DOKUZ EYLÜL UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

ON THE PERTURBATION THEORY FOR THE SCHRÖDINGER OPERATOR

by
Didem COŞKAN

January, 2011
İZMİR

ON THE PERTURBATION THEORY FOR THE SCHRÖDINGER OPERATOR

A Thesis Submitted to the

Graduate School of Natural and Applied Sciences of Dokuz Eylül University

In Partial Fulfilment of the Requirements for the Degree of Doctor of Philosophy in

Mathematics

by
Didem COŞKAN

January, 2011
İZMİR

Ph.D. THESIS EXAMINATION RESULT FORM

We have read the thesis entitled **" ON THE PERTURBATION THEORY FOR THE SCHRÖDINGER OPERATOR "** completed by **DİDEM COŞKAN** under supervision of **ASSISTANT PROF. SEDEF KARAKILIÇ** and we certify that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Doctor of Philosophy.

....

Supervisor

....

Thesis Committee Member

....

Thesis Committee Member

....

Examining Committee Member

....

Examining Committee Member

Prof. Dr. Mustafa SABUNCU
Director
Graduate School of Natural and Applied Sciences

ACKNOWLEDGEMENTS

I would like to express my gratitude to my supervisor Assistant Prof. Sedef KARAKILIÇ for her advice, encouragement, patience and belief in me. I want to thank Prof. Oktay VELIEV for his useful comments and advice.

Also, I would like to thank the Graduate School of Natural and Applied Sciences of Dokuz Eylül University for supporting this thesis by project number 2007 KB FEN 40.

Finally, I am grateful to my family for their never ending love, trust, encouragement throughout my life.

Didem COŞKAN

ON THE PERTURBATION THEORY FOR THE SCHRÖDINGER OPERATOR

ABSTRACT

In this thesis, we obtain asymptotic formulas for the eigenvalues of the Schrödinger operator with a matrix potential and the Neumann boundary condition.

Keywords: Schrödinger opeartor, matrix potential, Neumann condition, perturbation, asymptotic formulas.

SCHRÖDINGER OPERATÖRÜNÜN PERTURBASYON TEORİSİ ÜZERİNE

ÖZ

Bu tezde matris potensiyelli, Neumann sınır koşullu Schrödinger operatörünün özdeğerleri için asimptotik formüller elde edilmiştir.

Anahtar sözcükler: Schrödinger operatörü, matris potansiyel, Neumann koşulu, perturbasyon, asimptotik formüller.

CONTENTS

Page

CHAPTER ONE
INTRODUCTION

1.1 Introduction

This thesis deals with the study of perturbation of the time independent Schrödinger operator defined by the differential expression

$$L(\Psi(x)) = (-\Delta + V(x))\Psi(x)$$

which is introduced by Erwin Schrödinger. It is a fundamental operator of quantum physics. This operator can have the meaning of the energy operator of one or several particles depending on the form of the potential $V(x)$. It can also describe the beheviour of an electron in an atom in the case of a periodic potential $V(x)$. From a mathematician's point of view, the Schrödinger operator is as inexhaustiable as mathematics itself.

If the eigenvalues λ_n and the associated orthonormal eigenfuctions u_n of a self adjoint linear differential equation

$$L(u_n) + \lambda_n u_n = 0$$

are known for a prescribed domain (boundary conditions), then the eigenvalues and the eigenfunctions of an operator corresponding to a ”neighbouring” or ”perturbed” operator

$$L(\widetilde{u_n}) - \varepsilon\widetilde{u_n} + \widetilde{\lambda_n}\widetilde{u_n} = 0$$

can be calculated by methods of approximations which is important in applications, the so-called Perturbation Theory. It is understood that the boundary conditions and the domain remain unchanged.

From the late 1930s, originating in the works of F. Rellich and T. Kato, perturbation theory became a mighty tool to investigate both qualitative and quantitative properties of linear operators. If we consider the perturbation theory for the Schrödinger operator it can be easily applied for one dimensional case and asymptotic formulas for sufficiently large eigenvalues can be obtained. The crucial property in the analysis of the Sturm-Liouville problem is that the distance between consecutive eigenvalues becomes larger and larger, so that the perturbation theory can be applied and asymptotic formulas for sufficiently large eigenvalues can be obtained. However, in multi dimensional cases, the eigenvalues influence each other strongly and the regular perturbation theory does not work.

In this study, we consider the Schrödinger operator with a matrix potential $V(x)$ which is defined by the differential expression

$$L\Phi = -\Delta\Phi + V\Phi \tag{1.1}$$

and the Neumann boundary condition

$$\frac{\partial\Phi}{\partial n}\mid_{\partial Q}= 0, \tag{1.2}$$

in $L_2^m(Q)$ where Q is the d dimensional rectangle $Q = [0,a_1] \times [0,a_2] \times \cdots \times [0,a_d]$, ∂Q is the boundary of Q, $m \geq 2$, $d \geq 2$, Δ is a diagonal $m \times m$ matrix whose diagonal elements are the scalar Laplace operators $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_d^2}$, $x = (x_1, x_2, \ldots, x_d) \in R^d$, V is the operator of multiplication by a real valued symmetric matrix $V(x) = (v_{ij}(x))$, $i, j = 1, 2, \ldots, m$, $v_{ij}(x) \in L_2(Q)$, that is, $V^T(x) = V(x)$.

We denote the operator defined by the differential expression (1.1) and the boundary condition (1.2) by $L(V)$, the eigenvalues and the corresponding eigenfunctions of the operator $L(V)$ by Λ_N and Ψ_N, respectively.

In this thesis, we study how the eigenvalues of the unperturbed operator $L(0)$, that is, $V(x) = 0$ in equation (1.1), are effected under perturbation, by using energy as a large parameter and we obtain high energy asymptotics of "arbitrary order" for the eigenvalues Λ_N of the operator $L(V)$ in an arbitrary dimension. For this we use the methods in Veliev (1987)-Veliev (2008). This is one of the essential problems related to the Schrödinger operator and is being studied for a long time.

For the scalar case, $m = 1$, a method was first introduced by O. Veliev in Veliev (1987), Veliev (1988) to obtain the asymptotic formulas for the eigenvalues of the periodic Schrödinger operator with quasiperiodic boundary conditions. By some other methods, asymptotic formulas for quasiperiodic boundary conditions in two and three dimensional cases are obtained in Feldman, Knoerrer, & Trubowitz (1990), Feldman, Knoerrer, & Trubowitz (1991), Karpeshina (1992), Karpeshina (1996) and Friedlanger (1990). When this operator is considered with Dirichlet boundary condition in two dimensional rectangle, the asymptotic formulas for the eigenvalues are obtained in Hald, & McLaughlin (1996). The asymptotic formulas for the eigenvalues of the Schrödinger operator with Dirichlet or Neumann boundary conditions in an arbitrary dimension are obtained in Atılgan, Karakılıç, & Veliev (2002), Karalılıç, Atılgan, & Veliev (2005) and Karalılıç, Veliev, & Atılgan (2005).

For the matrix case asymptotic formulas for the eigenvalues of the Schrödinger operator with quasiperiodic boundary conditions are obtained in Karpeshina (2002).

In chapter one, we introduce some basic concepts for our further discussions. We give some properties of periodic functions for which the method of this study is applicable.

In chapter two, the operators $L(0)$ and $L(V_0)$ are introduced where V_0 is the matrix of $\int_Q V(x)dx$. We introduce the two domains: non-resonance and resonance domains with respect to which non-resonance and resonance eigenvalues of the operator $L(0)$ are defined.

Chapter three is the original part of this study, that is, high energy asymptotics for the eigenvalues of the operator $L(V)$ are obtained in non-resonance and resonance domains. In Section 3.1, we consider the operator $L(V)$ as the perturbation of $L(V_0)$ by $V(x) - V_0$. By the corollaries of this section, we emphisize that differing from the scalar case the eigenvalues of the matrix V_0 are essential for the study of the matrix case. In Section 3.2, the obtained formulas depend not only on the eigenvalues of the matrix $C(\gamma, \gamma_1, \ldots, \gamma_k)$ but also on the eigenvalues of the matrix V_0.

In chapter four, we summarize the main results of the study.

1.2 Basic Concepts

1.2.1 The Space of Vector Functions

Definition 1.1. Let R^m denote an m-dimensional real vector space. Let $x = (x_1, x_2, \ldots, x_d) \in R^d$. Then the function $y : R^d \to R^m$,

$$y(x) = (y_1(x), y_2(x), \ldots, y_m(x))$$

is called a *vector function*. Each of the scalar functions $y_r : R^d \to R$, $r = 1,2,\dots,m$ is called a component of the vector function $y(x)$.

Definition 1.2. A vector function $y : R^d \to R^m$ is said to be *continuous* at the point $x_0 \in R^d$ if all the components of the vector function are continuous at x_0. Similarly, a vector function $y(x)$ is said to be *differentiable* if its components are differentiable, and by definition,

$$\frac{\partial y}{\partial x_k} = (\frac{\partial y_1}{\partial x_k}, \frac{\partial y_2}{\partial x_k}, \dots, \frac{\partial y_m}{\partial x_k}), \quad k = 1,2,\dots,d.$$

By using the definitions 1.1 and 1.2, for vector functions y, z and a scalar function f it can be easily seen that

$$\frac{\partial(y+z)}{\partial x_k} = \frac{\partial y}{\partial x_k} + \frac{\partial z}{\partial x_k}, \quad k = 1,2,\dots,d,$$

$$\frac{\partial(fy)}{\partial x_k} = \frac{\partial f}{\partial x_k} y + f\frac{\partial y}{\partial x_k}, \quad k = 1,2,\dots,d,$$

$$\frac{\partial\{y \cdot z\}}{\partial x_k} = \frac{\partial y}{\partial x_k} \cdot z + y \cdot \frac{\partial z}{\partial x_k}, \quad k = 1,2,\dots,d.$$

Definition 1.3. Let $y_{ij} : R^d \to R$, $i, j = 1,2,\dots,m$ be scalar functions. Then we define an *operator function* by means of square matrices $Y(x) = (y_{ij}(x))$ whose elements are scalar functions y_{ij}, $i, j = 1,2,\dots,m$.

Definition 1.4. Let $Y(x)$ be an operator function. $Y(x)$ is said to be *continuous* at the point x_0 if all its elements $y_{ij}(x)$, $i, j = 1,2,\dots,m$ are continuous at x_0, and to be *differentiable* at the point x_0 if all the elements $y_{ij}(x)$, $i, j = 1,2,\dots,m$ are differentiable at x_0.

It follows from the Definition 1.4 that $\frac{\partial Y}{\partial x_k}$, $k = 1,2,\dots,m$ is the matrix whose elements are $\frac{\partial y_{ij}}{\partial x_k}$, $k = 1,2,\dots,d$, $i, j = 1,2,\dots,m$.

Similar to the properties of vector functions we may give the following properties. By using definitions 1.3 and 1.4, for operator functions Y, Z, a vector function $y : R^d \to R^m$ and a scalar function $f : R^d \to R$

$$\frac{\partial(Y+Z)}{\partial x_k} = \frac{\partial Y}{\partial x_k} + \frac{\partial Z}{\partial x_k}, \quad k = 1,2,\dots,d,$$

$$\frac{\partial(YZ)}{\partial x_k} = \frac{\partial Y}{\partial x_k} Z + Y\frac{\partial Z}{\partial x_k}, \quad k = 1,2,\dots,d,$$

$$\frac{\partial(fY)}{\partial x_k} = \frac{\partial f}{\partial x_k}Y + f\frac{\partial Y}{\partial x_k}, \quad k = 1,2,\ldots,d,$$
$$\frac{\partial(Zy)}{\partial x_k} = \frac{\partial Z}{\partial x_k}y + Z\frac{\partial y}{\partial x_k}, \quad k = 1,2,\ldots,d.$$

$L_2^m(Q)$ is the set of vector functions $u(x) = (u_1(x), u_2(x), \ldots, u_m(x))$ satisfying $u_i(x) \in L_2(Q)$ for all $i = 1,2,\ldots,m$ where $x = (x_1, x_2, \ldots, x_d) \in Q$ and Q is the d-dimensional rectangle $Q = [0,a_1] \times [0,a_2] \times \cdots \times [0,a_d]$. Let $f = (f_1, f_2, \ldots, f_m)$ and $g = (g_1, g_2, \ldots, g_m)$ be vector functions in $L_2^m(Q)$ where $f_k, g_k \in L_2(Q)$ for $k = 1,2,\ldots,m$. Then the norm and the inner product in $L_2^m(Q)$ are defined by the formulas

$$\| f \| = \left(\int_Q | f(x) |^2 \, dx \right)^{\frac{1}{2}}, \quad < f, g > = \int_Q (f(x) \cdot g(x)) dx,$$

respectively where $| \cdot |$ and "$\cdot$" denote the norm and the inner product in R^m, respectively. From now on for whole of the study to denote the relevant norm that we are using, we will use the notation $\| \cdot \|$ except for the norm in R^m, $m \geq 1$ which we denote by $| \cdot |$.

1.2.2 The Norms for Operators

Let $B(X,Y)$ denote the set of all linear operators from the finite dimensional vector space X, say $n = \dim X < \infty$, to a finite dimensional vector space Y, say $m = \dim Y < \infty$. If X and Y are normed spaces, then $B(X,Y)$ is defined to be a normed space with the norm given by

$$\| T \| = \sup_{\substack{u \in X \\ u \neq 0}} \frac{\| Tu \|}{\| u \|} = \sup_{\|u\|=1} \| Tu \| = \sup_{\|u\| \leq 1} \| Tu \|, \quad T \in B(X,Y).$$

If we introduce different norms in the given vector spaces X and Y, then $B(X,Y)$ acquires different norms accordingly. However, all these norms in $B(X,Y)$ are equivalent. By equivalence of norms in $B(X,Y)$ we mean that

$$c \| T \| \leq \| T \|' \leq c' \| T \|$$

holds for some positive constants c, c' and any two different norms $\| \cdot \|$, $\| \cdot \|'$ in $B(X,Y)$. Let (a_{ij}), $i = 1,2,\ldots,m$, $j = 1,2,\ldots,n$ denote the matrix of T with respect to the bases of X and Y. Then we have the following inequalities

$$| a_{ij} | \leq d \| T \|, \quad i = 1,2,\ldots,m, \quad j = 1,2,\ldots,n, \tag{1.3}$$

$$\| T \| \leq d' \max | a_{ij} |, \tag{1.4}$$

where the constants d, d' depend on the bases of X and Y, but are independent of the operator T.

To prove the inequalities (1.3) and (1.4), let $\{x_j\}_{j=1}^n$, $\{y_i\}_{i=1}^m$ denote the bases of X and Y, respectively and (a_{ij}), $i = 1,2,\ldots,m$, $j = 1,2,\ldots,n$ denote the matrix of T with respect to these bases. One may define a norm for T by $\| T \|' = \max_{i,j} | a_{ij} |$. Let $\| T \|$ be another norm for T with respect to the given bases. By equivalence of norms, we have $\max_{i,j} | a_{ij} | \leq c' \| T \|$ from which it follows that $| a_{ij} | \leq d \| T \|$ holds for some constant d. On the other hand, if $\| T \|$ denotes an arbitrary norm for T with respect to the given bases, then for each x_j, $j = 1,2,\ldots,n$ we have

$$\| Tx_j \| = \| \sum_{i=1}^m a_{ij} y_i \| \leq \sum_{i=1}^m | a_{ij} | \| y_i \| \leq \sum_{i=1}^m \Big(| a_{ij} | \max_i \| y_i \| \Big)$$

$$= \max_i \| y_i \| \sum_{i=1}^m | a_{ij} | \leq (\max_i \| y_i \|)(m \max_i | a_{ij} |)$$

from which it follows that

$$\frac{\| Tx_j \|}{\| x_j \|} \leq m \frac{\max_i \| y_i \|}{\| x_j \|} \max_i | a_{ij} |$$

for any $j = 1,2,\ldots,n$. By definition of norm, we have $\frac{\|Tx_j\|}{\|x_j\|} \leq \| T \|$ for any $j = 1,2,\ldots,n$. By definition of supremum, $\| T \| \leq m \frac{\max_i \|y_i\|}{\|x_j\|} \max_i | a_{ij} |$, or denoting $m \frac{\max_i \|y_i\|}{\|x_j\|}$ by d' we have $\| T \| \leq d' \max | a_{ij} |$.

$\alpha T + \beta S$ is a continuous function of the scalars α, β and the operators $T, S \in B(X,Y)$, and $\| T \|$ is a continuous function of T. Thus we have the inequality

$$\| TS \| \leq \| T \| \| S \| \quad for \quad T \in B(Y,Z) \quad and \quad S \in B(X,Y).$$

1.2.3 A Theorem of Lidskii

Perturbation theory is primarily interested in small changes of the various quantities involves. In chapter three, we need to estimate the relation between the eigenvalues of two symmetric operators A, B in terms of their difference $C = B - A$ which leads us to the well known theorem due to Lidskii.

Theorem 1.5. *Let α_n, β_n and γ_n, $n = 1,2,\ldots,N$ denote the repeated eigenvalues of the symmetric operators A, B, C where $C = B - A$. Then*

$$\sum_n |\beta_n - \alpha_n| \leq \sum_n |\gamma_n| \,.$$

Proof. For the proof see Kato (1980). □

1.3 Properties of Periodic Functions in R^d

In this section, we summarize some properties of periodic smooth functions in R^d. Thus we see that one of the class of functions which satisfies our assumption on the potential $V(x)$, (2.33), is the sufficiently periodic smooth functions.

Definition 1.6. A function $v(x)$ where $x \in R^d$ is said to be *periodic* if there are d linearly independent vectors $w_1, w_2, \ldots, w_d$ such that

$$v(x + w_i) = v(x), \quad i = 1,2,\ldots,d.$$

We note that the definition is equivalent to

$$v(x + w) = v(x) \quad \forall w \in \Omega, \tag{1.5}$$

where

$$\Omega = \{w : w = \sum_{i=1}^{d} m_i w_i, m_i \in Z, i = 1,2,\ldots,d\}$$

is the lattice generated by the vectors $w_1, w_2, \ldots, w_d$.
Hence the function $v(x)$ satisfying the condition (1.5) is said to be periodic with respect

to the lattice Ω and related with this lattice there is a d-dimensional parallelepiped

$$Q=\{\sum_{i=1}^{d}t_iw_i:0\leq t_i<1,i=1,2,\ldots,d\}$$

called the fundamental domain of Ω which is the period parallelepiped of $v(x)$.
We define the dual lattice Γ of Ω by

$$\Gamma=2\pi\Theta,$$

where the lattice

$$\Theta=\{\sum_{j=1}^{d}n_j\gamma_j:n_j\in Z,j=1,2,\ldots,d\}$$

is called the reciprocal lattice of Ω and the vectors $\gamma_1,\gamma_2,\ldots,\gamma_d$ are linearly independent vectors satisfying

$$w_i\cdot\gamma_j=\delta_{ij}=\begin{cases}1, & i=j,\\ 0, & i\neq j,\end{cases}$$

where "$\cdot$" denotes the inner product in R^d, $d\geq 2$.
For any $w\in\Omega$, $\gamma\in\Gamma$

$$w\cdot\gamma=(\sum_{i=1}^{d}m_iw_i)\cdot(\sum_{j=1}^{d}n_j\gamma_j)=\sum_{i=1}^{d}m_in_iw_i\gamma_i=2\pi k,$$

where $k\in Z$.
The functions $e^{i\{\gamma\cdot x\}}$ for $\gamma\in\Gamma$ are periodic with respect to Ω. Really,

$$e^{i\{\gamma\cdot(x+w)\}}=e^{i\{\gamma\cdot x\}}e^{i\{\gamma\cdot w\}}=e^{i\{\gamma\cdot x\}}e^{i2\pi k}=e^{i\{\gamma\cdot x\}}.$$

Let $v(x)$ be a real valued and periodic with respect to Ω function of the space

$$W_2^l(Q)=\{v:D^\alpha v\in L_2(Q),\forall\alpha\leq l\},$$

where $\alpha=(\alpha_1,\alpha_2,\ldots,\alpha_d)\in Z^d$, $|\alpha|=|\alpha_1|+|\alpha_2|+\cdots+|\alpha_d|$, $D^\alpha=\frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\ldots\partial x_d^{\alpha_d}}$, $l\in N$ and $l\geq\frac{(d+20)(d-1)}{2}+d+3$.

Since $\{e^{i\{\gamma\cdot x\}}\}_{\gamma\in\Gamma}$ is a basis for $L_2(Q)$, for a function $v\in L_2(Q)$ we have

$$v(x)=\sum_{\gamma\in\Gamma}v_\gamma e^{i\{\gamma\cdot x\}},$$

where $v_\gamma = (v(x), e^{i\{\gamma\cdot x\}}) = \int\limits_Q v(x)\overline{e^{i\{\gamma\cdot x\}}}dx$ are the Fourier coefficients of the function $v(x)$ with respect to the basis $\{e^{i\{\gamma\cdot x\}}\}_{\gamma\in\Gamma}$, Q is the d-dimensional rectangle $Q = [0,a_1]\times[0,a_2]\times\cdots\times[0,a_d]$, $(\cdot,\cdot)$ denotes the inner product in $L_2(Q)$.

Now, we give some properties of periodic smooth functions.

Property 1. Let $v(x)$ be a real-valued function which is periodic with respect to Ω. Then $v(x)$ is a function of $W_2^l(Q)$ if and only if the Fourier coefficients v_γ of $v(x)$ satisfy the relation

$$\sum_{\gamma\in\Gamma} |\, v_\gamma\,|^2\,(1+|\,\gamma\,|^{2l}) < \infty. \tag{1.6}$$

Proof. For the proof see Karakılıç (2004). □

Property 2. For a large parameter ρ we can write a periodic function $v(x)\in W_2^l(Q)$ as

$$v(x) = \sum_{\gamma\in\Gamma(\rho^{\alpha'})} v_\gamma e^{i\{\gamma\cdot x\}} + O(\rho^{-p\alpha'}), \tag{1.7}$$

where

$$\Gamma(\rho^{-p\alpha'}) = \{\gamma\in\Gamma : 0 < |\,\gamma\,| < \rho^{\alpha'}\},$$

$\alpha' > 0$, $p = l-d$ and $O(\rho^{-p\alpha'})$ is a function in $L_2(Q)$ with norm of order $\rho^{-p\alpha'}$. That is, $f(\xi) = O(g(\xi))$ if there exists a constant c such that $|\,\frac{f(\xi)}{g(\xi)}\,| < c$ at some neigborhood of infinity.

Proof. For the proof see Karakılıç (2004). □

Property 3. For a periodic function $v(x)\in W_2^l(Q)$, we have

$$\sum_{\gamma\in\Gamma} |\, v_\gamma\,| < \infty. \tag{1.8}$$

Proof. For the proof see Karakılıç (2004). □

CHAPTER TWO
PRELIMINARIES

2.1 The Operators $L(0)$ and $L(V_0)$

We first investigate the eigenvalues and the eigenfunctions of the operator which is defined by the differential expression (1.1) when $V(x) = 0$ and the boundary condition (1.2). We denote this operator by $L(0)$.

Lemma 2.7. *The eigenvalues and the corresponding eigenspaces of the operator* $L(0)$ *are* $|\gamma|^2$ *and* $E_\gamma = span\{\Phi_{\gamma,1}(x), \Phi_{\gamma,2}(x), \ldots, \Phi_{\gamma,m}(x)\}$*, respectively where*

$$\gamma = (\gamma^1, \gamma^2, \ldots, \gamma^d) \in \frac{\Gamma^{+0}}{2},$$

$$\frac{\Gamma^{+0}}{2} = \{(\frac{n_1\pi}{a_1}, \frac{n_2\pi}{a_2}, \cdots, \frac{n_d\pi}{a_d}) : \quad n_i \in Z^+ \bigcup \{0\}, \quad i = 1,2,\ldots,d\},$$

$$\Phi_{\gamma,j}(x) = (0,\ldots,0,u_\gamma(x),0,\ldots,0), \quad j = 1,2,\ldots,m,$$
$$u_\gamma(x) = cos\gamma^1 x_1 cos\gamma^2 x_2 \cdots cos\gamma^d x_d.$$

We note that the non-zero component $u_\gamma(x)$ *of* $\Phi_{\gamma,j}(x)$ *stands in the* j*th component.*

Proof. We use a standart method, that is, the method of separation of variables. Suppose that the solution $\Phi(x) = (\Phi_1(x), \Phi_2(x), \ldots, \Phi_m(x))$ of the operator $L(0)$ is of the form $\Phi_j(x) = \Phi_{j1}(x_1)\Phi_{j2}(x_2)\cdots\Phi_{jd}(x_d)$ for each $j = 1,2,\ldots,m$.

Then the differential expression $-\Delta\Phi(x) = \lambda\Phi(x)$ implies that

$$-\Phi''_{j1}(x_1)\cdots\Phi_{jd}(x_d) - \cdots - \Phi_{j1}(x_1)\cdots\Phi''_{jd}(x_d) = \lambda\Phi_{j1}(x_1)\cdots\Phi_{jd}(x_d) \qquad (2.9)$$

for all $j = 1,2,\ldots,m$. Dividing both sides of the equation (2.9) by $\Phi_{j1}(x_1)\Phi_{j2}(x_2)\cdots\Phi_{jd}(x_d)$, we get

$$-\frac{\Phi''_{j1}(x_1)}{\Phi_{j1}(x_1)} - \frac{\Phi''_{j2}(x_2)}{\Phi_{j2}(x_2)} - \cdots - \frac{\Phi''_{jd}(x_d)}{\Phi_{jd}(x_d)} = \lambda \qquad (2.10)$$

for all $j = 1,2,\ldots,m$. Letting λ_{ji} denote a scalar for all $j = 1,2,\ldots,m$, $i = 1,2,\ldots,d$

such that $\lambda = \lambda_{j1} + \lambda_{j2} + \cdots + \lambda_{jd}$ holds, we get from the equations (2.10) that

$$-\frac{\Phi''_{ji}(x_i)}{\Phi_{ji}(x_i)} = \lambda_{ji} \tag{2.11}$$

for all $j = 1, 2, \ldots, m$, $i = 1, 2, \ldots, d$.

On the other hand, from the boundary condition $\frac{\partial \Phi}{\partial n} \mid_{\partial Q} = 0$ we get

$$\frac{\partial \Phi_j}{\partial n} \mid_{\partial Q} = 0 \tag{2.12}$$

for all $j = 1, 2, \ldots, m$. Since $Q = [0, a_1] \times [0, a_2] \times \cdots \times [0, a_d]$, the boundary $\partial Q = \{(t_1 a_1, t_2 a_2, \ldots, t_d a_d) : t_j = 0 \text{ or } 1 \text{ at least for some } i,\ i = 1, 2, \ldots, d\}$ lies in the hyperplanes $\Pi_i = \{x \in R^d : x \cdot e_i = 0\}$ or its shifts $a_i e_i + \Pi_i$, $i = 1, 2, \ldots, m$ where $e_1 = (1, 0, \ldots, 0)$, $e_2 = (0, 1, 0, \ldots, 0)$,..., $e_d = (0, \ldots, 0, 1)$. So the normal vectors to the hyperplanes Π_i, $a_i e_i + \Pi_i$ are e_i, $-e_i$, $i = 1, 2, \ldots, d$, respectively. Hence it follows from the equation (2.12) that

$$\frac{\partial \Phi_{ji}}{\partial x_i} \mid_{x \in \Pi_i} = \Phi_{j1}(x_1) \cdots \Phi'_{ji}(x_i) \cdots \Phi_{jd}(x_d) \mid_{x_i = 0} = 0 \tag{2.13}$$

and

$$\frac{\partial \Phi_{ji}}{\partial x_i} \mid_{x \in a_i e_i + \Pi_i} = -\Phi_{j1}(x_1) \cdots \Phi'_{ji}(x_i) \cdots \Phi_{jd}(x_d) \mid_{x_i = a_i} = 0 \tag{2.14}$$

for all $j = 1, 2, \ldots, m$, $i = 1, 2, \ldots, d$. Since we supposed that $\Phi(x) \neq 0$, it follows from (2.13) and (2.14) that

$$\Phi'_{ji}(0) = 0, \quad \Phi'_{ji}(a_i) = 0 \tag{2.15}$$

for all $j = 1, 2, \ldots, m$, $i = 1, 2, \ldots, d$.

From the equations (2.11) and (2.15), we get the following Sturm-Liouville problems

$$-\Phi''_{ji}(x_i) = \lambda_{ji} \Phi_{ji}(x_i), \tag{2.16}$$

$$\Phi'_{ji}(0) = \Phi'_{ji}(a_i) = 0, \tag{2.17}$$

for all $j = 1, 2, \ldots, m$, $i = 1, 2, \ldots, d$. It can be easily calculated that the eigenvalues and the corresponding eigenfunctions of the problem (2.16)-(2.17) are $\lambda_{ji} = (\frac{n_i \pi}{a_i})^2$ and $\Phi_{ji}(x_i) = \cos(\frac{n_i \pi}{a_i} x_i)$, $n_i \in Z^+ \bigcup \{0\}$, respectively for all $j = 1, 2, \ldots, m$, $i = 1, 2, \ldots, d$.

Thus it follows from (2.10), (2.11) and the solution of (2.16)-(2.17) that the eigenvalues of the operator $L(0)$ satisfy $\lambda = (\frac{n_1 \pi}{a_1})^2 + (\frac{n_2 \pi}{a_2})^2 + \cdots + (\frac{n_d \pi}{a_d})^2$ where

$n_i \in Z^+ \bigcup\{0\}$, $i = 1, 2, \ldots, d$. Letting $\frac{\Gamma^{+0}}{2}$ denote the set $\{(\frac{n_1\pi}{a_1}, \frac{n_2\pi}{a_2}, \cdots, \frac{n_d\pi}{a_d}) : \quad n_i \in Z^+ \bigcup\{0\}, \quad i = 1, 2, \ldots, d\}$ and $\gamma = (\gamma^1, \gamma^2, \ldots, \gamma^d)$ the vectors of the set $\frac{\Gamma^{+0}}{2}$, we have that the eigenvalues of the operator $L(0)$ are $|\gamma|^2$.

On the other hand, it follows from $\Phi_j(x) = \Phi_{j1}(x_1)\Phi_{j2}(x_2)\cdots\Phi_{jd}(x_d)$ and the solution of (2.16)-(2.17) that $\Phi_j(x) = \cos(\frac{n_1\pi}{a_1}x_1)\cos(\frac{n_2\pi}{a_2}x_2)\cdots\cos(\frac{n_d\pi}{a_d}x_d)$, $j = 1, 2, \ldots, m$. Then since we assumed that $\Phi(x) = (\Phi_1(x), \Phi_2(x), \ldots, \Phi_m(x))$, the eigenfunctions of the operator $L(0)$ are from the span $span\{(\Phi_1(x), 0, \ldots, 0), (0, \Phi_2(x), 0, \ldots, 0), \ldots, (0, \ldots, 0, \Phi_m(x))\}$. Letting $u_\gamma(x)$ denote the function $cos\gamma^1 x_1 cos\gamma^2 x_2 \cdots cos\gamma^d x_d$ where $\gamma = (\gamma^1, \gamma^2, \ldots, \gamma^d) = (\frac{n_1\pi}{a_1}, \frac{n_2\pi}{a_2} \cdots, \frac{n_d\pi}{a_d}) \in \frac{\Gamma^{+0}}{2}$ and $\Phi_{\gamma,j}(x)$ the function $(0, \ldots, 0, u_\gamma(x), 0, \ldots, 0)$, $j = 1, 2, \ldots, m$ where the non-zero component $u_\gamma(x)$ of $\Phi_{\gamma,j}(x)$ stands in the jth component of $\Phi_{\gamma,j}(x)$, we have that the eigenfunctions $\Phi_\gamma(x)$ of the operator $L(0)$ corresponding to the eigenvalue $|\gamma|^2$ are from the span $span\{\Phi_{\gamma,1}(x), \Phi_{\gamma,2}(x), \ldots, \Phi_{\gamma,m}(x)\}$. □

To obtain asymptotic formulas for the non-resonance eigenvalues, we consider the operator $L(V)$ as the perturbation of $L(V_0)$, where $V_0 = \int_Q V(x)dx$, by $V(x) - V_0$. Therefore, we first consider the eigenvalues and the eigenfunctions of the operator $L(V_0)$. We denote the eigenvalues of V_0, counted with multiplicity, and the corresponding orthonormal eigenvectors by $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_m$ and $\omega_1, \omega_2, \ldots, \omega_m$, respectively. Thus

$$V_0\omega_i = \lambda_i\omega_i, \quad \omega_i \cdot \omega_j = \delta_{ij}.$$

Lemma 2.8. *The eigenvalues and the corresponding eigenfunctions of the operator $L(V_0)$ are*

$$\mu_{\gamma,i} = |\gamma|^2 + \lambda_i, \quad and \quad \varphi_{\gamma,i}(x) = \sum_{j=1}^{m} \omega_{ij}\Phi_{\gamma,j}(x), \tag{2.18}$$

respectively where $|\gamma|^2$ is an eigenvalue of the operator $L(0)$, λ_i, $i = 1, 2, \ldots, m$ is an eigenvalue of the matrix V_0, ω_{ij}, $i, j = 1, 2, \ldots, m$ are the components of the normalized eigenvector ω_i, $i = 1, 2, \ldots, m$ corresponding to the eigenvalue λ_i of the matrix V_0, $\Phi_{\gamma,j}(x)$, $j = 1, 2, \ldots, m$ is the function where $\Phi_\gamma(x) \in span\{\Phi_{\gamma,j}(x)\}_{j=1,2,\ldots,m}$ is the eigenfunction corresponding to the eigenvalue $|\gamma|^2$ of the operator $L(0)$.

Proof. We verify that

$$L(V_0)\varphi_{\gamma,i}(x) = \mu_{\gamma,i}\varphi_{\gamma,i}(x). \tag{2.19}$$

Substituting $\varphi_{\gamma,i}(x) = \sum_{j=1}^{m} \omega_{ij}\Phi_{\gamma,j}(x)$ into the differential expression (1.1) where

$V(x) = V_0$, and using $-\Delta\Phi_{\gamma,j}(x) = |\gamma|^2 \Phi_{\gamma,j}(x)$ for all $j = 1,2,\ldots,m$, we get

$$\begin{aligned}
-\Delta\varphi_{\gamma,i}(x) + V_0\varphi_{\gamma,i}(x) &= -\Delta(\sum_{j=1}^{m} \omega_{ij}\Phi_{\gamma,j}(x)) + V_0(\sum_{j=1}^{m} \omega_{ij}\Phi_{\gamma,j}(x)) \\
&= \sum_{j=1}^{m} \omega_{ij}(-\Delta\Phi_{\gamma,j}(x)) + \sum_{j=1}^{m} \omega_{ij}(V_0\Phi_{\gamma,j}(x)) \\
&= \sum_{j=1}^{m} \omega_{ij} |\gamma|^2 \Phi_{\gamma,j}(x) + \sum_{j=1}^{m} \omega_{ij}(V_0\Phi_{\gamma,j}(x)). \qquad (2.20)
\end{aligned}$$

On the other hand, using $\mu_{\gamma,i} = |\gamma|^2 + \lambda_i$ and $\varphi_{\gamma,i}(x) = \sum_{j=1}^{m} \omega_{ij}\Phi_{\gamma,j}(x)$, we have

$$\mu_{\gamma,i}\varphi_{\gamma,i}(x) = (|\gamma|^2 + \lambda_i)(\sum_{j=1}^{m} \omega_{ij}\Phi_{\gamma,j}(x)) = \sum_{j=1}^{m} \omega_{ij} |\gamma|^2 \Phi_{\gamma,j}(x) + \sum_{j=1}^{m} \omega_{ij}\lambda_i\Phi_{\gamma,j}(x). \qquad (2.21)$$

Now we show that the second sums in the equations (2.20) and (2.21) are equal. We have

$$V_0\Phi_{\gamma,j}(x) = \sum_{k=1}^{m} v_{kj0}\Phi_{\gamma,k}(x) \qquad (2.22)$$

from which it follows that

$$\sum_{j=1}^{m} \omega_{ij}(V_0\Phi_{\gamma,j}(x)) = \sum_{j=1}^{m} \omega_{ij}(\sum_{k=1}^{m} v_{kj0}\Phi_{\gamma,k}(x)). \qquad (2.23)$$

We also have from $V_0\omega_i = \lambda_i\omega_i$ that $\lambda_i\omega_{ij} = \sum_{k=1}^{m} v_{kj0}\omega_{ik}$ which together with (2.22) implies that

$$\sum_{j=1}^{m} \omega_{ij}\lambda_i\Phi_{\gamma,j}(x) = \sum_{j=1}^{m} (\sum_{k=1}^{m} v_{kj0}\omega_{ik})\Phi_{\gamma,j}(x). \qquad (2.24)$$

Since $V(x) = V^T(x)$, $v_{kj0} = v_{jk0}$ for all $j,k = 1,2,\ldots,m$. Then

$$\sum_{j=1}^{m} \omega_{ij}(\sum_{k=1}^{m} v_{kj0}\Phi_{\gamma,k}(x)) = \sum_{k=1}^{m} \omega_{ik}(\sum_{j=1}^{m} v_{jk0}\Phi_{\gamma,j}(x)) = \sum_{j=1}^{m}\sum_{k=1}^{m} v_{jk0}\omega_{ik}\Phi_{\gamma,j}(x)$$

which shows that (2.23) and (2.24) are equal. Thus the second sums in the equations (2.20) and (2.21) are equal.

Substituting $\varphi_{\gamma,i}(x) = \sum_{j=1}^{m} \omega_{ij}\Phi_{\gamma,j}(x)$ into the boundary condition (1.2), and using

$\frac{\partial \Phi_{\gamma,j}(x)}{\partial n} \mid_{\partial Q}= 0$ for all $j = 1,2,\ldots,m$, we get

$$\frac{\partial \varphi_{\gamma,i}(x)}{\partial n} \mid_{\partial Q}= \frac{\partial}{\partial n}[\sum_{j=1}^{m} \omega_{ij}\Phi_{\gamma,j}(x)] \mid_{\partial Q}= \sum_{j=1}^{m} \omega_{ij}\frac{\partial \Phi_{\gamma,j}(x)}{\partial n} \mid_{\partial Q}= 0.$$

Thus (2.19) holds. □

Lemma 2.9. *Let $\mid \gamma \mid^2$ be an eigenvalue of the operator $L(0)$ and $\Phi_{\gamma,j}(x)$ its corresponding eigenfunction. Let Λ_N be an eigenvalue of the operator $L(V)$ and $\Psi_N(x)$ its corresponding eigenfunction. Then the following formula holds*

$$(\Lambda_N - \mid \gamma \mid^2) < \Psi_N, \Phi_{\gamma,j} >=< \Psi_N, V\Phi_{\gamma,j} > . \tag{2.25}$$

Proof. Multiplying both sides of the equation $L(V)\Psi_N = \Lambda_N\Psi_N$ by $\Phi_{\gamma,j}$, using $V(x) = V^T(x)$ and the equation $L(0)\Phi_{\gamma,j} =\mid \gamma \mid^2 \Phi_{\gamma,j}$, we get

$$\begin{aligned} < L(V)\Psi_N, \Phi_{\gamma,j} > &= < (-\Delta + V(x))\Psi_N, \Phi_{\gamma,j} > \\ &= < \Psi_N, (-\Delta + V^T(x))\Phi_{\gamma,j} > \\ &= < \Psi_N(x), -\Delta\Phi_{\gamma,j} > + < \Psi_N(x), V(x)\Phi_{\gamma,j} > \\ &= < \Psi_N, \mid \gamma \mid^2 \Phi_{\gamma,j} > + < \Psi_N, V(x)\Phi_{\gamma,j} > \\ &= \mid \gamma \mid^2 < \Psi_N, \Phi_{\gamma,j} > + < \Psi_N, V(x)\Phi_{\gamma,j} > \end{aligned}$$

and

$$< \Lambda_N\Psi_N, \Phi_{\gamma,j} >= \Lambda_N < \Psi_N, \Phi_{\gamma,j} >$$

which together give

$$(\Lambda_N - \mid \gamma \mid^2) < \Psi_N, \Phi_{\gamma,j} >=< \Psi_N, V\Phi_{\gamma,j} > .$$

□

We call the formula (2.25) as the "binding formula".

Lemma 2.10. *Let $\mu_{\gamma,i}$ be an eigenvalue of the operator $L(V_0)$ and $\varphi_{\gamma,i}(x)$ its corresponding eigenfunction. Let Λ_N be an eigenvalue of the operator $L(V)$ and $\Psi_N(x)$ its corresponding eigenfunction. Then the following formula holds*

$$(\Lambda_N - \mu_{\gamma,i}) < \Psi_N, \varphi_{\gamma,i} >=< \Psi_N, (V(x) - V_0)\varphi_{\gamma,i} > . \tag{2.26}$$

Proof. Multiplying both sides of the equation $L(V)\Psi_N = \Lambda_N\Psi_N$ by $\varphi_{\gamma,i}$, using $V(x) = V^T(x)$ and the equation (2.19), we get

$$\begin{aligned} < L(V)\Psi_N, \varphi_{\gamma,i} > &= < (-\Delta + V(x))\Psi_N, \varphi_{\gamma,i} > \\ &= < \Psi_N, (-\Delta + V^T(x))\varphi_{\gamma,i} > \\ &= < \Psi_N, (-\Delta + V(x) - V_0 + V_0)\varphi_{\gamma,i} > \\ &= < \Psi_N(x), (-\Delta + V_0)\varphi_{\gamma,i} > + < \Psi_N(x), (V(x) - V_0)\varphi_{\gamma,i} > \\ &= < \Psi_N, \mu_{\gamma,i}\varphi_{\gamma,i} > + < \Psi_N, (V(x) - V_0)\varphi_{\gamma,i} > \\ &= \mu_{\gamma,i} < \Psi_N, \varphi_{\gamma,i} > + < \Psi_N, (V(x) - V_0)\varphi_{\gamma,i} > \end{aligned}$$

and

$$< \Lambda_N\Psi_N, \varphi_{\gamma,i} >= \Lambda_N < \Psi_N, \varphi_{\gamma,i} >$$

which together give

$$(\Lambda_N - \mu_{\gamma,i}) < \Psi_N, \varphi_{\gamma,i} >=< \Psi_N, (V(x) - V_0)\varphi_{\gamma,i} > .$$

□

We also call the formula (2.26) as the ”binding formula”.

2.2 Resonance and Non-Resonance Domains

As in papers Veliev (1987)-Veliev (2008), we divide the eigenvalues $|\gamma|^2$ of the operator $L(0)$ into two groups: Resonance and Non-Resonance eigenvalues. In order to classify the eigenvalues as resonance and non-resonance eigenvalues, we introduce resonance and non-resonance domains. In this section, we define these domains and give some estimations related to these domains.

We divide R^d into two domains: Resonance and Non-resonance domains. In order to define these domains, let us introduce the following sets.

Let $\alpha < \frac{1}{d+20}$, $\alpha_k = 3^k\alpha$, $k = 1, 2, \ldots, d-1$, ρ a large parameter and

$$V_b(\rho^{\alpha_1}) \equiv \{x \in R^d : \quad || x |^2 - | x + b |^2| < \rho^{\alpha_1}\},$$

$$E_1(\rho^{\alpha_1},p)\equiv\bigcup_{b\in\Gamma(p\rho^{\alpha})}V_b(\rho^{\alpha_1}),$$

$$U(\rho^{\alpha_1},p)\equiv R^d\setminus E_1(\rho^{\alpha_1},p),$$

$$E_k(\rho^{\alpha_k},p)\equiv\bigcup_{\gamma_1,\gamma_2,\ldots,\gamma_k\in\Gamma(p\rho^{\alpha})}(\bigcap_{i=1}^{k}V_{\gamma_i}(\rho^{\alpha_k})),$$

where $\Gamma(p\rho^{\alpha})\equiv\{b\in\frac{\Gamma}{2}:0<\mid b\mid<p\rho^{\alpha}\}$, the intersection $\bigcap_{i=1}^{k}V_{\gamma_i}(\rho^{\alpha_k})$ in E_k is taken over $\gamma_1,\gamma_2,\ldots,\gamma_k$ which are linearly independent vectors and the length of γ_i is not greater than the length of the other vectors in $\Gamma\bigcap\gamma_iR$. The set $U(\rho^{\alpha_1},p)$ is said to be a *non-resonance domain*, and the eigenvalue $\mid\gamma\mid^2$ of the operator $L(0)$ is called a non-resonance eigenvalue if $\gamma\in U(\rho^{\alpha_1},p)$. The domains $V_b(\rho^{\alpha_1})$ for all $b\in\Gamma(p\rho^{\alpha})$ are called *resonance domains*, and the eigenvalue $\mid\gamma\mid^2$ of the operator $L(0)$ is a resonance eigenvalue if $\gamma\in V_b(\rho^{\alpha_1})$.

The elements of the single resonance domain

$$V_b(\rho^{\alpha_1})=\{x\in R^d:\quad\mid\mid x\mid^2-\mid x+b\mid^2\mid<\rho^{\alpha_1}\}$$

are contained between the two hyperplanes

$$\Pi_1=\{x:\quad\mid\mid x\mid^2-\mid x+b\mid^2\mid=-\rho^{\alpha_1}\}$$

and

$$\Pi_2=\{x:\quad\mid\mid x\mid^2-\mid x+b\mid^2\mid=\rho^{\alpha_1}\}.$$

Π_1 and Π_2 are indeed the hyperplanes

$$\begin{aligned}\Pi_1=\{x:(x+\frac{b}{2}+\frac{\rho^{\alpha_1}b}{2\mid b\mid^2})\cdot b=0\}&=(\frac{b}{2}+\frac{\rho^{\alpha_1}b}{2\mid b\mid^2})+\Pi_b,\\ \Pi_2=\{x:(x+\frac{b}{2}-\frac{\rho^{\alpha_1}b}{2\mid b\mid^2})\cdot b=0\}&=(\frac{b}{2}-\frac{\rho^{\alpha_1}b}{2\mid b\mid^2})+\Pi_b,\end{aligned}$$

where $\Pi_b=\{x:x\cdot b=0\}$ is the hyperplane passing through the origin. This can be seen by using the following calculation

$$\mid x\mid^2-\mid x+b\mid^2=(x\cdot x)-[(x+b)\cdot(x+b)]=-2(x\cdot b)-\mid b\mid^2=\mp\rho^{\alpha_1},$$

$$x \cdot b + \frac{|\, b \,|^2}{2} \mp \frac{\rho^{\alpha_1}}{2} = 0.$$

We have the following lemma from Karakılıç (2004).

Lemma 2.11. *The non-resonance domain has asymptotically full measure on R^d, that is,*

$$\frac{\mu(U(\rho^{\alpha_1}, p) \bigcap B(\rho))}{\mu(B(\rho))} \to 1 \quad \text{as} \quad \rho \to \infty,$$

where $B(\rho) = \{x \in R^d : |\, x \,| \le \rho\}$.

Proof. It is clear that $V_b(\rho^{\alpha_1}) \bigcap B(\rho)$ is the part of $B(\rho)$ which is contained between the two parallel hyperplanes Π_1 and Π_2. Since the distance between these hyperplanes is $\frac{\rho^{\alpha_1}}{|b|}$, we have

$$\mu(V_b(\rho^{\alpha_1}) \bigcap B(\rho)) = O(\rho^{d-1+\alpha_1}).$$

The number of vectors in $\Gamma(p\rho^{\alpha})$ is $O(\rho^{d\alpha})$ and $\mu(B(\rho)) \sim \rho^d$, where $f(\rho) \sim g(\rho)$ means that there are positive independent of ρ constants c_1 and c_2 such that $c_1 \mid g(\rho) \mid < \mid f(\rho) \mid < c_2 \mid g(\rho) \mid$. Thus

$$\mu(\bigcup_{b \in \Gamma(p\rho^{\alpha})} V_b(\rho^{\alpha_1}) \bigcap B(\rho)) = O(\rho^{d-1+\alpha_1+d\alpha}) = \mu(B(\rho)) O(\rho^{d\alpha+\alpha_1-1}). \qquad (2.27)$$

Using that, $R^d = U(\rho^{\alpha_1}, p) \cup E_1$, and

$$R^d \bigcap B(\rho) = (U(\rho^{\alpha_1}, p) \bigcap B(\rho)) \bigcup (E_1 \bigcap B(\rho)),$$

we have

$$\mu(B(\rho)) = \mu(U(\rho^{\alpha_1}, p) \bigcap B(\rho)) + \mu(E_1 \bigcap B(\rho))$$

which together with (2.27) imply

$$\mu(U(\rho^{\alpha_1}, p) \bigcap B(\rho)) = \mu(B(\rho))(1 - O(\rho^{d\alpha+\alpha_1-1})).$$

Thus from (2.27) the result follows, since $\alpha_1 + d\alpha < 1$. That is, the domain $U(\rho^{\alpha_1}, p)$ has asymptotically full measure on R^d. □

Lemma 2.11 implies that the number of non-resonance eigenvalues is essentially greater than the number of resonance eigenvalues. Namely, if $N_n(\rho)$ and $N_r(\rho)$ denote the number of $\gamma \in U(\rho^{\alpha}, p) \bigcap (R(2\rho) \setminus R(\rho))$ and $\gamma \in \bigcup_{b \in \Gamma(p\rho^{\alpha})} V_b(\rho^{\alpha}) \bigcap (R(2\rho) \setminus R(\rho))$, respectively, then

$$\frac{N_r(\rho)}{N_n(\rho)} = O(\rho^{(d+1)\alpha-1}) = o(1) \tag{2.28}$$

for $(d+1)\alpha < 1$ where $R_\rho = \{x \in R^d :| x |= \rho\}$.

2.3 Preliminary Results

In this section, we give some relations on the eigenfunctions of the operator $L(0)$ and the expansion of the potential $V(x)$ with respect to these eigenfunctions which is obtained in Karakılıç, Atılgan, & Veliev (2005). These will help us to simplify our own proofs.

Consider the function $u_\gamma(x) = cos\gamma^1 x_1 cos\gamma^2 x_2 \cdots cos\gamma^d x_d$ where $\gamma = (\gamma^1, \gamma^2, \ldots, \gamma^d) \in \frac{\Gamma^{+0}}{2}$, $\frac{\Gamma^{+0}}{2} = \{(\frac{n_1\pi}{a_1}, \frac{n_2\pi}{a_2}, \cdots, \frac{n_d\pi}{a_d}) : \quad n_i \in Z^+ \bigcup\{0\}, \quad i = 1, 2, \ldots, d\}$. The norm of the function $u_\gamma(x)$ in $L_2(Q)$ is

$$\| u_\gamma(x) \| = \sqrt{\frac{a_1 a_2 \ldots a_d}{2^{d-k}}},$$

where k, $0 \le k \le d$ is the number of components γ^i of the vector $\gamma = (\gamma^1, \gamma^2, \ldots, \gamma^d)$ such that $\gamma^i = 0$. Equivalently,

$$\| u_\gamma(x) \| = \sqrt{\frac{\mu(Q)}{| A_\gamma |}},$$

where $\mu(Q)$ is the measure of Q, $A_\gamma = \{\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d) \in \frac{\Gamma}{2} : \quad | \alpha_i | = | \gamma^i |, \quad i = 1, 2, \ldots, d\}$, $\frac{\Gamma}{2} = \{(\frac{n_1\pi}{a_1}, \frac{n_2\pi}{a_2}, \ldots, \frac{n_d\pi}{a_d}) : \quad n_i \in Z, \quad i = 1, 2, \ldots, d\}$, $| A_\gamma |$ is the number of vectors in A_γ.

The function $u_\gamma(x) = cos\gamma^1 x_1 cos\gamma^2 x_2 \cdots cos\gamma^d x_d$ where $\gamma \in \frac{\Gamma^{+0}}{2}$ can be written as

$$u_\gamma(x) = \frac{1}{| A_\gamma |} \sum_{\alpha \in A_\gamma} e^{i\{\alpha \cdot x\}}. \tag{2.29}$$

For the sake of simplicity, from now on we will use $u_\gamma(x)$ of the form (2.29).

Lemma 2.12.

$$(\sum_{\widetilde{\gamma} \in A_a} e^{i\{\widetilde{\gamma} \cdot x\}})(\sum_{\alpha \in A_\gamma} e^{i\{\alpha \cdot x\}}) = \sum_{\widetilde{\gamma} \in A_a} \sum_{\alpha \in A_{\gamma+\widetilde{\gamma}}} e^{i\{\alpha \cdot x\}} \tag{2.30}$$

for all $\gamma, \widetilde{\gamma} \in \frac{\Gamma}{2}$.

Proof. For the proof see Karakılıç, Atılgan, & Veliev (2005). □

Lemma 2.13. *Let* $u_\gamma(x) = \frac{1}{|A_\gamma|} \sum\limits_{\alpha \in A_\gamma} e^{i\{\alpha \cdot x\}}$ *be the eigenfunction of the operator* (2.16)-(2.17) *for any* $j = 1, 2, \ldots, m$, *for all* $i = 1, 2, \ldots, d$. *Then*

$$u_a(x) u_\gamma(x) = \frac{1}{|A_a|} \sum_{\widetilde{\gamma} \in A_a} u_{\gamma + \widetilde{\gamma}}(x)$$

for all $\gamma \in \frac{\Gamma}{2}$, $\gamma \notin V_{e_k}(\rho^{\alpha_1})$, $k = 1, 2, \ldots, d$ *and* $a \in \Gamma(\rho^\alpha)$.

Proof. For the proof see Karakılıç, Atılgan, & Veliev (2005). □

It is clear that $\{u_\gamma(x) = \frac{1}{|A_\gamma|} \sum\limits_{\alpha \in A_\gamma} e^{i\{\alpha \cdot x\}}\}_{\gamma \in \frac{\Gamma+0}{2}}$ is a complete system in $L_2(Q)$. So for any $v(x)$ in $L_2(Q)$ we have

$$v(x) = \sum_{\gamma \in \frac{\Gamma+0}{2}} \frac{|A_\gamma|}{\mu(Q)} (v(x), u_\gamma(x)) u_\gamma(x). \tag{2.31}$$

Using the decomposition (2.31) and the obvious relations

$$u_\gamma(x) = u_\alpha(x), \quad (v(x), u_\gamma(x)) = (v(x), u_\alpha(x)), \quad \forall \alpha \in A_\gamma,$$

$$\frac{\Gamma}{2} = \bigcup_{\gamma \in \frac{\Gamma+0}{2}} A_\gamma, \quad (v(x), u_\gamma(x)) = \frac{1}{|A_\gamma|} \sum_{\alpha \in A_\gamma} (v(x), u_\alpha(x)),$$

we have

$$\begin{aligned} v(x) &= \sum_{\gamma \in \frac{\Gamma+0}{2}} \frac{|A_\gamma|}{\mu(Q)} (v(x), u_\gamma(x)) u_\gamma(x) \\ &= \sum_{\gamma \in \frac{\Gamma+0}{2}} \frac{|A_\gamma|}{\mu(Q)} \frac{1}{|A_\gamma|} \sum_{\alpha \in A_\gamma} (v(x), u_\alpha(x)) u_\alpha(x) \\ &= \sum_{\gamma \in \frac{\Gamma}{2}} \frac{1}{\mu(Q)} (v(x), u_\gamma(x)) u_\gamma(x). \end{aligned}$$

So one can write

$$v(x) = \sum_{\gamma \in \frac{\Gamma}{2}} v_\gamma u_\gamma(x), \tag{2.32}$$

where $v_\gamma = \frac{1}{\mu(Q)}(v(x), u_\gamma(x))$. Since the decompositions (2.31) and (2.32) are equivalent, for the sake of simplicity, we use the decomposition (2.32) instead of the decomposition (2.31). (Karakılıç, Atılgan, & Veliev (2005))

Hence, each entry $v_{ij}(x) \in L_2(Q)$ of the matrix $V(x)$ can be written in its Fourier series expansion

$$v_{ij}(x) = \sum_{\gamma \in \frac{\Gamma}{2}} v_{ij\gamma} u_\gamma(x)$$

for $i, j = 1, 2, \ldots, m$ where $v_{ij\gamma} = \frac{(v_{ij}(x), u_\gamma(x))}{\mu(Q)}$.

Assumption on the Potential $V(x)$**:** In this study, we assume that the Fourier coefficients $v_{ij\gamma}$ of $v_{ij}(x)$ satisfy

$$\sum_{\gamma \in \frac{\Gamma}{2}} | v_{ij\gamma} |^2 (1 + | \gamma |^{2l}) < \infty \qquad (2.33)$$

for each $i, j = 1, 2, \ldots, m$ where $l > \frac{(d+20)(d-1)}{2} + d + 3$ which implies

$$v_{ij}(x) = \sum_{\gamma \in \Gamma^{+0}(\rho^\alpha)} v_{ij\gamma} u_\gamma(x) + O(\rho^{-p\alpha}), \qquad (2.34)$$

where $\Gamma^{+0}(\rho^\alpha) = \{\gamma \in \frac{\Gamma}{2} : 0 \leq | \gamma | < \rho^\alpha\}$, $p = l - d$, $\alpha < \frac{1}{d+20}$, ρ is a large parameter and $O(\rho^{-p\alpha})$ is a function in $L_2(R^d)$ whose norm is big-oh of $\rho^{-p\alpha}$.

Indeed, we have

$$\| \sum_{\gamma \in \frac{\Gamma}{2} \setminus \Gamma^{+0}(\rho^\alpha)} v_{ij\gamma} u_\gamma(x) \|^2 = \| \sum_{|\gamma| > \rho^\alpha} v_{ij\gamma} u_\gamma(x) \|^2 = \sum_{|\gamma| > \rho^\alpha} | v_{ij\gamma} |^2 \| u_\gamma(x) \|^2$$

$$= \sum_{|\gamma| > \rho^\alpha} | v_{ij\gamma} |^2 \frac{a_1 a_2 \ldots a_d}{2^{d-k}} \leq a_1 a_2 \ldots a_d \sum_{|\gamma| > \rho^\alpha} | v_{ij\gamma} |^2 = a_1 a_2 \ldots a_d \sum_{|\gamma| > \rho^\alpha} \left[\frac{| v_{ij\gamma} | | \gamma |^l}{| \gamma |^l} \right]^2$$

$$\leq a_1 a_2 \ldots a_d \left[\sum_{|\gamma| > \rho^\alpha} \frac{| v_{ij\gamma} | | \gamma |^l}{| \gamma |^l} \right]^2 \leq a_1 a_2 \ldots a_d \left[\left(\sum_{|\gamma| > \rho^\alpha} (| v_{ij\gamma} | | \gamma |^l)^2 \right)^{\frac{1}{2}} \left(\sum_{|\gamma| > \rho^\alpha} \frac{1}{| \gamma |^{2l}} \right)^{\frac{1}{2}} \right]^2$$

$$= a_1 a_2 \ldots a_d \left(\sum_{|\gamma| > \rho^\alpha} | v_{ij\gamma} |^2 | \gamma |^{2l} \right) \left(\sum_{|\gamma| > \rho^\alpha} \frac{1}{| \gamma |^{2l}} \right).$$

The first sum in the last expression is convergent by (2.33). The second sum is big-oh of $\rho^{-p\alpha}$ by using the integral test. Thus (2.34) holds.

Furthermore, the assumption (2.33) implies

$$M_{ij} \equiv \sum_{\gamma \in \frac{\Gamma}{2}} \mid v_{ij\gamma} \mid < \infty \tag{2.35}$$

for all $i, j = 1, 2, \ldots, m$.

The series

$$\left(\sum_{\gamma \in \frac{\Gamma}{2}} \mid v_{ij\gamma} \mid^2 \mid \gamma \mid^{2l} \right)^{\frac{1}{2}}$$

converges by (2.33). Since $l > \frac{(d+20)(d-1)}{2} + d + 3$ and $d \geq 2$, we have $2l > 1$. So the series

$$\left(\sum_{\gamma \in \frac{\Gamma}{2}} \frac{1}{\mid \gamma \mid^{2l}} \right)^{\frac{1}{2}}$$

also converges. Then by using Cauchy-Schwarz inequality, we get

$$\sum_{\gamma \in \frac{\Gamma}{2}} \mid v_{ij\gamma} \mid = \sum_{\gamma \in \frac{\Gamma}{2}} \frac{\mid v_{ij\gamma} \mid\mid \gamma \mid^l}{\mid \gamma \mid^l} \leq \left(\sum_{\gamma \in \frac{\Gamma}{2}} \mid v_{ij\gamma} \mid^2 \mid \gamma \mid^{2l} \right)^{\frac{1}{2}} \left(\sum_{\gamma \in \frac{\Gamma}{2}} \frac{1}{\mid \gamma \mid^{2l}} \right)^{\frac{1}{2}}$$

from which (2.35) follows.

By means of the relation (2.35), we define the constants

$$M_i = \sum_{j=1}^{m} M_{ij}, \quad M_j = \sum_{i=1}^{m} M_{ij}, \quad M^2 = \max_{1 \leq i \leq m} M_i \max_{1 \leq j \leq m} M_j. \tag{2.36}$$

If $v(x) \in W_2^l(Q)$ and the support of $gradv(x) = (\frac{\partial v}{\partial x_1}, \frac{\partial v}{\partial x_2}, \ldots, \frac{\partial v}{\partial x_d})$ is contained in the interior of the domain Q, then $v(x)$ satisfies the condition (2.33) (see Hald, & McLaughlin (1996)). Another class of functions satisfying the condition (2.33) is the class of functions $v(x) \in W_2^l(Q)$ such that $v(x) = \sum_{\gamma \in \Gamma} v_\gamma u_\gamma(x)$ which is periodic with respect to Ω (see Section 1.3).

Lemma 2.14.

$$\sum_{\widetilde{\gamma} \in \Gamma(\rho^\alpha)} v_{\widetilde{\gamma}} u_{\widetilde{\gamma}}(x) u_\gamma(x) = \sum_{\widetilde{\gamma} \in \Gamma(\rho^\alpha)} v_{\widetilde{\gamma}} u_{\gamma + \widetilde{\gamma}}(x) \tag{2.37}$$

for all $\gamma \in \frac{\Gamma}{2}$, $\gamma \notin V_{e_k}(\rho^{\alpha_1})$.

Proof. For the proof see Karakılıç, Atılgan, & Veliev (2005). □

CHAPTER THREE
HIGH ENERGY ASYMPTOTICS
FOR THE EIGENVALUES OF THE OPERATOR $L(V)$

3.1 Asymptotic Formulas for the Eigenvalues in the Non-Resonance Domain

In this section, we improve the results in Coşkan, & Karakılıç (2009) which are also obtained during this study.

We consider the eigenvalues $|\gamma|^2$ of the operator $L(0)$ such that $|\gamma|\sim\rho$ where $|\gamma|\sim\rho$ means that $|\gamma|$ and ρ are asymptotically equal, that is, $c_1\rho\leq|\gamma|\leq c_2\rho$, c_i, $i=1,2,3,\ldots$ are positive real constants which do not depend on ρ and ρ is a large parameter.

We decompose $V(x)\Phi_{\gamma,j}(x)$ with respect to the basis $\{\Phi_{\gamma',i}(x)\}_{\gamma'\in\frac{\Gamma}{2},i=1,2,\ldots,m}$.
By definition of $\Phi_{\gamma,j}(x)$, it is obvious that

$$V(x)\Phi_{\gamma,j}(x)=(v_{1j}(x)u_\gamma(x),\ldots,v_{mj}(x)u_\gamma(x)). \tag{3.38}$$

Substituting the decomposition (2.34) of $v_{ij}(x)$ into (3.38), we get

$$V(x)\Phi_{\gamma,j}(x)=(\sum_{\gamma'\in\Gamma^{+0}(\rho^\alpha)}v_{1j\gamma'}u_{\gamma'}(x)u_\gamma(x),\ldots,\sum_{\gamma'\in\Gamma^{+0}(\rho^\alpha)}v_{mj\gamma'}u_{\gamma'}(x)u_\gamma(x))+O(\rho^{-p\alpha}). \tag{3.39}$$

Using (2.37) in (3.39), we obtain

$$\begin{aligned}V(x)\Phi_{\gamma,j}(x) &= (\sum_{\gamma'\in\Gamma^{+0}(\rho^\alpha)}v_{1j\gamma'}u_{\gamma+\gamma'}(x),\ldots,\sum_{\gamma'\in\Gamma^{+0}(\rho^\alpha)}v_{mj\gamma'}u_{\gamma+\gamma'}(x))+O(\rho^{-p\alpha})\\ &= \sum_{i=1}^{m}\sum_{\gamma'\in\Gamma^{+0}(\rho^\alpha)}v_{ij\gamma'}\Phi_{\gamma+\gamma',i}(x)+O(\rho^{-p\alpha}).\end{aligned} \tag{3.40}$$

The analogues of the following lemma can be found in Karakılıç (2004).

Lemma 3.15. *Let $\gamma\in U(\rho^{\alpha_1},p)$, that is, $|\gamma|^2$ be a non-resonance eigenvalue of the* *operator* $L(0)$, Λ_N *an eigenvalue of the operator* $L(V)$ *satisfying the inequality*

$$|\,\Lambda_N - |\,\gamma\,|^2| < \frac{1}{2}\rho^{\alpha_1}. \tag{3.41}$$

Then

$$|\,\Lambda_N - |\,\gamma + b\,|^2| > \frac{1}{2}\rho^{\alpha_1} \tag{3.42}$$

for all $b \in \Gamma(p\rho^{\alpha})$.

Proof. If $\gamma \in U(\rho^{\alpha_1}, p)$ then $||\,\gamma\,|^2 - |\,\gamma + b\,|^2| > \rho^{\alpha_1}$ for all $b \in \Gamma(p\rho^{\alpha})$ which together with $|\,\Lambda_N - |\,\gamma\,|^2| < \frac{1}{2}\rho^{\alpha_1}$ implies

$$|\,\Lambda_N - |\,\gamma + b\,|^2| \geq ||\,\Lambda_N - |\,\gamma\,|^2| - ||\,\gamma + b\,|^2 - |\,\gamma\,|^2|| > \frac{1}{2}\rho^{\alpha_1}.$$

□

We define the following $m \times m$ matrices.

$$D(\Lambda_N, \gamma) \equiv (\Lambda_N - |\gamma|^2)I - V_0,$$

$$S(a, p_1) \equiv \sum_{k=1}^{p_1} S^k(a),$$

where

$$S^k(a) = (s^k_{ji}(a)), \quad k = 1, 2, \ldots, p_1, \quad j, i = 1, 2, \ldots, m,$$

$$s^k_{ji}(a) = \sum_{i_1, i_2, \ldots, i_k = 1}^{m} \sum_{\substack{\gamma_1, \gamma_2, \ldots, \gamma_{k+1} \in \Gamma + 0(\rho^{\alpha}) \\ \gamma_1 + \gamma_2 + \cdots + \gamma_{k+1} = 0}} \frac{v_{i_1 j \gamma_1} v_{i_2 i_1 \gamma_2} \ldots v_{i i_k \gamma_{k+1}}}{(a - |\,\gamma + \gamma_1\,|^2) \ldots (a - |\,\gamma + \gamma_1 + \cdots + \gamma_k\,|^2)}.$$

We note that since $V(x)$ is symmetric, V_0 and $S(a, p_1)$ are symmetric real valued matrices. Hence $D(\Lambda_N, \gamma) - S(a, p_1)$ is a symmetric real valued matrix. We denote the eigenvalues and the corresponding normalized eigenvectors of the matrix $D(\Lambda_N, \gamma) - S(a, p_1)$ by $\beta_i \equiv \beta_i(\Lambda_N, \gamma, a)$ and $f_i \equiv f_i(\Lambda_N, \gamma, a)$, respectively. That is,

$$[D(\Lambda_N, \gamma) - S(a, p_1)] f_i = \beta_i f_i, \tag{3.43}$$

where $f_i \cdot f_j = \delta_{ij}$, $i, j = 1, 2, \ldots, m$.
We denote by $A(N, \gamma)$ the $m \times 1$ vector

$$A(N, \gamma) = (<\Psi_N, \Phi_{\gamma,1}>, <\Psi_N, \Phi_{\gamma,2}>, \ldots, <\Psi_N, \Phi_{\gamma,m}>).$$

Lemma 3.16. *Let* $|\gamma|^2$ *be a non-resonance eigenvalue of the operator* $L(0)$ *with* $|\gamma|\sim\rho$.
(a) *Let* β_i *be an eigenvalue of the matrix* $D(\Lambda_N,\gamma)-S(a,p_1)$ *and* $f_i=(f_{i_1},f_{i_2},\ldots,f_{i_m})$ *its corresponding normalized eigenvector. Then there exists an integer* $N\equiv N_i$ *such that* Λ_N *satisfies the inequality* (3.41) *and*

$$|A(N,\gamma)\cdot f_i|>c_3\rho^{\frac{-(d-1)}{2}}. \tag{3.44}$$

(b) *Let* Λ_N *be an eigenvalue of the operator* $L(V)$ *satisfying the inequality* (3.41). *Then there exists an eigenfunction* $\Phi_{\gamma,i}(x)$ *of the operator* $L(0)$ *such that*

$$|<\Phi_{\gamma,i},\Psi_N>|>c_4\rho^{\frac{-(d-1)}{2}} \tag{3.45}$$

holds.

Proof. We prove the lemma by using the same consideration as in Karakılıç (2004).
(a) We use a result from perturbation theory which states that the Nth eigenvalue of the operator $L(V)$ lies in M-neighborhood of the Nth eigenvalue of the operator $L(0)$. Let the Nth eigenvalues of $L(V)$ and $L(0)$ be Λ_N and $|\gamma|^2$, respectively. Then there is an integer N such that $|\Lambda_N-|\gamma|^2|<\frac{1}{2}\rho^{\alpha_1}$.
On the other hand, since $L(V)$ is a self adjoint operator, the eigenfunctions $\{\Psi_N(x)\}_{N=1}^{\infty}$ of $L(V)$ form an orthonormal basis for $L_2^m(Q)$. By Parseval's relation, we have

$$\begin{aligned}\|\sum_{j=1}^{m}f_{ij}\Phi_{\gamma,j}\|^2 &= \sum_{N:|\Lambda_N-|\gamma|^2|<\frac{1}{2}\rho^{\alpha_1}}|<\sum_{j=1}^{m}f_{ij}\Phi_{\gamma,j},\Psi_N>|^2\\ &+\sum_{N:|\Lambda_N-|\gamma|^2|\geq\frac{1}{2}\rho^{\alpha_1}}|<\sum_{j=1}^{m}f_{ij}\Phi_{\gamma,j},\Psi_N>|^2 .\end{aligned} \tag{3.46}$$

Now, we estimate the last expression in (3.46). By using the Cauchy-Schwarz inequality and the binding formula (2.25), we get

$$\sum_{N:|\Lambda_N-|\gamma|^2|\geq\frac{1}{2}\rho^{\alpha_1}}|<\sum_{j=1}^{m}f_{ij}\Phi_{\gamma,j},\Psi_N>|^2=\sum_{N:|\Lambda_N-|\gamma|^2|\geq\frac{1}{2}\rho^{\alpha_1}}|\sum_{j=1}^{m}f_{ij}<\Phi_{\gamma,j},\Psi_N>|^2$$

$$\begin{aligned}
&\leq \sum_{N:|\Lambda_N-|\gamma|^2|\geq\frac{1}{2}\rho^{\alpha_1}} [\sum_{j=1}^{m} | f_{ij} |^2 \sum_{j=1}^{m} |<\Psi_N,\Phi_{\gamma,j}>|^2] \\
&= \sum_{N:|\Lambda_N-|\gamma|^2|\geq\frac{1}{2}\rho^{\alpha_1}} \sum_{j=1}^{m} \frac{|<\Psi_N,V\Phi_{\gamma,j}>|^2}{|\Lambda_N-|\gamma|^2|^2} \\
&\leq (\frac{1}{2}\rho^{\alpha_1})^{-2} \sum_{N:|\Lambda_N-|\gamma|^2|\geq\frac{1}{2}\rho^{\alpha_1}} \sum_{j=1}^{m} |<\Psi_N,V\Phi_{\gamma,j}>|^2 \\
&\leq (\frac{1}{2}\rho^{\alpha_1})^{-2} \sum_{j=1}^{m} \| V\Phi_{\gamma,j} \|^2
\end{aligned}$$

from which together with the relation (2.35) we obtain

$$\sum_{N:|\Lambda_N-|\gamma|^2|\geq\frac{1}{2}\rho^{\alpha_1}} |<\sum_{j=1}^{m} f_{ij}\Phi_{\gamma,j},\Psi_N>|^2 = O(\rho^{-2\alpha_1}).$$

It follows from the last equation and (3.46) that

$$\sum_{N:|\Lambda_N-|\gamma|^2|<\frac{1}{2}\rho^{\alpha_1}} |<\sum_{j=1}^{m} f_{ij}\Phi_{\gamma,j},\Psi_N>|^2 = \sum_{N:|\Lambda_N-|\gamma|^2|<\frac{1}{2}\rho^{\alpha_1}} |A(N,\gamma)\cdot f_i|^2 = 1-O(\rho^{-2\alpha_1}). \tag{3.47}$$

On the other hand, if $a\sim\rho$, then the number of $\gamma\in\frac{\Gamma}{2}$ satisfying $||\gamma|^2-a^2|<1$ is less than $c_5\rho^{d-1}$. Therefore, the number of eigenvalues of $L(0)$ lying in (a^2-1,a^2+1) is less than $c_6\rho^{d-1}$. By this result and the result of perturbation theory, the number of eigenvalues Λ_N of $L(V)$ in the interval $[|\gamma|^2-\frac{1}{2}\rho^{\alpha_1},|\gamma|^2+\frac{1}{2}\rho^{\alpha_1}]$ is less than $c_7\rho^{d-1}$. Thus

$$1-O(\rho^{-2\alpha_1}) = \sum_{N:|\Lambda_N-|\gamma|^2|<\frac{1}{2}\rho^{\alpha_1}} |A(N,\gamma)\cdot f_i|^2 < c_7\rho^{d-1} |A(N,\gamma)\cdot f_i|^2 \tag{3.48}$$

from which we get the estimation (3.44).

(b) Since $L(0)$ is a self adjoint operator, the set of eigenfunctions $\{\Phi_{\gamma,i}(x)\}_{\gamma\in\frac{\Gamma}{2},i=1,2,\ldots,m}$ of $L(0)$ forms an orthonormal basis for $L_2^m(Q)$. By Parseval's relation, we have

$$\|\Psi_N\|^2 = \sum_{\gamma:|\Lambda_N-|\gamma|^2|<\frac{1}{2}\rho^{\alpha_1}} \sum_{i=1}^{m} |<\Psi_N,\Phi_{\gamma,i}>|^2 + \sum_{\gamma:|\Lambda_N-|\gamma|^2|\geq\frac{1}{2}\rho^{\alpha_1}} \sum_{i=1}^{m} |<\Psi_N,\Phi_{\gamma,i}>|^2 . \tag{3.49}$$

We estimate the last expression in (3.49). For a fixed $i=1,2,\ldots,m$ using the binding formula (2.25) together with the relation (2.35), we get

$$\sum_{\gamma:|\Lambda_N-|\gamma|^2|\geq\frac{1}{2}\rho^{\alpha_1}}\sum_{i=1}^{m}|<\Psi_N,\Phi_{\gamma,i}>|^2 = \sum_{\gamma:|\Lambda_N-|\gamma|^2|\geq\frac{1}{2}\rho^{\alpha_1}}\sum_{i=1}^{m}\frac{|<\Psi_N,V\Phi_{\gamma,i}>|^2}{|\Lambda_N-|\gamma|^2|^2}$$
$$\leq (\frac{1}{2}\rho^{\alpha_1})^{-2}\sum_{\gamma:|\Lambda_N-|\gamma|^2|\geq\frac{1}{2}\rho^{\alpha_1}}\sum_{i=1}^{m}|<V\Psi_N,\Phi_{\gamma,i}>|^2$$
$$\leq (\frac{1}{2}\rho^{\alpha_1})^{-2}\parallel V\Psi_N\parallel^2, \qquad (3.50)$$

that is,

$$\sum_{\gamma:|\Lambda_N-|\gamma|^2|\geq\frac{1}{2}\rho^{\alpha_1}}\sum_{i=1}^{m}|<\Psi_N,\Phi_{\gamma,i}>|^2=O(\rho^{-2\alpha_1}).$$

From the last equality and (3.49) we obtain

$$\sum_{\gamma:|\Lambda_N-|\gamma|^2|<\frac{1}{2}\rho^{\alpha_1}}\sum_{i=1}^{m}|<\Psi_N,\Phi_{\gamma,i}>|^2=1-O(\rho^{-2\alpha_1}).$$

Arguing as in the proof of part(a), we get

$$1-O(\rho^{-2\alpha_1})=\sum_{\gamma:|\Lambda_N-|\gamma|^2|<\frac{1}{2}\rho^{\alpha_1}}\sum_{i=1}^{m}|<\Psi_N,\Phi_{\gamma,i}>|^2\leq c_8\rho^{d-1}\,|<\Psi_N,\Phi_{\gamma,i}>|^2$$

from which the estimation (3.45) follows. □

Theorem 3.17. *Let* $|\gamma|^2$ *be a non-resonance eigenvalue of the operator* $L(0)$ *with* $|\gamma|\sim\rho$.
(a) *For each eigenvalue* λ_i, $i=1,2,\ldots,m$ *of the matrix* V_0 *there exists an eigenvalue* Λ_N *of the operator* $L(V)$ *satisfying*

$$\Lambda_N=|\gamma|^2+\lambda_i+O(\rho^{-\alpha_1}). \qquad (3.51)$$

(b) *For each eigenvalue* Λ_N *of the operator* $L(V)$ *satisfying the inequality* (3.41)*, there exists an eigenvalue* λ_i *of the matrix* V_0 *satisfying the formula* (3.51).

Proof. **(a)** We prove this part of the theorem by using the same consideration as in Karakılıç (2004). Let $|\gamma|^2$ be a non-resonance eigenvalue of the operator $L(0)$ with $|\gamma|\sim\rho$. By the result of perturbation theory, the Nth eigenvalue Λ_N of the operator $L(V)$ lies in $\frac{1}{2}\rho^{\alpha_1}$ neigborhood of the non-resonance eigenvalue $|\gamma|^2$ of the operator $L(0)$. That is, there exists an integer N such that Λ_N satisfies the inequality (3.41). We consider the binding formula (2.25) for these eigenvalues Λ_N and $|\gamma|^2$.

Substituting the decomposition (3.40) into the binding formula (2.25), we obtain

$$(\Lambda_N - |\gamma|^2) < \Psi_N, \Phi_{\gamma,j} >= \sum_{i_1=1}^{m} \sum_{\gamma_1 \in \Gamma^{+0}(\rho^\alpha)} v_{i_1 j \gamma_1} < \Psi_N, \Phi_{\gamma+\gamma_1, i_1} > + O(\rho^{-p\alpha}).$$

Isolating the terms with the coefficient $< \Psi_N, \Phi_{\gamma,i} >$, that is, $\gamma_1 = 0$, for each $i = 1, 2, \ldots, m$, we get

$$(\Lambda_N - |\gamma|^2) < \Psi_N, \Phi_{\gamma,j} >= \sum_{i=1}^{m} v_{ij0} < \Psi_N, \Phi_{\gamma,i} >$$
$$+ \sum_{i_1=1}^{m} \sum_{\gamma_1 \in \Gamma^{+0}(\rho^\alpha)} v_{i_1 j \gamma_1} < \Psi_N, \Phi_{\gamma+\gamma_1, i_1} > + O(\rho^{-p\alpha}).$$

In the second summation of the above equation, since Λ_N satisfies (3.41) and $\gamma \in U(\rho^{\alpha_1}, p)$, $\gamma_1 \in \Gamma^{+0}(\rho^\alpha)$ with $\gamma_1 \neq 0$, by the inequality (3.42), we obtain

$$(\Lambda_N - |\gamma|^2) < \Psi_N, \Phi_{\gamma,j} >= \sum_{i=1}^{m} v_{ij0} < \Psi_N, \Phi_{\gamma,i} >$$
$$+ \sum_{i_1, i_2=1}^{m} \sum_{\gamma_1, \gamma_2 \in \Gamma^{+0}(\rho^\alpha)} v_{i_1 j \gamma_1} v_{i_2 i_1 \gamma_2} \frac{< \Psi_N, \Phi_{\gamma+\gamma_1+\gamma_2, i_2} >}{(\Lambda_N - |\gamma+\gamma_1|^2)} + O(\rho^{-p\alpha}).$$

Again in the second summation of the above equation isolating the terms with the coefficient $< \Psi_N, \Phi_{\gamma,i} >$, that is, $\gamma_1 + \gamma_2 = 0$, $\gamma_1 \neq 0$ for each $i = 1, 2, \ldots, m$, we get

$$\begin{aligned}(\Lambda_N - |\gamma|^2) < \Psi_N, \Phi_{\gamma,j} >= \sum_{i=1}^{m} v_{ij0} < \Psi_N, \Phi_{\gamma,i} > \\ + \sum_{i_1, i=1}^{m} \sum_{\substack{\gamma_1, \gamma_2 \in \Gamma^{+0}(\rho^\alpha) \\ \gamma_1+\gamma_2=0}} \frac{v_{i_1 j \gamma_1} v_{i i_1 \gamma_2}}{(\Lambda_N - |\gamma+\gamma_1|^2)} < \Psi_N, \Phi_{\gamma,i} > \\ + \sum_{i_1, i_2=1}^{m} \sum_{\gamma_1, \gamma_2 \in \Gamma^{+0}(\rho^\alpha)} \frac{v_{i_1 j \gamma_1} v_{i_2 i_1 \gamma_2}}{(\Lambda_N - |\gamma+\gamma_1|^2)} < \Psi_N, \Phi_{\gamma+\gamma_1+\gamma_2, i_2} > \\ + O(\rho^{-p\alpha}). \qquad (3.52)\end{aligned}$$

Writing this equation for $j = 1, 2, \ldots, m$ and $i = 1, 2, \ldots, m$, after the first step of the iteration we obtain the following system.

$$[(\Lambda_N - |\gamma|^2)I - V_0]A(N,\gamma) = S^1(\Lambda_N)A(N,\gamma) + R^1 + O(\rho^{-p\alpha}),$$

where I is the $m\times m$ identity matrix, $S^1(\Lambda_N) = (s^1_{ji}(\Lambda_N))$ is the $m\times m$ matrix whose entries are

$$s^1_{ji}(\Lambda_N) = \sum_{i_1=1}^{m} \sum_{\substack{\gamma_1,\gamma_2\in\Gamma^{+0}(\rho^\alpha)\\ \gamma_1+\gamma_2=0}} \frac{v_{i_1 j\gamma_1} v_{ii_1\gamma_2}}{(\Lambda_N - |\gamma+\gamma_1|^2)},$$

$j,i = 1,2,\ldots,m$ and $R^1 = (r^1_j)$ is the $m\times 1$ vector whose components are

$$r^1_j = \sum_{i_1,i_2=1}^{m} \sum_{\gamma_1,\gamma_2\in\Gamma^{+0}(\rho^\alpha)} \frac{v_{i_1 j\gamma_1} v_{i_2 i_1\gamma_2}}{(\Lambda_N - |\gamma+\gamma_1+\gamma_2|^2)} < \Psi_N, \Phi_{\gamma+\gamma_1+\gamma_2,i_2} >,$$

$j = 1,2,\ldots,m.$

Now, we continue to iterate the equation (3.52). In the third summation of the equation (3.52), since Λ_N satisfies the inequality (3.41) and $\gamma\in U(\rho^{\alpha_1},p)$, $\gamma_1+\gamma_2\in\Gamma^{+0}(2\rho^\alpha)$ with $\gamma_1+\gamma_2\neq 0$, by the inequality (3.42), we obtain

$$\begin{aligned}(\Lambda_N - |\gamma|^2) < \Psi_N, \Phi_{\gamma,j} >= {} & \sum_{i=1}^{m} v_{ij0} < \Psi_N, \Phi_{\gamma,i} > \\ & + \sum_{i_1,i=1}^{m} \sum_{\substack{\gamma_1,\gamma_2\in\Gamma^{+0}(\rho^\alpha)\\ \gamma_1+\gamma_2=0}} \frac{v_{i_1 j\gamma_1} v_{ii_1\gamma_2}}{(\Lambda_N - |\gamma+\gamma_1|^2)} < \Psi_N, \Phi_{\gamma,i} > \\ & + \sum_{\substack{i_1,i_2,\\ i_3=1}}^{m} \sum_{\substack{\gamma_1,\gamma_2,\\ \gamma_3\in\Gamma^{+0}(\rho^\alpha)}} \frac{v_{i_1 j\gamma_1} v_{i_2 i_1\gamma_2} v_{i_3 i_2\gamma_3}}{(\Lambda_N - |\gamma+\gamma_1|^2)(\Lambda_N - |\gamma+\gamma_1+\gamma_2|^2)} \\ & < \Psi_N, \Phi_{\gamma+\gamma_1+\gamma_2+\gamma_3,i_3} > \\ & + O(\rho^{-p\alpha}).\end{aligned}$$

Isolating the terms with the coefficient $< \Psi_N, \Phi_{\gamma,i} >$ for each $i = 1,2,\ldots,m$, we get

$$(\Lambda_N - |\gamma|^2) < \Psi_N, \Phi_{\gamma,j} >= \sum_{i=1}^{m} v_{ij0} < \Psi_N, \Phi_{\gamma,i} >$$

$$
\begin{aligned}
&+ \sum_{i_1,i=1}^{m} \sum_{\substack{\gamma_1,\gamma_2\in\Gamma^{+0}(\rho^{\alpha})\\ \gamma_1+\gamma_2=0}} \frac{v_{i_1 j\gamma_1} v_{ii_1\gamma_2}}{(\Lambda_N - \mid \gamma+\gamma_1 \mid^2)} < \Psi_N, \Phi_{\gamma,i} > \\
&+ \sum_{i_1,i_2,i=1}^{m} \sum_{\substack{\gamma_1,\gamma_2,\gamma_3\in\Gamma^{+0}(\rho^{\alpha})\\ \gamma_1+\gamma_2+\gamma_3=0}} \frac{v_{i_1 j\gamma_1} v_{i_2 i_1\gamma_2} v_{ii_2\gamma_3}}{(\Lambda_N - \mid \gamma+\gamma_1 \mid^2)(\Lambda_N - \mid \gamma+\gamma_1+\gamma_2 \mid^2)} < \Psi_N, \Phi_{\gamma,i} > \\
&+ \sum_{\substack{i_1,i_2,\\ i_3=1}}^{m} \sum_{\substack{\gamma_1,\gamma_2,\\ \gamma_3\in\Gamma^{+0}(\rho^{\alpha})}} \frac{v_{i_1 j\gamma_1} v_{i_2 i_1\gamma_2} v_{i_3 i_2\gamma_3}}{(\Lambda_N - \mid \gamma+\gamma_1 \mid^2)(\Lambda_N - \mid \gamma+\gamma_1+\gamma_2 \mid^2)} \\
&\quad < \Psi_N, \Phi_{\gamma+\gamma_1+\gamma_2+\gamma_3,i_3} > \\
&+ O(\rho^{-p\alpha}).
\end{aligned}
$$

Again if we write this equation for $j = 1,2,\ldots,m$ and $i = 1,2,\ldots,m$ after the second step of the iteration we obtain the following system.

$$[(\Lambda_N - \mid \gamma \mid^2)I - V_0]A(N,\gamma) = (S^1(\Lambda_N) + S^2(\Lambda_N))A(N,\gamma) + R^2 + O(\rho^{-p\alpha}),$$

where this time $S^2(\Lambda_N) = (s^2_{ji}(\Lambda_N))$,

$$s^2_{ji}(\Lambda_N) = \sum_{i_1,i_2=1}^{m} \sum_{\substack{\gamma_1,\gamma_2,\gamma_3\in\Gamma^{+0}(\rho^{\alpha})\\ \gamma_1+\gamma_2+\gamma_3=0}} \frac{v_{i_1 j\gamma_1} v_{i_2 i_1\gamma_2} v_{ii_2\gamma_3}}{(\Lambda_N - \mid \gamma+\gamma_1 \mid^2)(\Lambda_N - \mid \gamma+\gamma_1+\gamma_2 \mid^2)},$$

$j,i = 1,2,\ldots,m$ and $R^2 = (r^2_j)$,

$$
\begin{aligned}
r^2_j &= \sum_{\substack{i_1,i_2,\\ i_3=1}}^{m} \sum_{\substack{\gamma_1,\gamma_2,\\ \gamma_3\in\Gamma^{+0}(\rho^{\alpha})}} \frac{v_{i_1 j\gamma_1} v_{i_2 i_1\gamma_2} v_{i_3 i_2\gamma_3}}{(\Lambda_N - \mid \gamma+\gamma_1 \mid^2)(\Lambda_N - \mid \gamma+\gamma_1+\gamma_2 \mid^2)} \\
&\quad < \Psi_N, \Phi_{\gamma+\gamma_1+\gamma_2+\gamma_3,i_3} >,
\end{aligned}
$$

$j = 1,2,\ldots,m.$

If we continue to iterate in this manner after the p_1st step where $p_1 = [\frac{p+1}{2}]$ and $[\cdot]$ is the integer function we obtain the following system.

$$[(\Lambda_N - \mid \gamma \mid^2)I - V_0]A(N,\gamma) = (\sum_{k=1}^{p_1} S^k(\Lambda_N))A(N,\gamma) + R^{p_1} + O(\rho^{-p\alpha}), \tag{3.53}$$

where

$$S^k(\Lambda_N) = (s^k_{ji}(\Lambda_N)), \quad k = 1,2,\ldots,p_1, \quad j,i = 1,2,\ldots,m, \tag{3.54}$$

$$s_{ji}^{k}(\Lambda_N)=\sum_{i_1,i_2,\dots,i_k=1}^{m}\sum_{\substack{\gamma_1,\gamma_2,\dots,\gamma_{k+1}\in\Gamma^{+0}(\rho^{\alpha})\\ \gamma_1+\gamma_2+\dots+\gamma_{k+1}=0}}\frac{v_{i_1j\gamma_1}v_{i_2i_1\gamma_2}\dots v_{ii_k\gamma_{k+1}}}{(\Lambda_N-\mid\gamma+\gamma_1\mid^2)\dots(\Lambda_N-\mid\gamma+\gamma_1+\dots+\gamma_k\mid^2)},$$

$$R^{p_1}=(r_j^{p_1}),\quad j=1,2,\dots,m, \tag{3.55}$$

$$r_j^{p_1}=\sum_{\substack{i_1,i_2,\dots,\\ i_{p_1+1}=1}}^{m}\sum_{\substack{\gamma_1,\gamma_2,\dots,\\ \gamma_{p_1+1}\in\Gamma^{+0}(\rho^{\alpha})}}\frac{v_{i_1j\gamma_1}\dots v_{i_{p_1+1}i_{p_1}\gamma_{p_1+1}}<\Psi_N,\Phi_{\gamma+\gamma_1+\dots+\gamma_{p_1+1},i_{p_1+1}}>}{(\Lambda_N-\mid\gamma+\gamma_1\mid^2)\dots(\Lambda_N-\mid\gamma+\gamma_1+\dots+\gamma_{p_1}\mid^2)}.$$

Since Λ_N satisfies the inequality (3.41), $\gamma\in U(\rho^{\alpha_1},p)$ and $\gamma_1+\gamma_2+\dots+\gamma_k\in\Gamma^{+0}(k\rho^{\alpha})$ with $\gamma_1+\gamma_2+\dots+\gamma_k\neq 0$, by the inequality (3.42) and the relation (2.35),

$$\begin{aligned}|r_j^{p_1}| &\leq \sum_{\substack{i_1,i_2,\dots,\\ i_{p_1+1}=1}}^{m}\sum_{\substack{\gamma_1,\gamma_2,\dots,\\ \gamma_{p_1+1}\in\Gamma^{+0}(\rho^{\alpha})}}\frac{|v_{i_1j\gamma_1}|\dots|v_{i_{p_1+1}i_{p_1}\gamma_{p_1+1}}||<\Psi_N,\Phi_{\gamma+\gamma_1+\dots+\gamma_{p_1+1},i_{p_1+1}}>|}{|(\Lambda_N-\mid\gamma+\gamma_1\mid^2)|\dots|(\Lambda_N-\mid\gamma+\gamma_1+\dots+\gamma_{p_1}\mid^2)|}\\ &\leq \frac{1}{(2\rho^{\alpha_1})^{p_1}}\sum_{i_1,i_2,\dots,i_{p_1+1}=1}^{m}M_{i_1j}M_{i_2i_1}\dots M_{i_{p_1+1}i_{p_1}},\end{aligned}$$

that is,

$$\parallel R^{p_1}\parallel=O(\rho^{-p_1\alpha_1}). \tag{3.56}$$

We have chosen $p_1=[\frac{p+1}{2}]$. So by definitions of α, α_1, l and p, we have the inequalities

$$p_1\geq\frac{p}{2},\quad p_1\alpha_1>p\alpha,\quad p>\frac{(d+20)(d-1)}{2}. \tag{3.57}$$

Thus it follows from the equation (3.53) together with the estimation (3.56) and (3.57) that

$$[D(\Lambda_N,\gamma)-S(\Lambda_N,p_1)]A(N,\gamma)=O(\rho^{-p\alpha}). \tag{3.58}$$

Now, let β_i be an eigenvalue of the matrix $D(\Lambda_N,\gamma)-S(\Lambda_N,p_1)$ and f_i its corresponding normalized eigenvector. By Lemma 3.16.a, there exists an integer N_i such that the eigenvalue Λ_{N_i} of the operator $L(V)$ satisfies the inequality (3.41) and the estimation (3.44) holds for $N=N_i$. So letting $N=N_i$ in (3.58) and multiplying both sides of (3.58) by f_i, we obtain

$$\beta_i[A(N,\gamma)\cdot f_i]=O(\rho^{-p\alpha}).$$

Using the estimation (3.44) in the above equation, we get

$$\beta_i=O(\rho^{-(p-\frac{d-1}{2\alpha})\alpha}). \tag{3.59}$$

On the other hand, since $D(\Lambda_N,\gamma)$ and $S(\Lambda_N,p_1)$ are symmetric real valued matrices, by Theorem of Lidskii in Section 1.3, $|\beta_i-(\Lambda_N-|\gamma|^2-\lambda_i)|\leq\|S(\Lambda_N,p_1)\|$ where we have $\|S(\Lambda_N,p_1)\|=O(\rho^{-\alpha_1})$. Because since Λ_N satisfies the inequality (3.41), $\gamma\in U(\rho^{\alpha_1},p)$ and $\gamma_1+\gamma_2+\cdots+\gamma_k\in\Gamma^{+0}(k\rho^{\alpha})$ with $\gamma_1+\gamma_2+\cdots+\gamma_k\neq 0$, by the inequality (3.42) and the relation (2.35),

$$|s^k_{ji}(\Lambda_N)|$$

$$\leq \sum_{i_1,i_2,\ldots,i_k=1}^{m}\sum_{\substack{\gamma_1,\gamma_2,\ldots,\gamma_{k+1}\in\Gamma^{+0}(\rho^{\alpha})\\ \gamma_1+\gamma_2+\cdots+\gamma_{k+1}=0}}\frac{|v_{i_1j\gamma_1}||v_{i_2i_1\gamma_2}|\ldots|v_{ii_k\gamma_{k+1}}|}{|(\Lambda_N-|\gamma+\gamma_1|^2)|\ldots|(\Lambda_N-|\gamma+\gamma_1+\cdots+\gamma_k|^2)|}$$

$$\leq \frac{1}{(2\rho^{\alpha_1})^k}\sum_{i_1,i_2,\ldots,i_k=1}^{m}M_{i_1j}M_{i_2i_1}\ldots M_{ii_k}$$

for each $k=1,2,\ldots,p_1$, $i,j=1,2,\ldots,m$. Thus

$$\|S^k(\Lambda_N)\|=O(\rho^{-k\alpha_1}),\quad\forall k=1,2,\ldots,p_1$$

which implies

$$\|\sum_{k=1}^{p_1}S^k(\Lambda_N)\|=O(\rho^{-\alpha_1}).\tag{3.60}$$

So we have

$$\beta_i=\Lambda_N-|\gamma|^2-\lambda_i+O(\rho^{-\alpha_1}).\tag{3.61}$$

Choosing $p>\frac{d-1}{2\alpha}+1$, using (3.59) and (3.61), we get the result.

(b) Let Λ_N be an eigenvalue of the operator $L(V)$ satisfying (3.41). By Lemma 3.16.b, there exists an eigenfunction $\Phi_{\gamma,i}(x)$ of the operator $L(0)$ satisfying the estimation (3.45) from which we have

$$|A(N,\gamma)|>c_9\rho^{\frac{-(d-1)}{2}}.\tag{3.62}$$

Let $|\gamma|^2$ be the eigenvalue of the operator $L(0)$ whose corresponding eigenfunction $\Phi_{\gamma,i}(x)$ satisfies the estimation (3.45). We consider the binding formula (2.25) for these eigenvalues Λ_N and $|\gamma|^2$. Arguing as in the proof of part(a), we get the equation (3.58)

$$[(\Lambda_N-|\gamma|^2)I-V_0]A(N,\gamma)=S(\Lambda_N,p_1)A(N,\gamma)+O(\rho^{-p\alpha}),$$

where $|\gamma|^2$ is a non-resonance eigenvalue of the operator with $|\gamma|\sim\rho$. Applying $\frac{1}{|A(N,\gamma)|}[(\Lambda_N-|\gamma|^2)I-V_0]^{-1}$ to both sides of the above equation, taking norm of both sides, and using the inequality (3.62), we obtain

$$1 \leq \|[(\Lambda_N - |\gamma|^2)I - V_0]^{-1}\| \| \sum_{k=1}^{p_1} S^k(\Lambda_N)\| + \|[(\Lambda_N - |\gamma|^2)I - V_0]^{-1}\| [O(\rho^{-(p\alpha - \frac{(d-1)}{2})}].$$

By using the estimation (3.60), we get

$$1 \leq \max_{i=1,2,\ldots,m} \frac{1}{|\Lambda_N - |\gamma|^2 - \lambda_i|} [O(\rho^{-\alpha_1}) + O(\rho^{-(p\alpha - \frac{d-1}{2})})].$$

Choosing $p > \frac{d-1}{2\alpha} + 1$, we obtain

$$\min_{i=1,2,\ldots,m} |\Lambda_N - |\gamma|^2 - \lambda_i| \leq c_{10}\rho^{-\alpha_1},$$

where minimum is taken over all eigenvalues of the matrix V_0 from which we obtain the result. □

Corollary 3.18. ***(a)*** *Let $\mu_{\gamma,i}$ be an eigenvalue of the operator $L(V_0)$ where $\gamma \in U(\rho^{\alpha_1}, p)$ with $|\gamma| \sim \rho$ and $i = 1, 2, \ldots, m$. Then there is an eigenvalue Λ_N of the operator $L(V)$ satisfying*

$$\Lambda_N = \mu_{\gamma,i} + O(\rho^{-\alpha_1}). \tag{3.63}$$

(b) *For each eigenvalue Λ_N of the operator $L(V)$ satisfying the inequality* (3.41) *there is an eigenvalue $\mu_{\gamma,i}$ of the operator $L(V_0)$ satisfying the formula* (3.63).

Proof. The proof follows from the proof of Theorem 3.17. □

Remark 3.19. We note that to obtain the estimations (3.60) and (3.56), we have only used the assumption that Λ_N satisfies the inequality (3.41), that is, $\Lambda_N \in J$ where $J = [|\gamma|^2 - \frac{1}{2}\rho^{\alpha_1}, |\gamma|^2 + \frac{1}{2}\rho^{\alpha_1}]$. Hence we may write

$$\| \sum_{k=1}^{p_1} S^k(a) \| = O(\rho^{-\alpha_1}), \qquad \forall a \in J. \tag{3.64}$$

Similarly, the estimation (3.56) holds for $a \in J$. So we may consider the equation (3.58) for any $a \in J$. That is, we may write

$$[D(\Lambda_N, \gamma) - S(a, p_1)]A(N, \gamma) = O(\rho^{-p\alpha}) \tag{3.65}$$

for any $a \in J$.

In the proof of Theorem 3.17, we have chosen $p > \frac{d-1}{2\alpha} + 1$. Now, we let

$c=[\frac{d-1}{2\alpha}]+1$. The estimations (3.44) and (3.45) can be written as

$$|A(N,\gamma)\cdot f_i|>c_{11}\rho^{-c\alpha} \tag{3.66}$$

and

$$|<\Phi_{\gamma,i},\Psi_N>|>c_{12}\rho^{-c\alpha}, \tag{3.67}$$

respectively. It follows from (3.67) that the estimation (3.62) can be written as

$$|A(N,\gamma)|>c_{13}\rho^{-c\alpha}. \tag{3.68}$$

We define the following $m\times m$ matrices.

$$F_0=0, \qquad F_1=S^1(|\gamma|^2+\lambda_s), \qquad F_j=S(|\gamma|^2+\lambda_s+\|F_{j-1}\|,j), \qquad j\geq 2. \tag{3.69}$$

Then we have

$$\|F_j\|=O(\rho^{-\alpha_1}) \tag{3.70}$$

for all $j=1,2,\ldots,p-c$. Indeed, since $F_0=0$, $\|F_0\|=0$ and if we assume that $\|F_{j-1}\|=O(\rho^{-\alpha_1})$, then since $|\gamma|^2+\lambda_s+\|F_{j-1}\|\in J$, by the estimation (3.64), we have $\|F_j\|=O(\rho^{-\alpha_1})$.

Theorem 3.20. *Let $|\gamma|^2$ be a non-resonance eigenvalue of the operator $L(0)$ with $|\gamma|\sim\rho$.*
(a) *For any eigenvalue λ_i, $i=1,2,\ldots,m$ of the matrix V_0, there exits an eigenvalue Λ_N of the operator $L(V)$ satisfying the formula*

$$\Lambda_N=|\gamma|^2+\lambda_i+\|F_{k-1}\|+O(\rho^{-k\alpha_1}), \tag{3.71}$$

where F_{k-1} is given by (3.69)*, $k=1,2,\ldots,p-c$.*
(b) *For any eigenvalue Λ_N of the operator $L(V)$ satisfying the inequality* (3.41)*, there is an eigenvalue λ_i of the matrix V_0 satisfying the formula* (3.71).

Proof. **(a)** We prove this part of the theorem by using the same consideration as in Karakılıç (2004). We use mathematical induction. For $k=1$ we obtain the result by Theorem 3.17.a.
Now, assume that for $k=j-1$ the formula (3.71) is true, that is,

$$\Lambda_N=|\gamma|^2+\lambda_i+\|F_{j-1}\|+O(\rho^{-j\alpha_1}). \tag{3.72}$$

By (3.70), we have $|\gamma|^2+\lambda_s+\|F_{j-1}\|+O(\rho^{-j\alpha_1})\in J$. Thus substituting $a\equiv|\gamma|^2+\lambda_s+\|F_{j-1}\|+O(\rho^{-j\alpha_1})$ into $S(a,p_1)$ in the equation (3.65), we get

$$[D(\Lambda_N,\gamma)-S(|\gamma|^2+\lambda_s+\|F_{j-1}\|+O(\rho^{-j\alpha_1}),p_1)]A(N,\gamma)=O(\rho^{-p\alpha}). \tag{3.73}$$

Adding and subtracting the term $F_jA(N,\gamma)=S(|\gamma|^2+\lambda_s+\|F_{j-1}\|,j)A(N,\gamma)$ into the left hand side of the equation (3.73), we obtain

$$[D(\Lambda_N,\gamma)-F_j]A(N,\gamma)-E_jA(N,\gamma)=O(\rho^{-p\alpha}), \tag{3.74}$$

where

$$\begin{aligned} E_j &= [S(|\gamma|^2+\lambda_s+\|F_{j-1}\|+O(\rho^{-j\alpha_1}),j)-S(|\gamma|^2+\lambda_s+\|F_{j-1}\|,j)] \\ &+ \Big(\sum_{k=j+1}^{p_1} S^k(|\gamma|^2+\lambda_s+\|F_{j-1}\|+O(\rho^{-j\alpha_1}))\Big). \end{aligned}$$

By the estimation (3.64), we have

$$\Big\|\sum_{k=j+1}^{p_1} S^k(|\gamma|^2+\lambda_s+\|F_{j-1}\|+O(\rho^{-j\alpha_1}))\Big\|=O(\rho^{-(j+1)\alpha_1}). \tag{3.75}$$

If we prove that

$$\|S(|\gamma|^2+\lambda_s+\|F_{j-1}\|+O(\rho^{-j\alpha_1}),j)-S(|\gamma|^2+\lambda_s+\|F_{j-1}\|,j)\|=O(\rho^{-(j+1)\alpha_1}), \tag{3.76}$$

then it follows from the estimations (3.75) and (3.76) that

$$\|E_j\|=O(\rho^{-(j+1)\alpha_1}). \tag{3.77}$$

Now, we prove the estimation (3.76). Since $|\gamma|^2+\lambda_s+\|F_{j-1}\|+O(\rho^{-j\alpha_1})\in J$ and $|\gamma|^2+\lambda_s+\|F_{j-1}\|\in J$ satisfy the inequality (3.41), by the inequality (3.42), we have

$$\begin{aligned} \big||\gamma|^2+\lambda_s+\|F_{j-1}\|+O(\rho^{-j\alpha_1})-|\gamma+\gamma_1+\cdots+\gamma_t|^2\big| &> \frac{1}{2}\rho^{\alpha_1}, \\ \big||\gamma|^2+\lambda_s+\|F_{j-1}\|-|\gamma+\gamma_1+\cdots+\gamma_t|^2\big| &> \frac{1}{2}\rho^{\alpha_1} \end{aligned} \tag{3.78}$$

for all $\gamma_t\in\Gamma(\rho^{\alpha})$ and $t=1,2,\ldots,p_1$. By its definition, $S(a,j)\equiv\sum_{k=1}^{j}S^k(a)$. Thus we first calculate the order of the first term of the summation in (3.76). To do this, we consider each entry of this term, and use the inequalities (3.78) and the relation (2.35).

$$
\begin{aligned}
&|s^1_{li}(|\gamma|^2+\lambda_s+\|F_{j-1}\|+O(\rho^{-j\alpha_1}))-s^1_{li}(|\gamma|^2+\lambda_s+\|F_{j-1}\|)| \\
&\leq \sum_{i_1=1}^{m}\sum_{\substack{\gamma_1,\gamma_2\in\Gamma^{+0}(\rho^{\alpha})\\ \gamma_1+\gamma_2=0}} \frac{|v_{i_1l\gamma_1}||v_{ii_1\gamma_2}|O(\rho^{-j\alpha_1})}{|(|\gamma|^2+\lambda_s+\|F_{j-1}\|+O(\rho^{-j\alpha_1})-|\gamma+\gamma_1|^2)||(|\gamma|^2+\lambda_s+\|F_{j-1}\|-|\gamma+\gamma_1|^2)|} \\
&\leq c_{14}\rho^{-(j+2)\alpha_1}
\end{aligned}
$$

for each $l,i=1,2,\ldots,m$ which implies

$$
\|S^1(|\gamma|^2+\lambda_s+\|F_{j-1}\|+O(\rho^{-j\alpha_1}))-S^1(|\gamma|^2+\lambda_s+\|F_{j-1}\|)\|=O(\rho^{-(j+2)\alpha_1}).
$$

If we consider each entry of the second term of the summation in (3.76), then again by the inequalities (3.78) and the relation (2.35), we see

$$
\begin{aligned}
&|s^2_{li}(|\gamma|^2+\lambda_s+\|F_{j-1}\|+O(\rho^{-j\alpha_1}))-s^2_{li}(|\gamma|^2+\lambda_s+\|F_{j-1}\|)| \\
&\leq \sum_{i_1,i_2=1}^{m}\sum_{\substack{\gamma_1,\gamma_2,\gamma_3\in\Gamma^{+0}(\rho^{\alpha})\\ \gamma_1+\gamma_2+\gamma_3=0}} |v_{i_1l\gamma_1}||v_{i_2i_1\gamma_2}||v_{ii_2\gamma_3}|O(\rho^{-j\alpha_1}) \\
&\quad \{\frac{1}{|(a'+O(\rho^{-j\alpha_1})-|\gamma+\gamma_1|^2)(a'+O(\rho^{-j\alpha_1})-|\gamma+\gamma_1+\gamma_2|^2)(a'-|\gamma+\gamma_1+\gamma_2|^2)|} \\
&+ \frac{1}{|(a'+O(\rho^{-j\alpha_1})-|\gamma+\gamma_1|^2)(a'-|\gamma+\gamma_1|^2)(a'+O(\rho^{-j\alpha_1})-|\gamma+\gamma_1+\gamma_2|^2)|}\} \\
&\leq c_{15}\rho^{-(j+3)\alpha_1}
\end{aligned}
$$

for each $l,i=1,2,\ldots,m$ where we use the notation $a'\equiv|\gamma|^2+\lambda_s+\|F_{j-1}\|$ for the sake of simplicity, which implies

$$
\|S^2(|\gamma|^2+\lambda_s+\|F_{j-1}\|+O(\rho^{-j\alpha_1}))-S^2(|\gamma|^2+\lambda_s+\|F_{j-1}\|)\|=O(\rho^{-(j+3)\alpha_1}).
$$

Therefore, by direct calculations, it can be easily seen that

$$
\|S^k(|\gamma|^2+\lambda_s+\|F_{j-1}\|+O(\rho^{-j\alpha_1}))-S^k(|\gamma|^2+\lambda_s+\|F_{j-1}\|)\|=O(\rho^{-(j+k+1)\alpha_1})
$$

from which we obtain the estimation (3.76).

Let β_i be an eigenvalue of the matrix $D(\Lambda_N,\gamma)-S(|\gamma|^2+\lambda_i+\|F_{j-1}\|+O(\rho^{-j\alpha_1}),p_1)$. If we multiply both sides of the equation (3.73) by its corresponding normalized eigenvector f_i, and use the estimation (3.66), then we obtain

$$\beta_i = O(\rho^{-(p-c)\alpha}). \tag{3.79}$$

On the other hand, the matrix $D(\Lambda_N,\gamma) - S(|\gamma|^2 + \lambda_i + \|F_{j-1}\| + O(\rho^{-j\alpha_1}), p_1)$ in (3.73) is decomposed as follows

$$D(\Lambda_N,\gamma) - S(|\gamma|^2 + \lambda_i + \|F_{j-1}\| + O(\rho^{-j\alpha_1}), p_1) = D(\Lambda_N,\gamma) - F_j - E_j.$$

Thus by (3.77), (3.79) and Theorem of Lidskii in Section 1.3,

$$|\beta_i - (\Lambda_N - |\gamma|^2 + \lambda_i)| \le \|F_j\| + O(\rho^{-(j+1)\alpha_1}),$$

where $1 \le j+1 \le p-c$, we get the proof of (3.71).

(b) Again we prove this part of the theorem by induction. For $j = 1$ we obtain the result by Theorem 3.17.b.

Now, assume that for $k = j-1$ the formula (3.71) is true. To prove (3.71) for $k = j$, we use the equation (3.74). By using the definition of the matrix $D(\Lambda_N,\gamma)$ and (3.74), we have

$$[(\Lambda_N - |\gamma|^2)I - D_j]A(N,\gamma) = E_j A(N,\gamma) + O(\rho^{-p\alpha}),$$

where $D_j = V_0 + F_j$. Applying $\frac{1}{|A(N,\gamma)|}[(\Lambda_N - |\gamma|^2)I - D_j]^{-1}$ to both sides of the above equation, taking norm of both sides, and using the estimations (3.68) and (3.77), we obtain

$$\begin{aligned} 1 &\le \|[(\Lambda_N - |\gamma|^2)I - D_j]^{-1}\|[O(\rho^{-(j+1)\alpha_1}] + \|[(\Lambda_N - |\gamma|^2)I - D_j]^{-1}\|[O(\rho^{-(p-c)\alpha})] \\ &\le \max_{i=1,2,\ldots,m} \frac{1}{|\Lambda_N - |\gamma|^2 - \widetilde{\lambda}_i(j)|}[O(\rho^{-(j+1)\alpha_1})], \end{aligned}$$

or

$$\min_{i=1,2,\ldots,m} |\Lambda_N - |\gamma|^2 - \widetilde{\lambda}_i(j)| \le c_{16}\rho^{-(j+1)\alpha_1},$$

where minimum is taken over all eigenvalues $\widetilde{\lambda}_i(j)$ of the matrix D_j, $1 \le j+1 \le p-c$. By the last inequality and the well known result in matrix theory, $|\widetilde{\lambda}_i(j) - \lambda_i| \le \|F_j\|$, we obtain the result. □

Corollary 3.21. ***(a)*** *Let $\mu_{\gamma,i}$ be an eigenvalue of the operator $L(V_0)$ where $\gamma \in U(\rho^{\alpha_1}, p)$ with $|\gamma| \sim \rho$ and $i = 1,2,\ldots,m$. Then there is an eigenvalue Λ_N of the operator $L(V)$*

satisfying

$$\Lambda_N = \mu_{\gamma,i} + \|F_{k-1}\| + O(\rho^{-k\alpha_1}), \tag{3.80}$$

where F_{k-1} is given by (3.69)*, $k = 1,2,\ldots,p-c$.*
(b) *For each eigenvalue Λ_N of the operator $L(V)$ satisfying the inequality* (3.41) *there is an eigenvalue $\mu_{\gamma,i}$ of the operator $L(V_0)$ satisfying the formula* (3.80).

Proof. The proof follows from the proof of Theorem 3.20. □

3.2 Asymptotic Formulas for the Eigenvalues in the Resonance Domain

We assume that $\gamma \notin V_{e_k}(\rho^{\alpha_1})$ for $k = 1,2,\ldots,d$ where $e_1 = (\frac{\pi}{a_1},0,\ldots,0)$, $e_2 = (0,\frac{\pi}{a_2},0,\ldots,0),\ldots,e_d = (0,\ldots,0,\frac{\pi}{a_d})$.

Let $|\gamma|^2$ be a resonance eigenvalue of the operator $L(0)$, that is, $\gamma \in (\bigcap_{i=1}^{k} V_{\gamma_i}(\rho^{\alpha_k})) \setminus E_{k+1}$, $k = 1,2,\ldots,d-1$, $\gamma_i \neq e_j$ for $i = 1,2,\ldots,k$ and $j = 1,2,\ldots,d-1$.

We define the following sets

$$B_k(\gamma_1,\gamma_2,\ldots,\gamma_k) = \{b : b = \sum_{i=1}^{k} n_i\gamma_i, n_i \in Z, |b| < \frac{1}{2}\rho^{\frac{1}{2}\alpha_{k+1}}\},$$

$$B_k(\gamma) = \gamma + B_k(\gamma_1,\gamma_2,\ldots,\gamma_k) = \{\gamma + b : b \in B_k(\gamma_1,\gamma_2,\ldots,\gamma_k)\},$$

$$B_k(\gamma,p_1) = B_k(\gamma) + \Gamma(p_1\rho^{\alpha}).$$

Let h_τ, $\tau = 1,2,\ldots,b_k$ denote the vectors of $B_k(\gamma,p_1)$, b_k the number of the vectors in $B_k(\gamma,p_1)$. We define the $mb_k \times mb_k$ matrix $C = C(\gamma,\gamma_1,\ldots,\gamma_k)$ by

$$C = \begin{bmatrix} |h_1|^2 I & V_{h_1-h_2} & \cdots & V_{h_1-h_{b_k}} \\ V_{h_2-h_1} & |h_2|^2 I & \cdots & V_{h_2-h_{b_k}} \\ \vdots & & & \\ V_{h_{b_k}-h_1} & V_{h_{b_k}-h_2} & \cdots & |h_{b_k}|^2 I \end{bmatrix}, \tag{3.81}$$

where $V_{h_\tau-h_\xi}$, $\tau,\xi=1,2,\ldots,b_k$ are the $m\times m$ matrices defined by

$$V_{h_\tau-h_\xi}=\begin{bmatrix} v_{11h_\tau-h_\xi} & v_{12h_\tau-h_\xi} & \cdots & v_{1mh_\tau-h_\xi} \\ v_{21h_\tau-h_\xi} & v_{22h_\tau-h_\xi} & \cdots & v_{2mh_\tau-h_\xi} \\ \vdots & & & \\ v_{m1h_\tau-h_\xi} & v_{m2h_\tau-h_\xi} & \cdots & v_{mmh_\tau-h_\xi} \end{bmatrix}. \tag{3.82}$$

The analogues of the following lemma can be found in Karakılıç (2004)(see Theorem 3.1.1.)

Lemma 3.22. *Let* $|\gamma|^2$ *be a resonance eigenvalue of the operator* $L(0)$*, that is,* $\gamma\in(\bigcap_{i=1}^{k}V_{\gamma_i}(\rho^{\alpha_k}))\setminus E_{k+1}$, $k=1,2,\ldots,d-1$ *where* $|\gamma|\sim\rho$, Λ_N *an eigenvalue of the operator* $L(V)$ *satisfying*

$$|\Lambda_N-|\gamma|^2|<\frac{1}{2}\rho^{\alpha_1}. \tag{3.83}$$

Then

$$|\Lambda_N-|h_\tau-\gamma\prime-\gamma_1-\gamma_2-\cdots-\gamma_s|^2|>\frac{1}{6}\rho^{\alpha_{k+1}} \tag{3.84}$$

where $h_\tau\in B_k(\gamma,p_1)$, $h_\tau-\gamma\prime\notin B_k(\gamma,p_1)$, $\gamma\prime\in\Gamma(\rho^\alpha)$, $\gamma_i\in\Gamma(\rho^\alpha)$, $i=1,2,\ldots,s$, $s=0,1,\ldots,p_1-1$.

Proof. The relations $h_\tau\in B_k(\gamma,p_1)$, $h_\tau-\gamma\prime\notin B_k(\gamma,p_1)$, $2p_1>p$ and $|\gamma\prime|,|\gamma_1|,\ldots,|\gamma_{p_1-1}|<\rho^\alpha$ imply that

$$a_s=h_\tau-\gamma\prime-\gamma_1-\gamma_2-\ldots-\gamma_s\in B_k(\gamma,p_1)\setminus B_k(\gamma)$$

for $s=0,1,\ldots,p_1-1$. To prove the inequality (3.84), we use the decomposition

$$a_s=\gamma+b+a,$$

where $b\in B_k$ and $a\in\Gamma(p_1\rho^\alpha)$. So $|b|<\frac{1}{2}\rho^{\frac{1}{2}\alpha_{k+1}}$ and $|a|<p_1\rho^\alpha$. First we show that

$$||\gamma+b+a|^2-|\gamma|^2|>\frac{1}{5}\rho^{\alpha_{k+1}}. \tag{3.85}$$

To prove the inequality (3.85), we consider the following cases.
Case1: If $a\in P=span\{\gamma_1,\gamma_2,\ldots,\gamma_k\}$, then $a+b\in P$ and $\gamma+b+a\notin B_k(\gamma)$ imply that $a+b\in P\setminus B_k$, that is,

$$|a+b|\geq\frac{1}{2}\rho^{\frac{1}{2}\alpha_{k+1}}.$$

Now, if we consider the orthogonal decomposition of γ as $\gamma = x+v$ where $v \in P$ and $x \perp v$, then by using $x \cdot a = x \cdot b = x \cdot v = 0$, $\mid a+b \mid \geq \frac{1}{2}\rho^{\frac{1}{2}\alpha_{k+1}}$ and $\mid v \mid < \rho^{\alpha_1}$, we get

$$\begin{aligned} || \gamma + b + a |^2 - | \gamma |^2| &= || x+v+b+a |^2 - | x+v |^2| \\ &= || v+b+a |^2 - | v |^2| > \frac{1}{5}\rho^{\frac{1}{2}\alpha_{k+1}}. \end{aligned}$$

Thus for Case1 the inequality (3.85) is true.

Case2: If $a \notin P$, then by definition of $\gamma \in (\bigcap_{i=1}^{k} V_{\gamma_i}(\rho^{\alpha_k})) \setminus E_{k+1}$, we have

$$|| \gamma + a |^2 - | \gamma |^2| > \rho^{\alpha_{k+1}}. \tag{3.86}$$

Consider the difference

$$|| \gamma + b + a |^2 - | \gamma |^2| = || \gamma + b + a |^2 - | \gamma + b |^2 + | \gamma + b |^2 - | \gamma |^2|,$$

where

$$d_1 = | \gamma + b + a |^2 - | \gamma + b |^2, d_2 = | \gamma + b |^2 - | \gamma |^2.$$

Since

$$d_1 = | \gamma + b + a |^2 - | \gamma + b |^2 = | \gamma + a |^2 - | \gamma |^2 + 2a \cdot b,$$

by the inequality (3.86) and $\mid 2a \cdot b \mid \leq 2 \mid a \mid\mid b \mid < p_1 \rho^{\alpha} \rho^{\frac{1}{2}\alpha_{k+1}} < \frac{1}{3}\rho^{\alpha_{k+1}}$,

$$| d_1 | > \frac{2}{3}\rho^{\alpha_{k+1}}.$$

On the other hand, using $\mid \gamma + b + a \mid^2 - \mid \gamma \mid^2 = \mid v+b+a \mid^2 - \mid v \mid^2$, and taking $a = 0$, we get

$$d_2 = | \gamma + b |^2 - | \gamma |^2 = | v+b |^2 - | v |^2 = (| v+b | - | v |)(| v+b | + | v |)$$

from which it follows that

$$| d_2 | < \frac{1}{3}\rho^{\alpha_{k+1}}.$$

Then

$$|| d_1 | - | d_2 || > \frac{1}{5}\rho^{\alpha_{k+1}}.$$

So in any case the inequality (3.85) is true. Therefore, the inequalities (3.83) and (3.85) imply that

$$|\Lambda_N - |\gamma+b+a|^2| = |\Lambda_N - |\gamma|^2 - |\gamma+b+a|^2 + |\gamma|^2| > \frac{1}{6}\rho^{\alpha_{k+1}}.$$

□

Theorem 3.23. *Let $|\gamma|^2$ be a resonance eigenvalue of the operator $L(0)$, that is, $\gamma \in (\bigcap_{i=1}^{k} V_{\gamma_i}(\rho^{\alpha_k})) \setminus E_{k+1}$, $k = 1,2,\ldots,d-1$ where $|\gamma| \sim \rho$, λ_i an eigenvalue of the matrix V_0, and Λ_N an eigenvalue of the operator $L(V)$ satisfying*

$$|\Lambda_N - |\gamma|^2| < \frac{1}{2}\rho^{\alpha_1} \tag{3.87}$$

and

$$|< \Phi_{\gamma,j}, \Psi_N >| > c_{17}\rho^{-c\alpha}. \tag{3.88}$$

Then there exists an eigenvalue $\eta_s(\gamma)$, $s = 1,2,\ldots,mb_k$ of the matrix C such that

$$\Lambda_N = \lambda_i + \eta_s(\gamma) + O(\rho^{-(p-c-\frac{d}{4}3^d)\alpha}).$$

Proof. We give the proof by using the same consideration as in Karakılıç (2004). The binding formula (2.25) for any $h_\tau \in B_k(\gamma, p_1)$, $\tau = 1,2,\ldots,b_k$ and the decomposition (3.40) give

$$(\Lambda_N - |h_\tau|^2) < \Psi_N, \Phi_{h_\tau,j} > = \sum_{i=1}^{m} \sum_{\gamma\prime \in \Gamma^{+0}(\rho^\alpha)} v_{ij\gamma\prime} < \Psi_N, \Phi_{h_\tau-\gamma\prime,i} > + O(\rho^{-p\alpha}). \tag{3.89}$$

We first show that

$$O(\rho^{-p\alpha}) = \sum_{i=1}^{m} \sum_{\substack{\gamma\prime \in \Gamma(\rho^\alpha) \\ h_\tau - \gamma\prime \notin B_k(\gamma, p_1)}} v_{ij\gamma\prime} < \Psi_N, \Phi_{h_\tau-\gamma\prime,i} > \tag{3.90}$$

for any $j = 1,2,\ldots,m$. Here we remark that $\gamma\prime \neq 0$. If it were the case, then we would have from $h_\tau - \gamma\prime \notin B_k(\gamma, p_1)$ that $h_\tau \notin B_k(\gamma, p_1)$ which is a contradiction.
Since Λ_N satisfies the inequality (3.87), by Lemma 3.22, we have $|\Lambda_N - |h_\tau - \gamma\prime|^2| > \frac{1}{6}\rho^{\alpha_{k+1}}$. Using this and the decomposition (3.89) for $h_\tau - \gamma\prime \notin B_k(\gamma, p_1)$, it follows that

$$\sum_{i=1}^{m}\sum_{\substack{\gamma'\in\Gamma(\rho^{\alpha})\\ h_\tau-\gamma'\notin B_k(\gamma,p_1)}} v_{ij\gamma'} < \Psi_N, \Phi_{h_\tau-\gamma',i} >$$

$$= \sum_{i=1}^{m}\sum_{\substack{\gamma'\in\Gamma(\rho^{\alpha})\\ h_\tau-\gamma'\notin B_k(\gamma,p_1)}} \frac{v_{ij\gamma'}}{\Lambda_N - \mid h_\tau-\gamma' \mid^2}\sum_{i_1=1}^{m}\sum_{\substack{\gamma_1\in\Gamma(\rho^{\alpha})\\ h_\tau-\gamma'\notin B_k(\gamma,p_1)}} v_{i_1 i\gamma_1} < \Psi_N, \Phi_{h_\tau-\gamma'-\gamma_1,i_1} >$$
$$+ \quad O(\rho^{-p\alpha}).$$

In this manner, iterating p_1 times, we get

$$\sum_{i=1}^{m}\sum_{\substack{\gamma'\in\Gamma(\rho^{\alpha})\\ h_\tau-\gamma'\notin B_k(\gamma,p_1)}} v_{ij\gamma'} < \Psi_N, \Phi_{h_\tau-\gamma',i} >$$

$$= \sum_{i,i_1,i_2,\ldots,i_{p_1}=1}^{m}\ \sum_{\substack{\gamma',\gamma_1,\gamma_2,\ldots,\gamma_{p_1}\in\Gamma(\rho^{\alpha})\\ h_\tau-\gamma'\notin B_k(\gamma,p_1)}}$$
$$\frac{v_{ij\gamma'}v_{i_1 i\gamma_1}\ldots v_{i_{p_1} i_{p_1-1}\gamma_{p_1}}}{(\Lambda_N - \mid h_\tau-\gamma' \mid^2)(\Lambda_N - \mid h_\tau-\gamma'-\gamma_1 \mid^2)\ldots(\Lambda_N - \mid h_\tau-\gamma'-\gamma_1-\cdots-\gamma_{p_1-1} \mid^2)}$$
$$< \Psi_N, \Phi_{h_\tau-\gamma'-\gamma_1-\cdots-\gamma_{p_1},i_{p_1}} > + O(\rho^{-p\alpha}).$$

Taking norm of both sides of the last equality, using Lemma 3.22, the relation (2.35) and the fact that $p_1\alpha_{k+1} \geq p_1\alpha_2 > p\alpha$, we obtain

$$\mid \sum_{i=1}^{m}\sum_{\substack{\gamma'\in\Gamma(\rho^{\alpha})\\ h_\tau-\gamma'\notin B_k(\gamma,p_1)}} v_{ij\gamma'} < \Psi_N, \Phi_{h_\tau-\gamma',i} > \mid$$

$$\leq \sum_{i,i_1,i_2,\ldots,i_{p_1}=1}^{m}\ \sum_{\substack{\gamma',\gamma_1,\gamma_2,\ldots,\gamma_{p_1}\in\Gamma(\rho^{\alpha})\\ h_\tau-\gamma'\notin B_k(\gamma,p_1)}}$$
$$\frac{\mid v_{ij\gamma'} \mid\mid v_{i_1 i\gamma_1} \mid \ldots \mid v_{i_{p_1} i_{p_1-1}\gamma_{p_1}} \mid}{\mid \Lambda_N - \mid h_\tau-\gamma' \mid^2\mid\mid \Lambda_N - \mid h_\tau-\gamma'-\gamma_1 \mid^2\mid \ldots \mid \Lambda_N - \mid h_\tau-\gamma'-\gamma_1-\cdots-\gamma_{p_1-1} \mid^2\mid}$$
$$\mid< \Psi_N, \Phi_{h_\tau-\gamma'-\gamma_1-\cdots-\gamma_{p_1},i_{p_1}} >\mid + O(\rho^{-p\alpha})$$
$$\leq (\frac{1}{6}\rho^{\alpha_{k+1}})^{-p_1} \sum_{\substack{\gamma',\gamma_1,\gamma_2,\ldots,\gamma_{p_1}\in\Gamma(\rho^{\alpha})\\ h_\tau-\gamma'\notin B_k(\gamma,p_1)}} \mid v_{ij\gamma'} \mid\mid v_{i_1 i\gamma_1} \mid \ldots \mid v_{i_{p_1} i_{p_1-1}\gamma_{p_1}} \mid$$
$$\mid< \Psi_N, \Phi_{h_\tau-\gamma'-\gamma_1-\cdots-\gamma_{p_1},i_{p_1}} >\mid + O(\rho^{-p\alpha})$$

$$
\begin{aligned}
&\leq \left(\frac{1}{6}\rho^{\alpha_{k+1}}\right)^{-p_1} \sum_{i,i_1,i_2,\ldots,i_{p_1}=1}^{m} M_{ij}M_{i_1 i}\ldots M_{i_{p_1}i_{p_1-1}} \mid <\Psi_N, \Phi_{h_\tau-\gamma\prime-\gamma_1-\cdots-\gamma_{p_1},i_{p_1}}> \mid \\
&+ O(\rho^{-p\alpha}) \\
&= O(\rho^{-p\alpha}).
\end{aligned}
$$

That is, the estimation (3.90) holds. Therefore, the decomposition (3.89) becomes

$$
(\Lambda_N - \mid h_\tau \mid^2) <\Psi_N, \Phi_{h_\tau,j}> = \sum_{i=1}^{m} \sum_{\substack{\gamma\prime\in\Gamma^{+0}(\rho^\alpha) \\ h_\tau-\gamma\prime\in B_k(\gamma,p_1)}} v_{ij\gamma\prime} <\Psi_N, \Phi_{h_\tau-\gamma\prime,i}> + O(\rho^{-p\alpha}). \quad (3.91)
$$

Since $h_\tau - \gamma\prime \in B_k(\gamma, p_1)$, using the notation $h_\xi = h_\tau - \gamma\prime$, the decomposition (3.91) can be written as

$$
(\Lambda_N - \mid h_\tau \mid^2) <\Psi_N, \Phi_{h_\tau,j}> = \sum_{i=1}^{m} \sum_{h_\tau-h_\xi\in\Gamma^{+0}(\rho^\alpha)} v_{ijh_\tau-h_\xi} <\Psi_N, \Phi_{h_\xi,i}> + O(\rho^{-p\alpha}).
$$

Isolating the terms where $h_\tau - h_\xi = 0$, we get

$$
\begin{aligned}
(\Lambda_N - \mid h_\tau \mid^2) <\Psi_N, \Phi_{h_\tau,j}> &= \sum_{i=1}^{m} v_{ij0} <\Psi_N, \Phi_{h_\tau,i}> \\
&+ \sum_{i=1}^{m} \sum_{h_\tau-h_\xi\in\Gamma(\rho^\alpha)} v_{ijh_\tau-h_\xi} <\Psi_N, \Phi_{h_\xi,i}> \\
&+ O(\rho^{-p\alpha}). \qquad (3.92)
\end{aligned}
$$

Considering the decomposition (3.92) for an arbitrary $h_\tau \in B_k(\gamma, p_1)$, $\tau = 1, 2, \ldots, b_k$ and for all $j = 1, 2, \ldots, m$, we get

$$
(\Lambda_N - \mid h_\tau \mid^2) I A(N, h_\tau) = V_0 A(N, h_\tau) + \sum_{\substack{\xi=1 \\ \xi\neq\tau}}^{b_k} V_{h_\tau-h_\xi} A(N, h_\xi) + O(\rho^{-p\alpha}), \qquad (3.93)
$$

or

$$
[(\Lambda_N - \mid h_\tau \mid^2) I - V_0] A(N, h_\tau) = \sum_{\substack{\xi=1 \\ \xi\neq\tau}}^{b_k} V_{h_\tau-h_\xi} A(N, h_\xi) + O(\rho^{-p\alpha}), \qquad (3.94)
$$

where I is an $m \times m$ identity matrix, $V_{h_\tau-h_\xi}$ is given by (3.82), $O(\rho^{-p\alpha})$ is an $m \times 1$ vector and $A(N, h_\xi)$ is the $m \times 1$ vector

$$
A(N, h_\xi) = (<\Psi_N, \Phi_{h_\xi,1}>, <\Psi_N, \Phi_{h_\xi,2}>, \ldots, <\Psi_N, \Phi_{h_\xi,m}>) \qquad (3.95)
$$

for any $\xi = 1, 2, \ldots, b_k$.

Let λ_i be an eigenvalue of the matrix V_0 and ω_i the corresponding normalized eigenvector. Multiplying both sides of the decomposition (3.94) by ω_i, we get

$$[(\Lambda_N - \mid h_\tau \mid^2)I - V_0]A(N,h_\tau)\cdot\omega_i = \sum_{\substack{\xi=1\\ \xi\neq\tau}}^{b_k} V_{h_\tau - h_\xi}A(N,h_\xi)\cdot\omega_i + O(\rho^{-p\alpha}). \tag{3.96}$$

For the left hand side of this last equality we have

$$\begin{aligned}[(\Lambda_N - \mid h_\tau \mid^2)I - V_0]A(N,h_\tau)\cdot\omega_i &= A(N,h_\tau)\cdot[(\Lambda_N - \mid h_\tau \mid^2)I - V_0]\omega_i \\ &= A(N,h_\tau)\cdot(\Lambda_N - \mid h_\tau \mid^2 - \lambda_i)\omega_i \\ &= (\Lambda_N - \mid h_\tau \mid^2 - \lambda_i)A(N,h_\tau)\cdot\omega_i. \end{aligned} \tag{3.97}$$

Letting $\lambda_{N,\tau,i} = \Lambda_N - \mid h_\tau \mid^2 - \lambda_i$, by the equation (3.97), we have from the decomposition (3.96) that

$$[\lambda_{N,\tau,i} IA(N,h_\tau) - \sum_{\substack{\xi=1\\ \xi\neq\tau}}^{b_k} V_{h_\tau - h_\xi}A(N,h_\xi)]\cdot\omega_i = O(\rho^{-p\alpha}). \tag{3.98}$$

Since the set of normalized eigenvectors $\{\omega_i\}_{i=1,2,\ldots,m}$ of the matrix V_0 forms a basis for R^m, for any vector $\lambda_{N,\tau,i} IA(N,h_\tau) - \sum_{\substack{\xi=1\\ \xi\neq\tau}}^{b_k} V_{h_\tau - h_\xi}A(N,h_\xi)$, $\tau = 1,2,\ldots,b_k$ in R^m by using Parseval's relation and the equation (3.98), we have

$$\mid \lambda_{N,\tau,i} IA(N,h_\tau) - \sum_{\substack{\xi=1\\ \xi\neq\tau}}^{b_k} V_{h_\tau - h_\xi}A(N,h_\xi) \mid^2$$

$$= \sum_{i=1}^{m} \mid [\lambda_{N,\tau,i} IA(N,h_\tau) - \sum_{\substack{\xi=1\\ \xi\neq\tau}}^{b_k} V_{h_\tau - h_\xi}A(N,h_\xi)]\cdot\omega_i \mid^2 = \sum_{i=1}^{m} \mid O(\rho^{-p\alpha}) \mid^2. \tag{3.99}$$

It follows from (3.99) that

$$\lambda_{N,\tau,i} IA(N,h_\tau) - \sum_{\substack{\xi=1\\ \xi\neq\tau}}^{b_k} V_{h_\tau - h_\xi}A(N,h_\xi) = O(\rho^{-p\alpha}). \tag{3.100}$$

Now, considering the equation (3.100) for all $h_\tau \in B_k(\gamma, p_1)$, $\tau = 1,2,\ldots,b_k$, we obtain the system

$$\begin{bmatrix} \lambda_{N,1,i}I & -V_{h_1-h_2} & \cdots & -V_{h_1-h_{b_k}} \\ -V_{h_2-h_1} & \lambda_{N,2,i}I & \cdots & -V_{h_2-h_{b_k}} \\ \vdots & & & \\ -V_{h_{b_k}-h_1} & -V_{h_{b_k}-h_2} & \cdots & \lambda_{N,b_k,i}I \end{bmatrix} \begin{bmatrix} A(N,h_1) \\ A(N,h_2) \\ \vdots \\ A(N,h_{b_k}) \end{bmatrix} = \begin{bmatrix} O(\rho^{-p\alpha}) \\ O(\rho^{-p\alpha}) \\ \vdots \\ O(\rho^{-p\alpha}) \end{bmatrix}. \quad (3.101)$$

We may write the system (3.101) as

$$[(\Lambda_N-\lambda_i)I-C]A(N,h_1,h_2,\ldots,h_{b_k}) = O(\rho^{-p\alpha}), \quad (3.102)$$

where I is an $mb_k \times mb_k$ identity matrix, C is given by (3.81), $A(N,h_1,h_2,\ldots,h_{b_k})$ is the $mb_k \times 1$ vector

$$A(N,h_1,h_2,\ldots,h_{b_k}) = (A(N,h_1),A(N,h_2),\ldots,A(N,h_{b_k}))$$

and $O(\rho^{-p\alpha})$ is an $mb_k \times 1$ vector. Multiplying both sides of the equation (3.102) by $[(\Lambda_N-\lambda_i)I-C]^{-1}$, and taking norm of both sides, we get

$$|\, A(N,h_1,h_2,\ldots,h_{b_k})\, | \leq \| \, [(\Lambda_N-\lambda_i)I-C]^{-1} \, \| \, | \, O(\rho^{-p\alpha}) \, | \, . \quad (3.103)$$

By the estimation (3.88), together with $b_k = O(\rho^{\frac{d}{2}3^d\alpha})$ we have the estimations

$$|\, A(N,h_1,h_2,\ldots,h_{b_k})\, | > c_{18}\rho^{-c\alpha}, \quad |\, O(\rho^{-p\alpha})\, | = O(\rho^{-(p-\frac{d}{4}3^d)\alpha}).$$

Thus it follows from the inequality (3.103) and the last estimations that

$$c_{18}\rho^{-c\alpha} \leq \| \, [(\Lambda_N-\lambda_i)I-C]^{-1} \, \| \, c_{19}\rho^{-(p-\frac{d}{4}3^d)\alpha},$$

$$\min_{s=1,2,\ldots,mb_k} |\, \Lambda_N-\lambda_i-\eta_s(\gamma)\, | \leq c_{20}\rho^{-(p-c-\frac{d}{4}3^d)\alpha},$$

$$\Lambda_N = \lambda_i+\eta_s(\gamma)+O(\rho^{-(p-c-\frac{d}{4}3^d)\alpha}).$$

□

Theorem 3.24. *Let* $|\,\gamma\,|^2$ *be a resonance eigenvalue of the operator* $L(0)$, *that is,* $\gamma \in (\bigcap_{i=1}^{k} V_{\gamma_i}(\rho^{\alpha_k})) \setminus E_{k+1}$, $k = 1,2,\ldots,d-1$ *where* $|\,\gamma\,| \sim \rho$, λ_i *an eigenvalue of the matrix* V_0, $\eta_s(\gamma)$ *an eigenvalue of the matrix C such that* $|\,\eta_s(\gamma) - |\,\gamma\,|^2| < \frac{3}{8}\rho^{\alpha_1}$. *Then there is* *an eigenvalue Λ_N of the operator $L(V)$ satisfying*

$$\Lambda_N = \lambda_i + \eta_s(\gamma) + O(\rho^{-p\alpha+\frac{d}{4}3^d\alpha+\frac{d-1}{2}}), \tag{3.104}$$

where $\eta_s(\gamma)$, $s = 1,2,\dots,mb_k$ is an eigenvalue of the matrix C which is given by (3.81).

Proof. We prove this theorem by using the same consideration as in Karakılıç (2004). By the general perturbation theory, there is an eigenvalue Λ_N of the operator $L(V)$ such that $\mid \Lambda_N - \mid \gamma \mid^2 \mid < \frac{1}{2}\rho^{2\alpha_1}$ holds. Thus one can use the system (3.102)

$$[(\Lambda_N - \lambda_i)I - C]A(N,h_1,h_2,\dots,h_{b_k}) = O(\rho^{-p\alpha}) \tag{3.105}$$

of Theorem 3.23. Let η_s, $s = 1,2,\dots,mb_k$ be an eigenvalue of the matrix C and $\theta_s = (\theta_s^1,\theta_s^2,\dots,\theta_s^{b_k})_{mb_k\times 1}$ the corresponding normalized eigenvector, $\mid \theta_s \mid = 1$, where $\theta_s^\tau = (\theta_s^{\tau 1},\theta_s^{\tau 2},\dots,\theta_s^{\tau m})_{m\times 1}$, $\tau = 1,2,\dots,b_k$. Multiplying the equation (3.105) by θ_s, we get

$$[(\Lambda_N - \lambda_i)I - C]A(N,h_1,h_2,\dots,h_{b_k}) \cdot \theta_s = O(\rho^{-p\alpha}) \cdot \theta_s. \tag{3.106}$$

From the left hand side of the equation (3.106) we get

$$\begin{aligned}
&[(\Lambda_N - \lambda_i)I - C]A(N,h_1,h_2,\dots,h_{b_k}) \cdot \theta_s \\
&= A(N,h_1,h_2,\dots,h_{b_k}) \cdot [(\Lambda_N - \lambda_i)I - C]\theta_s \\
&= A(N,h_1,h_2,\dots,h_{b_k}) \cdot [(\Lambda_N - \lambda_i)I\theta_s - \eta_s\theta_s] \\
&= A(N,h_1,h_2,\dots,h_{b_k}) \cdot (\Lambda_N - \lambda_i - \eta_s)\theta_s \\
&= (\Lambda_N - \lambda_i - \eta_s)A(N,h_1,h_2,\dots,h_{b_k}) \cdot \theta_s.
\end{aligned} \tag{3.107}$$

Using the equation (3.107) in the equation (3.106), and taking norm of both sides, we get

$$\mid \Lambda_N - \lambda_i - \eta_s \mid \mid A(N,h_1,h_2,\dots,h_{b_k}) \cdot \theta_s \mid = \mid O(\rho^{-p\alpha}) \cdot \theta_s \mid . \tag{3.108}$$

From the right hand side of the equation (3.108) by using $b_k = O(\rho^{\frac{d}{2}3^d\alpha})$, we have

$$\mid O(\rho^{-p\alpha}) \cdot \theta_s \mid \leq \mid O(\rho^{-p\alpha}) \mid \mid \theta_s \mid = \sqrt{mb_k(\rho^{-p\alpha})^2} = \sqrt{mb_k}\rho^{-p\alpha} = O(\rho^{-p\alpha+\frac{d}{4}3^d\alpha}). \tag{3.109}$$

The equation (3.108) and the estimation (3.109) give

$$\mid \Lambda_N - \lambda_i - \eta_s \mid\mid A(N,h_1,h_2,\ldots,h_{b_k}) \cdot \theta_s \mid = O(\rho^{-p\alpha+\frac{d}{4}3^d\alpha}). \tag{3.110}$$

Now, we estimate $\mid A(N,h_1,h_2,\ldots,h_{b_k}) \cdot \theta_s \mid$. Since

$$\mid A(N,h_1,h_2,\ldots,h_{b_k}) \cdot \theta_s \mid = \mid \sum_{\tau=1}^{b_k}\sum_{i=1}^{m} \theta_s^{\tau i} < \Psi_N, \Phi_{h_\tau,i} > \mid = \mid < \Psi_N, \sum_{\tau=1}^{b_k}\sum_{i=1}^{m} \theta_s^{\tau i} \Phi_{h_\tau,i} > \mid, \tag{3.111}$$

to estimate $\mid A(N,h_1,h_2,\ldots,h_{b_k}) \cdot \theta_s \mid$, we consider the Parseval's relation

$$\begin{aligned} 1 &= \| \sum_{\tau=1}^{b_k}\sum_{i=1}^{m} \theta_s^{\tau i} \Phi_{h_\tau,i} \|^2 = \sum_{N=1}^{\infty} \mid < \Psi_N, \sum_{\tau=1}^{b_k}\sum_{i=1}^{m} \theta_s^{\tau i} \Phi_{h_\tau,i} > \mid^2 \\ &= \sum_{N=1}^{\infty} \mid \sum_{\tau=1}^{b_k}\sum_{i=1}^{m} \theta_s^{\tau i} < \Psi_N, \Phi_{h_\tau,i} > \mid^2 \\ &= \sum_{N:|\Lambda_N - |\gamma|^2| \geq \frac{1}{2}\rho^{2\alpha_1}} \mid \sum_{\tau=1}^{b_k}\sum_{i=1}^{m} \theta_s^{\tau i} < \Psi_N, \Phi_{h_\tau,i} > \mid^2 \\ &+ \sum_{N:|\Lambda_N - |\gamma|^2| < \frac{1}{2}\rho^{2\alpha_1}} \mid \sum_{\tau=1}^{b_k}\sum_{i=1}^{m} \theta_s^{\tau i} < \Psi_N, \Phi_{h_\tau,i} > \mid^2 . \end{aligned} \tag{3.112}$$

We give an estimation for the first summation in the last expression.

$$\begin{aligned} &\sum_{N:|\Lambda_N - |\gamma|^2| \geq \frac{1}{2}\rho^{2\alpha_1}} \mid \sum_{\tau=1}^{b_k}\sum_{i=1}^{m} \theta_s^{\tau i} < \Psi_N, \Phi_{h_\tau,i} > \mid^2 \\ &= \sum_{N:|\Lambda_N - |\gamma|^2| \geq \frac{1}{2}\rho^{2\alpha_1}} \mid \sum_{\tau:|\eta_s - |h_\tau|^2| < \frac{1}{8}\rho^{\alpha_1}} \sum_{i=1}^{m} \theta_s^{\tau i} < \Psi_N, \Phi_{h_\tau,i} > \\ &+ \sum_{\tau:|\eta_s - |h_\tau|^2| \geq \frac{1}{8}\rho^{\alpha_1}} \sum_{i=1}^{m} \theta_s^{\tau i} < \Psi_N, \Phi_{h_\tau,i} > \mid^2 \\ &< 2 \sum_{N:|\Lambda_N - |\gamma|^2| \geq \frac{1}{2}\rho^{2\alpha_1}} \mid \sum_{\tau:|\eta_s - |h_\tau|^2| < \frac{1}{8}\rho^{\alpha_1}} \sum_{i=1}^{m} \theta_s^{\tau i} < \Psi_N, \Phi_{h_\tau,i} > \mid^2 \\ &+ 2 \sum_{N:|\Lambda_N - |\gamma|^2| \geq \frac{1}{2}\rho^{2\alpha_1}} \mid \sum_{\tau:|\eta_s - |h_\tau|^2| \geq \frac{1}{8}\rho^{\alpha_1}} \sum_{i=1}^{m} \theta_s^{\tau i} < \Psi_N, \Phi_{h_\tau,i} > \mid^2 . \end{aligned} \tag{3.113}$$

To estimate the term $2 \sum_{N:|\Lambda_N - |\gamma|^2| \geq \frac{1}{2}\rho^{2\alpha_1}} \mid \sum_{\tau:|\eta_s - |h_\tau|^2| \geq \frac{1}{8}\rho^{\alpha_1}} \sum_{i=1}^{m} \theta_s^{\tau i} < \Psi_N, \Phi_{h_\tau,i} > \mid^2$ in the inequality (3.113), we consider the matrix C as $C = A + B$ where

$$A = \begin{bmatrix} |h_1|^2 I & & 0 \\ & \ddots & \\ 0 & & |h_{b_k}|^2 I \end{bmatrix}, \quad B = \begin{bmatrix} 0 & V_{h_1-h_2} & \cdots & V_{h_1-h_{b_k}} \\ V_{h_2-h_1} & 0 & \cdots & V_{h_2-h_{b_k}} \\ \vdots & & \ddots & \vdots \\ V_{h_{b_k}-h_1} & V_{h_{b_k}-h_2} & \cdots & 0 \end{bmatrix}. \tag{3.114}$$

Let $\{e_{\tau,i}\}_{\tau=1,2,\ldots,b_k, i=1,2,\ldots,m}$ be a set of orthonormal vectors such that $e_{\tau,i} \cdot e_{\xi,k} = 1$ if $\tau = \xi$, $i = k$, $e_{\tau,i} \cdot e_{\xi,k} = 0$ otherwise. Multiplying $C\theta_s = (A+B)\theta_s$ by $e_{\tau,i}$, we get

$$C\theta_s \cdot e_{\tau,i} = (\eta_s \theta_s) \cdot e_{\tau,i} = \eta_s(\theta_s \cdot e_{\tau,i}) = \eta_s \theta_s^{\tau i},$$

and

$$(A+B)\theta_s \cdot e_{\tau,i} = \theta_s \cdot (A+B)e_{\tau,i} = \theta_s \cdot Ae_{\tau,i} + \theta_s \cdot Be_{\tau,i} = \theta_s^{\tau i} \mid h_\tau \mid^2 + \theta_s \cdot Be_{\tau,i}.$$

From the equality of the last two equations we have

$$(\eta_s - \mid h_\tau \mid^2)\theta_s^{\tau i} = \theta_s \cdot Be_{\tau,i} \tag{3.115}$$

for any $\tau = 1,2,\ldots,b_k$, $i = 1,2,\ldots,m$.
Using Bessel's inequality, Parseval's relation, orthogonality of the functions $\Phi_{h_\tau,i}(x)$, $\tau = 1,2,\ldots,b_k$, $i = 1,2,\ldots,m$, the binding formula (3.115) and $\| B \| \leq M$, we have

$$2 \sum_{N:|\Lambda_N - |\gamma|^2| \geq \frac{1}{2}\rho^{2\alpha_1}} \Big| \sum_{\tau:|\eta_s - |h_\tau|^2| \geq \frac{1}{8}\rho^{\alpha_1}} \sum_{i=1}^{m} \theta_s^{\tau i} < \Psi_N, \Phi_{h_\tau,i} > \Big|^2$$

$$\begin{aligned}
&= 2 \sum_{N:|\Lambda_N - |\gamma|^2| \geq \frac{1}{2}\rho^{2\alpha_1}} |< \Psi_N, \sum_{\tau:|\eta_s - |h_\tau|^2| \geq \frac{1}{8}\rho^{\alpha_1}} \sum_{i=1}^{m} \theta_s^{\tau i} \Phi_{h_\tau,i} >|^2 \\
&\leq 2 \sum_{N=1}^{\infty} |< \Psi_N, \sum_{\tau:|\eta_s - |h_\tau|^2| \geq \frac{1}{8}\rho^{\alpha_1}} \sum_{i=1}^{m} \theta_s^{\tau i} \Phi_{h_\tau,i} >|^2 \\
&= 2 \,\| \sum_{\tau:|\eta_s - |h_\tau|^2| \geq \frac{1}{8}\rho^{\alpha_1}} \sum_{i=1}^{m} \theta_s^{\tau i} \Phi_{h_\tau,i} \|^2 \\
&= 2 \sum_{\tau:|\eta_s - |h_\tau|^2| \geq \frac{1}{8}\rho^{\alpha_1}} \sum_{i=1}^{m} | \theta_s^{\tau i} |^2 \| \Phi_{h_\tau,i} \|^2
\end{aligned}$$

$$
\begin{aligned}
&= 2 \sum_{\tau:|\eta_s-|h_\tau|^2|\geq\frac{1}{8}\rho^{\alpha_1}} \sum_{i=1}^{m} |\theta_s^{\tau i}|^2 \\
&= 2 \sum_{\tau:|\eta_s-|h_\tau|^2|\geq\frac{1}{8}\rho^{\alpha_1}} \sum_{i=1}^{m} \frac{|\theta_s \cdot Be_{\tau,i}|^2}{|\eta_s - |h_\tau|^2|^2} \\
&\leq 2(\frac{1}{8}\rho^{\alpha_1})^{-2} \sum_{\tau:|\eta_s-|h_\tau|^2|\geq\frac{1}{8}\rho^{\alpha_1}} \sum_{i=1}^{m} |\theta_s|^2 \|B\|^2 |e_{\tau,i}|^2 \\
&= O(\rho^{-2\alpha_1}). \qquad (3.116)
\end{aligned}
$$

Now, we estimate the term $2 \sum_{N:|\Lambda_N-|\gamma|^2|\geq\frac{1}{2}\rho^{2\alpha_1}} | \sum_{\tau:|\eta_s-|h_\tau|^2|<\frac{1}{8}\rho^{\alpha_1}} \sum_{i=1}^{m} \theta_s^{\tau i} < \Psi_N, \Phi_{h_\tau,i} >|^2$ in the inequality (3.113). The assumption $|\eta_s - |\gamma|^2| < \frac{3}{8}\rho^{\alpha_1}$ of the theorem together with $|\Lambda_N - |\gamma|^2| \geq \frac{1}{2}\rho^{2\alpha_1}$ and $|\eta_s - |h_\tau|^2| < \frac{1}{8}\rho^{\alpha_1}$ imply that $|\Lambda_N - |h_\tau|^2| > \frac{1}{2}\rho^{\alpha_1}$ and $||\gamma|^2 - |h_\tau|^2| < \frac{1}{2}\rho^{\alpha_1}$. So one has

$$
\begin{aligned}
\frac{1}{\Lambda_N - |h_\tau|^2} &= \frac{1}{\Lambda_N - |\gamma|^2} \sum_{n=0}^{\infty} (\frac{|h_\tau|^2 - |\gamma|^2}{\Lambda_N - |\gamma|^2})^n \\
&= \frac{1}{\Lambda_N - |\gamma|^2} \{ \sum_{n=0}^{k} (\frac{|h_\tau|^2 - |\gamma|^2}{\Lambda_N - |\gamma|^2})^n + O(\rho^{-(k+1)\alpha_1}) \}. \qquad (3.117)
\end{aligned}
$$

Using the binding formula (2.25) for any $h_\tau \in B_k(\gamma, p_1)$, $|\Lambda_N - |h_\tau|^2| > \frac{1}{2}\rho^{\alpha_1}$ and the decomposition (3.117), we have

$$
2 \sum_{N:|\Lambda_N-|\gamma|^2|\geq\frac{1}{2}\rho^{2\alpha_1}} | \sum_{\tau:|\eta_s-|h_\tau|^2|<\frac{1}{8}\rho^{\alpha_1}} \sum_{i=1}^{m} \theta_s^{\tau i} < \Psi_N, \Phi_{h_\tau,i} >|^2
$$

$$
\begin{aligned}
&= 2 \sum_{N:|\Lambda_N-|\gamma|^2|\geq\frac{1}{2}\rho^{2\alpha_1}} | \sum_{\tau:|\eta_s-|h_\tau|^2|<\frac{1}{8}\rho^{\alpha_1}} \sum_{i=1}^{m} \theta_s^{\tau i} \frac{< \Psi_N, V\Phi_{h_\tau,i} >}{\Lambda_N - |h_\tau|^2} |^2 \\
&= 2 \sum_{N:|\Lambda_N-|\gamma|^2|\geq\frac{1}{2}\rho^{2\alpha_1}} | \sum_{\tau:|\eta_s-|h_\tau|^2|<\frac{1}{8}\rho^{\alpha_1}} \sum_{i=1}^{m} \frac{\theta_s^{\tau i} < \Psi_N, V\Phi_{h_\tau,i} >}{\Lambda_N - |\gamma|^2} \\
&\quad \{ \sum_{n=0}^{k} (\frac{|h_\tau|^2 - |\gamma|^2}{\Lambda_N - |\gamma|^2})^n + O(\rho^{-(k+1)\alpha_1}) \} |^2
\end{aligned}
$$

$$\leq \; 2\sum_{N:|\Lambda_N-|\gamma|^2|\geq\frac{1}{2}\rho^{2\alpha_1}}(k+1)\mid\sum_{\tau:|\eta_s-|h_\tau|^2|<\frac{1}{8}\rho^{\alpha_1}}\sum_{i=1}^{m}\frac{\theta_s^{\tau i}<\Psi_N,V\Phi_{h_\tau,i}>}{\Lambda_N-\mid\gamma\mid^2}\mid^2$$

$$+ \; 2\sum_{N:|\Lambda_N-|\gamma|^2|\geq\frac{1}{2}\rho^{2\alpha_1}}(k+1)\mid\sum_{\tau:|\eta_s-|h_\tau|^2|<\frac{1}{8}\rho^{\alpha_1}}\sum_{i=1}^{m}\frac{\theta_s^{\tau i}<\Psi_N,V\Phi_{h_\tau,i}>}{\Lambda_N-\mid\gamma\mid^2}\frac{\mid h_\tau\mid^2-\mid\gamma\mid^2}{\Lambda_N-\mid\gamma\mid^2}\mid^2$$

$$\vdots$$

$$+ \; 2\sum_{N:|\Lambda_N-|\gamma|^2|\geq\frac{1}{2}\rho^{2\alpha_1}}(k+1)\mid\sum_{\tau:|\eta_s-|h_\tau|^2|<\frac{1}{8}\rho^{\alpha_1}}\sum_{i=1}^{m}\frac{\theta_s^{\tau i}<\Psi_N,V\Phi_{h_\tau,i}>}{\Lambda_N-\mid\gamma\mid^2}[\frac{\mid h_\tau\mid^2-\mid\gamma\mid^2}{\Lambda_N-\mid\gamma\mid^2}]^k\mid^2$$

$$+ \; 2\sum_{N:|\Lambda_N-|\gamma|^2|\geq\frac{1}{2}\rho^{2\alpha_1}}(k+1)\mid\sum_{\tau:|\eta_s-|h_\tau|^2|<\frac{1}{8}\rho^{\alpha_1}}\sum_{i=1}^{m}\frac{\theta_s^{\tau i}<\Psi_N,V\Phi_{h_\tau,i}>}{\Lambda_N-\mid\gamma\mid^2}O(\rho^{-(k+1)\alpha_1})\mid^2 .$$

We estimate

$$2\sum_{N:|\Lambda_N-|\gamma|^2|\geq\frac{1}{2}\rho^{2\alpha_1}}(k+1)\mid\sum_{\tau:|\eta_s-|h_\tau|^2|<\frac{1}{8}\rho^{\alpha_1}}\sum_{i=1}^{m}\theta_s^{\tau i}<\Psi_N,V\Phi_{h_\tau,i}>\frac{(\mid h_\tau\mid^2-\mid\gamma\mid^2)^r}{(\Lambda_N-\mid\gamma\mid^2)^{r+1}}\mid^2,$$

where $r=0,1,2,\ldots,k$ and

$$2\sum_{N:|\Lambda_N-|\gamma|^2|\geq\frac{1}{2}\rho^{2\alpha_1}}(k+1)\mid\sum_{\tau:|\eta_s-|h_\tau|^2|<\frac{1}{8}\rho^{\alpha_1}}\sum_{i=1}^{m}\frac{\theta_s^{\tau i}<\Psi_N,V\Phi_{h_\tau,i}>}{\Lambda_N-\mid\gamma\mid^2}O(\rho^{-(k+1)\alpha_1})\mid^2 .$$

For an arbitrary $r=0,1,2,\ldots,k$ using Bessel's inequality, triangle inequality, $\mid\theta_s^{\tau i}\mid\leq 1$, $\mid\mid\gamma\mid^2-\mid h_\tau\mid^2\mid<\frac{1}{2}\rho^{\alpha_1}$ and the relations (2.35), (2.36), we have

$$2\sum_{N:|\Lambda_N-|\gamma|^2|\geq\frac{1}{2}\rho^{2\alpha_1}}(k+1)\mid\sum_{\tau:|\eta_s-|h_\tau|^2|<\frac{1}{8}\rho^{\alpha_1}}\sum_{i=1}^{m}\theta_s^{\tau i}<\Psi_N,V\Phi_{h_\tau,i}>\frac{(|h_\tau|^2-|\gamma|^2)^r}{(\Lambda_N-|\gamma|^2)^{r+1}}\mid^2$$

$$= \; 2\sum_{N:|\Lambda_N-|\gamma|^2|\geq\frac{1}{2}\rho^{2\alpha_1}}\frac{(k+1)}{\mid\Lambda_N-\mid\gamma\mid^2\mid^{2(r+1)}}$$

$$\mid\sum_{\tau:|\eta_s-|h_\tau|^2|<\frac{1}{8}\rho^{\alpha_1}}\sum_{i=1}^{m}\theta_s^{\tau i}<\Psi_N,V\Phi_{h_\tau,i}>(\mid h_\tau\mid^2-\mid\gamma\mid^2)^r\mid^2$$

$$\leq \; 2(\frac{1}{2}\rho^{2\alpha_1})^{-2(r+1)}(k+1)$$

$$\sum_{N:|\Lambda_N-|\gamma|^2|\geq\frac{1}{2}\rho^{2\alpha_1}}\mid<\Psi_N,\sum_{\tau:|\eta_s-|h_\tau|^2|<\frac{1}{8}\rho^{\alpha_1}}\sum_{i=1}^{m}\theta_s^{\tau i}(\mid h_\tau\mid^2-\mid\gamma\mid^2)^r V\Phi_{h_\tau,i}>\mid^2$$

$$\leq 2(\frac{1}{2}\rho^{2\alpha_1})^{-2(r+1)}(k+1) \parallel \sum_{\tau:|\eta_s-|h_\tau|^2|<\frac{1}{8}\rho^{\alpha_1}} \sum_{i=1}^{m} \theta_s^{\tau i}(|h_\tau|^2-|\gamma|^2)^r V\Phi_{h_\tau,i} \parallel^2$$

$$\leq 2(\frac{1}{2}\rho^{2\alpha_1})^{-2(r+1)}(k+1)(\sum_{\tau:|\eta_s-|h_\tau|^2|<\frac{1}{8}\rho^{\alpha_1}} \sum_{i=1}^{m} \parallel \theta_s^{\tau i}(|h_\tau|^2-|\gamma|^2)^r V\Phi_{h_\tau,i} > \parallel)^2$$

$$= 2(\frac{1}{2}\rho^{2\alpha_1})^{-2(r+1)}(k+1)(\sum_{\tau:|\eta_s-|h_\tau|^2|<\frac{1}{8}\rho^{\alpha_1}} \sum_{i=1}^{m} |\theta_s^{\tau i}|||h_\tau|^2-|\gamma|^2|^r \parallel V\Phi_{h_\tau,i} \parallel)^2$$

$$\leq 2(\frac{1}{2}\rho^{2\alpha_1})^{-2(r+1)}(\frac{1}{2}\rho^{\alpha_1})^{2r}(k+1)(\sum_{\tau:|\eta_s-|h_\tau|^2|<\frac{1}{8}\rho^{\alpha_1}} \sum_{i=1}^{m} \parallel V\Phi_{h_\tau,i} \parallel)^2.$$

Thus

$$O(\rho^{-4\alpha_1}) = \sum_{r=0}^{k} 2 \sum_{N:|\Lambda_N-|\gamma|^2|>\frac{1}{2}\rho^{2\alpha_1}} (k+1) \left| \sum_{\tau:|\eta_s-|h_\tau|^2|<\frac{1}{8}\rho^{\alpha_1}} \sum_{i=1}^{m} \theta_s^{\tau i} < \Psi_N, V\Phi_{h_\tau,i} > \frac{(|h_\tau|^2-|\gamma|^2)^r}{(\Lambda_N-|\gamma|^2)^{r+1}} \right|^2 \quad (3.118)$$

Similarly, we have

$$2 \sum_{N:|\Lambda_N-|\gamma|^2|\geq\frac{1}{2}\rho^{2\alpha_1}} (k+1) \left| \sum_{\tau:|\eta_s-|h_\tau|^2|<\frac{1}{8}\rho^{\alpha_1}} \sum_{i=1}^{m} \frac{\theta_s^{\tau i}<\Psi_N, V\Phi_{h_\tau,i}>}{\Lambda_N-|\gamma|^2} O(\rho^{-(k+1)\alpha_1}) \right|^2$$

$$= 2 \sum_{N:|\Lambda_N-|\gamma|^2|\geq\frac{1}{2}\rho^{2\alpha_1}} \frac{(k+1)}{|\Lambda_N-|\gamma|^2|^2} \left| \sum_{\tau:|\eta_s-|h_\tau|^2|<\frac{1}{8}\rho^{\alpha_1}} \sum_{i=1}^{m} \theta_s^{\tau i} < \Psi_N, V\Phi_{h_\tau,i} > O(\rho^{-(k+1)\alpha_1}) \right|^2$$

$$\leq 2(\frac{1}{2}\rho^{2\alpha_1})^{-2}(k+1) \sum_{N:|\Lambda_N-|\gamma|^2|\geq\frac{1}{2}\rho^{2\alpha_1}} |< \Psi_N, \sum_{\tau:|\eta_s-|h_\tau|^2|<\frac{1}{8}\rho^{\alpha_1}} \sum_{i=1}^{m} \theta_s^{\tau i} O(\rho^{-(k+1)\alpha_1}) V\Phi_{h_\tau,i} >|^2$$

$$\leq 2(\frac{1}{2}\rho^{2\alpha_1})^{-2}(k+1) \parallel \sum_{\tau:|\eta_s-|h_\tau|^2|<\frac{1}{8}\rho^{\alpha_1}} \sum_{i=1}^{m} \theta_s^{\tau i} O(\rho^{-(k+1)\alpha_1}) V\Phi_{h_\tau,i} \parallel^2$$

$$\leq 2(\frac{1}{2}\rho^{2\alpha_1})^{-2}(k+1)(\sum_{\tau:|\eta_s-|h_\tau|^2|<\frac{1}{8}\rho^{\alpha_1}} \sum_{i=1}^{m} \parallel \theta_s^{\tau i} O(\rho^{-(k+1)\alpha_1}) V\Phi_{h_\tau,i} > \parallel)^2$$

$$\leq 2(\frac{1}{2}\rho^{2\alpha_1})^{-2} O(\rho^{-2(k+1)\alpha_1})(k+1)(\sum_{\tau:|\eta_s-|h_\tau|^2|<\frac{1}{8}\rho^{\alpha_1}} \sum_{i=1}^{m} \parallel V\Phi_{h_\tau,i} \parallel)^2.$$

Thus

$$O(\rho^{-8\alpha_1}) = 2\sum_{N:|\Lambda_N-|\gamma|^2|\geq\frac{1}{2}\rho^{2\alpha_1}}(k+1)$$
$$|\sum_{\tau:|\eta_s-|h_\tau|^2|<\frac{1}{8}\rho^{\alpha_1}}\sum_{i=1}^{m}\frac{\theta_s^{\tau i}<\Psi_N,V\Phi_{h_\tau,i}>}{\Lambda_N-\mid\gamma\mid^2}O(\rho^{-(k+1)\alpha_1})\mid^2 . \quad (3.119)$$

By the inequality (3.113) and the estimations (3.116), (3.118) and (3.119), we have

$$O(\rho^{-2\alpha_1}) = \sum_{N:|\Lambda_N-|\gamma|^2|\geq\frac{1}{2}\rho^{2\alpha_1}} |\sum_{\tau=1}^{b_k}\sum_{i=1}^{m}\theta_s^{\tau i}<\Psi_N,\Phi_{h_\tau,i}>|^2 .$$

Therefore, from the decomposition (3.112) we have

$$1-O(\rho^{-2\alpha_1}) = \sum_{N:|\Lambda_N-|\gamma|^2|<\frac{1}{2}\rho^{2\alpha_1}} |\sum_{\tau=1}^{b_k}\sum_{i=1}^{m}\theta_s^{\tau i}<\Psi_N,\Phi_{h_\tau,i}>|^2 .$$

Since the number of indexes N satisfying $\mid\Lambda_N-\mid\gamma\mid^2|<\frac{1}{2}\rho^{2\alpha_1}$ is less then ρ^{d-1}, we have

$$1-O(\rho^{-2\alpha_1}) \leq \rho^{d-1}\mid\sum_{\tau=1}^{b_k}\sum_{i=1}^{m}\theta_s^{\tau i}<\Psi_N,\Phi_{h_\tau,i}>|^2$$

which implies together with the relation (3.111) that

$$\mid A(N,h_1,h_2,\ldots,h_{b_k})\cdot\theta_s\mid^2\geq\frac{1-O(\rho^{-2\alpha_1})}{\rho^{d-1}}. \quad (3.120)$$

It follows from the equation (3.110) and the estimation (3.120) that

$$\Lambda_N = \lambda_i+\eta_s+\frac{O(\rho^{-p\alpha+\frac{d}{4}3^d\alpha})}{O(\rho^{-\frac{d-1}{2}})}$$

from which we get the result. □

CHAPTER FOUR
CONCLUSION

In this study, we consider the Schrödinger operator with a matrix potential $V(x)$ which is defined by the differential expression

$$L\Phi = -\Delta\Phi + V\Phi \tag{4.121}$$

and the Neumann boundary condition

$$\frac{\partial\Phi}{\partial n}\mid_{\partial Q}= 0, \tag{4.122}$$

in $L_2^m(Q)$ where Q is the d dimensional rectangle $Q = [0,a_1] \times [0,a_2] \times \cdots \times [0,a_d]$, ∂Q is the boundary of Q, $m \geq 2$, $d \geq 2$, Δ is a diagonal $m \times m$ matrix whose diagonal elements are the scalar Laplace operators $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_d^2}$, $x = (x_1,x_2,\ldots,x_d) \in R^d$, V is the operator of multiplication by a real valued symmetric matrix $V(x) = (v_{ij}(x))$, $i, j = 1,2,\ldots,m$, $v_{ij}(x) \in L_2(Q)$, that is, $V^T(x) = V(x)$.

We denote the operator defined by the differential expression (4.121) and the boundary condition (4.122) by $L(V)$, the eigenvalues and the corresponding eigenfunctions of the operator $L(V)$ by Λ_N and Ψ_N, respectively.

We denote the operator defined by the differential expression (4.121) when $V(x) = 0$ and the boundary condition (4.122) by $L(0)$. The eigenvalues and the corresponding eigenspaces of the operator $L(0)$ are $\mid \gamma \mid^2$ and $E_\gamma = span\{\Phi_{\gamma,1}(x),\Phi_{\gamma,2}(x),\ldots,\Phi_{\gamma,m}(x)\}$, respectively where $\gamma \in \frac{\Gamma^{+0}}{2} = \{(\frac{n_1\pi}{a_1},\frac{n_2\pi}{a_2},\cdots,\frac{n_d\pi}{a_d}) : n_i \in Z^+ \bigcup\{0\}, \quad i = 1,2,\ldots,d\}$, $\Phi_{\gamma,j}(x) = (0,\ldots,0,u_\gamma(x),0,\ldots,0)$, $\quad j = 1,2,\ldots,m$, $u_\gamma(x) = cos\gamma^1 x_1 cos\gamma^2 x_2 \cdots cos\gamma^d x_d$. We note that the non-zero component $u_\gamma(x)$ of $\Phi_{\gamma,j}(x)$ stands in the jth component.

We denote the operator defined by the differential expression (4.121) when $V(x) = V_0$ where $V_0 = \int_Q V(x)dx$ and the boundary condition (4.122) by $L(V_0)$. Letting $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_m$ denote the eigenvalues, counted with multiplicity, of the matrix V_0 and $\omega_1,\omega_2,\ldots,\omega_m$ the corresponding normalized eigenvectors, the eigenvalues and the corresponding eigenfunctions of the operator $L(V_0)$ are $\mu_{\gamma,i} =\mid \gamma \mid^2 +\lambda_i$, $\varphi_{\gamma,i}(x) = \sum_{j=1}^{m} \omega_{ij}\Phi_{\gamma,j}(x)$.

As in papers Veliev (1987)-Veliev (2008), we divide the eigenvalues $|\gamma|^2$ of the operator $L(0)$ into two groups: Resonance and Non-Resonance eigenvalues. For this aim, first we divide R^d into two domains: Resonance and Non-resonance domains.
In order to define these domains, let us introduce the following sets.
Let $\alpha < \frac{1}{d+20}$, $\alpha_k = 3^k\alpha$, $k = 1,2,\ldots,d-1$, ρ a large parameter and

$$V_b(\rho^{\alpha_1}) \equiv \{x \in R^d : \quad ||x|^2 - |x+b|^2| < \rho^{\alpha_1}\},$$

$$E_1(\rho^{\alpha_1}, p) \equiv \bigcup_{b\in\Gamma(p\rho^{\alpha})} V_b(\rho^{\alpha_1}),$$

$$U(\rho^{\alpha_1}, p) \equiv R^d \setminus E_1(\rho^{\alpha_1}, p),$$

$$E_k(\rho^{\alpha_k}, p) \equiv \bigcup_{\gamma_1,\gamma_2,\ldots,\gamma_k\in\Gamma(p\rho^{\alpha})} (\bigcap_{i=1}^{k} V_{\gamma_i}(\rho^{\alpha_k})),$$

where $\Gamma(p\rho^{\alpha}) \equiv \{b \in \frac{\Gamma}{2} : 0 < |b| < p\rho^{\alpha}\}$, the intersection $\bigcap_{i=1}^{k} V_{\gamma_i}(\rho^{\alpha_k})$ in E_k is taken over $\gamma_1, \gamma_2, \ldots, \gamma_k$ which are linearly independent vectors and the length of γ_i is not greater than the length of the other vectors in $\Gamma\bigcap\gamma_i R$. The set $U(\rho^{\alpha_1}, p)$ is said to be a non-resonance domain, and the eigenvalue $|\gamma|^2$ of the operator $L(0)$ is called a non-resonance eigenvalue if $\gamma \in U(\rho^{\alpha_1}, p)$. The domains $V_b(\rho^{\alpha_1})$ for all $b \in \Gamma(p\rho^{\alpha})$ are called resonance domains, and the eigenvalue $|\gamma|^2$ of the operator $L(0)$ is a resonance eigenvalue if $\gamma \in V_b(\rho^{\alpha_1})$.

We have the following results in the non-resonance domain $U(\rho^{\alpha_1}, p)$.

Theorem 4.25. *Let $|\gamma|^2$ be a non-resonance eigenvalue of the operator $L(0)$ with $|\gamma| \sim \rho$.*
(a) *For each eigenvalue λ_i, $i = 1,2,\ldots,m$ of the matrix V_0, there exists an eigenvalue Λ_N of the operator $L(V)$ satisfying*

$$\Lambda_N = |\gamma|^2 + \lambda_i + O(\rho^{-\alpha_1}). \tag{4.123}$$

(b) *For each eigenvalue Λ_N of the operator $L(V)$ satisfying the inequality*

$$|\Lambda_N - |\gamma|^2| < \frac{1}{2}\rho^{\alpha_1}, \tag{4.124}$$

there exists an eigenvalue λ_i *of the matrix* V_0 *satisfying the formula* (4.123)*.*

Corollary 4.26. ***(a)*** *Let* $\mu_{\gamma,i}$ *be an eigenvalue of the operator* $L(V_0)$ *where* $\gamma \in U(\rho^{\alpha_1}, p)$ *with* $|\gamma| \sim \rho$ *and* $i = 1,2,\dots,m$*. Then there is an eigenvalue* Λ_N *of the operator* $L(V)$ *satisfying*

$$\Lambda_N = \mu_{\gamma,i} + O(\rho^{-\alpha_1}). \tag{4.125}$$

(b) *For each eigenvalue* Λ_N *of the operator* $L(V)$ *satisfying the inequality* (4.124) *there is an eigenvalue* $\mu_{\gamma,i}$ *of the operator* $L(V_0)$ *satisfying the formula* (4.125)*.*

We define the following $m \times m$ matrices.

$$S(a, p_1) \equiv \sum_{k=1}^{p_1} S^k(a),$$

where

$$S^k(a) = (s^k_{ji}(a)), \quad k = 1,2,\dots,p_1, \quad j,i = 1,2,\dots,m,$$

$$s^k_{ji}(a) = \sum_{i_1,i_2,\dots,i_k=1}^{m} \sum_{\substack{\gamma_1,\gamma_2,\dots,\gamma_{k+1} \in \Gamma^{+0}(\rho^{\alpha})\\ \gamma_1+\gamma_2+\dots+\gamma_{k+1}=0}} \frac{v_{i_1 j \gamma_1} v_{i_2 i_1 \gamma_2} \dots v_{i i_k \gamma_{k+1}}}{(a - |\gamma+\gamma_1|^2)\dots(a - |\gamma+\gamma_1+\dots+\gamma_k|^2)}$$

and

$$F_0 = 0, \qquad F_1 = S^1(|\gamma|^2 + \lambda_s), \qquad F_j = S(|\gamma|^2 + \lambda_s + \|F_{j-1}\|, j), \qquad j \geq 2. \tag{4.126}$$

Theorem 4.27. *Let* $|\gamma|^2$ *be a non-resonance eigenvalue of the operator* $L(0)$ *with* $|\gamma| \sim \rho$*.*
(a) *For any eigenvalue* λ_i*,* $i = 1,2,\dots,m$ *of the matrix* V_0*, there exits an eigenvalue* Λ_N *of the operator* $L(V)$ *satisfying the formula*

$$\Lambda_N = |\gamma|^2 + \lambda_i + \|F_{k-1}\| + O(\rho^{-k\alpha_1}), \tag{4.127}$$

where F_{k-1} *is given by* (4.126)*,* $k = 1,2,\dots,p-c$*.*
(b) *For any eigenvalue* Λ_N *of the operator* $L(V)$ *satisfying the inequality* (4.124)*, there is an eigenvalue* λ_i *of the matrix* V_0 *satisfying the formula* (4.127)*.*

Corollary 4.28. ***(a)*** *Let* $\mu_{\gamma,i}$ *be an eigenvalue of the operator* $L(V_0)$ *where* $\gamma \in U(\rho^{\alpha_1}, p)$ *with* $|\gamma| \sim \rho$ *and* $i = 1,2,\dots,m$*. Then there is an eigenvalue* Λ_N *of the operator* $L(V)$ *satisfying*

$$\Lambda_N = \mu_{\gamma,i} + \|F_{k-1}\| + O(\rho^{-k\alpha_1}), \tag{4.128}$$

where F_{k-1} is given by (3.69), $k=1,2,\dots,p-c$.
(b) *For each eigenvalue Λ_N of the operator $L(V)$ satisfying the inequality* (4.124) *there is an eigenvalue $\mu_{\gamma,i}$ of the operator $L(V_0)$ satisfying the formula* (4.128).

We have the following results in the resonance domain $(\bigcap_{i=1}^{k} V_{\gamma_i}(\rho^{\alpha_k}))\setminus E_{k+1}$, $k=1,2,\dots,d-1$, $\gamma_i\neq e_j$ for $i=1,2,\dots,k$ and $j=1,2,\dots,d-1$ where $e_j=(0,\dots,0,\frac{\pi}{a_j},0,\dots,0)$ for $j=1,2,\dots,d-1$.

We define the $mb_k\times mb_k$ matrix $C=C(\gamma,\gamma_1,\dots,\gamma_k)$ by

$$C=\begin{bmatrix} |h_1|^2 I & V_{h_1-h_2} & \cdots & V_{h_1-h_{b_k}} \\ V_{h_2-h_1} & |h_2|^2 I & \cdots & V_{h_2-h_{b_k}} \\ \vdots & & & \\ V_{h_{b_k}-h_1} & V_{h_{b_k}-h_2} & \cdots & |h_{b_k}|^2 I \end{bmatrix}, \tag{4.129}$$

where $V_{h_\tau-h_\xi}$, $\tau,\xi=1,2,\dots,b_k$ are the $m\times m$ matrices defined by

$$V_{h_\tau-h_\xi}=\begin{bmatrix} v_{11h_\tau-h_\xi} & v_{12h_\tau-h_\xi} & \cdots & v_{1mh_\tau-h_\xi} \\ v_{21h_\tau-h_\xi} & v_{22h_\tau-h_\xi} & \cdots & v_{2mh_\tau-h_\xi} \\ \vdots & & & \\ v_{m1h_\tau-h_\xi} & v_{m2h_\tau-h_\xi} & \cdots & v_{mmh_\tau-h_\xi} \end{bmatrix}, \tag{4.130}$$

h_τ, $\tau=1,2,\dots,b_k$ are the vectors of the set $B_k(\gamma,p_1)$, b_k is the number of the vectors in $B_k(\gamma,p_1)$. The set $B_k(\gamma,p_1)$ is defined by $B_k(\gamma,p_1)=B_k(\gamma)+\Gamma(p_1\rho^{\alpha})$ where $B_k(\gamma)=\gamma+B_k(\gamma_1,\gamma_2,\dots,\gamma_k)$, $B_k(\gamma_1,\gamma_2,\dots,\gamma_k)=\{b: b=\sum_{i=1}^{k} n_i\gamma_i, n_i\in Z, |b|<\frac{1}{2}\rho^{\frac{1}{2}\alpha_{k+1}}\}$.

Theorem 4.29. *Let $|\gamma|^2$ be a resonance eigenvalue of the operator $L(0)$, that is, $\gamma\in(\bigcap_{i=1}^{k} V_{\gamma_i}(\rho^{\alpha_k}))\setminus E_{k+1}$, $k=1,2,\dots,d-1$ where $|\gamma|\sim\rho$, λ_i an eigenvalue of the matrix V_0, and Λ_N an eigenvalue of the operator $L(V)$ satisfying*

$$|\Lambda_N-|\gamma|^2|<\frac{1}{2}\rho^{\alpha_1} \tag{4.131}$$

and

$$|<\Phi_{\gamma,j},\Psi_N>|>c_{17}\rho^{-c\alpha}. \tag{4.132}$$

Then there exists an eigenvalue $\eta_s(\gamma)$, $s=1,2,\dots,mb_k$ of the matrix C such that

$$\Lambda_N=\lambda_i+\eta_s(\gamma)+O(\rho^{-(p-c-\frac{d}{4}3^d)\alpha}).$$

Theorem 4.30. *Let* $|\gamma|^2$ *be a resonance eigenvalue of the operator* $L(0)$*, that is,* $\gamma \in (\bigcap_{i=1}^{k} V_{\gamma_i}(\rho^{\alpha_k})) \setminus E_{k+1}$*,* $k = 1, 2, \ldots, d-1$ *where* $|\gamma| \sim \rho$*,* λ_i *an eigenvalue of the matrix* V_0*,* $\eta_s(\gamma)$ *an eigenvalue of the matrix* C *such that* $|\eta_s(\gamma) - |\gamma|^2| < \frac{3}{8}\rho^{\alpha_1}$*. Then there is an eigenvalue* Λ_N *of the operator* $L(V)$ *satisfying*

$$\Lambda_N = \lambda_i + \eta_s(\gamma) + O(\rho^{-p\alpha + \frac{d}{4}3^d\alpha + \frac{d-1}{2}}), \tag{4.133}$$

where $\eta_s(\gamma)$*,* $s = 1, 2, \ldots, mb_k$ *is an eigenvalue of the matrix* C *which is given by* (4.129).

REFERENCES

Atılgan, Ş., Karakılıç, S., & Veliev.O. A. (2002). Asymptotic Formulas for the Eigenvalues of the Schrödinger Operator. *Turkish J. Math.*, *26*, 215-227.

Berezin, F. A., Shubin, M. A. (1991). *The Schrödinger Equation*. Kluwer Academic Publishers, Dordrecht.

Coşkan, D., & Karakılıç, S. (2009). On the Spectral Properties of the Schrödinger Operator with a Matrix Potential. *Adv. Stud. Contemp. Math.*, *19(2)*, 249-259.

Eastham, M. S. P. (1973). *The Spectral Theory of Periodic Differential Equations*. Scottish Academic Press, Edinburg.

Feldman, J., Knoerrer, H., & Trubowitz, E. (1990). The Perturbatively Stable Spectrum of the Periodic Schrödinger Operator. *Invent. Math.*, *100*, 259-300.

Feldman, J., Knoerrer, H., & Trubowitz, E. (1991). The Perturbatively Unstable Spectrum of the Periodic Schrödinger Operator. *Comment. Math. Helv.*, *66*, 557-579.

Friedlanger, L. (1990). On the Spectrum for the Periodic Problem for the Schrödinger Operator. *Comm. Partial Differential Equations*, *15*, 1631-1647.

Hald, O. H., & McLaughlin, J.R. (1996). Inverse Nodal Problems: Finding the Potential from Nodal Lines. *Mem. Amer. Math. Soc.*, *572*, 119, 0075-9266.

Horn, R. A., & Johnson, C.R. (1985). *Matrix Analysis*. Cambridge: Cambridge University Press.

Karakılıç, S. (2004). *On the Perturbation Theory for the Multidimensional Schrödinger Operator*. İzmir.

Karakılıç, S., Atılgan, Ş., & Veliev. O. A. (2005). Asymptotic Formulas for the Eigenvalues of the Schrödinger Operator with Dirichlet and Neumann Boundary Conditions. *Rep. Math. Phys.*, *55*(2), 221-239.

Karakılıç, S., Veliev. O. A., & Atılgan, Ş. (2005). Asymptotic Formulas for the Resonance Eigenvalues of the Schrödinger Operator. *Turkish J. Math.*,*29*(4), 323-347.

Karpeshina, Y. (1992). Perturbation Theory for the Schrödinger Operator with a non-smooth Periodic Potential. *Math. USSR-Sb*, *71*, 701-123.

Karpeshina, Y. (1996). Perturbation series for the Schrödinger Operator with a Periodic Potential near Planes of Diffraction. *Comm. Anal. Geom.*, *4*, 3, 339-413.

Karpeshina, Y. (2002). On the Spectral Properties of Periodic Polyharmonic Matrix Operators. *Indian Acad. Sci.(Math. Sci.)*, *112*(1), 117-130.

Kato, T. (1980). *Perturbation Theory for Linear Operators*. Springer, Berlin.

Reed, M., Simon, B. (1987). *Methods of Modern Mathematical Physics, vol. IV.* (3rd ed). New York, San Francisco, London: Academic Press.

Veliev, O. A. (1987). Asimptotic Formulas for the Eigenvalues of the Periodic Schrödinger Operator and the Bethe-Sommerfeld Conjecture. *Functsional Anal. i Prilozhen*, *21*, 2, 1-15.

Veliev, O. A. (1988). The Spectrum of Multidimensional Periodic Operators. *Teor. Funktsional Anal. i Prilozhen*, *49*, 17-34.

Veliev, O. A. (2006). Asymptotic Formulae for the Bloch Eigenvalues Near Planes of Diffraction. *Rep. Math. Phys.*, *58(3)*, 445-464.

Veliev, O. A. (2008). On the Constructive Determination of Spectral Invariants of the Periodic Schrödinger Operator with Smooth Potentials. *J. Phys. A*, *41*, 365206(26 pp) (doi:10.1088/1751-8113/41/36/365206).