (2000).

2.2.3 Myopic Policy

As a dictionary meaning myopic means, "short-sighted". In the context of this study it should be understood that the word "myopic" corresponds to an approach, which looks at only the "current" one-period problem. "*What we seek is a form of a planning-horizon theorem: you need only to solve a one-period problem to know you have the optimal decision rule for that period, regardless of the planning horizon of the actual problem. This result can be valid both deterministic and stochastic problems, stationary and non-stationary.*" (Porteus E., 2002)

As stated by Porteus, most dynamic inventory problems that are nonstationary and have long planning horizon provide better information about the near term than the long term. In general, when the size of the state space gets large, the problem becomes intractable. Bellman (1957) calls this phenomenon the *curse of* *dimensionality.* There are several approaches dealing with this curse. One of these approaches is to use an exploit special structure.

Myopic policy approach attempts to prove the form of the policy is simple. The problem of searching over a huge number of decision rules for each period reduces to the problem of finding a small number of parameters, such as the base stock level *S* of an optimal base stock policy. One of the most important and practical approach is to attempt to prove that a myopic policy is optimal: the decision rule that optimizes the return in a single- period problem with an easily identifiable terminal value function for that period is optimal.

2.3 Game Theory: Decision Making With Conflicting Objectives

Game theory studies the behavior of rational players in interaction with other rational players. Players are considered to be *rational* if they maximize their objective functions given their thoughts about the environment. They act in an environment where other players' decisions influence their payoffs.

Games can be classified according to number of players, number of strategies and behaviors of players. A classification is given in the Figure 2.9. There may be two or more players. When summation of the players' payoffs equals to zero, there is a zero-sum game. The number of strategies applicable for each player may be assumed to be finite or infinite.

Since von Neumann and Morgenstern's (1947) fundamental work on game theory, it has become tradition to distinguish between cooperative and noncooperative game theory. The main difference between these two branches lies in the type of questions they try to answer.

“Cooperative game theory is concerned with the kind of coalitions a group of players will form if different coalitions produce different outcomes and if these joint outcomes then have to be shared among the members.” (Jürgen E.,1993)

In contrast, “*noncooperative game theory* focuses on strategies players will choose. In the noncooperative game, attention is focused on the actions that each player is able to take, and how these actions jointly determine each player’s payoff.” (Friedman J.W.,1986) In noncooperative games players are unable to make contractual agreements with one another.

Noncooperative games are often expressed in a fashion that exposes each individual move a player can make; this is called the *extensive form*. Or they are expressed in a way that suppresses individual moves but highlights the overall plans, or strategies, which are available for players. This form is called the *normal form*.

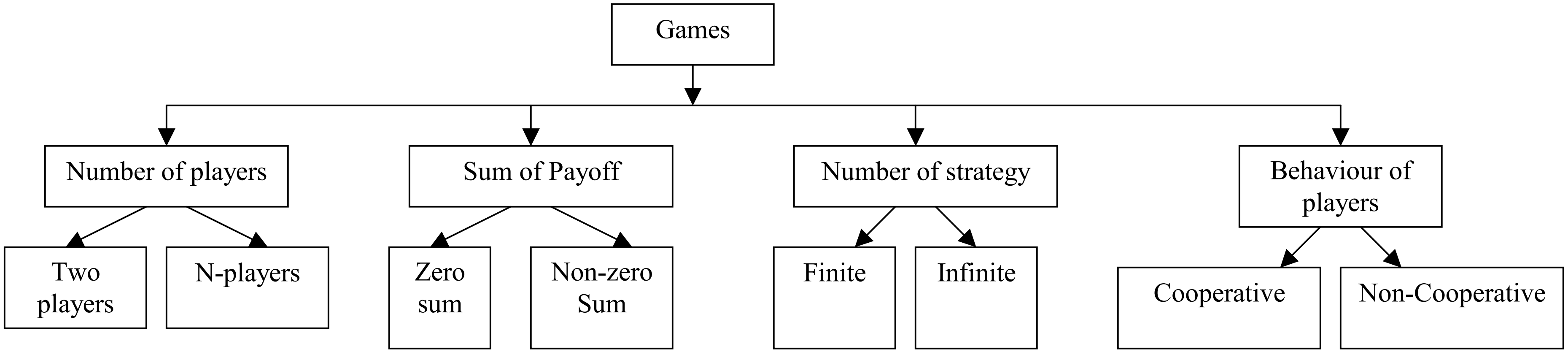


Figure 2.9 Classification of Games

2.3.1 Some Basic Concepts for All Kind of Games

In this section, after defining the basic elements of a noncooperative game, the solution concepts will be given. In addition to defining notation and some basic terms, assumptions usually made in noncooperative game theory will be also stated.

2.3.1.1 Game Setup

Every game has a set of rational decision makers, called *players*, whose decisions are central to the study of game. Players are indexed by $i=1,...,n$ and the set of players are denoted by N where $N = \{1,2,...,n\}$. Each player has strategies to apply for each possible situation of the game. A *strategy* can be defined as the pre-determined rules, which tell responses of each player to each possible situation in every period of the game. In some cases, it is perfectly easy to write down all the different possible situations, which may arise and specify what will be done in each case. Such a detail specification of actions is called a *pure strategy*. However in some cases, it is not so easy to choose the best strategy. For these cases, it is considered the probability of each strategy. The vector, which specifies the probabilities for each strategy, is called *mixed strategy* vector.

$s_i \in S_i$ denotes a *strategy* of player i, where S_i is called a *strategy space* for player i. The strategy space for player i includes all possible strategies that player i has.

Cartesian product of the individual strategy spaces, $S = S_1 \times S_2 \times ... \times S_n$, is called the strategy space for the game and denoted by S.

$s = (s_1, s_2, ..., s_n) \in S$ is called a *combination*, or more formally, a *strategy combination*, and it consists of n strategies, one for each player.

There is a payoff function for each player which is scalar valued. $P_i(s) \in \Re$ is the payoff function of player *i*. And also the payoff vector can be expressed as $\boldsymbol{P}(\boldsymbol{s}) = [P_1(s),...,P_n(s)] \in \Re^n$.

A player's strategy can be thought of as the complete instruction for which actions to take in a game. For example, a player can give his or her strategy to a person that has absolutely no knowledge of the player's payoff or preferences and that person should be able to use the instructions contained in the strategy to choose the actions the player desires. As a result, each player's set of strategies must be independent of the strategies chosen by the other players. The strategy choice by one player is not allowed to limit the feasible strategies of another player. Otherwise the game is ill defined and any analytical results obtained from the game are questionable.

Another important issue is to be clear concerning the information, which a player possesses in a game, and several kinds of information must be distinguished. First, concepts of complete information and incomplete information will be described. *Complete information* is obtained when each player knows (a) who the set of players is, (b) all actions available to all players, and (c) all potential outcomes to all players. By contrast if player *i* only knows his own payoffs, then there is *incomplete information*.

Secondly, the games in which each player knows exactly what has happened in previous moves are called *games with perfect information*. Games in which there is some uncertainty about previous moves are called *games with imperfect information*.

2.3.1.2 Solution Concept

The first existence proof for equilibrium points in *n*-person noncooperative games is due to Nash (1951). The concept of equilibrium point is a natural generalization of von Neumann's (1928) saddle point equilibrium for zero-sum games.

An *equilibrium point* is a combination, which is feasible (i.e. is contained in strategy space *S*) and for which each player maximizes his own payoff with respect to his own strategy choice, given the strategy choices of the other players. The equilibrium point was first introduced by Nash (1951). More formally:

DEFINITION 2.1. An equilibrium point is a combination $s^* \in S$ that satisfies $P_i(s^*) \geq P_i(s^* | s_i)$ for all $s_i \in S_i$ and for all $i \in N$. (Friedman J.W.,1986)

There is a set of common assumptions, which have to be made guarantee to find an equilibrium point. Before the assumptions some definitions will be given:

DEFINITION 2.2. A function $y = f(x)$ is *concave* if, for any x^1 and x^2 in the domain of the function, and any scalar $\lambda \in [0,1]$, if the following inequality holds: (Friedman J.W.,1986)

$$f\left[\lambda x^1 + (1-\lambda)x^2\right] \geq \lambda f(x^1) + (1-\lambda) f(x^2) \tag{2.13}$$

DEFINITION 2.3. A *compact set* in $\Re^n$ is a set that is both closed (i.e., contains its own boundary) and bounded (i.e., can be contained within a ball of a finite radius). A *convex set* has the property that the straight line segment connecting any two points in the set is also in the set. (Friedman J.W.,1986)

The common assumptions are:

ASSUMPTION 2.1. $S_i \subset \Re^m$ *is compact and convex for each* $i \in N$.

ASSUMPTION 2.2. $P_i(s) \in \Re$ *is defined, continuous, and bounded for all* $s \in S$ *and all* $i \in N$.

ASSUMPTION 2.3. $P_i(s|t_i)$ *is concave with respect to* $t_i \in S_i$ *for all* $s \in S$ *and all* $i \in N$.(Friedman J.W.,1986)

Assumptions 2.1 to 2.3 pertain to the structure of the game. There are additional conditions relating to the rules of the game and to the information conditions.

RULE 2.1. The players are not able to make binding agreements.
RULE 2.2. The strategy choice made by each player is made prior to the beginning of the play of the game, and without prior knowledge of the strategy choices made by other players. (Friedman J.W.,1986)

DEFINITION 2.4. A *game of complete information* is a game in which each player *i* knows all the strategy sets $S_j, j \in N$, each knows all the payoff functions $P_j(s), j \in N$, all players know that this information is in the possession of each of them, and all players know that everyone in the game knows all of these things. (Friedman J.W.,1986)

Rule 2.1. defines a noncooperative game. Rule 2.2. states that players may be thought of as choosing their strategies simultaneously; however this places no restriction on the structure of the game. In this study the games of complete information is examined and these games satisfy Assumptions 2.1 to 2.3 and Rules 2.1 and 2.2. As it is stated in Friedman J.W. 1986, all *n*-person noncooperative games of complete information that satisfy Assumptions 2.1 to 2.3, and Rules 2.1 and 2.2 have equilibrium points.

2.3.1.2.1 Best Reply Mappings and Their Relationship to Equilibrium Points

The *best reply mapping* can be called the optimal strategy mapping. Another name for this concept is *the best response function*.

DEFINITION 2.5. The *best reply mapping for player i* is a set-valued relationship associating each strategy combination $s \in S$ with a subset of S_i according to the following rule: (Friedman J.W.,1986)

$$r_i(s) = \left\{ t_i \in S_i \middle| P_i(s|t_i) = \max_{s_i' \in S} P_i(s|s_i') \right\} \tag{2.14}$$

In words, the strategy t_i is a best reply for player *i* to the strategy combination *s* if t_i maximizes the payoff of player *i*, given the strategy choices of the others. In general, t_i need not to be unique. The strategy combination $t \in S$ is a best reply to $s \in S$ if each component, t_i, of *t* is a best reply for player *i*.

DEFINITION 2.6. The *best reply mapping* is a set valued relationship associating each strategy combination $s \in S$ with a subset of *S* according to the rule $t \in r(s)$ if and only if $t_i \in r_i(s), i \in N$. Thus $r(s)$ is the Cartesian product $r_1(s) \times r_2(s) \times \ldots \times r_n(s)$. (Friedman J.W.,1986)

The best reply mapping provides a natural way to think about equilibrium points, because equilibrium points which satisfy the condition that s^* is an equilibrium point if and only if $s^* \in r(s)$. That is, an equilibrium point is a best reply to itself, and any strategy combination that is a best reply to itself is an equilibrium point. This argument can be expressed formally as follows:

LEMMA 2.1. Let $\Gamma = (N, S, P)$ be a noncooperative game. $s \in S$ is an equilibrium point of Γ if and only if $s \in r(s)$. (Friedman J.W.,1986, page 36)

2.3.1.2.2 Existence of Equilibrium

Non-existence of an equilibrium is potentially a conceptual problem since in this case it is not clear what the outcome of the game will be. However, in many games a Nash Equilibrium (NE) does exists and there are some reasonably simple ways to show that at least one NE exists. The simplest and the most widely used technique for demonstrating the existence of NE is through verifying concavity of the player's payoffs.

THEOREM 2.1. (Debreu 1952): Suppose that for each player the strategy space is compact and convex and the payoff function is continuous and quasi-concave with respect to each player's own strategy. Then there exists at least one pure strategy NE in the game.

In addition to the Theorem 2.1, it is possible to prove that a noncooperative *n*-person game has an equilibrium point according to the following theorem.

THEOREM 2.2. (Friedman,1986): Let $\Gamma = (N,S,P)$ be a game of complete information that satisfies Assumptions 2.1 to 2.3 and Rules 2.1 and 2.2. Then Γ has at least one equilibrium point.

2.3.1.2.3 Uniqueness of Equilibrium

It is quite useful to have a game with a unique NE from the perspective of generating qualitative insights. If there is only one equilibrium, then one can characterize equilibrium actions without much ambiguity. In the case of multiple equilibria, it is not clear that players can be expected to coordinate to play an equilibrium strategy combination. If they have no means of communication, even if each one selects a strategy associated with an equilibrium point, the resulting combination may not be an equilibrium point. If the players can communicate, then they could agree on a particular equilibrium point, but then we have a problem like to figure out which equilibrium point would be selected.

Hence it is important to show the uniqueness of equilibrium. Unfortunately, demonstrating uniqueness is generally much harder than demonstrating existence of equilibrium. There are several methods for proving uniqueness. No single method dominates; all may have to be tried to find the one that works. Furthermore, one should be careful to recognize that these methods assume existence; existence of NE must be shown separately.

One of the methods to prove the uniqueness of NE requires that Assumption 2.3 be modified so that the best reply mapping is a single-valued function.

ASSUMPTION $2.3'$: $P_i(s|t_i)$ is strictly concave with respect to $t_i \in S_i$ for all $s \in S$ and all $i \in N$.

Strictly concavity means that for any $s \in S$, any $t_i, t_i^{'} \in S_i$ with $t_i \neq t_i^{'}$, and any $\lambda \in (0,1)$, the following inequality holds:

$$P_i\left[s\left|\left(\lambda t_i + (1-\lambda)t_i^{'}\right)\right.\right] \geq \lambda P_i(s|t_i) + (1-\lambda)P_i(s|t_i^{'})$$

LEMMA 2.2. For a game $\Gamma = (N, S, P)$ satisfying Assumptions 2.1, 2.2, and $2.3'$, the set

$$\left\{t_i \in S_i \middle| P_i(s|t_i) \geq P_i(s|t_i^{'}) \; for \; all \; t_i^{'} \in S_i\right\} \qquad (2.16)$$

consists of exactly one element for each $s \in S$. (Friedman J.W.,1986 -page 43)

Second method to prove the uniqueness of NE is called *contraction mapping approach*. This approach will be defined briefly.

DEFINITION 2.7. Let $x, y \in \Re^m$. The *distance* from *x* to *y*, denoted either $d(x,y)$ or $\|x-y\|$ is $d(x,y) = \max_i |x_i - y_i|$. (Friedman J.W.,1986)

DEFINITION 2.8. Let $f(x)$ be a function with domain $A \subset \Re^m$ and range $B \subset \Re^n$. If there is a positive scalar $\lambda < 1$ such that for any $x, x^{'} \in A$, $d\left(f(x), f(x^{'})\right) \leq \lambda d(x, x^{'})$ then $f(x)$ is a contraction. (Friedman J.W.,1986)

In other words, a contraction leaves the images of the two points closer than were the original points themselves.

THEOREM 2.3. (Friedman, 1986): Let $\Gamma = (N, S, P)$ be a game of complete information that satisfies Assumptions 2.1, 2.2 and 2.3, and Rules 2.1 and 2.2. If the best reply function, $r(s)$, is a contraction, then Γ has exactly one equilibrium point.

Another method to prove the uniqueness of NE is called *univalent mapping approach*. This method by contrast to the previous one, does not restrict the best reply mapping, but it requires differentiability and places some other restrictions. Let $\overset{o}{S}$ denote the interior of *S*.

DEFINITION 2.9. $\Gamma = (N,S,P)$ is a smooth game if the following derivatives exist and are continuous on $\overset{o}{S}$: (Friedman J.W.,1986)

$$\frac{\partial P_i}{\partial s_{jk}}, \quad k=1,...,m, \quad j \in N \text{ and}$$

$$\frac{\partial^2 P_i}{\partial s_{ik} \partial s_{jl}}, \quad k,l=1,...,m, \quad i,j \in N \tag{2.17}$$

exist for all sequences of points in $\overset{o}{S}$ converging to s' on the boundary of *S*.

The second partial derivatives that exist everywhere in *S* for strictly smooth game are precisely the derivatives that appear in the Jacobian of the systems

$$\frac{\partial P_i}{\partial S_{ik}} = 0, \quad k = 1,...,m \quad i \in N \tag{2.18}$$

This Jacobian, denoted *J(s)*, is a square matrix with *m x n* rows and columns. Its elements are $\partial^2 P_i / \partial S_{ik} \partial S_{jl} \quad (i, j \in N \text{ and } k,l = 1,...,m)$. It is the Jacobian of the implicit form of the best reply function, which must obey a special condition everywhere on its domain, *S*.

DEFINITION 2.10. Let *A* be an $m \times n$ matrix. A is *negative quasi-definitive* if $B = A + A^T$ is negative definite. (Friedman J.W.,1986)

The uniqueness theorem requires that *J(s)* be negative quasi-definite for all $s \in S$, and the theorem allowing the proof of uniqueness is the following theorem:

THEOREM 2.4 (Gale–Nikaodo, 1965 univalence theorem): Let *f(x)* be a function from a convex set $X \subset \Re^m$ to $\Re^m$. If the Jacobian of *f* is negative quasi-definite for all $x \in X$, the *f* is one to one. (That is, if $f(x') = y'$, then, for all $x \neq x', f(x) \neq y'$).

THEOREM 2.5. Let $\Gamma = (N, S, P)$ be a smooth game of complete information that satisfies Assumptions 2.1, 2.2, and 2.3", and Rules 2.1 and 2.2 Assume that *J(s)*, the Jacobian of the implicit from of the best reply function, is negative quasi-definite for all $s \in \overset{o}{S}$, and that, for any $s \in S, r(s) \in \overset{o}{S}$. Then Γ has a unique equilibrium point. (Friedman J.W.,1986)

2.3.2 Game Theory In Supply Chain Analysis

As explained in the previous sections, game theory is a powerful tool for analyzing situations in which the decisions of multiple agents affect each agent's payoff. Since game theory deals with the interactive optimization problems, it plays an important role in supply chain analysis. Figure 2.10 illustrates the types of games which have found large applications in supply chain area. In this part, the most important game types for this study will be explained in detail.

Since the nature of everything is dynamic, dynamic models can reflect the real life applications in best way. Thus a significant portion of the Supply Chain Management (SCM) literature is devoted to dynamic models in which decisions are made over time. In most cases the solution concept for these games is similar to the backward induction used when solving dynamic programming problems.

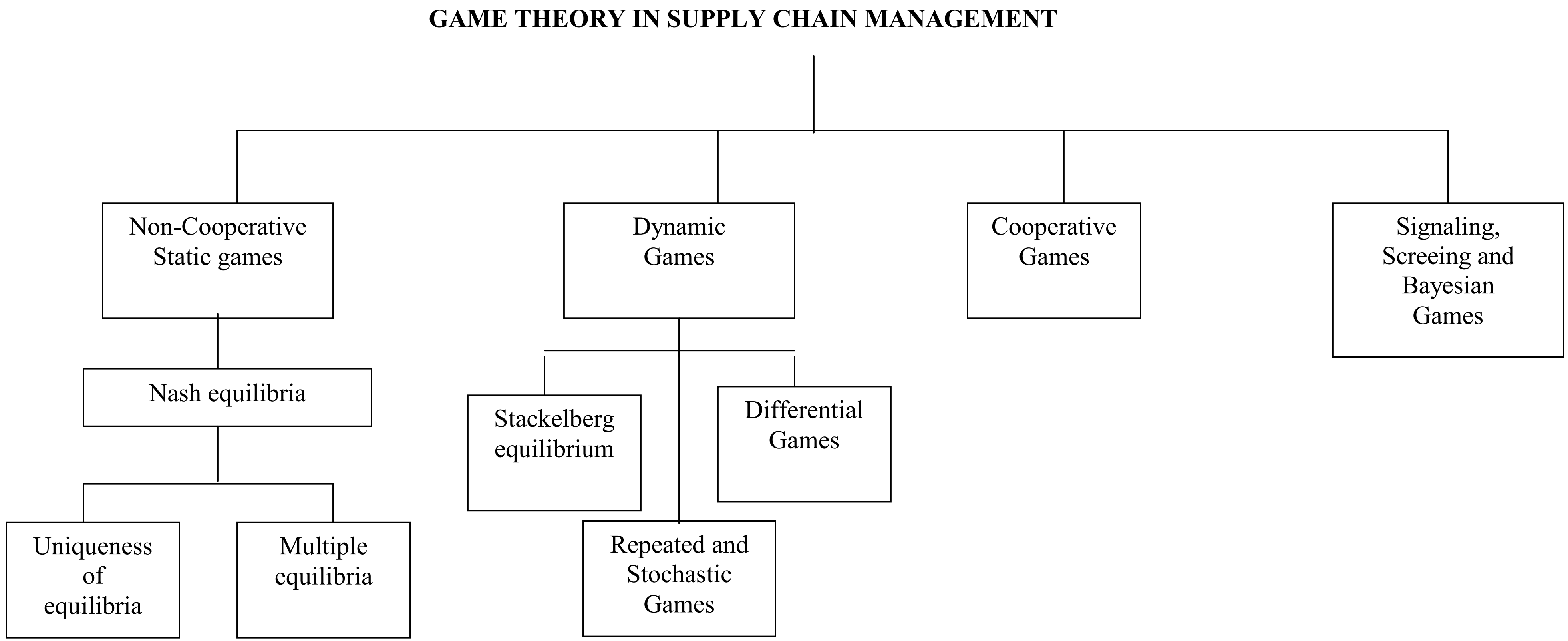


Figure 2.10 Classification of Game Theory in Supply Chain Analysis

One type of dynamic game arises when both players take actions in multiple periods. Since inventory models used in SCM literature often involve inventory replenishment decisions that are made over and over again, multi-period games should be a logical extension of these inventory models. Two major types of multiple-period games exist: without and with time dependence. In the following two sections, after defining multi-period or multi-stage games, two types of multi-period games, which are interested in this study, will be explained.

2.3.3 Multi-Period (Multi-stage) Game Setup

Although this is not always the case, it is often natural to identify the "stages" of the game with time periods, as in this study. In the first period of a multi-period game (period 0), all players $i \in N$ simultaneously choose actions from choice sets $A_i(h^0)$. It is assumed that $h^0 = \varnothing$ be the "history" at the start of the play. At the end of each period, all players observe the period's action profile. Let $a^0 \equiv (a_1^0, a_2^0, ..., a_n^0)$ be the period–0 action profile. At the beginning of period 1, players know history h^1, which can be identified with a^0 given that h^0 is trivial. In general, the actions player *i* has available in period 1 may depend on what has happened previously, so we let $A_i(h^1)$ denote the possible second-period actions when the history is h^1. Continuing iteratively, we define h^{k+1}, the history at the end of period k, to be the sequence of actions in the previous periods,

$$h^{k+1} = \left(a^0, a^1, ..., a^k\right) \tag{2.19}$$

and let $A_i(h^{k+1})$ denote the player *i*'s feasible actions in period *k+1* when the history is h^{k+1}. Let *K+1* denote the total number of periods in the game, with the understanding that in some applications $K = +\infty$, corresponding to an infinite number of periods; in this case the "outcome" when the game is played will be an infinite history, h^{∞}. Since each h^{k+1} by definition describes an entire sequence of actions from the beginning of the game on, the set H^{K+1} of all "terminal histories" is the same as the set of possible outcomes when the game is played.

In this setting, a pure strategy for player *i* is simply a contingent plan of how to play in each period *k* for possible history h^k. If we let H^k denote the set of all period–*k* histories, and let

$$A_i(H^k) = \bigcup_{h^k \in H^k} A_i(h^k) \tag{2.20}$$

a pure strategy for player *i* is a sequence of maps $\{s_i^k\}_{k=0}^{K}$, where each s_i^k maps H^k to the set of player *i*'s feasible actions $A_i(H^k)$ (i.e., satisfies $s_i^k(h^k) \in A_i(h^k)$ for all h^k). The sequence of actions generated by a profile of such strategies can be found as the following way: The period-0 actions are $a^0 = s^0(h^0)$, the period-1 actions are $a^1 = s^1(a^0)$, the period-2 actions are $a^2 = s^2(a^0, a^1)$, and so on. This is called the path of strategy profile. Since the terminal histories represent an entire sequence of play, we can represent each player *i*'s payoff as a function $P_i : H^{K+1} \to \Re$.

2.3.3.1 Repeated Games

In the multi-period game without time dependence, the exact same game is played over and over again; hence the term *repeated* games is used. Time dependence means that the payoff associated with a particular time period depends only on the actions of that time period. Since there is no time dependence, each period is considered independent each other.

A *repeated game* is a multi-period game in which the same (ordinary) game is played at each time period. The strategy for each player is a sequence of actions taken in all periods. In this case, there are no links between successive periods other than the player's memory about the actions taken in all previous periods. All players realize they will play a sequence of games and know that this is a common knowledge.

In each time period, the n players simultaneously select moves. These moves can be interpreted as strategies in a game confined to the current time period. That is, each time period has a payoff function associated with it and a strategy set (set of available moves) for each player. A repeated game is a game in which the circumstances of the initial time period (payoff functions and sets of available moves) repeat themselves identically in each succeeding period. Of course, the player does not merely consider each occurrence in isolation; he is interested in his overall payoff over the whole time horizon, and he considers strategies that direct the choices of each individual period's action from this global perspective.

Suppose that $\Gamma = (N,S,P)$ is game in strategic from satisfying Assumptions 2.1 to 2.3, and Rules 2.1 and 2.2; and imagine that the player will engage in this game in each of the time periods $t= 0,1,2,...,T$. Assume, too, that each player $i \in N$ discounts the future using the discount parameter $\alpha_i = 1/(1+r_i)$, where $r_i > 0$ is the discount rate. Letting s_i^t be the strategy chosen by player i in the tth play of the game, and $s^t = (s_1^t, s_2^t, ..., s_n^t)$, the discounted payoff stream to player i is

$$G_i(\sigma) = \sum_{t=0}^{T} \alpha_i^t P_i(s^t), \tag{2.21}$$

Allowing the possibility that $T=\infty$, the number of plays can be finite or infinite.

DEFINITION 2.11. A repeated game is $\Gamma = (N,S,P,\alpha,T)$ where (N,S,P) satisfies Assumptions 2.1 to 2.3 and Rules 2.1 and 2.2, $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ satisfies $\alpha_i \in (0,1]$ for all $i \in N$ and T is nonnegative. T can be finite or equal to $+\infty$. (Friedman J.W.,1986)

Something must be specified concerning the flow of information over the time horizon. The player is not actually committed to s_i^t until time t occurs. That is, even if a sequence $s_i^0, s_i^1, s_i^2, ...$ is decided at *time 0*, the player retains the right at any time t to select at that moment any elements of S_i^t. Two natural assumptions are:

(a) At each play of the game, Rule 2.2 holds, and after s_i^t is chosen for all $i \in N$, then all players are informed of s^t.

(b) s_i^t must be chosen for all $i \in N$ and $t = 0,1,\ldots,T$ before any player is informed of the choices made by the other players.

These two assumptions are stated as Rule 2.3 and Rule $2.3'$ respectively.

RULE 2.3: At each time t, the s_i^t $i \in N$, are chosen simultaneously; however, for $t>0$, s^τ, $\tau=0,1,\ldots t-1$ is known to all players. (Friedman J.W.,1986)

RULE $2.3'$: At each time t, the s_i^t $i \in N$ are chosen simultaneously. For $t>0$, player i knows s_i^τ but does not know $s_j^\tau (j \neq i), \tau = 0,1,\ldots,t-1$. The realized value of $P_i(s^t)$ is not revealed to the players until after all choices are made. (Friedman J.W.,1986)

Rules 2.3 and $2.3'$ have different implications regarding the way the players' strategy spaces should be modeled. Under Rule $2.3'$, a player accumulates no information as time passes therefore; her choice at any time t cannot be a function of the previous actions of the other players. Thus her strategy space is $\Sigma_i = \times_0^T S_i^t$. Let elements of Σ_i be denoted σ_i and let $\sigma = (\sigma_1,\ldots,\sigma_n) \in \Sigma$ where $\Sigma = \times_{i \in N} \Sigma_i$. Under Rule 2.3, the information revealed to each player is the past component game pure strategy combination that was chosen.

THEOREM 2.6 (Friedman, 1986): A repeated game $\Gamma = (N,S,P,\alpha,T)$ satisfying rule $2.3'$ has a noncooperative equilibrium. $\hat{\sigma} = (\hat{s}^0,\hat{s}^1,\ldots,\hat{s}^T)$ is a noncooperative equilibrium of Γ if and only if $\hat{s}^t$ is a noncooperative equilibrium for the game $\Gamma = (N,S,P)$, $t = 0,1,\ldots,T$.

Theorem 2.6. states that the repeated play of a Nash equilibrium forms a Nash equilibrium of the repeated game. Hence there is no existence problem for Nash equilibria in repeated games provided that the one-period game has an equilibrium.

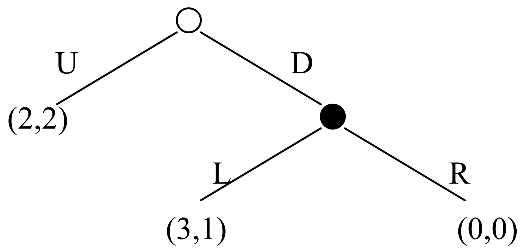
The concept of Nash equilibrium can be applied to all games however; many game theorists doubt that Nash equilibrium is the right solution concept for repeated games. The many–move nature of the repeated game introduces a need to refine the Nash equilibrium because certain conceivable Nash equilibria are not plausible. The fundamental refinement, called *perfect equilibrium*, applies to both single-period and many-period games and this concept will be discussed in the following section.

2.3.3.1.1 Perfect Equilibrium Points

There are certain equilibrium points that, on examination, appear implausible. In games in which each player has more than one move, some unsatisfactory equilibrium points can be interpreted as utilizing threats that are not credible. In games in strategic form or games of one move per player, an equilibrium point may appear unsatisfactory, because the equilibrium strategy of (at least) one player would be far from optimal if the game perturbed in a very slight way. To avoid such equilibria, a refinement of Nash equilibria called *perfect equilibrium* is used. A perfect equilibrium point is a Nash equilibrium that satisfies some additional properties.

"Selten (1965) was the first to argue that in general extensive games some of the Nash equilibria are "more reasonable" than others. He began with the example illustrated in Figure 2.11. This is a finite game of perfect information, and the solution is that player 2 should play L if his information set is reached, and so player 1 should play D. Inspection of the strategic form corresponding to this game shows that there is another Nash equilibrium, where player 1 plays U

and player 2 plays R. The profile (U,R) is a Nash equilibrium because, given that player 1 plays U, player 2's information set is not reached, and player 2 loses nothing by playing R. But Selten argued that this equilibrium is suspect. After all, if player 2's information set is reached, then, as long as player 2 is convinced that his payoffs are as specified in the figure, player 2 should play L. And if we were player 2, this is how we would play. Moreover, if we were player 1, we would expect player 2 to play L, and so we would play D. Thus, the equilibrium (U,R) is not credible because it relies on an empty threat by player 2 to play R. The threat is empty because player 2 would never wish to carry it out." (Fudenberg D. and Tirole J, 1991)



	L	R
U	2,2	2,2
D	3,1	0,0

Figure 2.11. An example

There are two versions of perfect equilibrium; *subgame perfect equilibrium* and *trembling hand perfect equilibrium*. The first one, which is used in this study, will be explained briefly in the following section.

2.3.3.1.2 Subgame Perfect Equilibrium

As it is explained in the previous section, to avoid incredible threads, "Selten (1975) has proposed that the strategies of the players should be best replies to one another for each subgame in the game, and a strategy combination obeying this condition is called a *subgame perfect equilibrium*." (Friedman J.W.,1986).

In a game, there are certain moments such that, from that moment onward, the reminder of the game is, itself, a game. Such a game is a *subgame* of the original game.

DEFINITION 2.12: Let $\Gamma = (N, S, P)$ be a noncooperative game. Then the *subgame* of Γ at time *t* with history h^t is described by

(a) set of players *N*

(b) the strategy spaces $S_{ih^t} = \times_{\tau=0}^{\infty} A_i^{\tau}$

(c) the payoff functions $P_{i_{h^t}} = \sum_{\tau=t}^{\infty} \alpha_i^{\tau} P_i^{\tau}\left(a_i^{\tau}\right)$

(d) the history h^{τ}.

This subgame is denoted $\Gamma_{h^t} = \left(N, S_{h^t}, P_{h^t}\right)$. (Friedman J.W.,1986)

For example, there are five subgames in the game consisting of two plays of the game in Figure 2.11; the original game and four additional games. Each additional subgame is defined to the four possible histories *(a_1,b_1), (a_1,b_2), (a_2,b_1),* and *(a_2,b_2)*.

Repeated games have perfect information at the beginning of each period. This suggests applying the concept of a subgame perfect equilibrium to repeated games to eliminate equilibria that rely on incredible threats. In repeated games a subgame begins after each history h^t. In other words, a subgame begins at the beginning of each one-period game.

DEFINITION 2.13: $\hat{\sigma} \in S$ is a subgame perfect equilibrium point of Γ if $\hat{\sigma}$ is an equilibrium point of Γ_{h^t} for $t = 0,1,...,\infty$ and for all $h^t \in S^t$. (Friedman J.W.,1986)

According to definition 2.13, $\hat{\sigma}$ is subgame perfect if it is an equilibrium point for any possible subgame of the original game. That is, the strategy combination induced on Γ_{h^t} by $\hat{\sigma}$ must be an equilibrium point in each subgame Γ_{h^t}, even if the subgame Γ_{h^t} would never be encountered when $\hat{\sigma}$ is actually played.

A repeated game strategy combination that has each agent in each one-period game, play a Nash equilibrium strategy of the one-period game will be subgame perfect. Hence, there is no existence problem for subgame perfect equilibria in repeated games.

2.3.3.1.3 Backward Induction

Backward induction can be applied to any finite game of perfect information, where finite means that the number of periods is finite and the number of feasible actions at any period is finite. And also it can be extended to infinite games of perfect information, where there is no last period from which to backward.

The algorithm begins by determining the optimal choices in the final period K for each history h^K; that is, the action for the player on move, given history h^K, that maximizes that player's payoff conditional on h^K is being reached. Then we work back to period K-1, and determine the optimal action for the player on move there, given that the player on move at period K with history h^K will play the action we determined previously. The algorithm proceeds to roll back, just as in solving decision problem, until the initial period is reached. At this point we have constructed a strategy profile, and it is easy to verify that this profile is a Nash equilibrium. Moreover, each player's actions are optimal at every possible history, which is called subgame perfect equilibrium.

2.3.3.2 Stochastic Games

In time-dependent multi-period games, players' payoffs in each period depend on the actions in the previous as well as current periods. Typically the payoff structure does not change from period to period (so called stationary payoffs). Clearly, such setup closely resembles multi-period inventory models in which time periods are connected through the transfer of inventories and backlogs. Due to this similarity, time-dependent games have found applications in SCM literature. We will only discuss one type of time-dependent multi-period games, which is called *stochastic games* or *Markov games,* due to their wide applicability in SCM.

The basic feature that distinguishes stochastic games from other games is that, in each time period, the particular payoff functions to be faced by the players are chosen randomly, and the exact probability distribution depends on the actions of the players in the previous period as well as on the particular payoff functions that were drawn at that time. At the start of each period one of the one-period games (one-shot game) is selected at random. The selection is made known to the players.

The idea behind a stochastic game is that the history at each period can be summarized by a state. Current payoffs depend on this state and on current actions. The state follows a Markov process; that is, the probability distribution on tomorrow's state is determined by today's state and actions.

The setup of the stochastic game is essentially a combination of a static game and Markovian decision process: in addition to the set of players with strategies which is now a vector of strategies, one for each period, and payoffs, we have a set of states and transition mechanism $p(k'|k,s)$, probability that transition from state k to state k' given action s. Transition probabilities are typically defined through random demand occurring in each period.

The probability distribution governing the choice among the one-shot games to be played in period $t+1$, depends on the actual game played in period t and on the actions selected by the players in period t. This probability distribution is called the *transition mechanism*. Since a one-period game corresponds to a state, to say that game s was encountered at time t is exactly the same as to say the state at time t was state s.

DEFINITION 2.14: The transition mechanism is $p(k'|k,s)$, which is the probability that the next state will be k' when the current state is k and the current action is s. (Friedman J.W.,1986)

ASSUMPTION 2.4: The set of states $\Omega = \{1,2,...,K\}$ is finite.

ASSUMPTION 2.5: The transition mechanism satisfies the following conditions:

a) $p(k'|k,s) \geq 0$ for all $k,k' \in \Omega$ and all $s \in S_k$

b) $\sum_{k' \in \Omega} p(k'|k,s) = 1$

DEFINITION 2.15: A stochastic game is denoted $\Gamma = (N,\Omega,\{S_k\},\{P_k\},p,\alpha)$ where (N,S_k,P_k) is for each $k \in \Omega$, a game satisfying Assumptions 2.1 to 2.3 and Rules 2.1 and 2.2. Γ has p as its transition mechanism, Γ satisfies Assumptions 2.4 and 2.5, and for each player $i \in N$, the objective function is expected discounted payoff using the discount parameter $\alpha_i \in [0,1)$ (Friedman J.W.,1986).

The difficulties inherent in considering non-stationary inventory models are passed over to the game-theoretic extensions of these models, so a standard simplifying assumption is that demands are independent and identical across periods. When only a single decision-maker is involved, such an assumption leads to a unique stationary solution (e.g. stationary inventory policy of some form: order-up-to, S-s, etc.). In game theory setting, however, things get more complicated; just as in the repeated games described in the previous section, non-stationary equilibria are possible. A standard approach is to consider just one class of equilibria –e.g. stationary- since non-stationary policies are hard to implement in practice and they are not always intuitively appealing. Hence, with the assumption that the policy is stationary the stochastic game reduces to an equivalent static game and equilibrium is found as a sequence of NE in an appropriately modified single-period game. Another approach is to focus on "Markov" or "state space" strategies in which the past influences the future through the state variables but not through the history of the play.

A *Markov Perfect equilibrium* (MPE) is a profile of Markov strategies that yields a Nash equilibrium in every proper subgame. Since the state captures the influence of past play on the strategies and payoff functions for each subgame, if a player's opponents use Markov strategies, that player has a best response that is Markov as well.

Suppose that there are $t+1$ periods $(t=0,\dots,T)$ where T can be finite or infinite. At date t, player i $(i=1,\dots,N)$ knows the history $h^t(a^0,\dots,a^{t-1})$ (where $a^\tau \equiv (a_1^\tau,\dots,a_n^\tau)$) and chooses an action a_i^t in a finite action set $A_i^t(h^t)$. A Markov strategy for player i may be conditioned on less than player i's information. Then the summaries or partitions of the history $\{H^t(h^t)\}_{t=0,\dots,T}$ which, for each date, are mappings from the set of histories into a set of disjoint and exhaustive subsets of the set of possible histories at that date. Suppose for instance that there are four possible histories, at the beginning of date 2. One partition is $H^2(h)=H^2(h')=A$, $H^2(h'')=B$ and $H^2(h''')=C$, in which the first two histories are lumped in the same summary. The partition can also be written $\{(h,h'),(h''),(h''')\}$.

While summarizing the history, a partition must not be too coarse. That is, at each date, the players must be able to recover the strategic elements of the ensuing subgame from the element of the partition to which h^t belongs.

DEFINITION 2.16: A partition $\{H^t(.)\}_{t=0,\dots,T}$ is *sufficient* if, for all t, h^t, and $\tilde{h}^t$ such that $H^t(h^t)=H^t(\tilde{h}^t)$, the subgames starting at date t after histories h^t, and $\tilde{h}^t$ are strategically equivalent:

(i) The action spaces (defined on conditionally on action taken from date t on) are identical.

(ii) The players' von Neumann-Morgenstern utility functions (or payoff functions) conditional on h^t, and $\tilde{h}^t$ are representations of the same preferences. (Fudenberg D. and Tirole J, 1991)

DEFINITION 2.17: The *payoff-relevant history* is the minimal sufficient partition. (Fudenberg D. and Tirole J, 1991)

In our example, if the subgames starting at date 2 after histories h, h', and h'' (but not h''') are strategically equivalent, the partition $\{(h,h'),(h''),(h''')\}$ is sufficient but not minimal. The coarsest sufficient partition is $\{(h,h',h''),(h''')\}$.

DEFINITION 2.18: A *Markov Perfect Equilibrium* (MPE) is a profile of strategies that are perfect equilibrium and measurable with respect to the payoff-relevant history. (Fudenberg D. and Tirole J, 1991)

THEOREM 2.7. (Fudenberg D. and Tirole J, 1991): Suppose either that $T < \infty$, or that $T = \infty$ and the objective functions are continuous at infinity. Then there exists an MPE.

Having given some information about multi-echelon inventory systems and game theoretic models, related work, which was done in the literature, will be given in the following section.

2.4. Related Work in Literature

There are several issues that determine the structure of an inventory model and make multi-echelon inventory systems challenging.

The first issue is the depot *demand process*. It is a unique issue of the multi echelon inventory systems. The demand process at the depot (upper stage) is the summation of retailers' (lower stages') ordering process. That is, the demand process at bases (lower echelon -retailer-), together with the ordering policy followed at each base; decide the demand process at the depot (higher echelon).

Second, the *type of items* affects the complexity of the analysis. Products with a limited lifetime are perishable items. If perishable items have been stored in the inventory system, after certain periods, those perishable items have to be discarded; which will increase the cost of the inventory system. Computer software and fashion items can be considered as perishable items. Repairable items also complicate the analysis of the inventory system. To deal with repairable items, the inventory system has to consider the repair facility, the waiting time and the service time of failed items. That is the reason why perishable or repairable items make inventory systems more difficult to analyze than do consumable items.

The third major issue in multi-echelon inventory systems is the *lateral transshipment* among retailers. It is possible that, a retailer runs out of stock and is expecting to receive its orders from the depot while customers who arrive at retailers are backlogged. At the same time, any other retailer may have inventory. It is reasonable to supply those waiting customers with other retailer's inventory. This is called "lateral transshipment". Lateral transshipment results in customer satisfaction. However, lateral transshipment has its transportation cost and greater coordination of retailer's inventory management is required. With lateral transshipment, the demand at any retailer should include the possible demand from other retailer's customers if they run out of stock.

The fourth issue is the assumption of backorders. When a retailer runs out of stock, some customers may wait and their demands are backordered while other customers may go to its competitors. With backorders allowed, the demand rate at a retailer will change according to its inventory level.

One of the earliest multi echelon models was developed by *Clark* and *Scarf (1960).* They assumed that the system structure consisted of several installations, *1,2,…,N*; with installation 1 receiving stock from installation 2, installation 2 receiving stock from installation 3, etc., and with demands originating at echelon 1 only. They also assumed that the cost of ordering and shipping from any installation to the next is a linear function of the amount shipped without any set-up cost, except at the highest echelon. Finally, positive lead times for shipping between echelons and full backordering is assumed.

The Clark and Scarf study was significant for several reasons. It was the first to depict the form of an optimal policy for a stochastic demand, multi-period, multi-echelon model. It also was important for introducing the concepts of *echelon stock* and implied shortage cost, which form the basis for the analysis of more complex systems. Clark and Scarf (1960) provides a theoretical foundation for much of the research in the increasingly important area of supply chain management. However, the assumptions of this model make it unlikely that it would be used to manage a real system. For instance, the assumption that all replenishment costs are proportional to the size of the replenishment order is somewhat unrealistic. When fixed ordering costs were applied at all locations, Clark and Scarf (1960) were able to provide only approximately optimal policies. Also the simple serial system considered, where items flow through echelons each with a single location, has limited applicability in practice since few actual retail distribution systems have this type of structure.

2.4.1. Fluctuating Demand in Single-Period Setting

As it is mentioned before, one of the issues which makes the system structure more complex and which makes also this study different from the existing literature is the demand process. In this section demand is assumed to be stochastic or fluctuating over time. The essential paper belongs to Samuel Karlin.

Karlin (1960) formulated a dynamic inventory model in which the demand distributions may change from period to period. In other words, the demand in each period is assumed to be independent but not necessarily identically distributed. Several costs are incurred during each period such as purchase cost, holding cost and shortage cost which are assumed linear. There is no fixed cost of ordering. Both the cases where excess demand is lost and backlogged were considered. Most studies of dynamic inventory models are concerned with determining the characteristics of the optimal policy which means the policy that minimizes the total expected costs in the future periods are properly discounted. If the cost functions of the model are suitably convex, and if demands that arise in successive periods are independent and identically distributed random variables with known distribution functions, then it is clear that the optimal policy in each period is characterized by a single critical number or at most two such numbers. The main importance of Karlin's paper is that he developed qualitative results describing the variation of the critical number which describes the optimal policy over time as a function of the demand densities in all future periods. Cost functions are assumed to be the same in all periods.

Iglehart and Karlin (1962) considered again an inventory model with a stochastic demand process but this study is different from Karlin's paper in that this time the distributions of demand in successive periods are correlated. The relationship of demands in successive periods regarded as a generalized Markov process and the demand process is described by discrete time Markov chain. Iglehart and Karlin also consider the case in which costs are nonconvex.

Models have been developed for the situation where the warehouse holds no inventory. In these cases, retailers use the warehouse merely as a distribution point through which items flow. The first study of this type of system was from Eppen and Schrage (1981). Eppen and Schrage's model is essentially the warehouse/store system of two-echelon inventory system. The assumption that the warehouse does not hold stock does not mean that the warehouse serves no purpose. By ordering centrally, more advantageous quantity discounts can be sought. There are also "statistical economies of scale" as observed by Eppen (1979) in which savings are achieved by aggregating orders rather than operating

N individual inventory systems. Eppen and Schrage show that unless the demand at the stores is perfectly correlated, the coefficient of variation of demand (σ/μ) for the aggregate system is smaller than for the demand at the individual stores.

Eppen and Schrage (1981) assume deterministic lead times from both the supplier to the warehouse and from the warehouse to the stores. Demand originates only at the store level and is assumed to follow a normal distribution with parameters allowed to differ between stores. Eppen and Schrage call their ordering policy an *(m,y)*, where every *m* periods, the inventory positions raised to a base stock of *y*. The warehouse must have enough stock to resupply the stores so that the probability of a stockout at each store is the same. This is shown likely to hold when the fixed cost of ordering from the warehouse is high and/or the coefficient of variation of demand at the stores is moderate.

Federgruen and Zipkin (1984a) consider the same problem but approach the solution differently. Their paper addresses the computational issues of the Clark and Scarf model. They showed that the optimal policy established by Clark and Scarf for the finite horizon problem, can be extended to the infinite horizon versions of the problem under the criterion of discounted cost and for long-term average cost. They also establish simpler computational formulas in the infinite horizon case.

Federgruen and Zipkin formulate the problem as a dynamic program with a state space of very large dimension. The dimension of the problem is at least $N+L$ (the number of outlets+the supplier to warehouse lead time), which makes it impractical to solve except for small values of *N* and *L*. To avoid the "curse of dimensionality" they showed that the model can be systematically approximated by a single location inventory problem.

Federgruen and Zipkin (1984c) consider other approaches to the problem of approximating optimal policies where they assume that the penalty and holding costs are proportional. The results of this paper deal with the problem of unequal coefficients of variations of demand at the stores. The approximation techniques and results are similar to those of Federgruen and Zipkin (1984a).

Zipkin (1989) considered an infinite-horizon problem with stochastic demands. This study is an extension of Karlin (1960) in that Zipkin developed an alternative and simpler approach. Zipkin proved the optimality of such policies for the average cost case.

*Song and Zipkin'*s paper *(1993)* is similar to Iglehart and Karlin (1962). However, Song and Zipkin offer simpler computations. And also their demand model is specialized and simpler to specify than Iglehart and Karlin's model because for each state Song and Zipkin's model requires the demand rate whereas the others' requires the full demand density. Many randomly changing environmental factors, such as fluctuating economic conditions and uncertain market conditions can have a major effect on demand. Song and Zipkin called the variables representing the environment as the state of the world (demand state); and they modeled the world as a continuous time Markov chain. When the world is in state *i*, demand follows a Poisson process with rate λ_i. Then they called the demand process as Markov-modulated Poisson process. The demand state can affect other parameters of the inventory system such as the cost functions.

Sethi and Cheng (1997), gives a generalization of classical inventory models that exhibit *(s,S)* policies. In other words, this paper is a generalization of Song and Zipkin (1993). In the model in Sethi and Cheng (1997), the distribution of demands in successive periods is dependent on a Markov chain. The model includes the case of cyclic or seasonal demands. Sethi and Cheng considered the presence of various constraints on ordering decisions and inventory levels which can be seen easily in real life applications. For example periods such as weekends and holidays is considered. Furthermore they extended their model to the cases in which there is no ordering periods storage and service level constraints. In this study both finite and infinite nonstationary problems are considered. It is proved that *(s,S)* policies are optimal for their model.

Chen and Song (2001) examine for the optimal policy with fluctuating demand in multi-period setting. In this study Chen and Song considered a multistage serial inventory system with Markov-modulated demand. Random demand arises at retailer stage. The demand distribution in each period is determined by the current state of an exogenous Markov chain. Backlogging is allowed. All costs incurred were assumed to be linear. They found a policy which minimizes the long-run average costs in the system. Chen and Song (2001) showed that the optimal policy is an echelon base-stock policy with state dependent order-up-to levels.

2.4.2. Time Series Models in Single Location Models

The papers that will be mentioned in this part, examine the demand process which are related among time periods. In other words, the assumption of independence is not relevant any more. Then it is suitable to use time series models.

The first and one of the fundamental papers is belong to Veinott. *Veinott (1965)* examine a multi-product, dynamic, nonstationary inventory problem. The demand process has no stationarity or independence assumptions. Unfilled demand can be backlogged. The costs are linear and also may vary over time. Veinott aimed in his study to determine an ordering policy that minimizes the expected discounted costs over an infinite time horizon. It is proven that the base-stock ordering policy is optimal and also that base stock levels in each period are easy to calculate. Veinott's paper is quite significant in that; it was the first study which attempt to develop sufficient conditions ensuring that the optimal policy takes a simple form and to compute the parameters of that policy easily. This approach was going to be called myopic policy. The paper derived the solutions under the assumption of independence of demand across periods. Also the case of dependent demands over time was examined. In this case, the distribution of demand in period *i* depends upon the past history of the system.

Johnson and Thompson (1975) examined the myopic replenishment rules for periodic inventory systems operating under certain dependent demand process. This paper shows that the myopic policies are optimal under certain demand processes even when demand is not stochastically increasing over time. The demand process was characterized by means of the models described and analyzed by Box and Jenkins.

Miller (1986) assumed in his study that, the expected value of demand is given by exponential smoothing formula and that uncertainty is multiplicative. Miller considered a finite horizon inventory model with linear holding, shortage, and ordering costs. The demand random variables are dependent, and average demand is described by an exponential smoothing formula. This model is formulated as a two-state variable (inventory level, weighted past demands) dynamic program. It is proven that two-state variable dynamic program can be reduced to one state. Also it is shown that dependent demand model orders less than or equal to the amount ordered by a comparable independent demand model.

Erkip, Hausmann and Nahmias (1990) extended the studies in which various time series models used for the demand process in multi location models. The authors consider an extension of the Eppen and Schrage (1981) model to the case of correlated demands over time. This work differs from a number of other studies in that it is allowed that item demands to be correlated both across warehouses and also in time. They observed both high correlations between successive monthly demands (around 0.70) and correlations between demands for an item at different locations (also about 0.70) in a given time period. They derive an explicit expression for the optimal safety stock as a function of the level of correlation through time. Erkip et.al's analysis requires two assumptions:

1. the allocation assumption
2. the equal coefficient of variation assumption

2.4.3. Competition in One-Period Setting

Eppen (1979) considered a multi-location newsboy problem with linear holding and penalty cost functions at each location with normal demand. He assumed *N* identical retail outlets that order independently according to a simple order-up-to point model obtained by minimizing one period holding and penalty costs, and derived an expression for the expected cost each facility. The model is used to demonstrate: the expected holding and penalty costs in a decentralized system; the magnitude of the savings depends on the correlation of demands, and if demands are identical and uncorrelated, the costs increase as the square root of the number of consolidated demands.

In many production/inventory related decision problems the presence of several decision makers with competing objectives can be observed. When more than one decision maker is involved in a decision situation, the classical optimization concepts may no longer applicable. Instead, game theoretic ideas may be suitable to analyze the possible decision making strategies of different players. Using game theory in inventory problems provides a better analysis when a decision maker's problem cannot be treated in isolation from the others' decisions and objectives.

Parlar's article is the first one which analyzes an inventory problem using game theory. *Parlar (1988)* developed a two-firm competitive newsboy problem where the firms face independent random demands. He examined the substitutable product problem using concepts from two person continuous games. The decision makers were called players who choose order quantities of the products. As the demands are random and substitution may exist, the objective function of each player depends on the decision variables chosen by the players. The players are assumed to have knowledge of the demand densities and all other information related with the game and each other. He proved the existence of a unique Nash equilibrium in inventory levels when demands are represented by strictly increasing and continuous cumulative distributions and showed that the expected profit when the firms cooperate exceeds the sum of the competitive profits.

Lippmann and McCardle (1997) considered a competitive version of classical newsboy problem in which a firm must choose an inventory or production level for a perishable good with random demand. They also investigate the effect of competition on inventory. Prior to realization of demand, firms (players) choose an inventory level of perishable good to be sold at a predetermined price. Each firm's strategy is its level of inventory. There is no price competition because each charges the preset price. Instead, competition among the firms emanates from the fact that they share the industry demand and that an increase in one firm's inventory stochastically reduces the other firm's sales. Aggregate industry demand was allocated among the firms and individual firm demands were considered to be correlated in each other. Lippmann and McCardle proved in their study that there is an equilibrium and showed that each firm's equilibrium inventory ordering strategy is represented as fractile of the effective demand distribution.

2.4.4. Decentralized Cost Structure

In multi-echelon inventory system environment, inventory control policies may take one of two approaches. One approach is local inventory control policy where each warehouse is responsible for their own stocking policies, independent of each other. This approach can be stated as decentralized control scheme. The second approach is echelon inventory control policy where inventory control parameters are determined simultaneously, taking into account the interrelationship between the depot and the warehouses. This approach can be called as centralized control scheme.

Muckstadt and Thomas (1980) compare the performance of these two approaches. They concluded that it may be worthwhile to implement the optimal multi-echelon model, or in other words centralized control policy.

Hausmann and Erkip (1994) considered a multi-echelon inventory system which has a central depot and *M* warehouses. At each location, a continuous review inventory policy *(S-1,S)* is used. Hausman and Erkip examined the amount of suboptimization, which can occur if multi-echelon inventory systems are managed as independent single echelon systems or in other words, managed locally. They studied Muckstadt and Thomas's optimal multi-echelon policy and an improved version of the local inventory control model.

Axsäter (2001) considered two-echelon inventory system consisting of a central warehouse and a number of retailers and gave a framework for decentralized control. The final demand occurs at the retailer level. In this paper a cost structure that can be used for decentralized control of a multi-echelon inventory system is provided. This cost structure means that the warehouse, in addition to its local costs, pays a penalty cost for a delay at the warehouse to the retailer facing the delay. By minimizing its local costs according to the suggested cost structure, an installation can reduce its costs. Then the total system costs are reduced by the same amount.

Lee and Whang (1999) mainly examined the incentive problems arising in the supply chain system when each decision maker maximizes his/her performance metric and discussed performance measures in decentralized supply chain. This paper also discussed the cost conservation, incentive compatibility, and informational decentralizability properties of the supply chain system. They concluded by giving a particular performance measurement scheme.

Chen (1999) considered a series system, which consists of N divisions. Customer demand occurs at the last stage, again. The demand is assumed to be independent in different periods and has the same probability distribution. Backlogging is allowed. The paper considers two models for the system. First the model for centralized control scheme is identified. Each installation manager has local information. It is proved that for the centralized model, for each installation base-stock policy with a determined level is optimal. Secondly, each installation is considered as cost centers. Each manager makes his/her decision based on the costs incurred in his/her own installation. It was found that decentralized decision-making is very beneficial when the owner of the firm does not have perfect knowledge about the demand distribution or when the firm faces a fluctuating demand environment.

Among applications of stochastic games, one of the papers is written by *Cachon and Zipkin (1999)*, which analyzed a two-echelon game with the wholesaler and the retailer making stocking decisions. Cachon and Zipkin investigate a two-stage serial supply chain with stationary stochastic demand in each period. Inventory holding costs are charged in each stage, and each stage may incur a consumer backorder penalty cost. It is considered two games such as local and echelon games. In both, the stages independently choose base stock policies to minimize their costs. The games differ in how the firms track their inventory levels. The policies chosen under this competitive regime is compared to those selected to minimize total supply chain costs. Cachon and Zipkin showed that the games have a unique Nash equilibrium, and it differs from the optimal solution. Competition reduces efficiency. And also the authors showed that the system optimal solution can be achieved as a Nash equilibrium using simple linear contracts. Cachon and Zipkin (1999) also discuss Stackelberg equilibria.

2.4.5. Stochastic Games and Dynamic Oligopoly

Kirman and Sobel (1974) developed a dynamic model of oligopoly and discussed the existence and characteristics of optimal polices for firms in such a model. Inventories play an essential role as the link between successive periods. This leads to develop ideas of sequential policies and of an equilibrium point for a stochastic game. Inventories may be held from one period to the next, thus a firm's position in the next period depends on its action in the present one. So, the authors express the oligopoly model as a stochastic game. Demand was considered to be stochastic. The existence of equilibrium is showed. It is the first paper which considers a dynamic oligopoly model as a stochastic game.

In the pair of papers *Maskin and Tirole (1988a)* and *Maskin and Tirole (1988b)*, the authors present a theory of oligopolistic firms that behave over time. They present a class of infinite horizon sequential duopoly games. Firms want to maximize their sum of single-period profits, and the authors characterized the perfect equilibria. The dynamic programming equations associated with an equilibrium; Markov perfect equilibrium is derived. The importance of this paper is that the authors introduced the concept of Markov strategies and Markov perfect equilibrium for dynamic oligopoly problem. Maskin and Tirole (1988a) considered quantitiy competition whereas Maskin and Tirole (1988b) considered price competition.

Maskin and Tirole (1995) defined Markov strategy and Markov perfect equilibrium for games with observable actions, in other words games of perfect information or games of perfect monitoring. In such games all players know the history of the game before making a decision. The authors considered a sequential game (stochastic game) and defined the Markov strategies which only depend on payoff-relevant past states of the system. And also the authors defined Markov perfect equilibrium (MPE) for the game. It is quite pragmatic to use MPE for the solution because of the curse of dimensionality. They proved that MPE is successful in reducing a large multiplicity of equilibria in dynamic games, and thus enhancing the predictive power of the model. MPE, by preventing non-payoff-relevant variables from affecting strategic behavior, has allowed researchers to identify the impact of state variables on outcomes. A second relevant reason for focusing on MPE is that Markov strategies substantially reduce the number of parameters to be estimated in dynamic econometric models.

CHAPTER THREE
CENTRALIZED CONTROL SCHEME

As it is mentioned in previous sections, there are several ways to manage a supply chain inventory. Supply chain members may desire to optimize the whole system performance, and they try to minimize the overall system cost. In other words they prefer a *centralized control scheme*. Although centralized control of supply chain is preferred by many organizations, this approach may not yield an optimal solution which minimizes each supply chain member's own costs. In that case, firms may behave more personal and prefer a control scheme so as to minimize their own costs rather than the overall system cost. This competitive approach is a *decentralized control scheme*.

In the following section, the problem that has been focused on will be described. Chapter 3 includes the centralized solution for the described system. The basic formulation is given by Zipkin (2000). The defined problem is formulated by using this basic formulation.

The part of the study that differs from the existing literature is related with the demand process. Customer demand, which occur at retailer, is assumed to be stochastic, independent across periods and non-stationary. Most of the studies in literature, which assume that demand is non-stationary, express the system via time series modeling. What is different from them in our study is that, game theory is used and the problem is modeled using two different game setups. Finally solution concepts are discussed for each game setup. In this chapter the problem is considered and analyzed in a centralized manner, whereas in the following chapter, the decentralized control scheme and game behavior of the firms is considered.

3.1. Problem Description

In this study, we investigate a two-stage serial supply chain with nonstationary demand. We consider a one-product inventory system with one supplier and one retailer. Stage 2 refers to the retailer whereas stage 1 refers to the supplier. The echelon structure for the problem, which will be analyzed in this study, is illustrated in Figure 3.1.

There is no fixed cost for placing or processing an order. Purchasing cost does not change as the quantity of order changes; in other words there is no quantity discounts. The supplier incurs a holding cost per period for each unit in its stock. As it is required by echelon relationships, the retailer also incurs a holding cost, which is considered in addition to the supplier stage.

Backordering is allowed and all backordered demands are ultimately filled. Both the retailer and supplier incur a backorder cost.

There is a fixed lead time for shipments from the external source to the supplier, and also a lead time from the supplier to the retailer.

Time is divided into discrete periods. The sequence of events during a period can be summarized as follows:

(1) A replenishment order -if any- is placed

(2) Replenishments arrive

(3) Demand occurs during the period

(4) Inventory and shortage costs are charged at the end of the period

Each firm uses a base-stock policy. At the beginning of each period, the firm orders a sufficient amount that increases the inventory position to the predetermined optimal inventory level.

The goal is to analyze non-cooperative game behavior and search an optimum inventory policy for two echelon supply chain under non-stationary demand.

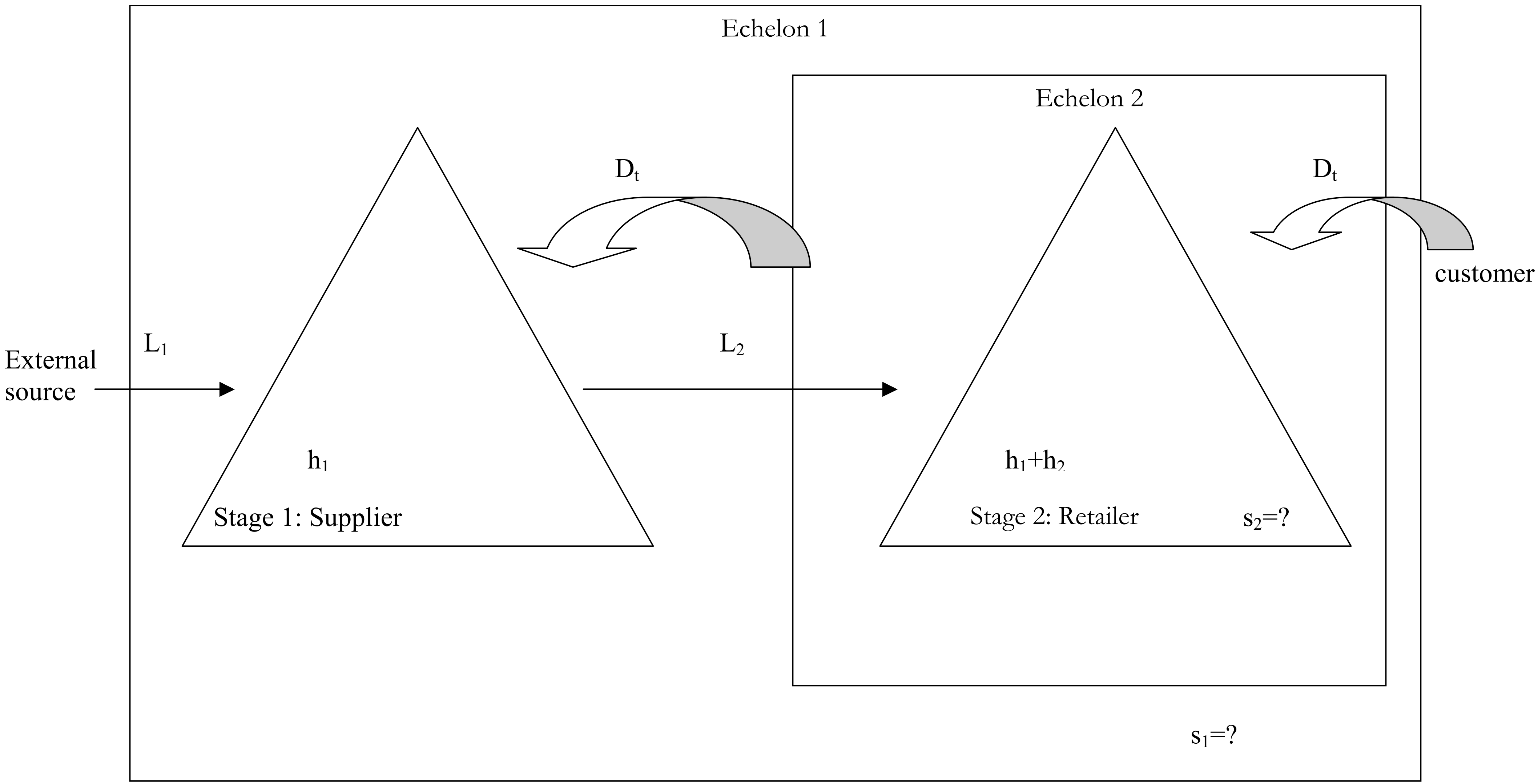


Figure 3.1 Echelon structure for defined problem

Base Stock Policy Under Centralized Control Scheme

In a centralized system, all decisions are made at one particular point and all the information flows to that point. The decisions are made by one stage and implemented by all stages in the network. While studying the centralized system, the planning horizon is considered in two parts as suggested by Zipkin (2000). In the first part, in which finite horizon is considered, customer demand is assumed to be nonnegative and independent across periods but not necessarily identically distributed. However, in the second part of the horizon, which is considered to be infinite, random demand is assumed to be independent and identically distributed.

Formulation

We focus on a two-stage series system ($J=2$). Information and control are fully centralized. Shipments sent to stage j arrive after a lead time of L_j.

$d(t)$: demand at time t, $t=0,\dots,T-1$

$\hat{x}_j(t)$: echelon net inventory at stage j at time t, $t=0,\dots,T$

$x_j(t)$: echelon inventory-transit position at stage j at time t

Demand during period t is a random variable. Echelon net inventory at stage j equals the difference of echelon inventory at stage j and backorders whereas echelon inventory-transit position at stage j equals the summation of echelon net inventory and inventory in transit to stage j.

$z_j(t)$: shipments sent to stage j at time t

$z_j(t)$ is a shipment; not an order. The distinction is not important for the supplier (stage 1), but it does matter for the retailer (stage 2). The order amount may be different from the actual shipments, if there is a stockout. Under centralized control there is no place for stage 2 orders that stage 1 cannot fill immediately. Only actual or feasible shipments count.

The state of the system can be characterized by two state variables which are $x_j(t)$ and $z_j(t)$. The echelon inventory-transit position at stage *j* at time *(t+1)* equals the echelon inventory-transit position at stage *j* at time *t* plus shipments sent to stage *j* at time *t* minus demand at time *t*. Then the dynamics can be expressed as follows:

$$x_j(t+1) = x_j(t) + z_j(t) - d(t) \tag{3.1}$$

We desire to determine the inventory position after we order. Inventory level after ordering at time *t* would be equal to echelon inventory-transit position plus shipments sent to stage *j* at time *t*.

$y_j(t)$: inventory level after ordering at time *t*.

$$y_j(t) = x_j(t) + z_j(t) \tag{3.2}$$

Then inventory position at time *(t+1)* will be

$$x_j(t+1) = x_j(t) + z_j(t) - d(t)$$
$$= y_j(t) - d(t) \tag{3.3}$$

If we consider the lead time $L_j > 0$ then we should consider the orders in transition; that means orders placed but not received yet.

$z_{jl}(t)$: shipment in transit to stage *j* sent *l* periods ago, i.e. at time *t-l*, $l = 1, \ldots, L_j - 1$

Shipments due to stages will be represented by a vector $\mathbf{z}_j(\mathbf{t})$

$$\mathbf{z}_j(\mathbf{t}) = (z_{jl}(t))_l$$

$$\mathbf{z}_j(\mathbf{t}+1) = \lfloor z_j(t), z_{j1}(t), \ldots, z_{j,L_j-1}(t) \rfloor$$

(3.4)

So the echelon inventory transit position at time t will be:

$$x_j(t) = \hat{x}_j(t) + \sum_{l>0} z_{jl}(t) \tag{3.5}$$

Because of the lead-time, the state of the system can be characterized by two state variables which are $\hat{x}_j(t)$ and $z_{jl}(t)$. The dynamics are

$$\hat{x}_j(t+1) = \hat{x}_j(t) + z_{j,L_j-1}(t) - d(t) \tag{3.6}$$

The cost structure is as follows:

$c_j(t)$: unit variable order cost at time t, for stage j

$h_j(t)$: echelon holding cost per unit for stage j time t.

$$h_j(t) = h'_j - h'_{j-1}$$

$b(t)$: system unit backorder-penalty cost rate at time t.

Since we write a dynamic programming equation, future costs are discounted at rate γ, so we also define a discount factor.

γ : discount factor ($0<\gamma<1$)

The inventory holding and backorder-penalty cost is assessed on $x(t)$ and it is represented by $\hat{C}(t,x)$, where

$$\hat{C}(t,x) = h(t)[x]^+ + b(t)[x]^- \tag{3.7}$$

Since we define $y(t)$ as the inventory position after ordering, $C(t,y(t))$ measures the expected inventory-backorder cost. $C(t,y(t))$ can be called as one-period cost function. This function is defined by Zipkin (2000) as follows:

$$C(t,y) = E\left[\hat{C}(t+1, y-d(t))\right] \tag{3.8}$$

For two-stage system if we define;

$TSC(t)$: Total shipment cost at time t

$IC(t)$: Inventory and backorder cost at time t

Then,

$$TSC(t) = c_1(t)z_1(t) + c_2(t)z_2(t) \quad (3.9)$$

From the equation (2.5), the inventory and backorder cost at time t can be found as below:

$$IC(t) = h_1(t)\hat{x}_1(t) + h_2(t)\hat{x}_2(t) + [b(t) + h_2'(t)].[\hat{x}_2(t)]^-$$

$$= h_1(t)\hat{x}_1(t) + \hat{C}_2(t, \hat{x}_2(t)) \quad (3.10)$$

where $\hat{C}_2(t,x) = h_2(t)[x] + [b(t) + h_2'(t)].[x]^-$

Solution

Dynamic programming is a systematic technique for multi-stage problem solving. Since it is a useful mathematical technique for making a sequence of interrelated decisions, it has been extensively used. There are several interrelated decisions to make in the problem that we have defined. After formulating as a dynamic programming problem, the base-stock level for the defined problem can be found by working backward. Dynamic programming formulation of the defined problem will be given in this section.

Figure 3.2 illustrates the time horizon for T periods; when there is a lead time for the shipments both to the supplier and to the retailer.

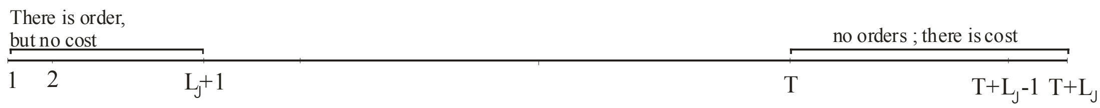


Figure 3.2 Time horizon with lead time

When we assume that shipments sent to stage j arrive after a lead time of L_j, the last shipment to stage 1 is sent at time *(T-1)* and received by stage 1 at time

$(T+L_1-1)$. Then stage 1 sends stage 2's shipments at time $(T+L_1-1)$, and this shipment is received by stage 2 at time $T+L_1+L_2-1$. Hence, the horizon extends to time $T+L_1+L_2-1$; demands continue to occur and costs to accrue until then. We can formulate the dynamic program as follows: Define;

$\tilde{V}(t,\hat{x}_1, \mathbf{z}_1, \hat{x}_2, \mathbf{z}_2)$: minimal expected discounted costs in periods $t, t+1, ..., T$, assuming period t begins with $\hat{x}_j(t)=\hat{x}_j$ and $\mathbf{z}_j(\mathbf{t})=\mathbf{z}_j$

The terminal value will be the discounted cost, which will be found from infinite horizon problem. Denote the discounted cost for infinite horizon problem by $\mathrm{V}(t,\hat{x}_1, \mathbf{z}_1, \hat{x}_2, \mathbf{z}_2)$. Then;

$$V(T+L_1+L_2-1,\ \hat{x}_1, \mathbf{z}_1, \hat{x}_2, \mathbf{z}_2) = \mathrm{V}(t,\hat{x}_1, \mathbf{z}_1, \hat{x}_2, \mathbf{z}_2) \tag{3.11}$$

and for $t < T+L_1+L_2-1$, the $\tilde{V}(t,\hat{x}_1, \mathbf{z}_1, \hat{\mathbf{x}}_2, \mathbf{z}_2)$ satisfy the recursion

$$\begin{aligned}&\tilde{V}(t,\hat{x}_1, \mathbf{z}_1, \hat{\mathbf{x}}_2, \mathbf{z}_2) = min\{c_1(t)z_1 + c_2(t)z_2 + h_1(t)\hat{x}_1 + \hat{C}_2(t,\hat{x}_2) + \\ &\gamma \mathrm{E}\left[V(t+1, \hat{x}_1 + z_{1,L_1-1} - d(t), (z_1, z_{11}, ..., z_{1,L_1-2}), \hat{x}_2 + z_{2,L_2-1} - d(t), (z_2, z_{21,...,}z_{2,L_2-2}))\right]: \\ &z_j \geq 0, x_2 + z_2 \leq \hat{x}_1\ \}\end{aligned} \tag{3.12}$$

Evidently (3.12) is a very complex dynamic program with a large state space. It may be simplified by using *myopic policy*. As it is described by Zipkin (2000), under nonstationary demand, the myopic policy treats the current data as stationary over an infinite horizon, and non-stationary over the finite horizon. Hence, the solution procedure is as follows: First, we think about the finite horizon problem and the data are treated to be nonstationary.

One-Period Problem

For a single period; $T=1$, single order decision is made at time 0. During period 0 a random demand occurs and the order arrives. At time 1, the inventory-backorder cost $\hat{C}(1,x(1))$ is assessed and the terminal cost is realized. Because there is only one period, we suppress the time index *t*. That is;

$$d(0)=d,\ y(0)=y,\ c(0)=c,\ h(1)=h,\ b(1)=b,$$
$$\hat{C}(1,x)=\hat{C}(x) \qquad (3.13)$$
$$C(0,y)=E\left[\hat{C}(y-d)\right]=C(y)$$

This may seem an artificially simple scenario, but it portrays situations where the product has a short useful life. Newspapers, magazines, many foods and beverages may be examples.

Sometimes, when the planning horizon is finite, it is convenient to include a special mechanism to "settle accounts" at the end. When $x(T)<0$, we must purchase stock to fill the remaining backorders $-x(T)$ at the unit purchase price $c(T)$. Also, if $x(T)>0$, we can sell the leftover stock at the same price; thus, we receive total revenue $c(T)x(T)$, or equivalently, we pay a cost of $-c(T)x(T)$. In sum, regardless of the sign of *x(T)*, there is a terminal cost $c(T)x(T)$. The terminal cost factor *c(T)* is called the *salvage value*.

The following optimization problem is faced in the single variable *y*:

$$\min \ c(y-x)+C(y)-\gamma c(1)(y-E[d])$$
$$s.t. \qquad (3.14)$$
$$y \geq x$$

This is easy to solve for any given *x*. To determine an optimal policy, we must solve this optimization problem for every value of *x*. If we define the following function as in Zipkin (2000);

$$H(y)=cy+C(y)-\gamma c(1)(y-E[d])$$
$$=\gamma c(1)E[d]-\gamma c(1)y+cy+C(y) \qquad c^{+}=c-\gamma c(1); \qquad (3.15)$$
$$=\gamma c(1)E[d]-c^{+}y+C(y)$$

So the objective of the optimization problem given in (3.14) is $H(y) - cx$. Since the term *cx* is constant, we can equivalently minimize *H(y)* to minimize the objective function. Because *C* is a convex function and the other terms in *H* are linear, *H* is also convex. Thus the smallest *y* that minimizes *H(y)* which is denoted by s^* can be found easily.

The optimal solution to (3.14) depends on the relation between *x* and s^*. If $x \leq s^*$, then $y = s^*$ is feasible and hence optimal. If $x > s^*$, then the optimal solution is $y = x$, because *H(y)* is nondecreasing over the entire feasible range $y \geq x$. This is a base stock policy with base stock level s^*. If the initial inventory is above s^*, don't order; if it is below s^*, order the difference; just enough to raise the inventory to s^*.

To compute s^*, we must solve $H'(y) = 0$.

$$H'(y) = (c^+ + h) - (b + h)F^0(y) \tag{3.16}$$

where F^0 is the ccdf of *d*. And s^* solves the following:

$$F^0(y) = \frac{c^+ + h}{b + h} \tag{3.17}$$

Let $V(x)$ denote the optimal cost of problem (3.10) regarded as a function of *x*.

$$V(x) = -cx + V^+(x) \tag{3.18}$$

where

$$V^+(x) = H(max\{s^*, x\}) \tag{3.19}$$

Finite Horizon Problem

In this section, finite horizon problem in which demand is assumed to be nonstationary is considered. In other words, finite *T* periods will be considered. The terminal value will be the discounted cost, which will be found for infinite horizon problem. The discounted cost for infinite horizon problem is denoted by $\underline{C}_1^+(y)$; so the terminal value of the finite horizon problem is set to $\underline{C}_1^+(y)$ and finite horizon problem is solved.

The formulation is derived stage by stage. Customer demand occurs at retailer level; in stage 2. Retailer meets the customer demand and he places an order that increases his inventory position to a certain base-stock level. After that, supplier (stage 1) takes the retailer's orders as a customer demand and makes a shipment to stage 2. Therefore, the formulation is derived for stage 2 first, and then stage 1 formulation is given.

In this case, there are several interrelated decisions to be made. In choosing *y(0)*, the costs in future periods must be considered. And also the decision to be made at time *t=1* will affect the future costs but at *t=0*, this decision hasn't been made yet. As it is mentioned in the previous section, dynamic programming is a useful technique in these types of situations.

First, time point $(T-1)$ is considered. Once we arrive period $(T-1)$, no matter what may have happened in the past, a single-period problem is faced. This problem is defined in section 3.3.2.1, and the solution is found the way defined. Next, period $(T-2)$ is considered and two-period problem is faced. Assuming that it will be acted optimally at $(T-1)$ according to the policy obtained already, the best decision is made. Working backward in this way the initial period is reached. This approach is called principle of optimality. Define

$V_j(t,x)$: minimal expected discounted cost for stage *j* in periods $t,t+1,...,T$, assuming period *t* begins with $x(t)=x$

These functions are computed recursively and optimal policy is obtained. Suppose we have determined $V_j(t+1,x)$ along with the optimal policies for points $(t+1)$ through $(T-1)$. To address the problem at time *t,* given $x(t)=x$, define

$$H_j(t,y) = c_j(t)y + C_j(t,y) + \gamma E[V_j(t+1), y - d(t)] \quad (3.20)$$

This quantity measures all the relevant costs if we choose $y(t) = y$. The first two terms represent the costs at time *t* itself. The last term is the expected value of all future costs, assuming we act optimally in the future, since $x(t+1) = y - d(t)$. The problem at time *t* is to choose *y* to minimize $H_j(t,y)$, subject to $y \geq x$. The solution to this problem for each *x* gives the optimal policy at time *t*. Then

$$V_j(t,x) = -c_j(t)x + min\{H_j(t,y) : y \geq x\} \quad (3.21)$$

For stage 2 (retailer);

$$V_2(T,x) = \underline{C}_1^+(y) \quad (3.22)$$

$$H_2(t,y) = c_2(t)y + C_2(t,y) + \gamma E[V_2(t+1), y - d(t)] \quad (3.23)$$

Set $C_2(t,y)$ to $C_2^l(t,y)$ with $l = L_2$ where

$C_2^l(t,y) = \gamma^l E\left[\hat{C}_2(t+l, y - D[t,t+l))\right]$; then,

$$V_2(t,x) = -c_2(t)x + min\{H_2(t,y) : y \geq x\} \quad (3.24)$$

Since $C_2(t,y)$ is convex in x in *y,* base stock policy is optimal. Let $s_2^*(t)$ denote the optimal base stock levels and it is found by $H_2^{'}(t,y) = 0$. Related with the base-stock level chosen, the cost $\underline{C}_2(t,y)$ which can be defined as follows is incurred by the retailer stage.

$$\underline{C}_2(t,y) = H_2(t, min\{s_2^*(t), y\}) \quad (3.25)$$

For stage 1 (supplier stage) the optimal cost will be;

$$\hat{C}_1(t,x) = h_1(t)x + \underline{C}_2(t,x) \quad (3.26)$$

$$C_1^l(t,y) = \gamma^l E\left[\hat{C}_1(t+l, y - D(t,t+l))\right] \quad (3.27)$$

The dynamic programming formulations are given as follows for stage 1:

$$V_1(T,x) = \underline{C}_1^+(y) \quad (3.28)$$

$$H_1(t,y) = c_1(t)y + C_1(t,y) + \gamma E[V_1(t+1, y - d(t)] \quad (3.29)$$

$$V_1(t,x) = -c_1(t)x + min\{H_1(t,y) : y \geq x\} \quad (3.30)$$

Since $C_1(t,y)$ is convex in x in *y,* base stock policy is optimal. Let $s_1^*(t)$ denote the optimal base stock levels and it is found by $H_1'(t,y) = 0$. Related with the base-stock level chosen, the cost $\underline{C}_1(t,y)$ which can be defined as follows is incurred by the supplier stage.

$$\underline{C}_1(t,y) = H_1(t,min\{s_1^*(t),y\}) \quad (3.31)$$

The echelon base stock policy with parameters $s_j^*(t)$ is optimal for the system as a whole. In other words, to solve the complex dynamic program (3.12), we need simpler ones, which are given by (3.23) and (3.29).

This policy describes actual shipments, not orders. For stage 2, it is shipped nothing if $x_2(t) \geq s_2^*(t)$, and otherwise it is set $y_2(t) = x_2(t) + z_2(t)$ to $min\{s_2^*(t), \hat{x}_1(t)\}$.

Since the demand is nonstationary, it is better to view each period separately, independent of the future periods. Now, *myopic policy* will be constructed for the defined finite horizon problem. Define

$c_j^+(t) = c_j(t) - \gamma c_j(t+1)$ and (3.32)

$$C_2^+(t,y) = \gamma c_2(t+1)E[d(t)] + c_2^+(t)y + C_2(t,y) \quad (3.33)$$

This equality is found from the following equations:

$$C_2^+(t,y) = \mathrm{E}[c_2(t)(y-x) + C_2(t,y) | x = x(t), y = y(t)] \quad (3.34)$$

$$= c_2(t)y + \mathrm{E}[c_2(t)(y(t-1) - d(t-1) + C_2(t,y)]$$

$$= c_2(t)y - \gamma c_2(t+1)y + \gamma c_2(t+1)\mathrm{E}[d(t)] + C_2(t,y)$$

Since $c_j^+(t) = c_j(t) - \gamma c_j(t+1)$,

$$C_2^+(t,y) = c_2^+(t)y + \gamma c_2(t+1)E[d(t)] + C_2(t,y) \quad (3.35)$$

$V_j^+(t,x)$ denotes the optimal cost for stage j for the finite horizon problem. And also the terminal value of the finite horizon is the optimal value for the infinite horizon, which is as calculated previously:

$$V_2(T,x) = \underline{C}_1^+(y) \tag{3.36}$$

$$H_2(t,y) = C_2^+(t,y) + \gamma E\left[V_2^+(t+1, y-d(t)\right] \tag{3.37}$$

$$V_2^+(t,x) = min\left\{H_2(t,y) : y \geq x\right\} \tag{3.38}$$

That means;

$$V_2^+(t,x) = min\left\{C_2^+(t,y) + \gamma E\left[V_2^+(t+1, x+1)\right]\right\}$$

$$= min\left\{C_2^+(t,y) + \gamma \mathrm{E}\left[\underline{C}_1^+(y)\right] \right\} \tag{3.39}$$

$$V_2^+(t,x) = min\left\{c_2(t)(y-x) + C_2(t,y) + \gamma \mathrm{E}\left[V_2^+(t+1, y-d(t))\right] : y \geq x \right\}$$

$$s_2^+(t) = arg\,min\left\{C_2^+(t,y)\right\} \tag{3.40}$$

Minimizing $V_2^+(t,x)$ to minimizing $C_2^+(t,y)$ over y. Let $s_2^+(t)$ minimize $C_2^+(t,y)$ over y. The corresponding base-stock policy minimizes the current cost while ignoring the future, so myopic policy is used. Since $C_2^+(t,y)$ is a convex function it is easy to optimize. Then the optimal cost function for stage 2 can be expressed as following:

$$\underline{C}_2^+(t,y) = C_2^+\left(t, min\left\{s_2^*(t), y\right\}\right) \tag{3.41}$$

Now, the base-stock policy for stage 1 can be considered. The cost function for echelon 1;

$$\hat{C}_1^+(t,y) = h_1(t)x + \underline{C}_2^+(t,x) \tag{3.42}$$

$$C_1^{+^l}(t,y) = \gamma^l E\left[\hat{C}_1^+(t+l, y-d[t,t+l))\right]; \tag{3.43}$$

set $C_1^{+^l}(t,y)$ for $C_1(t,y)$ and write

$$C_1^+(t,y) = \gamma c_2[t+1)E\left[d(t)\right] + c_1^+(t)y + C_1(t,y) \tag{3.44}$$

$$s_1^+(t) = arg\,min\left\{C_1^+(t,y)\right\} \tag{3.45}$$

and let $s_1^+(t)$ minimize $C_1^+(t,y)$ over y. then the myopic policy is the echelon base-stock policy with base-stock levels $s_j^+(t)$. In particular, $s_j^+(t)$ solves an equation given in (3.46).

$$F^0_{d(t)}(y) = \frac{c_j^+(t) + h_j(t+1)}{b(t+1) + h_j(t+1)} \tag{3.46}$$

where $F^0(x) = 1 - F(x)$ is the complementary cumulative distribution function. And optimal cost function for stage 1 is given as follows:

$$\underline{C}_1^+(t,y) = C_1^+\left[t, min\left\{s_1^+(t), y\right\}\right] \tag{3.47}$$

Infinite Horizon Problem

The second part of the horizon is considered to be infinite and the demand along this period is assumed to be independent and identically distributed. The formulation will be given by myopic policy again. Since the time horizon is considered to be infinite, time index is eliminated.

$$c_j^+ = c_j - \gamma c_j \tag{3.48}$$

and

$$C_2^+(y) = \gamma c_2 E\left[d(t)\right] + c_2^+ y + C_2(y) \tag{3.49}$$

This equality is found from the following equations:

$$C_2^+(y) = \mathrm{E}\left[c_2(y - x) + C_2(y) \middle| x = x(t), y = y(t)\right] \tag{3.50}$$

$$= \mathrm{E}\left[c_2 y - c_2(y(t-1) - d(t-1)) + C_2(y)\right]$$

$$= c_2 y - \gamma c_2 y + \gamma c_2 \mathrm{E}\left[d(t)\right] + C_2(y)$$

$$= (1-\gamma) c_2 y + \gamma c_2 E\left[d(t)\right] + C_2(y) \qquad c_2^+ = c_2 - \gamma c_2;$$

$$C_2^+(y) = \gamma c_2 \mathrm{E}\left[d(t)\right] + c_2^+ y + C_2(y) \tag{3.51}$$

where $C_2(y) = \gamma^{L2} \mathrm{E}\left[\hat{C}_2(y - d(L_1, L_2 + L_2))\right]$

$V_j^+(x)$ denotes the optimal cost for the infinite horizon problem and (3.52) is defined by Zipkin (2000) as:

$$V_j^+(x) = cx + V_j(x) \tag{3.52}$$

and the terminal value is again assumed to be zero while finding myopic policy:

$$H_2(y) = C_2^+(y) + \gamma E\left[V_2^+(y - d(t))\right] \tag{3.53}$$

$$V_2^+(x) = min\{H_2(y) : y \geq x\} \tag{3.54}$$

That means;

$$V_2^+(x) = min\{C_2^+(y) + \gamma E\left[V_2^+(x+1)\right]\} \tag{3.55}$$

$$s_2^+ = arg\ min\{C_2^+(y)\} \tag{3.56}$$

Minimizing $V_2^+(x)$ means minimizing $C_2^+(y)$ over *y*. Let s_2^+ minimize $C_2^+(y)$ over *y*. The corresponding base-stock policy minimizes the current cost while ignoring the future, so it is called as *myopic policy*. Since $C_2^+(y)$ is a convex function it is easy to optimize. Then the following cost function can be defined for stage 2, as it is given in (2.12);

$$\underline{C}_2^+(y) = C_2^+\left[min\{s_2^+, y\}\right] \tag{3.57}$$

Now the base-stock policy for stage 1 can be considered. The cost function for echelon 1;

$$\hat{C}_1^+(x) = h_1 x + \underline{C}_2^+(x) \tag{3.58}$$

$C_1^{+^l}(y) = \gamma^l E\left[\hat{C}_1^+(y - d(t, t+l))\right]$; set $C_1^{+^l}(y)$ for $C_1(y)$ and write

$$C_1^+(y) = c_1^+ y + \gamma c_1 \mathrm{E}\left[d(t)\right] + C_1(y). \tag{3.59}$$

and let s_1^+ minimize $C_1^+(y)$ over *y*.

$$s_1^+ = arg\,min\{C_1^+(y)\}. \tag{3.60}$$

Then the myopic policy is the echelon base-stock policy with base-stock levels s_j^+. In particular, s_j^+ solves an equation given in (3.61)

$$F_{d(t)}^0(y) = \frac{c_j^+ + h_j}{b + h_j} \tag{3.61}$$

where $F^0(x) = 1 - F(x)$ is the complementary cumulative distribution function. And optimal cost function for stage 1 is given as follows:

$$\underline{C}_1^+(y) = C_1^+\left[min\{s_1^+, y\}\right] \tag{3.62}$$

CHAPTER FOUR
DECENTRALIZED CONTROL SCHEME

In a decentralized system, information and control are distributed throughout the network. There is no central decision maker. Each stocking point observes his own demand and inventory position and makes his own decision to minimize his own costs rather than the whole supply chain's costs. In this case, according to defined problem in chapter 3; the two stages, which refer to the supplier and retailer, can be considered as independent firms and they make their own decisions. At this point, as it is expressed in chapter 2, the decisions of stages may be conflicting. So, firms can be called as "players" and the behaviors of the firms can be best expressed in game theoretic models.

In this chapter, the game theoretic framework will be given for the defined problem. After giving some general information and definitions for the decentralized framework in section 4.2, the game that is played in a single period will be defined in section 4.3. When the time horizon consists of finite number of periods, and this one-period game is played in each period repeatedly, therefore a repeated game, which is defined in section 4.4, is occurred. And finally the Markovian game version of the defined problem will be given in section 4.5.

Base Stock Policy Under Decentralized Control Scheme

In a decentralized system, decisions are made locally; in other words, decisions are made at a stocking point by the decision-maker at that socking point. A local base-stock policy is a decentralized control scheme, where each stocking point monitors its own local inventory position, places orders with its predecessor, and responds to orders from its successors. Each stage j follows a single stage base-stock policy with parameter which is a non-negative integer.

s_j' : local base-stock level for stage j

The policy works as follows: Stage J monitors its own inventory position. It experiences demands and places orders with stage $J-1$ just like a single location operating alone, using a standard base-stock policy with base-stock level s_J'. Stage $J-1$ treats these incoming orders as its own demands, filling them when it has stock available and otherwise logging backorders to be filled later. Stage $J-1$ too follows a standard base-stock policy with parameter s_{J-1}' to determine the orders it places with stage $J-2$. Stage $J-2$ treats these orders as its demands, etc. Stage 1's orders go to external source, which fills them immediately.

Hence, each customer demand triggers a demand at stage $J-1$, which in turn generates a demand at stage $J-2$, and so on. In this way demand propagates backward through the system, all the way to the external source. Thus, every stage experiences the original demand process.

A local base stock policy can be interpreted in echelon terms. Given $\boldsymbol{s}'$; the vector which consists of base stock levels for each echelon, it is known that $s_j = \sum_{i \geq j} s_i'$. The echelon base-stock policy says that every stage order one unit when a demand occurs. Thus, this policy is entirely equivalent to the original local one.

Conversely, every echelon base-stock policy is equivalent to a local one. Given $\boldsymbol{s}$, if the $\boldsymbol{s}_j$ are nondecreasing, the local base stock level for echelon j equals the difference of echelon base stock levels for two echelons. In other words, $s_j' = s_j - s_{j+1}$ (where $s_{J+1} = 0$).

"To see this, consider a two-stage system (J=2), and assume $s_1 < s_2$. Suppose the system starts with no inventory at stage 1 and inventory s_1 at stage 2. Stage 2 immediately orders $s_2 - s_1$, but stage 1 has no inventory, so it backlogs those orders. In fact, stage 1's echelon inventory position is already at its base stock level s_1, so it orders only in response to subsequent demands. Thus, the initial backlog at stage 1 stays there forever, that is it remains at least $s_2 - s_1$. Stage 1 never holds inventory,

and the inventory at stage 2 never exceeds s_1. *In the local policy,* $s_1' = 0$ *and* $s_2' = s_1$ *" (Zipkin P., 2000).*

As a summary, every local base stock policy is equivalent to some echelon base stock policy, and the opposite is also true.

In this section, decentralized base-stock policy will be evaluated in echelon terms and dynamic programming formulations will be given for the problem defined in section 3.2. The planning horizon is considered finite in this chapter. The following notation will be used in addition to the previous chapter:

$h_j'(t)$: inventory holding cost for stage *j* at time *t*

$b_j'(t)$: backorder-penalty cost rate for stage *j* at time *t*.

In addition, recall the following notation defined in chapter 3:

$\hat{x}_j(t)$: echelon net inventory at stage *j* at time *t*

$x_j(t)$: echelon inventory-transit position at stage *j* at time *t*

In each period, related with this inventory, the retailer is charged inventory holding cost and backorder cost, which can be shown as $h_2'(t)$ and $b_2'(t)$ respectively. Then $\hat{C}_2[t, \hat{x}_2(t) - d(t)]$ can be defined as the sum of these costs in period *t*, where

$$\hat{C}_2(t, y) = h_2'(t)[y]^+ + b_2'(t)[y]^- \qquad (4.1)$$

$C_2[t, x_2(t)]$ is defined as the retailer's expected cost in period $t + L_2$, where;

$$C_2[t, y] = E[\hat{C}_2(t, y - d[0, L_1))] \qquad (4.2)$$

$s_2(t)$ is the echelon base stock level for retailer at time *t* that minimizes the retailer's costs, $C_2[t, y]$;

$$s_2(t) = \arg\min C_2[t, y] \tag{4.3}$$

It can be verified that $C_2[t, y]$ is strictly convex, so $s_2(t)$ is determined by $C_2'[t, s_2(t)] = 0$.

In decentralized case, since both stage holds inventory, base stock levels cannot be determined independently. Stage 2's cost depends on both s_1 and s_2. In other words, the retailer's expected cost depends on both its own base stock as well as the supplier 2's base stock. If we define,

$s_j(t)$: echelon base stock level for stage *j* at time *t*

If $s_1(t) - d[t, t + L_1) \geq s_2(t)$, then orders of stage 2 (retailer) can be filled completely by stage 1 (supplier). This means the echelon inventory position at stage 2 is $x_2(t) = s_2(t)$.

If $s_1(t) - d[t, t + L_1) < s_2(t)$ then orders cannot be filled so the echelon inventory position of stage 2 is $x_2(t) = s_1(t) - d[t, t + L_1)$.

Because the echelon inventory position of stage 2 can be at most the echelon inventory position of stage 1, the retailer's cost function will depend on both base stock levels and can be expressed as follows:

$$G_2(s_1(t), s_2(t)) = E[C_2(\min\{s_1(t) - d[t, t + L_1), s_2(t)\}] \tag{4.4}$$

$\hat{C}_1[\hat{x}_1(t) - d(t)]$ is defined as the supplier's backorder cost at period *t*, where

$$\hat{C}_I(y) = b_I'(t)[y]^- \tag{4.5}$$

$C_I[t, x_I(t)]$ is defined as the supplier's expected backorder cost in period $t + L_I$, where;

$$C_I(t, y) = E[\hat{C}_I(t, y - d[t, t + L_I))] \tag{4.6}$$

Then the supplier's expected cost will be:

$$G_1(s_1(t), s_2(t)) = E\left[\hat{G}_1\left(s_2(t), s_1(t) - s_2(t) - d[t, t + L_1)\right)\right] \tag{4.7}$$

where

$$\hat{G}_1(s_1(t), x) = h_1^{'}(t)E\left[d[t, t + L_1)\right] + h_1^{'}(t)[x]^+ + C_1\left(s_2^+(t) + min\{x, 0\}\right) \tag{4.8}$$

The first term in equation (4.8) is the expected holding cost for the units in transit to the retailer, the second term is the expected cost for inventory held at the supplier, and the final term corresponds to the expected backorder cost charged to the supplier.

In the following section the game, which is played between two firms in one period, will be defined. The cost functions found in this section is going to be used in game theoretic frame of the defined problem.

One-period Game

Since there are two stages in the problem setting, there will be two players in the game theoretic framework; supplier and retailer. Both players move simultaneously in each time period *t*. Actions chosen refer to the base stock levels.

Formulation

In this section, the notation for *one-period game* will be introduced.

I : set of players.

$$I = \{1, 2\}$$

a_i : action chosen by player *i*

Actions chosen refer to the base stock level preferred by each player. The base stock levels which are denoted by "*s*", will be denoted by "*a*" from now on. Actions chosen by the players will be the action profile of the game.

a : action profile of the game

$$a = (a_1, a_2)$$

A_i: action space for player i

Action space is assumed to be finite but is sufficiently large that it never constraints the players: [0,S]

A : action space for the game

$$A = A_1 \times A_2$$

$G_i(a)$: cost function of one-period game for player i ; $G_i : A \to \Re$

Cost functions for supplier and retailer are defined precisely below, respectively.

$$G_1(a_1,a_2) = h_1E[d] + h_1E[a_1 - a_2 - d] + E[C_1(a_2 + min\{a_1 - a_2 - d, 0\})] \quad (4.9)$$

$$= h_1E[d] + h_1 \int_0^{a_1 - a_2} (a_1 - a_2 - x) f(x) dx + C_1(a_2) F(a_1 - a_2) + \int_{a_1 - a_2}^{\infty} C_1(a_1 - x) f(x) dx$$

$$G_2(a_1,a_2) = E[C_2(min\{a_1 - d,\ a_2\})]$$

$$= C_2(a_2)F(a_1 - a_2) + \int_{a_1 - a_2}^{\infty} C_2(a_1 - x) f(x) dx \quad (4.10)$$

The one-period game can be denoted by $\Gamma = \{I, A, G\}$

Equilibrium

In period *t*, players choose their base-stock levels a_i and obtain the action profile $a = (a_1, a_2)$. The action space is limited by a sufficiently large number. After their choices, the players implement their policies for only associated period. For each period they may have different action profile.

Supplier and retailer choose their action so that his own cost function is minimized. For the one-period game the best reply mappings for players are introduced below:

$$r_1(a_2) = \{a_1 \in A | G_1(a_1, a_2) = \min_{x \in A} G_1(x, a_2)\} \tag{4.11}$$

$$r_2(a_1) = \{a_2 \in A | G_2(a_1, a_2) = \min_{x \in A} G_2(a_1, x)\} \tag{4.12}$$

$a = (a_1, a_2)$ will be the pure strategy Nash Equilibrium which is a pair of echelon base stock levels in the one-period game.

Cachon and Zipkin (1999) showed that supplier's cost function is strictly convex for $a_1 \geq 0$ and $a_2 \geq 0$. So, supplier's best response can be determined by first order conditions. When we set equal to zero the first derivative of the cost function, we find that supplier's echelon base stock level is always grater than or equal to that of retailer's:

$$a_1 = r_1(a_2) \geq a_2 \tag{4.13}$$

It is also shown in Cachon and Zipkin (1999) that retailer's cost function is quasiconvex. Retailer's echelon base-stock level is determined according to the supplier's echelon base-stock level. Recall that s_2 is the base stock level, which minimizes the retailer's cost function, found in chapter 3.

$$a_2 = r_2(a_1) = \begin{cases} s_2, & a_1 \geq s_2 \\ [a_1, S], & a_1 < s_2 \end{cases} \tag{4.14}$$

(4.14) says briefly as follows: If the base stock level (s_2) that minimizes the retailer's cost is less than the supplier's base stock level preference for that period, then all the orders from supplier can be met. Hence the retailer set his base stock level to s_2. Otherwise, the orders of retailer cannot be met by supplier, so the base stock level is equal to the amount that the supplier has; which means a_1.

THEOREM 4.1 (Fudenberg and Tirole): If the cost functions of a strategic form game, whose strategy space A_i are non-empty compact convex subsets of Euclidean space, are continuous in a and quasiconvex in a_i, there exists a pure strategy Nash equilibrium.

Since $G_1(a_1,a_2)$ is strictly convex in a_1, and $G_2(a_1,a_2)$ is quasiconvex in a_2, there is at least one equilibrium. As it is given in (4.13), there is a requirement such as $a_1 = r_1(a_2) \geq a_2$. This condition and the second situation in which $a_1 \leq s_2$ in (4.14), is a contradiction. Hence $a_1 > s_2$; and this implies that $a_2 = r_2(a_1) = s_2$. Therefore there is a unique Nash equilibrium of the one-period game.

Since the planning horizon is assumed to occur T successive periods in the problem setting, the defined one-period game is repeated in each period. Thus, a repeated game needs to be defined.

Repeated Game

It is assumed that the time horizon is finite and there are T periods. The players choose actions simultaneously in each time period. All players know the actions chosen at all previous periods *1,2,…t-1* when choosing their actions in period *t*.

As it is defined in section 2.3.3.1, a repeated game is a multi-period game in which the same (ordinary) game is played at each time period. For finite horizon, because the one period game is played over and over again in each time period, this can be formulized by repeated game.

Since the repeated game consists of one-period games, a strategy for player i in the repeated game consists of action choices in each one-period game Γ.

Formulation

The notation related with the repeated game and the repeated game formulation will be given in this section.

σ_i : pure strategy chosen by player *i*

$\sigma_i : H^t \rightarrow A_i$

H^t denotes the set of all period–*t* histories,

So we can write a strategy of player *i* for the repeated game as;

$\sigma_i = (a_i^1, a_i^2, \ldots, a_i^T)$

σ : strategy profile of the repeated game

$\sigma = (\sigma_1, \sigma_2)$

Σ_i : strategy space for player *i*

$$\Sigma_i = A_i \times \underbrace{A_i \times \ldots \times A_i}_{T \text{ times}}$$

Σ : strategy space for the repeated game

$\Sigma = \Sigma_1 \times \Sigma_2$

Recall that we assume the discount factor $\gamma \leq 1$, we can compute the cost function of the repeated game as follows:

π_i : cost function of the repeated game; $\pi_i : H^T \rightarrow \Re$

$$\pi_i = \sum_{t=1}^{T} \gamma^t G_i(a^t) \tag{4.15}$$

So we can denote the repeated game with *T* periods as $\Gamma^T = (I, \Sigma, \pi)$.

Equilibrium

Since we know that there is a unique equilibrium for one-period game, it is expected that it is possible to find an equilibrium point for the repeated game also.

When exactly the same game is played at each period, the following theorem states that there is a Nash equilibrium for the repeated game.

THEOREM 4.2 (Friedman J.W.,1986): Let $\Gamma^T = (I,\Sigma,\pi)$ be a repeated game with finite T and at each time t the $a_i^t, i \in I$ are chosen simultaneously, however for $t > 0$ $a_\tau, \tau = 1,2,...,t-1$ is known to all players. Let $\hat{a} \in A$ be the unique equilibrium point of $\Gamma = \{I, A, G\}$. Then the only equilibrium point of $\Gamma^T = (I,\Sigma,\pi)$ is $\sigma = (\hat{a},\hat{a},...,\hat{a})$.

However in the defined problem, there is only one equilibrium point for one-period game but it may be found different equilibrium point in each period. In other words equilibrium point is unique, but may change from period to period according to the cost functions, since the cost functions may be different in different periods. Therefore, it is not possible to use the theorem above to find an equilibrium for the repeated game in our problem.

In this context, it is useful to handle the problem as subgames. Just like dynamic programming formulation, defining subgames for the whole game makes the problem easy to solve because the problem is divided into different parts; subgames. In repeated games, a subgame begins at the beginning of each one-period game. Hence, a subgame begins after each history h^t. The formal definition is given by Fudenberg and Tirole, 1999.

DEFINITION 4.1 (Fudenberg and Tirole): A proper subgame of an extensive form game consists of a single node and all its successors with the property that the information sets and costs of the subgame inherited from the original game.

Subgame concept was defined in section 2.1.3.1.2, also. The subgame is denoted $\Gamma_{h^t} = (I,\Sigma_{i_{h^t}},\pi_{i_{h^t}})$.

Repeated games have perfect information at the beginning of each period when an action, that is a strategy of the one-period game is chosen. This suggests applying the concept of a subgame perfect equilibrium to repeated games to eliminate equilibria that rely on incredible threats. Having defined the subgames for the repeated game, the solution concept, subgame perfect equilibrium should also be defined.

DEFINITION 4.2 (Fudenerg and Tirole): A strategy profile σ of a multi-period game with observed actions is a subgame perfect equilibrium if for every h^t, the restriction to Γ_{h^t} is a Nash equilibrium of Γ_{h^t}.

In our problem, since the game has fixed number of periods T, we can characterize the subgame perfect equilibrium using backward induction. The strategies in the last period must be a Nash equilibrium of the corresponding one-shot simultaneous move game.

$$\left.\begin{aligned} \sigma_1^* &= a_1^T \\ \sigma_2^* &= a_2^T \end{aligned}\right\} \quad \Rightarrow \sigma^* = (a_1^T, a_2^T)$$

By moving backward the equilibrium can be found. In a general notation,

$$\sigma_i = (a_i^T)$$

$$\sigma_i = (a_i^{T-1}, a_i^T)$$

$$\sigma_i = (a_i^{T-2}, a_i^{T-1}, a_i^T)$$

$$\vdots$$

$$\sigma_i = (a_i^2, \ldots, a_i^{T-2}, a_i^{T-1}, a_i^T)$$

$$\sigma_i = (a_i^1, a_i^2, \ldots, a_i^{T-2}, a_i^{T-1}, a_i^T) \tag{4.16}$$

When we reach the initial period, we also reach the equilibrium of the repeated game, which is given by (4.16).

Markovian Game

As a result of demand uncertainty or some other reasons, price wars and other problems related with companies continues to be a challenge. From Friedman (1977) and earlier, such kinds of problems have been analyzed as repeated games. A second approach was developed by Maskin and Tirole (2001) that of Markov strategies.

The current action of a player could affect his future costs in two ways: first, the effect the action has on the environment in which future decisions must be made and, second, its impact on the behavior of other players. Markovian games thus extend dynamic programming problems (which include only the first effect) and repeated games (which include only the second effect); Markovian games can include both.

In the previous section, we studied the case in which the decisions made independently in each period, for a finite horizon. That is; in period *t,* players play a one-period game and apply the obtained base stock levels in that period. In period *t+1,* another one-period game is played and base stock levels which does not depend on previous period's decisions for associated period is obtained. Since the games in each period played in the same conditions, it was formulated as a repeated game.

In this section, we assume that the game played in period *t* affects the next period's decisions. In other words, the decisions made in the previous period have an influence on current decisions. Since the same game does not repeat across periods and the system gains a dynamic structure, this setting of the problem requires Markovian game formulation. Markovian game formulation will be introduced in the following section.

Formulation

The setup of a Markovian game can be considered a combination of a static game theory and a Markov decision process. In addition to the elements of a static game, which includes players, strategies, and cost function, a set of states and a transition mechanism should be defined. Besides, in this setting "strategies" is a vector of strategies, which refers to one for each period, as a difference from a static game. Vectors are denoted by bold characters.

Planning horizon consisting of T periods is considered as discrete, as it is assumed in previous sections. At the beginning of each period, players simultaneously make a decision that determines their base stock levels.

Players:

There are two players; the supplier and the retailer. Index $i = 1,2$ will be used for the players, respectively. The set of players will be denoted by I. $I = \{1,2\}$

States:

Markovian games posit the existence of a "state" variable that is designed to capture the environment of the game at each point in time, but that changes through time in response to the actions taken by the players in the game.

Since it gives the state of the system, state variable is defined related with the echelon inventory position.

s^t : echelon inventory position of player i in period t

$\boldsymbol{s}^t$: vector of individual state variables

$\boldsymbol{s}^t = (s_1^t, s_2^t)$

S : state space (includes all possible states)

In Markovian game setting, states refer to one-period games. In each period t, one state (one-period game) occurs according to a probability rule, and an action chosen for that state in that period.

Actions:

Since the base stock levels are desired to be determined, actions to be chosen are defined related with echelon inventory position after ordering for each player.

a_i^t : echelon inventory position after ordering for player *i*

$\boldsymbol{a^t}$: vector of individual actions (action profile for period *t*)

$\boldsymbol{a^t} = (a_1^t, a_2^t)$

$A_i^t(s)$: all possible actions for player *i* in period *t* for state *s*.

Since actions are determined according to state that occurs in that period and in each period *t* the state space is the same, we can denote the action space as $A_i(s)$

A : Action space for the whole game.

$$A = \underset{i \in I}{\times} \underset{s \in S}{\cup} A_i(s)$$

$$A = \underset{s \in S}{\cup} A_1(s) \times A_2(s) \quad (4.17)$$

Both state space and action space are assumed to be finite. It is also assumed that, in period *t* the history h^t is known to all players before they choose period *t* actions. The history at time *t* can be denoted as follows:

$$h^t = (s^1, a^1, s^2, a^2, \ldots, s^{t-1}, a^{t-1}, s^t) \quad (4.18)$$

Transition Function:

In each period *t*, one state occurs according to a transition function. For our problem, transition probabilities are obtained from probability distribution of the demand.

D_i^t : demand for player *i* during period *t*

$\boldsymbol{D^t}$: vector of demands for period *t*

$\boldsymbol{D^t} = (D_1^t, D_2^t)$

The state of the system in period *t+1*, depends on the action chosen in period *t* and the demand during period *t*. That is;

$$\boldsymbol{s^{t+1}} = \boldsymbol{a^t} - \boldsymbol{D^t}$$

$p^{a^t}_{s^{t+1},s'}$: the probability of being in the state $\boldsymbol{s}^{t+1}$ which is less than $\boldsymbol{s}'$ if the current state is $\boldsymbol{s}^t$ and the current action is $\boldsymbol{a}^t$.

$$p^{a^t}_{s^{t+1},s'} = \Pr\{\boldsymbol{s}^{t+1} < \boldsymbol{s}' | \boldsymbol{s}^t, \boldsymbol{a}^t\} = \Pr\{\boldsymbol{a}^t - \boldsymbol{D}^t < \boldsymbol{s}' | \boldsymbol{s}^t, \boldsymbol{a}^t\}$$

$$= Pr\{\boldsymbol{D}^t > \boldsymbol{a}^t - \boldsymbol{s}' | \boldsymbol{s}^t, \boldsymbol{a}^t\} = 1\text{-F}(\boldsymbol{a}^t - \boldsymbol{s}', \boldsymbol{a}^t) \quad (4.19)$$

At the beginning of each period *t*, player *i* observes the history h^t and then chooses an action a_i^t. A decision rule for player *i* specifies an action $a_i^t(s)$ for each state $s \in S$. The decision rule is the equilibrium in the one-period game that occurs in period *t*.

A strategy for player *i* (π_i) determines a decision rule for each period. A strategy π_i is a function that assigns a probability distribution to the action space for each history h^t. In other words, a strategy is a specification of a probability distribution, at each period and state, over the available actions, conditional on the history of the game up to that period.

Cost Function:

Retailer's and supplier's cost functions for the one-stage game are given below, respectively:

$$G_1(a_1,a_2) = h_1^t E[D_1^t] + h_1^t \int_0^{a_1-a_2} (a_1 - a_2 - x) f(x)dx + C_1(a_2)F(a_1 - a_2) + \int_{a_1-a_2}^{\infty} C_1(a_1 - x) f(x)dx$$

$$G_2(a_1,a_2) = C_2(a_2)F(a_1 - a_2) + \int_{a_1-a_2}^{\infty} C_2(a_1 - x) f(x)dx \quad (4.20)$$

In a Markovian game, each one-period game refers to a state. According to a transition mechanism, one of these one-period games (or one of the states) is played (or encountered) and according to the action profile in that period, the associated cost $G_i^t(a_1^t, a_2^t)$ is incurred.

Since a Markovian game is a combination of a static game and a Markov decision process, the cost structure must be defined as in Markov decision process.

G_i^t : inventory holding and shortage cost incurred in period t for player i

$g_i(\boldsymbol{s},\boldsymbol{a})$: single period cost function

$$g_i(\boldsymbol{s},\boldsymbol{a}) = E[G_i^t \mid \boldsymbol{s^t} = \boldsymbol{s}, \boldsymbol{a^t} = \boldsymbol{a}], \quad \boldsymbol{s} \in S,\ a \in A,\ i \in I \tag{4.21}$$

$L_i(\boldsymbol{s})$: terminal value

$$L_i(\boldsymbol{s}) = E[G_i^{T+1} \mid s^{T+1} = s] \qquad \boldsymbol{s} \in S, \quad i \in I \tag{4.22}$$

$V_i(\boldsymbol{s})$: the sum of discounted costs

$$V_i(\boldsymbol{s}) = \sum_{t=1}^{T} \gamma^{t-1} g_i\ (\boldsymbol{s^t}, \boldsymbol{a^t}) + \gamma^T L_i(\boldsymbol{s^{T+1}}) \tag{4.23}$$

$v_i^t(\pi,\boldsymbol{s})$: expected present value of the costs incurred in periods $t, t+1, \ldots, T$ given that the game starts period t in state s and strategy π is implemented.

$$v_i^t(\pi,\boldsymbol{s}) = E[V_i(\pi,\boldsymbol{s}) \mid \boldsymbol{s}^1 = \boldsymbol{s}]$$

$$= \boldsymbol{E}\left\{\sum_{\tau=t}^{T} \gamma^{\tau-1} \boldsymbol{g_i}\ (\boldsymbol{s}^\tau, \boldsymbol{a}^\tau(\boldsymbol{s})) + \gamma^{\boldsymbol{T}-\tau+1} \boldsymbol{L_i}\ (\boldsymbol{s}^{\tau+1})\right\} \tag{4.24}$$

$f_i^t(\boldsymbol{s})$: optimal value function of player i for period t

$$f_i^t(\boldsymbol{s}) = \min_{\pi \in \Pi} v_i^t(\pi,\boldsymbol{s})$$

$$= \min_{\boldsymbol{a_i^t} \in \boldsymbol{A}} \{\ g_i(\boldsymbol{s^t}, \boldsymbol{a^t}) + \gamma \sum_{j \in S} p_{sj}^a f^{t+1}(j)\} \tag{4.25}$$

where $f^{T+1}(\boldsymbol{s}) = L_i(\boldsymbol{s})$ for each $s \in S,\ i \in I$.

Having defined the game setup and the cost structure, the solution concept will be discussed in the following section.

Equilibrium

When demand is considered non-stationary, there may be many stationary and non-stationary equilibria but since the solution is subgame perfect, there is a unique Markov Perfect Equilibrium (MPE). An MPE can be characterized as follows: starting from any point a player selects the action that minimizes its intertemporal cost, given the subsequent moves of itself and its rivals. That is the term "Markov perfect" arises from the simple observation that all Nash equilibria in Markovian situations are also subgame perfect.

Maskin and Tirole (2001), by making the Markovian assumption, consider only those equilibria whose strategies depend on the "payoff-relevant" history. Such strategies make behavior in any period dependent on only relatively small set of variables rather than on the entire history of the play.

Markov perfect equilibrium is a profile of Markov strategies that yields a Nash equilibrium in every proper subgame. A Markov strategy is one that doesn't depend at all on variables that are functions of the history of the game except those that affect payoffs.

"The MPE concept's popularity stems in part from several practical considerations. First, MPE is often quite successful in eliminating or reducing large multiplicity of equilibria in dynamic games, and thus in enhancing the predictive power of the model. Relatedly, MPE, by preventing non-payoff-relevant variables from affecting strategic behavior, has allowed researchers to identify the impact of state variables on outcomes; it for example has permitted researchers to obtain a clean, unobstructed analysis of strategic positioning in industrial organization." Maskin and Tirole (2001).

In a Markovian game, as we have defined in the previous section, in every period t player i's single period cost function depends on both vector of player's actions and the (payoff relevant) state of the system.

In period t the history of the game is the sequence of the previous actions and states $h^t = (s^1, a^1, s^2, a^2, \ldots, s^{t-1}, a^{t-1}, s^t)$. But the only aspect of history that directly affects player i's costs and action sets starting in period t is the state $\boldsymbol{s^t}$. Hence a Markov strategy in this model should make player i's period t action dependent only on the state $\boldsymbol{s^t}$, rather than on the whole history h^t.

The state space S is again assumed to be finite. Recall the state of the system as it is defined in the previous section:

$\boldsymbol{s^t}$: echelon inventory position for player i

Then the state profile of period t can be denoted as $\boldsymbol{s^t} = (s_1^t, s_2^t)$.

At each period t, each player i chooses an action a_i^t from his finite action space A. Actions are defined as in previous section;

a_i^t : echelon inventory position after ordering for player i

Then, again denote the action profile of period t as $\boldsymbol{a^t} = (a_1^t, a_2^t)$.

Markov strategy definition requires players to make their strategies measurable with respect to a certain partition of possible histories. More specifically, this partitions, one for each player, must include at each point of time, a player's preferences over his continuation strategies which are the same for any history in a given element of his partition provided that the other players use strategies that are measurable with respect to their own partitions. Strategies that are measurable with respect to this partition are called Markovian, and a subgame perfect equilibrium in Markov strategies is called Markov perfect equilibrium. For multiperiod games in which the action spaces are finite in any period an MPE exists if the number of periods is finite or infinite.

Markov strategies are defined as those that are measurable with respect to the maximally coarse consistent partition; in other words the Markov partition. Although this is the right definition conceptually, it is a bit cumbersome practically. In practice, "how does one go about finding this partition" is still an important question.

Maskin and Tirole (2001) claim that for a broad class of games there is a readily checked conditions that enable us to determine whether or not two histories belong to the same element of the Markov partition. These conditions will be explained below.

Since the action profile of period t is $\boldsymbol{a}^t = (a_1^t, a_2^t)$, the action profile for the whole horizon is denoted as $\boldsymbol{a} = (\boldsymbol{a}^1, \boldsymbol{a}^2, ..., \boldsymbol{a}^T)$. The history in period t is the sequence of actions chosen before period t: $h^t = (\boldsymbol{a}^1, \boldsymbol{a}^2, ..., \boldsymbol{a}^{t-1})$. Let H^t be the set of all possible period t histories. History h^t is common knowledge in period t. The future in period t is the sequence of current and future actions: $f^t = (\boldsymbol{a}^t, \boldsymbol{a}^{t+1}, ..., \boldsymbol{a}^T)$. Thus, player i's cost function can be representable by $G_i(\boldsymbol{a}) = G_i(h^t, f^t)$. And also let $\widetilde{H}^{\cdot}(.)$ be the collection (partition) defined so that, for all t and for all $h^t, \widetilde{h}^t \in H^t$, $\widetilde{h}^t \in \widetilde{H}^t(h^t)$ if and only if

(i) $\overline{H}^t(h^t) = \overline{H}^t(\widetilde{h}^t)$

(ii) for all i there exist scalar $\alpha > 0$ and function $\beta : A_{-i}^t(h^t) \rightarrow \Re$ such that

$$G_i(\widetilde{h}^t, f^t) = \alpha G(h^t, f^t) + \beta(\boldsymbol{a}_{-i}^t) \qquad (4.26)$$

for all f^t.

(i) requires only to verify that action spaces following h^t and $\widetilde{h}^t$ are the same, whereas *(ii)* involves checking that continuation cost functions are appropriate affine transformations of one another.

Since each history begins at the beginning of each period t, and in each period the action space is the same, the first condition is hold. In other words, the action spaces following two possible different histories h^t and $\widetilde{h}^t$, will be the same. And also it can be easily shown that the cost functions given in (4.20) can be transformable to each other in the problem framework. Thus, all strategies are Markovian and an MPE is the same thing as a subgame perfect equilibrium.

A Markov strategy of player *i* is denoted by m_i which is a function that assigns to each feasible state an action.

m_i: Markov strategy for player *i* ;

$m_i : S \to A$

Markov strategy combination can be denoted by $\boldsymbol{m} = (m_1, m_2)$.

M_i: Markov strategy space for player *i* ;

$M = M_1 \times M_2$

Recall the dynamics given previously,

$$\boldsymbol{s}^{t+1} = \boldsymbol{s}^t + \boldsymbol{z}^t - \boldsymbol{D}^t = \boldsymbol{a}^t - \boldsymbol{D}^t \qquad (4.27)$$

Then; $\boldsymbol{a}^t = \boldsymbol{s}^t + \boldsymbol{z}^t$ and an action is defined as the function of the payoff-relevant state:

$\alpha(\boldsymbol{s}^t) = \boldsymbol{s}^t + \boldsymbol{z}^t$;

$$\alpha(\boldsymbol{s}) = S \to A \qquad (4.28)$$

$g_i\,[\boldsymbol{s}, \alpha(\boldsymbol{s})]$: single period cost function

$\boldsymbol{v}_i^t(\boldsymbol{m}, \boldsymbol{s})$: expected present value of the costs incurred in periods $t, \ldots, T$ given that the game starts period *t* in state ***s*** and a Markov strategy *m* is implemented.

$$\boldsymbol{v}_i^t(\boldsymbol{m}, \boldsymbol{s}) = \boldsymbol{E}\left\{\sum_{\tau=t}^{T} \gamma^{\tau-1} \boldsymbol{g}_i\,(\boldsymbol{s}^\tau, \alpha(\boldsymbol{s}^\tau)) + \gamma^{T-\tau+1} L_i\,(\boldsymbol{s}^{\tau+1})\right\} \qquad (4.29)$$

As stated in Maskin and Tirole (2001), a Markov perfect equilibrium is a subgame perfect equilibrium in which all players use Markov strategies. In other words, MPE is a profile of Markov strategies that yields a Nash equilibrium in every subgame. The following definition expresses this idea formally.

DEFINITION 4.3 (Fudenberg and Tirole): A Markov perfect equilibrium is a profile of strategies those are a perfect equilibrium and are measurable with respect to the payoff-relevant history.

THEOREM 4.3 (Fudenberg and Tirole): Suppose either $T < \infty$ or $T = \infty$ and the objective functions are continuous at infinity. Then there exists an MPE.

In the case of infinite horizon game, if the game is continuous at infinity, then there exists an MPE. The game is continuous at infinity if players discount future payoffs at a constant rate; $\gamma > 0$.

DEFINITION 4.4 (Fudenberg and Tirole): A game is continuous at infinity if for each player *i* the cost function v_i satisfies

$$\sup_{h,\tilde{h} \text{ s.t. } h^t = \tilde{h}^t} \left| v_i(h) - v_i(\tilde{h}) \right| \to 0 \quad \text{as} \quad t \to \infty \tag{4.30}$$

This condition says that events in the distant future are relatively unimportant. It will be satisfied if the overall costs are a discounted sum of the per-period costs $g_i[s,\alpha(s)]$ are uniformly bounded. In other words, there is a *B* such that

$$\max_{t,a^t} \left| g_i[s,\alpha(s)] \right| < B \tag{4.31}$$

Since the action space is bounded, per-period costs and the discounted sum of per-period costs are uniformly bounded. So we can say that the game is continuous. Hence, there exists an MPE and it is found by backward induction.

As stated in Maskin and Tirole (2001), to prove the existence of an MPE for a finite horizon game, one can work backwards, and select the same Nash equilibrium for all histories in an equivalence class so as to obtain Markov measurability.

At date *T*, a Nash equilibrium is selected. This Nash equilibrium is the same for all histories h^T in the same payoff-relevant history $H^T(h^T)$ which means in the same Markov partition. The reason that, for all histories with the same payoff relevant history, the last period subgames are strategically equivalent and the sets of Nash equilibria are the same.

Folding back, the subgame at period *T-1* becomes a one-period game, and the Nash equilibrium is selected depending on the payoff-relevant history. By using backward induction, the fist period is achieved.

CHAPTER FIVE
CONCLUSION

In this chapter a summary of the dissertation and also some recommendations for future work will be given.

5.1 Summary

The focus of this dissertation is on developing a base stock policy in a serial two echelon supply chain, under the consideration of non-stationary demand. The main contribution of this thesis is about the demand distribution across periods. The demand distribution is assumed to be independent across period, but not necessarily identically distributed. Firstly, the problem formulation is done for a centralized supply chain. The optimal base stock policy is found by myopic policy. Second, in decentralized case since the firms are considered as independent decision makers, game theoretic framework is needed.

One of the research objectives was to express the behaviors of the decision makers (the supplier and the retailer) under a non-cooperative game. With this consideration, this dissertation provides the game theoretic formulation. Game theoretic framework is derived for two cases. In the first case, the base stock policy chosen in a period hasn't an effect on the other period's base stock policy decision and cost function. In this case, a repeated game is formulated for the defined problem and the solution concept is subgame perfect equilibrium. The second case occurs when the base stock decisions made in the past has an influence on the future base stock decisions. This requires a stochastic game or Markovian game formulation. The solution concept for the stochastic game is Markov perfect equilibrium. In the study, the formulations for the solutions of the two different games are derived.

5.2 Limitations

In this dissertation, we focus on the game theoretic framework for two echelon serial supply chain under the consideration of non-stationary demand. The two echelon serial supply chain contains only a single supplier and a single retailer. Although this research may shed lights on the related literature, it must be admitted that modeling of the dissertation has been limited to the simplest supply chain structure.

5.3 Further Research

In this section some new directions for exploring further research avenues will be proposed.

As it is addressed just before, in this study the formulation is derived for the simplest supply chain. As an extension, an assembly system may be considered. Since the optimal policy is quite complex even under stationary data, the shipments costs may assume to be linear.

As another extension, a distribution system in which there is one supplier and N retailers may be also considered. This also makes the system more complex when the stock relationships among retailers are considered.

In the game theoretic framework, some other extensions are also possible. In a more complex supply chain design, the supplier may be considered as a leader and Stackelberg game framework may be constructed.

Besides, in this dissertation, the order cost is assumed to be linear; depending on the order amount, rather than to be fixed. Under the consideration of two-stage system, there may be assumed that there is a fixed order cost.

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