DOKUZ EYLÜL UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

PROBLEMS FOR HYPERBOLIC EQUATION SYSTEMS

by
Ali SEVİMLİCAN

March, 2007
İZMİR

PROBLEMS FOR HYPERBOLIC EQUATION SYSTEMS

A Thesis Submitted to the
Graduate School of Natural And Applied Sciences of Dokuz Eylül University
In Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy in
Mathematiccs

by
Ali SEVİMLİCAN

March, 2007
İZMİR

Ph.D. THESIS EXAMINATION RESULT FORM

We have read the thesis entitled ”Problems for Hyperbolic Equation Systems” completed by Ali SEVİMLİCAN under supervision of Prof. Dr. Valery YAKHNO and we certify that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Doctor of Philosophy.

Prof. Dr. Valery YAKHNO

Supervisor

Prof. Dr. Şennur SOMALI	Prof. Dr. Çoşkun SARI
Thesis Committee Member	Thesis Committee Member
Prof. Dr.Abdullah ALTIN	Prof. Dr. Gonca ONARGAN
Examining Committee Member	Examining Committee Member

Prof. Dr. Cahit HELVACI
Director
Graduate School of Natural and Applied Sciences

ACKNOWLEDGEMENTS

I would like to express my deepest gratitude to my supervisor Valery YAKHNO for his guidance and endless patience during the study. He has always encouraged and supported me to participate in both national and international conferences. I would like to to thank his invaluable contribution to this thesis[1]. He has also helped me improve my background in mathematics. He is not only a very good scientist, but also a very good teacher. I am proud to be his PhD student.

I would like to express my gratitude to all lecturers and research assistants at Department of Mathematics Especially, I would like to thank Engin MERMUT for his helping me on Latex.

Finally, I would like to thank my family for their confidence in me all my through my life. I dedicate this thesis to my mother and my father.

[1]This thesis supported by under research grant 03.KB.FEN.049

PROBLEMS FOR HYPERBOLIC EQUATION SYSTEMS

ABSTRACT

Initial value problems for two systems of partial differential equations of hyperbolic type are main object of this thesis. New methods for solving these problems are suggested and justified in the thesis. In addition, theorems about existence and uniqueness of these problems are proved. The considered systems of partial differential equations describe electric and magnetic wave propagations in electrically an magnetically anisotropic media (crystals, dielectrics etc) and in media with an anisotropic conductivity (biological tissue and earth materials). The results, obtained in the thesis, can find their applications in the theory of electromagnetic waves.

Keywords: Partial differential equations, hyperbolic systems, Initial value problem, Maxwell's system, Telegraph equation, Anisotropic media, Green's function, Fourier transform.

HİPERBOLİK DENKLEM SİSTEMLERİ İÇİN PROBLEMLER

ÖZ

Tezin ana konusu hiperbolik türdeki iki kısmi diferansiyel denklemler sistemleri için başlangıç değer problemleridir. Tezde bu problemlerin çözümü için yeni yötemler önerildi ve doğrulandı. Ek olarak, bu problemlerin varlık ve teklik teoremleri ispatlandı. Ele alınan kısmi diferansiyel denklemler sistemleri elektiriksel ve manyetiksel izotrop olmayan ortamda (kristaller, dielektirikler, v.b.) ve iletkenliği izotrop olmayan ortamda (biyolojik doku, yeryüzü malzemeleri) elektirik ve manyetik dalga dağılımlarını tanımlar. Tezde elede edilen sonuçların uygulamaları
elektromanyetik dalgalar teorisinde bulunabilir.

Anahtar Sözcükler: Kısmi diferansiyel denklemler, Hiperbolik sistemler, Başlangıç değer problemleri, Maxwell sistemi, Telgraf denklemi, İzotrop olmayan ortam, Green's fonksiyonu, Fourier dönüşümü.

CONTENTS

Page

CHAPTER ONE
INTRODUCTION

1.1 Problems Set-up

In this thesis we consider two partial differential operators $\mathcal{P}$ and $\mathcal{L}$ defined by

$$\mathcal{P}\mathbf{E} = \mathcal{E}\frac{\partial^2 \mathbf{E}}{\partial t^2} + \mathbf{curl}_x(\mathcal{M}^{-1}\mathbf{curl}_x\mathbf{E}), \tag{1.1.1}$$

$$\mathcal{L}\mathbf{u} = \frac{\partial^2 \mathbf{u}}{\partial t^2} - a^2\Delta_x\mathbf{u} + 2\mathcal{Q}\frac{\partial \mathbf{u}}{\partial t}, \tag{1.1.2}$$

where $x = (x_1, x_2, x_3) \in R^3$, $t \in R$; $\mathbf{u}(x,t) = \Big(u_1(x,t), u_2(x,t), u_3(x,t)\Big)$, $\mathbf{E}(x,t) = \Big(E_1(x,t), E_2(x,t), E_3(x,t)\Big)$ are vector functions,

$$\mathbf{curl}_x\mathbf{E} = \Big(\frac{\partial E_3}{\partial x_2} - \frac{\partial E_2}{\partial x_3}, \frac{\partial E_1}{\partial x_3} - \frac{\partial E_3}{\partial x_1}, \frac{\partial E_2}{\partial x_1} - \frac{\partial E_1}{\partial x_2}\Big),$$

$\mathcal{E} = \big(\varepsilon_{ij}(x)\big)_{3\times 3}$, $\mathcal{M}^{-1} = \big(m_{ij}(x)\big)_{3\times 3}$, $\mathcal{Q} = \big(q_{ij}(x)\big)_{3\times 3}$ are matrices of 3×3 order, a is a given positive constant.

In the Section 1.2.1 we show that operators $\mathcal{P}$ and $\mathcal{L}$ are hyperbolic if the matrices $\mathcal{E}$ and $\mathcal{M}^{-1}$ are symmetric positive definite. The main problems of this thesis are the following initial value problems.

Problem 1. Let $\mathcal{P}$ be the hyperbolic operator defined by (1.1.1), $\mathbf{f}(x,t) = \big(f_1(x,t),\ f_2(x,t),\ f_3(x,t)\big)$ be a given vector function for $x = (x_1, x_2, x_3) \in R^3$, $t \geq 0$. Find vector function $\mathbf{E}(x,t) = \Big(E_1(x,t), E_2(x,t), E_3(x,t)\Big)$ satisfying

$$\mathcal{P}\mathbf{E} = \mathbf{f}, \tag{1.1.3}$$

$$\mathbf{E}|_{t=0} = 0, \quad \frac{\partial \mathbf{E}}{\partial t}|_{t=0} = 0. \tag{1.1.4}$$

Problem 2. Let $\mathcal{L}$ be the hyperbolic operator defined by (1.1.2), $\mathbf{f}(x,t) = \Big(f_1(x,t),$ $f_2(x,t),\ f_3(x,t)\Big)$, $\varphi(x) = \Big(\varphi_1(x,t), \varphi_2(x,t), \varphi_3(x,t)\Big)$, $\psi(x) = \Big(\psi_1(x),\ \psi_2(x),$

$\psi_3(x)\Big)$ be given vector functions for $x = (x_1, x_2, x_3) \in R^3$, $t \geq 0$. Find vector function $\mathbf{u}(x,t) = \Big(u_1(x,t), u_2(x,t), u_3(x,t)\Big)$ satisfying

$$\mathcal{L}\mathbf{u} = \mathbf{f}, \tag{1.1.5}$$

$$\mathbf{u}|_{t=0} = \varphi(x), \quad \frac{\partial \mathbf{u}}{\partial t}|_{t=0} = \psi(x). \tag{1.1.6}$$

Nowadays in the view of growing interest to development of new anisotropic materials the analysis of the electromagnetic waves is an important issue and the study of different problems for (1.1.3) becomes actual. Many problems for (1.1.3) in homogeneous isotropic and anisotropic media have been studied and their applications have been made, see, for example, (Kong (1990), Ramo et al. (1994), Monk (2003), Cohen (2002), Lindell (1990), Haba (2004), Wijinands & Pendry (1997), Li et al. (2001), Gottis & Konddylis (1995), Ortner & Wagner (2004), Yakhno (2005), Zienkiewicz & Taylor (2000), Cohen et al. (2003), Yakhno et al. (2006)). In particular, decomposition method for the case of homogeneous isotropic materials ($\mathcal{M} = \mu\mathcal{I}$, $\mathcal{E} = \varepsilon\mathcal{I}$, μ, ε are positive constants; $\mathcal{I}$ is the identity matrix) has been studied in (Lindell (1990)). Analytic methods of Green's function constructions have been studied for the case of homogeneous isotropic materials in (Haba (2004), Wijinands & Pendry (1997)); for homogeneous uniaxial anisotropic media ($\mathcal{M} = \mu\mathcal{I}$, $\mathcal{E} = diag(\varepsilon_{11}, \varepsilon_{11}, \varepsilon_{33})$) in (Li et al. (2001), Gottis & Konddylis (1995)); for homogeneous biaxial anisotropic crystals ($\mathcal{M} = \mu\mathcal{I}$, $\mathcal{E} = diag(\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33})$) in (Ortner & Wagner (2004), Burridge & Qian (2006)); for arbitrary non-dispersive homogeneous anisotropic dielectrics ($\mathcal{M} = \mu\mathcal{I}$, $\mathcal{E} = (\varepsilon_{ij})_{3\times 3}$ is a symmetric positive definite matrix) in (Yakhno (2005)). Most of the studies and modeling electromagnetic waves had been made by numerical methods, in particular finite element method (Monk (2003), Cohen (2002), Zienkiewicz & Taylor (2000), Cohen et al. (2003)). The initial value problem for the system (1.1.3) has been studied in (Courant & Hilbert (1989), page 603-612) for the case $\mathcal{E} = diag(\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33})$, $\mathcal{M} = \mu\mathcal{I}$, where ε_{ij}, $j = 1, 2, 3$; μ are constants. This problem was reduced (see Courant & Hilbert (1989), page 603-612) to initial value problem for a fourth order partial differential equation and an explicit formula for the solution of the last problem was obtained. Using the plane wave approach IVP for (1.1.3) was investigated in (Ortner & Wagner (2004), Burridge & Qian (2006)) when $\mathcal{E} = diag(\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33})$, $\mathcal{M} = \mu\mathcal{I}$, where ε_{jj}, $j = 1, 2, 3$; μ are positive constants. In (Burridge & Qian (2006)) paper the presentation of a solution of IVP was given in an integral form. The paper contains also an analysis of the structure of the domain of the integration and the numerical calculation of an approximate solution. On the other hand nowadays computers can perform very complicated symbolic computations (in addition to numerical calculations) and this opens new possibilities in modeling and the simulation of the wave propagation phenomena. Symbolic computations can be considered as a useful tool for analytical methods which can provide exact solutions of IVP for (1.1.3). In (Yakhno (2005), Yakhno et al. (2006)) a new analytical method for constructing explicit formula of IVP for (1.1.3) inside different anisotropic non-dispersive homogeneous materials was obtained. In general case of dielectrics ($\mathcal{E} = (\varepsilon_{ij})_{3\times3}$ is a symmetric positive definite, $\mathcal{M} = \mu\mathcal{I}$) an explicit formula is very cumbersome and it has been computed using symbolic computation in MATLAB. Applying this explicit formula the simulation of the electric waves was obtained in (Yakhno et al. (2006)). Unfortunately the exact solution can not be found for all complex equations and systems. So, for example, there is no explicit formula for (1.1.3) in the case where $\mathcal{E}$ and $\mathcal{M}$ depend on one or all space variables.

One of the goals of this thesis is to find a method of solving Problem 1 for $t \in [0, T]$ in the case when T is a given positive number, $\mathcal{E}$ and $\mathcal{M}$ are given diagonal matrices with positive elements depending on x_3.

The Chapter 2 of the thesis is devoted to the study of Problem 1 in which the matrices $\mathcal{E}$ and $\mathcal{M}$ have the form

$$\mathcal{E} = diag(\varepsilon_{11}, \varepsilon_{11}, \varepsilon_{33}), \quad \mathcal{M} = diag(\mu_{11}, \mu_{11}, \mu_{33}),$$

and their elements are twice continuously differentiable functions depending on x_3 only and such that $\varepsilon_{jj} > 0$, $\mu_{jj} > 0$ for $x_3 \in R$, $j = 1, 3$. We suppose in the Chapter 2 that the Fourier transform of the vector function **f** with respect to variables x_1, x_2 has components which are continuous relative to all variables simultaneously. We note that such type of $\mathcal{E}$, $\mathcal{M}$ correspond to uniaxial anisotropic media (see Subsection 1.3.2).

In Chapter 3 of the thesis Problem 1 is studied for the case when

$$\mathcal{E} = diag(\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}), \quad \mathcal{M} = diag(\mu_{11}, \mu_{22}, \mu_{33}),$$

and the following assumptions are used. Let α, β, T be given positive numbers, $\alpha \le \beta$, $c = \sqrt{\beta/\alpha}$, Δ be the triangle given by

$$\Delta = \{(x_3, t): \ 0 \le t \le T, \ -c(T-t) \le x_3 \le c(T-t)\}.$$

We suppose that components of the Fourier transform of the vector function **f** with respect to variables x_1, x_2 are such that $\tilde{f}_j(\nu, x_3, t) \in C(R^2 \times \Delta), \ j = 1, 2, 3; \ \nu = (\nu_1, \nu_2) \in R^2$. We assume also that elements of diagonal positive definite matrices $\mathcal{E}$, $\mathcal{M}$ are twice continuously differentiable functions depending on x_3 variable only over $[-cT, cT]$ and such that $0 < \alpha \le \varepsilon_{jj}(x_3) \le \beta$, $0 < \alpha \le \dfrac{1}{\mu_{jj}(x_3)} \le \beta$, $j = 1, 2, 3$. We note that such type of $\mathcal{E}$, $\mathcal{M}$ corresponds to biaxial anisotropic vertical inhomogeneous media (see Section 1.3.2).

The main results of the Chapter 2 and Chapter 3 are methods of solving Problem 1 under above mentioned assumptions. These methods consist of the following. First of all Problem 1 is written in terms of the Fourier transform with respect to the space lateral variables . After that the obtained problem is transformed into an equivalent second kind vector integral equation of the Volterra type. Applying the successive approximations method to this integral equation we have constructed its solution. At last using the equivalence of this vector integral equation to IVP obtained after the Fourier transformation and the real Paley-Wiener theorem we found a solution of Problem 1. At the same time theorems about the existence and uniqueness of the solution were proved in the Chapter 2 and Chapter 3.

The problem 2 is studied in the Chapter 4. We note that the matrix $\mathcal{Q} = (g_{ij})_{3\times 3}$,

appearing in (1.1.2), corresponds to an anisotropy of electrical conductivity (see Section 1.3.3). The effect of the anisotropy of electrical conductivity is well known. So, for example, the fact that most biological tissues and earth materials have anisotropic conductivity values is well known (Seo et al. (2004), Weiss & Newmann (2003), and Wolters et al. (2005)). The mathematical models of these media are described by the Maxwell's system with anisotropic (matrix) conductivity (Seo et al. (2004), Weiss & Newmann (2003), and Wolters et al. (2005)). In the Section 1.3 and Chapter 4 we consider the Maxwell's system with a matrix conductivity and a constant dielectric
permittivity and a constant magnetic permeability. We rewrite this system in terms of the scalar and vector potentials and as a result of it we obtain the operator $\mathcal{L}$ defined by (1.1.2).

The second goal of the thesis is to construct a solution of Problem 2. In Chapter 4 the Green's function method is used for solving Problem 2. This method consists in constructing the Green's matrix of IVP for $\mathcal{L}$ (see Section 4.1) and then finding an explicit formula for a solution of Problem 2 using this Green's matrix (see Section 4.2). As an application of an explicit formula for a Green's matrix of IVP for Maxwell's operator with constant dielectric permittivity and magnetic permeability, and a matrix conductivity has been constructed (see Subsection 4.3.1) and generalized initial value problem has been solved (see Subsection 4.3.2).

1.2 Hyperbolicity of $\mathcal{P}$ and $\mathcal{L}$

In this section we give a definition of the hyperbolicity for the second order partial differential operators from (Ikawa (1999)). Using this definition we will show that the vector operators $\mathcal{P}$ and $\mathcal{L}$ are hyperbolic.

1.2.1 Definition of the Hyperbolicty for a Second Order System of the Partial Differential Operators

Consider the following 3×3 matrices,

$$A_{jl}(t,x), \quad j,l = 1,2,3, \quad H_j(t,x), \quad j = 0,1,2,3,$$

$$A_j(t,x), \quad j = 0,1,2,3,$$

where $x \in R^3$, $t \in R$.

Further, suppose that $u_j(t,x)$, $j = 1,2,3$ are unknown functions, and set $\mathbf{u}(t,x) = (u_1(t,x), u_2(t,x), u_3(t,x))$. Finally, let us consider the partial differential operator P that acts on $\mathbf{u}$ in the following form

$$\begin{aligned} \mathcal{P}\mathbf{u} &= \frac{\partial^2 \mathbf{u}}{\partial t^2} + \sum_{j=1}^{3}\sum_{l=1}^{3} A_{jl} \frac{\partial^2 \mathbf{u}}{\partial x_j \partial x_l} + \sum_{j=1}^{3} A_j \frac{\partial \mathbf{u}}{\partial x_j} \\ &+ 2\sum_{j=1}^{3} H_j \frac{\partial^2 \mathbf{u}}{\partial x_j \partial t} + H_0 \frac{\partial \mathbf{u}}{\partial t} + A_0 \mathbf{u}. \end{aligned} \tag{1.2.1}$$

For a second order partial differential operator $\mathcal{P}$, we will say that the principal part of $\mathcal{P}$ is

$$\mathcal{P}_0 = \frac{\partial^2}{\partial t^2} I_3 + \sum_{j=1}^{3}\sum_{l=1}^{3} A_{jl} \frac{\partial^2}{\partial x_j \partial x_l} + 2\sum_{j=1}^{3} H_j \frac{\partial^2}{\partial x_j \partial t}.$$

For $\lambda \in \mathcal{C}$ and $\xi = (\xi_1, \xi_2, \xi_3) \in R^3$, let

$$p_0(t,x,\lambda,\xi) = det\Big(\lambda^2 I_3 + 2\sum_{j=1}^{3} H_j(t,x)\xi_j\lambda + \sum_{j=1}^{3}\sum_{l=1}^{3} A_{jl}(t,x)\xi_j\xi_l\Big).$$

$p_0(t,x,\lambda,\xi)$ is called the characteristic polynomial of the partial differential operator $\mathcal{P}$. If we consider $(t,x,\lambda) \in (R \times R^3 \times R)$ to be a parameter and p_0 to be a polynomial in λ, then the degree of characteristic polynomial is 6. We denote the roots of $p_0(t,x,\lambda,\xi)$=0 by $\lambda_k(t,x,\xi)$, $(k = 1,2,\ldots,6)$; we call these roots the characteristic roots of the partial differential operators $\mathcal{P}$.

Definition 1.2.1. (Ikawa (1999)) The second order partial differential operator $\mathcal{P}$ given in (1.2.1) is said to be of hyperbolic type (or simply just hyperbolic) in the t direction if for an arbitrary parameter (t, x, ξ) the characteristic roots of P, $\lambda_k(t, x, \xi)$, $k = 1, 2, \ldots, 6$ are all real. Further, if

$$inf\ |\lambda_k(t, x, \xi) - \lambda_j(t, x, \xi)| > 0, \quad k \neq j,$$

then $\mathcal{P}$ is said to be regularly hyperbolic. In the above, we assume that the infimum is taken over $(t, x) \in (R \times R^3)$ and $|\xi| = 1$.

1.2.2 Hyperbolicty of $\mathcal{P}$

Lemma 1.2.2. *Let $\mathcal{E} = (\varepsilon_{ij}(x))_{3\times3}$ and $\mathcal{M}^{-1} = (m_{ij}(x))_{3\times3}$ be symmetric positive definite matrices. Then the operator $\mathcal{P}$ defined in (1.1.1) is hyperbolic.*

Proof. The operator $\mathbf{curl}_x$ that acts on $\mathbf{E}$ can be written in matrix form as follows

$$\mathbf{curl}_x\mathbf{E} = \mathcal{S}(\mathbf{D}_x)\mathbf{E}, \tag{1.2.2}$$

where $\mathbf{D}_x = (D_{x_1}, D_{x_2}, D_{x_3}), \quad (D_{x_j} = \dfrac{\partial}{\partial x_j}, \quad j = 1, 2, 3)$

$$\mathcal{S}(\mathbf{D}_x) = \begin{pmatrix} 0 & -D_{x_3} & D_{x_2} \\ D_{x_3} & 0 & -D_{x_1} \\ -D_{x_2} & D_{x_1} & 0 \end{pmatrix}.$$

The operator $\mathbf{curl}_x(\mathcal{M}^{-1}\mathbf{curl}_x) = \mathcal{S}(\mathbf{D}_x)\mathcal{M}^{-1}\mathcal{S}(\mathbf{D}_x)$ may be written in matrix form including second order derivatives only as follows

$$\mathbf{curl}_x(\mathcal{M}^{-1}\mathbf{curl}_x) = \Big(a_{jl}(D_x)\Big)_{3\times3} \tag{1.2.3}$$

where

$$\begin{aligned}
a_{11}(D_x) &= 2m_{23}D_{x_2}D_{x_3} - m_{33}D^2_{x_2^2} - m_{22}D^2_{x_3^2},\\
a_{12}(D_x) = a_{21}(D_x) &= -m_{13}D_{x_2}D_{x_3} + m_{33}D_{x_1}D_{x_2} + m_{12}D^2_{x_3^2} - m_{23}D_{x_1}D_{x_3},\\
a_{13}(D_x) = a_{31}(D_x) &= m_{13}D^2_{x_2^2} - m_{23}D_{x_1}D_{x_2} - m_{12}D_{x_2}D_{x_3} + m_{22}D_{x_1}D_{x_3},\\
a_{22}(D_x) &= -m_{11}D^2_{x_3^2} + 2m_{13}D_{x_1}D_{x_3} - m_{33}D^2_{x_1^2},\\
a_{23}(D_x) = a_{32}(D_x) &= m_{11}D_{x_2}D_{x_3} - m_{12}D_{x_1}D_{x_3} - m_{13}D_{x_1}D_{x_2} + m_{23}D^2_{x_1^2},\\
a_{33}(D_x) &= 2m_{12}D_{x_1}D_{x_2} - m_{22}D^2_{x_1^2} - m_{11}D^2_{x_2^2}.
\end{aligned}$$

The principal part of the operator $\mathcal{P}$ may be written in another form as follows

$$\mathcal{P}_0 = \mathcal{E}\frac{\partial^2}{\partial t^2} + \sum_{j=1}^{3}\sum_{l=1}^{3} A_{jl}(x)\frac{\partial^2}{\partial x_j \partial x_l}, \tag{1.2.4}$$

where

$$A_{11} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -m_{33} & m_{23} \\ 0 & m_{23} & -m_{22} \end{pmatrix}, \quad 2A_{12} = 2A_{21} = \begin{pmatrix} 0 & m_{33} & -m_{23} \\ m_{33} & 0 & -m_{13} \\ -m_{23} & -m_{13} & 2m_{12} \end{pmatrix},$$

$$A_{22} = \begin{pmatrix} -m_{33} & 0 & m_{13} \\ 0 & 0 & 0 \\ m_{13} & 0 & -m_{11} \end{pmatrix}, \quad 2A_{13} = 2A_{31} = \begin{pmatrix} 0 & -m_{23} & m_{22} \\ -m_{23} & 2m_{13} & -m_{12} \\ m_{22} & -m_{12} & 0 \end{pmatrix},$$

$$A_{33} = \begin{pmatrix} -m_{22} & m_{12} & 0 \\ m_{12} & -m_{11} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad 2A_{23} = 2A_{32} = \begin{pmatrix} 2m_{23} & -m_{13} & -m_{12} \\ -m_{13} & 0 & m_{11} \\ -m_{12} & m_{11} & 0 \end{pmatrix}.$$

Setting

$$\frac{\partial}{\partial t} \leftrightarrow \lambda, \quad D_{x_j} = \frac{\partial}{\partial x_j} \leftrightarrow \xi_j, \quad D_{x_j}D_{x_l} = \frac{\partial^2}{\partial x_j x_l} \leftrightarrow \xi_j\xi_l \quad j = 1,2,3; \;\; k = 1,2,3;$$

$\lambda \in \mathcal{C}$, $\xi = (\xi_1, \xi_2, \xi_3) \in R^3$ in the equation (1.2.4) we obtain the characteristic polynomial as follow

$$\det(\lambda^2\mathcal{E} - \mathcal{A}(x,\xi)), \tag{1.2.5}$$

where
$$\mathcal{A}(x,\xi) = -\sum_{j=1}^{3}\sum_{l=1}^{3} A_{jl}(x)\xi_j\xi_l = -\mathcal{S}(\xi)\mathcal{M}^{-1}\mathcal{S}(\xi).$$

We note that $\mathcal{A}(x,\xi)$ is symmetric. According to the definition 1.2.1 we need to show that all roots of (1.2.5) are real. For this we use the following theorem from (Goldberg (1992), page 383):

If $\mathcal{E}$ is symmetric positive definite and $\mathcal{A}$ is symmetric and positive semi-definite, then there exists a nonsingular matrix $\mathcal{T}$ such that

$$\mathcal{T}^T\mathcal{E}\mathcal{T} = \mathcal{I}, \quad \mathcal{T}^T\mathcal{A}\mathcal{T} = \mathcal{D},$$

where $\mathcal{I}$ and $\mathcal{D}$ are identity and diagonal matrices respectively. ($\mathcal{T}^T$ is the transpose of the matrix $\mathcal{T}$).

By the theorem 8.7.1 from (Goldberg (1992), page 382) there exists $\mathcal{E}^{1/2}$ such that $(\mathcal{E}^{1/2})^{1/2} = \mathcal{E}$. Moreover since $\mathcal{E}^{1/2}$ is positive definite, $\mathcal{E}^{-1/2}$ exists and symmetric, so that $(\mathcal{E}^{-1/2})^T = \mathcal{E}^{-1/2}$. The symmetric matrix $\mathcal{E}^{-1/2}\mathcal{A}\mathcal{E}^{-1/2}$ is unitarily similar to a diagonal matrix of its eigenvalues; that is, there exists an orthogonal matrix $\mathcal{Q}$ such that $\mathcal{Q}^T(\mathcal{E}^{-1/2}\mathcal{A}\mathcal{E}^{-1/2})\mathcal{Q} = \mathcal{D}$. Set $\mathcal{T} = \mathcal{E}^{-1/2}\mathcal{Q}$. Then

$$\mathcal{T}^T\mathcal{E}\mathcal{T} = (\mathcal{E}^{-1/2}\mathcal{Q})^T\mathcal{E}(\mathcal{E}^{-1/2}\mathcal{Q}) = \mathcal{Q}^T(\mathcal{E}^{-1/2}\mathcal{E}\mathcal{E}^{-1/2})\mathcal{Q} = \mathcal{Q}^T\mathcal{I}\mathcal{Q} = \mathcal{Q}^T\mathcal{Q} = \mathcal{I},$$

and
$$\mathcal{T}^T\mathcal{A}\mathcal{T} = (\mathcal{E}^{-1/2}\mathcal{Q})^T\mathcal{A}(\mathcal{E}^{-1/2}\mathcal{Q}) = \mathcal{Q}^T(\mathcal{E}^{-1/2}\mathcal{A}\mathcal{E}^{-1/2})\mathcal{Q} = \mathcal{D}.$$

The relations, written below show that the matrices $\mathcal{A}(x,\xi) = -\mathcal{S}(\xi)\mathcal{M}^{-1}\mathcal{S}(\xi)$ and $\mathcal{E}^{-1/2}\mathcal{A}\mathcal{E}^{-1/2}$ are positive semi definite:

$$\begin{aligned}
\Big(-(\mathcal{S}(\xi)\mathcal{M}^{-1}\mathcal{S}(\xi))\eta,\eta\Big) &= -\Big((\mathcal{M}^{-1}\mathcal{S}(\xi))\eta, \mathcal{S}^T(\xi)\eta\Big)\\
&= -\Big((\mathcal{M}^{-1}\mathcal{S}(\xi))\eta, -\mathcal{S}(\xi)\eta\Big)\\
&= \Big(\mathcal{M}^{-1}(\mathcal{S}(\xi)\eta), (\mathcal{S}(\xi)\eta)\Big) \geq 0.
\end{aligned}$$

$$\begin{aligned}
\Big(\mathcal{E}^{-1/2}(-\mathcal{S}(\xi)\mathcal{A}\mathcal{S}(\xi))\mathcal{E}^{-1/2}\eta,\eta\Big) &= \Big((-\mathcal{S}(\xi)\mathcal{A}\mathcal{S}(\xi))\mathcal{E}^{-1/2}\eta, \mathcal{E}^{-1/2}\eta\Big)\\
&= \Big(-\mathcal{A}(\mathcal{S}(\xi)\mathcal{E}^{-1/2}\eta), -(\mathcal{S}(\xi)\mathcal{E}^{-1/2}\eta)\Big) \geq 0.
\end{aligned}$$

Here we used $\mathcal{S}^T(\xi) = -\mathcal{S}(\xi)$, ($\mathcal{S}(\xi)$ is skew-symmetric). Positive semi-definiteness of the matrix $\mathcal{E}^{-1/2}\mathcal{A}\mathcal{E}^{-1/2}$ implies that the eigenvalues of $\mathcal{D}$ are nonnegative by theorem 8.4.2 from (Goldberg (1992), page 366). We have

$$\mathcal{T}^T(\lambda^2\mathcal{E} - \mathcal{A})\mathcal{T} = \lambda^2\mathcal{T}^T\mathcal{E}\mathcal{T} - \mathcal{T}^T\mathcal{A}\mathcal{T} = \lambda^2\mathcal{I} - \mathcal{D}.$$

It follows from the above equality that

$$\det(\mathcal{T}^T(\lambda^2\mathcal{E} - \mathcal{A})\mathcal{T}) = \det(\lambda^2\mathcal{I} - \mathcal{D}).$$

Further, since $\mathcal{T}$ is nonsingular we can get

$$\det(\lambda^2\mathcal{E} - \mathcal{A})) = \frac{\det(\lambda^2\mathcal{I} - \mathcal{D})}{\det(\mathcal{T}^T)\det(\mathcal{T})}.$$

The last equality implies that roots of $\det(\lambda^2\mathcal{E}-\mathcal{A})$ are equal to roots of $\det(\lambda^2\mathcal{I}-\mathcal{D})$. As a result characteristic roots of (1.2.5) are all real. This shows that the operator $\mathcal{P}$ defined by (1.1.1) is hyperbolic according to the definition 1.2.1. □

1.2.3 Hyperbolicty of $\mathcal{L}$

Lemma 1.2.3. *Let a be given positive number, $\mathcal{Q} = (q_{ij}(x))_{3\times 3}$ be a matrix, $\mathcal{L}$ be the operator defined by (1.1.2). Then the operator $\mathcal{L}$ is hyperbolic.*

Proof. We find that the principal part of the operator $\mathcal{L}$ may be defined in another form as follows

$$\mathcal{L}_0 = \mathcal{I}\frac{\partial^2}{\partial t^2} - a\mathcal{I}\Delta_x. \tag{1.2.6}$$

Further $\mathcal{I}\Delta_x$ can be written in the form

$$\mathcal{I}\Delta_x = \sum_{j=1}^{3}\sum_{l=1}^{3}\mathcal{A}_{jl}\frac{\partial^2}{\partial x_j \partial x_l},$$

where

$$\mathcal{A}_{jl} = \delta_{ij}\mathcal{I}, \quad \delta_{ij} = \begin{cases} 1 & j = l; \\ 0 & j \neq l \end{cases} \quad j, l = 1, 2, 3.$$

Setting

$$\frac{\partial}{\partial t} \leftrightarrow \lambda, \quad \frac{\partial}{\partial x_j} \leftrightarrow \xi_j, \quad \frac{\partial^2}{\partial x_j x_l} \leftrightarrow \xi_j \xi_l \quad j = 1, 2, 3; \;\; k = 1, 2, 3;$$

$\lambda \in \mathcal{C}$, $\xi = (\xi_1, \xi_2, \xi_3) \in R^3$ in the equation (1.2.6) we obtain the characteristic polynomial as follows

$$\det(\lambda^2 \mathcal{I} - a^2 \mathcal{A}(\xi)), \tag{1.2.7}$$

where

$$\mathcal{A}(\xi) = \sum_{j=1}^{3} \sum_{l=1}^{3} A_{jl} \xi_j \xi_l.$$

Roots of the characteristic polynomial are $\lambda_j = a|\xi|$; $\lambda_{j+1} = -a|\xi|$, $j = 1, 2, 3$. Since the roots of (1.2.6) are all real we conclude that the operator $\mathcal{L}$ defined in (1.1.2) is hyperbolic according to the definition 1.2.1. □

1.3 Application to Electrodynamics

1.3.1 Equation (1.1.3) as an Equation of the Electric Field in Anisotropic Materials

Time dependent Maxwell equations in three dimensional ($3D$ case) can be written as follows (see for example Cohen (2002), Ramo et al. (1994))

$$\mathbf{curl}_x \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J}, \tag{1.3.1}$$

$$\mathbf{curl}_x \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{1.3.2}$$

$$div_x \mathbf{B} = 0, \tag{1.3.3}$$

$$div_x \mathbf{D} = \rho, \tag{1.3.4}$$

where $x = (x_1, x_2, x_3)$ is a space variable from R^3; t is a time variable from R, $\mathbf{E} = (E_1, E_2, E_3)$ and $\mathbf{H} = (H_1, H_2, H_3)$ are electric and magnetic fields,

$E_k = E_k(x,t)$, $H_k = H_k(x,t)$, $k = 1,2,3$; $\mathbf{D} = (D_1, D_2, D_3)$ and $\mathbf{B} = (B_1, B_2, B_3)$ are electric an magnetic inductions, $D_k = D_k(x,t)$, $B_k = B_k(x,t)$, $k = 1,2,3$; $\mathbf{J} = (J_1, J_2, J_3)$ is the density of the electric current $J_k = J_k(x,t)$, $k = 1,2,3$; ρ is the density of electric charges. The values ρ and $\mathbf{J}$ satisfy the relation

$$\frac{\partial \rho}{\partial t} + div_x \mathbf{J} = 0, \tag{1.3.5}$$

and hence the equations (1.3.1) and (1.3.2) are related to each other. The relation (1.3.5) expresses the law of the conservation of the electric charge.
In general there are constitutive relations that express $\mathbf{D}$, $\mathbf{B}$ and $\mathbf{J}$ in terms of $\mathbf{E}$ and $\mathbf{H}$. These equations are

$$\mathbf{D} = \mathcal{E}\mathbf{E} \quad \mathbf{B} = \mathcal{M}\mathbf{H}, \quad \mathbf{J} = \sigma\mathbf{E} + \mathbf{j}, \tag{1.3.6}$$

where $\mathcal{E}$ is the dielectric permittivity $\mathcal{M}$ is the magnetic permeability, σ is the conductivity and $\mathbf{j}$ is the density of the currents arising from the action of the external electromagnetic forces. Moreover we suppose that

$$\mathbf{E} = 0, \quad \mathbf{H} = 0, \quad \rho = 0, \quad \mathbf{j} = 0 \quad \text{for t} \leq 0. \tag{1.3.7}$$

This means that there is no electric charges and currents at the time $t \leq 0$; electric and magnetic fields vanish $t \leq 0$.

Remark 1.3.1. We note that equation (1.3.3) follows immediately from (1.3.5), (1.3.6), and equation (1.3.4) can be obtained from (1.3.1), (1.3.5), (1.3.6). So equalities (1.3.1), (1.3.2), (1.3.6) with conditions (1.3.5) imply (1.3.3), (1.3.4).

Remark 1.3.2. We note that ρ can be defined as a solution of the initial value problem for the ordinary differential equation (1.3.5) with respect to t, subject to $\rho|_{t\leq 0} = 0$. Here $div_x\mathbf{J}$ is given.

Let us consider the equations (1.3.1)-(1.3.4) for the case:

$$\mathcal{E} = \mathcal{E}(x) = (\varepsilon_{ij}(x))_{3\times 3}, \quad \mathcal{M} = \mathcal{M}(x) = (\mu_{ij}(x))_{3\times 3} \quad \sigma = 0. \tag{1.3.8}$$

We shall use (1.3.1), (1.3.2) and (1.3.6) to eliminate $\mathbf{D}$ and $\mathbf{B}$ from Maxwell's equations. Hence we shall generally deal with equations involving $\mathbf{E}$ and $\mathbf{H}$. By combining (1.3.1)-(1.3.4), we obtain the second form the Maxwell's equations:

$$\mathcal{P}\mathbf{E} = \mathbf{f},$$

where $\mathcal{P}$ is the vector operator defined by (1.1.1), $\mathbf{f} = -\dfrac{\partial \mathbf{J}}{\partial t}$, and

$$\mathcal{R}\mathbf{H} = \mathbf{f}^{\star},$$

where $\mathbf{f}^{\star} = \mathbf{curl}_x(\mathcal{E}^{-1}\mathbf{J})$, and $\mathcal{R}$ is the vector operator defined by

$$\mathcal{R}\mathbf{H} = \mathcal{M}\frac{\partial^2 \mathbf{H}}{\partial t^2} + \mathbf{curl}_x(\mathcal{E}^{-1}\mathbf{curl}_x\mathbf{H}).$$

We note that the vector operator $\mathcal{R}$ is defined similarly to the operator $\mathcal{P}$ (see formula (1.1.1)). To define $\mathcal{R}$ we can replace the matrices $\mathcal{E}$ and $\mathcal{M}$.

1.3.2 Different Types of Anisotropic Materials

We note that if the characteristics of the material do not depend on position, the material is said to be homogeneous, otherwise inhomogeneous. (For instance the atmosphere is inhomogeneous). If the characteristics of the material are independent from the direction of the vectors, the material is isotropic, otherwise anisotropic. (For instance some important crystals are anisotropic). The matrices $\mathcal{E}$ and $\mathcal{M}$ describe electric and magnetic properties (characteristics) of a material. Materials can be classified according to electric and magnetic properties of the media. (see, Kong (1990), Herbert & Neff (1987))

Isotropic Homogeneous Media

$$\mathcal{E} = \varepsilon\mathcal{I}, \quad \mathcal{M} = \mu\mathcal{I},$$

where $\mathcal{I}$ is the identity matrix of order 3×3, ε, μ are positive constants.

Electrically Anisotropic Inhomogeneous Media

$$\mathcal{E} = (\varepsilon_{ij}(x))_{3\times 3}, \quad \mathcal{M} = \mu\mathcal{I},$$

where $\mathcal{I}$ is the identity matrix of order 3×3, μ is positive constant.

Magnetically Anisotropic Inhomogeneous Media

$$\mathcal{E} = \varepsilon\mathcal{I}, \quad \mathcal{M} = (\mu_{ij}(x))_{3\times 3},$$

where $\mathcal{I}$ is the identity matrix of order 3×3, ε is a positive constant.

Electrically and Magnetically Anisotropic Inhomogeneous Media

$$\mathcal{E} = (\varepsilon_{ij}(x))_{3\times 3}, \quad \mathcal{M} = (\mu_{ij}(x))_{3\times 3}.$$

Crystals are described in general by symmetric permittivity tensors. There always exists a coordinate transformation that transforms a symmetric matrix into a diagonal matrix. For cubic crystals $\mathcal{E} = diag(\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33})$, $\varepsilon_{11} = \varepsilon_{22} = \varepsilon_{33}$, and they are isotropic. In tetragonal, hexagonal crystals, two out of three parameters are equal (for instance, $\varepsilon_{11} = \varepsilon_{22} \neq \varepsilon_{33}$). Such crystals are uniaxial. In orthorhombic, monoclinic, and triclinic crystals, $\varepsilon_{11} \neq \varepsilon_{22} \neq \varepsilon_{33}$, and the medium is biaxial.

1.3.3 Equation (1.1.5) as an Equation for the Electric Vector Potential in Media with Electric Conductivity

Let us consider now equations (1.3.1)-(1.3.8) for the case:

$$\mathcal{E} = \varepsilon\mathcal{I}, \quad \mathcal{M} = \mu\mathcal{I}, \quad \sigma = (\sigma_{ij})_{3\times 3},$$

where ε, μ and σ_{ij} are constants, $\mathcal{I}$ is the identity matrix. Using the reasoning of Section 1.3.1, remark 1.3.1 and remark 1.3.2 we find that equations

$$\mathbf{curl}_x\mathbf{H} = \frac{\partial(\varepsilon\mathbf{E})}{\partial t} + \sigma\mathbf{E} + \mathbf{j}, \tag{1.3.9}$$

$$\mathbf{curl}_x\mathbf{E} = -\frac{\partial(\mu\mathbf{H})}{\partial t} \tag{1.3.10}$$

are basic equations under conditions (1.3.7). Let us consider the following presentation for **H** and **E**

$$\mathbf{H} = \frac{1}{\mu}\mathbf{curl}_x\mathbf{A}, \tag{1.3.11}$$

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} + \nabla_x\varphi, \tag{1.3.12}$$

where **A** is the vector function φ is the scalar function which are called the vector and scalar potentials respectively. Substituting (1.3.11), (1.3.12) into the equation (1.3.9) using the property $\mathbf{curl}_x(\mathbf{curl}_x\mathbf{A}) = \nabla_x div_x\mathbf{A} - \Delta_x\mathbf{A}$, we find the following equality

$$\frac{1}{\mu}\nabla_x div_x\mathbf{A} - \varepsilon\frac{\partial\phi}{\partial t} - \sigma\phi + \varepsilon\frac{\partial^2\mathbf{A}}{\partial t^2} - \frac{1}{\mu}\Delta_x\mathbf{A} + \sigma\frac{\partial\mathbf{A}}{\partial t} = \mathbf{j}, \tag{1.3.13}$$

where $\phi = \nabla_x\varphi$. Let us choose the vector function **A** from

$$\mathcal{L}\mathbf{A} = \mathbf{f}, \tag{1.3.14}$$

Let $\mathcal{L}$ be the operator defined by (1.1.2), $a = \frac{1}{\sqrt{\mu\varepsilon}}$, $2\mathcal{Q} = \frac{1}{\varepsilon}\sigma$, $\mathbf{f} = a^2\mu\mathbf{j}$. Then we find from (1.3.13), (1.3.14) that has to satisfy

$$\frac{\partial\phi}{\partial t} + 2\mathcal{Q}\phi = a^2\nabla_x div_x\mathbf{A}. \tag{1.3.15}$$

For holding (1.3.7) the conditions $\phi|_{t\leq 0} = 0$, $\mathbf{A}|_{t\leq 0} = 0$ are sufficient.
If the vector potential **A** is found then the scalar potential can be defined by

$$\phi(x,t) = a^2\int_{-\infty}\int_{\infty}\theta(t-\tau)\exp(-2\mathcal{Q}t)\nabla_x div_x\mathbf{A}(x,\tau)d\tau. \tag{1.3.16}$$

CHAPTER TWO
INITIAL VALUE PROBLEM FOR THE VECTOR EQUATION OF ELECTRIC FIELD IN UNIAXIAL MATERIALS

The time dependent electric field $\mathbf{E}$ in electrically and magnetically anisotropic media is governed by the following vector equation (see Section 1.3)

$$\mathcal{E}\frac{\partial^2 \mathbf{E}}{\partial t^2} + curl_x(\mathcal{M}^{-1} curl_x \mathbf{E}) \;\; = \;\; \mathbf{f}, \tag{2.0.1}$$

where $x = (x_1, x_2, x_3) \in R^3$ is the space variable, $t \in R$ is the time variable, $\mathbf{E} = (E_1, E_2, E_3)$ is the vector function with components $E_k = E_k(x,t),\; k = 1,2,3$; $\mathbf{f} = -\partial \mathbf{j}(x,t)/\partial t$, $\mathbf{j}(x,t) = (j_1(x,t), j_2(x,t), j_3(x,t))$ is the density of electric current; $\mathcal{M}^{-1}$ is the inverse matrix of the positive definite matrix $\mathcal{M}$ of the magnetic permeability; $\mathcal{E}$ is the positive definite matrix of electric permittivity. The main object of the this Chapter is Problem 1 which consists of finding the vector function $\mathbf{E}$(x,t) satisfying (2.0.1) and conditions

$$\mathbf{E}|_{t=0} = 0, \quad \frac{\partial \mathbf{E}}{\partial t}|_{t=0} = 0. \tag{2.0.2}$$

We suppose that the Fourier transform of the vector function $\mathbf{f}$ with respect to variables x_1, x_2 has components which are continuous relative to all variables simultaneously. We assume also that $\mathcal{E}$ and $\mathcal{M}^{-1}$ are diagonal matrices of the form

$$\mathcal{E} = diag(\varepsilon_{11}, \varepsilon_{11}, \varepsilon_{33}),\; \mathcal{M}^{-1} = diag(m_{11}, m_{11}, m_{33})$$

and the elements of these matrices are twice continuously differentiable functions depending on x_3 variable only and such that $\varepsilon_{jj}(x_3) > 0$, $m_{jj}(x_3) > 0$ for $x_3 \in R$, $j = 1,3$. We note that such type of $\mathcal{E}$ and $\mathcal{M}^{-1}$ corresponds to uniaxial anisotropic vertical inhomogeneous media. The main result of this Chapter is a new method for solving the stated IVP. This method has several steps. On the first step the original initial value problem is written in terms of the Fourier transform with respect to lateral variables x_1, x_2. After that the obtained problem is transformed into an equivalent second kind vector integral equation of the

Volterra type. A solution of this integral equation is constructed by successive approximations. At last, using the real Paley-Wiener theorem, a solution of the original IVP is found. In addition, theorem about existence and uniqueness of the IVP (2.0.1), (2.0.2) is proved.

2.1 FIVP and Its Reduction to a Vector Integral Equation

In this section IVP (2.0.1), (2.0.2) is written in terms of the Fourier images with respect to the space variables x_1, x_2. We show that FIVP is equivalent to a second kind vector integral equation of Volterra type. Properties of this vector integral equation are described.

This Section is organized as follows. In Subsection 2.1.1 FIVP is stated. This FIVP consists of a system of three partial differential equations with two independent variables x_3, t. The two-dimensional Fourier transform parameter $\nu = (\nu_1, \nu_2) \in R^2$ is appeared in the obtained system. The principal part of this system contains function-coefficients depending on x_3. In Subsection 2.1.2 the obtained system is simplified to a 'canonical' form. The important part of this simplification consists in the following. The principal part of the first two equations of the simplified system has the form of the simplest one-dimensional wave equation and the last third equation is an ordinary differential equation. Using D'Alambert formula for the wave equation and an explicit formula for a linear ordinary differential equation in Subsection 2.1.3 we have transformed the obtained simplified IVP to a second kind vector integral equation of the Volterra type. Essential properties of this vector integral equation are described in the Subsection 2.1.4.

2.1.1 Statement of FIVP

Let components of vector functions $\tilde{\mathbf{E}}(\nu, x_3, t) = (\tilde{E}_1(\nu, x_3, t), \tilde{E}_2(\nu, x_3, t),$ $\tilde{E}_3(\nu, x_3, t))$, and $\tilde{\mathbf{f}}(\nu, x_3, t) = (\tilde{f}_1(\nu, x_3, t), \tilde{f}_2(\nu, x_3, t), \tilde{f}_3(\nu, x_3, t))$ be defined by

$$\tilde{E}_j(\nu, x_3, t) = \mathcal{F}_{x_1x_2}[E_j](\nu, x_3, t), \;\; \tilde{f}_j(\nu, x_3, t) = \mathcal{F}_{x_1x_2}[f_j](\nu, x_3, t),$$

$$j = 1, 2, 3, \;\; \nu = (\nu_1, \nu_2) \in R^2,$$

where $\mathcal{F}_{x_1x_2}$ is the Fourier transform with respect to x_1, x_2, i.e.

$$\mathcal{F}_{x_1x_2}[\mathbf{E}](\nu, x_3, t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathbf{E}(x, t) e^{i(\nu_1 x_1 + \nu_2 x_2)} dx_1 dx_2, \quad i^2 = -1,$$

$\nu = (\nu_1, \nu_2) \in R^2$ is the Fourier transform parameter.
Applying the operator $\mathcal{F}_{x_1x_2}$ to (2.0.1), (2.0.2) and using the properties of the Fourier transform we can write the problem (2.0.1), (2.0.2) in terms of the Fourier image $\tilde{\mathbf{E}}(\nu, x_3, t)$ as follows

$$\begin{aligned} \varepsilon_{11}(x_3)\frac{\partial^2 \tilde{E}_j}{\partial t^2} - \frac{\partial}{\partial x_3}\Big(m_{11}(x_3)\frac{\partial \tilde{E}_j}{\partial x_3}\Big) &= -\nu_k^2 m_{33}(x_3)\tilde{E}_j + \nu_j \nu_k m_{33}(x_3)\tilde{E}_k \\ &+ (i\nu_j)\frac{\partial}{\partial x_3}\Big(m_{11}(x_3)\tilde{E}_3\Big) + \tilde{f}_j, \quad (2.1.1) \end{aligned}$$

$$\begin{aligned} \varepsilon_{33}(x_3)\frac{\partial^2 \tilde{E}_3}{\partial t^2} + (\nu_1^2 + \nu_2^2) m_{11}(x_3)\tilde{E}_3 &= m_{11}(x_3)[(i\nu_1)\frac{\partial \tilde{E}_1}{\partial x_3} \\ &+ (i\nu_2)\frac{\partial \tilde{E}_2}{\partial x_3}] + \tilde{f}_3, \qquad (2.1.2) \end{aligned}$$

$$\tilde{\mathbf{E}}|_{t=0} = 0, \qquad \frac{\partial \tilde{\mathbf{E}}}{\partial t}|_{t=0} = 0, \qquad (2.1.3)$$

where $j = 1, 2$; k is different from j and runs values $1, 2$.

2.1.2 FIVP (2.1.1) - (2.1.3) in terms of 'Canonical' Variables

Let us consider the following transformation

$$y = \tau(x_3), \;\; \tau(x_3) = \int_0^{x_3} c(\xi) d\xi, \quad c^2(\xi) = \frac{\varepsilon_{11}(\xi)}{m_{11}(\xi)}. \qquad (2.1.4)$$

We note that the function $y = \tau(x_3)$ has the inverse function which we denote as $x_3 = \tau^{-1}(y)$. Let us denote

$$\tilde{W}_l(\nu, y, t) = \tilde{E}_l(\nu, x_3, t)|_{x_3=\tau^{-1}(y)},\ l = 1, 2, 3, \tag{2.1.5}$$

then the following relations hold

$$\frac{\partial \tilde{E}_m}{\partial x_3}(\nu, x_3, t)|_{x_3=\tau^{-1}(y)} = c(\tau^{-1}(y))\frac{\partial \tilde{W}_m}{\partial y}(\nu, y, t),\ m = 1, 2, 3. \tag{2.1.6}$$

Equations (2.1.1), (2.1.2) may be written in terms of y and $\tilde{W}_l(\nu, y, t),\ l = 1, 2, 3,$ as follows

$$\frac{\partial^2 \tilde{W}_j}{\partial t^2} - \frac{\partial^2 \tilde{W}_j}{\partial y^2} = -K(y)\frac{\partial \tilde{W}_j}{\partial y} - \frac{m_{33}(\tau^{-1}(y))}{\varepsilon_{11}(\tau^{-1}(y))}\Big[\nu_k^2 \tilde{W}_j - \nu_j \nu_k \tilde{W}_k\Big]$$
$$+\frac{(i\nu_j)}{\varepsilon_{11}(\tau^{-1}(y))}\Big[m'_{11}(\tau^{-1}(y))\tilde{W}_3 + m_{11}(\tau^{-1}(y))c(\tau^{-1}(y))\frac{\partial \tilde{W}_3}{\partial y}\Big]$$
$$+\frac{\tilde{f}_j(\nu, \tau^{-1}(y), t)}{\varepsilon_{11}(\tau^{-1}(y))}, \tag{2.1.7}$$

where

$$K(y) = \frac{d}{dy}\Big(lnA(y)\Big), \quad A(y) = \frac{1}{\sqrt{m_{11}(x_3)\varepsilon_{11}(x_3)}}|_{x_3=\tau^{-1}(y)}, \tag{2.1.8}$$

$$j = 1, 2;\ k \neq j,\ k = 1, 2;$$

$$\frac{\partial^2 \tilde{W}_3}{\partial t^2} + \frac{(\nu_1^2 + \nu_2^2)m_{11}(x_3)}{\varepsilon_{33}(x_3)}|_{x_3=\tau^{-1}(y)}\tilde{W}_3 = \frac{m_{11}(x_3)c(x_3)}{\varepsilon_{33}(x_3)}|_{x_3=\tau^{-1}(y)}\Big[i\nu_1\frac{\partial \tilde{W}_1}{\partial y}$$

$$+i\nu_2\frac{\partial \tilde{W}_2}{\partial y}\Big] + \frac{\tilde{f}_3(\nu, x_3, t)}{\varepsilon_{33}(x_3)}|_{x_3=\tau^{-1}(y)}. \tag{2.1.9}$$

We seek a solution of (2.1.7), (2.1.9) in the following form

$$\tilde{W}_l(\nu, y, t) = S(y)\tilde{V}_l(\nu, y, t),\ l = 1, 2, 3, \tag{2.1.10}$$

where the function $S(y)$ is defined by

$$S(y) = exp(\frac{1}{2}\int_0^y K(\xi)d\xi). \tag{2.1.11}$$

Substituting (2.1.10) into (2.1.7) and (2.1.9) we find

$$\frac{\partial^2 \tilde{V}_j}{\partial t^2} - \frac{\partial^2 \tilde{V}_j}{\partial y^2} = [q(y) - \nu_k^2 L_3(y)]\tilde{V}_j + \nu_j \nu_k L_3(y) \tilde{V}_k$$

$$+i\nu_j \Big[L_2(y)\tilde{V}_3 + L_1(y)\frac{\partial \tilde{V}_3}{\partial y} \Big] + F_j(\nu, y, t), \quad j = 1,2;\ k \neq j,\ k = 1,2; \quad (2.1.12)$$

$$\frac{\partial^2 \tilde{V}_3}{\partial t^2} + (\nu_1^2 + \nu_2^2) L_4(y) \tilde{V}_3 = L_5(y) \Big[i\nu_1 \Big(\frac{K(y)}{2}\tilde{V}_1 + \frac{\partial \tilde{V}_1}{\partial y} \Big) +$$

$$+ i\nu_2 \Big(\frac{K(y)}{2}\tilde{V}_2 + \frac{\partial \tilde{V}_2}{\partial y} \Big) \Big] + F_3(\nu, y, t). \quad (2.1.13)$$

Here the following notations were used

$$q(y) = \frac{1}{2}K'(y) - \frac{1}{4}K^2(y),$$
$$L_1(y) = \frac{M_1(y)C(y)}{N_1(y)}, \quad L_2(y) = \frac{M_1'(y)C(y)}{N_1(y)} + \frac{M_1(y)C(y)K(y)}{2N_1(y)},$$
$$L_3(y) = \frac{M_3(y)}{N_1(y)}, \quad F_j(\nu, y, t) = \frac{\tilde{f}_j(\nu, \tau^{-1}(y), t)}{S(y)N_1(y)},\ j = 1,2, \quad (2.1.14)$$

$$L_4(y) = \frac{M_1(y)}{N_3(y)}, \quad L_5(y) = \frac{M_1(y)C(y)}{N_3(y)},$$
$$F_3(\nu, y, t) = \frac{\tilde{f}_3(\nu, \tau^{-1}(y), t)}{S(y)N_3(y)}, \quad (2.1.15)$$

where $K(y)$, $S(y)$ are defined by (2.1.8), (2.1.11) and $C(y)$, $N_n(y)$, $M_n(y)$, $l = 1,3$ are defined by

$$C(y) = c(x_3)|_{x_3=\tau^{-1}(y)}, \quad N_n(y) = \varepsilon_{nn}(x_3)|_{x_3=\tau^{-1}(y)},$$
$$M_n(y) = m_{nn}(x_3)|_{x_3=\tau^{-1}(y)},\ n = 1,3. \quad (2.1.16)$$

Initial data (2.1.3) in terms of $\tilde{V}_l(\nu, y, t)$ are written as

$$\tilde{V}_l|_{t=0} = 0, \quad \frac{\partial \tilde{V}_l}{\partial t}|_{t=0} = 0, \quad l = 1,2,3. \quad (2.1.17)$$

We note that the problem (2.1.12), (2.1.13), (2.1.17) is FIVP in terms of variables y, t, and unknown functions $\tilde{V}_l(\nu, y, t)$, $l = 1,2,3$ depending on y, t and the parameter $\nu \in R^2$.

2.1.3 Reduction of IVP (2.1.12), (2.1.13), (2.1.17) to a Vector Integral Equation

Using D'Alambert formula (Vladimirov (1971), see also Appendix A) we can show that equation (2.1.12) with zero initial data (2.1.17) is equivalent to the following integral equation

$$\tilde{V}_j(\nu, y, t) = \frac{1}{2}\int_0^t \int_{y-(t-\tau)}^{y+(t-\tau)} \Big\{ \Big[q(\xi) - \nu_k^2 L_3(\xi)\Big] \tilde{V}_j(\nu, \xi, \tau)$$
$$+\nu_j \nu_k L_3(\xi) \tilde{V}_k(\nu, \xi, \tau) + i\nu_j \Big[L_2(\xi)\tilde{V}_3(\nu, \xi, \tau) + L_1(\xi)\frac{\partial \tilde{V}_3}{\partial \xi}(\nu, \xi, \tau)\Big]$$
$$+F_j(\nu, \xi, \tau)\Big\} d\xi d\tau, \quad j = 1, 2;\ k \neq j,\ k = 1, 2. \quad (2.1.18)$$

Using the formula

$$L_1(y)\frac{\partial \tilde{V}_3}{\partial y}(\nu, y, t) = \frac{\partial}{\partial y}\Big(L_1(y)\tilde{V}_3(\nu, y, t)\Big) - L_1'(y)\tilde{V}_3(\nu, y, t)$$

equation (2.1.18) may be written as follows

$$\tilde{V}_j(\nu, y, t) = \frac{1}{2}\int_0^t \int_{y-(t-\tau)}^{y+(t-\tau)} \Big\{ \Big[q(\xi) - \nu_k^2 L_3(\xi)\Big] \tilde{V}_j(\nu, \xi, \tau)$$

$$+\nu_j \nu_k L_3(\xi)\tilde{V}_k(\nu, \xi, \tau) + i\nu_j \Big[L_2(\xi) - L_1'(\xi)\Big]\tilde{V}_3(\nu, \xi, \tau)\Big\} d\xi d\tau$$

$$+\frac{i\nu_j}{2}\int_0^t \Big[L_1(y + (t-\tau))\tilde{V}_3(\nu, y + (t-\tau), \tau)$$

$$-L_1(y - (t-\tau))\tilde{V}_3(\nu, y - (t-\tau), \tau)\Big] d\tau$$

$$+\frac{1}{2}\int_0^t \int_{y-(t-\tau)}^{y+(t-\tau)} F_j(\nu, \xi, \tau) d\xi d\tau, \;\; j = 1, 2; k = 1, 2; j \neq k. \quad (2.1.19)$$

After changing a variable in the second integral, the equation (2.1.19) has the form

$$\tilde{V}_j(\nu, y, t) \;\; = \;\; \frac{1}{2}\int_0^t \int_{y-(t-\tau)}^{y+(t-\tau)} \Big\{ \Big[q(\xi) - \nu_k^2 L_3(\xi)\Big] \tilde{V}_j(\nu, \xi, \tau)$$

$$+\nu_j\nu_k L_3(\xi)\tilde{V}_k(\nu,\xi,\tau) + i\nu_j\Big[L_2(\xi) - L_1'(\xi)\Big]\tilde{V}_3(\nu,\xi,\tau)\Big\}d\xi d\tau$$

$$+\frac{i\nu_j}{2}\Big\{\int_y^{y+t} L_1(\eta)\tilde{V}_3(\nu,\eta,y+t-\eta)d\eta$$

$$-\int_{y-t}^{y} L_1(\mu)\tilde{V}_3(\nu,\mu,-y+t+\mu)d\mu\Big\}$$

$$+\frac{1}{2}\int_0^t\int_{y-(t-\tau)}^{y+(t-\tau)} F_j(\nu,\xi,\tau)d\xi d\tau,\ j=1,2;\ k=1,2;\ j\neq k. \qquad (2.1.20)$$

Differentiating (2.1.20) with respect to y we get equations the left hand sides of which contain $\frac{\partial\tilde{V}_j}{\partial y}$, $j=1,2$. These are the following equations

$$\frac{\partial\tilde{V}_j}{\partial y}(\nu,y,t) = \frac{1}{2}\int_0^t\Big\{\Big[q(\xi)-\nu_k^2L_3(\xi)\Big]\tilde{V}_j(\nu,\xi,\tau)$$

$$+\nu_j\nu_kL_3(\xi)\tilde{V}_k(\nu,\xi,\tau) + i\nu_j\Big[L_2(\xi)-L_1'(\xi)\Big]\tilde{V}_3(\nu,\xi,\tau)$$

$$+i\nu_jL_1(\xi)\frac{\partial\tilde{V}_3}{\partial t}(\nu,\xi,\tau)\Big\}\Big|_{\xi=y-(t-\tau)}^{\xi=y+(t-\tau)}d\tau - i\nu_jL_1(y)\tilde{V}_3(\nu,y,t)$$

$$+\frac{1}{2}\int_0^t\Big\{F_j(\nu,\xi,\tau)\Big\}\Big|_{\xi=y-(t-\tau)}^{\xi=y+(t-\tau)}d\tau, \qquad (2.1.21)$$

$$j=1,2;\ k\neq j,\ k=1,2.$$

The notation $\Big\{...\Big\}\Big|_{\xi=y-(t-\tau)}^{\xi=y+(t-\tau)}$ means the difference of the expression which is inside brackets for $\xi=y+(t-\tau)$ and $\xi=y-(t-\tau)$.

Integrating the equation (2.1.13) twice with respect to t and using zero initial data (2.1.17) we find

$$\tilde{V}_3(\nu,y,t) = \int_0^t\Big\{L_5(y)\Big[i\nu_1\Big(\frac{K(y)}{2}\tilde{V}_1(\nu,y,\tau) + \frac{\partial\tilde{V}_1}{\partial y}(\nu,y,\tau)\Big)$$

$$+i\nu_2\Big(\frac{K(y)}{2}\tilde{V}_2(\nu,y,\tau) + \frac{\partial\tilde{V}_2}{\partial y}(\nu,y,\tau)\Big)\Big]$$

$$+F_3(\nu,y,\tau)\Big\}\frac{\sin\big(d(\nu,y)(t-\tau)\big)}{d(\nu,y)}d\tau, \tag{2.1.22}$$

where

$$d(\nu,y)=\sqrt{(\nu_1^2+\nu_2^2)L_4(y)}. \tag{2.1.23}$$

Differentiating (2.1.22) with respect to t we find a relation containing $\frac{\partial\tilde{V}_3}{\partial t}$ in the left-hand side:

$$\frac{\partial\tilde{V}_3}{\partial t}(\nu,y,t)=\int_0^t\Big\{L_5(y)\Big[i\nu_1\Big(\frac{K(y)}{2}\tilde{V}_1(\nu,y,\tau)+\frac{\partial\tilde{V}_1}{\partial y}(\nu,y,\tau)\Big)$$

$$+i\nu_2\Big(\frac{K(y)}{2}\tilde{V}_2(\nu,y,\tau)+\frac{\partial\tilde{V}_2}{\partial y}(\nu,y,\tau)\Big)\Big]$$

$$+F_3(\nu,y,\tau)\Big\}\cos\big(d(\nu,y)(t-\tau)\big)d\tau. \tag{2.1.24}$$

Substituting $\tilde{V}_3(\nu,y,t)$ in the equation (2.1.21) we find

$$\frac{\partial\tilde{V}_j}{\partial y}(\nu,y,t) \;=\; \frac{1}{2}\int_0^t\Big\{\Big[q(\xi)-\nu_k^2L_3(\xi)\Big]\tilde{V}_j(\nu,\xi,\tau)$$

$$+\nu_j\nu_kL_3(\xi)\tilde{V}_k(\nu,\xi,\tau)+i\nu_j\Big[L_2(\xi)-L_1'(\xi)\Big]\tilde{V}_3(\nu,\xi,\tau)$$

$$+i\nu_jL_1(\xi)\frac{\partial\tilde{V}_3}{\partial t}(\nu,\xi,\tau)\Big\}\Big|_{\xi=y-(t-\tau)}^{\xi=y+(t-\tau)}d\tau$$

$$-i\nu_jL_1(y)L_5(y)\int_0^t\Big[i\nu_1\Big(\frac{K(y)}{2}\tilde{V}_1(\nu,y,\tau)+\frac{\partial\tilde{V}_1}{\partial y}(\nu,y,\tau)\Big)$$

$$+i\nu_2\Big(\frac{K(y)}{2}\tilde{V}_2(\nu,y,\tau)+\frac{\partial\tilde{V}_2}{\partial y}(\nu,y,\tau)\Big)\Big]\frac{\sin\big(d(\nu,y)(t-\tau)\big)}{d(\nu,y)}d\tau$$

$$+G_j(\nu,y,t), \tag{2.1.25}$$

where

$$G_j(\nu,y,t) \;=\; \frac{1}{2}\int_0^t\Big\{F_j(\nu,\xi,\tau)\Big\}\Big|_{\xi=y-(t-\tau)}^{\xi=y+(t-\tau)}d\tau$$

$$-i\nu_j L_1(y)\int_0^t F_3(\nu,y,\tau)\frac{\sin\big(d(\nu,y)(t-\tau)\big)}{d(\nu,y)}d\tau,$$

$$j=1,2;\; k\neq j,\; k=1,2.$$

Equations (2.1.18), (2.1.21), (2.1.24), (2.1.25) represent the closed system of integral equations with respect to unknown $\tilde{V}_j$, $\dfrac{\partial \tilde{V}_j}{\partial y}, j=1,2;\; \tilde{V}_3,\; \dfrac{\partial \tilde{V}_3}{\partial t}$. This system can be written in the form

$$\mathbf{V}(\nu,y,t)=\mathbf{G}(\nu,y,t)+\int_0^t \big(\mathbf{K}\mathbf{V}\big)(\nu,y,t,\tau)d\tau, \tag{2.1.26}$$

where $\mathbf{V}=(V_1,V_2,V_3,V_4,V_5,V_6)$ is unknown vector-function whose components are

$$V_1=\tilde{V}_1, V_2=\tilde{V}_2, V_3=\tilde{V}_3, V_4=\frac{\partial \tilde{V}_1}{\partial y}, V_5=\frac{\partial \tilde{V}_2}{\partial y}, V_6=\frac{\partial \tilde{V}_3}{\partial t}; \tag{2.1.27}$$

$\mathbf{G}=(G_1,G_2,G_3,G_4,G_5,G_6)$ is the given vector-function whose components are defined by

$$G_j(\nu,y,t)=\frac{1}{2}\int_0^t\int_{y-(t-\tau)}^{y+(t-\tau)} F_j(\nu,\xi,\tau)d\xi d\tau,\;\; j=1,2, \tag{2.1.28}$$

$$G_3(\nu,y,t)=\int_0^t F_3(\nu,y,\tau)\frac{\sin\big(d(\nu,y)(t-\tau)\big)}{d(\nu,y)}d\tau, \tag{2.1.29}$$

$$G_{3+j}(\nu,y,t)=\frac{1}{2}\int_0^t \Big\{F_j(\nu,\xi,\tau)\Big\}\Big|_{\xi=y-(t-\tau)}^{\xi=y+(t-\tau)}d\tau$$

$$-i\nu_j L_1(y)\int_0^t F_3(\nu,y,\tau)\frac{\sin\big(d(\nu,y)(t-\tau)\big)}{d(\nu,y)}d\tau,\;\; j=1,2; \tag{2.1.30}$$

$$G_6(\nu,y,t)=\int_0^t F_3(\nu,y,\tau)\cos\big(d(\nu,y)(t-\tau)\big)d\tau, \tag{2.1.31}$$

where $F_j(\nu,y,t),\; j=1,2,3$ and $L_m(y),\; m=1,2,...,5$ are defined in (2.1.14), (2.1.15).

The components of the vector-operator $\mathbf{K}=(\mathcal{K}_1,\mathcal{K}_2,\mathcal{K}_3,\mathcal{K}_4,\mathcal{K}_5,\mathcal{K}_6)$ are defined by

$$\big(\mathcal{K}_j\mathbf{V}\big)(\nu,y,t,\tau)=\frac{1}{2}\int_{y-(t-\tau)}^{y+(t-\tau)}\Big\{\Big[q(\xi)-\nu_k^2 L_3(\xi)\Big]V_j(\nu,\xi,\tau)$$

$$+\nu_j\nu_k L_3(\xi)V_k(\nu,\xi,\tau)+i\nu_j\Big[L_2(\xi)-L_1'(\xi)\Big]\tilde{V}_3(\nu,\xi,\tau)\Big\}d\xi$$

$$+\Big\{\frac{i\nu_j}{2}L_1(\xi)V_3(\nu,\xi,\tau)\Big\}\Big|_{\xi=y-(t-\tau)}^{\xi=y+(t-\tau)},\quad j=1,2;\ k\neq j,\ k=1,2, \tag{2.1.32}$$

$$\big(\mathcal{K}_3\mathbf{V}\big)(\nu,y,t,\tau)=L_5(y)\Big\{i\nu_1\Big[\frac{K(y)}{2}V_1(\nu,y,\tau)+V_4(\nu,y,\tau)\Big]$$

$$+i\nu_2\Big[\frac{K(y)}{2}V_2(\nu,y,\tau)+V_5(\nu,y,\tau)\Big]\Big\}\frac{\sin\big(d(\nu,y)(t-\tau)\big)}{d(\nu,y)}, \tag{2.1.33}$$

$$\big(\mathcal{K}_{3+j}\mathbf{V}\big)(\nu,y,t,\tau)=\frac{1}{2}\Big\{\Big[q(\xi)-\nu_k^2L_3(\xi)\Big]V_j(\nu,\xi,\tau)$$

$$+\nu_j\nu_kL_3(\xi)V_k(\nu,\xi,\tau)++i\nu_j\Big[L_2(\xi)-L_1'(\xi)\Big]V_3(\nu,\xi,\tau)$$

$$+i\nu_j\Big\{L_1(\xi)V_6(\nu,\xi,\tau)\Big\}\Big|_{\xi=y-(t-\tau)}^{\xi=y+(t-\tau)}$$

$$-i\nu_jL_1(y)L_5(y)\Big[i\nu_1\Big(\frac{K(y)}{2}V_1(\nu,y,\tau)+V_4(\nu,y,\tau)\Big)+i\nu_2\Big(\frac{K(y)}{2}V_2(\nu,y,\tau)$$

$$+V_5(\nu,y,\tau)\Big)\Big]\frac{\sin\big(d(\nu,y)(t-\tau)\big)}{d(\nu,y)},j=1,2;\ k\neq j,\ k=1,2, \tag{2.1.34}$$

$$\big(\mathcal{K}_6\mathbf{V}\big)(\nu,y,t,\tau)=L_5(y)\Big\{i\nu_1\Big[\frac{K(y)}{2}V_1(\nu,y,\tau)+V_4(\nu,y,\tau)\Big]$$

$$+i\nu_2\Big[\frac{K(y)}{2}V_2(\nu,y,\tau)+V_5(\nu,y,\tau)\Big]\Big\}\cos\big(d(\nu,y)(t-\tau)\big); \tag{2.1.35}$$

$$\int_0^t\Big(\mathbf{KV}\Big)(\nu,y,t,\tau)d\tau=\Big(\int_0^t\Big(\mathcal{K}_1\mathbf{V}\Big)(\nu,y,t,\tau)d\tau,...,\int_0^t\Big(\mathcal{K}_6\mathbf{V}\Big)(\nu,y,t,\tau)d\tau\Big).$$

As a result we conclude that the initial value problem (2.1.1)–(2.1.3) is equivalent to the operator integral equation (2.1.26).

2.1.4 Properties of the Vector Integral Equation (2.1.26)

In this Subsection we study the properties of inhomogeneous term and kernel of (2.1.26) in the forms convenient to prove the existence and uniqueness theorems for (2.1.26). We state these problems by the following propositions.

Proposition 1. *Let T be a fixed positive number,*

$$\Delta(T) = \{(y,t)|\ 0 \le t \le T - |y|\}, \tag{2.1.36}$$

components of $\boldsymbol{G} = (G_1, G_2, ..., G_6)$ be defined by (2.1.28)-(2.1.31). Then under above assumptions $G_j(\nu, y, t),\ j = 1, 2, ..., 6$ are continuous functions for $\nu \in R^2, (y,t) \in \Delta(T)$.

Proof. Let the functions $\varepsilon_{jj}(x_3)$, $m_{jj}(x_3)$ satisfy assumptions at the beginning of the chapter 2; the function τ defined in (2.1.4) is monotonic increasing function and has a monotonic inverse function τ^{-1} satisfies the properties; $\tau(0) = 0$, $\tau(x_3) \in C^3(R)$, $\tau^{-1}(y) \in C^3(R)$. We also assumed that the Fourier transform of the vector function $\mathbf{f}$ with respect to variables x_1, x_2 has components which are continuous relative to all variables simultaneously. Using the formulas (2.1.8), (2.1.11), (2.1.14) we find that the functions A, S, C, N_i, M_i ; $i = 1, 3$ are twice continuously differentiable on R; the function K is one times continuously differentiable. Using these result we conclude that the functions $F_j(\nu, y, t)$, $j = 1, 2, 3$ are continuous the functions with respect to $(y,t) \in \Delta(T)$, $\nu \in R^2$; $L_1(y)$ is is twice continuously differentiable with respect to $y \in R$. The function $d(\nu, y)$ defined by (2.1.23) is twice continuously differentiable with respect to $y \in R$ for any $\nu \in R^2$ and $\dfrac{\sin\Big(d(\nu,y)(t-\tau)\Big)}{d(\nu,y)}$ is bounded and twice continuously differentiable with respect to $(y,t) \in \Delta$ for any $\nu \in R^2$, $o \le \tau \le t$. Consequently using properties of τ we find that $G_j(\nu, y, t)$, $j = 1, 2, \ldots, 6$ are continuous function for $(y,t) \in \Delta(T)$ and $\nu \in R^2$. $\square$

Proposition 2. *Let T be a fixed positive number and components of the vector operator $\boldsymbol{K} = (\mathcal{K}_1, \mathcal{K}_2, ..., \mathcal{K}_6)$ be defined by (2.1.32)-(2.1.35). Then under above assumptions the expression $\int_0^t \big(\mathcal{K}_j \boldsymbol{V}\big)(\nu, y, t, \tau)d\tau$ is a continuous function for $\nu \in R^2, (y,t) \in \Delta(T)$ and for any $j = 1, 2, ..., 6$ and any vector function $\boldsymbol{V}(\nu, y, t)$ with continuous components for $\nu \in R^2, (y,t) \in \Delta(T)$.*

Proof. Using the reasoning made in the proof of Proposition 1 and formulae (2.1.32)-(2.1.35) we find that

$$\int_0^t \big(\mathcal{K}_j \mathbf{V}\big)(\nu, y, t, \tau)d\tau,\ j = 1, 2, \dots, 6$$

are continuous functions with respect to $(y,t) \in \Delta(T)$ for any $\nu \in R^2$ and any vector function $\mathbf{V} = (V_1, V_2, \dots, V_6)$ with continuous components $V_j(\nu, y, t)$ for $(y,t) \in \Delta(T)$ and $\nu \in R^2$. □

Proposition 3. Let T be a fixed, Ω be an arbitrary positive numbers and $\mathbf{K}$ be the operator defined by (2.1.32)-(2.1.35). Then under above assumptions the following inequalities are satisfied

$$|\int_0^t \Big(\mathcal{K}_j \mathbf{V}\Big)(\nu, y, t, \tau)d\tau| \le M \int_0^t \|\mathbf{V}\|(\nu, \tau)d\tau,\ j = 1, 2, ...6; \tag{2.1.37}$$

where $(y,t) \in \Delta(T)$, $|\nu| \le \Omega$,v M is a positive number depending on T, Ω; and

$$\|\mathbf{V}\|(\nu, \tau) = \max_{j=1,2,\dots,6} \max_{\xi \in [-(T-\tau),(T-\tau)]} |V_j(\nu, \xi, \tau)|. \tag{2.1.38}$$

Proof. Let T be a given positive number, $\Delta(T)$ be the triangle defined by (2.1.36), (y,t) be arbitrary point from $\Delta(T)$; $q(y), L_j(y),\ j = 1, 2, 3$ be functions defined in (2.1.14),

$$Q(y,t) = \max_{y-(t-\tau) \le \xi \le y+(t-\tau)} \max_{j=1,2,3} \Big\{|q(\xi)|, |L_j(\xi)|, |L_1'(\xi)|\Big\}.$$

We can obtain the following inequality from the equation (2.1.32)

$$\Big|(\mathcal{K}_j \mathbf{V})(\nu, y, t, \tau)\Big| \le \frac{1}{2} \int_{y-(t-\tau)}^{y+(t-\tau)} \Big\{Q(y,t)(1 + |\nu|^2)|V_j(\nu, \xi, \tau)|$$

$$+|\nu|^2 Q(y,t)|V_k(\nu,\xi,\tau)| + |\nu|Q(y,t)|V_3(\nu,\xi,\tau)|\Big\}d\xi + |\nu|Q(y,t)\|V\|(\nu,\tau)$$

$$\leq M_j(T,\Omega)\|V\|(\nu,\tau), \quad j = 1,2;$$

where

$$M_j(T,\Omega) = \max_{(y,t)\in\Delta(T)} \Big\{T(Q(y,t)(1 + 2|\Omega|^2 + |\Omega|)) + Q(y,t)|\Omega|\Big\}.$$

Using the equation (2.1.33) we find the following inequality

$$\Big|\big(\mathcal{K}_3\mathbf{V}\big)(\nu,y,t,\tau)\Big| \leq M_3(T,\Omega)\|V\|(\nu,\tau),$$

where

$$M_3(T,\Omega) = 3T|\nu|P(T), \qquad P(T) = \max_{y\in[-T,T]} \Big\{|L_5(y)|, |L_5(y)K(y)|\Big\}.$$

.

Similarly using the equations (2.1.34), (2.1.35) we can define $M_j(T,\Omega)$, $j = 4,5,6$ such that the following inequalities are satisfied

$$\Big|(\mathcal{K}_j\mathbf{V})(\nu,y,t,\tau)\Big| \leq M_j(T,\Omega)\|V\|(\nu,\tau), \quad j = 4,5,6.$$

Proof of the Proposition 3 is completed by choosing M as

$$M = \max_{j=1,2,\ldots,6} M_j(T,\Omega).$$

□

2.2 Uniqueness and Existence Theorems for the Vector Integral Equation (2.1.26)

Uniqueness and existence theorems of the operator integral equation (2.1.26) are proved in this section.

2.2.1 Uniqueness Theorem

Theorem 2.2.1. *Let T be a fixed positive number; $\boldsymbol{G} = (G_1, G_2, ..., G_6)$ be a vector function such that $G_j = G_j(\nu, y, t) \in C(R^2 \times \Delta(T)), j = 1, 2, ..., 6$; $\boldsymbol{K} = (\mathcal{K}_1, \mathcal{K}_2, ..., \mathcal{K}_6)$ be the vector operator defined by (2.1.32)-(2.1.35). Then there can exist only one solution $\boldsymbol{V} = (V_1, V_2, ..., V_6)$ of the operator integral equation (2.1.26) such that $V_j \in C(R^2 \times \Delta(T)),\ j = 1, 2, ..., 6$.*

Proof. Let Ω be an arbitrary positive number, $\mathbf{V}(\nu, y, t)$ and $\mathbf{V}^*(\nu, y, t)$ be two solution of (2.1.26) with continuous components for $(y, t) \in \Delta(T),\ |\nu| \leq \Omega$. Letting $\hat{\mathbf{V}}(\nu, y, t) = \mathbf{V}(\nu, y, t) - \mathbf{V}^*(\nu, y, t)$ we find from (2.1.26)

$$\hat{\mathbf{V}}(\nu, y, t) = \int_0^t \Big(\mathbf{K}\hat{\mathbf{V}}\Big)(\nu, y, t, \tau)d\tau. \tag{2.2.1}$$

Using Proposition 3 we find from (2.2.1)

$$\|\hat{\mathbf{V}}\|(\nu, t) \leq M \int_0^t \|\hat{\mathbf{V}}\|(\nu, \tau)d\tau, \tag{2.2.2}$$

where $|\nu| \leq \Omega,\ t \in [0, T]$; $\|.\|(\nu, t)$ and M are defined in Proposition 3.
Applying Grownwall's lemma (see Nagle et al. (2004)) to (2.2.2) we find

$$\|\hat{\mathbf{V}}\|(\nu, t) = 0, \quad t \in [0, T],\ |\nu| \leq \Omega. \tag{2.2.3}$$

Using the continuity of $\hat{\mathbf{V}}(\nu, y, t)$ we conclude that

$$\hat{\mathbf{V}}(\nu, y, t) \equiv 0, \quad (y, t) \in \Delta(T),\ |\nu| \leq \Omega.$$

Since Ω is an arbitrary positive number we find that $\mathbf{V}(\nu, y, t) \equiv \mathbf{V}^*(\nu, y, t)$ for $(y, t) \in \Delta(T),\ \nu \in R^2$. Theorem is proved. □

2.2.2 Existence Theorem and Method of Solving

Applying successive approximations we prove the existence theorem in this Subsection. We note that the proof of this theorem contains a method of solving (2.1.26)

Theorem 2.2.2. *Let T be a fixed positive number; $\boldsymbol{K} = (\mathcal{K}_1, \mathcal{K}_2, ..., \mathcal{K}_6)$ be the vector operator defined by (2.1.32)-(2.1.35). Then for any $\boldsymbol{G} = (G_1, G_2, ..., G_6)$ such that $G_j = G_j(\nu, y, t) \in C(R^2 \times \Delta(T)), j = 1, 2, ..., 6$ there exists a solution $\boldsymbol{V} = (V_1, V_2, ..., V_6)$ of the operator integral equation (2.1.26) such that $V_j \in C(R^2 \times \Delta(T)),\ j = 1, 2, ..., 6$.*

Proof. Let Ω be an arbitrary positive number. Let us consider the integral equation (2.1.26) for $(y,t) \in \Delta(T)$, $|\nu| \le \Omega$. For finding a solution of this equation we apply the following successive approximations

$$\begin{aligned} \mathbf{V}^{(0)}(\nu, y, t) &= \mathbf{G}(\nu, y, t), \\ \mathbf{V}^{(n)}(\nu, y, t) &= \int_0^t \big(\mathbf{K}\mathbf{V}^{(n-1)}\big)(\nu, y, t, \tau) d\tau,\ n = 1, 2 \ldots . \end{aligned} \tag{2.2.4}$$

Our goal is to show that for $(y,t) \in \Delta(T)$, $|\nu| \le \Omega$ the series $\sum_{n=0}^{\infty} \mathbf{V}^{(n)}(\nu, y, t) = \Big(\sum_{n=1}^{\infty} V_1^{(n)}(\nu, y, t), \ldots, \sum_{n=1}^{\infty} V_6^{(n)}(\nu, y, t)\Big)$ is uniformly convergent to a vector function $\mathbf{V}(\nu, x_3, t) = \Big(V_1(\nu, y, t), V_2(\nu, x_3, t), \ldots, V_6(\nu, y, t)\Big)$ with continuous components and this vector function is a solution of (2.1.26).

Indeed, we find from (2.2.4) and Propositions 1, 2 of Section 2.1.4 that for $(y,t) \in \Delta(T)$, $|\nu| \le \Omega$ the vector function $\mathbf{V}^{(n)}(\nu, y, t)$, $n = 0, 1, 2 \ldots$ have continuous components and

$$|V_j^{(n)}(\nu, y, t)| \le M \int_0^t \|\mathbf{V}^{(n-1)}\|(\nu, \tau) d\tau, \tag{2.2.5}$$

where $\|.\|(\nu, \tau)$ and M are defined in Proposition 3.

It follows from (2.2.5) that

$$|V_j^{(n)}(\nu, y, t)| \le \frac{(MT)^n}{n!} \max_{|\nu| \le \Omega} \|\mathbf{G}\|(\nu, T), \tag{2.2.6}$$
$$j = 1, 2, \ldots, 6,\ n = 0, 1, 2 \ldots .$$

The uniform convergence of $\sum_{n=0}^{\infty} V_j^{(n)}(\nu, y, t)$ to a continuous function $V_j(\nu, y, t)$ follows from inequality (2.2.6) and the first Weierstrass theorem (Apostol (1967), page 425). Let us show that the vector function $\mathbf{V}(\nu, y, t)$ is a solution of (2.1.26). Summing the equation (2.2.4) with respect to n from 1 to N we have

$$\sum_{n=1}^{N} \mathbf{V}^{(n)}(\nu, y, t) = \sum_{n=0}^{N-1} \int_0^t (K\mathbf{V}^{(n)})(\nu, y, t, \tau) d\tau, \tag{2.2.7}$$

where

$$\sum_{n=1}^{N} \mathbf{V}^{(n)}(\nu, y, t) = \Big(\sum_{n=1}^{N} V_1^{(n)}(\nu, y, t), \ldots, \sum_{n=1}^{N} V_6^{(n)}(\nu, y, t) \Big).$$

Adding both sides of (2.2.7) the vector function $\mathbf{G}(\nu, y, t)$ we find

$$\sum_{n=0}^{N} \mathbf{V}^{(n)}(\nu, y, t) = \mathbf{G}(\nu, y, t) + \int_0^t \sum_{n=0}^{N-1} (K\mathbf{V}^{(n)})(\nu, y, t, \tau) d\tau. \tag{2.2.8}$$

Approaching N the infinity and using the second Weierstrass theorem (Apostol (1967), page 426) we find that the vector function $\mathbf{V}(\nu, y, t)$ satisfies (2.1.26) for $(y, t) \in \Delta(T)$, $|\nu| \leq \Omega$. Since Ω is an arbitrary positive number we find that the vector function $\mathbf{V}(\nu, y, t)$ with continuous components is a solution of (2.1.26) for $(y, t) \in \Delta(T)$, $\nu \in R^2$. □

2.3 Initial Value Problem (2.0.1), (2.0.2) Solving

The existence and uniqueness theorem of the initial value problem (2.0.1), (2.0.2) is the main result of this section. We show also that if the solution $\mathbf{V}(\nu, y, t)$ of the operator integral equation (2.1.26) is constructed then a solution $\mathbf{E}(x, t) = (E_1(x, t), E_2(x, t), E_3(x, t))$ of (2.0.1), (2.0.2) and derivatives $\frac{\partial E_j}{\partial x_3}(x, t)$, $\frac{\partial E_3}{\partial t}(x, t)$, $j = 1, 2$ may be found by explicit formulae.

In this section we will use the following notions and notations. For the exponent $\alpha = (\alpha_1, \alpha_2)$ with $\alpha_j \in \{0, 1, 2, ...\}$ and $|\alpha| = \alpha_1 + \alpha_2$, the partial derivatives of higher order

$$\frac{\partial^{|\alpha|}}{\partial \nu_j} \tilde{f}_k(\nu, y, t), \quad \frac{\partial^{|\alpha|}}{\partial \nu_j} V_l(\nu, y, t), \;\; j = 1, 2; \; k = 1, 2, 3; \; l = 1, 2, ..., 6,$$

will be denoted by

$$D_\nu^\alpha \tilde{f}_k(\nu, y, t), \;\; D_\nu^\alpha V_l(\nu, y, t).$$

For vector functions $\mathbf{V} = (V_1, V_2, ..., V_6)$, $\tilde{\mathbf{f}} = (\tilde{f}_1, \tilde{f}_2, \tilde{f}_3)$ and each α we define $D_\nu^\alpha \mathbf{V}$ and $D_\nu^\alpha \tilde{\mathbf{f}}$ by

$$D_\nu^\alpha \mathbf{V} = (D_\nu^\alpha V_1, D_\nu^\alpha V_2, ..., D_\nu^\alpha V_6),\ D_\nu^\alpha \tilde{\mathbf{f}} = (D_\nu^\alpha \tilde{f}_1, D_\nu^\alpha \tilde{f}_2, D_\nu^\alpha \tilde{f}_3).$$

We denote by $C(R^2)$ the class consisting of all continuous functions that are defined on R^2, then for $m = 0, 1, 2, ...$ we define $C^m(R^2)$ by $C^0(R^2) = C(R^2)$ and otherwise by

$$C^m(R^2) = \{\varphi(\nu) \in C(R^2) : D_\nu^\alpha \varphi(\nu) \in C(R^2) \quad \text{for all } |\alpha| \le m\},$$

$$C^\infty(R^2) = \bigcap_{m=1}^{\infty} C^m(R^2).$$

Further, $C_c(R^2)$ is the class of all functions from $C(R^2)$ with compact supports; $\mathcal{L}_2(R^2)$ is the class of all square integrable functions over R^2; $\|\varphi\|_2$ is defined for each $\varphi(\nu) \in \mathcal{L}_2(R^2)$ by

$$\|\varphi\|_2^2 = \int_{R^2} |\varphi(\nu)|^2 d\nu.$$

The Paley-Wiener space $PW(R^2)$ is a space consisting of all functions $\varphi(x_1, x_2) \in C^\infty(R^2)$ satisfying (Andersen (2004), see also Appendix B):

$$(a) \quad (1 + \sqrt{x_1^2 + x_2^2})^m \Delta^n \varphi(x_1, x_2) \in \mathcal{L}_2(R^2) \text{ for all } m, n \in \{0, 1, 2... \quad \},$$

$$(b) \quad R_\varphi^\Delta = \lim_{n\to\infty} \|\Delta^n \varphi(x_1, x_2)\|_2^{1/2n} < \infty,$$

where $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$ is the Laplace operator on R^2. Let T be a fixed positive number; $\Delta(T)$ be defined by (2.1.36); $y = \tau(x_3)$ defined by (2.1.4) for $x_3 \in R$; $D(T)$ be a set of R^2 defined by

$$D(T) = \{(x_3, t) :\ 0 \le t \le T - |\tau(x_3)|\};$$

$C(D(T); C_c(R^2))$ is a class of all continuous mappings of $(x_3, t) \in D(T)$ into the class $C(R^2)$ of functions $\nu = (\nu_1, \nu_2) \in R^2$; $C(D(T); PW(R^2))$ is a class of all continuous mappings of $D(T)$ into $PW(R^2)$.

The main result of this section is the following theorem.

Theorem 2.3.1. *Let T be a fixed positive number; $\tilde{\boldsymbol{f}} = (\tilde{f}_1, \tilde{f}_2, \tilde{f}_3)$ be the Fourier transform with respect to x_1, x_2 of the inhomogeneous term $\boldsymbol{f}$ in (2.0.1) and such that for each α*

$$D_\nu^\alpha \tilde{f}_k \in C(R^2 \times D(T)) \cap C(D(T); C_c(R^2)),\ k = 1, 2, 3.$$

Then under assumptions for $\mathcal{E}$, $\mathcal{M}^{-1}$ mentioned at beginning of the Chapter 2, there exists a unique generalized solution $\boldsymbol{E}(x,t) = (E_1(x,t), E_2(x,t), E_3(x,t))$ of (2.0.1), (2.0.2) such that

$$E_l(x,t), \frac{\partial}{\partial x_3} E_j(x,t), \frac{\partial}{\partial t} E_3(x,t) \in C(R^2 \times D(T)) \cap C(D(T); PW(R^2)),$$
$$l = 1, 2, 3;\ j = 1, 2.$$

Proof. We note that under hypothesis of theorem 2.3.1 the functions $G_k(\nu, y, t)$, $k = 1, 2, 3$ defined by (2.1.28), (2.1.29) for any α satisfy the following conditions

$$D_\nu^\alpha G_k(\nu, y, t) \in C(R^2 \times \Delta(T)) \cap C(\Delta(T); C_c(R^2)), \tag{2.3.1}$$

$\alpha = (\alpha_1, \alpha_2),\ \alpha_j \in \{0, 1, 2, \dots\ \},\ \nu = (\nu_1, \nu_2) \in R^2,\ (y, t) \in \Delta(T).$
Applying D_ν^α to the vector integral equation (2.1.26) we obtain

$$D_\nu^\alpha \mathbf{V}(\nu, y, t) = D_\nu^\alpha \mathbf{G}(\nu, y, t) + \int_0^t \Big(\mathbf{K} D_\nu^\alpha \mathbf{V}\Big)(\nu, y, t, \tau) d\tau, \tag{2.3.2}$$

$$\nu \in R^2,\ (y, t) \in \Delta(T).$$

Equation (2.3.2) has the same form as (2.1.26). Theorems 2.2.1 and 2.2.2 are hold for (2.3.2) with an arbitrary α. Therefore the solution $\mathbf{V}(\nu, y, t)$ of (2.1.26), which is found by the method of successive approximations described in Subsection 2.2.2, satisfies for any α the property:

$$D_\nu^\alpha \mathbf{V}(\nu, y, t) \in C(R^2 \times \Delta(T)).$$

Using (2.3.2) and (2.1.37) for any α we obtain the following inequality

$$\|D_\nu^\alpha \mathbf{V}\|(\nu, t) \le \|D_\nu^\alpha \mathbf{G}\|(\nu, t) + M \int_0^t \|\mathbf{V}\|(\nu, \tau) d\tau, \tag{2.3.3}$$

where M, $\|\cdot\|(\nu,t)$ are defined in the Proposition 3 (see formula (2.1.38)). Applying the Grownwall's lemma Nagle et al. (2004) for the inequality (2.3.3) we find

$$\|D^\alpha_\nu \mathbf{V}\|(\nu,t) \le \|D^\alpha_\nu \mathbf{G}\|(\nu,t)e^{MT}, \;\; \nu \in R^2, \; t \in [0,T]. \tag{2.3.4}$$

From (2.3.1), (2.3.4) we have that the solution $\mathbf{V}(\nu,y,t)$ of (2.1.26) satisfies for any α the following property

$$D^\alpha_\nu \mathbf{V}(\nu,y,t) \in C(\Delta(T); C_c(R^2)). \tag{2.3.5}$$

Using proposition 1, theorem 2.2.1, theorem 2.2.2 and formulae (2.1.5), (2.1.6), (2.1.10), (2.1.27) we find that there exists a unique generalized solution $\tilde{\mathbf{E}}(\nu,x_3,t) = (\tilde{E}_1(\nu,x_3,t), \tilde{E}_2(\nu,x_3,t), \tilde{E}_3(\nu,x_3,t))$ of (2.1.1) - (2.1.3) such that for any α

$$D^\alpha_\nu \tilde{E}_l, \frac{\partial}{\partial x_3} D^\alpha_\nu \tilde{E}_j, \frac{\partial}{\partial t} D^\alpha_\nu \tilde{E}_3 \in C(R^2 \times D(T)) \cap C(D(T); C_c(R^2)), \;\; j = 1,2$$

and

$$\tilde{E}_l(\nu,x_3,t) = S(\tau(x_3))V_l(\nu,\tau(x_3),t), \quad l = 1,2,3; \tag{2.3.6}$$

$$\begin{aligned} \frac{\partial \tilde{E}_j}{\partial x_3}(\nu,x_3,t) &= c(x_3)\Big[S'(\tau(x_3))V_j(\nu,\tau(x_3),t) \\ &+ S(\tau(x_3))V_{j+3}(\nu,\tau(x_3),t)\Big], \;\; j = 1,2; \end{aligned} \tag{2.3.7}$$

$$\frac{\partial \tilde{E}_3}{\partial t}(\nu,x_3,t) = S(\tau(x_3))V_6(\nu,\tau(x_3),t), \tag{2.3.8}$$

where $\tau(x_3)$, $c(x_3)$, $S(y)$ are defined by (2.1.4), (2.1.11); $V_j(\nu,y,t), \; j = 1,2,...,6$ are components of the solution $\mathbf{V}(\nu,y,t)$ of (2.1.26).
Therefore the generalized solution $\tilde{\mathbf{E}}(\nu,x_3,t) = (\tilde{E}_1(\nu,x_3,t), \tilde{E}_2(\nu,x_3y,t), \tilde{E}_3(\nu,x_3,t))$ of (2.1.1), (2.1.2) satisfies the following condition:

$$\tilde{E}_l(\nu,x_3,t), \frac{\partial}{\partial x_3}\tilde{E}_j(\nu,x_3,t), \frac{\partial}{\partial t}\tilde{E}_3(\nu,x_3,t) \in C(R^2 \times D(T)) \cap C(D(T); C^\infty_c(R^2)),$$
$$l = 1,2,3; \; j = 1,2.$$

Applying the inverse Fourier transform with respect to ν_1, ν_2 variables to the equation (2.1.1) - (2.1.3) using the real version of the Paley-Wiener theorem Andersen (2004) (see also Appendix B) we find that $\mathbf{E}(x,t) = \mathcal{F}_\nu^{-1}[\tilde{\mathbf{E}}]$ is a unique generalized solution of (2.0.1), (2.0.2) such that $E_l(x,t), \dfrac{\partial}{\partial x_3}E_j(x,t), \dfrac{\partial}{\partial t}E_3(x,t)$ belong to the class $C(R^2 \times D(T)) \cap C(D(T); PW(R^2))$, $l = 1,2,3;\ j = 1,2.$ $\square$

Remark 2.3.2. We note that if the solution $\mathbf{V}(\nu, y, t)$ of (2.1.26) is found for $\nu \in R^2$, $(y,t) \in \Delta(T)$ then the solution $\mathbf{E}(x,t)$ of the initial value problem (2.0.1), (2.0.2) and the derivatives $\dfrac{\partial}{\partial t}E_3(x,t)$, $\dfrac{\partial}{\partial x_3}E_j(x,t)$, $j = 1,2$ are given by formulae (2.3.6) - (2.3.8) for $(x,t) \in R^2 \times D(T)$.

2.4 IVP of Vector Equation for Electric Field in Electrically Anisotropic Media (Crystals)

Let us consider the Problem 1 in which permittivity and permeability matrices of the form

$$\mathcal{E} = diag(\varepsilon_{11}, \varepsilon_{11}, \varepsilon_{33}), \quad \mathcal{M} = \mu\mathcal{I},$$

where μ is a positive constant, $\mathcal{I}$ is the identity matrix of the order 3×3. The system (2.0.1) for these $\mathcal{E}$, $\mathcal{M}$ related to crystal optics (see, for example Cohen (2002), Lindell (1990), Yakhno (2005). The different methods for solving this problem when elements of the matrix $\mathcal{E}$ and μ are positive constants may be found in Cohen (2002), Lindell (1990), Yakhno (2005). In this section we show that the method, described in Sections 2.1–2.3, can be successfully applied for solving Problem 1 in the case when elements of the matrix $\mathcal{M}$ are functions of depending on x_3. We suppose that the Fourier transform the vector function $\mathbf{f}$ with respect to variables x_1, x_2 has components which are continuous relative to all variables simultaneously. We assume also that elements of the matrix $\mathcal{E} = diag(\varepsilon_{11}, \varepsilon_{11}, \varepsilon_{33})$ are twice continuously differentiable functions depending on x_3 variable only and such that $\mu\varepsilon_{jj}(x_3) = a_j^2(x_3) > 0$ for $x_3 \in R$, $j = 1,3$. The main problem is the Initial Value Problem (IVP) for finding electric field

$\mathbf{E}$ satisfying (2.0.1), (2.0.2) if the vector function $\mathbf{f}(x,t) = -\mu\partial\mathbf{j}/\partial t$, and the matrix $\mathcal{E}$ are given. We note that such type of $\mathcal{E}$, $\mathcal{M}$ corresponds to electrically anisotropic vertical inhomogeneous media (see Subsection 1.3.2).

2.5 Reduction to Vector Integral Equation

Problem 1 under assumptions of Section (2.4) can be transformed into an equivalent second kind vector integral equation of the Volterra type. For this aim we use the following steps. On the first step Problem 1 is written in terms of the Fourier transform with respect to lateral variables x_1, x_2. Then the obtained equations are written in terms of the new functions $\tilde{U}_l(\nu,y,t)$ using the following transformation

$$y = \tau(x_3),\ \tau(x_3) = \int_0^{x_3} a_1(\xi)d\xi$$

and the equalities

$$\tilde{U}_l(\nu,y,t) = \tilde{E}_l(\nu,x_3,t)|_{x_3=\tau^{-1}(y)},\ l = 1,2,3,$$
$$\frac{\partial\tilde{E}_m}{\partial x_3}(\nu,x_3,t)|_{x_3=\tau^{-1}(y)} = a_1(\tau^{-1}(y))\frac{\partial\tilde{U}_m}{\partial y}(\nu,y,t),\ m = 1,2,3.$$

After that equations involving these functions are written in terms of $\tilde{W}_l(\nu,y,t)$, where

$$\tilde{U}_l(\nu,y,t) = S(y)\tilde{W}_l(\nu,y,t),\ l = 1,2,3,$$
$$S(y) = exp(-\frac{1}{2}\int_0^y A(\xi)d\xi), \quad A(\xi) = \frac{a_1'(x_3)}{a_1^2(x_3)}\Big|_{x_3=\tau^{-1}(\xi)}.$$

As a result the obtained integral equalities represent system of integral equations with respect to unknowns $\tilde{W}_j$, $\frac{\partial\tilde{W}_j}{\partial y}, j = 1,2$; $\tilde{W}_3$, $\frac{\partial\tilde{W}_3}{\partial t}$. This system can be written in the form (see explanation in detail in (Yakhno & Sevimlican (2007))).

$$\mathbf{V}(\nu,y,t) = \mathbf{G}(\nu,y,t) + \int_0^t \big(\mathbf{K}\mathbf{V}\big)(\nu,y,t,\tau)d\tau, \tag{2.5.1}$$

where $\mathbf{V} = (V_1,V_2,V_3,V_4,V_5,V_6)$ is unknown vector-function whose components are

$$V_1 = \tilde{W}_1, V_2 = \tilde{W}_2, V_3 = \tilde{W}_3, V_4 = \frac{\partial\tilde{W}_1}{\partial y}, V_5 = \frac{\partial\tilde{W}_2}{\partial y}, V_6 = \frac{\partial\tilde{W}_3}{\partial t}; \tag{2.5.2}$$

$\mathbf{G} = (G_1, G_2, G_3, G_4, G_5, G_6)$ is the given vector-function whose components are defined by

$$G_j(\nu, y, t) = \frac{1}{2}\int_0^t \int_{y-(t-\tau)}^{y+(t-\tau)} \frac{\tilde{f}_j(\nu, \tau^{-1}(\xi), \tau)}{a_1^2(\tau^{-1}(\xi))S(\xi)} d\xi d\tau, \quad j = 1, 2, \tag{2.5.3}$$

$$G_3(\nu, y, t) = \frac{C(y)}{S(y)} \int_0^t \tilde{f}_3(\nu, \tau^{-1}(y), \tau) \frac{\sin\big(d(\nu, y)(t-\tau)\big)}{d(\nu, y)} d\tau, \tag{2.5.4}$$

$$G_{3+j}(\nu, y, t) \quad = \quad \frac{1}{2}\int_0^t \Big\{ \frac{\tilde{f}_j(\nu, \tau^{-1}(\xi), \tau)}{a_1^2(\tau^{-1}(\xi))S(\xi)} \Big\}\Big|_{\xi=y-(t-\tau)}^{\xi=y+(t-\tau)} d\tau$$

$$+\frac{B(y)}{a_3^2(\tau^{-1}(y))} \int_0^t \tilde{f}_3(\nu, \tau^{-1}(y), \tau) \frac{\sin\big(d(\nu, y)(t-\tau)\big)}{d(\nu, y)} d\tau, j = 1, 2 \tag{2.5.5}$$

$$G_6(\nu, y, t) = \frac{C(y)}{S(y)} \int_0^t \tilde{f}_3(\nu, \tau^{-1}(y), \tau) \cos\big(d(\nu, y)(t-\tau)\big) d\tau. \tag{2.5.6}$$

The components of the vector-operator $\mathbf{K} = (\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3, \mathcal{K}_4, \mathcal{K}_5, \mathcal{K}_6)$ are defined by

$$\big(\mathcal{K}_j \mathbf{V}\big)(\nu, y, t, \tau) \quad = \quad \frac{1}{2}\int_{y-(t-\tau)}^{y+(t-\tau)} \Big\{ \Big[q(\xi) - \nu_k^2 B^2(\xi)\Big] V_j(\nu, \xi, \tau)$$

$$+\nu_j \nu_k B^2(\xi) V_k(\nu, \xi, \tau) + \frac{i\nu_j}{2} B(\xi)\Big[-\frac{1}{2}A(\xi) + B'(\xi)\Big] \tilde{V}_3(\nu, \xi, \tau)\Big\} d\xi$$

$$+\Big\{\frac{i\nu_j}{2} B(\xi) V_3(\nu, \xi, \tau)\Big\}\Big|_{\xi=y-(t-\tau)}^{\xi=y+(t-\tau)}, \; j = 1, 2; \; k \neq j, \; k = 1, 2, \tag{2.5.7}$$

$$\big(\mathcal{K}_3 \mathbf{V}\big)(\nu, y, t, \tau) \quad = \quad \frac{1}{a_3^2(\tau^{-1}(y))}\Big\{ i\nu_1 a_1(\tau^{-1}(y))\Big[\frac{-A(y)}{2} V_1(\nu, y, \tau)$$

$$+V_4(\nu, y, \tau)\Big] + i\nu_2 a_1(\tau^{-1}(y))\Big[\frac{-A(y)}{2} V_2(\nu, y, \tau)$$

$$+V_5(\nu, y, \tau)\Big]\Big\} \frac{\sin\big(d(\nu, y)(t-\tau)\big)}{d(\nu, y)}, \tag{2.5.8}$$

$$\big(\mathcal{K}_{3+j} \mathbf{V}\big)(\nu, y, t, \tau) = \frac{1}{2}\Big\{ \Big[q(\xi) - \nu_k^2 B^2(\xi)\Big] V_j(\nu, \xi, \tau)$$

$$+\nu_j\nu_k B^2(\xi)V_k(\nu,\xi,\tau)+B(\xi)V_6(\nu,\xi,\tau)+\frac{i\nu_j}{2}B(\xi)\Big[-\frac{1}{2}A(\xi)$$

$$+B'(\xi)\Big]V_3(\nu,\xi,\tau)\Big\}\Big|_{\xi=y-(t-\tau)}^{\xi=y+(t-\tau)}-\frac{B(y)}{a_3^2(\tau^{-1}(y))}\Big[i\nu_1 a_1(\tau^{-1}(y))\Big(\frac{-A(y)}{2}$$

$$\times V_1(\nu,y,\tau)+V_4(\nu,y,\tau)\Big)+i\nu_2 a_1(\tau^{-1}(y))\Big(\frac{-A(y)}{2}V_2(\nu,y,\tau)$$

$$+V_5(\nu,y,\tau)\Big)\Big]\frac{\sin\big(d(\nu,y)(t-\tau)\big)}{d(\nu,y)},\, j=1,2;\; k\neq j,\; k=1,2, \tag{2.5.9}$$

$$\big(\mathcal{K}_6\mathbf{V}\big)(\nu,y,t,\tau)=\frac{1}{a_3(\tau^{-1}(y))}\Big\{i\nu_1 a_1(\tau^{-1}(y))\Big[\frac{-A(y)}{2}V_1(\nu,y,\tau)$$

$$+V_4(\nu,y,\tau)\Big]+i\nu_2 a_1(\tau^{-1}(y))\Big[\frac{-A(y)}{2}V_2(\nu,y,\tau)$$

$$+V_5(\nu,y,\tau)\Big]\Big\}\cos\big(d(\nu,y)(t-\tau)\big). \tag{2.5.10}$$

Reasonings of Subsections 2.1.1 and 2.1.2 are used to prove the existence and uniqueness theorems for (2.5.1). Fourier images of the electric field components $\tilde{E}_l(\nu,x_3,t)$ $l=1,2,3$; and their derivatives $\dfrac{\partial\tilde{E}_j}{\partial x_3}(\nu,x_3,t)$, $j=1,2$; $\dfrac{\partial\tilde{E}_3}{\partial t}(\nu,x_3,t)$ are found by the formulas (2.3.6)–(2.3.8) . Applying the inverse Fourier transform $\mathcal{F}_\nu^{-1}$ with respect to for ν_1, ν_2 variables to the obtained solution of the integral equation (2.5.1) we find electric field components $E_l(\nu,x_3,t)$, $l=1,2,3$ and their derivatives $\dfrac{\partial E_j}{\partial x_3}(\nu,x_3,t)$, $j=1,2$; $\dfrac{\partial E_3}{\partial t}(\nu,x_3,t)$ for $(x_1,x_2)\in R^2$, $(x_3,t)\in\tilde{\Delta}(T)$. Here

$$D(T)=\{(x_3,t)|\;0\le t\le T-|\tau(x_3)|\},$$

$$\mathcal{F}_\nu^{-1}[\tilde{E}_l](x,t)=\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\tilde{E}_l(\nu,x_3,t)e^{-i(\nu_1x_1+\nu_2x_2)}d\nu_1 d\nu_2\;\; i^2=-1.$$

For proving theorem about the existence of a unique solution of the stated IVP the reasonings made in the proof of theorem are used.

CHAPTER THREE
INITIAL VALUE PROBLEM FOR THE VECTOR EQUATION OF ELECTRIC FIELD IN BIAXIAL MATERIALS

This chapter is focused on the biaxial anisotropic medium, where permittivity and permeability are positive definite diagonal matrices of the form

$$\mathcal{E} = diag(\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}),\ \mathcal{M} = diag(\mu_{11}, \mu_{22}, \mu_{33}),$$

respectively. The electric field $\mathbf{E}$ in these media satisfies the vector partial differential equation (see Section 1.3)

$$\mathcal{E}\frac{\partial^2 \mathbf{E}}{\partial t^2} + curl_x(\mathcal{M}^{-1} curl_x \mathbf{E}) \quad = \quad \mathbf{f}, \tag{3.0.1}$$

where $x = (x_1, x_2, x_3) \in R^3$ is a space variable, $t \in R$ is the time variable, $\mathbf{E} = (E_1, E_2, E_3)$ is a vector function with components $E_k = E_k(x,t),\ k = 1,2,3;$ $\mathbf{f} = -\partial \mathbf{j}/\partial t$, $\mathbf{j}(x,t) = (j_1(x,t), j_2(x,t), j_3(x,t))$ is the density of electric current; $\mathcal{M}^{-1} = diag(m_{11}, m_{22}, m_{33})$ is the inverse matrix of $\mathcal{M}$, i.e. $m_{jj} = 1/\mu_{jj}$, $j = 1,2,3$.

The main object of this chapter is Problem 1 which consists of finding the vector function $\mathbf{E}$ satisfying (3.0.1) and initial data

$$\mathbf{E}|_{t=0} = 0, \quad \frac{\partial \mathbf{E}}{\partial t}|_{t=0} = 0. \tag{3.0.2}$$

The following assumptions will be needed throughout the chapter. Let α, β, T be given positive numbers, $\alpha \leq \beta$, $c = \sqrt{\beta/\alpha}$, Δ be the triangle given by

$$\triangle = \{(x_3, t) :\ 0 \leq t \leq T,\ -c(T-t) \leq x_3 \leq c(T-t)\}. \tag{3.0.3}$$

We suppose that components of the Fourier transform of the vector function $\mathbf{f}$ with respect to variables x_1, x_2 such that $\tilde{f}_j(\nu, x_3, t) \in C(R^2 \times \Delta),\ j = 1,2,3;\ \nu = (\nu_1, \nu_2) \in R^2$. We assume also that elements of diagonal positive definite matrices $\mathcal{E}$, $\mathcal{M}^{-1}$ are twice continuously differentiable functions depending on x_3 variable only over $[-cT, cT]$ and such that $0 < \alpha \leq \varepsilon_{jj}(x_3) \leq \beta$, $0 < \alpha \leq m_{jj}(x_3) \leq \beta$,

$j = 1, 2, 3$. In the present paper we assume that $(x_1, x_2) \in R^2$ and $(x_3, t) \in \Delta$, i.e. IVP (3.0.1), (3.0.2) is studied here for $(x_1, x_2) \in R^2$ and $(x_3, t) \in \Delta$. We note that such type of $\mathcal{E}$, $\mathcal{M}$ corresponds to biaxial anisotropic vertical inhomogeneous media.

This chapter is organized as follows. IVP (3.0.1), (3.0.2) is written in terms of the Fourier transform with respect to lateral variables x_1, x_2 in Section 3.1. We denote this problem as FTIVP. The reduction of FTIVP to an equivalent operator integral equation is given in Section 3.2. The properties of the inhomogeneous term and the kernel of the operator integral equation are described in Section 3.3. Using these properties the uniqueness and existence theorems of the operator integral equation are proved in Section 3.4. A class of vector functions and the existence of a unique solution of IVP (3.0.1), (3.0.2) in this class are described in Section 3.5.

The main result of this Chapter is anew method for solving the stated IVP. This method follows throughout the Sections 3.1–3.5. In addition, theorem about existence and uniqueness of the IVP (3.0.1), (3.0.2) is proved.

3.1 Set-up of FTIVP

Let components of vector functions $\tilde{\mathbf{E}}(\nu, x_3, t) = (\tilde{E}_1(\nu, x_3, t), \tilde{E}_2(\nu, x_3, t), \tilde{E}_3(\nu, x_3, t))$ and $\tilde{\mathbf{f}}(\nu, x_3, t) = (\tilde{f}_1(\nu, x_3, t), \tilde{f}_2(\nu, x_3, t), \tilde{f}_3(\nu, x_3, t))$ be defined by

$$\tilde{E}_j(\nu, x_3, t) = \mathcal{F}_{x_1x_2}[E_j](\nu, x_3, t), \quad \tilde{f}_j(\nu, x_3, t) = \mathcal{F}_{x_1x_2}[f_j](\nu, x_3, t),$$

$$j = 1, 2, 3, \ \nu = (\nu_1, \nu_2) \in R^2,$$

where $\mathcal{F}_{x_1x_2}$ is the operator of the Fourier transform with respect to x_1, x_2, i.e.

$$\mathcal{F}_{x_1x_2}[\mathbf{E}](\nu, x_3, t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathbf{E}(x, t) e^{i(\nu_1 x_1 + \nu_2 x_2)} dx_1 dx_2, \quad i^2 = -1,$$

$\nu = (\nu_1, \nu_2) \in R^2$ is the Fourier transform parameter.

Applying the operator $\mathcal{F}_{x_1x_2}$ to (3.0.1), (3.0.2) and using the properties of the Fourier transform we can write the problem (3.0.1), (3.0.2) in terms of the Fourier image $\tilde{\mathbf{E}}(\nu, x_3, t)$ as follows

$$\varepsilon_{jj}(x_3)\frac{\partial^2 \tilde{E}_j}{\partial t^2} - \frac{\partial}{\partial x_3}\Big(m_{kk}(x_3)\frac{\partial \tilde{E}_j}{\partial x_3}\Big) = -\nu_k^2 m_{33}(x_3)\tilde{E}_j + \nu_j\nu_k m_{33}(x_3)\tilde{E}_k + (i\nu_j)\frac{\partial}{\partial x_3}\Big(m_{kk}(x_3)\tilde{E}_3\Big) + \tilde{f}_j, \quad (3.1.1)$$

$$\varepsilon_{33}(x_3)\frac{\partial^2 \tilde{E}_3}{\partial t^2} + (\nu_1^2 m_{22}(x_3) + \nu_2^2 m_{11}(x_3))\tilde{E}_3 = (i\nu_1)m_{22}(x_3)\frac{\partial \tilde{E}_1}{\partial x_3} + (i\nu_2)m_{11}(x_3)\frac{\partial \tilde{E}_2}{\partial x_3} + \tilde{f}_3, \quad (3.1.2)$$

$$\tilde{\mathbf{E}}|_{t=0} = 0, \quad \frac{\partial \tilde{\mathbf{E}}}{\partial t}|_{t=0} = 0, \quad (3.1.3)$$

where $j = 1, 2$; k is different from j and runs values $1, 2$.

3.2 Reduction of FTIVP to Operator Integral Equation

The main aim of this section is to show that FTIVP is equivalent to a second kind operator integral equation of the Volterra type. This section organized as follows. In Subsection 3.2.1 we obtain the equivalence of (3.1.1) under data (3.1.3) to some integral equalities for $\tilde{E}_j(\nu, x_3, t)$, $j = 1, 2$. The equivalence of (3.1.2) under data (3.1.3) to an integral equality for $\tilde{E}_3(\nu, x_3, t)$ is described in the Subsection 3.2.2. Integral equality for $\frac{\partial \tilde{E}_3}{\partial t}(\nu, x_3, t)$ is written in subsection 3.2.2 also. Subsection 3.2.3 contains integral equalities for $\frac{\partial \tilde{E}_j}{\partial x_3}(\nu, x_3, t)$, $j = 1, 2$ in the forms which are necessary to get a closed system of integral equations for unknowns $\tilde{E}_j$, $\tilde{E}_3$, $\frac{\partial \tilde{E}_3}{\partial t}$, $\frac{\partial \tilde{E}_j}{\partial x_3}$, $j = 1, 2$. This system of integral equations is written in the from of a second kind of an operator integral equation of the Volterra type in the subsection 3.2.4.

3.2.1 Equivalence of (3.1.1), (3.1.3) to Integral Equalities

Now we show that for each $j = 1, 2$ the equation (3.1.1) is written in the terms of new function $V_j(\nu, y_j, t)$ depending on ν, t and a new variable y_j. The obtained equation is a partial differential equation with constant coefficients in the principal part. We find integral equality for $V_j(\nu, y_j, t)$ by inverting the principal part of the obtained differential equation. As a next step the integral equality is written in the term of $\tilde{E}_j(\nu, x_3, t)$.

Let us consider the following transformation

$$y_j = \tau_j(x_3),\ \tau_j(x_3) = \int_0^{x_3} c_j(\xi)\, d\xi, \tag{3.2.1}$$

where

$$c_1^2(\xi) = \frac{\varepsilon_{11}(\xi)}{m_{22}(\xi)},\ \ c_2^2(\xi) = \frac{\varepsilon_{22}(\xi)}{m_{11}(\xi)}.$$

Remark 3.2.1. We note that under assumptions mentioned at the beginning of Chapter 3 the function $\tau_j(x_3)$, defined by (3.2.1) for each $j = 1, 2$, has the following properties:

(a) $\tau_j(x_3)$ is monotonic increasing function mapping $[-cT, cT]$ into $[Y_j^-, Y_j^+]$, where $Y_j^- = \tau_j(-cT)$, $Y_j^+ = \tau_j(cT)$;

(b) $\tau_j(x_3)$ has a monotonic increasing inverse function $\tau_j^{-1}(y_j)$ mapping $[Y_j^-, Y_j^+]$ into $[-cT, cT]$;

(c) $\tau_j(0) = 0,\ \tau_j^{-1}(0) = 0$;

(d) $\tau_j(x_3) \in C^3[-cT, cT],\ \ \tau_j^{-1}(y_j) \in C^3[Y_j^-, Y_j^+]$.

Let

$$W_j(\nu, y_j, t) = \tilde{E}_j(\nu, x_3, t)|_{x_3=\tau_j^{-1}(y_j)}, \tag{3.2.2}$$

then we have

$$\frac{\partial \tilde{E}_j}{\partial x_3}(\nu, x_3, t)|_{x_3=\tau_j^{-1}(y_j)} = c_j(\tau_j^{-1}(y_j))\frac{\partial W_j}{\partial y_j}(\nu, y_j, t). \tag{3.2.3}$$

The equation (3.1.1) may be written in terms of y_j and $W_j(\nu, y_j, t)$ as follows

$$\frac{\partial^2 W_j}{\partial t^2} - \frac{\partial^2 W_j}{\partial y_j^2} = -K_j(y_j)\frac{\partial W_j}{\partial y_j} - \nu_k^2 \frac{m_{33}(x_3)}{\varepsilon_{jj}(x_3)}|_{x_3=\tau_j^{-1}(y_j)} W_j$$

$$+\nu_j\nu_k \frac{m_{33}(x_3)}{\varepsilon_{jj}(x_3)}|_{x_3=\tau_j^{-1}(y_j)} \tilde{E}_k + (i\nu_j)\Big[\frac{1}{\varepsilon_{jj}(x_3)}\frac{\partial}{\partial x_3}\Big(m_{kk}(x_3)\tilde{E}_3(\nu, x_3, t)\Big)\Big]_{x_3=\tau_j^{-1}(y_j)}$$

$$+\frac{\tilde{f}_j(\nu, x_3, t)}{\varepsilon_{jj}(x_3)}|_{x_3=\tau_j^{-1}(y_j)}, \qquad (3.2.4)$$

where

$$K_j(y_j) = \frac{d}{dy_j}\Big(lnA_j(y_j)\Big), \quad A_j(y_j) = \frac{1}{\sqrt{m_{kk}(x_3)\varepsilon_{jj}(x_3)}}|_{x_3=\tau_j^{-1}(y_j)}, \qquad (3.2.5)$$

$$j = 1, 2; \ k \neq j, \ k = 1, 2.$$

Let us introduce the function $V_j(\nu, y_j, t)$ by the following equality

$$W_j(\nu, y_j, t) = S_j(y_j)V_j(\nu, y_j, t), \qquad (3.2.6)$$

where the function $S_j(y_j)$ is defined by

$$S_j(y_j) = exp(\frac{1}{2}\int_0^{y_j} K_j(\xi)d\xi). \qquad (3.2.7)$$

Substituting (3.2.6) into (3.2.4) we find

$$\frac{\partial^2 V_j}{\partial t^2} - \frac{\partial^2 V_j}{\partial y_j^2} \;=\; [q_j(y_j) - \nu_k^2 M_{3j}(y_j)N_j(y_j)]V_j + \nu_j\nu_k M_{3j}(y_j)L_j(y_j)$$

$$\times\tilde{E}_k(\nu, \tau_j^{-1}(y_j), t) + (i\nu_j)\frac{\partial}{\partial y_j}\Big[C_j(y_j)L_j(y_j)M_{kj}(y_j)\tilde{E}_3(\nu, \tau_j^{-1}(y_j), t)\Big]$$

$$-(i\nu_j)M_{kj}(y_j)\frac{\partial}{\partial y_j}\Big[C_j(y_j)L_j(y_j)\Big]\tilde{E}_3(\nu, \tau_j^{-1}(y_j), t) + F_j(\nu, y_j, t), \qquad (3.2.8)$$

$$j = 1, 2; \ k \neq j, \ k = 1, 2.$$

Here the following notations were used:

$$\begin{aligned} q_j(y_j) &= \frac{1}{2}K_j'(y_j) - \frac{1}{4}K_j^2(y_j), \quad C_j(y_j) = c_j(x_3)|_{x_3=\tau_j^{-1}(y_j)}, \\ N_j(y_j) &= \frac{1}{\varepsilon_{jj}(x_3)}|_{x_3=\tau_j^{-1}(y_j)}, \quad M_{lj}(y_j) = m_{ll}(x_3)|_{x_3=\tau_j^{-1}(y_j)}, \ l = 1, 2, 3; \\ L_j(y) &= \frac{N_j(y)}{S_j(y)}, \quad F_j(\nu, y_j, t) = \tilde{f}_j(\nu, \tau_j^{-1}(y_j), \tau)L_j(y_j), \qquad (3.2.9) \end{aligned}$$

where $K_j(y_j)$, $S_j(y_j)$ are defined by (3.2.5), (3.2.7). Using D'Alambert formula (Vladimirov (1971), see also appendix A) we can show that equation (3.2.8) with zero initial data is equivalent to the following integral equation

$$V_j(\nu, y_j, t) \;=\; \frac{1}{2}\int_0^t \int_{y_j-(t-\tau)}^{y_j+(t-\tau)} \Big\{\Big[q_j(\xi) - \nu_k^2 M_{3j}(\xi)N_j(\xi)\Big]V_j(\nu, \xi, \tau)$$

$$+\nu_j\nu_k M_{3j}(\xi)L_j(\xi)\tilde{E}_k(\nu,\tau_j^{-1}(\xi),\tau) - (i\nu_j)M_{kj}(\xi)\frac{\partial}{\partial\xi}\Big[C_j(\xi)L_j(\xi)\Big]$$

$$\times\tilde{E}_3(\nu,\tau_j^{-1}(\xi),\tau) + F_j(\nu,\xi,\tau)\Big\}d\xi d\tau$$

$$+\frac{i\nu_j}{2}\int_0^t \Big\{C_j(\xi)L_j(\xi)M_{kj}(\xi)\tilde{E}_3(\nu,\tau_j^{-1}(\xi),\tau)\Big\}_{\xi=y_j-(t-\tau)}^{\xi=y_j+(t-\tau)}d\tau, \qquad (3.2.10)$$

$$j=1,2;\ k\neq j,\ k=1,2.$$

The notation $\Big\{...\Big\}\Big|_{\xi=y-(t-\tau)}^{\xi=y+(t-\tau)}$ means the difference of the expression which is inside bracket for $\xi = y+(t-\tau)$ and $\xi = y-(t-\tau)$. Using (3.2.2), (3.2.6) equation (3.2.10) may be written as follows

$$\tilde{E}_j(\nu,x_3,t) \;=\; \frac{S_j(\tau_j(x_3))}{2}\int_0^t\int_{\tau_j(x_3)-(t-\tau)}^{\tau_j(x_3)+(t-\tau)}\Big\{\Big[q_j(\xi)-\nu_k^2 M_{3j}(\xi)N_j(\xi)\Big]$$

$$\times\frac{\tilde{E}_j(\nu,\tau_j^{-1}(\xi),\tau)}{S_j(\xi)} + \nu_j\nu_k M_{3j}(\xi)L_j(\xi)\tilde{E}_k(\nu,\tau_j^{-1}(\xi),\tau)$$

$$-(i\nu_j)M_{kj}(\xi)\frac{\partial}{\partial\xi}\Big[C_j(\xi)L_j(\xi)\Big]\tilde{E}_3(\nu,\tau_j^{-1}(\xi),\tau) + F_j(\nu,\xi,\tau)\Big\}d\xi d\tau$$

$$+\frac{i\nu_j}{2}S_j(\tau_j(x_3))\int_0^t\Big\{C_j(\xi)L_j(\xi)M_{kj}(\xi)\tilde{E}_3(\nu,\tau_j^{-1}(\xi),\tau)\Big\}_{\xi=\tau_j(x_3)-(t-\tau)}^{\xi=\tau_j(x_3)+(t-\tau)}d\tau,$$

$$j=1,2;\ k\neq j,\ k=1,2. \qquad (3.2.11)$$

3.2.2 Integral Equalities for $\tilde{E}_3$, $\dfrac{\partial\tilde{E}_3}{\partial t}$

Integrating the equation (3.1.2) with respect to t with zero initial data, we find the integral equality for $\tilde{E}_3(\nu,x_3,t)$:

$$\tilde{E}_3(\nu,x_3,t) = \frac{1}{\varepsilon_{33}(x_3)}\int_0^t\Big[i\nu_1 m_{22}(x_3)\frac{\partial\tilde{E}_1}{\partial x_3}(\nu,x_3,\tau)$$

$$+i\nu_2 m_{11}(x_3)\frac{\partial \tilde{E}_2}{\partial x_3}(\nu, x_3, \tau) + \tilde{f}_3(\nu, x_3, \tau)\Big]\frac{\sin\Big(d(\nu, x_3)(t-\tau)\Big)}{d(\nu, x_3)}d\tau, \quad (3.2.12)$$

where

$$d(\nu, x_3) = \sqrt{\frac{\nu_1^2 m_{22}(x_3) + \nu_2^2 m_{11}(x_3)}{\varepsilon_{33}(x_3)}}. \quad (3.2.13)$$

Differentiating (3.2.12) with respect to t we find the integral equality for $\frac{\partial \tilde{E}_3}{\partial t}(\nu, x_3, t)$:

$$\frac{\partial \tilde{E}_3}{\partial t}(\nu, x_3, t) = \frac{1}{\varepsilon_{33}(x_3)}\int_0^t \Big[i\nu_1 m_{22}(x_3)\frac{\partial \tilde{E}_1}{\partial x_3}(\nu, x_3, \tau)$$

$$+i\nu_2 m_{11}(x_3)\frac{\partial \tilde{E}_2}{\partial x_3}(\nu, x_3, \tau) + \tilde{f}_3(\nu, x_3, \tau)\Big]\cos\Big(d(\nu, x_3)(t-\tau)\Big)d\tau \quad (3.2.14)$$

3.2.3 Integral Equalities for $\frac{\partial \tilde{E}_j}{\partial x_3}$, $j = 1, 2$

equalities for $\frac{\partial \tilde{E}_j}{\partial x_3}(\nu, x_3, t)$, $j = 1, 2$ in the form containing functions $\tilde{E}_j(\nu, x_3, t)$, $\frac{\partial \tilde{E}_j}{\partial x_3}(\nu, x_3, t)$, $j = 1, 2$, $\tilde{E}_3(\nu, x_3, t)$, $\frac{\partial \tilde{E}_3}{\partial t}(\nu, x_3, t)$. A starting point here is the equation (3.2.11) which can be written in the form

$$\tilde{E}_j(\nu, x_3, t) = \frac{S_j(\tau_j(x_3))}{2}\int_0^t \int_{\tau_j(x_3)-(t-\tau)}^{\tau_j(x_3)+(t-\tau)} \Big\{\Big[q_j(\xi) - \nu_k^2 M_{3j}(\xi)N_j(\xi)\Big]$$

$$\times\frac{\tilde{E}_j(\nu, \tau_j^{-1}(\xi), \tau)}{S_j(\xi)} + \nu_j\nu_k M_{3j}(\xi)L_j(\xi)\tilde{E}_k(\nu, \tau_j^{-1}(\xi), \tau)$$

$$-(i\nu_j)M_{kj}(\xi)\frac{\partial}{\partial \xi}\Big[C_j(\xi)L_j(\xi)\Big]\tilde{E}_3(\nu, \tau_j^{-1}(\xi), \tau) + F_j(\nu, \xi, \tau)\Big\}d\xi d\tau$$

$$+\frac{i\nu_j}{2}S_j(\tau_j(x_3))\Big\{\int_{\tau_j(x_3)}^{\tau_j(x_3)+t} C_j(\eta)L_j(\eta)M_{kj}(\eta)\tilde{E}_3(\nu, \tau_j^{-1}(\eta), \tau_j(x_3) + (t-\eta))d\eta,$$

$$-\int_{\tau_j(x_3)-t}^{\tau_j(x_3)} C_j(\mu)L_j(\mu)M_{kj}(\mu)\tilde{E}_3(\nu, \tau_j^{-1}(\mu), -\tau_j(x_3) + (t+\mu))d\mu\Big\}, \quad (3.2.15)$$

$$j = 1,2;\ k \neq j,\ k = 1,2.$$

Differentiating (3.2.15) with respect to x_3 we find

$$\frac{\partial \tilde{E}_j}{\partial x_3}(\nu, x_3, t) = \frac{c_j(\tau_j(x_3))S_j'(\tau_j(x_3))}{2}\Bigg\{\int_0^t \int_{\tau_j(x_3)-(t-\tau)}^{\tau_j(x_3)+(t-\tau)} \Big\{\Big[q_j(\xi)$$

$$-\nu_k^2 M_{3j}(\xi)N_j(\xi)\Big]\frac{\tilde{E}_j(\nu, \tau_j^{-1}(\xi), \tau)}{S_j(\xi)} + \nu_j\nu_k M_{3j}(\xi)L_j(\xi)\tilde{E}_k(\nu, \tau_j^{-1}(\xi), \tau) +$$

$$-(i\nu_j)M_{kj}(\xi)\frac{\partial}{\partial \xi}\Big[C_j(\xi)L_j(\xi)\Big]\tilde{E}_3(\nu, \tau_j^{-1}(\xi), \tau) + F_j(\nu, \xi, \tau)\Big\}d\xi d\tau$$

$$+(i\nu_j)\Big[\int_{\tau_j(x_3)}^{\tau_j(x_3)+t} C_j(\eta)L_j(\eta)M_{kj}(\eta)\tilde{E}_3(\nu, \tau_j^{-1}(\eta), \tau_j(x_3) + (t-\eta))d\eta$$

$$-\int_{\tau_j(x_3)-t}^{\tau_j(x_3)} C_j(\mu)L_j(\mu)M_{kj}(\mu)\tilde{E}_3(\nu, \tau_j^{-1}(\mu), -\tau_j(x_3) + (t+\mu))d\mu\Big]\Bigg\}$$

$$+\frac{S_j(\tau_j(x_3))}{2}\int_0^t \Big\{\Big[q_j(\xi) - \nu_k^2 M_{3j}(\xi)N_j(\xi)\Big]\frac{\tilde{E}_j(\nu, \tau_j^{-1}(\xi), \tau)}{S_j(\xi)}$$

$$+\nu_j\nu_k M_{3j}(\xi)L_j(\xi)\tilde{E}_k(\nu, \tau_j^{-1}(\xi), \tau) - (i\nu_j)M_{kj}(\xi)\frac{\partial}{\partial \xi}\Big[C_j(\xi)L_j(\xi)\Big]$$

$$\times\tilde{E}_3(\nu, \tau_j^{-1}(\xi), \tau) + F_j(\nu, \xi, \tau)\Big\}_{\xi=\tau_j(x_3)-(t-\tau)}^{\xi=\tau_j(x_3)+(t-\tau)} d\tau + \frac{i\nu_j c_j(x_3)}{2}S_j(\tau_j(x_3))\Big\{$$

$$\int_{\tau_j(x_3)}^{\tau_j(x_3)+t} C_j(\eta)L_j(\eta)M_{kj}(\eta)\frac{\partial \tilde{E}_3}{\partial t}(\nu, \tau_j^{-1}(\eta), \tau_j(x_3) + (t-\eta))d\eta$$

$$+\int_{\tau_j(x_3)-t}^{\tau_j(x_3)} C_j(\mu)L_j(\mu)M_{kj}(\mu)\frac{\partial \tilde{E}_3}{\partial t}(\nu, \tau_j^{-1}(\mu), -\tau_j(x_3) + (t+\mu))d\mu\Big\}$$

$$-(i\nu_j)c_j(x_3)S_j(\tau_j(x_3))C_j(\tau_j(x_3))L_j(\tau_j(x_3))M_{kj}(\tau_j(x_3))\tilde{E}_3(\nu, x_3, t). \quad (3.2.16)$$

$$j = 1,2;\ k \neq j,\ k = 1,2.$$

Using (3.2.12) the equation (3.2.16) can be written as follows

$$
\begin{aligned}
\frac{\partial \tilde{E}_j}{\partial x_3}(\nu, x_3, t) \;=\; & \frac{c_j(\tau_j(x_3))S_j'(\tau_j(x_3))}{2}\Bigg\{\int_0^t \int_{\tau_j(x_3)-(t-\tau)}^{\tau_j(x_3)+(t-\tau)} \Big\{\Big[q_j(\xi) \\
& -\nu_k^2 M_{3j}(\xi)N_j(\xi)\Big]\frac{\tilde{E}_j(\nu, \tau_j^{-1}(\xi), \tau)}{S_j(\xi)} + \nu_j\nu_k M_{3j}(\xi)L_j(\xi)\tilde{E}_k(\nu, \tau_j^{-1}(\xi), \tau) \\
& -(i\nu_j)M_{kj}(\xi)\frac{\partial}{\partial \xi}\Big[C_j(\xi)L_j(\xi)\Big]\tilde{E}_3(\nu, \tau_j^{-1}(\xi), \tau) + F_j(\nu, \xi, \tau)\Big\}d\xi d\tau \\
& +(i\nu_j)\int_0^t \Big\{C_j(z)L_j(z)M_{kj}(z)\tilde{E}_3(\nu, \tau_j^{-1}(z), \tau)\Big\}_{z=\tau_j(x_3)-(t-\tau)}^{z=\tau_j(x_3)+(t-\tau)} d\tau\Bigg\} \\
& +\frac{S_j(\tau_j(x_3))}{2}\int_0^t \Big\{\Big[q_j(\xi) - \nu_k^2 M_{3j}(\xi)N_j(\xi)\Big]\frac{\tilde{E}_j(\nu, \tau_j^{-1}(\xi), \tau)}{S_j(\xi)} \\
& +\nu_j\nu_k M_{3j}(\xi)L_j(\xi)\tilde{E}_k(\nu, \tau_j^{-1}(\xi), \tau) - (i\nu_j)M_{kj}(\xi)\frac{\partial}{\partial \xi}\Big[C_j(\xi)L_j(\xi)\Big] \\
& \times\tilde{E}_3(\nu, \tau_j^{-1}(\xi), \tau) + F_j(\nu, \xi, \tau)\Big\}_{\xi=\tau_j(x_3)-(t-\tau)}^{\xi=\tau_j(x_3)+(t-\tau)} d\tau \\
& +\frac{i\nu_j c_j(x_3)}{2}S_j(\tau_j(x_3))\Bigg\{\int_0^t \Big\{C_j(z)L_j(z)M_{kj}(z)\frac{\partial \tilde{E}_3}{\partial t}(\nu, \tau_j^{-1}(z), \tau)\Big\}_{z=\tau_j(x_3)+(t-\tau)} \\
& +\Big\{C_j(z)L_j(z)M_{kj}(z)\frac{\partial \tilde{E}_3}{\partial t}(\nu, \tau_j^{-1}(z), \tau)\Big\}_{z=\tau_j(x_3)-(t-\tau)} d\tau\Bigg\} \\
& -\frac{(i\nu_j)c_j(x_3)}{\varepsilon_{33}(x_3)}S_j(\tau_j(x_3))C_j(\tau_j(x_3))L_j(\tau_j(x_3))M_{kj}(\tau_j(x_3)) \\
& \times\int_0^t \Big[i\nu_1 m_{22}(x_3)\frac{\partial \tilde{E}_1}{\partial x_3}(\nu, x_3, \tau) + i\nu_2 m_{11}(x_3)\frac{\partial \tilde{E}_2}{\partial x_3}(\nu, x_3, \tau) \\
& +\tilde{f}_3(\nu, x_3, \tau)\Big]\frac{\sin\Big(d(\nu, x_3)(t-\tau)\Big)}{d(\nu, x_3)}d\tau, \quad j = 1,2;\ k \neq j,\ k = 1,2. \qquad (3.2.17)
\end{aligned}
$$

3.2.4 An Operator Integral Equation

Equations (3.2.11), (3.2.12), (3.2.14), (3.2.17) represent a system of integral equations with respect to unknowns $\tilde{E}_j$, $\tilde{E}_3$, $\dfrac{\partial \tilde{E}_3}{\partial t}$, $\dfrac{\partial \tilde{E}_j}{\partial x_3}$, $j = 1, 2$. Reasonings of Subsections 3.2.1-3.2.3 show that this system is equivalent of (3.1.1), (3.1.2) under condition (3.1.3). The system (3.2.11)-(3.2.14), (3.2.17) can be written in the form of the following operator integral equation.

$$\mathbf{V}(\nu, x_3, t) = \mathbf{G}(\nu, x_3, t) + \int_0^t \big(\mathbf{KV}\big)(\nu, x_3, t, \tau)d\tau, \qquad (3.2.18)$$

where $\mathbf{V} = (V_1, V_2, V_3, V_4, V_5, V_6)$ is unknown vector-function whose components are

$$V_1 = \tilde{E}_1, V_2 = \tilde{E}_2, V_3 = \tilde{E}_3, V_4 = \frac{\partial \tilde{E}_3}{\partial t}, V_5 = \frac{\partial \tilde{E}_1}{\partial x_3}, V_6 = \frac{\partial \tilde{E}_2}{\partial x_3}; \qquad (3.2.19)$$

$\mathbf{G} = (G_1, G_2, G_3, G_4, G_5, G_6)$ is the given vector-function whose components are defined by

$$G_j(\nu, x_3, t) = \frac{S_j(\tau_j(x_3))}{2} \int_0^t \int_{\tau_j(x_3)-(t-\tau)}^{\tau_j(x_3)+(t-\tau)} F_j(\nu, \xi, \tau)d\xi d\tau, \quad j = 1, 2, \qquad (3.2.20)$$

$$G_3(\nu, x_3, t) = \frac{1}{\varepsilon_{33}(x_3)} \int_0^t \tilde{f}_3(\nu, x_3, \tau)\frac{\sin\Big(d(\nu, x_3)(t-\tau)\Big)}{d(\nu, x_3)}d\tau, \qquad (3.2.21)$$

$$G_4(\nu, x_3, t) = \frac{1}{\varepsilon_{33}(x_3)} \int_0^t \tilde{f}_3(\nu, x_3, \tau)\cos\Big(d(\nu, x_3)(t-\tau)\Big)d\tau \qquad (3.2.22)$$

$$G_{4+j}(\nu, x_3, t) = \frac{c_j(\tau_j(x_3))S_j'(\tau_j(x_3))}{2} \int_0^t \Bigg\{ \int_{\tau_j(x_3)-(t-\tau)}^{\tau_j(x_3)+(t-\tau)} F_j(\nu, \xi, \tau)d\xi$$

$$+(i\nu_j)\Big\{F_j(\nu, \xi, \tau)\Big\}_{\xi=\tau_j(x_3)-(t-\tau)}^{\xi=\tau_j(x_3)+(t-\tau)}\Bigg\}d\tau - (i\nu_j)\frac{c_j(x_3)}{\varepsilon_{33}(x_3)}S_j(\tau_j(x_3))L_j(\tau_j(x_3))$$

$$\times M_{kj}(\tau_j(x_3)) \int_0^t \tilde{f}_3(\nu, x_3, \tau)\frac{\sin\Big(d(\nu, x_3)(t-\tau)\Big)}{d(\nu, x_3)}d\tau, \quad j = 1, 2; \qquad (3.2.23)$$

The components of the vector-operator $\mathbf{K} = (\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3, \mathcal{K}_4, \mathcal{K}_5, \mathcal{K}_6)$ are defined by

$$\Big(\mathcal{K}_j \mathbf{V}\Big)(\nu, x_3, t, \tau) \;=\; \frac{S_j(\tau_j(x_3))}{2} \int_{\tau_j(x_3)-(t-\tau)}^{\tau_j(x_3)+(t-\tau)} \Big\{ \Big[q_j(\xi) - \nu_k^2 M_{3j}(\xi) N_j(\xi) \Big]$$

$$\times \frac{V_j(\nu, \tau_j^{-1}(\xi), \tau)}{S_j(\xi)} + \nu_j \nu_k M_{3j}(\xi) L_j(\xi) V_k(\nu, \tau_j^{-1}(\xi), \tau)$$

$$-(i\nu_j) M_{kj}(\xi) \frac{\partial}{\partial \xi} \Big[C_j(\xi) L_j(\xi) \Big] V_3(\nu, \tau_j^{-1}(\xi), \tau) \Big\} d\xi$$

$$+\frac{i\nu_j}{2} S_j(\tau_j(x_3)) \Big\{ C_j(\xi) L_j(\xi) M_{kj}(\xi) V_3(\nu, \tau_j^{-1}(\xi), \tau) \Big\}_{\xi=\tau_j(x_3)-(t-\tau)}^{\xi=\tau_j(x_3)+(t-\tau)}, \quad (3.2.24)$$

$$j = 1, 2; \;\; k \neq j, \;\; k = 1, 2.$$

$$\Big(\mathcal{K}_3 \mathbf{V}\Big)(\nu, x_3, t, \tau) \;=\; \frac{1}{\varepsilon_{33}(x_3)} \Big[i\nu_1 m_{22}(x_3) V_5(\nu, x_3, \tau)$$

$$+ i\nu_2 m_{11}(x_3) V_6(\nu, x_3, \tau) \Big] \frac{\sin\Big(d(\nu, x_3)(t-\tau)\Big)}{d(\nu, x_3)}, \qquad (3.2.25)$$

$$\Big(\mathcal{K}_4 \mathbf{V}\Big)(\nu, x_3, t, \tau) \;=\; \frac{1}{\varepsilon_{33}(x_3)} \Big[i\nu_1 m_{22}(x_3) V_5(\nu, x_3, \tau)$$

$$+ i\nu_2 m_{11}(x_3) V_6(\nu, x_3, \tau) \Big] \cos\Big(d(\nu, x_3)(t-\tau)\Big), \qquad (3.2.26)$$

$$\Big(\mathcal{K}_{4+j} \mathbf{V}\Big)(\nu, x_3, t, \tau) \;=\; \frac{c_j(\tau_j(x_3)) S_j'(\tau_j(x_3))}{2} \Bigg\{ \int_{\tau_j(x_3)-(t-\tau)}^{\tau_j(x_3)+(t-\tau)} \Big\{ \Big[q_j(\xi)$$

$$-\nu_k^2 M_{3j}(\xi) N_j(\xi) \Big] \frac{V_j(\nu, \tau_j^{-1}(\xi), \tau)}{S_j(\xi)} + \nu_j \nu_k M_{3j}(\xi) L_j(\xi) V_k(\nu, \tau_j^{-1}(\xi), \tau)$$

$$-(i\nu_j) M_{kj}(\xi) \frac{\partial}{\partial \xi} \Big[C_j(\xi) L_j(\xi) \Big] V_3(\nu, \tau_j^{-1}(\xi), \tau) \Big\} d\xi$$

$$+(i\nu_j)\Big\{C_j(z)L_j(z)M_{kj}(z)V_3(\nu,\tau_j^{-1}(z),\tau)\Big\}_{z=\tau_j(x_3)-(t-\tau)}^{z=\tau_j(x_3)+(t-\tau)}\Bigg\}$$

$$+\frac{S_j(\tau_j(x_3))}{2}\Big\{\Big[q_j(\xi)-\nu_k^2M_{3j}(\xi)N_j(\xi)\Big]\frac{V_j(\nu,\tau_j^{-1}(\xi),\tau)}{S_j(\xi)}$$

$$+\nu_j\nu_kM_{3j}(\xi)L_j(\xi)V_k(\nu,\tau_j^{-1}(\xi),\tau)-(i\nu_j)M_{kj}(\xi)\frac{\partial}{\partial\xi}\Big[C_j(\xi)L_j(\xi)\Big]$$

$$\times V_3(\nu,\tau_j^{-1}(\xi),\tau)\Big\}_{\xi=\tau_j(x_3)-(t-\tau)}^{\xi=\tau_j(x_3)+(t-\tau)}$$

$$+\frac{i\nu_jc_j(x_3)}{2}S_j(\tau_j(x_3))\Bigg\{\Big\{C_j(z)L_j(z)M_{kj}(z)V_4(\nu,\tau_j^{-1}(z),\tau)\Big\}_{z=\tau_j(x_3)+(t-\tau)}$$

$$+\Big\{C_j(z)L_j(z)M_{kj}(z)V_4(\nu,\tau_j^{-1}(z),\tau)\Big\}_{z=\tau_j(x_3)-(t-\tau)}\Bigg\}$$

$$-\frac{(i\nu_j)c_j(x_3)}{\varepsilon_{33}(x_3)}S_j(\tau_j(x_3))C_j(\tau_j(x_3))L_j(\tau_j(x_3))M_{kj}(\tau_j(x_3))\Big[i\nu_1m_{22}(x_3)$$

$$\times V_5(\nu,x_3,\tau)+i\nu_2m_{11}(x_3)V_6(\nu,x_3,\tau)\Big]\frac{\sin\Big(d(\nu,x_3)(t-\tau)\Big)}{d(\nu,x_3)},\qquad(3.2.27)$$

$$j=1,2;\ k\neq j,\ k=1,2.$$

3.3 Properties of the Operator Integral Equation (3.2.18)

In this section the properties of the inhomogeneous term and the kernel of (3.2.18) are described in forms convenient to prove the existence and uniqueness theorems for (3.2.18). We state these properties by the following propositions.

Proposition 1. *Let components of $\boldsymbol{G}=(G_1,G_2,\ldots,G_6)$ be defined by (3.2.20)-(3.2.23). Then under assumptions mentioned at the beginning of Chapter 3 these components are continuous functions for $(x_3,t)\in\Delta$, $\nu\in R^2$.*

Proposition 2. *Let components of the vector operator* $\boldsymbol{K} = (\mathcal{K}_1, \mathcal{K}_2, \dots, \mathcal{K}_6)$ *be defined by (3.2.24)-(3.2.27). Then under assumptions mentioned in Section 1*
(i) *the expressions*

$$\int_0^t \big(\mathcal{K}_m \boldsymbol{V}\big)(\nu, x_3, t, \tau) d\tau, \ m = 1, 2, \dots, 6$$

are continuous functions for $(x_3, t) \in \Delta$, $\nu \in R^2$ *and any vector function* $\boldsymbol{V}(\nu, x_3, t) = (V_1(\nu, x_3, t), V_2(\nu, x_3, t), \dots, V_6(\nu, x_3, t))$ *with continuous components for* $(x_3, t) \in \Delta$, $\nu \in R^2$.
(ii) *for any positive number* Ω *the following inequalities are satisfied*

$$|\int_0^t \Big(\mathcal{K}_m \boldsymbol{V}\Big)(\nu, x_3, t, \tau) d\tau| \leq B \int_0^t \| \boldsymbol{V} \|(\nu, \tau) d\tau, \ m = 1, 2, \dots, 6; \qquad (3.3.1)$$

where $(x_3, t) \in \Delta$, $|\nu| \leq \Omega$, B *is a positive number depending on* α, β, T, Ω;

$$\| \boldsymbol{V} \|(\nu, \tau) = \max_{m=1,2,\dots,6} \ \max_{\xi \in [-c(T-\tau), c(T-\tau)]} |V_m(\nu, \xi, \tau)|. \qquad (3.3.2)$$

Proof of Proposition 1. Let numbers α, β, T, c, the set Δ, and functions $\varepsilon_{jj}(x_3)$, $m_{jj}(x_3)$ satisfy assumptions of Section 1, numbers Y_j^-, Y_j^+ and functions τ_j, τ_j^{-1} satisfy the properties of Remark 3.2.1. Using the formula (3.2.5), (3.2.7), (3.2.9) we find that the functions A_j, S_j, C_j, N_j, L_j, M_{lj} are twice continuously differentiable on $[Y_j^-, Y_j^+]$; the functions K_j are one time continuously differentiable on $[Y_j^-, Y_j^+]$, and q_j, F_j are continuous on $[Y_j^-, Y_j^+]$. The function $d(\nu, x_3)$ defined by (3.2.13) is twice continuously differentiable with respect to $x_3 \in [-cT, cT]$ for any $\nu \in R^2$ and $\dfrac{\sin\Big(d(\nu, x_3)(t - \tau)\Big)}{d(\nu, x_3)}$ is bounded and twice continuously differentiable with respect to $(x_3, t) \in \Delta$ for any $\nu \in R^2$, $o \leq \tau \leq t$. Using properties τ_j described in Remark 3.2.1 we find that $G_m(\nu, x_3, t)$, $m = 1, 2, \dots, 6$ are continuous function for $(x_3, t) \in \Delta$ and $\nu \in R^2$. Proposition 1 is proved.

Proof of Proposition 2. Using the reasoning made in the proof of Proposition 1 and formulae (3.2.24)-(3.2.27) we find that

$$\int_0^t \big(\mathcal{K}_m \mathbf{V}\big)(\nu, x_3, t, \tau) d\tau, \ m = 1, 2, \dots, 6$$

are continuous functions with respect to $(x_3, t) \in \Delta$ for any $\nu \in R^2$ and any vector function $\mathbf{V} = (V_1, V_2, \dots, V_6)$ with continuous components $V_j(\nu, x_3, t)$ for

$(x_3, t) \in \Delta$ and $\nu \in R^2$. Hence the affirmation (i) of proposition 2 is proved. To prove (ii) we need the following lemma.

Lemma 3.3.1. *Let c, T be numbers and Δ be triangle defined in the introduction, τ_j be the function defined in (3.2.1), τ_j^{-1} be inverse function to τ_j; Y_j^-, Y_j^+ be numbers defined in Remark 3.2.1. Then for any $(x_3, t) \in \Delta$ and $\tau \in [0, t]$, $\xi \in [\tau_j(x_3) - (t-\tau), \tau_j(x_3) + (t-\tau)]$ the following relations are satisfied $\tau_j^{-1}(\xi) \in [-c(T-\tau), c(T-\tau)]$, $\xi \in [Y_j^-, Y_j^+]$.*

Proof of Lemma. Let $y = \tau_j(x_3)$. Using Remark 3.2.1 we find

$$y = \int_0^{\tau_j^{-1}(y)} \sqrt{\frac{\varepsilon_{jj}(z)}{m_{jj}(z)}} dz. \tag{3.3.3}$$

Differentiating both sides of (3.3.3) with respect to y we find

$$\frac{d\tau_j^{-1}(y)}{dy} = \sqrt{\frac{\varepsilon_{jj}(\tau_j^{-1}(y))}{m_{jj}(\tau_j^{-1}(y))}} \tag{3.3.4}$$

Integrating (3.3.4) from 0 to y and using $\tau_j^{-1}(0) = 0$ we find

$$\tau_j^{-1}(y) = \int_0^{y} \sqrt{\frac{\varepsilon_{jj}(\tau_j^{-1}(z))}{m_{jj}(\tau_j^{-1}(z))}} dz. \tag{3.3.5}$$

Using (3.3.5) we find that

$$\begin{aligned} \tau_j^{-1}(\tau_j(x_3) - (t-\tau)) &= \int_0^{\tau_j(x_3)-(t-\tau)} \sqrt{\frac{\varepsilon_{jj}(\tau_j^{-1}(z))}{m_{jj}(\tau_j^{-1}(z))}} dz = \\ &= x_3 - \int_{\tau_j(x_3)-(t-\tau)}^{\tau_j(x_3)} \sqrt{\frac{\varepsilon_{jj}(\tau_j^{-1}(z))}{m_{jj}(\tau_j^{-1}(z))}} dz. \end{aligned} \tag{3.3.6}$$

We have from (3.3.6)

$$\tau_j^{-1}(\tau_j(x_3) - (t-\tau)) \geq x_3 - c(t-\tau). \tag{3.3.7}$$

Using $(x_3, t) \in \Delta$ we find

$$x_3 - c(t-\tau) \geq -c(T-t) - c(t-\tau) = -c(T-t)$$

and therefore

$$\tau_j^{-1}(\tau_j(x_3) - (t-\tau)) \geq -c(T-\tau). \tag{3.3.8}$$

Similarly we find

$$\tau_j^{-1}(\tau_j(x_3) + (t-\tau)) \leq c(T-\tau). \tag{3.3.9}$$

Using monotonic increasing $\tau_j^{-1}(\xi)$ (see Remark 3.2.1) and (3.3.8), (3.3.9) we obtain

$$-c(T-\tau) \leq \tau_j^{-1}(\xi) \leq c(T-\tau) \tag{3.3.10}$$

for any $\xi \in [\tau_j(x_3) - (t-\tau), \tau_j(x_3) + (t-\tau)]$. It follows from (3.3.10) that

$$-cT \leq \tau_j^{-1}(\xi) \leq cT$$

and therefore, using monotonic increasing τ_j we find

$$Y_j^- = \tau_j(-cT) \leq \xi \leq \tau_j(cT) = Y_j^+ \tag{3.3.11}$$

for any $\xi \in [\tau_j(x_3) - (t-\tau), \tau_j(x_3) + (t-\tau)]$. Lemma is proved.

Let the number Q is defined by

$$Q = \max_{j=1,2} \max_{y\in[Y_j^-, Y_j^+]} \{|q_j(y)|, |L_j(y)|, |N_j(y)|, |S_j(y)|, |C_j(y)|$$

$$|\frac{\partial}{\partial y}(C_j(y)L_j(y))|, \max_{l=1,2,3} |M_{lj}(y)| \}.$$

Using the lemma we find that for any $(x_3, t) \in \Delta$ and $\tau \in [0, t]$, $\xi \in [\tau_j(x_3) - (t - \tau), \tau_j(x_3) + (t-\tau)]$ the following inequalities are satisfied

$$|q_j(\xi)| \leq Q,\ |L_j(\xi)| \leq Q,\ |N_j(\xi)| \leq Q,\ |S_j(\xi)| \leq Q,\ |C_j(\xi)| \leq Q$$

$$|\frac{\partial}{\partial \xi}(C_j(\xi)L_j(\xi))| \leq Q,\ \max_{l=1,2,3} |M_{lj}(\xi)| \leq Q,\ |V_m(\nu, \tau_j^{-1}(\xi), \tau)| \leq \|\mathbf{V}\|(\nu, \tau),$$

$$j = 1, 2;\ m = 1, 2, \ldots, 6.$$

We find from above mentioned inequalities and the equation (3.2.24) the following relation

$$\Big|(\mathcal{K}_j \mathbf{V})(\nu, x_3, t, \tau)\Big| \leq \frac{Q}{2} \int_{\tau_j(x_3)-(t-\tau)}^{\tau_j(x_3)+(t-\tau)} \Big\{(Q + |\nu|^2 Q^2)|V_j(\nu, \tau_j^{-1}(\xi), \tau)|$$

$$+|\nu|^2Q^2|V_k(\nu,\tau_j^{-1}(\xi),\tau)|+|\nu|Q^2|V_3(\nu,\tau_j^{-1}(\xi),\tau)|\Big\}d\xi$$

$$+|\nu|Q^4\|V\|(\nu,\tau),\quad j=1,2;k=1,2;k\neq j.$$

Let Ω be any positive number. Then we find from the last relation

$$\Big|(\mathcal{K}_j\mathbf{V})(\nu,x_3,t,\tau)\Big|\leq B_j(T,\Omega)\|V\|(\nu,\tau)$$

where $(x_3,t)\in\Delta$, $|\nu|\leq\Omega$,

$$B_j(T,\Omega)=\max_{(x_3,t)\in\Delta}\{Q^2T\Big(1+2\Omega^2Q+\Omega Q\Big)+\Omega Q^4\},\ j=1,2.$$

We find from the equation (3.2.25)

$$\Big|(\mathcal{K}_3\mathbf{V})(\nu,x_3,t,\tau)\Big|\leq B_3(T,\Omega)\|V\|(\nu,\tau)$$

where $(x_3,t)\in\Delta$, $|\nu|\leq\Omega$, $B_3(T,\Omega)=2c^2|\Omega|T$.
Using the similar reasoning we can define constants $B_m(T,\Omega)$ for $m=4,5,6$ such that

$$\Big|(\mathcal{K}_m\mathbf{V})(\nu,x_3,t,\tau)\Big|\leq B_m(T,\Omega)\|V\|(\nu,\tau),$$

where $(x_3,t)\in\Delta$, $|\nu|\leq\Omega$.
Choosing B as follows

$$B=\max_{m=1,2,\ldots,6}B_m(T,\Omega)$$

we complete the proof of the Proposition 2.

3.4 Uniqueness and Existence Theorems for the Operator Integral Equation (3.2.18)

Uniqueness and existence theorems of the operator integral equation (3.2.18) are proved in this section.

3.4.1 Uniqueness Theorem

Theorem 3.4.1. *Under the assumptions stated at the beginning of Chapter 3 there can exist only one solution* $\boldsymbol{V} = (V_1, V_2, \ldots, V_6)$ *of the operator integral equation (3.2.18) such that* $V_m \in C(R^2 \times \Delta),\ m = 1, 2, \ldots, 6$.

Proof. Let Ω be an arbitrary positive number, $\mathbf{V}(\nu, x_3, t)$ and $\mathbf{V}^*(\nu, x_3, t)$ be two solution of (3.2.18) with continuous components for $(x_3, t) \in \Delta$, $|\nu| \le \Omega$. Letting $\hat{\mathbf{V}}(\nu, x_3, t) = \mathbf{V}(\nu, x_3, t) - \mathbf{V}^*(\nu, x_3, t)$ we find from (3.2.18)

$$\hat{\mathbf{V}}(\nu, x_3, t) = \int_0^t \left(\mathbf{K}\hat{\mathbf{V}}\right)(\nu, x_3, t, \tau) d\tau. \tag{3.4.1}$$

Using Proposition 2 we find from (3.4.1)

$$\|\hat{\mathbf{V}}\|(\nu, t) \le B \int_0^t \|\hat{\mathbf{V}}\|(\nu, \tau) d\tau, \tag{3.4.2}$$

where $|\nu| \le \Omega,\ t \in [0, T]$; $\|.\|(\nu, t)$ and B are defined in the statement of Proposition 2.

Applying Grownwall's lemma (Nagle et al. (2004)) to (3.4.2) we find

$$\|\hat{\mathbf{V}}\|(\nu, t) = 0, \quad t \in [0, T],\ |\nu| \le \Omega. \tag{3.4.3}$$

Using the continuity of $\hat{\mathbf{V}}(\nu, x_3, t)$ we conclude that

$$\hat{\mathbf{V}}(\nu, x_3, t) \equiv 0, \quad (x_3, t) \in \Delta,\ |\nu| \le \Omega.$$

Since Ω is an arbitrary positive number we find that $\mathbf{V}(\nu, x_3, t) \equiv \mathbf{V}^*(\nu, x_3, t)$ for $(x_3, t) \in \Delta,\ \nu \in R^2$. Theorem is proved. □

3.4.2 Existence Theorem and Method of Solving

Applying successive approximations we prove the existence theorem in this Subsection. We note that the proof of this theorem contains a method of solving (3.2.18)

Theorem 3.4.2. *Under the assumptions stated at the beginning of Chapter 3 there exists a solution* $\boldsymbol{V} = (V_1, V_2, \ldots, V_6)$ *of the operator integral equation (3.2.18) such that* $V_m \in C(R^2 \times \Delta),\ m = 1, 2, \ldots, 6.$

Proof. Let Ω be an arbitrary positive number. Let us consider the integral equation (3.2.18) for $(x_3, t) \in \Delta$, $|\nu| \leq \Omega$. For finding a solution of this equation we apply the following successive approximations

$$\begin{aligned} \mathbf{V}^{(0)}(\nu, x_3, t) &= \mathbf{G}(\nu, x_3, t), \\ \mathbf{V}^{(n)}(\nu, x_3, t) &= \int_0^t \big(\mathbf{K}\mathbf{V}^{(n-1)}\big)(\nu, x_3, t, \tau) d\tau,\ n = 1, 2 \ldots . \end{aligned} \tag{3.4.4}$$

Our goal is to show that for $(x_3, t) \in \Delta$, $|\nu| \leq \Omega$ the series $\sum_{n=0}^{\infty} \mathbf{V}^{(n)}(\nu, x_3, t) = \Big(\sum_{n=1}^{\infty} V_1^{(n)}(\nu, x_3, t), \ldots, \sum_{n=1}^{\infty} V_6^{(n)}(\nu, x_3, t)\Big)$ is uniformly convergent to a vector function $\mathbf{V}(\nu, x_3, t) = \Big(V_1(\nu, x_3, t), V_2(\nu, x_3, t), \ldots, V_6(\nu, x_3, t)\Big)$ with continuous components and this vector function is a solution of (3.2.18).

Indeed, we find from (3.4.4) and Propositions 1, 2 of Section 3.3 that for $(x_3, t) \in \Delta$, $|\nu| \leq \Omega$ the vector function $\mathbf{V}^{(n)}(\nu, x_3, t)$, $n = 0, 1, 2 \ldots$ have continuous components and

$$|V_j^{(n)}(\nu, x_3, t)| \leq B \int_0^t \|\mathbf{V}^{(n-1)}\|(\nu, \tau) d\tau, \tag{3.4.5}$$

where $\|.\|(\nu, \tau)$ and B are defined in Proposition 2.

It follows from (3.4.5) that

$$|V_m^{(n)}(\nu, x_3, t)| \leq \frac{(BT)^n}{n!} \max_{|\nu| \leq \Omega} \|\mathbf{G}\|(\nu, T), \tag{3.4.6}$$
$$m = 1, 2, \ldots, 6,\ n = 0, 1, 2 \ldots .$$

The uniform convergence of $\sum_{n=0}^{\infty} V_m^{(n)}(\nu, x_3, t)$ to a continuous function $V_m(\nu, x_3, t)$ follows from inequality (3.4.6) and the first Weierstrass theorem (Apostol (1967), page 425). Let us show that the vector function $\mathbf{V}(\nu, x_3, t)$ is a solution of (3.2.18). Summing the equation (3.4.4) with respect to n from 1 to N we have

$$\sum_{n=1}^{N} \mathbf{V}^{(n)}(\nu, x_3, t) = \sum_{n=0}^{N-1} \int_0^t (K\mathbf{V}^{(n)})(\nu, x_3, t, \tau) d\tau, \tag{3.4.7}$$

where

$$\sum_{n=1}^{N} \mathbf{V}^{(n)}(\nu, x_3, t) = \Big(\sum_{n=1}^{N} V_1^{(n)}(\nu, x_3, t), \dots, \sum_{n=1}^{N} V_6^{(n)}(\nu, x_3, t)\Big).$$

Adding both sides of (3.4.7) the vector function $\mathbf{G}(\nu, x_3, t)$ we find

$$\sum_{n=0}^{N} \mathbf{V}^{(n)}(\nu, x_3, t) = \mathbf{G}(\nu, x_3, t) + \int_0^t \sum_{n=0}^{N-1} (K\mathbf{V}^{(n)})(\nu, x_3, t, \tau) d\tau. \qquad (3.4.8)$$

Approaching N the infinity and using the second Weierstrass theorem (Apostol (1967), page 426) we find that the vector function $\mathbf{V}(\nu, x_3, t)$ satisfies (3.2.18) for $(x_3, t) \in \Delta$, $|\nu| \leq \Omega$. Since Ω is an arbitrary positive number we find that the vector function $\mathbf{V}(\nu, x_3, t)$ with continuous components is a solution of (3.2.18) for $(x_3, t) \in \Delta$, $\nu \in R^2$. $\square$

3.5 Initial Value Problem (3.0.1), (3.0.2) Solving

In this Section we show that a generalized solutions of (3.0.1), (3.0.2) may be found by the inverse Fourier transform of the first three components of $\mathbf{V}(\nu, x_3, t)$, where $\mathbf{V}(\nu, x_3, t)$ is the generalized solution of (3.2.18) found in Section 3.4. We describe a class of the vector functions where solution of (3.0.1), (3.0.2) is unique. We will use the following notions and notations. For the exponent $\alpha = (\alpha_1, \alpha_2)$ with $\alpha_j \in \{0, 1, 2, ...\}$ and $|\alpha| = \alpha_1 + \alpha_2$, the partial derivatives of higher order,

$$\frac{\partial^{|\alpha|}}{\partial \nu_1^{\alpha_1} \partial \nu_2^{\alpha_2}} \tilde{f}_k(\nu, x_3, t), \quad \frac{\partial^{|\alpha|}}{\partial \nu_1^{\alpha_1} \partial \nu_2^{\alpha_2}} V_l(\nu, x_3, t), \quad k = 1, 2, 3; \ l = 1, 2, ..., 6,$$

will be denoted by

$$D_\nu^\alpha \tilde{f}_k(\nu, x_3, t), \quad D_\nu^\alpha V_l(\nu, x_3, t).$$

For vector functions $\mathbf{V} = (V_1, V_2, \dots, V_6)$, $\tilde{\mathbf{f}} = (\tilde{f}_1, \tilde{f}_2, \tilde{f}_3)$ and each α we defined $D_\nu^\alpha \mathbf{V}$ and $D_\nu^\alpha \tilde{\mathbf{f}}$ by

$$D_\nu^\alpha \mathbf{V} = (D_\nu^\alpha V_1, D_\nu^\alpha V_2, \dots, D_\nu^\alpha V_6), \ D_\nu^\alpha \tilde{\mathbf{f}} = (D_\nu^\alpha \tilde{f}_1, D_\nu^\alpha \tilde{f}_2, D_\nu^\alpha \tilde{f}_3).$$

We denote by $C(R^2)$ the class consisting of all continuous functions that are defined on R^2, then for $m = 0, 1, 2, ...$ we define $C^m(R^2)$ by $C^0(R^2) = C(R^2)$ and otherwise by

$$C^m(R^2) = \{\varphi(\nu) \in C(R^2) : \text{for all } |\alpha| \le m \;\; D_\nu^\alpha \varphi(\nu) \in C(R^2)\},$$
$$C^\infty(R^2) = \bigcap_{m=1}^{\infty} C^m(R^2).$$

Further, $C_c(R^2)$ is the class of all functions from $C(R^2)$ with compact supports; $\mathcal{L}_2(R^2)$ is the class of all square integrable functions over R^2; $\|\varphi\|_2$ is defined for each $\varphi(\nu) \in \mathcal{L}_2(R^2)$ by

$$\|\varphi\|_2^2 = \int_{R^2} |\varphi(\nu)|^2 d\nu.$$

The Paley-Wiener space $PW(R^2)$ is the space consisting of all functions $\varphi(x_1, x_2) \in C^\infty(R^2)$ satisfying (see Appendix B)

$$(a) \quad (1 + \sqrt{x_1^2 + x_2^2})^m \Delta^n \varphi(x_1, x_2) \in \mathcal{L}_2(R^2) \text{ for all } m, n \in \{0, 1, 2 \ldots \quad \},$$
$$(b) \quad R_\varphi^\Delta = \lim_{n \to \infty} \|\Delta^n \varphi(x_1, x_2)\|_2^{1/2n} < \infty,$$

where $\Delta^n = (\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2})^n$. $C(\triangle; C_c(R^2))$ is the class of all continuous mappings of $(x_3, t) \in \triangle$ into the class $C(R^2)$ of functions $\nu = (\nu_1, \nu_2) \in R^2$; $C(\triangle; PW(R^2))$ is the class of all continuous mappings of $\triangle$ into $PW(R^2)$.

The main result of this section is the following theorem.

Theorem 3.5.1. *Let assumptions at the beginning of Chapter 3 hold and* $\tilde{\boldsymbol{f}} = (\tilde{f}_1, \tilde{f}_2, \tilde{f}_3)$ *be the Fourier transform with respect to* x_1, x_2 *of the inhomogeneous term* $\boldsymbol{f}$ *in (3.0.1) such that for each* α

$$D_\nu^\alpha \tilde{f}_k \in C(R^2 \times \triangle) \cap C(\triangle; C_c(R^2)),\; k = 1, 2, 3.$$

Then there exists a unique generalized solution $\boldsymbol{E}(x, t) = (E_1(x, t), E_2(x, t), E_3(x, t))$ *of (3.0.1), (3.0.2) such that*

$$E_l, \frac{\partial}{\partial x_3} E_j, \frac{\partial}{\partial t} E_3 \in C(R^2 \times \triangle) \cap C(\triangle; PW(R^2)),\; l = 1, 2, 3;\; j = 1, 2.$$

Proof. Let us consider the problem (3.1.1) - (3.1.3) (FTIVP). It was shown in Section 3.3 that this problem is equivalent to the operator integral equation

(3.2.18). It was proved the existence and uniqueness theorems for this operator integral equation in Section 3.4. Using these theorems we find that there exists a unique vector function $\tilde{\mathbf{E}}(\nu, x_3, t) = (\tilde{E_1}(\nu, x_3, t), \tilde{E_2}(\nu, x_3, t), \tilde{E_3}(\nu, x_3, t))$ such that $\tilde{E}_l, \frac{\partial}{\partial x_3}\tilde{E}_j, \frac{\partial}{\partial t}\tilde{E}_3 \in C(R^2 \times \triangle) \cap C(\triangle; C_c(R^2)),\ l = 1,2,3;\ j = 1,2;$ and $\tilde{\mathbf{E}}(\nu, x_3, t)$ is a generalized solution of FTIVP. We are going to show now that it is possible to apply the inverse Fourier transform with respect to ν_1, ν_2 to the generalized solution $\tilde{\mathbf{E}}(\nu, x_3, t)$ of (3.1.1) - (3.1.3).

Using the Proposition 1 and the hypothesis of theorem 3.5.1 we find that $G_m(\nu, x_3, t),\ m = 1, 2, \ldots, 6$ defined by (3.2.20)-(3.2.23) for any α satisfy the following conditions

$$D^\alpha_\nu G_m(\nu, x_3, t) \in C(R^2 \times \triangle) \cap C(\triangle; C_c(R^2)), \tag{3.5.1}$$

$\alpha = (\alpha_1, \alpha_2),\ \alpha_j \in \{0, 1, 2, \ldots\ \},\ \nu = (\nu_1, \nu_2) \in R^2,\ (x_3, t) \in \triangle.$

Applying D^α_ν to (3.2.18) we obtain

$$\begin{aligned} D^\alpha_\nu \mathbf{V}(\nu, x_3, t) &= D^\alpha_\nu \mathbf{G}(\nu, x_3, t) + \int_0^t \Big(\mathbf{K} D^\alpha_\nu \mathbf{V}\Big)(\nu, x_3, t, \tau) d\tau, \\ &\quad \nu \in R^2,\ (x_3, t) \in \triangle. \end{aligned} \tag{3.5.2}$$

Equation (3.5.2) has the same form as (3.2.18). According to theorem 3.4.1 and theorem 3.4.2 the solution $\mathbf{V}(\nu, y, t)$ of (3.2.18), which is found by the method of successive approximations described in Subsection 3.4.2, satisfies for any α the following property:

$$D^\alpha_\nu \mathbf{V}(\nu, x_3, t) \in C(R^2 \times \triangle).$$

Using (3.5.2) and (3.3.1) we obtain the following inequality for any positive number Ω and any α

$$\|D^\alpha_\nu \mathbf{V}\|(\nu, t) \leq \|D^\alpha_\nu \mathbf{G}\|(\nu, t) + B \int_0^t \|\mathbf{V}\|(\nu, \tau) d\tau, \tag{3.5.3}$$

where $|\nu| \leq \Omega,\ t \in [0, T];\ B$ and $\|.\|(\nu, t)$ are defined in the statement of Proposition 2 (see formula (3.3.2)).

Applying the Grownwall's lemma (Nagle et al. (2004)) for the inequality (3.5.3) we find

$$\|D^\alpha_\nu \mathbf{V}\|(\nu,t) \le \|D^\alpha_\nu \mathbf{G}\|(\nu,t)e^{BT}, \ \ |\nu| \le \Omega, \ t \in [0,T]. \tag{3.5.4}$$

It follows from (3.5.1), (3.5.4) that the solution of (3.2.18) satisfies the following property $D^\alpha_\nu \mathbf{V}(\nu,x_3,t) \in C(\triangle; C_c(R^2))$ for any α. Hence the components of the generalized solution $\tilde{\mathbf{E}}(\nu,x_3,t) = (\tilde{E}_1(\nu,x_3,t), \tilde{E}_2(\nu,x_3,t), \tilde{E}_3(\nu,x_3,t))$ of (3.1.1) - (3.1.3) satisfy the conditions
$\tilde{E}_l, \dfrac{\partial}{\partial x_3}\tilde{E}_j, \dfrac{\partial}{\partial t}\tilde{E}_3$ belong to $C(R^2 \times \triangle) \cap C(\triangle; C^\infty_c(R^2)),\ l = 1,2,3;\ j = 1,2.$
Applying the inverse Fourier transform with respect to ν_1, ν_2 to (3.1.1) - (3.1.3) using the real version of the Paley-Wiener theorem (Andersen (2004)) (see also Appendix B) we find that $\mathbf{E}(x,t) = \mathcal{F}^{-1}_\nu[\tilde{\mathbf{E}}]$ is a unique generalized solution of (3.0.1), (3.0.2) such that $E_l(x,t), \dfrac{\partial}{\partial x_3}E_j(x,t), \dfrac{\partial}{\partial t}E_3(x,t)$ belong to the class $C(R^2 \times \triangle) \cap C(\triangle; PW(R^2)),\ l = 1,2,3;\ j = 1,2.$ □

CHAPTER FOUR
SOLVING INITIAL VALUE PROBLEM FOR VECTOR TELEGRAPH EQUATION. GREEN'S FUNCTION METHOD

In this chapter we consider the vector operator

$$\mathcal{L} \equiv \frac{\partial^2}{\partial t^2} - a^2 \mathcal{I} \Delta_x + 2\mathcal{Q}\frac{\partial}{\partial t}, \tag{4.0.1}$$

where $x = (x_1, x_2, x_3) \in R^3$, $t \in R$, a is a positive constant, $\mathcal{I}$ is identity matrix of the order 3×3, $\mathcal{Q}$ is a matrix of the order 3×3 with constant elements. The main problem of the study is IVP related to this operator. The Green's function method is used for solving this problem. This method consists in constructing the Green's function of the IVP and finding an explicit formula for a solution of IVP using the Green's function. At the end of this Chapter an application of the obtained formulae is given for constructing the Green's function of IVP for the Maxwell's system of electrodynamics. This Chapter is started with elements of generalized functions which are actively used.

4.1 Elements of Generalized Functions

In this chapter we use the notions and notations from (Vladimirov (1971)). We denote by $D(R^n)$ the class of test functions: all infinitely differentiable functions in R^n with compact support. We shall define convergence in $D(R^n)$ as follows. The sequence of the functions $\varphi_1, \varphi_2, \ldots$ from $D(R^n)$ converges to the function $\varphi \in D(R^n)$ if
(i) there exists a number $M > 0$ such that $supp\ \varphi_k \subset \{x \in R^n : \ |x| \leq M\}$;
(ii) for each $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ with nonnegative integer components α_j the following relation holds:

$$\sup_{x \in R^n} |D^\alpha \varphi_k(x) - D^\alpha \varphi_k(x)| \to 0, \quad k \to \infty.$$

Here *supp* φ_k is a closure of the set $\{x \in R^n : \ \varphi_k(x) \neq 0\}$,

$$D^\alpha = \frac{\partial^|\alpha|}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_n^{\alpha_n}}, \quad |\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_n.$$

Further, $D'(R^n)$ is the space of generalized functions: each linear continuous functional over the space $D(R^n)$ in Sobolev-Schwartz sense. The generalized function f becomes zero in the region G if $(f, \varphi) = 0$ for all $\varphi \in D(R^n)$ such that *supp* $\varphi \subset G$. In correspondence with this definition, the generalized functions f and g are said to be equal in the region G if $f - g = 0$ for all $x \in G$; in the case we can write $f = g$ for all $\varphi \in G$; $(f, \varphi) = (g, \varphi)$.

The simplest example of a generalized function is the functional generated by the function $f(x)$ locally integrable in R^n:

$$(f, \varphi) = \int_{R^n} f(x)\varphi(x)dx, \quad \varphi(x) \in D(R^n). \tag{4.1.1}$$

Generalized functions which are definable in terms of functions locally integrable in R^n according to formula (4.1.1) are said to be regular generalized functions.

Du Bois Reymond lemma. In order that the function $f(x)$, locally integrable in G, should become zero in the region G in the sense of generalized functions , it is necessary and sufficient that $f(x) = 0$ almost everywhere in G.

It follows from *Du Bois Reymond's lemma* that each regular generalized function is defined by a unique (with accuracy as far as the values on a set of measure zero) function locally integrable in R^n. Consequently there is a mutual one to one correspondence between the functions locally integrable in R^n and regular generalized functions.

The remaining generalized functions are said to be singular generalized functions. The simplest example of singular generalized function is the Dirac delta (δ) function defined by the formula

$$(\delta(x), \varphi(x)) = \varphi(0), \quad \varphi(x) \in D(R^n).$$

Evidently, $\delta(x) \in D'(R^n)$, $\delta(x) = 0$ for $x \neq 0$, so that *supp* $\delta(x) = \{0\}$.

Let us now consider some important properties and operations over generalized functions by brief description only.

Operation 1. (Change of variables in generalized functions)

Let $f(y) \in D'(R^n)$, $y = w(x)$ be an infinitely differentiable one-to-one transformation of R^n onto itself with nonzero determinant of the Jacobian $\dfrac{\partial w}{\partial x}$, *i.e.* $det(\dfrac{\partial w}{\partial x}) = \left|\dfrac{\partial w}{\partial x}\right| \neq 0$. $x = w^{-1}(y)$ be the inverse transformation to $y = w(x)$. Then for any $\varphi(x) \in D(R^n)$ the relation

$$\Big(f(w(x)), \varphi(x)\Big) = \Big(f(y), \varphi(w^{-1}(y))|\frac{\partial w^{-1}(y)}{\partial y}|\Big),$$

defines the generalized function $f(w(x))$ for any $f(y)$.

Using this definition we can show the following properties of the Dirac delta function:

$$\Big(\delta(x - x^0), \varphi(x)\Big) = \varphi(x^0), \quad \varphi(x) \in D(R^n), \tag{4.1.2}$$

$$\delta(w(x)) = \frac{\delta(x - x^0)}{|w'(x^0)|}, \quad w(x^0) = 0, \tag{4.1.3}$$

Operation 2. (Multiplication of generalized functions)

Let $f(x) \in D'(R^n)$, $a(x) \in C^\infty(R^n)$. Then the relation for any $\varphi(x) \in D(R^n)$

$$\Big(a(x)f(x), \varphi(x)\Big) = \Big(f(x), a(x)\varphi(x)\Big),$$

defines the product $a(x) \in C^\infty(R^n)$ for any $f(x) \in D'(R^n)$.

Using this definition we can show the following properties of the Dirac delta function:

$$a(x)\delta(x) = a(0)\delta(x), \quad a(x)\delta(x - x^0) = a(x^0)\delta(x - x^0).$$

Operation 3. (Multiplication of generalized functions)

Let $f(x) \in D'(R^n)$, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ be a vector with nonnegative integer components α_j;

$$D^\alpha f(x) = \frac{\partial^{|\alpha|} f(x)}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_n^{\alpha_n}}, \quad D^0 f(x) = f(x), \quad |\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_n.$$

Then the relation for any $\varphi(x) \in D(R^n)$

$$\Big(D^\alpha f(x), \varphi(x)\Big) = (-1)^\alpha \Big(f(x), D^\alpha \varphi(x)\Big),$$

defines a generalized derivative $D^\alpha f(x)$ for any $f(x) \in D'(R^n)$.

From the definition of the generalized derivative the following properties hold:

- If $f(x) \in D'(R^n)$ then $D^\alpha f(x) \in D'(R^n)$ for any α
- If $f(x) \in C^p(G)$, $\{D^\alpha f(x)\}$ is the classical derivatives (where it exists) then $D^\alpha f(x) = \{D^\alpha f(x)\}$, $x \in G$, $|\alpha| \leq p$.
- Any generalized function is infinitely differentiable.
- The result of the differentiation does not depend on the order of differentiation.
- If $f(x) \in D'(R^n)$ and $a(x) \in C^\infty(R^n)$ then the Leibnitz's formula differentiation of the product $a(x)f(x)$ is valid.
- If the generalized function $f(x) = 0$ for $x \in G$, then also $D^\alpha f(x) = 0$ for $x \in G$, so that *supp* $D^\alpha f(x) \subset$ *supp* $f(x)$.
- If the function $f(x)$ has isolated discontinues of the first kind at the points $\{x_k\}$, then
$$\frac{df}{dx} = \left\{\frac{df}{dx}\right\} + \sum_k [f]|_{x_k}\delta(x - x_k),$$
where $[f]|_{x_k} = f(x_k + 0) - f(x_k - 0)$.

Operation 4. (Convolution of generalized functions)

Let $f(x)$ and $g(x)$ be locally integrable functions in R^n. The function

$$\begin{aligned}(f * g)(x) &= \int_{R^n} f(y)g(x-y)dy \\ &= \int_{R^n} g(y)f(x-y)dy = (g * f)(x)\end{aligned}$$

is known as the convolution $f * g$ of these functions. The function $(f * g)(x)$ is locally integrable functions in R^n and therefore defines a regular generalized function, acting on the test functions $\varphi(x) \in D(R^n)$ according to the rule:

$$\begin{aligned}\Big((f * g)(x), \varphi(x)\Big) &= \Big(f(x)g(y), \varphi(x+y)\Big) \\ &= \int_{R^{2n}} f(x)g(y)\varphi(x+y)dxdy.\end{aligned}$$

From the definition the convolution of the generalized functions the following properties hold:

- If $f * g$ exists then $g * f$ exists, and $f * g = g * f$.
- For any $f(x) \in D'(R^n)$, $f * \delta = \delta * f = f$.
- If $f * g$ exists, then If $D^\alpha f * g$ and $f * D^\alpha g$ exist, and moreover

$$D^\alpha (f * g) = D^\alpha f * g = f * D^\alpha g.$$

We will also use well known generalized function called Heaviside step-function and its properties throughout the chapter.

- $\theta_0(t) = \theta(t)$ is the Heaviside step-function, $\theta_0(t) = 1$ for $t \geq 0$, $\theta_0(t) = 0$ for $t \leq 0$.
- $\theta'(t) = \delta(t)$, $\theta_k(t) = \dfrac{t^k}{k!}\theta_0(t)$ for $k = 0, 1, 2, \ldots$; $\theta_{-1}(t) = \delta(t)$.

4.2 Green's Function of IVP for $\mathcal{L}$

This section deals with constructing an explicit formula for the Green's function (fundamental solution) of the initial value problem for the vector operator $\mathcal{L}$ defined in (4.0.1).

Definition 4.2.1. **(Green's function of IVP for $\mathcal{L}$)** A matrix

$$\mathcal{G}(x,t) = \begin{pmatrix} G_1^1(x,t) & G_1^2(x,t) & G_1^3(x,t) \\ G_2^1(x,t) & G_2^2(x,t) & G_2^3(x,t) \\ G_3^1(x,t) & G_3^2(x,t) & G_3^3(x,t) \end{pmatrix}$$

the jth column $\mathcal{G}^j = (G_1^j, G_2^j, G_3^j)^T$ of which satisfies the following equalities

$$\mathcal{L}\mathcal{G}^j = \mathbf{e}_j \delta(x)\delta(t), \tag{4.2.1}$$

$$\mathcal{G}^j|_{t<0} = 0, \tag{4.2.2}$$

is called the Green's function (the Green's matrix) of the initial value problem for the vector operator $\mathcal{L}$. Here $\mathbf{e}_1 = (1,0,0)^T$, $\mathbf{e}_2 = (0,1,0)^T$, $\mathbf{e}_3 = (0,0,1)^T$. The upper index T means the operation of the transposition.

In the following theorem an explicit formula for the Green's function of the initial value problem the vector operator $\mathcal{L}$ is given. The proof of the theorem contains a construction of this function. The result of this section corresponds to the paper (Yakhno & Sevimlican (2001)).

4.2.1 Constructing the Green's Function of IVP for $\mathcal{L}$: An Explicit Formula

Theorem 4.2.2. *Let a is a given positive number, $\mathcal{Q} = (q_{mn})_{3\times 3}$ be a matrix with constant elements; $x = (x_1, x_2, x_3) \in R^3$ be the space variable, $t \in R$ be time variable, $\Gamma = t^2 - |x|^2/a^2$, $|x|^2 = x_1^2 + x_2^2 + x_3^2$; $\exp(\mathcal{Q}t)$, $\mathcal{I}_1(\mathcal{Q}t)$ are matrices defined by*

$$\exp(\mathcal{Q}t) = \sum_{k=0}^{\infty} \frac{(\mathcal{Q}t)^k}{k!}, \quad \mathcal{I}_1(\mathcal{Q}t) = \sum_{k=0}^{\infty} \frac{(\mathcal{Q}t)^{2k+1}}{k!(k+1)!}. \tag{4.2.3}$$

Then the matrix $\mathcal{G}(x,t)$ defined by

$$\begin{aligned} \mathcal{G}(x,t) &= \frac{1}{2\pi a^3}\theta(t)\delta(\Gamma)\exp(-\mathcal{Q}t) \\ &+ \frac{1}{4\pi a^3\sqrt{\Gamma}}\theta(t-|x|/a)\mathcal{Q}\exp(-\mathcal{Q}t)\mathcal{I}_1(\mathcal{Q}\sqrt{\Gamma}/2), \end{aligned} \tag{4.2.4}$$

is the Green's function of IVP for $\mathcal{L}$.

Proof. We seek the jth column of the Green's function of initial value problem for the vector operator $\mathcal{L}$ in the following expansion

$$\mathcal{G}^j = \theta(t) \sum_{k=-1}^{\infty} \alpha_k^j(x,t)\theta_k(\Gamma), \tag{4.2.5}$$

where $\alpha_k^j(x,t)$ are unknown vector functions which we have to determine. For this aim we use the following properties of generalized functions:

$$\Gamma\theta_{k-2}(\Gamma) = (k-1)\theta_{k-1}(\Gamma), \quad \theta_k'(\Gamma) = \theta_{k-1}(\Gamma),$$
$$\delta(t)\theta_k(\Gamma) = 0, \ k \geq -1, \quad -\delta'(t)\theta_{-1}(\Gamma) = 2\pi a^3\delta(x,t),$$

and the following expressions for $\mathcal{G}_t^j$, $\mathcal{G}_{tt}^j$, $\nabla_x\mathcal{G}^j$, $\Delta_x\mathcal{G}^j$:

$$\begin{aligned}\frac{\partial\mathcal{G}^j}{\partial t} &= \delta(t)\sum_{k=-1}^{\infty}\alpha_k^j\theta_k(\Gamma) + \theta(t)\sum_{k=-1}^{\infty}\frac{\partial\alpha_k^j}{\partial t}\theta_k(\Gamma) + \theta(t)\sum_{k=-1}^{\infty}\alpha_k^j\frac{\partial}{\partial t}\theta_k(\Gamma)\\ &= \theta(t)\sum_{k=-1}^{\infty}\Big[\frac{\partial\alpha_{k-1}^j}{\partial t} + \frac{\partial\Gamma}{\partial t}\alpha_k^j\Big]\theta_{k-1}(\Gamma),\end{aligned} \tag{4.2.6}$$

$$\begin{aligned}\frac{\partial^2\mathcal{G}^j}{\partial t^2} &= -\delta'(t)\sum_{k=-1}^{\infty}\alpha_k^j\theta_k(\Gamma) + 2\frac{\partial}{\partial t}\Big(\delta(t)\sum_{k=-1}^{\infty}\alpha_k^j\theta_k(\Gamma)\Big)\\ &+ \theta(t)\frac{\partial^2}{\partial t^2}\Big(\sum_{k=-1}^{\infty}\alpha_k^j\theta_k(\Gamma)\Big)\\ &= 2\pi a^3\alpha_{-1}^j(0,0)\delta(x,t) + \theta(t)\sum_{k=-1}^{\infty}\theta_{k-1}(\Gamma)\Big[\frac{\partial^2\alpha_{k-1}^j}{\partial t^2} + 2\frac{\partial\Gamma}{\partial t}\frac{\partial\alpha_k^j}{\partial t}\\ &+ \alpha_k^j\frac{\partial^2\Gamma}{\partial t^2} + (\frac{\partial\Gamma}{\partial t})^2\alpha_k^j\frac{(k-1)}{\Gamma}\Big],\end{aligned} \tag{4.2.7}$$

$$\nabla_x\mathcal{G}^j = \theta(t)\sum_{k=-1}^{\infty}\Big[\nabla_x\alpha_{k-1}^j + \alpha_k^j\nabla_x\Gamma\Big]\theta_{k-1}(\Gamma), \tag{4.2.8}$$

$$\begin{aligned}\Delta_x\mathcal{G}^j &= \theta(t)\sum_{k=-1}^{\infty}\Big[\Delta_x\alpha_{k-1}^j + 2\nabla_x\alpha_k^j\nabla_x\Gamma\\ &+ \alpha_k^j(\Delta_x\Gamma)^2\frac{(k-1)}{\Gamma}\Big]\theta_{k-1}(\Gamma),\end{aligned} \tag{4.2.9}$$

where $\alpha_{-2}^j = 0$, $\dfrac{\partial\Gamma}{\partial t} = 2t$, $\nabla_x\Gamma = -\dfrac{2x}{a^2}$, $\Delta_x\Gamma = -\dfrac{6}{a^2}$.
Substituting into (4.2.5) into (4.2.1) we get

$$\begin{aligned}&\Big[2\pi a^3\alpha_{-1}^j(0,0) - \mathbf{e}_j\Big]\delta(x,t) + \theta(t)\sum_{k=-1}^{\infty}\Big[\mathcal{L}\alpha_{k-1}^j + 2\frac{\partial\alpha_k^j}{\partial t}\frac{\partial\Gamma}{\partial t}\\ &-2a^2\nabla_x\alpha_k^j\nabla_x\Gamma + \Big(\mathcal{L}\Gamma + 4(k-1)\alpha_k^j\Big)\Big]\theta_{k-1}(\Gamma) = 0.\end{aligned} \tag{4.2.10}$$

Equating to zero the expressions by $\delta(x,t)$ and $\theta_{k-1}(\Gamma)$ for $k \geq -1$ in (4.2.10) we obtain the following relations

$$\alpha^j_{-1}(0,0) = \frac{1}{2\pi a^3}\mathbf{e}_j, \tag{4.2.11}$$

$$x\frac{\partial \alpha^j_k}{\partial x} + t\frac{\partial \alpha^j_k}{\partial t} + \big((k+1)\mathcal{Q}t\big)\alpha^j_k = -\frac{1}{4}\mathcal{L}\alpha^j_{k-1}. \tag{4.2.12}$$

Considering (4.2.12) along the curve defined by

$$\frac{dx}{d\tau} = \frac{x(\tau)}{\tau}, \quad t = p\tau,$$

where p is a constant, τ is a parameter and multiplying (4.2.12) by τ^k, k=-1,0,1,..., the relation (4.2.12) may be written as follows

$$\frac{d}{d\tau}\Big[\tau^{k+1}\alpha^j_k(x(\tau),p\tau)\Big] + \mathcal{Q}p\tau^{k+1}\alpha^j_k = -\frac{\tau^k}{4}\mathcal{L}\alpha^j_{k-1}\Big|_{x=x(\tau),\, t=p\tau}. \tag{4.2.13}$$

Integrating (4.2.13) from 0 to $\tau(x)$ and using (4.2.11) we find

$$\alpha^j_{-1}(x,t) \;=\; \exp(-\mathcal{Q}t)\alpha^j_{-1}(0,0),$$

$$\begin{aligned}\tau^{k+1}\alpha^j_k(x(\tau),p\tau) = -\frac{1}{4}\exp(-\mathcal{Q}t)\int_0^{\tau(x)} \tau^k \exp(\mathcal{Q}p\tau)\\ \times\mathcal{L}\alpha^j_{k-1}(\xi,z)|_{\xi=x(\tau),\, z=p\tau}d\tau, \quad k=0,1,\ldots.\end{aligned} \tag{4.2.14}$$

Making the change of variable $\tau = \tau(x)s$, the equation (4.2.14) may be written as follows

$$\alpha^j_k(x,t) = -\frac{1}{4}\int_0^1 s^k \exp(\mathcal{Q}t(s-1))\mathcal{L}\alpha^j_{k-1}(\xi,z)|_{\xi=sx,\, z=st}\, ds, \quad k=0,1,.. \tag{4.2.15}$$

We can show that

$$\begin{aligned}\mathcal{L}\alpha^j_{k-1}(\xi,z)|_{\xi=sx,\, z=st} &= -\mathcal{Q}^2\exp(-\mathcal{Q}st)\alpha^j_{-1}(0,0),\\ \alpha^j_0(x,t) &= (\frac{\mathcal{Q}}{2})^2\alpha^j_{-1}(x,t);\end{aligned} \tag{4.2.16}$$

$$\begin{aligned}\mathcal{L}\alpha^j_0(\xi,z)|_{\xi=sx,\, z=st} &= -\frac{\mathcal{Q}^4}{4}\exp(-\mathcal{Q}st)\alpha^j_{-1}(0,0),\\ \alpha^j_1(x,t) &= -\frac{1}{4}\int_0^1 s\exp(\mathcal{Q}t(s-1))\mathcal{L}\alpha^j_0(\xi,z)|_{\xi=sx,\, z=st}ds\\ &= \frac{1}{2!}(\frac{\mathcal{Q}}{2})^4\alpha^j_{-1}(x,t);\end{aligned} \tag{4.2.17}$$

and so on continuing the reasoning, for arbitrary natural k we have:

$$\begin{aligned}
\mathcal{L}\alpha^j_{k-1}(\xi,z)|_{\xi=sx,\, z=st} &= -\frac{1}{(k+1)!}(\frac{\mathcal{Q}}{2})^{2k+2}\exp(-\mathcal{Q}st)\alpha^j_{-1}(0,0),\\
\alpha^j_k(x,t) &= -\frac{1}{4}\int_0^1 s^k\exp(\mathcal{Q}t(s-1))\mathcal{L}\alpha^j_{k-1}(\xi,z)|_{\xi=sx,\, z=st}\, ds\\
&= \frac{1}{(k+1)!}(\frac{\mathcal{Q}}{2})^{2k+2}\alpha^j_{-1}(x,t). \qquad (4.2.18)
\end{aligned}$$

Hence we found $\alpha^j_k(x,t)$, $k=-1,0,1,\dots$, by means of (4.2.15)–(4.2.18) therefore the equation (4.2.5) may be written as follows

$$\mathcal{G}(x,t) = \theta(t)\theta_{-1}(\Gamma)\alpha^j_{-1}(x,t) + \theta(t)\sum_{k=0}^{\infty}\frac{\frac{\mathcal{Q}}{2}^{2k+2}}{(k+1)!}\alpha^j_{-1}(x,t)\theta_k(\Gamma). (4.2.19)$$

Using the following properties

$$\theta_{-1}(\Gamma)=\delta(t), \quad \theta(\Gamma)\delta(\Gamma)=\theta(t-|x|/a), \quad \theta_k(\Gamma)=\frac{\Gamma^k}{k!}\theta_0(\Gamma),\ k=0,1,2,\dots;$$

and the (4.2.11), the equation (4.2.19) maybe written as follows

$$\begin{aligned}
\mathcal{G}(x,t) &= \frac{1}{2\pi a^3}\theta(t)\delta(t)\exp(-\mathcal{Q}t)\\
&+ \frac{1}{4\pi a^3\sqrt{\Gamma}}\theta(t-|x|/a)\mathcal{Q}\exp(-\mathcal{Q}t)\mathcal{I}_1(\mathcal{Q}\sqrt{\Gamma}/2),
\end{aligned}$$

where $\exp(-\mathcal{Q}t)$ and $\mathcal{I}_1(\mathcal{Q}t)$ are defined by (4.2.3), $\mathcal{I}_1$ is the modified Bessel function defined by (4.2.3). □

4.2.2 IVP for the Vector Operator $\mathcal{L}$: An Explicit Formula for a Solution

Let $\mathcal{L}$ be the vector operator operator defined by (4.0.1), $f(x,t)\in C(R^3\times[0,\infty))$, $\varphi(x)\in C^1(R^3)$, $\psi(x)\in C(R^3)$ are given functions. Let us consider the following equation

$$\mathcal{L}u(x,t)=f(x,t), \quad x=(x_1,x_2,x_3)\in R^3, t>0, \qquad (4.2.20)$$

subject to following initial data

$$u|_{t=+0}=\varphi(x), \quad \frac{\partial u}{\partial t}|_{t=+0}=\psi(x), \quad x=(x_1,x_2,x_3)\in R^3. \qquad (4.2.21)$$

The problem is to construct $u(x,t)$ satisfying (4.2.20), (4.2.21)

Lemma 4.2.3. *Let* $u(x,t) \in C^2(R^3 \times (0,\infty)) \bigcap C^1(R^3 \times [0,\infty))$ *be a solution of (4.2.20), (4.2.21) then* $V(x,t) = \theta(t)u(x,t)$ *is a solution of the following problem*

$$\mathcal{L}V(x,t) = F(x,t), \tag{4.2.22}$$

$$V(x,t)|_{t<0} = 0, \tag{4.2.23}$$

where

$$F(x,t) = \theta(t)f(x,t) + \delta(t)\Big(\psi(x) + 2\mathcal{Q}\varphi(x)\Big) + \delta'(t)\varphi(x). \tag{4.2.24}$$

Proof. Let $V(x,t) = \theta(t)u(x,t)$. Differentiating with respect to t, using the property (see Section 4.1 Operation 2), $\delta(t)u(x,t) = \delta(t)u(x,0)$. We find the expressions for $\dfrac{\partial V}{\partial t}$ and $\dfrac{\partial^2 V}{\partial t^2}$ the following relations:

$$\begin{aligned} \frac{\partial V}{\partial t} &= \delta(t)u(x,t) + \theta(t)\frac{\partial u}{\partial t} \\ &= \delta(t)\varphi(x) + \theta(t)\frac{\partial u}{\partial t}, \end{aligned} \tag{4.2.25}$$

$$\begin{aligned} \frac{\partial^2 V}{\partial t^2} &= \delta'(t)\varphi(x) + \delta(t)\frac{\partial u}{\partial t} + \theta(t)\frac{\partial^2 u}{\partial t^2} \\ &= \delta'(t)\varphi(x) + \delta(t)\psi(x) + \theta(t)\frac{\partial^2 u}{\partial t^2}. \end{aligned} \tag{4.2.26}$$

Using formulas (4.2.25), (4.2.26) we find that

$$\begin{aligned} \mathcal{L}V(x,t) &= \delta'(t)\varphi(x) + \delta(t)\Big(\psi(x) + 2\mathcal{Q}\varphi(x)\Big) + \theta(t)\Big(\frac{\partial^2 u}{\partial t^2} - a^2\Delta_x u \\ &+ 2\mathcal{Q}\frac{\partial u}{\partial t}\Big) \\ &= \delta'(t)\Big(\psi(x) + 2\mathcal{Q}\varphi(x)\Big) + \delta'(t)\varphi(x) + \theta(t)f(x,t) \\ &= F(x,t). \end{aligned}$$

It is clear that $V(x,t)|_{t<0} = 0$. □

Remark 4.2.4. Let $F(x,t) \in D'(R^4)$, $F(x,t)|_{t<0} = 0$. Then the problem of finding a generalized function $V(x,t) \in D'(R^4)$ satisfying (4.2.22), (4.2.23) is called generalized initial value problem for the vector operator $\mathcal{L}$.

Lemma 4.2.5. *Let* $F(x,t) \in D'(R^4)$, $F(x,t)|_{t<0} = 0$ *and* $\mathcal{G}(x,t)$ *be Green's function for initial value problem for the vector operator* $\mathcal{L}$ *then* $V(x,t) = (\mathcal{G} * F)(x,t)$ *is a solution of the initial value problem (4.2.22), (4.2.23).*

Proof. We need to show that $V(x,t) = (\mathcal{G} * F)(x,t)$ satisfies (4.2.22), (4.2.23). Indeed we have

$$\mathcal{L}V(x,t) = \mathcal{L}(\mathcal{G} * F)(x,t) = \mathcal{L}\mathcal{G} * F(x,t) = \delta(x,t) * F(x,t) = F(x,t).$$

The proof of the lemma 4.2.5 will be completed by checking that $V(x,t) = 0$ for $t < 0$. □

Theorem 4.2.6. *Let* $\mathcal{G}(x,t)$ *be Green's function of IVP for* $\mathcal{L}$ *defined by (4.2.4) and* $F(x,t)$ *be a function defined by (4.2.24) for any* $f(x,t) \in C(R^3 \times [0,\infty))$, $\varphi(x) \in C^1(R^3)$, $\psi(x) \in C(R^3)$. *Then a solution of IVP (4.2.20), (4.2.21) is given by*

$$\begin{aligned} u(x,t) = {} & \frac{1}{4\pi a^2}\int\int\int_{|x-\xi|\le at} \frac{1}{|x-\xi|}\exp(-\mathcal{Q}(\frac{|x-\xi|}{a}))f(\xi, t-\frac{|x-\xi|}{a})d\xi \\ & + \frac{1}{4\pi a^3}\int_0^t\int\int_{|x-\xi|\le a(t-\tau)} \frac{1}{\sqrt{(t-\tau)^2 - \frac{|x-\xi|^2}{a^2}}}\mathcal{Q}\exp(-\mathcal{Q}(t-\tau)) \\ & \times \mathcal{I}_1\Big(\frac{\mathcal{Q}\sqrt{(t-\tau)^2-\frac{|x-\xi|^2}{a^2}}}{2}\Big)f(\xi,\tau)d\xi d\tau \\ & + \frac{\exp(-\mathcal{Q}t)}{4\pi a^4 t}\int\int_{|x-\xi|=at}\Big(\psi(\xi)+2\mathcal{Q}\varphi(\xi)\Big)ds \\ & + \frac{\mathcal{Q}\exp(-\mathcal{Q}t)}{4\pi a^3}\int\int\int_{|x-\xi|\le at}\frac{1}{\sqrt{t^2-\frac{|x-\xi|^2}{a^2}}}\mathcal{I}_1\Big(\frac{\mathcal{Q}\sqrt{t^2-\frac{|x-\xi|^2}{a^2}}}{2}\Big) \\ & \times\Big(\psi(\xi)+2\mathcal{Q}\varphi(\xi)\Big)d\xi + \frac{\partial}{\partial t}\Big\{\frac{\exp(-\mathcal{Q}t)}{4\pi a^4 t}\int\int_{|x-\xi|=at}\varphi(\xi)ds + \frac{\mathcal{Q}\exp(-\mathcal{Q}t)}{4\pi a^3} \\ & \times\int\int\int_{|x-\xi|\le at}\frac{1}{\sqrt{t^2-\frac{|x-\xi|^2}{a^2}}}\mathcal{I}_1\Big(\frac{\mathcal{Q}\sqrt{t^2-\frac{|x-\xi|^2}{a^2}}}{2}\Big)\varphi(\xi)d\xi\Big\}, \quad t>0, \end{aligned} \tag{4.2.27}$$

where $\exp(\mathcal{Q}t)$ *and* $\mathcal{I}_1(\mathcal{Q}t)$ *are defined by (4.2.3).*

Proof. It follows from lemma 4.2.5 that a solution $u(x,t)$ of (4.2.20), (4.2.21) may be constructed by $u(x,t) = (\mathcal{G} * F)(x,t)$ for $t > 0$. The convolution $(\mathcal{G} * F)(x,t)$ may be found by the following formula (see Section 4.1 Operation 4)

$$\begin{aligned}(\mathcal{G} * F)(x,t) &= \int_{R^4} \mathcal{G}(x-\xi, t-\tau)\Big[\theta(\tau) f(\xi,\tau) + \delta(\tau)\Big(\psi(\xi) \\ &+ 2\mathcal{Q}\varphi(\xi)\Big) + \delta'(\tau)\varphi(\xi)\Big] d\xi d\tau. \end{aligned} \tag{4.2.28}$$

Using the following property (see Section 4.1 Operation 4)

$$\begin{aligned}(\mathcal{G}(x,t) * \delta'(t)) &= \frac{\partial}{\partial t}(\mathcal{G}(x,t) * \delta(t)) \\ &= \frac{\partial}{\partial t}\Big(\int_{R^4} \mathcal{G}(x-\xi, t-\tau)\delta(\tau) d\xi d\tau\Big) \\ &= \frac{\partial}{\partial t}\Big(\int_{R^3} \mathcal{G}(x-\xi, t) d\xi\Big),\end{aligned}$$

the equation (4.2.28) may be written sum of three integrals as follows

$$(\mathcal{G} * F)(x,t) = I^1(x,t) + I^2(x,t) + I^3(x,t), \tag{4.2.29}$$

where

$$I^1(x,t) = \int_{R^4} \mathcal{G}(x-\xi, t-\tau)\theta(\tau) f(\xi,\tau) d\xi d\tau, \tag{4.2.30}$$

$$I^2(x,t) = \int_{R^3} \mathcal{G}(x-\xi, t)\Big(\psi(\xi) + 2\mathcal{Q}\varphi(\xi)\Big) d\xi, \tag{4.2.31}$$

$$I^3(x,t) = \frac{\partial}{\partial t}\int_{R^3} \Big(\mathcal{G}(x-\xi, t)\psi(\xi) d\xi\Big). \tag{4.2.32}$$

Substituting (4.2.4) into (4.2.30) we find

$$\begin{aligned} I^1(x,t) &= \frac{1}{2\pi a^3}\int_{R^4} \theta(t-\tau)\delta((t-\tau)^2 - \frac{|x-\xi|^2}{a^2}) \\ &\times \exp(-\mathcal{Q}(t-\tau))\theta(\tau) f(\xi,\tau) d\xi d\tau \\ &+ \frac{1}{4\pi a^3}\int_{R^4} \theta(t-\tau-\frac{|x-\xi|}{a})\frac{1}{\sqrt{(t-\tau)^2 - \frac{|x-\xi|^2}{a^2}}} \\ &\times \mathcal{Q}\exp(-\mathcal{Q}(t-\tau))\mathcal{I}_1\Big(\frac{\mathcal{Q}\sqrt{(t-\tau)^2 - \frac{|x-\xi|^2}{a^2}}}{2}\Big)\theta(\tau) f(\xi,\tau) d\xi d\tau. \end{aligned} \tag{4.2.33}$$

Using the change of variable of generalized function $w(\tau) = (t-\tau)^2 - \frac{|x-\xi|^2}{a^2}$, the following formula

$$\delta(w(\tau)) = \frac{\delta(\tau - \tau^\star)}{|w'(\tau^\star)|}, \quad \tau^\star = (t-\tau) - \frac{|x-\xi|}{a},$$

and the property (4.1.2) (see Section 4.1 Operation 1) in the first integral of (4.2.33); then the equation (4.2.33) may be written as follows

$$\begin{aligned} I^1(x,t) &= \frac{\theta(t)}{4\pi a^2}\int\int\int_{|x-\xi|\leq at} \frac{1}{|x-\xi|}\exp(-\mathcal{Q}\frac{|x-\xi|}{a})f(\xi, t - \frac{|x-\xi|}{a})d\xi \\ &+ \frac{\theta(t)}{4\pi a^3}\int_0^t\int\int\int_{|x-\xi|\leq a(t-\tau)} \frac{1}{\sqrt{(t-\tau)^2 - \frac{|x-\xi|^2}{a^2}}}\mathcal{Q}\exp(-\mathcal{Q}(t-\tau)) \\ &\times \mathcal{I}_1\Big(\frac{\mathcal{Q}\sqrt{(t-\tau)^2 - \frac{|x-\xi|^2}{a^2}}}{2}\Big)f(\xi,\tau)d\xi d\tau \end{aligned} \tag{4.2.34}$$

Substituting into we find

$$\begin{aligned} I^2(x,t) &= \frac{\theta(t)\exp(-\mathcal{Q}t)}{2\pi a^3}\int_{R^3}\delta(t^2 - \frac{|x-\xi|^2}{a^2})h(\xi)d\xi \\ &+ \frac{1}{4\pi a^3}\int_{R^3}\theta(t - \frac{|x-\xi|}{a})\frac{1}{\sqrt{t^2 - \frac{|x-\xi|^2}{a^2}}}\mathcal{Q}\exp(-\mathcal{Q}t) \\ &\times \mathcal{I}_1\Big(\frac{\mathcal{Q}\sqrt{t^2 - \frac{|x-\xi|^2}{a^2}}}{2}\Big)h(\xi)d\xi d\tau, \end{aligned} \tag{4.2.35}$$

where $h(\xi) = \Big(\psi(\xi) + 2\mathcal{Q}\varphi(\xi)\Big)$.

Using the property (4.1.2) in both integrals of the equation (4.2.35) and introducing spherical coordinates in the first integral of the equation (4.2.35); r, θ, φ of ξ as follows:

$$\xi = x + r\nu, \quad \nu = (\nu_1, \nu_2, \nu_3) \quad |\nu| = 1,$$

$$\nu_1 = sin\theta\cos\varphi, \quad \nu_2 = \sin\theta\sin\varphi, \quad \nu_3 = \cos\theta,$$

$$d\xi = r^2\sin\theta drd\theta d\varphi;\ 0 \leq \theta \leq \pi,\ 0 \leq \varphi \leq 2\pi,\ r > 0;$$

then the equation (4.2.35) becomes

$$\begin{aligned} I^2(x,t) &= \frac{\theta(t)\exp(-\mathcal{Q}t)}{2\pi a^3}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty}\delta(t^2 - \frac{r^2}{a^2})h(x + r\nu\xi)r^2\sin\theta drd\theta d\varphi \\ &+ \frac{\theta(t)\mathcal{Q}\exp(-\mathcal{Q}t)}{4\pi a^3}\int\int\int_{|x-\xi|\leq at}\frac{1}{\sqrt{t^2 - \frac{|x-\xi|^2}{a}^2}}\mathcal{I}_1\Big(\frac{\mathcal{Q}\sqrt{t^2 - \frac{|x-\xi|^2}{a^2}}}{2}\Big) \\ &\times h(\xi)d\xi. \end{aligned} \tag{4.2.36}$$

Using the change of variable of generalized function $w(r) = t^2 - \dfrac{r^2}{a^2}$, the formula

$$\delta(w(r)) = \frac{\delta(r - r^\star)}{|w'(r^\star)|}, \quad r^\star = at,$$

and the property (4.1.2) (see Section 4.1 Operation 1) in the first integral of (4.2.36) then equation (4.2.36) becomes

$$\begin{aligned} I^2(x,t) &= \frac{\theta(t)\exp(-\mathcal{Q}t)}{4\pi a^4}\int\int_{|x-\xi|=at}\Big(\psi(\xi) + 2\mathcal{Q}\varphi(\xi)\Big)ds \\ &+ \frac{\theta(t)\mathcal{Q}\exp(-\mathcal{Q}t)}{4\pi a^3}\int\int\int_{|x-\xi|\le at}\frac{1}{\sqrt{t^2 - \frac{|x-\xi|^2}{a^2}}}\mathcal{I}_1\Big(\frac{\mathcal{Q}\sqrt{t^2 - \frac{|x-\xi|^2}{a^2}}}{2}\Big) \\ &\times \Big(\psi(\xi) + 2\mathcal{Q}\varphi(\xi)\Big)d\xi. \end{aligned} \tag{4.2.37}$$

Using (4.2.37) and (4.2.32) we find

$$\begin{aligned} I^3(x,t) &= \theta(t)\frac{\partial}{\partial t}\Big\{\frac{\exp(-\mathcal{Q}t)}{4\pi a^4 t}\int\int_{|x-\xi|=at}\varphi(\xi)ds + \frac{\mathcal{Q}\exp(-\mathcal{Q}t)}{4\pi a^3} \\ &\times \int\int\int_{|x-\xi|\le at}\frac{1}{\sqrt{t^2 - \frac{|x-\xi|^2}{a^2}}}\mathcal{I}_1\Big(\frac{\mathcal{Q}\sqrt{t^2 - \frac{|x-\xi|^2}{a^2}}}{2}\Big)\varphi(\xi)d\xi\Big\}. \end{aligned} \tag{4.2.38}$$

The equation (4.2.27) follows from substituting the formulas (4.2.34), (4.2.37) and (4.2.38) into (4.2.29) for $t > 0$. □

Remark 4.2.7. Using the spherical coordinates for the integration in (4.2.27) we find

$$u|_{t=+0} = \varphi(x), \quad \frac{\partial u}{\partial t}|_{t=+0} = \psi(x).$$

4.3 Application to Electrodynamics

Let us consider Maxwell's system (1.3.1)-(1.3.4) (see Section 1.3) for the case:

$$\mathcal{E} = \varepsilon\mathcal{I}, \quad \mathcal{M} = \mu\mathcal{I}, \quad \sigma = (\sigma_{ij})_{3\times 3},$$

where σ_{ij} are constants and ε, μ are positive constants. Let us assume that conditions in (1.3.7) are satisfied. Using the reasoning of Subsections 1.3.1 and 1.3.3 we find

$$\mathbf{curl}_x\mathbf{H} = \varepsilon\frac{\partial\mathbf{E}}{\partial t} + \sigma\mathbf{E} + \mathbf{j}, \tag{4.3.1}$$

$$\mathbf{curl}_x\mathbf{E} = -\mu\frac{\partial\mathbf{H}}{\partial t}, \tag{4.3.2}$$

$$\mathbf{E}|_{t<0} = 0, \quad \mathbf{H}|_{t<0} = 0. \tag{4.3.3}$$

Let us consider the following relations for $\mathbf{H}$ and $\mathbf{E}$

$$\mathbf{H} = \frac{1}{\mu}\mathbf{curl}_x\mathbf{A}, \tag{4.3.4}$$

$$\mathbf{E} = -\frac{\partial\mathbf{A}}{\partial t} + \nabla_x\varphi, \tag{4.3.5}$$

where $\mathbf{A}$ is the vector function, φ is the scalar function (vector and scalar electric potentials). Substituting (4.3.4), (4.3.5) into (4.3.1) we find

$$\frac{1}{\mu}\nabla_x div_x\mathbf{A} - \varepsilon\frac{\partial\phi}{\partial t} - \sigma\phi + \varepsilon\frac{\partial^2\mathbf{A}}{\partial t^2} - \frac{1}{\mu}\Delta_x\mathbf{A} + \sigma\frac{\partial\mathbf{A}}{\partial t} = \mathbf{j}, \tag{4.3.6}$$

where $\phi = \nabla_x\varphi$. We choose the vector function $\mathbf{A}$ from

$$\mathcal{L}\mathbf{A} = \mathbf{f}, \tag{4.3.7}$$

where $\mathcal{L}$ is the operator defined by (4.0.1), $a = \frac{1}{\sqrt{\mu\varepsilon}}$, $2\mathcal{Q} = \frac{1}{\varepsilon}\sigma$, $\mathbf{f} = a^2\mu\mathbf{j}$. Then we find from (4.3.6), (4.3.7) that ϕ has to satisfy

$$\frac{\partial\phi}{\partial t} + 2\mathcal{Q}\phi = a^2\nabla_x div_x\mathbf{A}. \tag{4.3.8}$$

For holding (4.3.3) the conditions $\mathbf{A}|_{t\le 0} = 0$, $\phi|_{t\le 0} = 0$, are sufficient.

4.3.1 Green's Function of IVP for Maxwell's Operator and Its Construction

This Subsection deals with constructing an explicit formula for the Green's function (fundamental solution) of the initial value problem for Maxwell's operator. The Maxwell's operator M is defined by means of the following relations

$$M = \begin{pmatrix} \mathbf{curl}_x & -\varepsilon\mathcal{I}\dfrac{\partial}{\partial t} - \sigma \\ \mu\mathcal{I}\dfrac{\partial}{\partial t} & \mathbf{curl}_x \end{pmatrix}, \quad x = (x_1, x_2, x_3) \in R^3,\ t \in R, \tag{4.3.9}$$

$$M\begin{pmatrix} \mathbf{H} \\ \mathbf{E} \end{pmatrix} = \begin{pmatrix} \mathbf{curl}_x\mathbf{H} - \varepsilon\dfrac{\partial \mathbf{E}}{\partial t} - \sigma\mathbf{E} \\ \mathbf{curl}_x\mathbf{E} + \mu\dfrac{\partial \mathbf{H}}{\partial t} \end{pmatrix}.$$

Definition 4.3.1. (Green's function of IVP for Maxwell's operator)

$$\mathcal{E}(x,t) = \begin{pmatrix} H_1^1(x,t) & H_1^2(x,t) & H_1^3(x,t) \\ H_2^1(x,t) & H_2^2(x,t) & H_2^3(x,t) \\ H_3^1(x,t) & H_3^2(x,t) & H_3^3(x,t) \\ E_1^1(x,t) & H_1^2(x,t) & E_1^3(x,t) \\ E_2^1(x,t) & E_2^2(x,t) & E_2^3(x,t) \\ E_3^1(x,t) & E_3^2(x,t) & E_3^3(x,t) \end{pmatrix}_{6\times 3} = \begin{pmatrix} \mathbf{H}^j \\ \mathbf{E}^j \end{pmatrix}_{j=1,2,3} \tag{4.3.10}$$

the jth column of which satisfies the following equalities

$$M\begin{pmatrix} \mathbf{H}^j \\ \mathbf{E}^j \end{pmatrix} = \begin{pmatrix} \mathbf{e}_j\delta(x)\delta(t) \\ 0 \end{pmatrix}, \tag{4.3.11}$$

$$\mathbf{E}^j|_{t<0} = 0, \quad \mathbf{H}^j|_{t<0} = 0, \tag{4.3.12}$$

is called a Green's function of the initial value problem for the operator M defined in (4.3.9). Here $\mathbf{e}_1 = (1,0,0)^T$, $\mathbf{e}_2 = (0,1,0)^T$, $\mathbf{e}_3 = (0,0,1)^T$.

Thus by the reasoning which we used for obtaining (4.3.4)-(4.3.7) if we choose $\mathbf{j} = \mathbf{e}_j\delta(x)\delta(t)$ we can get the following equalities

$$\mathbf{H}^j = \frac{1}{\mu}\mathbf{curl}_x\mathbf{A}^j, \tag{4.3.13}$$

$$\mathbf{E}^j \quad = \quad -\frac{\partial \mathbf{A}^j}{\partial t} + \phi^j, \tag{4.3.14}$$

$$\frac{\partial \phi^j}{\partial t} + 2\mathcal{Q}\phi^j = a^2 \nabla_x div_x \mathbf{A}^j, \quad \phi^j|_{t\leq 0} = 0, \tag{4.3.15}$$

$$\frac{\partial^2 \mathbf{A}^j}{\partial t^2} - a^2 \Delta_x \mathbf{A}^j + \frac{1}{\varepsilon}\sigma \frac{\partial \mathbf{A}^j}{\partial t} = a^2 \mu \mathbf{e}_j \delta(x)\delta(t), \quad \mathbf{A}^j|_{t\leq 0} = 0. \tag{4.3.16}$$

Theorem 4.3.2. *Let a be a positive number and $\mathcal{Q}$ be a matrix with constant elements. Then a Green's function of IVP for the Maxwell's operator is given by $\mathcal{E} = (\boldsymbol{H}^j(x,t), \boldsymbol{E}^j(x,t))^T$,*

$$\boldsymbol{H}^j = \boldsymbol{H}^j_{-2}\delta'(t - \frac{|x|}{a}) + \boldsymbol{H}^j_{-1}\delta(t - \frac{|x|}{a}) + \boldsymbol{H}^j_0\theta(t - \frac{|x|}{a}); \tag{4.3.17}$$

$$\boldsymbol{E}^j = \boldsymbol{E}^j_{-2}\delta'(t - \frac{|x|}{a}) + \boldsymbol{E}^j_{-1}\delta(t - \frac{|x|}{a}) + \boldsymbol{E}^j_0\theta(t - \frac{|x|}{a}); \tag{4.3.18}$$

where $x = (x_1, x_2, x_3) \in R^3$, $t \in R$,

$$\boldsymbol{H}^j_{-2} \quad = \quad \frac{-1}{4\pi a|x|}\theta(t)\exp(-\mathcal{Q}t)[\nabla_x(|x|) \times \boldsymbol{e}_j],$$

$$\begin{aligned} \boldsymbol{H}^j_{-1} \quad &= \quad \frac{1}{4\pi|x|}\theta(t)\exp(-\mathcal{Q}t)\Big\{[\nabla_x(\frac{1}{|x|}) \times \boldsymbol{e}_j] \\ &- \quad \frac{1}{a^2}\tilde{\mathcal{I}}_1\Big(\frac{\mathcal{Q}\sqrt{t^2 - \frac{|x|^2}{a^2}}}{2}\Big)[\nabla_x(|x|) \times \boldsymbol{e}_j]\Big\}, \end{aligned}$$

$$\boldsymbol{H}^j_0 \quad = \quad \frac{1}{4\pi a}\mathcal{Q}\exp(-\mathcal{Q}t)\Big\{[\nabla_x\tilde{\mathcal{I}}_1\Big(\frac{\mathcal{Q}\sqrt{t^2 - \frac{|x|^2}{a^2}}}{2}\Big) \times \boldsymbol{e}_j]\Big\};$$

$$\boldsymbol{E}^j_{-2} \quad = \quad \frac{\mu\boldsymbol{e}_j}{4\pi}\Big\{\frac{-1}{|x|}\theta(t)\exp(-\mathcal{Q}t) + g(x,t)(\nabla_x(|x|))^2\Big\},$$

$$\begin{aligned} \boldsymbol{E}^j_{-1} \quad &= \quad \frac{\mu\boldsymbol{e}_j}{4\pi}\Big\{\mathcal{Q}\exp(-\mathcal{Q}t)\Big[\frac{1}{|x|} - \tilde{\mathcal{I}}_1\Big(\frac{\mathcal{Q}\sqrt{t^2 - \frac{|x|^2}{a^2}}}{2}\Big)\Big] \\ &- \quad ag(x,t)\nabla^2_x(|x|) - 2a\nabla_x g(x,t)\nabla_x(|x|)\Big\}, \end{aligned}$$

$$\boldsymbol{E}^j_0 \quad = \quad \frac{\mu\boldsymbol{e}_j}{4\pi}\Big\{\frac{-1}{a}\mathcal{Q}\frac{\partial}{\partial t}\Big[\exp(-\mathcal{Q}t)\tilde{\mathcal{I}}_1\Big(\frac{\mathcal{Q}\sqrt{t^2 - \frac{|x|^2}{a^2}}}{2}\Big)\Big] + a^2\nabla^2_x g(x,t)\Big\},$$

$$\tilde{\mathcal{I}}_1(az) = \frac{\mathcal{I}_1(az)}{z}, \quad g(x,t) = \frac{1}{|x|}\exp(-2\mathcal{Q}(t-\frac{|x|}{a}))\exp(-\mathcal{Q}\frac{|x|}{a})$$
$$+\int_{\frac{|x|}{a}}^{t}\exp(-2\mathcal{Q}(t-\tau))\mathcal{Q}\exp(-\mathcal{Q}\tau)\tilde{\mathcal{I}}_1\Big(\frac{\mathcal{Q}\sqrt{\tau^2-\frac{|x|^2}{a^2}}}{2}\Big)d\tau. \quad (4.3.19)$$

Proof. Let $\frac{1}{\varepsilon}\sigma = 2\mathcal{Q}$. Using a Green's function for IVP of the vector operator $\mathcal{L}$ given by (4.2.4), the solution of (4.3.16) can be represented by the following formula

$$\begin{aligned}\mathbf{A}^j(x,t) &= a^2\mu\mathcal{G}(x,t)\mathbf{e}_j, \quad j=1,2,3\\ &= a^2\mu\Big\{\frac{1}{2\pi a^3}\theta(t^2-\frac{|x|^2}{a^2})\delta(t)\exp(-\mathcal{Q}t)\\ &+ \frac{1}{4\pi a^3\sqrt{t^2-\frac{|x|^2}{a^2}}}\theta(t-\frac{|x|}{a})\mathcal{Q}\exp(-\mathcal{Q}t)\mathcal{I}_1\Big(\frac{\mathcal{Q}\sqrt{t^2-\frac{|x|^2}{a^2}}}{2}\Big)\Big\}\mathbf{e}_j.\end{aligned} \quad (4.3.20)$$

Using the following equalities

$$\delta(t^2-\frac{|x|^2}{a^2}) = \frac{a}{2|x|}\delta(t-\frac{|x|}{a}), \quad \tilde{\mathcal{I}}_1(az) = \frac{\mathcal{I}_1(az)}{z}$$

then equation (4.3.20) becomes

$$\begin{aligned}\mathbf{A}^j(x,t) &= a^2\mu\Big\{\frac{1}{4\pi a^2|x|}\theta(t-\frac{|x|}{a})\delta(t)\exp(-\mathcal{Q}t)\\ &+ \frac{1}{4\pi a^3}\theta(t-\frac{|x|}{a})\mathcal{Q}\exp(-\mathcal{Q}t)\tilde{\mathcal{I}}_1\Big(\frac{\mathcal{Q}\sqrt{t^2-\frac{|x|^2}{a^2}}}{2}\Big)\Big\}\mathbf{e}_j. \end{aligned} \quad (4.3.21)$$

We may find the solution of (4.3.15) by means of the formula

$$\phi^j(x,t) = a^2\int_{-\infty}^{\infty}\theta(t-\tau)\exp(-2\mathcal{Q}(t-\tau))\nabla_x div_x\mathbf{A}^j(x,\tau)d\tau, \; j=1,2,3.$$

Using the properties of Dirac delta and Heaviside functions and the formulas (4.2.4), (4.3.21); the last equation may be written as follows:

$$\phi^j(x,t) = \frac{a^2\mu}{4\pi}\nabla_x div_x\Big\{\theta(t-\frac{|x|}{a})g(x,t)\mathbf{e}_j\Big\}, \quad j=1,2,3. \quad (4.3.22)$$

where $g(x,t)$ is given by (4.3.19). Finding $div_x\Big\{\theta(t-\frac{|x|}{a})g(x,t)\mathbf{e}^j\Big\}$, then the equation (4.3.22) may be written as follows;

$$\phi^j(x,t) = \frac{a^2\mu}{4\pi}\nabla_x\Big\{\theta(t-\frac{|x|}{a})\nabla_x g(x,t) - \frac{1}{a}\delta(t-\frac{|x|}{a})\nabla_x(|x|)g(x,t)\Big\}\mathbf{e}_j.$$

The last equation may be written as follows

$$\phi^j(x,t) = \frac{a^2\mu}{4\pi}\nabla_x\Big\{\frac{1}{a^2}(\nabla_x(|x|))^2 g(x,t)\delta'(t-\frac{|x|}{a})$$
$$-\frac{1}{a}\Big[\nabla_x^2(|x|)g(x,t) + 2\nabla_x(|x|)\nabla_x g(x,t)\Big]\delta(t-\frac{|x|}{a})$$
$$-\nabla_x^2 g(x,t)\theta(t-\frac{|x|}{a})\Big\}\mathbf{e}_j. \qquad (4.3.23)$$

Differentiating the equation (4.3.21) with respect to t we find

$$\frac{\partial \mathbf{A}^j}{\partial t}(x,t) = a^2\mu\Big\{\frac{-1}{4\pi a^2|x|}\theta(t)\mathcal{Q}\exp(-\mathcal{Q}t)\delta(t-\frac{|x|}{a})$$
$$+\frac{1}{4\pi a^2|x|}\theta(t)\exp(-\mathcal{Q}t)\delta'(t-\frac{|x|}{a})$$
$$+\frac{1}{4\pi a^3}\mathcal{Q}\exp(-\mathcal{Q}t)\tilde{\mathcal{I}}_1\Big(\frac{\mathcal{Q}\sqrt{t^2-\frac{|x|^2}{a^2}}}{2}\Big)\delta(t-\frac{|x|}{a})$$
$$+\frac{1}{4\pi a^3}\mathcal{Q}\frac{\partial}{\partial t}\Big[\exp(-\mathcal{Q}t)\tilde{\mathcal{I}}_1\Big(\frac{\mathcal{Q}\sqrt{t^2-\frac{|x|^2}{a^2}}}{2}\Big)\Big]\theta(t-\frac{|x|}{a})\Big\}\mathbf{e}_j. \qquad (4.3.24)$$

(4.3.18) can be obtained by substituting (4.3.23) and (4.3.22) into (4.3.14). Finding $\mathbf{H}^j$ from (4.3.13) we get

$$\mathbf{H}^j = \frac{1}{4\pi}\theta(t)\exp(-\mathcal{Q}t)\Big\{\frac{-1}{a|x|}[\nabla_x(|x|)\times\mathbf{e}_j]\delta'(t-\frac{|x|}{a})$$
$$+[\nabla_x(\frac{1}{|x|})\times\mathbf{e}_j]\delta(t-\frac{|x|}{a})\Big\}$$
$$+\frac{1}{4\pi a}\mathcal{Q}\exp(-\mathcal{Q}t)\Big\{\frac{-1}{a}\tilde{\mathcal{I}}_1\Big(\frac{\mathcal{Q}\sqrt{t^2-\frac{|x|^2}{a^2}}}{2}\Big)[\nabla_x(|x|)\times\mathbf{e}_j]\delta(t-\frac{|x|}{a})$$
$$+[\nabla_x\tilde{\mathcal{I}}_1\Big(\frac{\mathcal{Q}\sqrt{t^2-\frac{|x|^2}{a^2}}}{2}\Big)\times\mathbf{e}_j]\theta(t-\frac{|x|}{a})\Big\}. \qquad (4.3.25)$$

□

4.3.2 A Generalized IVP for Maxwell's System

Theorem 4.3.3. *Let $\boldsymbol{j}(x,t)$ be a vector function with components $j_k(x,t) \in D'(R^4)$, $j_k(x,t)|_{t<0} = 0$; $\mathcal{E}$ be Green's function of the initial value problem for Maxwell's operator. Then the vector function*

$$\begin{pmatrix} \boldsymbol{H} \\ \boldsymbol{E} \end{pmatrix} = \int_{R^4} \mathcal{E}(x-\xi, t-\tau)\theta(\tau)\boldsymbol{j}(\xi,\tau)d\xi d\tau \tag{4.3.26}$$

is a solution of the initial value problem for Maxwell's system (4.3.1)-(4.3.3).

Proof. The proof of the theorem is based on checking that (4.3.26) satisfies the equations (4.3.1)-(4.3.3). Let M be the differential vector operator defined in (4.3.9). Applying the operator M to (4.3.26) and using (4.3.11) we have

$$\begin{aligned} M\begin{pmatrix} \mathbf{H} \\ \mathbf{E} \end{pmatrix} &= \int_{R^4} M\mathcal{E}(x-\xi, t-\tau)\theta(\tau)\mathbf{j}(\xi,\tau)d\xi d\tau \\ &= \int_{R^4} \theta(\tau)\delta(x-\xi, t-\tau)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} j_1(\xi,\tau) \\ j_2(\xi,\tau) \\ j_3(\xi,\tau) \end{pmatrix} d\xi d\tau \\ &= \begin{pmatrix} \mathbf{j} \\ 0 \end{pmatrix}. \end{aligned}$$

This shows that (4.3.26) satisfies the equations (4.3.1), (4.3.2). Using the similar reasoning the proof of lemma 4.2.5 we find $\mathbf{E}|_{t<0} = 0$ and $\mathbf{H}|_{t<0} = 0$. This shows that (4.3.26) satisfies the equation (4.3.3).

□

CHAPTER FIVE
CONCLUSION

The main results of the thesis are the following:

- A new algorithm is suggested and justified for solving Problem 1 which is related to recovery the electric field in uniaxial and biaxial electrically and magnetically anisotropic vertical inhomogeneous media.
- Theorems about existence and uniqueness of the solutions of Problem 1 for the cases uniaxial and biaxial anisotropic vertical inhomogeneous media are proved.
- An explicit formula for Green's function of initial value problem for the vector telegraph operator $\mathcal{L}$ is obtained.
- An explicit formula for a solution of Problem 2 is obtained.
- An explicit formula for a Green's matrix of initial value problem for the Maxwell's operator with constant dielectric permittivity and magnetic permeability, and a matrix conductivity with constant elements is constructed.
- Generalized initial value problem for the Maxwell system with anisotropic conductivity is solved.

The main results of the thesis were published in the following papers:

- Yakhno, V. G., Sevimlican, A., (2007). A method for the recovery of the electric field vibration inside vertical inhomogeneous anisotropic dielectrics. Mathematical Methods in Engineering, Tas, K. at al., Springer, 455-466.
- Yakhno, V. G., Sevimlican, A., (2001). Fundamental solution of the Cauchy problem for an anisotropic electrodynamic system, Selçuk Journal of Applied Mathematics, 1(2), 83-94.

The results of this dissertation were presented and discussed on the following conferences and symposium:

- Workshop on Differential Equations and Applications, 8-10, February, 2007; Middle East Technical University, Ankara, TURKEY.
- XIX National Mathematical Symposium, 22-25 August, 2006; Dumlupınar University, Kütahya.
- International conference on Mathematical Methods in Engineering, 27-29, April, 2006 Çankaya University, Ankara, TURKEY.
- International conference on Mathematical Modeling and Scientific Computing, 2-6, April, 2001, Middle East Technical University, Ankara, TURKEY.

REFERENCES

Andersen, N. B. (2004). Real Paley-Wiener theorems for the inverse Fourier transform on a Riemannian symmetric space. *Pacific J. Math.*, 213(1), 1–13.

Apostol, T. M. (1967). *Calculus. Vol. I: One-variable calculus, with an introduction to linear algebra*. Second edition, Blaisdell Publishing Co. Ginn and Co., Waltham, Mass.-Toronto, Ont.-London.

Bang, H. H. (1995). Functions with bounded spectrum. *Trans. Amer. Math. Soc.*, 37, 1067–1080.

Burridge, R., & Qian, J. (2006). The fundamental solution of the time-dependent system of crystal optics. *European J. Appl. Math.*, 17(1), 63–94.

Cohen, G. C. (2002). *Higher-order numerical methods for transient wave equations*. Scientific Computation, Berlin: Springer-Verlag. With a foreword by R. Glowinski.

Cohen, G. C., Heikola, E., Joly, P., & Neitaan, M. P. (2003). *Mathematical and numerical aspects of wave propagation*. Scientific Computation, Berlin: Springer-Verlag.

Courant, R., & Hilbert, D. (1989). *Methods of mathematical physics. Vol. II*. Wiley Classics Library, New York: John Wiley & Sons Inc. Partial differential equations, Reprint of the 1962 original, A Wiley-Interscience Publication.

Goldberg, J. L. (1992). *Matrix theory with applications*. Churchhill Brown Series, New York: McGraw-Hill.

Gottis, P. G., & Konddylis, G. D. (1995). Properties of dyadic Green's function for unbounded anisotropic medium. *IEEE Trans. Antennas and Propagation*, 45, 154–161.

Haba, Z. (2004). Green functions and propagation of waves in strongly inhomogeneous media. *J. Phys. A*, 37(39), 9295–9302.

Herbert, P., & Neff, T. (1987). *Basic electromagnetic fields.* New York: Harper and row.

Ikawa, M. (1999). *Hyperbolic partial differential equations and wave phenomena.* American Mathematical Society, Rhode Island: Providence, 2nd ed.

Kong, J. A. (1990). *Electromagnetic wave theory.* A Wiley-Interscience Publication, New York: John Wiley & Sons Inc., 2nd ed.

Li, L. W., Liu, S., Leong, M. S., & Yeo, T. S. (2001). Circular cylindiracal waveguide filled with uniaxial anisotropicmedia-electromagnetic fields and dyadic Green's function. *IEEE Transactions on Microwave and Techniques*, 49(7), 1361–1364.

Lindell, I. V. (1990). Time-domain TE/TM decomposition of electromagnetic sources. *IEEE Trans. Antennas and Propagation*, 38(3), 353–358.

Monk, P. (2003). *Finite element methods for Maxwell's equations.* Numerical Mathematics and Scientific Computation, New York: Oxford University Press.

Nagle, R. K., Saff, E. B., & Snider, A. D. (2004). *Fundamentals of differential equations and boundary value problems.* New York: Pearson Education-Addison Wesley.

Ortner, N., & Wagner, P. (2004). Fundamental matrices of homogeneous hyperbolic systems. Applications to crystal optics, elastodynamics, and piezoelectromagnetism. *ZAMM Z. Angew. Math. Mech.*, 84(5), 314–346.

Ramo, S., Whinnery, J. R., & Duzer, T. (1994). *Fields and waves in communication electronics.* New York: John Wiley and Sons.

Seo, J. K., Pyo, H. C., Park, C., & Woo, E. J. (2004). Image reconstruction of anisotropic conductivity distribution in MREIT: computre simulation study. *Physics in medicine and biology*, 49, 4371–4382.

Tuan, V. K., & Zayed, A. I. (2002). Real Paley-Wiener-type theorems for a class of integral transforms. *J. Math. Anal. Appl.*, pp. 200–266.

Vladimirov, V. S. (1971). *Equations of mathematical physics*, vol. 3 of *Translated from the Russian by Audrey Littlewood. Edited by Alan Jeffrey. Pure and Applied Mathematics*. New York: Marcel Dekker Inc.

Weiss, C. J., & Newmann, G. A. (2003). Electromagnetic induction in a generalized 3D anisotropic earth, Part 2: The LIN preconditioner. *Geophysics*, 68(3), 922–924.

Wijinands, F., & Pendry, J. B. (1997). Green's functions for Maxwell's equations: application to spontaneous emission. *Optical and Quantum Electronics*, 29, 199–216.

Wolters, C. H., Anwander, A., Tricoche, X., Weinstein, D., Koch, M. A., & MacLeod, R. S. (2005). Influence of tissue conductivty ansiotropy on EEG/MEG field and return current computation in a realistic head model: A simulation and visualization study using high-resolution finite element modelling. *Elsevier*, 30, 813–826.

Yakhno, V. G. (2005). Constructing Green's function for the time-dependent Maxwell system in anisotropic dielectrics. *J. Phys. A*, 38(10), 2277–2287.

Yakhno, V. G., & Sevimlican, A. (2001). Fundamental solution of the cauchy problem for an anisotropic electrodynamic system. *Selçuk Journal of Applied Mathematics*, 2(1), 83–94.

Yakhno, V. G., & Sevimlican, A. (2007). A method for the recovery of the electric field vibration inside vertical inhomogeneous anisotropic dielectrics. *Mathematical Methods in Engineering.*

Yakhno, V. G., Yakhno, T. M., & Kasap, M. (2006). A novel approach for modeling and simulation of electromagnetic waves in anisotropic dielectrics. *Internat. J. Solids Structures*, 43(20), 6261–6276.

Zienkiewicz, O. C., & Taylor, R. L. (2000). *The finite element method. Vol. 1.* Oxford: Butterworth-Heinemann, 5th ed. The basis.

APPENDIX A
GENERALIZED CAUCHY PROBLEM FOR THE WAVE EQUATION

Let us consider IVP for the following wave equation with two independent variables

$$\Big(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial y^2}\Big)w(y,t) = f(y,t),\ y \in R,\ t > 0, \tag{6.0.1}$$

$$w(y,t)|_{t=+0} = 0, \quad \frac{\partial w}{\partial t}(y,t)|_{t=+0} = 0. \tag{6.0.2}$$

If $f(y,t) \in C^1(R \times [0,\infty))$ then there exists a unique solution $w(y,t) \in C^2(R \times [0,\infty))$ which can be given by the D'Alambert formula (see, for example Vladimirov (1971), page176).

Let now assume that $f(y,t) \in C(R \times [0,\infty))$. In this case the problem (6.0.1), (6.0.2) will understand as the generalized Cauchy problem (Vladimirov (1971), page 171-178). According to the theorem from (Vladimirov (1971), page 174) there exists an inverse operator $(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial y^2})^{-1}$ such that the function $w(y,t)$ defined by

$$w(y,t) = \Big(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial y^2}\Big)^{-1} f(y,t) \equiv$$
$$\int\int_{R^2} \theta((t-\tau) - |y-\xi|)\theta(\tau) f(\xi,\tau) d\xi d\tau, \tag{6.0.3}$$

is a unique generalized solution of the generalized Cauchy problem (6.0.1), (6.0.2) for any $f(y,t) \in C(R\times[0,\infty))$. This means that the equality (6.0.3) is equivalent to (6.0.1), (6.0.2), where (6.0.1) is understood as the equality of generalized functions (Vladimirov (1971)).

Remark 6.0.4. We note that (6.0.3) may be written as the D'Alambert formula

$$w(y,t) = \frac{\theta(t)}{2}\int_0^t \int_{y-(t-\tau)}^{y+(t-\tau)} f(\xi,\tau) d\xi d\tau, \tag{6.0.4}$$

or

$$w(y,t) = \frac{\theta(t)}{2}\int_{y-t}^{y+t}\int_{0}^{t-|\xi-y|} f(\xi,\tau)d\xi d\tau, \quad y \in R, t \in R. \tag{6.0.5}$$

It follows from (6.0.4) that for $y \in R, t > 0$ the derivatives $\frac{\partial w}{\partial t}, \frac{\partial w}{\partial y}$ can be found by

$$\frac{\partial w}{\partial t}(y,t) = \frac{1}{2}\int_0^t [f(y+(t-\tau),\tau) - f(y-(t-\tau),\tau)]d\tau,$$

$$\frac{\partial w}{\partial y}(y,t) = \frac{1}{2}\int_0^t [f(y+(t-\tau),\tau) - f(y-(t-\tau),\tau)]d\tau.$$

This means that a generalized solution $w(y,t)$ of (6.0.1), (6.0.2) belongs to $C^1(R\times [0,\infty))$ for any $f(y,t) \in C(R \times [0,\infty))$.

Remark 6.0.5. Let us consider IVP consider IVP for the wave equation with two independent variables

$$\Big(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial y^2}\Big)w(y,t) = \mathcal{F}(y,t,w(y,t),\frac{\partial w}{\partial t}(y,t),\frac{\partial w}{\partial y}(y,t)),\ y \in R, t > 0, \tag{6.0.6}$$

$$w(y,t)|_{t=+0} = 0, \quad \frac{\partial w}{\partial t}(y,t)|_{t=+0} = 0, \tag{6.0.7}$$

where

$$\mathcal{F}(y,t,w,\frac{\partial w}{\partial t},\frac{\partial w}{\partial y}) = p_2(y,t)\frac{\partial w}{\partial t} + p_1(y,t)\frac{\partial w}{\partial y} + p_0(y,t)w + f(y,t),$$

$p_k(y,t),\ f(y,t) \in C^1(R \times [0,\infty)),\ k = 0,1,2$ are given functions. Using the reasoning made above we find that the generalized Cauchy problem (6.0.6), (6.0.7) is equivalent to the following equation

$$w(y,t) = \Big(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial y^2}\Big)^{-1}\mathcal{F}(y,t,w(y,t),\frac{\partial w}{\partial t}(y,t),\frac{\partial w}{\partial y}(y,t)) \equiv$$
$$\int\int_{R^2} \theta((t-\tau)-|y-\xi|)\theta(\tau)\mathcal{F}(y,t,w(\xi,\tau),\frac{\partial w}{\partial t}(\xi,\tau),\frac{\partial w}{\partial y}(\xi,\tau))d\xi d\tau, \tag{6.0.8}$$

The equation (6.0.8) may be written in the form

$$w(y,t) = \frac{\theta(t)}{2}\int_0^t\int_{y-(t-\tau)}^{y+(t-\tau)} \mathcal{F}(y,t,w(\xi,\tau),\frac{\partial w}{\partial t}(\xi,\tau),\frac{\partial w}{\partial y}(\xi,\tau))d\xi d\tau. \tag{6.0.9}$$

APPENDIX B
PALEY-WIENER SPACE AND THE REAL VERSION OF THE PALEY-WIENER THEOREM

The result here have been taken from the paper Andersen (2004) (see also Bang (1995), Tuan & Zayed (2002)).
As well known (see for example, Andersen (2004), Bang (1995), Tuan & Zayed (2002)) the classical Fourier transform $\mathcal{F}$ is an isomorphism of the Schwartz space $S(R^k)$ onto itself. The space $C_c^\infty(R^k)$ of the smooth functions with compact support is dense in $S(R^k)$, and the classical Paley-Wiener theorem characterizes the image of $C_c^\infty(R^k)$ under $\mathcal{F}$ as rapidly decreasing function having an holomorphic extension to C^k of exponential type. In this appendix we will define the Paley-Wiener space and consider the real version of the Paley-Wiener theorem follow the nice work (Andersen (2004)).

Definition 7.0.6. We define Paley-Wiener space $PW(R^k)$ as the space of all functions $\varphi(x) \in C^\infty(R^k)$ satisfying:

$$(a) \quad (1+|x|)^m \Delta^n \varphi(x) \in \mathcal{L}_2(R^2) \text{ for all } m, n \in \{0, 1, 2\ldots\},$$
$$(b) \quad R_\varphi^\Delta = \lim_{n\to\infty} \|\Delta^n \varphi(x)\|_2^{1/2n} < \infty,$$

where $\mathcal{L}_2(R^2)$ is the space of square integrable functions with norm $\|\varphi\|_2 = \left(\int_{R^2} |\varphi(x)|^2 dx\right)^{1/2}$ for any $\varphi(x) \in \mathcal{L}_2(R^2)$; $\Delta = \frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_k^2}$ denotes the Laplacian on R^k. Further $PW_B(R^k) = \{\varphi(x) \in PW(R^k) |\ R_\varphi^\Delta = B\}$ for $B \geq 0$.

Theorem 7.0.7. *The inverse Fourier transform $\mathcal{F}^{-1}$ is a bijection on $C_c^\infty(R^k)$ onto $PW(R^k)$, mapping $C_B^\infty(R^k)$ onto $PW(R^k)$.*

Here $C_B^\infty(R^k)$ is defined as

$$C_B^\infty(R^k) = \{\varphi(x) \in C_B^\infty(R^k) |\, R_\varphi = B\},$$

where $R_\varphi = \sup_{x \in supp\varphi} |x|$ is the radius of the support of $\varphi(x)$.

We note that the work (Andersen (2004)) contains the proof of this theorem.