DOKUZ EYLÜL UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

A GENERALIZATION OF THE RATIONAL BEZIER SURFACES

by
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June, 2009
İZMİR

A GENERALIZATION OF THE RATIONAL BEZIER SURFACES

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Ph.D. THESIS EXAMINATION RESULT FORM

We have read the thesis entitled **"A GENERALIZATION OF THE RATIONAL BÉZIER SURFACES"** completed by **ÇETİN DİŞİBÜYÜK** under supervision of **ASSOC. PROF. HALİL ORUÇ** and we certify that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Doctor of Philosophy.

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Çetin Dişibüyük

A GENERALIZATION OF THE RATIONAL BÉZIER SURFACES

ABSTRACT

In this thesis, we introduce a generalization of rational Bezier surfaces using q-Bernstein Bezier polynomilas. We generate these surfaces by a new de Casteljau type algorithm, which is in affine form. The explicit formula of intermediate points of de Casteljau algorithm is obtained. These points of the algorithm are expressed in terms of q-differences and consequently rational $q-$Bernstein Bezier surfaces are also expressed in terms of $q-$differences. The change of basis matrix between tensor product Bernstein Bezier basis and tensor product $q-$Bernstein Bezier basis is given. We study the degree elevation procedure for $q-$Bernstein Bezier surfaces. Finally, the convergence properties of tensor product $q-$Bernstein Bezier surfaces and $q-$Bezier triangles are studied.

Keywords: Rational $q-$Bernstein Bézier surfaces, $q-$Bernstein polynomials, de Casteljau algorithm, multivariate approximation.

RASYONEL BÉZİER YÜZEYLERİNİN BİR GENELLEŞTİRMESİ

ÖZ

q-Bernstein Bezier polinomları kullanılarak rasyonel Bezier yüzeyleri genelleştirildi. Bu yüzeyler, affine formda olan yeni bir de Casteljau tipi algoritma kullanılarak elde edildi. de Casteljau algoritmasının ara noktaları $q-$farklar ile ifade edildi ve bunun sonucunda da $q-$Bernstein Bezier yüzeyleri de $q-$farklar ile ifade edildi. Tensör çarpım Bernstein Bezier tabanı ve tensör çarpım $q-$Bernstein Bezier tabanı arasındaki dönüşüm matrisi verildi. $q-$Bernstein Bezier yüzeylerinin derecesi yükseltildi. Son olarak, tensör çarpım $q-$Bernstein Bezier yüzeyleri ve $q-$Bezier üçgenlerinin yakınsaklık özellikleri çalışıldı.

Anahtar sözcükler: Rasyonel $q-$Bernstein Bézier yüzeyleri, $q-$Bernstein polinomları, de Casteljau algoritması, çok değişkenli yaklaşım.

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CHAPTER ONE
INTRODUCTION

We first give some basics of Bernstein Bézier polynomials which may be found in (Farin, 2002). In section 1.2, a generalization of Bernstein Bézier polynomials introduced by G. M. Phillips is given. Using $q-$Bernstein Bézier polynomials, one parameter family of Bézier curves and one parameter family of rational Bézier curves are given in section 1.3. We investigate certain geometric properties of these curves. We also obtain a second de Casteljau type algorithm for computing $q-$Bernstein Bézier curves that can be found in (Dişibüyük & Oruç, 2008).

1.1 Bézier Curves

One of the most important mathematical representation of curves and surfaces used in computer graphics and computer-aided geometric design (CAGD) is Bézier representation. Bézier curves are first publicized by French engineer Pierre Bézier in 1962. These curves are first used to design automobile bodies. A parametric Bézier curve of degree n is defined by

$$P(t) = \sum_{i=0}^{n} \mathsf{b}_i \binom{n}{i} t^i (1-t)^{n-i}, \quad t \in [0,1], \quad \mathsf{b}_i \in \mathbb{E}^2 \text{ or } \mathbb{E}^3 \tag{1.1.1}$$

where $\mathbb{E}^n$ denotes $n-$dimensional Euclidean space. The points b_i are called the control points and the polygon obtained by joining the control point b_i with the control point b_{i+1} for $i = 0, 1, \ldots, n-1$ is called the control polygon. The reason of the popularity of Bézier curves in CAGD is that the points b_i give information about the shape of the polynomial curve $P(t)$. The shape of $P(t)$ can be predicted using the shape of its control polygon.

The basis functions

$$B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i}, \quad i = 0, \ldots, n, \tag{1.1.2}$$

are called Bernstein Bézier polynomials of degree n. These polynomials are first introduced by S. Bernstein to give a proof of approximation theorem of Weierstrass which asserts that for each continuous function $f(x)$, on a closed interval $[a,b]$, and a given $\varepsilon > 0$ there is a polynomial $P(x)$ approximating $f(x)$ uniformly:

$$|f(x) - P(x)| < \varepsilon.$$

Bernstein shows that for a function $f(x)$ bounded on $[0,1]$, the relation

$$\lim_{n\to\infty} B_n(f;x) = f(x)$$

holds at each point of continuity of x of f; and the relation holds uniformly on $[0,1]$ if $f(x)$ is continuous on this interval (see Lorentz, 1986). Here the polynomial $B_n(f;x)$ is called the Bernstein polynomial of order n of the function $f(x)$ and defined by

$$B_n(f;x) = \sum_{i=0}^{n} f\left(\frac{i}{n}\right)\binom{n}{i} x^i (1-x)^{n-i}.$$

For the other proofs of theorem of Weierstrass see (Lorentz, 1986).

There is another approach to theorem of Weierstrass type which uses sequence of positive linear operators. An operator U that maps $C[a,b]$ into itself is positive if $f \geqslant 0$ implies $U(f) \geqslant 0$. If in addition, when $f \leqslant g$ we have $U(f) \leqslant U(g)$ then U is a positive linear operator (see DeVore & Lorentz, 1993). Bohman-Korovkin theorem states that for a sequence U_n, $n = 1,2,\ldots$ of positive linear operators, convergence $U_n(f) \to f$ in uniform norm follows for all $f \in C[a,b]$, if it holds for test functions $f = 1, x, x^2$. It can easily verified that the operators B_n, $n = 1,2,\ldots$ are linear monotone operator on $[0,1]$ and satisfy the conditions of Bohman-Korovkin theorem which gives the uniform convergence of $B_n f$ to f for all $f \in C[0,1]$.

In CAGD applications, the choice of basis used for designing parametric curves and surfaces is important. The most suitable bases for this purpose is the normalized totaly positive bases. A system of functions $\{\phi_0, \phi_1, \ldots, \phi_n\}$ is called totaly positive if all its collocation matrices $\left(\phi_j(x_i)_{i,j=0}^{n}\right)$ are totaly positive, that is all their minors are nonnegative. In addition if $\{\phi_0, \phi_1, \ldots, \phi_n\}$ is totally positive basis and $\sum_{i=0}^{n} \phi_i = 1$ then $\{\phi_0, \phi_1, \ldots, \phi_n\}$ is called normalized totally positive basis. Goodman (Goodman,

1996) shows that the power basis

$$(1, x, x^2, \ldots, x^n), x \geqslant 0$$

is totaly positive. Moreover, using this fact he shows that Bernstein basis functions $(B_0^n(x), B_1^n(x), \ldots, B_n^n(x))$ is totaly positive basis. I. J. Schoenberg discovered that if A is a totaly positive matrix then it has variation diminishing property, that is the number of sign changes in a vector does not change upon multiplicity by A. Total positivity provides a technique for discussing shape properties of approximations, due to the variation diminishing properties of totaly positive functions, bases and matrices.

Bernstein Bézier polynomials have the following properties that lead to some geometric properties of Bernstein Bézier curves. The Bernstein Bézier polynomials have partition of unity property,

$$1 = ((1-t)+t)^n = \sum_{i=0}^{n} \binom{n}{i} t^i (1-t)^{n-i},$$

which follows from the Binomial Theorem. The end point conditions are

$$B_i^n(0) = \delta_{i,0}, \quad B_i^n(1) = \delta_{i,n}$$

and

$$B_i^n(t) = B_{n-i}^n(1-t)$$

shows symmetry of the basis functions. Figure 1.1 is the figure of cubic Bernstein Bézier polynomials for $t \in [0,1]$.

The properties of Bézier curves are

1. Convex hull property: The Bernstein Bézier polynomials have partition of unity property. Furthermore, for $t \in [0,1]$ these polynomials are nonnegative. Hence Bézier curve $P(t)$ is a convex combination of its control points which geometrically means that $P(t)$ lies in the convex hull of the control points. Convex hull of a set of points is the smallest region formed by all convex combination of points.

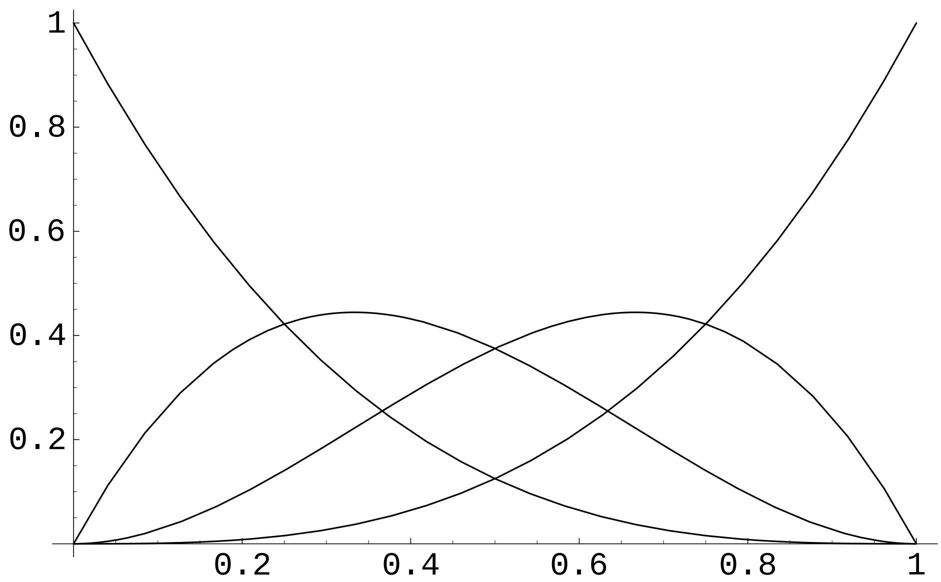

Figure 1.1 Cubic Bernstein Bézier polynomials.

2. Affine invariance property: Since the Bernstein Bézier polynomials sum to one, the Bézier curves are barycentric (affine) combinations of its control points. Thus the curve is invariant under affine transformations. This means that the following two procedures give the same result:

i) Compute $P(t)$ and then apply an affine map to it.

ii) Apply the map to the control points then evaluate $P(t)$.

3. Endpoint interpolation property: The curve interpolate endpoints b_0 and b_n. That is

$$P(0) = \mathsf{b}_0, \quad P(1) = \mathsf{b}_n.$$

4. Variation diminishing property: It comes from the totally positivity of the Bernstein basis functions and geometrically means that the number of times that any line intersects the curve is bounded by the number of times the line intersects the control polygon. Namely the curve does not oscillate about any straight line more often than the control polygon does.

5. Symmetry property: Let $\mathsf{b}_0, \ldots, \mathsf{b}_n$ and $\mathsf{c}_i = \mathsf{b}_{n-i}, \quad i = 0, \ldots, n$ be two control

polygons. Since the Bernstein Bézier polynomials have symmetry property these two polygons trace out the same Bézier curve. They differ only in the direction in which they are traversed,

$$\sum_{i=0}^{n} \mathsf{b}_i B_i^n(t) = \sum_{i=0}^{n} \mathsf{b}_{n-i} B_i^n(1-t).$$

As a result of these properties, the shape of the curve mimics the shape of its control polygon.

Although Bézier curves are first publicized in 1962, Paul de Casteljau is the first one who developed them in 1959 by using an algorithm that gives a point on the curve.

For the given points $\mathsf{b}_0, \ldots, \mathsf{b}_n$ and $t \in \mathbb{R}$, this algorithm is

Algorithm 1.1: (de Casteljau Algorithm)

$$\mathsf{b}_i^r(t) = (1-t)\mathsf{b}_i^{r-1}(t) + t\mathsf{b}_{i+1}^{r-1}(t), \qquad \begin{cases} r = 1, \ldots, n \\ i = 0, \ldots, n-r \end{cases} \tag{1.1.3}$$

where $\mathsf{b}_i^0(t) = \mathsf{b}_i$ for all i. Then it can be shown by induction on n that $\mathsf{b}_0^n(t)$ is the point with the parameter value t on the Bézier curve $P(t)$. Hence by continuity $\mathsf{b}_0^n(t) = P(t)$.

There are two important technique, that aim to increase the flexibility of Bézier curves, subdivision and degree elevation. Subdivision is an application of the de Casteljau algorithm. We can subdivide a Bézier curve into two Bézier curve segments which join together at a point $t_0 \in (0,1)$. The part of the curve that corresponds to the interval $[0,t_0]$ have the control points $\mathsf{b}_0^i(t_0)$, $i = 0,1,\ldots,n$. It follows from the symmetry property that the control points for the part corresponding to $[t_0, 1]$ are given by $\mathsf{b}_i^{n-i}(t_0)$, $i = 0,1,\ldots,n$,(See Farin, 2002).

Thus the curve segments are

$$P_{[0,t_0]}(t) = \sum_{i=0}^{n} \mathsf{b}_i^{(l)} B_i^n(t), \qquad P_{[t_0,1]}(t) = \sum_{i=0}^{n} \mathsf{b}_i^{(r)} B_i^n(t)$$

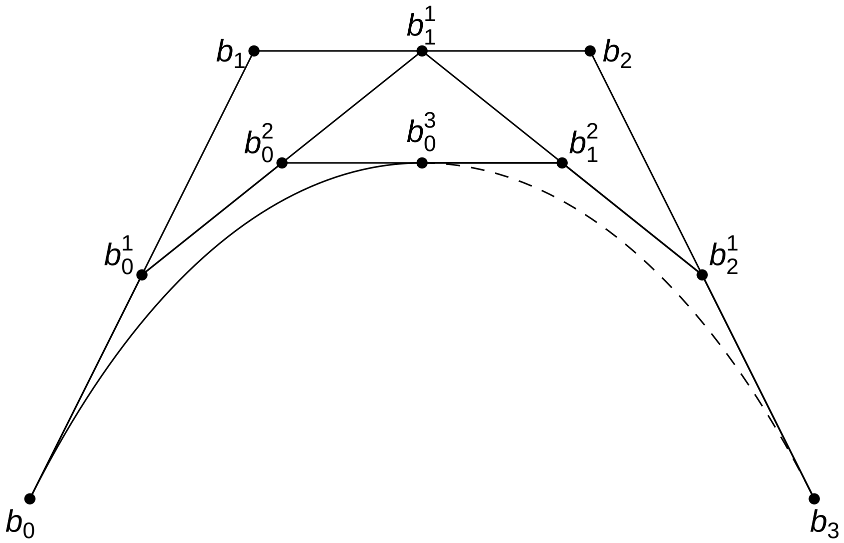

Figure 1.2 Subdivision of cubic Bézier curve in the de Casteljau algorithm.

where $\mathsf{b}_i^{(l)}$ denotes $\mathsf{b}_0^i(t_0)$ and $\mathsf{b}_i^{(r)}$ denotes $\mathsf{b}_i^{n-i}(t_0)$, and

$$P_{[0,1]}(t) = P_{[0,t_0]}(t) \cup P_{[t_0,1]}(t) = \sum_{i=0}^{n} \mathsf{b}_i B_i^n(t).$$

Degree elevation is a method which enables us to have more flexible curve by obtaining a new set of control points. For a given Bézier curve of degree n we can express the same curve as one of more degree. For this purpose write

$$P(t) = (1-t)P(t) + tP(t). \tag{1.1.4}$$

Since $(1-t)B_i^n(t) = \frac{n+1-i}{n+1} B_i^{n+1}(t)$ and $tB_i^n(t) = \frac{i+1}{n+1} B_{i+1}^{n+1}(t)$ we have

$$P(t) = \sum_{i=0}^{n} \frac{n+1-i}{n+1} \mathsf{b}_i B_i^{n+1}(t) + \sum_{i=0}^{n} \frac{i+1}{n+1} \mathsf{b}_i B_{i+1}^{n+1}(t).$$

Extending the upper limit of the first sum to $n+1$, shifting the index of the second sum to the limits 1 to $n+1$ and then extending the lower limit to 0 we obtain

$$P(t) = \sum_{i=0}^{n+1} \frac{n+1-i}{n+1} \mathsf{b}_i B_i^{n+1}(t) + \sum_{i=0}^{n+1} \frac{i}{n+1} \mathsf{b}_{i-1} B_i^{n+1}(t).$$

Then

$$P(t) = \sum_{i=0}^{n+1} \left(\frac{n+1-i}{n+1} \mathsf{b}_i + \frac{i}{n+1} \mathsf{b}_{i-1} \right) B_i^{n+1}(t). \tag{1.1.5}$$

Thus, the new control points denoted by b_i^1 are

$$\mathsf{b}_i^1 = \frac{i}{n+1} \mathsf{b}_{i-1} + \left(1 - \frac{i}{n+1}\right) \mathsf{b}_i, \qquad i = 0, \ldots, n+1. \tag{1.1.6}$$

Notice that control points $\mathsf{b}_0^1, \ldots, \mathsf{b}_{n+1}^1$ and $\mathsf{b}_0, \ldots, \mathsf{b}_n$ describe the same Bézier curve with the bases $B_i^{n+1}(t)$ and $B_i^n(t)$ respectively. Degree elevation process interpolates the end points, that is $\mathsf{b}_0^1 = \mathsf{b}_0$ and $\mathsf{b}_{n+1}^1 = \mathsf{b}_n$. Further, if $C^k = \{\mathsf{b}_0^k, \mathsf{b}_1^k, \ldots, \mathsf{b}_{n+k}^k\}$ is the set of control points obtained from k times repeated application of degree elevation then as $k \to \infty$, setting $\frac{i}{n+k} = t$ yields $\mathsf{b}_{i+k}^k \to P(t)$, a point on the curve with parameter value t (see Farin, 2002).

Bézier curves can be used to represent a wide variety of curves. But the conic sections which are important in geometric design cannot be represented in Bézier form. In order to be able to include conic sections in the set of representable curves in Bézier form, we turn to rational Bézier curves.

A rational Bézier curve of degree n in $\mathbb{E}^d, d = 2, 3$ is obtained by projecting an nth degree Bézier curve in $\mathbb{E}^{d+1}$ into the hyperplane $w = 1$. Rational Bézier curve $R(t)$ is defined by

$$R(t) = \frac{\sum_{i=0}^n w_i \mathsf{b}_i B_i^n(t)}{\sum_{i=0}^n w_i B_i^n(t)}, \text{ where } \mathsf{b}_i \in \mathbb{E}^d. \tag{1.1.7}$$

The positive real values w_i are called weights and the points b_i are the control points which is the projection of the $d+1$ dimensional control points $[w_i \mathsf{b}_i \quad w_i]^T$. If the weights are set to $w_i = 1$ for all i, then we obtain polynomial Bézier curves.

Rational Bézier curves inherit the following properties of Bézier curves.

1. Convex hull property holds when all $w_i > 0$.

2. Endpoint interpolation property; $R(0) = \mathsf{b}_0, R(1) = \mathsf{b}_n$.

3. Variation diminishing property holds when all $w_i > 0$.

4. Affine invariance property

In addition to above properties, $R(t)$ satisfies projective invariance property. Projective invariance property means that the following procedures give the same result:

i) Compute $P(t)$ in $\mathbb{E}^{d+1}$ and then project it to the hyperplane $w = 1$ to find $R(t)$ in $\mathbb{E}^d$.

ii) Project the control polygon points of $P(t)$ to the hyperplane and then evaluate rational Bézier curve.

Weights add more flexibility to the curves so that if we increase the weight w_i then all points on the curve move towards the control point b_i, if we decrease w_i then all points of the curve move away from b_i. Hence one can change the shape of the curve without changing the control points.

Note that the de Casteljau algorithm can be extended to compute rational Bézier curves by applying it to the homogeneous coordinates $[w_i\mathsf{b}_i \quad w_i]^T$ and projecting each intermediate point to the hyperplane $w = 1$.

1.2 q-Bernstein Bézier Polynomials

A great deal of research papers have appeared on $q-$Bernstein Bézier polynomials since it is first introduced by G.M. Phillips in (Phillips, 1997) as a generalization of Bernstein polynomials. In general they fall into two categories; works that display geometric properties and investigation on its convergence properties. See full details in a recent survey paper by G. M. Phillips (Phillips, 2008). One parameter family (q, the parameter) of Bernstein Bézier polynomials (called q-Bernstein Bézier polynomials) are defined by

$$B_i^{n,q}(t) = \begin{bmatrix} n \\ i \end{bmatrix} t^i \prod_{s=0}^{n-i-1} (1 - q^s t), \quad t \in [0,1], \quad 0 \leqslant i \leqslant n, \tag{1.2.1}$$

where an empty product denotes one and the parameter q is positive real number. The $q-$binomial coefficient $\begin{bmatrix} n \\ i \end{bmatrix}$, which is also called a Gaussian polynomial (See Andrews, 1998), is defined as

$$\begin{bmatrix} n \\ i \end{bmatrix} = \frac{[n][n-1]\cdots[n-i+1]}{[i][i-1]\cdots[1]} \tag{1.2.2}$$

for $0 \leqslant i \leqslant n$, and has the value 0 otherwise. Here $[i]$ denotes a q-integer, defined by

$$[i] = \begin{cases} (1-q^i)/(1-q), & q \neq 1, \\ i, & q = 1. \end{cases} \tag{1.2.3}$$

When $q = 1$ the $q-$binomial coefficients reduce to the usual binomial coefficients. They satisfy the following recurrence relations

$$\begin{bmatrix} n \\ i \end{bmatrix} = q^{n-i} \begin{bmatrix} n-1 \\ i-1 \end{bmatrix} + \begin{bmatrix} n-1 \\ i \end{bmatrix} \tag{1.2.4}$$

and

$$\begin{bmatrix} n \\ i \end{bmatrix} = \begin{bmatrix} n-1 \\ i-1 \end{bmatrix} + q^i \begin{bmatrix} n-1 \\ i \end{bmatrix}. \tag{1.2.5}$$

Using (1.2.4) it is easily shown by induction on n that

$$(1-t)(1-qt)\cdots(1-q^{n-1}t) = \sum_{i=0}^{n} (-1)^i q^{i(i-1)/2} \begin{bmatrix} n \\ i \end{bmatrix} t^i \tag{1.2.6}$$

It follows from putting (1.2.4) and (1.2.5) in (1.2.1) that $q-$Bernstein polynomials computed recursively by

$$B_i^{n,q}(t) = q^{n-i} t B_{i-1}^{n-1,q}(t) + (1 - q^{n-i-1}t) B_i^{n-1,q}(t). \tag{1.2.7}$$

and

$$B_i^{n,q}(t) = t B_{i-1}^{n-1,q}(t) + (q^i - q^{n-1}t) B_i^{n-1,q}(t), \tag{1.2.8}$$

(See Oruç & Phillips, 2003).

Using $q-$Bernstein Bézier polynomials Phillips proposed the following generalization of the Bernstein polynomials, based on the $q-$integers (see Phillips, 1997). For each positive integer n, he defines

$$B_n(f;x) = \sum_{r=0}^{n} f_r \begin{bmatrix} n \\ r \end{bmatrix} x^r \prod_{s=0}^{n-r-1} (1 - q^s x)$$

where $f_r = f\left(\frac{[r]}{[n]}\right)$. It is shown in (Phillips, 1997) that $B_n(f;x)$ can be expressed in terms of $q-$differences, in the form

$$B_n(f;x) = \sum_{r=0}^{n} \begin{bmatrix} n \\ r \end{bmatrix} \Delta^r f_0 x^r. \tag{1.2.9}$$

which gives the difference form of the classical Bernstein polynomials when we set $q = 1$. It follows from (1.2.9) that for any polynomial f of degree m, $B_n(f;x)$ is a polynomial of degree $\min(m,n)$. It is also clear from (1.2.9) that

$$B_n(1;x) = 1,\, B_n(x;x) = x \text{ and } B_n(x^2;x) = x^2 + \frac{x(1-x)}{[n]}.$$

For a fixed value of $q \in (0,1)$ the polynomial $B_n(x^2;x)$ does not converge to x^2. Thus, although $B_n f, n = 1,2,\ldots$ are positive linear operators, when $0 < q < 1$ is fixed the Bohman-Korovkin theorem is not applicable and $B_n(f;x) \to f$ requires that f be a linear function (see Il'inskii & Ostrovska, 2002). In (Phillips, 1997) it is shown that the generalized Bernstein polynomials of a function $f(x)$ converges to $f(x)$ for all $f(x) \in C[0,1]$. For this purpose Phillips choose $q-$integers depend on the degree of the $B_n f$ such that $[r] = \frac{1-q_n^r}{1-q_n}$. Hence taking a sequence $q = q_n$ such that $[n] \to \infty$ as $n \to \infty$ follows that $B_n(x^2;x) \to x^2$. Thus, using Bohman-Korovkin theorem, $B_n f \to f$ for all $f \in C[0,1]$.

Converge properties of generalized Bernstein polynomials are also investigated for the case $q > 1$ (see, for example, (Oruç & Tuncer, 2002), (Ostrovska, 2003)). In this case Bohman-Korovkin theorem does not applicable, since $B_n(f;x)$ does not generate positive linear operators when $q > 1$. In (Goodman, Oruç & Phillips, 1999)

$q-$Bernstein polynomial of a monomial is given in terms of Stirling polynomials of the second kind such that

$$B_n(x^i;x) = \sum_{j=0}^{i} \lambda_j [n]^{j-i} S_q(i,j) x^j,$$

where

$$\lambda_j = \prod_{r=0}^{j-1} \left(1 - \frac{[r]}{[n]}\right)$$

with empty product denotes 1, and

$$S_q(i,j) = \frac{1}{[j]!q^{j(j-1)/2}} \sum_{r=0}^{j} (-1)^r q^{r(r-1)/2} \begin{bmatrix} j \\ r \end{bmatrix} [j-r]^i, \, 0 \leqslant j \leqslant i$$

or recursively

$$S_q(i+1,j) = S_q(i,j-1) + [j] S_q(i,j), \tag{1.2.10}$$

with $S_q(0,0) = 1$, $S_q(i,0) = 0$ for $i > 0$ and $S_q(i,j) = 0$ for $j > i$ (see Oruç, 1998).

Using Stirling polynomial form of $B_n(x^i;x)$, it is shown in (Oruç & Tuncer, 2002) that for a fixed real number $q \geqslant 1$ and any polynomial p

$$\lim_{n\to\infty} B_n(p;x) = p(x).$$

It is shown in (Ostrovska, 2003) that when $q > 1$ the approximating properties of $q-$Bernstein polynomials may be better than in the case $q \leqslant 1$, so that in the case $q > 1$, $B_n(f;x)$ converges uniformly to $f(x)$ when $f(x)$ has analytic expansion that is

$$f(x) = \sum_{i=0}^{\infty} a_i x^i \text{ with } \sum_{i=0}^{\infty} |a_i| < \infty.$$

Recently another direction to $q-$Bernstein polynomials is given by (Dişibüyük & Oruç, 2007), (Lewanowicz & Woźny, 2004) and (Nowak, 2009), the first one gives their rational counterpart and the latter two define more general polynomials in which the second leads to a connection with $q-$Jacobi polynomials and the latter is a generalization of Stancu operators that gives $q-$Bernstein polynomials in a special case.

1.3 One Parameter Family of Bézier Curves

One parameter family of Bézier curves, called $q-$Bernstein Bézier curves, of degree n is introduced in (Oruç & Phillips, 2003) and defined by

$$P(t) = \sum_{i=0}^{n} \mathsf{b}_i \begin{bmatrix} n \\ i \end{bmatrix} t^i \prod_{j=0}^{n-i-1} (1 - q^j t). \tag{1.3.1}$$

Note that if we set the parameter q to the value 1, we obtain standard Bézier curves. The properties of $q-$Bernstein Bézier curves are as follows:

1. Convex hull property holds when $0 < q \leqslant 1$ and the Bézier polygon approximately describe the shape of the curve.

2. Affine invariance property holds.

3. The curve passes through the endpoints b_0 and b_n.

4. If $q \in (0,1]$ then the variation diminishing property holds.

Figure 1.3 depicts two cubic q-Bernstein Bézier curves with the same control polygon but different values of q.

It is shown in (Phillips, 1996) that $q-$Bernstein Bézier curves may evaluated by the following de Casteljau type algorithm:

Algorithm 1.2: For the given control points $\mathsf{b}_0, \ldots, \mathsf{b}_n \in \mathbb{E}^2$ or $\mathbb{E}^3$ compute

$$\hat{\mathsf{b}}_i^r(t) = (q^i - q^{r-1}t)\hat{\mathsf{b}}_i^{r-1}(t) + t\hat{\mathsf{b}}_{i+1}^{r-1}(t), \qquad \begin{cases} r = 1, 2, \ldots, n \\ i = 0, 1, \ldots, n-r \end{cases} \tag{1.3.2}$$

where $\hat{\mathsf{b}}_i^0(t) = \mathsf{b}_i$ for all i.

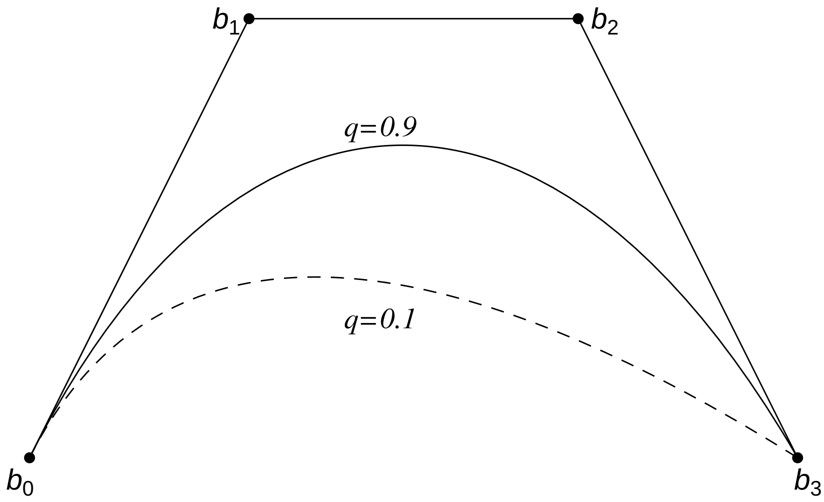

Figure 1.3 Two q-Bernstein Bézier curves with different values of q.

Note that Algorithm 1.2 does not consist only of convex combinations. Thus although the $q-$Bernstein Bézier curve lies in the convex hull of the control points, the intermediate points of Algorithm 1.2 may not lie in the convex hull of the control polygon. We now give a second de Casteljau type algorithm for computing the $q-$Bernstein Bézier curves. This algorithm is an affine combination and it will enable us to construct rational $q-$Bernstein Bézier curves and $q-$Bernstein Bézier surfaces.

Algorithm 1.3: For the given control points $\mathsf{b}_0, \ldots, \mathsf{b}_n \in \mathbb{E}^2$ or $\mathbb{E}^3$ compute

$$\mathsf{b}_i^r(t) = (1 - q^{r-i-1}t)\mathsf{b}_i^{r-1}(t) + q^{r-i-1}t\mathsf{b}_{i+1}^{r-1}(t), \qquad \begin{cases} r = 1, 2, \ldots, n \\ i = 0, 1, \ldots, n-r \end{cases} \tag{1.3.3}$$

Algorithm 1.2 differs from Algorithm 1.3 since each step of the latter is in barycentric (affine) form which evantually make up a curve that remains invariant under affine maps. Note that in CAGD systems it is desirable to express curves and surfaces in barycentric form (Farin, 2002). Furthermore $q = 1$ recovers the standard de Casteljau algorithm for both of the above algorithms. Further results on Algorithm 1.3 can be read in (Dişibüyük & Oruç, 2008).

1.3.1 One Parameter Family of Rational Bézier Curves

The $q-$Bernstein Bézier curves are generalized to their rational counterparts as one parameter family of rational Bernstein Bézier curves in (Dişibüyük & Oruç, 2007). A rational $q-$Bernstein Bézier curve of degree n is defined by

$$R(t) = \frac{\sum_{i=0}^{n} w_i \mathsf{b}_i B_i^{n,q}(t)}{\sum_{i=0}^{n} w_i B_i^{n,q}(t)} \tag{1.3.4}$$

where the points $\mathsf{b}_i, \quad i = 0, \ldots, n \in \mathbb{E}^2$ or $\mathbb{E}^3$ form the control polygon of rational curve $R(t)$ and the number w_i is called the weight of the associated point b_i. Restricting all $w_i > 0$ guarantees that the bases functions are nonnegative and the curve does not have any singularities. The properties of rational $q-$Bernstein Bézier curves are

1. Convex hull property holds when $w_i > 0$ and $0 < q \leqslant 1$

2. Endpoint interpolation property

3. Variation diminishing property

4. Affine invariance property

5. Projective invariance property

As an illustration, it is shown in (Dişibüyük & Oruç, 2007) that quadratic rational $q-$Bernstein Bézier curves can be used to represent conic sections. The classification of conic sections is as follows:
Let the weights are $w_0 = w_2 = 1$ and $w_1 = w$.

If $q = -1$ then $R(t)$ is a straight line for any w. $R(t)$ is a parabola for any $q \neq 1$ when $w = 1$. $R(t)$ is an ellipse when $q < -1$ and $w > 1$ or when $q > -1$ and $w < 1$. $R(t)$ is an hyperbola if $q < -1$ and $w > 1$ or when $q > -1$ and $w > 1$.

The following figure shows a hyperbola, a line and an ellipse using same control points but different parameter values q.

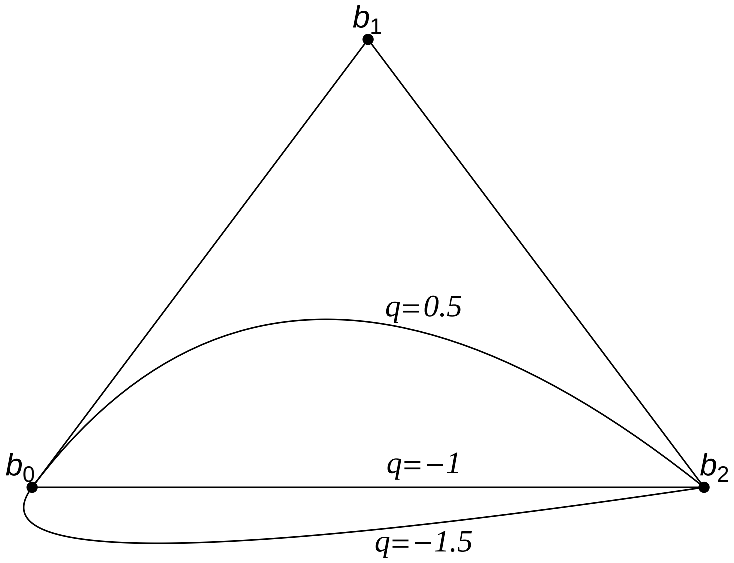

Figure 1.4 $q = 0.5$ a hyperbola, $q = -1$ a line and $q = 1.5$ an ellipse.

CHAPTER TWO
BÉZIER SURFACES

Surfaces have a fundamental role in computer graphics and in CAGD. A generalization of Bézier curves to higher dimension are called Bézier surfaces. Similar to a control polygon for curves, Bernstein Bézier surfaces are defined by a control net. These surfaces are parametrized in two directions where $u \in [0,1]$ and $v \in [0,1]$. In this chapter we investigate two such generalizations, tensor product Bézier surfaces and Bézier triangles. The de Casteljau type algorithms for these surfaces are given and using the difference form of the intermediate points of the algorithms, the difference forms of two type of Bézier surfaces are given. We also obtain the same surfaces by surfaces of higher degree. In section 2.3 rational Bézier surfaces and their properties are obtained.

2.1 Tensor Product Bézier Surfaces

Bézier curves can be evaluated by de Casteljau algorithm using repeated linear interpolation (Farin, 2002). Using bilinear interpolation which is an extension of linear interpolation, Bézier curves can be extended to the Bézier surfaces. As an example consider four distinct points, $\mathsf{b}_{0,0}, \mathsf{b}_{0,1}, \mathsf{b}_{1,0}, \mathsf{b}_{1,1}$ in $\mathbb{E}^3$ the bilinear interpolant $X(u,v)$ passing through the points $\mathsf{b}_{i,j}$; $i,j = 0,1$ is

$$X(u,v) = \begin{bmatrix} 1-u & u \end{bmatrix} \begin{bmatrix} \mathsf{b}_{0,0} & \mathsf{b}_{0,1} \\ \mathsf{b}_{1,0} & \mathsf{b}_{1,1} \end{bmatrix} \begin{bmatrix} 1-v \\ v \end{bmatrix}.$$

Initially, we obtain a tensor product Bézier surface $S(u,v)$ of degree (n,n) by using repeated bilinear interpolation. Suppose that we are given a rectangular array of points $\mathsf{b}_{i,j} \in \mathbb{E}^3$ $i,j = 0,\ldots,n$ and parameter values (u,v) compute

$$\mathsf{b}_{i,j}^{r,r} = \begin{bmatrix} 1-u & u \end{bmatrix} \begin{bmatrix} \mathsf{b}_{i,j}^{r-1,r-1} & \mathsf{b}_{i,j+1}^{r-1,r-1} \\ \mathsf{b}_{i+1,j}^{r-1,r-1} & \mathsf{b}_{i+1,j+1}^{r-1,r-1} \end{bmatrix} \begin{bmatrix} 1-v \\ v \end{bmatrix}, \quad \begin{array}{l} r = 1,2,\ldots,n \\ i,j = 0,1,\ldots,n-r, \end{array}$$

where $\mathsf{b}_{i,j}^{0,0} = \mathsf{b}_{i,j}$. Then $\mathsf{b}_{0,0}^{n,n}(u,v)$ is a point on the surface with the parameter values (u,v). The net that formed by $\mathsf{b}_{i,j}$ is called control net or Bézier net of the surface. The Bernstein Bézier form of the surfece is

$$S(u,v) = \sum_{i=0}^{n}\sum_{j=0}^{n} \mathsf{b}_{i,j} B_i^n(u) B_j^n(v)$$

where $0 \leqslant u,v \leqslant 1$ and $B_i^n(u), B_j^n(v)$ are Bernstein polynomials in u and in v respectively. This representation can be extended to a tensor product Bézier surface of degree (m,n). Let the control net points given by $\mathsf{b}_{i,j} \in \mathbb{E}^3$, $i = 0,\ldots,m$ and $j = 0,\ldots,n$, then the tensor product Bézier surface of degree (m,n) is given by

$$S(u,v) = \sum_{i=0}^{m}\sum_{j=0}^{n} \mathsf{b}_{i,j} B_i^m(u) B_j^n(v), \quad 0 \leqslant u,v \leqslant 1. \tag{2.1.1}$$

Note that the set of basis functions

$$\{B_i^m(u)B_0^n(v), B_i^m(u)B_1^n(v), \ldots, B_i^m(u)B_n^n(v)\},\ i = 0,\ldots,m$$

is obtained by tensor product of the sets $\{B_0^m(u),\ldots,B_m^m(u)\}$ and $\{B_0^n(v),\ldots,B_n^n(v)\}$. Properties of tensor product Bézier surfaces are as follows:

1. Affine invariance property: Since $\sum_{i=0}^{m}\sum_{j=0}^{n} B_i^m(u)B_j^n(v) = 1$, $S(u,v)$ is an affine combinations of its control net points. Thus $S(u,v)$ is affinely invariant.

2. Convex hull property: The basis form partition of unity and additionally they are nonnegative for the parameter values $0 \leqslant u,v \leqslant 1$. Hence $S(u,v)$ is a convex combination of $\mathsf{b}_{i,j}$ and lies in the convex hull of its control net points.

3. Boundary curves: Boundary curves of $S(u,v)$ are evaluated by $S(u,0)$, $S(u,1)$, $S(0,v)$ and $S(1,v)$. The first two curves are Bernstein Bézier curves in u and the last two curves are Bernstein Bézier curves in v.

4. Corner point interpolation: The control points of the boundary curves are the boundary points of the control net of $S(u,v)$. Thus it follows from the end point interpolation property of Bézier curves that the corner control net points coincide with

the four corners of the surface. Namely,

$$S(0,0) = \mathsf{b}_{0,0}, S(0,1) = \mathsf{b}_{0,n}, S(1,0) = \mathsf{b}_{m,0} \text{ and } S(1,1) = \mathsf{b}_{m,n}.$$

What follow is the de Casteljau algorithm to compute $S(u,v)$ of degree (m,n).

Algorithm 2.1: Given the control net $\mathsf{b}_{i,j} \in \mathbb{E}^3, i = 0,\ldots,m, j = 0,\ldots,n$. Compute

$$\mathsf{b}_{i,j}^{r,r} = \begin{bmatrix} 1-u & u \end{bmatrix} \begin{bmatrix} \mathsf{b}_{i,j}^{r-1,r-1} & \mathsf{b}_{i,j+1}^{r-1,r-1} \\ \mathsf{b}_{i+1,j}^{r-1,r-1} & \mathsf{b}_{i+1,j+1}^{r-1,r-1} \end{bmatrix} \begin{bmatrix} 1-v \\ v \end{bmatrix} \tag{2.1.2}$$

for $r = 1,\ldots,k$, $i = 0,\ldots,m-r$, $j = 0,\ldots,n-r$ where $k = \min(m,n)$.

Since $m \neq n$, performing the de Casteljau algorithm k times will not give a point on the surface. Then to get a point on the surface after kth application of Algorithm 2.1 we perform Algorithm 1.1 for the intermediate points $\mathsf{b}_{i,j}^{k,k}$ with suitable parameter value (see Farin, 2002).

We can extend degree elevation procedure for the surfaces. Let $S(u,v)$ be a surface of degree (m,n). To have the same surface with of degree $(m+1,n)$ we first write tensor product Bézier patches in the form

$$S(u,v) = \sum_{j=0}^{n} b_j B_j^n(u) \tag{2.1.3}$$

where $b_j = \sum_{i=0}^{m} \mathsf{b}_{i,j} B_i^m(v)$. Thus the problem is reduced to expressing an mth degree Bézier curve b_j by a curve of $(m+1)$th degree. From the degree elevation procedure for b_j in the latter equation we obtain

$$S(u,v) = \sum_{i=0}^{m+1} \sum_{j=0}^{n} \mathsf{b}_{i,j}^{(1,0)} B_i^{m+1}(u) B_j^n(v),$$

where

$$\mathsf{b}_{i,j}^{(1,0)} = \left(1 - \frac{m+1-i}{m+1}\right) \mathsf{b}_{i-1,j} + \frac{m+1-i}{m+1} \mathsf{b}_{i,j}, \quad i = 0,\ldots,m+1, j = 0,\ldots,n.$$

Similarly, to obtain the same surface as one of degree $(m, n+1)$ we need new control points such that

$$\mathsf{b}_{i,j}^{(0,1)} = \left(1 - \frac{n+1-j}{n+1}\right)\mathsf{b}_{i,j-1} + \frac{n+1-j}{n+1}\mathsf{b}_{i,j}, \quad i = 0, \ldots, m, j = 0, \ldots, n+1.$$

Finally, to obtain $S(u,v)$ as a surface of degree $(m+1, n+1)$, evaluate the new control points from the product

$$\mathsf{b}_{i,j}^{(1,1)} = \begin{bmatrix} 1 - \frac{m+1-i}{m+1} & \frac{m+1-i}{m+1} \end{bmatrix} \begin{bmatrix} \mathsf{b}_{i-1,j-1} & \mathsf{b}_{i-1,j} \\ \mathsf{b}_{i,j-1} & \mathsf{b}_{i,j} \end{bmatrix} \begin{bmatrix} 1 - \frac{n+1-j}{n+1} \\ \frac{n+1-j}{n+1} \end{bmatrix}.$$

The repeated degree elevation procedure can be used to obtain higher degree surfaces and when we apply it infinitely many times the control net will converge to the surface. We also can express $S(u,v)$ in terms of differences, where we define differences in the $u-$direction by

$$\Delta_1^{k+1}\mathsf{b}_{i,j} = \Delta_1^k \mathsf{b}_{i+1,j} - \Delta_1^k \mathsf{b}_{i,j}$$

for all $k \geqslant 0$ with $\Delta_1^0 \mathsf{b}_{i,j} = \mathsf{b}_{i,j}$, and the differences in the $v-$direction by

$$\Delta_2^{k+1}\mathsf{b}_{i,j} = \Delta_2^k \mathsf{b}_{i,j+1} - \Delta_2^k \mathsf{b}_{i,j}$$

for all $k \geqslant 0$ and $\Delta_2^0 \mathsf{b}_{i,j} = \mathsf{b}_{i,j}$. Then we also define

$$\Delta_1 \Delta_2 \mathsf{b}_{i,j} = \Delta_1(\Delta_2 \mathsf{b}_{i,j}).$$

Note that Δ_1^i and Δ_2^j commute, that is

$$\Delta_1^i \Delta_2^j \mathsf{b}_{i,j} = \Delta_2^j \Delta_1^i \mathsf{b}_{i,j}.$$

Theorem 2.1.1. *$S(u,v)$ can be expressed in terms of differences by*

$$S(u,v) = \sum_{i=0}^{m} \sum_{j=0}^{n} \binom{m}{i} \binom{n}{j} \Delta_1^i \Delta_2^j \mathsf{b}_{0,0} u^i v^j. \tag{2.1.4}$$

Proof. Since (2.1.3) is in Bézier form, using differences form of Bézier curves (see

Farin, 2002) with differences taken in the $v-$direction yields

$$S(u,v) = \sum_{j=0}^{n} \binom{n}{j} \Delta_2^j b_0 v^j.$$

Then with differences in $u-$direction for b_0 and commutativity property of $\Delta_1\Delta_2$ give the desired result. $\square$

2.2 Bézier Triangles

Another generalization of Bézier curves to Bézier surfaces is by Bézier triangles. A Bézier triangle of degree n is defined by

$$S(u,v) = \sum_{i=0}^{n} \sum_{j=0}^{n-i} \mathsf{b}_{i,j} B_{i,j}^n(u,v), \quad 0 \leqslant u,v \leqslant 1 \text{ and } 0 \leqslant u+v \leqslant 1,$$

where $\mathsf{b}_{i,j} \in \mathbb{E}^3$ are control points and $B_{i,j}^n(u,v)$ are Bernstein polynomial defined by

$$B_{i,j}^n(u,v) = \binom{n}{i\ ,\ j} u^i v^j (1-u-v)^{n-i-j} \tag{2.2.1}$$

with the multinomial $\binom{n}{i\ ,\ j} = \frac{n!}{i!j!(n-i-j)!}$. Note that to obtain an nth degree Bézier triangle we need $\frac{(n+1)(n+2)}{2}$ control points. For constructing a triangular patch we use repeated triangular bivariate interpolation. The control net in triangular de Casteljau algorithm for surfaces is of a triangular structure (see Farin, 2002). The structure of control net for a Bézier triangle of degree 2 is in the form

$$\begin{array}{ccccc} & & \mathsf{b}_{0,0} & & \\ & \mathsf{b}_{0,1} & & \mathsf{b}_{1,0} & \\ \mathsf{b}_{0,2} & & \mathsf{b}_{1,1} & & \mathsf{b}_{2,0} \end{array}$$

The de Casteljau algorithm for Bézier triangle is

Algorithm 2.2: Let $\mathsf{b}_{i,j}$, $i = 0,\ldots,n$, $j = 0,\ldots,n-i$ be the control points of the surface. Then compute

$$\mathsf{b}_{i,j}^{r}(u,v) = (1-u-v)\mathsf{b}_{i,j}^{r-1}(u,v) + u\mathsf{b}_{i+1,j}^{r-1}(u,v) + v\mathsf{b}_{i,j+1}^{r-1}(u,v) \tag{2.2.2}$$

for $r = 1,2,\ldots,n$; $i = 0,1,\ldots,n-r$; $j = 0,1,\ldots,n-r-i$ where $0 \leqslant u+v \leqslant 1$ and $\mathsf{b}_{i,j}^{0}(u,v) = \mathsf{b}_{i,j}$. Bézier triangles have the following properties:

1. Affine invariance property: Since

$$\sum_{i=0}^{n}\sum_{j=0}^{n-i} B_{i,j}^{n}(u,v) = (u+v+(1-u-v))^{n} = 1,$$

$S(u,v)$ is an affine combination of its control points and every affine map L leaves the barycentric combinations invariant, that is

$$L\left(\sum_{i=0}^{n}\sum_{j=0}^{n-i} \mathsf{b}_{i,j}B_{i,j}^{n}(u,v)\right) = \sum_{i=0}^{n}\sum_{j=0}^{n-i} L(\mathsf{b}_{i,j})B_{i,j}^{n}(u,v).$$

2. Convex hull property: $S(u,v)$ is in the convex hull of its control points since each basis function $B_{i,j}^{n}(u,v)$ is nonnegative for the parameter values $0 \leqslant u+v \leqslant 1$.

3. Boundary curves: Boundary curves of the surface are determined by the boundary control points. These curves are $\sum\limits_{i=0}^{n} \mathsf{b}_{i,0}B_{i}^{n}(t)$, $\sum\limits_{i=0}^{n} \mathsf{b}_{0,i}B_{i}^{n}(t)$ and $\sum\limits_{i=0}^{n} \mathsf{b}_{i,n-i}B_{i}^{n}(t)$.

4. Corner point interpolation: Since

$$B_{i,j}^{n}(0,0) = \delta_{0,i}\delta_{0,j},\ B_{i,j}^{n}(1,0) = \delta_{n,i}\delta_{0,j},\ B_{i,j}^{n}(0,1) = \delta_{0,i}\delta_{n,j}$$

we have $S(0,0) = \mathsf{b}_{0,0}, S(1,0) = \mathsf{b}_{n,0}$ and $S(0,1) = \mathsf{b}_{0,n}$.

The following theorem gives the difference form of the intermediate points of Algorithm 2.2

Theorem 2.2.1. *The intermediate points of the Algorithm 2.2 can be expressed in terms* *of differences as*

$$\mathsf{b}^m_{r,s} = \sum_{i=0}^{m}\sum_{j=0}^{m-i}\binom{m}{i\ ,\ j}\Delta_1^i\Delta_2^j\mathsf{b}_{r,s}u^iv^j \tag{2.2.3}$$

As a corollary of the theorem one can deduce that a Bézier triangle $S(u,v)$ can be expressed by

$$S(u,v) = \sum_{i=0}^{n}\sum_{j=0}^{n-i}\binom{n}{i\ ,\ j}\Delta_1^i\Delta_2^j\mathsf{b}_{0,0}u^iv^j. \tag{2.2.4}$$

It is also possible to use degree elevation procedure for the Bézier triangles. Take an nth degree Bézier triangle $S(u,v)$,

$$S(u,v) = \sum_{i=0}^{n}\sum_{j=0}^{n-i}\mathsf{b}_{i,j}\binom{n}{i\ ,\ j}u^iv^j(1-u-v)^{n-i-j},$$

multiply both sides of the equation by $(u+v+(1-u-v))$ to get

$$\begin{aligned} S(u,v) = &\sum_{i=0}^{n}\sum_{j=0}^{n-i}\mathsf{b}_{i,j}\binom{n}{i\ ,\ j}u^{i+1}v^j(1-u-v)^{n-i-j} \\ &+\sum_{i=0}^{n}\sum_{j=0}^{n-i}\mathsf{b}_{i,j}\binom{n}{i\ ,\ j}u^iv^{j+1}(1-u-v)^{n-i-j} \\ &+\sum_{i=0}^{n}\sum_{j=0}^{n-i}\mathsf{b}_{i,j}\binom{n}{i\ ,\ j}u^iv^j(1-u-v)^{n+1-i-j}. \end{aligned}$$

Shifting and expanding the index of the summations yield

$$\begin{aligned} S(u,v) = &\sum_{i=0}^{n+1}\sum_{j=0}^{n+1-i}\mathsf{b}_{i-1,j}\binom{n}{i-1\ ,\ j}u^iv^j(1-u-v)^{n+1-i-j} \\ &+\sum_{i=0}^{n+1}\sum_{j=0}^{n+1-i}\mathsf{b}_{i,j-1}\binom{n}{i\ ,\ j-1}u^iv^j(1-u-v)^{n+1-i-j} \\ &+\sum_{i=0}^{n+1}\sum_{j=0}^{n+1-i}\mathsf{b}_{i,j}\binom{n}{i\ ,\ j}u^iv^j(1-u-v)^{n+1-i-j}. \end{aligned}$$

Since $\binom{n}{i-1\ ,\ j} = \frac{i}{n+1}\binom{n+1}{i\ ,\ j}$, $\binom{n}{i\ ,\ j-1} = \frac{j}{n+1}\binom{n+1}{i\ ,\ j}$, and $\binom{n}{i\ ,\ j} = \frac{n+1-i-j}{n+1}\binom{n+1}{i\ ,\ j}$ we have

$$S(u,v) = \sum_{i=0}^{n+1}\sum_{j=0}^{n+1-i}\left\{\frac{i}{n+1}\mathsf{b}_{i-1,j} + \frac{j}{n+1}\mathsf{b}_{i,j-1} + \frac{n+1-i-j}{n+1}\mathsf{b}_{i,j}\right\}B_{i,j}^{n+1}(u,v).$$

Thus $S(u,v)$ can be expressed as a surface of degree $n+1$ with the control points $\mathsf{b}_{i,j}^1$, where

$$\mathsf{b}_{i,j}^1 = \frac{i}{n+1}\mathsf{b}_{i-1,j} + \frac{j}{n+1}\mathsf{b}_{i,j-1} + \left(1 - \frac{i+j}{n+1}\right)\mathsf{b}_{i,j}$$

for $i = 0,\ldots,n+1$, $j = 0,\ldots,n+1-i$.

2.3 Rational Bézier Surfaces

As in the rational Bézier curves, rational Bézier surfaces are obtained as the projection of $4D$ Bézier surface.

2.3.1 Rational tensor product Bézier surfaces

Rational tensor product Bézier surface of degree (m,n) is defined by

$$R(u,v) = \frac{\sum_{i=0}^{m}\sum_{j=0}^{n} w_{i,j}\mathsf{b}_{i,j}B_i^m(u)B_j^n(v)}{\sum_{i=0}^{m}\sum_{j=0}^{n} w_{i,j}B_i^m(u)B_j^n(v)}, \quad 0 \leqslant u,v \leqslant 1. \tag{2.3.1}$$

The control points $\mathsf{b}_{i,j} \in \mathbb{E}^3$ with the weights $w_{i,j} \in \mathbb{R}$ are obtained by projecting the points $[w_{i,j}\mathsf{b}_{i,j} \quad w_{i,j}]^T \in \mathbb{E}^4$ to the hyperplane $w = 1$. Rational tensor product Bézier surfaces have the following properties of their nonrational counterparts.

1. Affine invariance property: The basis functions of rational tensor product Bézier surfaces are

$$\phi_{i,j} = \frac{w_{i,j}B_i^m(u)B_j^n(v)}{\sum_{r=0}^{m}\sum_{s=0}^{n} w_{r,s}B_r^m(u)B_s^n(v)}, \quad i = 0,\ldots,m,\ j = 0,\ldots,n.$$

Since the basis functions sum to one, $R(u,v)$ is affinely invariant.

2. Convex hull property: For the parameter values $0 \leqslant u, v \leqslant 1$ and positive weights the basis functions $\phi_{i,j}$ are nonnegative. Since $\sum_{i=0}^{m}\sum_{j=0}^{n}\phi_{i,j} = 1$, $R(u,v)$ is a convex combination of its control net points. Thus if $w_{i,j} > 0$ then $R(u,v)$ lies in the convex hull of the control net.

3. Boundary curves: Boundary curves of $R(u,v)$ are obtained by projection of boundary curves of projected tensor product Bézier surface. This curves are $R(u,0)$, $R(u,1)$, $R(0,v)$ and $R(1,v)$.

4. Corner point interpolation: Since four corner points of a tensor product Bézier surface coincide with the corner points of its control polygon, their projection also coincide with the control points of the control net of $R(u,v)$. That is

$$R(0,0) = \mathsf{b}_{0,0}, R(0,1) = \mathsf{b}_{0,n}, R(1,0) = \mathsf{b}_{m,0} \text{ and } R(1,1) = \mathsf{b}_{m,n}.$$

In addition to above properties of tensor product Bézier surfaces, rational tensor product Bézier surfaces inherit the projective invariance property.

Note that, although rational tensor product Bézier surfaces are obtained by projection of tensor product surfaces, they are not tensor product surfaces. It comes from the fact that, the basis functions $\phi_{i,j}(u,v)$ cannot be factored in the form $\phi_{i,j}(u,v) = A_i(u)B_j(v)$, (see Farin, 2002).

Projective invariance property allow us to modify Algorithm 2.1 for the rational tensor product Bézier surface. The algorithm is

Algorithm 2.3: Given the control net $\mathsf{b}_{i,j} \in \mathbb{E}^3$, and the corresponding weights $w_{i,j} \in \mathbb{R}$; $i = 0,\ldots,m$, $j = 0\ldots,n$. Compute

$$w_{i,j}^{r,r}\mathsf{b}_{i,j}^{r,r} = \begin{bmatrix} 1-u & u \end{bmatrix} \begin{bmatrix} w_{i,j}^{r-1,r-1}\mathsf{b}_{i,j}^{r-1,r-1} & w_{i,j+1}^{r-1,r-1}\mathsf{b}_{i,j+1}^{r-1,r-1} \\ w_{i+1,j}^{r-1,r-1}\mathsf{b}_{i+1,j}^{r-1,r-1} & w_{i+1,j+1}^{r-1,r-1}\mathsf{b}_{i+1,j+1}^{r-1,r-1} \end{bmatrix} \begin{bmatrix} 1-v \\ v \end{bmatrix} \quad (2.3.2)$$

for $r = 1, \ldots, k$, $i = 0, \ldots, m-r$, $j = 0, \ldots, n-r$, where $k = \min(m,n)$ and

$$w_{i,j}^{r,r} = \begin{bmatrix} 1-u & u \end{bmatrix} \begin{bmatrix} w_{i,j}^{r-1,r-1} & w_{i,j+1}^{r-1,r-1} \\ w_{i+1,j}^{r-1,r-1} & w_{i+1,j+1}^{r-1,r-1} \end{bmatrix} \begin{bmatrix} 1-v \\ v \end{bmatrix}$$

As in nonrational case we turn to de Casteljau algorithm of rational Bézier curves after kth application of Algorithm 2.3.

The degree elevation procedure for tensor product surfaces can be extended for rational tensor product Bézier surfaces. Because the similarity we will not give the required points to obtain a rational tensor product Bézier surface of degree (m,n) as one of more degree. The following theorem gives the difference form of rational tensor product Bézier surfaces.

Theorem 2.3.1. *$R(u,v)$ can be expressed in terms of differences by*

$$R(u,v) = \frac{\sum_{i=0}^{m}\sum_{j=0}^{n}\binom{m}{i}\binom{n}{j}\Delta_1^i\Delta_2^j(w_{0,0}\mathsf{b}_{0,0})u^iv^j}{\sum_{i=0}^{m}\sum_{j=0}^{n}\binom{m}{i}\binom{n}{j}\Delta_1^i\Delta_2^j w_{0,0}u^iv^j}. \tag{2.3.3}$$

Proof. Let $R(u,v)$ obtained by projection of tensor product Bézier surface $S(u,v)$. The control points of $S(u,v)$ are $\mathsf{c}_{i,j} = [w_{i,j}\mathsf{b}_{i,j} \quad w_{i,j}]^T$, $i = 0, \ldots, m$, $j = 0, \ldots, n$. Then from Theorem 2.1.1 we have

$$S(u,v) = \sum_{i=0}^{m}\sum_{j=0}^{n}\binom{m}{i}\binom{n}{j}\Delta_1^i\Delta_2^j\mathsf{c}_{0,0}u^iv^j. \tag{2.3.4}$$

Projecting (2.3.4) to the hyperplane gives

$$R(u,v) = \frac{\sum_{i=0}^{m}\sum_{j=0}^{n}\binom{m}{i}\binom{n}{j}\Delta_1^i\Delta_2^j(w_{0,0}\mathsf{b}_{0,0})u^iv^j}{\sum_{i=0}^{m}\sum_{j=0}^{n}\binom{m}{i}\binom{n}{j}\Delta_1^i\Delta_2^j w_{0,0}u^iv^j}.$$

$\square$

2.3.2 Rational Bézier Triangles

Rational Bézier triangle of degree n is defined by

$$R(u,v) = \frac{\sum_{i=0}^{n}\sum_{j=0}^{n-i} w_{i,j}\mathsf{b}_{i,j}B_{i,j}^{n}(u,v)}{\sum_{i=0}^{n}\sum_{j=0}^{n-i} w_{i,j}B_{i,j}^{n}(u,v)}, \quad 0 \leqslant u,v \leqslant 1 \text{ and } 0 \leqslant u+v \leqslant 1. \tag{2.3.5}$$

These surfaces have the following properties:

1. Affine invariance property
2. Convex hull property
3. Boundary curves
4. Corner point interpolation property
5. Projective invariance property

As in rational tensor product Bézier surfaces, projective invariance property is important that will lead us to a de Casteljau algorithm for computing rational $q-$ Bernstein Bézier triangles. Furthermore, using projective invariance property, we are able to express each intermediate point of de Casteljau algorithm and consequently rational Bézier triangle in terms of differences. The following is the de Casteljau type algorithm for rational $q-$Bernstein Bézier triangles.

Algorithm 2.4: Let $\mathsf{b}_{i,j}$, $i = 0,\ldots,n$, $j = 0,\ldots,n-i$ be the control points and the real values $w_{i,j}$ be associated weights. Compute

$$w_{i,j}^{r}\mathsf{b}_{i,j}^{r} = (1-u-v)w_{i,j}^{r-1}\mathsf{b}_{i,j}^{r-1} + uw_{i+1,j}^{r-1}\mathsf{b}_{i+1,j}^{r-1} + vw_{i,j+1}^{r-1}\mathsf{b}_{i,j+1}^{r-1} \tag{2.3.6}$$

for $r = 1,\ldots,n$, $i = 0,\ldots,n-r$, $j = 0,\ldots,n-r-i$ where $0 \leqslant u+v \leqslant 1$ and

$$w_{i,j}^{r} = (1-u-v)w_{i,j}^{r-1} + uw_{i+1,j}^{r-1} + vw_{i,j+1}^{r-1}.$$

The following theorem can be proved by using the projective invariance property

Theorem 2.3.2. *The intermediate points of Algorithm 2.4 can be expressed as*

$$\mathsf{b}_{r,s}^{m} = \frac{\sum_{i=0}^{m}\sum_{j=0}^{m-i}\binom{n}{i\,,\,j}\Delta_1^i\Delta_2^j(w_{r,s}\mathsf{b}_{r,s})u^iv^j}{\sum_{i=0}^{m}\sum_{j=0}^{m-i}\binom{n}{i\,,\,j}\Delta_1^i\Delta_2^j w_{r,s}u^iv^j}. \tag{2.3.7}$$

Corollary 2.3.1. *The rational Bézier triangle is*

$$R(u,v) = \mathsf{b}_{0,0}^{n} = \frac{\sum_{i=0}^{n}\sum_{j=0}^{n-i}\binom{n}{i\,,\,j}\Delta_1^i\Delta_2^j(w_{0,0}\mathsf{b}_{0,0})u^iv^j}{\sum_{i=0}^{n}\sum_{j=0}^{n-i}\binom{n}{i\,,\,j}\Delta_1^i\Delta_2^j w_{0,0}u^iv^j}.$$

The degree elevation procedure can be used for rational Bézier triangles. To find new control points first, we degree elevate the projected Bézier triangle $S(u,v) \in \mathbb{E}^4$ and then project each new control point. Thus, to express $R(u,v)$ as a surface of degree $n+1$, we need the following control points

$$\mathsf{b}_{i,j}^{1} = \frac{\frac{i}{n+1}w_{i-1,j}\mathsf{b}_{i-1,j} + \frac{j}{n+1}w_{i,j-1}\mathsf{b}_{i,j-1} + \frac{n+1-i-j}{n+1}w_{i,j}\mathsf{b}_{i,j}}{w_{i,j}^{1}}$$

where the weights $w_{i,j}^1$ are

$$w_{i,j}^{1} = \frac{i}{n+1}w_{i-1,j} + \frac{j}{n+1}w_{i,j-1} + \left(1 - \frac{i+j}{n+1}\right)w_{i,j}.$$

CHAPTER THREE
A GENERALIZATION of BÉZIER SURFACES

First, a two-parameter family of tensor product Bézier surfaces is defined. Then we give the change of basis matrix for tensor product $q-$Bernstein Bézier surfaces. In section 3.2 the generalization of Bézier triangles is given. The rational counterparts of generalized Bézier surfaces are given in section 3.3. Finally, the convergence properties of tensor product $q-$Bernstein Bézier surfaces and $q-$Bézier triangles are investigated in section 3.4.

3.1 Tensor Product $q-$Bernstein Bézier Surfaces

In this section we now introduce a two-parameter family of tensor product Bézier surfaces using $q-$Bernstein polynomials defined in Chapter 1. A special case of this surfaces, when the parameter values equal to one, it gives standard tensor product Bézier surfaces. We define a two-parameter tensor product Bézier surfaces, we will call tensor product $q-$Bernstein Bézier surface of degree (m,n) by

$$S(u,v) = \sum_{i=0}^{m} \sum_{j=0}^{n} \mathsf{b}_{i,j} B_i^{m,q_1}(u) B_j^{n,q_2}(v) \tag{3.1.1}$$

where $\mathsf{b}_{i,j} \in \mathbb{E}^3$, $i = 0,\ldots,m$, $j = 0,\ldots,n$ are control points, $B_i^{m,q_1}(u)$ are $q-$Bernstein polynomials of degree m in u with parameter value q_1 and $B_j^{n,q_2}(v)$ are $q-$Bernstein polynomials of degree n in v, with the parameter value q_2. It is not surprising that the parameters q_1 and q_2 add extra flexibility to the basis functions and hence they vary the shape of the Bézier surfaces. A change in q_1, q_2 results a different surface with the same control net. In Figure 3.1 we have two surfaces with same control net but different parameter values.

Properties:

1. Affine invariance property: Since $\sum_{i=0}^{m} \sum_{j=0}^{n} B_i^{m,q_1}(u) B_j^{n,q_2}(v) = 1$, $S(u,v)$ is an affine combination of its control net points. Thus $S(u,v)$ is affinely invariant.

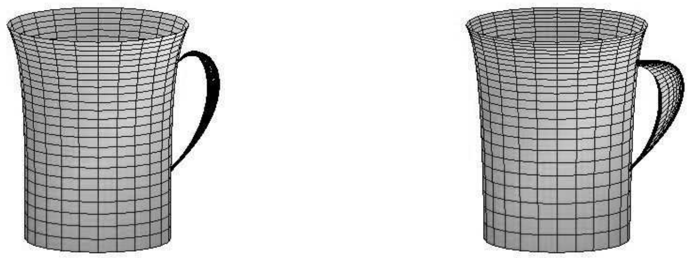

Figure 3.1 Two tensor product q-Bernstein Bézier surfaces
with different values of q.

2. Convex hull property: When $0 < q_1, q_2 \leqslant 1$, the basis polynomials are nonnegative and form partition of unity property. Thus, $S(u,v)$ is a convex combination of $\mathsf{b}_{i,j}$ and lies in the convex hull of its control net points.

3. Boundary curves: Boundary curves of $S(u,v)$ are evaluated by $S(u,0)$, $S(u,1)$, $s(0,v)$ and $S(1,v)$.

4. Corner point interpolation: The corner control net points coincide with the four corners of the surface.

Algorithm 2.1 can be modified by using Algorithm 1.3 to compute tensor product $q-$Bernstein Bézier surface by a de Casteljau type algorithm. This algorithm is

Algorithm 3.1: Given the control net $\mathsf{b}_{i,j} \in \mathbb{E}^3$; $i = 0\ldots,m$, $j = 0,\ldots,n$. Compute

$$\mathsf{b}_{i,j}^{r,r} = \left[\begin{array}{cc} 1-q_1^{r-i-1}u & q_1^{r-i-1}u \end{array} \right] \left[\begin{array}{cc} \mathsf{b}_{i,j}^{r-1,r-1} & \mathsf{b}_{i,j+1}^{r-1,r-1} \\ \mathsf{b}_{i+1,j}^{r-1,r-1} & \mathsf{b}_{i+1,j+1}^{r-1,r-1} \end{array} \right] \left[\begin{array}{c} 1-q_2^{r-j-1}v \\ q_2^{r-j-1}v \end{array} \right] \quad (3.1.2)$$

for $r = 1,\ldots,k$, $i = 0,\ldots,m-r$, $j = 0,\ldots,n-r$ where $k = \min(m,n)$.

Another way to evaluate a point on the surface $S(u,v)$ is that, first for each $j = 0,1,\ldots,n$ use Algorithm 1.3 in $u-$direction with parameter value q_1 and control points $\mathsf{b}_{0,j}, \mathsf{b}_{1,j}, \ldots, \mathsf{b}_{m,j}$ to obtain a $q-$Bernstein Bézier curve $\mathsf{b}^m_{0,j}$; $j = 0, \ldots, n$. Then apply Algorithm 1.3 in $v-$direction with parameter value q_2 to the control points $\mathsf{b}^m_{0,j}$; $j = 0, \ldots, n$.

Using this idea it is possible to express each intermediate point of Algorithm 3.1 explicitly

Theorem 3.1.1. *The intermediate points of Algorithm 3.1 are*

$$\mathsf{b}^{r,r}_{i,j} = \sum_{k=0}^{r} \sum_{l=0}^{r} q_1^{-ri} q_2^{-rj} \mathsf{b}_{i+k,j+l} \begin{bmatrix} r \\ k \end{bmatrix}_{q_1} u^k \prod_{s=0}^{r-k-1} (q_1^i - q_1^s u) \begin{bmatrix} r \\ l \end{bmatrix}_{q_2} v^l \prod_{s=0}^{r-l-1} (q_2^j - q_2^s v). \quad (3.1.3)$$

where $\begin{bmatrix} r \\ k \end{bmatrix}_{q_1}$ *and* $\begin{bmatrix} r \\ l \end{bmatrix}_{q_2}$ *are $q-$binomial coefficients* $\begin{bmatrix} r \\ k \end{bmatrix}$ *and* $\begin{bmatrix} r \\ l \end{bmatrix}$ *with replacing q by q_1 and q_2 respectively.*

Proof. First, apply r steps of Algorithm 1.3 in $u-$direction and parameter value q_1 to the points $\mathsf{b}_{0,j}, \ldots, \mathsf{b}_{m,j}$ for $j = 0, 1 \ldots, n$. Hence the resulting points are $\mathsf{b}^r_{0,j}, \ldots, \mathsf{b}^r_{m-r,j}$; $j = 0, \ldots, n$. Now apply r steps of Algorithm 1.3 in $v-$direction and parameter value q_2 to the points $\mathsf{b}^r_{i,0}, \ldots, \mathsf{b}^r_{i,n}$; $i = 0, \ldots, m-r$ to obtain the point $\mathsf{b}^{r,r}_{i,j}$. Since $\mathsf{b}^{r,r}_{i,j}$ is obtained by Algorithm 1.3 it can be expressed, (see Dişibüyük & Oruç, 2008), as

$$\mathsf{b}^{r,r}_{i,j} = \sum_{l=0}^{r} q_2^{-rj} \mathsf{b}^r_{i,j+l} \begin{bmatrix} r \\ l \end{bmatrix}_{q_2} v^l \prod_{s=0}^{r-l-1} (q_2^j - q_2^s v) \quad (3.1.4)$$

But the points $\mathsf{b}^r_{i,j+l}$ are also obtained from Algorithm 1.3 which may expressed as

$$\mathsf{b}^r_{i,j+l} = \sum_{k=0}^{r} q_1^{-ri} \mathsf{b}_{i+k,j+l} \begin{bmatrix} r \\ k \end{bmatrix}_{q_1} u^k \prod_{s=0}^{r-k-1} (q_1^i - q_1^s u). \quad (3.1.5)$$

Substituting the last equation in (3.1.4) we obtain

$$\mathsf{b}^{r,r}_{i,j} = \sum_{k=0}^{r} \sum_{l=0}^{r} q_1^{-ri} q_2^{-rj} \mathsf{b}_{i+k,j+l} \begin{bmatrix} r \\ k \end{bmatrix}_{q_1} u^k \prod_{s=0}^{r-k-1} (q_1^i - q_1^s u) \begin{bmatrix} r \\ l \end{bmatrix}_{q_2} v^l \prod_{s=0}^{r-l-1} (q_2^j - q_2^s v).$$

□

We now give $q-$differences form of tensor product $q-$Bernstein Bézier surfaces.

$q-$difference form will lead us some properties on convergence of tensor product $q-$Bernstein Bézier surfaces. These properties will be discussed in the following sections. First we define $q-$differences in $u-$direction by

$$\Delta_{q_1}^{k+1}\mathsf{b}_{i,j} = \Delta_{q_1}^{k}\mathsf{b}_{i+1,j} - q_1^k\Delta_{q_1}^{k}\mathsf{b}_{i,j}$$

for all $k \geqslant 0$ with $\Delta_{q_1}^{0}\mathsf{b}_{i,j} = \mathsf{b}_{i,j}$ and the $q-$differences in $v-$direction by

$$\Delta_{q_2}^{k+1}\mathsf{b}_{i,j} = \Delta_{q_2}^{k}\mathsf{b}_{i,j+1} - q_2^k\Delta_{q_2}^{k}\mathsf{b}_{i,j}$$

for all $k \geqslant 0$ with $\Delta_{q_2}^{0}\mathsf{b}_{i,j} = \mathsf{b}_{i,j}$ (see Phillips, 2003). Then we also define

$$\Delta_{q_1}\Delta_{q_2}\mathsf{b}_{i,j} = \Delta_{q_1}(\Delta_{q_2}\mathsf{b}_{i,j}).$$

Note that $\Delta_{q_1}^{i}$ and $\Delta_{q_2}^{j}$ commute, that is

$$\Delta_{q_1}^{i}\Delta_{q_2}^{j}\mathsf{b}_{i,j} = \Delta_{q_2}^{j}\Delta_{q_1}^{i}\mathsf{b}_{i,j}.$$

Theorem 3.1.2. *$S(u,v)$ can be expressed in terms of $q-$differences by*

$$S(u,v) = \sum_{i=0}^{m}\sum_{j=0}^{n}\begin{bmatrix} m \\ i \end{bmatrix}_{q_1}\begin{bmatrix} n \\ j \end{bmatrix}_{q_2}\Delta_{q_1}^{i}\Delta_{q_2}^{j}\mathsf{b}_{0,0}u^iv^j. \tag{3.1.6}$$

Proof. First, write tensor product Bézier surface in the form

$$S(u,v) = \sum_{j=0}^{n} b_j B_j^{n,q_2}(v) \text{ where } b_j = \sum_{i=0}^{m}\mathsf{b}_{i,j}B_i^{m,q_1}(u).$$

Using $q-$difference form of Bézier curves we write b_j in the form, (see Dişibüyük & Oruç, 2008)

$$b_j = \sum_{i=0}^{m}\begin{bmatrix} m \\ i \end{bmatrix}_{q_1}\Delta_{q_1}^{i}\mathsf{b}_{0,j}u^i.$$

Thus

$$S(u,v) = \sum_{j=0}^{n}\left\{\sum_{i=0}^{m}\begin{bmatrix} m \\ i \end{bmatrix}_{q_1}\Delta_{q_1}^{i}\mathsf{b}_{0,j}u^i\right\}B_j^{n,q_2}(v)$$

and after one more rearrangement we will have

$$S(u,v) = \sum_{i=0}^{m} \begin{bmatrix} m \\ i \end{bmatrix}_{q_1} \left\{ \sum_{j=0}^{n} \Delta_{q_1}^{i} \mathsf{b}_{0,j} B_j^{n,q_2}(v) \right\} u^i.$$

For $i = 0, \ldots, m$, the expression in the curly brackets are $q-$Bernstein Bézier curves in v with the control points $\Delta_{q_1}^{i} \mathsf{b}_{0,j}$, writing $q-$difference form of these curves and using the commutativity property we obtain (3.1.6). $\square$

It is also possible to degree elevate a two-parameter tensor product Bézier surface $S(u,v)$ of degree (m,n) as in the standard case. The control points

$$\mathsf{b}_{i,j}^{(1,0)} = \left(1 - \frac{[m+1-i]_{q_1}}{[m+1]_{q_1}}\right) \mathsf{b}_{i-1,j} + \frac{[m+1-i]_{q_1}}{[m+1]_{q_1}} \mathsf{b}_{i,j}, \ i = 0, \ldots, m+1, \ j = 0, \ldots, n$$

gives $S(u,v)$ as a surface of degree $(m+1,n)$, where $[i]_{q_1}$ denotes the $q-$integer $[i]$ with the parameter value q_1. Similarly, to obtain the same surface of degree $(m,n+1)$ we need new control points $\mathsf{b}_{i,j}^{(0,1)}$ such that

$$\mathsf{b}_{i,j}^{(0,1)} = \left(1 - \frac{[n+1-j]_{q_2}}{[n+1]_{q_2}}\right) \mathsf{b}_{i,j-1} + \frac{[n+1-j]_{q_2}}{[n+1]_{q_2}} \mathsf{b}_{i,j}, \ i = 0, \ldots, m, \ j = 0, \ldots, n+1.$$

Finally, to obtain $S(u,v)$ as a surface of degree $(m+1,n+1)$ new control points evaluated from the product

$$\mathsf{b}_{i,j}^{(1,1)} = \begin{bmatrix} 1 - \frac{[m+1-i]_{q_1}}{[m+1]_{q_1}} & \frac{[m+1-i]_{q_1}}{[m+1]_{q_1}} \end{bmatrix} \begin{bmatrix} \mathsf{b}_{i-1,j-1} & \mathsf{b}_{i-1,j} \\ \mathsf{b}_{i,j-1} & \mathsf{b}_{i,j} \end{bmatrix} \begin{bmatrix} 1 - \frac{[n+1-j]_{q_2}}{[n+1]_{q_2}} \\ \frac{[n+1-j]_{q_2}}{[n+1]_{q_2}} \end{bmatrix}.$$

The repeated degree elevation procedure can be computed by following the univariate case described in (Oruç & Phillips, 2003).

3.1.1 Matrix Form and Change of Basis

The tensor product $q-$Bernstein Bézier patch can be written in matrix form as

$$S(u,v) = [B_0^{m,q_1}(u),\ldots,B_m^{m,q_1}(u)]\begin{bmatrix} \mathsf{b}_{0,0} & \cdots & \mathsf{b}_{0,n} \\ \vdots & & \vdots \\ \mathsf{b}_{m,0} & \cdots & \mathsf{b}_{m,n} \end{bmatrix}\begin{bmatrix} B_0^{n,q_2}(v) \\ \vdots \\ B_n^{n,q_2}(v) \end{bmatrix}$$

The basis of the tensor product polynomial space $\mathbb{P}_m \otimes \mathbb{P}_n$ has dimension $(m+1)(n+1)$ and each basis element may be in the form $u^i v^j,\ i = 0,1,\ldots,m,\ j = 0,1,\ldots,n$. For simplicity we take $m = n$ and $q_1 = q_2 = q$. Let $C = [\mathbf{c}_0,\mathbf{c}_1,\ldots,\mathbf{c}_n]^T$ be a $(n+1)^2 \times 1$ block vector with elements $\mathbf{c}_i = [u^i, u^i v,\ldots,u^i v^n]^T$ and let $B^q = [\mathbf{b}_0,\mathbf{b}_1,\ldots,\mathbf{b}_n]^T$ be a block matrix with block elements

$$\mathbf{b}_i = [B_i^{n,q}(u)B_0^{n,q}(v), B_i^{n,q}(u)B_1^{n,q}(v),\ldots,B_i^{n,q}(u)B_n^{n,q}(v)]^T$$

for $i = 0,\ldots,n$. Since the tensor product $q-$Bernstein Bézier surfaces span the space of tensor product polynomials, there exists a transformation matrix $M^{n,q}$ such that

$$B^q = M^{n,q}C.$$

Let us consider

$$B_i^{n,q}(u)B_j^{n,q}(v) = \begin{bmatrix} n \\ i \end{bmatrix} u^i \prod_{s=0}^{n-i-1}(1-q^s u)\begin{bmatrix} n \\ j \end{bmatrix} v^j \prod_{s=0}^{n-j-1}(1-q^s v).$$

Using the property (1.2.6) we deduce that

$$B_i^{n,q}(u)B_j^{n,q}(v) = \sum_{k=i}^{n}(-1)^{k-i}q^{\binom{k-i}{2}}\begin{bmatrix} n \\ i \end{bmatrix}\begin{bmatrix} n-i \\ k-i \end{bmatrix}u^k \sum_{l=j}^{n}(-1)^{l-j}q^{\binom{l-j}{2}}\begin{bmatrix} n \\ j \end{bmatrix}\begin{bmatrix} n-j \\ l-j \end{bmatrix}v^l.$$

Rearranging the terms using the definition (1.2.2) we have

$$B_i^{n,q}(u)B_j^{n,q}(v) = \sum_{k=0}^{n}\sum_{l=0}^{n}(-1)^{(k+l)-(i+j)}q^{\binom{k-i}{2}}q^{\binom{l-j}{2}}\begin{bmatrix} n \\ i \end{bmatrix}\begin{bmatrix} n-i \\ k-i \end{bmatrix}\begin{bmatrix} n \\ j \end{bmatrix}\begin{bmatrix} n-j \\ l-j \end{bmatrix}u^k v^l.$$

Since

$$\begin{bmatrix} n-i \\ k-i \end{bmatrix} = \frac{\begin{bmatrix} n \\ k \end{bmatrix}\begin{bmatrix} k \\ i \end{bmatrix}}{\begin{bmatrix} n \\ i \end{bmatrix}} \tag{3.1.7}$$

we obtain

$$B_i^{n,q}(u)B_j^{n,q}(v) = \sum_{k=0}^{n}\sum_{l=0}^{n}(-1)^{(k+l)-(i+j)}q^{\binom{k-i}{2}+\binom{l-j}{2}}\begin{bmatrix} n \\ k \end{bmatrix}\begin{bmatrix} k \\ i \end{bmatrix}\begin{bmatrix} n \\ l \end{bmatrix}\begin{bmatrix} l \\ j \end{bmatrix}u^k v^l.$$

As a consequence, one may write $B^q = M^{n,q}C$ where $M^{n,q}$ is an upper triangular block matrix with a generic element

$$\left((M_{ij}^{n,q})_{k,l=0}^{n}\right)_{i,j=0}^{n} = (-1)^{(j+l)-(i+k)}q^{\binom{j-i}{2}+\binom{l-k}{2}}\begin{bmatrix} n \\ j \end{bmatrix}\begin{bmatrix} j \\ i \end{bmatrix}\begin{bmatrix} n \\ l \end{bmatrix}\begin{bmatrix} l \\ k \end{bmatrix}.$$

Conversely, to express the monomial basis in terms of the $q-$Bernstein basis we multiply the equation

$$\sum_{k=0}^{n-i}B_k^{n-i,q}(u)\sum_{l=0}^{n-j}B_j^{n-l,q}(v) = 1$$

by $u^i v^j$. Then we have

$$u^i v^j = \sum_{k=0}^{n-i}\begin{bmatrix} n-i \\ k \end{bmatrix}u^{i+k}\prod_{s=0}^{n-i-k-1}(1-q^s u)\sum_{l=0}^{n-j}\begin{bmatrix} n-j \\ l \end{bmatrix}v^{j+l}\prod_{s=0}^{n-j-l-1}(1-q^s v).$$

Shifting the limits of the sums and rearranging the terms using the equation (3.1.7) yields

$$u^i v^j = \sum_{k=i}^{n}\frac{\begin{bmatrix} k \\ i \end{bmatrix}}{\begin{bmatrix} n \\ i \end{bmatrix}}B_k^{n,q}(u)\sum_{l=j}^{n}\frac{\begin{bmatrix} l \\ j \end{bmatrix}}{\begin{bmatrix} n \\ j \end{bmatrix}}B_l^{n,q}(v).$$

From definition (1.2.2) one may write

$$u^i v^j = \sum_{k=0}^{n}\sum_{l=0}^{n}\frac{\begin{bmatrix} k \\ i \end{bmatrix}\begin{bmatrix} l \\ j \end{bmatrix}}{\begin{bmatrix} n \\ i \end{bmatrix}\begin{bmatrix} n \\ j \end{bmatrix}}B_k^{n,q}(u)B_l^{n,q}(v).$$

It follows that $C = \tilde{M}^{n,q}B^q$ where $\tilde{M}^{n,q}$ is a block matrix with a generic element

$$\left((\tilde{M}_{i,j}^{n,q})_{k,l=0}^{n}\right)_{i,j=0}^{n} = \frac{\begin{bmatrix} j \\ i \end{bmatrix}\begin{bmatrix} l \\ k \end{bmatrix}}{\begin{bmatrix} n \\ i \end{bmatrix}\begin{bmatrix} n \\ k \end{bmatrix}}.$$

Note that $\tilde{M}^{n,q}$ is an upper triangular block matrix and $(\tilde{M}^{n,q})^{-1} = M^{n,q}$. Now, we find a transformation matrix between the $q-$Bernstein basis and the standard Bernstein basis. Since $C = \tilde{M}^{n,q}B^q$, we express the monomial basis in terms of standard Bernstein basis when $q = 1$. So, we have

$$C = \tilde{M}^{n,1}B^1,$$

where B^1 is the standard tensor product Bernstein basis and the matrix $\tilde{M}^{n,1}$ is a block matrix with a generic element

$$\left((\tilde{M}^{n,1}_{i,j})^n_{k,l=0}\right)^n_{i,j=0} = \frac{\binom{j}{i}\binom{l}{k}}{\binom{n}{i}\binom{n}{k}}.$$

Thus

$$\tilde{M}^{n,q}B^q = \tilde{M}^{n,1}B^1.$$

Premultiplying both sides by the matrix $M^{n,q}$, we obtain

$$B^q = T^{n,q,1}B^1,$$

where $T^{n,q,1} = M^{n,q}\tilde{M}^{n,1}$. It is worth noting that the transformation matrix $T^{n,q,1}$ makes it possible to exchange $q-$Bernstein Bézier and standard Bézier representations of the surface $S(u,v)$.

3.2 A Generalization of Bézier Triangles

In order to construct a one-parameter family Bézier triangles, called $q-$Bézier triangles we generalize Algorithm 2.2 using Algorithm 1.3. For a given triangular array of points $\mathsf{b}_{i,j}$, $i = 0,\ldots,n$, $j = 0,\ldots,n-i$ and a fixed real q, $0 < q \leqslant 1$ we modify the de Casteljau type algorithm as follows:

Algorithm 3.2: Given triangular array of points $\mathsf{b}_{i,j}$, compute

$$\mathsf{b}^m_{r,s} = (1 - q^{m-r-1}u - q^{m-s-1}v)\mathsf{b}^{m-1}_{r,s} + q^{m-r-1}u\mathsf{b}^{m-1}_{r+1,s} + q^{m-s-1}v\mathsf{b}^{m-1}_{r,s+1}$$

for $m = 1,2,\ldots,n$, $r = 0,1,\ldots,n-m$, $s = 0,1,\ldots,n-m-r$ where $0 \leqslant u+v \leqslant 1$.

Many properties of $q-$Bézier triangles can be stated based on this algorithm. These properties are:

1) Affine invariance property: At each step the intermediate points of Algorithm 3.2 is affine combination of the intermediate points that obtained in the previous step. Thus the $q-$Bézier triangles are affinely invariant.

2) Boundary curves: Boundary curves of the surfaces are obtained as $q-$Bernstein Bézier curves which control points are the boundary points of the control net.

3) Corner point interpolation: If we take $u = v = 0$ in Algorithm 3.2 then each intermediate point will be equal to the point $\mathsf{b}_{0,0}$. Hence $\mathsf{b}^n_{0,0}(0,0) = \mathsf{b}_{0,0}$. If we take $u = 0$ the the algorithm will be in the form

$$\mathsf{b}^m_{r,s} = (1 - q^{m-s-1}v)\mathsf{b}^{m-1}_{r,s} + q^{m-s-1}v\mathsf{b}^{m-1}_{r,s+1}$$

and when $r = 0$ the above expression turn into the form Algorithm 1.3 with the control points $\mathsf{b}_{0,j}$, $j = 0, \ldots, n$ and from the end point interpolation property of $q-$Bernstein Bézier curves we have $\mathsf{b}^n_{0,0}(0,1) = \mathsf{b}_{0,n}$. Similarly we can say that $\mathsf{b}_{n,0}$ is on the surface. Namely $\mathsf{b}^n_{0,0}(1,0) = \mathsf{b}_{n,0}$.

The following result shows that each intermediate point of Algorithm 3.2 can be written explicitly in terms of $q-$differences where we define $q-$differences by

$$\Delta^k_1\mathsf{b}_{r,s} = \Delta^{k-1}_1\mathsf{b}_{r+1,s} - q^{k-1}\Delta^{k-1}_1\mathsf{b}_{r,s}$$

and

$$\Delta^k_2\mathsf{b}_{r,s} = \Delta^{k-1}_2\mathsf{b}_{r,s+1} - q^{k-1}\Delta^{k-1}_2\mathsf{b}_{r,s}$$

for $k \geqslant 0$ and $\Delta^0_1\mathsf{b}_{i,j} = \Delta^0_1\mathsf{b}_{i,j} = \mathsf{b}_{i,j}$. Note that Δ^k_1 and Δ^l_2 commute.

The next result generalizes (2.2.4), the difference form of standard Bézier triangles.

Theorem 3.2.1. *The intermediate points of Algorithm 3.2 are*

$$\mathsf{b}_{r,s}^{m} = \sum_{i=0}^{m}\sum_{j=0}^{m-i} q^{ij} q^{-ri-sj} \begin{bmatrix} m \\ i+j \end{bmatrix} \binom{i+j}{i} \Delta_1^i \Delta_2^j \mathsf{b}_{r,s} u^i v^j. \tag{3.2.1}$$

Proof. We use induction on m. When $m = 0$ we have

$$\mathsf{b}_{r,s}^{0} = \sum_{i=0}^{0}\sum_{j=0}^{0-i} q^{ij} q^{-ri-sj} \begin{bmatrix} m \\ i+j \end{bmatrix} \binom{i+j}{i} \Delta_1^i \Delta_2^j \mathsf{b}_{r,s} u^i v^j = \Delta_1^0 \Delta_2^0 \mathsf{b}_{r,s} = \mathsf{b}_{r,s}.$$

Now assume that (3.2.1) holds for $m-1$. That is

$$\mathsf{b}_{r,s}^{m-1} = \sum_{i=0}^{m-1}\sum_{j=0}^{m-1-i} q^{ij} q^{-ri-sj} \begin{bmatrix} m-1 \\ i+j \end{bmatrix} \binom{i+j}{i} \Delta_1^i \Delta_2^j \mathsf{b}_{r,s} u^i v^j.$$

Putting $\mathsf{b}_{r,s}^{m-1}$ on the right of Algorithm 3.2 gives

$$\begin{aligned} \mathsf{b}_{r,s}^{m} = (1 - q^{m-r-1}u - q^{m-s-1}v) &\sum_{i=0}^{m-1}\sum_{j=0}^{m-i-1} q^{ij} q^{-ri-sj} \begin{bmatrix} m-1 \\ i+j \end{bmatrix} \binom{i+j}{i} \Delta_1^i \Delta_2^j \mathsf{b}_{r,s} u^i v^j \\ + q^{m-r-1}u &\sum_{i=0}^{m-1}\sum_{j=0}^{m-i-1} q^{ij} q^{-(r+1)i-sj} \begin{bmatrix} m-1 \\ i+j \end{bmatrix} \binom{i+j}{i} \Delta_1^i \Delta_2^j \mathsf{b}_{r+1,s} u^i v^j \\ + q^{m-s-1}v &\sum_{i=0}^{m-1}\sum_{j=0}^{m-i-1} q^{ij} q^{-ri-(s+1)j} \begin{bmatrix} m-1 \\ i+j \end{bmatrix} \binom{i+j}{i} \Delta_1^i \Delta_2^j \mathsf{b}_{r,s+1} u^i v^j. \end{aligned}$$

Let us split the last equation into three by

$$\mathsf{b}_{r,s}^{m} = \alpha + \beta + \gamma$$

where

$$\alpha = \sum_{i=0}^{m-1}\sum_{j=0}^{m-i-1} q^{ij}q^{-ri-sj}\begin{bmatrix} m-1 \\ i+j \end{bmatrix}\binom{i+j}{i}\Delta_1^i\Delta_2^j\mathsf{b}_{r,s}u^iv^j,$$

$$\beta = q^{m-r-1}u\sum_{i=0}^{m-1}\sum_{j=0}^{m-i-1} q^{ij}q^{-ri-sj}\begin{bmatrix} m-1 \\ i+j \end{bmatrix}\binom{i+j}{i}(q^{-i}\Delta_1^i\Delta_2^j\mathsf{b}_{r+1,s}-\Delta_1^i\Delta_2^j\mathsf{b}_{r,s})u^iv^j$$

and

$$\gamma = q^{m-s-1}v\sum_{i=0}^{m-1}\sum_{j=0}^{m-i-1} q^{ij}q^{-ri-sj}\begin{bmatrix} m-1 \\ i+j \end{bmatrix}\binom{i+j}{i}(q^{-j}\Delta_1^i\Delta_2^j\mathsf{b}_{r,s+1}-\Delta_1^i\Delta_2^j\mathsf{b}_{r,s})u^iv^j.$$

Now rearrange $\alpha, \beta,$ and γ independently. Since $\left[\begin{smallmatrix} m-1 \\ i+j \end{smallmatrix}\right] = 0$ for $i = m$ and for $j = m-i$, we can write

$$\alpha = \sum_{i=0}^{m}\sum_{j=0}^{m-i} q^{ij}q^{-ri-sj}\begin{bmatrix} m-1 \\ i+j \end{bmatrix}\binom{i+j}{i}\Delta_1^i\Delta_2^j\mathsf{b}_{r,s}u^iv^j.$$

Shifting the index of the first summation in β gives

$$\beta = \sum_{i=1}^{m}\sum_{j=0}^{m-i} q^{(i-1)j}q^{-ri-sj}q^{m-1}\begin{bmatrix} m-1 \\ i+j-1 \end{bmatrix}\binom{i+j-1}{i-1}$$

$$\times(q^{-i+1}\Delta_1^{i-1}\Delta_2^j\mathsf{b}_{r+1,s}-\Delta_1^{i-1}\Delta_2^j\mathsf{b}_{r,s})u^iv^j.$$

$\binom{i+j-1}{i-1} = 0$ for $i = 0$, thus the last equation written as

$$\beta = \sum_{i=0}^{m}\sum_{j=0}^{m-i} q^{ij}q^{-ri-sj}q^{m-j-1}\begin{bmatrix} m-1 \\ i+j-1 \end{bmatrix}\binom{i+j-1}{i-1}$$

$$\times(q^{-i+1}\Delta_1^{i-1}\Delta_2^j\mathsf{b}_{r+1,s}-\Delta_1^{i-1}\Delta_2^j\mathsf{b}_{r,s})u^iv^j.$$

Since

$$\begin{aligned} q^{-i+1}\Delta_1^{i-1}\Delta_2^j\mathsf{b}_{r+1,s}-\Delta_1^{i-1}\Delta_2^j\mathsf{b}_{r,s} &= q^{-i+1}(\Delta_1^{i-1}\Delta_2^j\mathsf{b}_{r+1,s}-q^{i-1}\Delta_1^{i-1}\Delta_2^j\mathsf{b}_{r,s}) \\ &= q^{-i+1}\Delta_1^i\Delta_2^j\mathsf{b}_{r,s}, \end{aligned}$$

we have

$$\beta = \sum_{i=0}^{m}\sum_{j=0}^{m-i} q^{ij} q^{-ri-sj} q^{m-i-j} \begin{bmatrix} m-1 \\ i+j-1 \end{bmatrix} \binom{i+j-1}{i-1} \Delta_1^i \Delta_2^j \mathsf{b}_{r,s} u^i v^j.$$

Rearranging and shifting the index of the second summation of γ we obtain

$$\gamma = \sum_{i=0}^{m-1}\sum_{j=1}^{m-i} q^{i(j-1)} q^{-ri-sj} q^{m-1} \begin{bmatrix} m-1 \\ i+j-1 \end{bmatrix} \binom{i+j-1}{i}$$

$$\times (q^{-j+1} \Delta_1^i \Delta_2^{j-1} \mathsf{b}_{r,s+1} - \Delta_1^i \Delta_2^{j-1} \mathsf{b}_{r,s}) u^i v^j.$$

Using $q^{-j+1}\Delta_1^i\Delta_2^{j-1}\mathsf{b}_{r,s+1} - \Delta_1^i\Delta_2^{j-1}\mathsf{b}_{r,s} = q^{-j+1}\Delta_1^i\Delta_2^j\mathsf{b}_{r,s}$, $\left[\begin{smallmatrix} m-1 \\ i+j-1 \end{smallmatrix}\right] = 0$ for $i = m$ and $\binom{i+j-1}{i} = 0$ for $j = 0$ we have

$$\gamma = \sum_{i=0}^{m}\sum_{j=0}^{m-i} q^{ij} q^{-ri-sj} q^{m-i-j} \begin{bmatrix} m-1 \\ i+j-1 \end{bmatrix} \binom{i+j-1}{i} \Delta_1^i \Delta_2^j \mathsf{b}_{r,s} u^i v^j.$$

Hence

$$\mathsf{b}_{r,s}^m = \sum_{i=0}^{m}\sum_{j=0}^{m-i} q^{ij} q^{-ri-sj} w \Delta_1^i \Delta_2^j \mathsf{b}_{r,s} u^i v^j$$

where

$$w = \left\{ \begin{bmatrix} m-1 \\ i+j \end{bmatrix} \binom{i+j}{i} + q^{m-i-j} \begin{bmatrix} m-1 \\ i+j-1 \end{bmatrix} \left\{ \binom{i+j-1}{i-1} + \binom{i+j-1}{i} \right\} \right\}.$$

Using the pascal identity we have

$$w = \left\{ \binom{i+j}{i} \left\{ \begin{bmatrix} m-1 \\ i+j \end{bmatrix} + q^{m-i-j} \begin{bmatrix} m-1 \\ i+j-1 \end{bmatrix} \right\} \right\}.$$

From the pascal type identity (1.2.4) of $q-$binomial coefficients we obtain

$$\mathsf{b}_{r,s}^m = \sum_{i=0}^{m}\sum_{j=0}^{m-i} q^{ij} q^{-ri-sj} \begin{bmatrix} m \\ i+j \end{bmatrix} \binom{i+j}{i} \Delta_1^i \Delta_2^j \mathsf{b}_{r,s} u^i v^j$$

and the proof is completed. □

As a consequence of theorem 3.2.1 one may express $q-$Bézier triangles explicitly.

Corollary 3.2.1. *$q-$Bézier triangle of degree n is*

$$S(u,v) = \mathsf{b}^n_{0,0} = \sum_{i=0}^{n}\sum_{j=0}^{n-i} q^{ij} \begin{bmatrix} n \\ i+j \end{bmatrix} \binom{i+j}{i} \Delta_1^i \Delta_2^j \mathsf{b}_{0,0} u^i v^j. \tag{3.2.2}$$

Note that $q = 1$ recovers the difference form of standard Bézier triangles (2.2.4), since $\binom{n}{i+j}\binom{i+j}{i} = \binom{n}{i\ ,\ j}$.

3.3 A Generalization of Rational Bézier Surfaces

3.3.1 Rational Tensor Product $q-$Bernstein Bézier Surfaces

Using analogous technique that is used in obtaining rational tensor product Bézier surface we introduce a generalization of tensor product $q-$Bernstein Bézier surfaces by projecting tensor product q-Bernstein Bézier surfaces to the hyperplane $w = 1$. Let $R(u,v) \in \mathbb{E}^3$ be a point on rational tensor product $q-$Bernstein Bézier surface of degree (m,n). The point $R(u,v)$ may be identified as $[R(u,v) \quad 1]^T \in \mathbb{E}^4$. For $0 \leqslant u,v \leqslant 1$, this point is the projection of the point $[w(u,v)R(u,v) \quad w(u,v)]^T$ which is a point on the projecting surface $S(u,v)$ of degree (m,n) in $4D$. Hence $w(u,v)$ is a polynomial in u and v of degree (m,n) and may be expressed in terms of tensor product $q-$Bernstein Bézier polynomials of degree (m,n) by

$$w(u,v) = \sum_{i=0}^{m}\sum_{j=0}^{n} w_{i,j} B_i^{m,q_1}(u) B_j^{n,q_2}(v), \quad w_{i,j} \in \mathbb{R}.$$

It follows from the homogeneous form $[w(u,v)R(u,v) \quad w(u,v)]^T$ that

$$\begin{pmatrix} R(u,v)\sum_{i=0}^{m}\sum_{j=0}^{n} w_{i,j} B_i^{m,q_1}(u) B_j^{n,q_2}(v) \\ \sum_{i=0}^{m}\sum_{j=0}^{n} w_{i,j} B_i^{m,q_1}(u) B_j^{n,q_2}(v) \end{pmatrix} = \sum_{i=0}^{m}\sum_{j=0}^{n} \begin{pmatrix} c_{i,j} \\ w_{i,j} \end{pmatrix} B_i^{m,q_1}(u) B_j^{n,q_2}(v).$$

Namely,

$$R(u,v)\sum_{i=0}^{m}\sum_{j=0}^{n} w_{i,j}B_i^{m,q_1}(u)B_j^{n,q_2}(v)=\sum_{i=0}^{m}\sum_{j=0}^{n} c_{i,j}B_i^{m,q_1}(u)B_j^{n,q_2}(v).$$

Then

$$R(u,v)=\frac{\sum_{i=0}^{m}\sum_{j=0}^{n} c_{i,j}B_i^{m,q_1}(u)B_j^{n,q_2}(v)}{\sum_{i=0}^{m}\sum_{j=0}^{n} w_{i,j}B_i^{m,q_1}(u)B_j^{n,q_2}(v)}.$$

Finally setting $c_{i,j}=w_{i,j}\mathsf{b}_{i,j}$ gives

$$R(u,v)=\frac{\sum_{i=0}^{m}\sum_{j=0}^{n} w_{i,j}\mathsf{b}_{i,j}B_i^{m,q_1}(u)B_j^{n,q_2}(v)}{\sum_{i=0}^{m}\sum_{j=0}^{n} w_{i,j}B_i^{m,q_1}(u)B_j^{n,q_2}(v)}. \tag{3.3.1}$$

The points $\mathsf{b}_{i,j}$ form the control net and the numbers $w_{i,j}>0$ are called weights associated with $\mathsf{b}_{i,j}$. The following figure depicts a rational tensor product $q-$Bernstein Bézier surface of degree $(4,2)$.

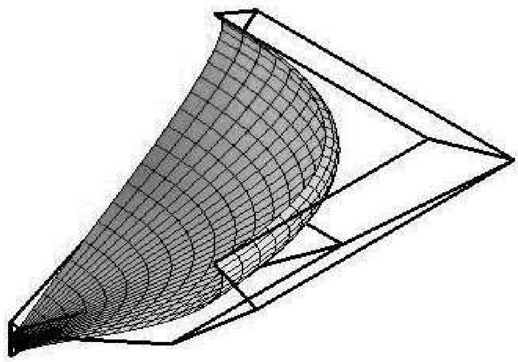

Figure 3.2 Rational tensor product q-Bernstein Bézier surface of degree $(4,2)$.

The basis functions for rational tensor product $q-$Bernstein Bézier surfaces are

$$\phi_{i,j} = \frac{w_{i,j}B_i^{m,q_1}(u)B_j^{n,q_2}(v)}{\sum_{i=0}^{m}\sum_{j=0}^{n} w_{i,j}B_i^{m,q_1}(u)B_j^{n,q_2}(v)}$$

which form partition of unity and are positive for the parameter values $0 < q_1, q_2 \leqslant 1$. Using the basis functions we find out some of properties of $R(u,v)$ listed below.

1. Affine invariance property: It is a consequence of the basis functions $\phi_{i,j}$ sum to one.

2. Convex hull property: For $0 < q_1, q_2 \leqslant 1$, the basis functions are positive. Furthermore they sum to one and hence $R(u,v)$ has convex hull property.

3. Boundary curves: Boundary curves of $R(u,v)$ are evaluated by $R(u,0)$, $R(u,1)$, $R(0,v)$, and $R(1,v)$ where the first two are tensor product $q-$Bernstein Bézier surface in u and the latter two are tensor product $q-$Bernstein Bézier surface in v.

4. Corner point interpolation: The corner points of the surface and the corner points of the control net coincide:

$$R(0,0) = \mathsf{b}_{0,0}, R(1,0) = \mathsf{b}_{m,0}, R(0,1) = \mathsf{b}_{0,n}, \text{ and } R(1,1) = \mathsf{b}_{m,n}.$$

5. Projective invariance property: $R(u,v)$ has also the projective invariance property which is explained in Chapter 1.

The equation (3.3.1) defines a more general Bernstein Bézier surface. If we set $q = 1$ then we obtain standard rational tensor product Bernstein Bézier surface (2.3.1). Taking all weights equal reveals tensor product $q-$Bernstein Bézier surface (3.1.1). Moreover, if we set $q = 1$ and take all weights equal then we obtain standard tensor product Bernstein Bézier surface (2.1.1).

Figure 3.2 is a part of a hourglass. Using the symmetry of the surface about the

$xy-$plane, we obtain a quarter of the hourglass. Since taking symmetry of an object is an affine map, we can obtain the control net of the resulting surface by symmetry of the control net of the Figure 3.2. Then using the symmetry of the resulting surface about the $xz-$plane, $yz-$plane and the plane $x+y=0$ and combining the results we obtain Figure 3.3.

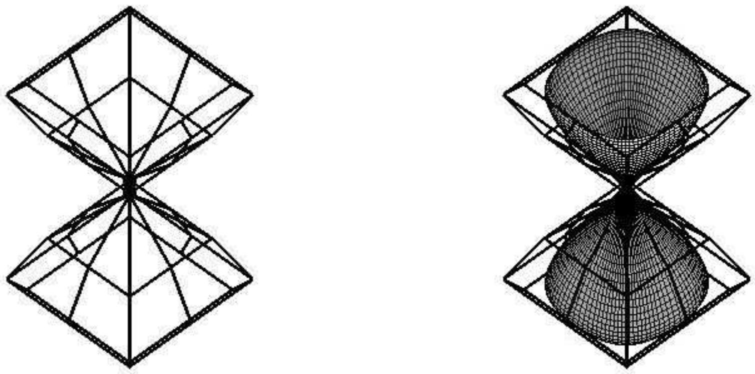

Figure 3.3 A hourglass and its control net.

We also modify Algorithm 3.1 to obtain rational tensor product $q-$Bernstein Bézier surfaces. For this purpose we project each intermediate point of Algorithm 3.1 into $\mathbb{E}^3$. The intermediate points for $4D$ surface are

$$p_{i,j}^{r,r} = \begin{bmatrix} 1-q_1^{r-i-1}u & q_1^{r-i-1}u \end{bmatrix} \begin{bmatrix} p_{i,j}^{r-1,r-1} & p_{i,j+1}^{r-1,r-1} \\ p_{i+1,j}^{r-1,r-1} & p_{i+1,j+1}^{r-1,r-1} \end{bmatrix} \begin{bmatrix} 1-q_2^{r-j-1}v \\ q_2^{r-j-1}v \end{bmatrix}$$

for $r=1,2,\ldots,k;\ i=0,1,\ldots,m-r;\ j=0,1,\ldots,n-r;$ where $k=\min(m,n)$ and $p_{i,j}^{0,0}=p_{i,j}=[w_{i,j}\mathrm{b}_{i,j} \quad w_{i,j}]^T$. Thus the algorithm is

Algorithm 3.3: Given the control net $\mathrm{b}_{i,j}\in\mathbb{E}^3$ and the corresponding weights $w_{i,j}\in$

$\mathbb{R}$; $i = 0, \ldots, m$; $j = 0, \ldots, n$. Compute

$$w_{i,j}^{r,r}\mathsf{b}_{i,j}^{r,r} = \begin{bmatrix} 1 - q_1^{r-i-1}u & q_1^{r-i-1}u \end{bmatrix} \begin{bmatrix} w_{i,j}^{r-1,r-1}\mathsf{b}_{i,j}^{r-1,r-1} & w_{i,j+1}^{r-1,r-1}\mathsf{b}_{i,j+1}^{r-1,r-1} \\ w_{i+1,j}^{r-1,r-1}\mathsf{b}_{i+1,j}^{r-1,r-1} & w_{i+1,j+1}^{r-1,r-1}\mathsf{b}_{i+1,j+1}^{r-1,r-1} \end{bmatrix} \times$$

$$\begin{bmatrix} 1 - q_2^{r-j-1}v \\ q_2^{r-j-1}v \end{bmatrix} \tag{3.3.2}$$

for $r = 1, 2, \ldots, k$; $i = 0, \ldots, m - r$; $j = 0, \ldots, n - r$; where $k = \min(m, n)$ and

$$w_{i,j}^{r,r} = \begin{bmatrix} 1 - q_1^{r-i-1}u & q_1^{r-i-1}u \end{bmatrix} \begin{bmatrix} w_{i,j}^{r-1,r-1} & w_{i,j+1}^{r-1,r-1} \\ w_{i+1,j}^{r-1,r-1} & w_{i+1,j+1}^{r-1,r-1} \end{bmatrix} \begin{bmatrix} 1 - q_2^{r-j-1}v \\ q_2^{r-j-1}v \end{bmatrix} \tag{3.3.3}$$

The following theorem gives the explicit form of intermediate points

Theorem 3.3.1. *The intermediate points of Algorithm 3.3 expressed explicitly as*

$$\mathsf{b}_{i,j}^{r,r} = \frac{\sum_{k=0}^{r}\sum_{l=0}^{r} w_{i+k,j+l}\mathsf{b}_{i+k,j+l} {r \brack k}_{q_1} u^k \prod_{s=0}^{r-k-1}(q_1^i - q_1^s u) {r \brack l}_{q_2} v^l \prod_{s=0}^{r-l-1}(q_2^j - q_2^s v)}{\sum_{k=0}^{r}\sum_{l=0}^{r} w_{i+k,j+l} {r \brack k}_{q_1} u^k \prod_{s=0}^{r-k-1}(q_1^i - q_1^s u) {r \brack l}_{q_2} v^l \prod_{s=0}^{r-l-1}(q_2^j - q_2^s v)} \tag{3.3.4}$$

Proof. We will use projective invariance property. Take control points $p_{i,j} = [w_{i,j}\mathsf{b}_{i,j} \quad w_{i,j}]^T \in \mathbb{E}^4$. Now apply Algorithm 3.1 for $p_{i,j}^{r,r}$. We see from the Theorem3.1.1 that

$$p_{i,j}^{r,r} = \sum_{k=0}^{r}\sum_{l=0}^{r} q_1^{-ri} q_2^{-rj} p_{i+k,j+l} \begin{bmatrix} r \\ k \end{bmatrix}_{q_1} u^k \prod_{s=0}^{r-k-1} (q_1^i - q_1^s u) \begin{bmatrix} r \\ l \end{bmatrix}_{q_2} v^l \prod_{s=0}^{r-l-1} (q_2^j - q_2^s v)$$

and

$$w_{i,j}^{r,r} = \sum_{k=0}^{r}\sum_{l=0}^{r} q_1^{-ri} q_2^{-rj} w_{i+k,j+l} \begin{bmatrix} r \\ k \end{bmatrix}_{q_1} u^k \prod_{s=0}^{r-k-1} (q_1^i - q_1^s u) \begin{bmatrix} r \\ l \end{bmatrix}_{q_2} v^l \prod_{s=0}^{r-l-1} (q_2^j - q_2^s v).$$

Projecting $p_{i,j}^{r,r}$ in $\mathbb{E}^4$ onto $\mathbb{E}^3$ yields

$$\mathsf{b}_{i,j}^{r,r} = \frac{\sum_{k=0}^{r}\sum_{l=0}^{r} w_{i+k,j+l}\mathsf{b}_{i+k,j+l} {r \brack k}_{q_1} u^k \prod_{s=0}^{r-k-1}(q_1^i - q_1^s u) {r \brack l}_{q_2} v^l \prod_{s=0}^{r-l-1}(q_2^j - q_2^s v)}{\sum_{k=0}^{r}\sum_{l=0}^{r} w_{i+k,j+l} {r \brack k}_{q_1} u^k \prod_{s=0}^{r-k-1}(q_1^i - q_1^s u) {r \brack l}_{q_2} v^l \prod_{s=0}^{r-l-1}(q_2^j - q_2^s v)},$$

and this completes the proof. $\square$

By using the projective invariance property, it is also possible to express the intermediate points of Algorithm 3.3 explicitly in terms of $q-$differences.

Theorem 3.3.2. *The $q-$differences form of the intermediate points of Algorithm 3.3 is*

$$\mathsf{b}_{i,j}^{r,r} = \frac{\sum_{k=0}^{r}\sum_{l=0}^{r} q_1^{-ik} q_2^{-jl} \begin{bmatrix} r \\ k \end{bmatrix}_{q_1} \begin{bmatrix} r \\ l \end{bmatrix}_{q_2} \Delta_{q_1}^{k} \Delta_{q_2}^{l} (w_{i,j}\mathsf{b}_{i,j}) u^k v^l}{\sum_{k=0}^{r}\sum_{l=0}^{r} q_1^{-ik} q_2^{-jl} \begin{bmatrix} r \\ k \end{bmatrix}_{q_1} \begin{bmatrix} r \\ l \end{bmatrix}_{q_2} \Delta_{q_1}^{k} \Delta_{q_2}^{l} w_{i,j} u^k v^l}. \tag{3.3.5}$$

Proof. Take $p_{i,j} = [w_{i,j}\mathsf{b}_{i,j} \quad w_{i,j}]^T$ and apply Algorithm 3.1. We have from theorem (3.1.1) that

$$p_{i,j}^{r,r} = \sum_{l=0}^{r} q_2^{-rj} p_{i,j+l}^{r} \begin{bmatrix} r \\ l \end{bmatrix}_{q_2} v^l \prod_{s=0}^{r-l-1} (q_2^j - q_2^s v).$$

The above expression is the intermediate points of Algorithm 1.3 with control points $p_{i,j}^{r}$; $j = 0, 1, \ldots, n$. Hence using the $q-$difference form of the intermediate points of Algorithm 1.3 we obtain

$$p_{i,j}^{r,r} = \sum_{l=0}^{r} q_2^{-jl} \begin{bmatrix} r \\ l \end{bmatrix}_{q_2} v^l \Delta_{q_2}^{l} p_{i,j}^{r}.$$

We also know that $p_{i,j}^{r}$; $j = 0, 1, \ldots, n$ are the intermediate points of Algorithm 1.3 with the control points $p_{0,j}, p_{1,j}, \ldots, p_{m,j}$ for each j and hence $p_{i,j}^{r}$ can be written as

$$p_{i,j}^{r} = \sum_{k=0}^{r} q_1^{-ik} \begin{bmatrix} r \\ k \end{bmatrix}_{q_1} u^k \Delta_{q_1}^{k} p_{i,j}.$$

Substituting the last equation in $p_{i,j}^{r,r}$ and projecting it onto $\mathbb{E}^3$ gives

$$\mathsf{b}_{i,j}^{r,r} = \frac{\sum_{k=0}^{r}\sum_{l=0}^{r} q_1^{-ik} q_2^{-jl} \begin{bmatrix} r \\ k \end{bmatrix}_{q_1} \begin{bmatrix} r \\ l \end{bmatrix}_{q_2} \Delta_{q_1}^{k} \Delta_{q_2}^{l} (w_{i,j}\mathsf{b}_{i,j}) u^k v^l}{\sum_{k=0}^{r}\sum_{l=0}^{r} q_1^{-ik} q_2^{-jl} \begin{bmatrix} r \\ k \end{bmatrix}_{q_1} \begin{bmatrix} r \\ l \end{bmatrix}_{q_2} \Delta_{q_1}^{k} \Delta_{q_2}^{l} w_{i,j} u^k v^l}.$$

$\square$

We now give the $q-$difference form of rational tensor product $q-$Bernstein Bézier surface $R(u,v)$.

Theorem 3.3.3. *The surface $R(u,v)$ may also be computed by*

$$R(u,v) = \frac{\sum_{i=0}^{m}\sum_{j=0}^{n}\begin{bmatrix} m \\ i \end{bmatrix}_{q_1}\begin{bmatrix} n \\ j \end{bmatrix}_{q_2}\Delta_{q_1}^{i}\Delta_{q_2}^{j}(w_{0,0}\mathsf{b}_{0,0})u^i v^j}{\sum_{i=0}^{m}\sum_{j=0}^{n}\begin{bmatrix} m \\ i \end{bmatrix}_{q_1}\begin{bmatrix} n \\ j \end{bmatrix}_{q_2}\Delta_{q_1}^{i}\Delta_{q_2}^{j}w_{0,0}u^i v^j}. \tag{3.3.6}$$

Proof. First write $R(u,v)$ in the form

$$R(u,v) = \frac{\sum_{j=0}^{n} d_j B_j^{n,q_2}(v)}{\sum_{j=0}^{n} w_j B_j^{n,q_2}(v)}$$

where $d_j = \sum_{i=0}^{m} w_{i,j}\mathsf{b}_{i,j}B_i^{m,q_1}(u)$ and $w_j = \sum_{i=0}^{m} w_{i,j}B_i^{m,q_1}(u)$. Since d_j and w_j are $q-$ Bernstein Bézier curves we may write

$$d_j = \sum_{i=0}^{m}\begin{bmatrix} m \\ i \end{bmatrix}_{q_1}\Delta_{q_1}^{i}(w_{0,j}\mathsf{b}_{0,j})u^i; \quad w_j = \sum_{i=0}^{m}\begin{bmatrix} m \\ i \end{bmatrix}_{q_1}\Delta_{q_1}^{i}w_{0,j}u^i.$$

Thus,

$$R(u,v) = \frac{\sum_{j=0}^{n}\left\{\sum_{i=0}^{m}\begin{bmatrix} m \\ i \end{bmatrix}_{q_1}\Delta_{q_1}^{i}(w_{0,j}\mathsf{b}_{0,j})u^i\right\}B_j^{n,q_2}(v)}{\sum_{j=0}^{n}\left\{\sum_{i=0}^{m}\begin{bmatrix} m \\ i \end{bmatrix}_{q_1}\Delta_{q_1}^{i}w_{0,j}u^i\right\}B_j^{n,q_2}(v)}$$

and

$$R(u,v) = \frac{\sum_{i=0}^{m}\begin{bmatrix} m \\ i \end{bmatrix}_{q_1}\left\{\sum_{j=0}^{n}\Delta_{q_1}^{i}(w_{0,j}\mathsf{b}_{0,j})B_j^{n,q_2}(v)\right\}u^i}{\sum_{i=0}^{m}\begin{bmatrix} m \\ i \end{bmatrix}_{q_1}\left\{\sum_{j=0}^{n}\Delta_{q_1}^{i}w_{0,j}B_j^{n,q_2}(v)\right\}u^i}.$$

The expressions in curly brackets are $q-$Bernstein Bézier curves. Then writing the $q-$difference form of these curves and using the commutativity property of $q-$differences we have

$$R(u,v) = \frac{\sum_{i=0}^{m}\sum_{j=0}^{n}\begin{bmatrix} m \\ i \end{bmatrix}_{q_1}\begin{bmatrix} n \\ j \end{bmatrix}_{q_2}\Delta_{q_1}^{i}\Delta_{q_2}^{j}(w_{0,0}\mathsf{b}_{0,0})u^i v^j}{\sum_{i=0}^{m}\sum_{j=0}^{n}\begin{bmatrix} m \\ i \end{bmatrix}_{q_1}\begin{bmatrix} n \\ j \end{bmatrix}_{q_2}\Delta_{q_1}^{i}\Delta_{q_2}^{j}w_{0,0}u^i v^j}.$$

□

It can easily be shown using the projective invariance property that rational tensor product $q-$Bernstein Bézier surface $R(u,v)$ of degree (m,n) can be expressed as one of higher degree. Hence the required points are as follows:

i) express $R(u,v)$ as a surface of degree $(m+1,n)$:

$$\mathsf{b}_{i,j}^{(1,0)} = \left(1 - \frac{[m+1-i]_{q_1}}{[m+1]_{q_1}}\right) \frac{\mathsf{b}_{i-1,j}}{w_{i,j}^{(1,0)}} + \frac{[m+1-i]_{q_1}}{[m+1]_{q_1}} \frac{\mathsf{b}_{i,j}}{w_{i,j}^{(1,0)}},$$

where the corresponding weights are

$$w_{i,j}^{(1,0)} = \left(1 - \frac{[m+1-i]_{q_1}}{[m+1]_{q_1}}\right) w_{i-1,j} + \frac{[m+1-i]_{q_1}}{[m+1]_{q_1}} w_{i,j},\ i = 0,\dots,m+1;\ j = 0,\dots,n.$$

ii) express $R(u,v)$ as a surface of degree $(m,n+1)$:

$$\mathsf{b}_{i,j}^{(0,1)} = \left(1 - \frac{[n+1-j]_{q_2}}{[n+1]_{q_2}}\right) \frac{\mathsf{b}_{i,j-1}}{w_{i,j}^{(1,0)}} + \frac{[n+1-j]_{q_2}}{[n+1]_{q_2}} \frac{\mathsf{b}_{i,j}}{w_{i,j}^{(1,0)}},$$

where

$$w_{i,j}^{(0,1)} = \left(1 - \frac{[n+1-j]_{q_2}}{[n+1]_{q_2}}\right) w_{i-1,j} + \frac{[n+1-j]_{q_2}}{[n+1]_{q_2}} w_{i,j},\ i = 0,\dots,m;\ j = 0,\dots,n+1.$$

3.3.2 *Rational $q-$Bézier Triangles*

We define rational $q-$Bézier triangle of degree n by

$$R(u,v) = \frac{\sum_{i=0}^{n}\sum_{j=0}^{n-i} q^{ij} \begin{bmatrix} n \\ i+j \end{bmatrix} \binom{i+j}{i} \Delta_1^i \Delta_2^j (w_{0,0}\mathsf{b}_{0,0}) u^i v^j}{\sum_{i=0}^{n}\sum_{j=0}^{n-i} q^{ij} \begin{bmatrix} n \\ i+j \end{bmatrix} \binom{i+j}{i} \Delta_1^i \Delta_2^j w_{0,0} u^i v^j}, \quad 0 \leqslant u+v \leqslant 1$$

where the $q-$differences Δ_1 and Δ_2 are defined in section 3.2. Note that if we choose the weights $w_{i,j} = w$, $i = 0,1,\dots,n$, $j = 0,1,\dots,n-i$, where $w > 0$ is a fixed real then we have $\Delta_1^i \Delta_2^j w_{0,0} = 0$ for any $i,j > 1$. Thus, we turn to $q-$Bézier triangles when all weights are equal.

One can also compute rational $q-$Bézier triangle by a de Casteljau type algorithm obtained by projection of each intermediate point of Algorithm 3.2 onto $\mathbb{E}^3$. The algorithm is

Algorithm 3.4: Let $\mathsf{b}_{i,j}$, $i = 0,1,\ldots,n$, $j = 0,1,\ldots,n-i$ be the control points and the real values $w_{i,j}$ be associated weights. Compute

$$w^m_{r,s}\mathsf{b}^m_{r,s} = (1-q^{m-r-1}u - q^{m-s-1}v)w^{m-1}_{r,s}\mathsf{b}^{m-1}_{r,s} + q^{m-r-1}uw^{m-1}_{r+1,s}\mathsf{b}^{m-1}_{r+1,s} + q^{m-s-1}vw^{m-1}_{r,s+1}\mathsf{b}^{m-1}_{r,s+1}$$

where

$$w^m_{r,s} = (1-q^{m-r-1}u - q^{m-s-1}v)w^{m-1}_{r,s} + q^{m-r-1}uw^{m-1}_{r+1,s} + q^{m-s-1}vw^{m-1}_{r,s+1}$$

for $m = 1,2,\ldots,n$, $r = 0,1,\ldots,n-m$, $s = 0,1,\ldots,n-m-r$ where $0 \leqslant u+v \leqslant 1$.

The properties of rational $q-$Bézier triangles are

1. Affine invariance property comes from the coefficients of Algorithm 3.4 sum to one.

2. Boundary curves: $q-$Bernstein Bézier curves, whose control points are the boundary net points of $R(u,v)$, form the boundary curves of $q-$Bézier triangle.

3. Corner point interpolation: $R(u,v)$ coincide with the corner points of its control net.

$$R(0,0) = \mathsf{b}_{0,0},\ R(1,0) = \mathsf{b}_{n,0},\ \text{and } R(0,1) = \mathsf{b}_{0,n}.$$

4. $R(u,v)$ satisfies the projective invariance property

The following result gives the $q-$difference form of Algorithm 3.4 and can be proved by induction or using projective invariance property.

Theorem 3.3.4. *The intermediate points $\mathsf{b}^m_{r,s}$ in Algorithm 3.4 are*

$$\mathsf{b}^m_{r,s} = \frac{\sum_{i=0}^{m}\sum_{j=0}^{m-i} q^{ij}q^{-ri-sj}\begin{bmatrix} m \\ i+j \end{bmatrix}\binom{i+j}{i}\Delta_1^i\Delta_2^j(w_{r,s}\mathsf{b}_{r,s})u^iv^j}{\sum_{i=0}^{m}\sum_{j=0}^{m-i} q^{ij}q^{-ri-sj}\begin{bmatrix} m \\ i+j \end{bmatrix}\binom{i+j}{i}\Delta_1^i\Delta_2^j w_{r,s}u^iv^j}$$

Corollary 3.3.1. *The intermediate point* $\mathsf{b}^n_{0,0}$ *is a point on the surface*

$$R(u,v) = \mathsf{b}^n_{0,0} = \frac{\sum_{i=0}^{n}\sum_{j=0}^{n-i} q^{ij} \begin{bmatrix} n \\ i+j \end{bmatrix} \binom{i+j}{i} \Delta_1^i \Delta_2^j (w_{0,0}\mathsf{b}_{0,0}) u^i v^j}{\sum_{i=0}^{n}\sum_{j=0}^{n-i} q^{ij} \begin{bmatrix} n \\ i+j \end{bmatrix} \binom{i+j}{i} \Delta_1^i \Delta_2^j w_{0,0} u^i v^j}.$$

3.4 Multivariate Bernstein Polynomials

Convergence properties of $q-$Bernstein polynomials are studied in (Oruç & Tuncer, 2002), (Il'inskii & Ostrovska, 2002), (Ostrovska, 2003) and (Wang, 2008). It is shown in (Oruç & Tuncer, 2002) when $0 < q < 1$ that the uniform convergence of f by the sequence $\{B_n(f;x)\}$ requires that f be a linear function. It is also shown in (Oruç & Tuncer, 2002) that when $q \geqslant 1$, a one parameter family of Bernstein polynomials converge to f as $n \to \infty$ if f is a polynomial. On the other hand in (Ostrovska, 2003) it is shown that when $q > 1$ $B_n f \to f$ if f is analytic. We now aim at finding analogous results found in (Oruç & Tuncer, 2002) and (Ostrovska, 2003).

Let us recall that tensor product $q-$Bernstein Bézier surfaces can be expressed in terms of $q-$differences by

$$S(u,v) = \sum_{i=0}^{m}\sum_{j=0}^{n} \begin{bmatrix} m \\ i \end{bmatrix}_{q_1} \begin{bmatrix} n \\ j \end{bmatrix}_{q_2} \Delta_{q_1}^i \Delta_{q_2}^j \mathsf{b}_{0,0} u^i v^j.$$

A nonparametric surface of the form $z = f(x,y)$ has the parametric representation

$$S(x,y) = (x,y,f(x,y)).$$

It is shown in (Phillips, 2003) that for the $q-$Bernstein Bézier polynomial the following identity holds:

$$\sum_{i=0}^{n} \frac{[i]}{[n]} B_i^{n,q}(t) = t. \tag{3.4.1}$$

Hence, if we choose the control net points in the form $\mathsf{b}_{i,j} = \begin{bmatrix} \frac{[i]}{[m]} & \frac{[j]}{[n]} & b_{i,j} \end{bmatrix}^T$, where $b_{i,j} \in \mathbb{R}$ and use the identity (3.4.1) we obtain a nonparametric patch.

We now define $q-$Bernstein polynomials for a function $f(x,y)$ whose domain is $[0,1]\times[0,1]$ by

$$B_{m,n}(f;x,y)=\sum_{i=0}^{m}\sum_{j=0}^{n} f_{i,j}B_i^{m,q_1}(x)B_j^{n,q_2}(y), \tag{3.4.2}$$

where $f_{i,j}=f(\frac{[i]}{[m]},\frac{[j]}{[n]})$. Note that the $B_{m,n}(f;x,y)$ is a monotone linear operator for $0<q_1,q_2\leqslant 1$, and as a consequence of identity (3.4.1), $B_{m,n}(f;x,y)$ reproduce any polynomial in the form $f(x,y)=axy+bx+cy+d$. Using (3.1.6) it is clear from (3.4.2) that

$$B_{m,n}(f;x,y)=\sum_{i=0}^{m}\sum_{j=0}^{n}\begin{bmatrix} m \\ i \end{bmatrix}_{q_1}\begin{bmatrix} n \\ j \end{bmatrix}_{q_2}\Delta_{q_1}^{i}\Delta_{q_2}^{j}f_{0,0}x^iy^j. \tag{3.4.3}$$

Let $f(x,y)=x^ry^s$. Since the operator $\Delta_{q_1}^{i}\Delta_{q_2}^{j}$ annihilates any polynomial of total degree less than $i+j$, we see from (3.4.3) that $B_{m,n}(x^ry^s;x,y)$ is a polynomial of total degree $\min(m+n,r+s)$.

In (Goodman, Oruç & Phillips, 1999) $q-$Bernstein polynomial of univariate monomial functions are given in terms of Stirling polynomial of the second kind, $S_q(i,j)$ which is defined recursively by

$$S_q(i+1,j)=S_q(i,j-1)+[j]S_q(i,j)$$

for $i\geqslant 1$ and $j\geqslant 1$ with $S_q(0,0)=1$, $S_q(i,0)=0$ for $i>0$ and $S_q(i,j)=0$ for $j>i$. This polynomial can be expressed explicitly as

$$S_q(i,j)=\frac{1}{[j]!q^{j(j-1)/2}}\sum_{r=0}^{j}(-1)^rq^{r(r-1)/2}\begin{bmatrix} j \\ r \end{bmatrix}[j-r]^i.$$

The Stirling polynomial form of the $q-$Bernstein polynomials leads to some results on convergence (see, Oruç & Tuncer (2002)). Thus, $q-$Bernstein polynomial of the function $f(x,y)=x^ry^s$ is

$$B_{m,n}(x^ry^s;x,y)=\sum_{i=0}^{r}\sum_{j=0}^{s}\lambda_i^{m,q_1}\lambda_j^{n,q_2}[m]_{q_1}^{i-r}[n]_{q_2}^{j-s}S_{q_1}(r,i)S_{q_2}(s,j)x^iy^j \tag{3.4.4}$$

where $\lambda_i^{n,q}=\prod_{r=0}^{i-1}\left(1-\frac{[r]}{[n]}\right)$.

Theorem 3.4.1. *Let $q_1, q_2 \geqslant 1$ be fixed real numbers. Then, for any polynomial $p(x,y)$,*

$$\lim_{m,n\to\infty} B_{m,n}(p;x,y) = p(x,y).$$

Proof. Let $p(x,y)$ be a polynomial of total degree k.

$$p(x,y) = \sum_{r=0}^{k}\sum_{s=0}^{k-r} a_{r,s}x^r y^s.$$

Using linearity property of the operator $B_{m,n}(f;x,y)$ we obtain

$$B_{m,n}(p;x,y) = \sum_{r=0}^{k}\sum_{s=0}^{k-r} a_{r,s}B_{m,n}(x^r y^s;x,y).$$

Let $m+n > k \geqslant r+s$ and consider $B_{m,n}(x^r y^s;x,y)$ for $r = 0,1,\ldots,k$, $s = 0,1,\ldots,k-r$. Since $m+n \geqslant r+s$, using (3.4.4) yields

$$B_{m,n}(x^r y^s;x,y) = \sum_{i=0}^{r}\sum_{j=0}^{s} \lambda_i^{m,q_1}\lambda_j^{n,q_2}[m]_{q_1}^{i-r}[n]_{q_2}^{j-s}S_{q_1}(r,i)S_{q_2}(s,j)x^i y^j.$$

It is easily seen that, since $q_1, q_2 \geqslant 1$ the term $[m]_{q_1}^{i-r}$ converges to 1 as $m \to \infty$ for $i = r$ and converges to 0 for other values of i. Similarly, as $n \to \infty$ the term $[n]_{q_2}^{j-s}$ converges to 1 for $j = s$ and converges to 0 for all other values of j. On the other hand λ_i^{m,q_1} and λ_j^{n,q_2} both converge to 1 as $m,n \to \infty$ for all values of $0 \leqslant i \leqslant r$ and $0 \leqslant j \leqslant s$. Hence we have

$$B_{m,n}(x^r y^s;x,y) \to S_{q_1}(r,r)S_{q_2}(s,s)x^r y^s$$

as $m,n \to \infty$. Using $S_q(k,l) = 1$ when $k = l$, we have

$$B_{m,n}(x^r y^s;x,y) \to x^r y^s.$$

Thus we obtain

$$B_{m,n}(p;x,y) = \sum_{r=0}^{k}\sum_{s=0}^{k-r} a_{r,s}B_{m,n}(x^r y^s;x,y) \to \sum_{r=0}^{k}\sum_{s=0}^{k-r} a_{r,s}x^r y^s$$

as $m,n \to \infty$, and this completes the proof. $\square$

We also define $q-$Bernstein polynomials for the function $f(x,y)$ whose domain is $0 \leqslant x+y \leqslant 1$, $x,y \geqslant 0$ by

$$B_n(f(x,y);x,y) = \sum_{i=0}^{n} \sum_{j=0}^{n-i} q^{ij} \begin{bmatrix} n \\ i+j \end{bmatrix} \binom{i+j}{i} \Delta_1^i \Delta_2^j f_{0,0} x^i y^j \tag{3.4.5}$$

To investigate the convergence properties of $B_n(f;x,y)$ we need to express (3.4.5) in terms of Stirling polynomials of the second kind. For this purpose the following identities, which can easily shown, will be useful

$$[n]-[j] = q^j[n-j]$$

for $0 \leqslant j \leqslant n$ and

$$\begin{bmatrix} n \\ j \end{bmatrix} = \frac{[n]^j}{[j]!q^{j(j-1)/2}} \lambda_j^{n,q}$$

for $0 \leqslant j \leqslant n$. Since, (see Phillips, 2003)

$$\Delta_1^k f_{i,j} = \sum_{r=0}^{k} (-1)^k q^{r(r-1)/2} \begin{bmatrix} k \\ r \end{bmatrix} f_{i+k-r,j}$$

and

$$\Delta_2^k f_{i,j} = \sum_{r=0}^{k} (-1)^k q^{r(r-1)/2} \begin{bmatrix} k \\ r \end{bmatrix} f_{i,j+k-r}$$

we can write

$$\Delta_1^i \Delta_2^j f_{r,s} = \sum_{k=0}^{i} \sum_{l=0}^{j} (-1)^{k+l} q^{k(k-1)/2} q^{l(l-1)/2} \begin{bmatrix} i \\ k \end{bmatrix} \begin{bmatrix} j \\ l \end{bmatrix} f_{r+i-k,s+j-l}.$$

Substituting this in (3.4.5) we will get
$B_n(f(x,y);x,y) =$

$$= \sum_{i=0}^{n} \sum_{j=0}^{n-i} q^{ij} \begin{bmatrix} n \\ i+j \end{bmatrix} \binom{i+j}{i} x^i y^j \sum_{k=0}^{i} \sum_{l=0}^{j} (-1)^{k+l} q^{k(k-1)/2} q^{l(l-1)/2} \begin{bmatrix} i \\ k \end{bmatrix} \begin{bmatrix} j \\ l \end{bmatrix} f_{i-k,j-l}.$$

Let $f(x,y) = x^r y^s$. Then

$$B_n(x^ry^s;x,y) = \sum_{i=0}^{n}\sum_{j=0}^{n-i} q^{ij} \begin{bmatrix} n \\ i+j \end{bmatrix} \binom{i+j}{i} x^i y^j$$
$$\times \sum_{k=0}^{i}\sum_{l=0}^{j} (-1)^{k+l} q^{k(k-1)/2} q^{l(l-1)/2} \begin{bmatrix} i \\ k \end{bmatrix} \begin{bmatrix} j \\ l \end{bmatrix} \frac{[i-k]^r [j-l]^s}{[n]^{r+s}}.$$

On the other hand we have

$$\begin{bmatrix} n \\ i+j \end{bmatrix} = \frac{\begin{bmatrix} n \\ i \end{bmatrix} \begin{bmatrix} n-i \\ j \end{bmatrix}}{\begin{bmatrix} i+j \\ i \end{bmatrix}} = \frac{[n]^i \lambda_i^{n,q} [n-i]^j \lambda_j^{n-i,q}}{[i]! q^{i(i-1)/2} [j]! q^{j(j-1)/2} \begin{bmatrix} i+j \\ i \end{bmatrix}}.$$

Substituting the last equation in $B_n(x^ry^s;x,y)$ and rearranging it we will have

$$B_n(x^ry^s;x,y) = \sum_{i=0}^{n}\sum_{j=0}^{n-i} \Bigg\{ q^{ij} \lambda_i^{n,q} \lambda_j^{n-i,q} [n]^{i-r-s} [n-i]^j \frac{\binom{i+j}{i}}{\begin{bmatrix} i+j \\ i \end{bmatrix}}$$
$$\times \frac{\sum_{k=0}^{i} (-1)^k q^{k(k-1)/2} \begin{bmatrix} i \\ k \end{bmatrix} [i-k]^r}{[i]! q^{i(i-1)/2}} \frac{\sum_{l=0}^{j} (-1)^l q^{l(l-1)/2} \begin{bmatrix} j \\ l \end{bmatrix} [j-l]^s}{[j]! q^{j(j-1)/2}} x^i y^j \Bigg\}.$$

Thus, using the explicit form of Stirling polynomial of the second kind we obtain

$$B_n(x^ry^s;x,y) = \sum_{i=0}^{n}\sum_{j=0}^{n-i} q^{ij} \lambda_i^{n,q} \lambda_j^{n-i,q} [n]^{i-r-s} [n-i]^j \frac{\binom{i+j}{i}}{\begin{bmatrix} i+j \\ i \end{bmatrix}} S_q(r,i) S_q(s,j) x^i y^j. \quad (3.4.6)$$

Theorem 3.4.2. *Let $q > 1$ be a fixed real number. Then, for any polynomial $p(x,y)$*

$$\lim_{n\to\infty} B_n(p(x,y);x,y) = p(x,y)$$

if and only if $p(x,y)$ is of total degree 1.

Proof. It is clear from (3.4.6) that $B_n(p(x,y);x,y)$ reproduce polynomials of the form $p(x,y) = ax+by+c$. So, we only need to show if $\lim_{n\to\infty} B_n(p(x,y);x,y) = p(x,y)$ then $p(x,y)$ is of total degree 1. Let $p(x,y)$ be a polynomial of total degree m. Since the monomials in x and y of total degree m are

$$x^m, x^{m-1}y, \ldots, xy^{m-1}, y^m,$$

we can write $p(x,y)$ in the form

$$p(x,y)=\sum_{r=0}^{m}\sum_{s=0}^{m-r}a_{r,s}x^ry^s.$$

Since $B_n(f(x,y);x,y)$ is a linear operator we can write

$$B_n(p(x,y);x,y)=\sum_{r=0}^{m}\sum_{s=0}^{m-r}a_{r,s}B_n(x^ry^s;x,y).$$

Let $n>m\geqslant r+s$ consider $B_n(x^ry^s;x,y)$ for $r=0,\dots,m,\quad s=0,\dots,m-r$ since $n\geqslant r+s$ we have

$$B_n(x^ry^s;x,y)=\sum_{i=0}^{r}\sum_{j=0}^{s}q^{ij}\lambda_i^{n,q}\lambda_j^{n-i,q}[n]^{i-r-s}[n-i]^j\frac{\binom{i+j}{i}}{\left[{i+j \atop i}\right]}S_q(r,i)S_q(s,j)x^iy^j.$$

Using $[n-i]=\dfrac{[n]-[i]}{q^i}$ we have $[n-i]^j=\dfrac{([n]-[i])^j}{q^{ij}}$ and the last equation will be in the form

$$B_n(x^ry^s;x,y)=\sum_{i=0}^{r}\sum_{j=0}^{s}\lambda_i^{n,q}\lambda_j^{n-i,q}\frac{[n]^i}{[n]^r}\frac{([n]-[i])^j}{[n]^s}\frac{\binom{i+j}{i}}{\left[{i+j \atop i}\right]}S_q(r,i)S_q(s,j)x^iy^j.$$

It is easily seen that

$$\begin{aligned}
&\frac{[n]^i}{[n]^r}\to 0 \text{ when } i<r,\ \frac{([n]-[i])^j}{[n]^s}\to 0 \text{ when } j<s,\\
&\frac{[n]^i}{[n]^r}\to 1 \text{ when } i=r,\ \frac{([n]-[i])^j}{[n]^s}\to 1 \text{ when } j=s,\\
&\lambda_i^{n,q}\to 1\ \text{ for all } i \text{ and } \lambda_j^{n-i,q}\to 1\ \text{ for all } j.
\end{aligned}$$

Thus,

$$B_n(x^ry^s;x,y)=\frac{\binom{r+s}{r}}{\left[{r+s \atop r}\right]}S_q(r,r)S_q(s,s)x^ry^s.$$

Using $S_q(i,j)=1$ when $i=j$, we will have

$$B_n(x^ry^s;x,y)=\frac{\binom{r+s}{r}}{\left[{r+s \atop r}\right]}x^ry^s.$$

Thus, we get

$$\sum_{r=0}^{m}\sum_{s=0}^{m-r} a_{r,s}\frac{\binom{r+s}{r}}{\left[{r+s \atop r}\right]}x^r y^s = \sum_{r=0}^{m}\sum_{s=0}^{m-r} a_{r,s}x^r y^s.$$

Hence we must have $\frac{\binom{r+s}{r}}{\left[{r+s \atop r}\right]} = 1$. But this is true if and only if $r+s=1$ or $q=1$. This completes the proof. □

As a result of Theorem 3.4.1 and Theorem 3.4.2, we see that when $q_1, q_2 \geqslant 1$ the two-parameter Bernstein polynomial $B_{m,n}(f;x,y)$ converges to $f(x,y)$ if $f(x,y)$ is a polynomial. Furthermore, in the case $q > 1$, $B_n(f;x,y)$ converges to $f(x,y)$ if $f(x,y)$ is a polynomial of total degree 1 and in the case $q = 1$, $B_n(f;x,y)$ converges to $f(x,y)$ if $f(x,y)$ is a polynomial of any total degree. Moreover, the results on convergence of univariate $q-$Bernstein polynomials on $C[0,1]$ can be carried over multivariate $q-$Bernstein polynomials $B_{m,n}f$. For example when $q_1, q_2 \geqslant 1$, in order to achieve uniform convergence of $B_{m,n}f$ on $C[0,1] \times C[0,1]$ we need to assume f has multivariate analytic expansion such that

$$f(x,y) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_{i,j}x^i y^j \text{ with } \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} |a_{i,j}| < \infty.$$

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