DOKUZ EYLÜL UNIVERSITY

GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

COMBINATORICS AND TOPOLOGY OF CONIC-LINE ARRANGEMENTS

by

Celal Cem SARIOĞLU

October, 2009

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COMBINATORICS AND TOPOLOGY OF CONIC-LINE ARRANGEMENTS

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Ph.D. THESIS EXAMINATION RESULT FORM

We have read the thesis entitled "**COMBINATORICS AND TOPOLOGY OF CONIC-LINE ARRANGEMENTS**" completed by **CELAL CEM SARIOĞLU** under supervision of **ASSIST. PROF. DR. BEDİA AKYAR MØLLER** and **ASSOC. PROF. DR. A. MUHAMMED ULUDAĞ** and we certify that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Doctor of Philosophy.

. . . .

Assist. Prof. Dr. Bedia Akyar MØLLER

Supervisor

. . . .

Assoc. Prof. Dr. A. Muhammed ULUDAĞ

Co-supervisor

. . . .

Prof. Dr. Gonca ONARGAN

Thesis Committee Member

. . . .

Prof. Dr. Hamza POLAT

Thesis Committee Member

. . . .

Assoc. Prof. Dr. Oya ÖZBAKIR

Examining Committee Member

. . . .

Assoc. Prof. Dr. Meral TOSUN

Examining Committee Member

. . . .

Assist. Prof. Dr. İlhan KARAKILIÇ

Examining Committee Member

Prof. Dr. Cahit HELVACI
Director
Graduate School of Natural and Applied Sciences

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COMBINATORICS AND TOPOLOGY OF CONIC-LINE ARRANGEMENTS

ABSTRACT

In this thesis, we have concentrated on quadric-line arrangements. First we are interested with the combinatorics of line arrangements and also quadric arrangements. Next, we have studied the branched coverings of complex projective plane and two dimensional orbifolds. In addition to this, we have explicitly exhibited the covering relations among orbifold germs, observed by Yoshida. Finally, by using orbifold Chern numbers we have discovered new orbifolds uniformized by two dimensional complex ball and studied the covering relations among them.

Keywords : quadric-line arrangements, orbifold.

KONİK-DOĞRU DÜZENLEMELERİNİN TOPOLOJİSİ VE KATIŞIMI

ÖZ

Bu tezde kuadrik-doğru düzenlemeleri üzerine yoğunlaştık. İlk olarak doğru düzenlemelerinin ve konik düzenlemelerinin katışımını inceledik. Daha sonra karmaşık projektif düzlemin dallanmış örtülerini ve iki boyutlu orbifoldları çalıştık. Bunun yanı sıra, Yoshida'nın elde ettiği orbifold tohumları arasındaki örtü ilişkilerini açıkça sergiledik. Son olarak, orbifold Chern sayılarını kullanarak iki boyutlu karmaşık top tarafından uniform edilen yeni orbifoldlar keşfettik ve bunlar arasındaki örtü ilişkilerini inceledik.

Anahtar Sözcükler : kuadrik-doğru düzenlemeleri, orbifold.

CONTENTS

CHAPTER ONE

INTRODUCTION

The study of arrangements was begun by Swiss mathematician Jakob Steiner, who proved the first bounds on the maximum number of features of different types that an arrangement in Euclidean plane might have. An arrangement with n lines has at most $\frac{n(n-1)}{2}$ vertices, one per pair of crossing lines. This maximum is achieved for *simple arrangements*, those in which each two lines have a distinct pair of crossing points. In any arrangement there will be n infinite-downward rays, one per line; these rays separate $n+1$ cells of the arrangement that are unbounded in the downward direction. The remaining cells all have a unique bottommost vertex (choose the bottommost vertex to be the right endpoint of the horizontal bottom edge), and each vertex is bottommost for a unique cell, so the number of cells in an arrangement is the number of vertices plus $1+n$, or at most $1+n+\binom{n}{2}$. This was generalized by Schläfli (1901) as " n cuts can divide an m-dimensional cheese into as many as $\sum_{k=0}^{m}\binom{n}{k}$". However the bounds are known for the cheese cutting problem, there is no general answer. Since Steiner's works, it has become a popular object not only in combinatorics but also in geometry and topology, and have been studied by thousands of researchers.

Projective plane is a compactification of Euclidean plane by the simple expedient of adjoining the "line at infinity". So, we shall concentrate our attention on arrangements in the projective plane. We collect some basic but important facts of projective geometry in chapter 2.

In chapter 3, we will study the line arrangements combinatorially. First of all, we will interest in simplicial line arrangements. The simplicial arrangements not only often provide optimal solutions for various problems related with polytopes, graphs, and complexes, but also important objects of Geometry and Topology for the point of algebraic surfaces. It is known that, if an algebraic surface associated to arrangement has $\mathbf{B}_2$ as universal cover, then underlying arrangement have to be rigid. Furthermore, the simplicial line arrangements are the candidates for being rigid. For this reason, in the the light of the facts in (Grünbaum, 1967, 1971, 1972, 2009), we will first deal with the isomorphism types of line arrangements. Secondly, we will introduce the Füredi & Palásti (1984)'s method to construct an arrangement of lines with maximum number of triangles; and solution of orchard problem due to Burr et al. (1974). Then by using the torsion subgroup of an Elliptic curve, we give the complete solution of orchard problem and also for the maximum number of triple points in an arrangements of n-lines in $\mathbb{CP}^2$.

Compared the case of lines, very little is known about the question: "What kind of configurations of quadrics are possible in the complex projective plane?". This problem was originally motivated by the problem of finding interesting abelian covers of $\mathbb{CP}^2$ branched over several quadrics. Naruki (1983) obtained some results for this problem by excluding any kind of triple intersection points and contacts of order higher then 2. He described the parameter space (the moduli) for some elementary configurations.

Suppose, configuration of n quadrics has only nodes and tacnodes (A_1 and A_3 type singularities.), but no other types of singularities. Let $t(n)$ be the maximal number of tacnodes for given n. Obviously $t(n) \leq n(n-1)$. (Hirzebruch, 1986, Sec. 9) mentions the problem whether $\limsup_{n\to\infty} \frac{t(n)}{n^2}$ is positive. By considering the double cover of $\mathbb{CP}^2$ branched along the union of quadrics, and applying the Miyoka-Yau inequality to the double cover, he gave the inequality

$$t(n) \leq \frac{4}{9}n^2 + \frac{4}{3}n \tag{1.0.1}$$

If equality held, the double cover X of $\mathbb{CP}^2$ branched along the union of quadrics would be a surface for which Miyaoka-Yau equality holds for singular surfaces, and if Y were smooth surface with covering $Y \to X$ étale outside the singularities of X, then we would have $c_1^2(Y) = 3c_2(Y)$ (Megyesi, 1999). That is why this problem is interesting in algebraic geometry.

Smooth quadrics in $\mathbb{CP}^2$ are parametrized by an open subset of $(\mathbb{CP}^5)^\star$, each tacnode imposes one condition and $\dim \mathrm{Aut}(\mathbb{CP}^2) = 8$, so by a naive dimension count, one would expect $5n - t - 8$ dimensional family of configurations modulo projective equivalence for n quadrics with t tacnodes. But, examples in (Hirzebruch, 1986) show that there exist configurations with negative expected dimension. By applying the results in Megyesi (1993) Megyesi & Szabó (1996) proved that the inequality (1.0.1) is not sharp, $t(n) < \lfloor \frac{4}{9} n(n+3) \rfloor$ in for $n = 8, 9, 12$ and for $n \geq 15$, and in fact $t(n) \leq cn^{2-\frac{1}{7633}}$ for a suitable constant c. So, in (Megyesi, 2000) he studied on possible and impossible configurations of conics with many tacnodes and derive equations for them. In chapter 4, we also studied the same problem and obtain some partial results for possible or impossible configuration of quadrics, and derive the equations for these possible arrangements.

Zariski van-Kampen theorem is a tool for computing fundamental groups of complements to curves (germs of curve singularities, affine or projective plane curves). It gives us the fundamental groups in terms of generators and relations. Roughly speaking, the generators can be taken in a generic line and the relations consist of identifying these generators with their images by some monodromies. In the chapter 6, we will investigate the braid monodromy and give the statement of the Zariski van-Kampent theorem based on the lecture notes of Shimada (2007). In addition, we will also compute the local fundamental groups of the germs in Figure 6.1, and fundamental groups of some quadric arrangements related to line arrangements.

An orbifold is a space locally modeled on a smooth manifold modulo a finite group action, which is said to be uniformizable if it is a global quotient. They were first studied in the 50's by Satake under the name *V-manifold* and renamed by Thurston in 70's. Orbifolds appear naturally in various fields of mathematics and physics and they are studied from several points of view. In chapter 5, we focus on the uniformization problem and consider orbifolds with a complex projective space as base space. From this perspective, orbifolds can be viewed as a refinement of double covering construction of special algebraic varieties. The first steps in this refinement were taken by Hirzebruch (1983) culminating in the monograph Barthel et al. (1987) devoted to Kummer coverings of $\mathbb{CP}^2$ branched along line arrangements. Kobayashi (1990) studied more general coverings with non-linear branch loci with non-nodal singularities.

Chern classes are characteristic classes. They are topological invariants associated to vector bundles on a smooth manifold. If you describe the same vector bundle on a manifold in two different ways, the Chern classes will be the same. Then, the Chern classes provide a simple test: if the Chern classes of a pair of vector bundles do not agree, then the vector bundles are different. Depending on the partition of n such that $\sum_{i=1}^{n} ia_i = n$, there are Chern forms $c_I[V] := c_1^{a_1}[V]c_2^{a_2}[V]\cdots c_n^{a_n}[V]$ in terms of wedge product of Chern classes, where $I := (a_1, a_2, \cdots a_n)$. The integral of these Chern forms on manifold M takes values in $\mathbb{Z}$ and they are called *Chern numbers* of V, and denoted by $c_I := c_1^{a_1}c_2^{a_2}\cdots c_n^{a_n}$. In case of $n = 1$, there is only one Chern number, c_1, that is the Euler number e. If $n = 2$, the Chern numbers are c_1^2 and $c_2 = e$. Chern numbers are numerical invariants of manifolds.

Many basic topological invariants such as the fundamental group and Chern numbers has an orbifold version, and the usual notion of Galois covering is extended to orbifolds. It was observed by Yoshida (1987) that orbifold germs are related via covering maps, In the Section 6.2.3, we have explicitly exhibited the covering relations among orbifold germs, observed by Yoshida. Uludağ (2003, 2005, 2004, 2007) exploit these coverings to find infinitely many interesting orbifolds uniformized by the complex 2-ball $\mathbf{B}_2$, and products of Poincaré discs $\mathbf{B}_1 \times \mathbf{B}_1$. By using orbifold Chern numbers we have discovered new orbifolds and studied their covering relations together with known orbifolds uniformized by $\mathbf{B}_2$, which is the main part of this thesis.

CHAPTER TWO
PRELIMINARIES

In this chapter we will investigate well known but required facts of complex projective geometry, such as complex projective line, complex projective plane, complex projective transformations, cross ratio, projective conics, duality, intersection and parametrization of conics, cubic curves and the parametrization of elliptic curves via Weierstraß $\wp$ function.

2.1 Complex Projective Space

An n dimensional complex projective space is defined by

$$\mathbb{CP}^n = \left(\mathbb{C}^{n+1} \setminus \{\mathbf{0}\}\right) / \sim \tag{2.1.1}$$

with the equivalence relation $(z_0, z_1, \cdots, z_n) \sim (\lambda z_0, \lambda z_1, \cdots, \lambda z_n)$, where λ is an arbitrary non-zero complex number. The equivalence classes are denoted by $[z_0 : z_1 : \cdots : z_n]$ and known as *homogeneous coordinates*. Equivalently, $\mathbb{CP}^n$ is the set of all complex lines in $\mathbb{C}^{n+1}$ passing through the origin $\mathbf{0} := (0, \cdots, 0)$. Since $\lambda \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$, one may also regard $\mathbb{CP}^n$ as a quotient of $\mathbb{C}^{n+1} \setminus \{\mathbf{0}\} \doteq S^{2n+1}$ under the action of $\mathbb{C}^*$:

$$\mathbb{CP}^n = \left(\mathbb{C}^{n+1} \setminus \{\mathbf{0}\}\right) / \mathbb{C}^*. \tag{2.1.2}$$

Notice that any point $[z_0 : z_1 : \cdots : z_n]$ with $z_n \neq 0$ is equivalent to $[\frac{z_0}{z_n} : \frac{z_1}{z_n} : \cdots : \frac{z_{n-1}}{z_n} : 1]$. So there are two open disjoint subsets of the projective space: first one consists of the points $[\frac{z_0}{z_n} : \frac{z_1}{z_n} : \cdots : \frac{z_{n-1}}{z_n} : 1]$ for $z_n \neq 0$ and the second one consists of the remaining points $[z_0 : z_1 : \cdots : z_{n-1} : 0]$. The open set consisting of the points $[z_0 : z_1 : \cdots : z_{n-1} : 0]$ can be divided into two disjoint subsets with points $[\frac{z_0}{z_{n-1}} : \frac{z_1}{z_{n-1}} : \cdots : \frac{z_{n-2}}{z_{n-1}} : 1 : 0]$ for $z_{n-1} \neq 0$ and $[z_0 : z_1 : \cdots : z_{n-2} : 0 : 0]$. In a similar way, if one continues to subdivision then reaches to open sets containing the points $[\frac{z_0}{z_1} : 1 : 0 : \cdots : 0]$ for $z_1 \neq 0$ and $[z_0 : 0 : \cdots : 0] = [1 : 0 : \cdots : 0]$, respectively. Note that these last two open sets are complex line, the first is called line at infinity, and second is the point at infinity. Geometrically, the open subsets of $\mathbb{CP}^n$ obtained by subdivision are isomorphic (not only as a set, but also as a manifold) to $\mathbb{C}^p$, where $p = 0, 1, \cdots, n$. We thus have a cell decomposition

$$\mathbb{CP}^n = \mathbb{C}^n \sqcup \mathbb{C}^{n-1} \sqcup \cdots \sqcup \mathbb{C} \sqcup \{\infty\} \tag{2.1.3}$$

and it can be used to calculate some topological invariants such as the singular cohomology or the Euler characteristic of a complex projective space. As it is seen from this decomposition that a complex projective space is a compact topological space.

The above definition of complex projective space gives a set. For purposes of differential geometry, which deals with manifolds, it is useful to endow this set with a complex manifold structure. Namely consider the following subsets:

$$U_i = \{[z_0 : z_1 : \cdots : z_n] \mid \quad z_i \neq 0\}, \qquad i = 0, 1, 2, \cdots, n.$$

By the definition of complex projective space, their union is the whole complex projective space. Further, U_i is in bijection to $\mathbb{C}^n$ via

$$[z_0 : z_1 : \cdots : z_n] \mapsto \Big(\frac{z_0}{z_i}, \frac{z_1}{z_i}, \cdots, \frac{\widehat{z_i}}{z_i}, \cdots, \frac{z_n}{z_i}\Big). \tag{2.1.4}$$

Here, the hat means that the i-th entry is missing. It is clear that $\mathbb{CP}^n$ is a complex manifold of complex dimension n, so it has real dimension 2n.

In general context, $\mathbb{CP}^1$ is called as the *complex projective line*, which is also known as the *Riemann sphere*, and $\mathbb{CP}^2$ is called as the *complex projective plane*.

For the simplicity, from now on unless otherwise indicated we will use the term “projective” instead of “complex projective”.

2.2 Complex Projective Transformations

Let V and V' be two complex vector spaces, $p : V \setminus \{0\} \to \mathbb{P}_V$ and $p' : V' \setminus \{0\} \to \mathbb{P}_{V'}$ two projections. A projective transformation $g : \mathbb{P}_V \to \mathbb{P}_{V'}$ is a mapping such that there exists a linear isomorphism $f : V \to V'$ with $p' \circ f = g \circ p$, in other words such that the following diagram

$$\begin{tikzcd} V \setminus \{0\} \arrow[r, "f"] \arrow[d, "p"] & V' \setminus \{0\} \arrow[d, "p'"] \\ \mathbb{P}_V \arrow[r, "g"] & \mathbb{P}_{V'} \end{tikzcd} \tag{2.2.1}$$

commutes.

Since f is a linear isomorphism, it maps the set of lines passing through the origin to itself. Therefore, the image under g of a point L of $\mathbb{P}_V$ (line of V through the origin) is the point $L' = f(L)$ of $\mathbb{P}_{V'}$.

If $V = V' = \mathbb{C}^2$ then the automorphisms of $\mathbb{C}^2$ are just the 2×2 invertible matrices with complex entries and these automorphisms forms a group under ordinary matrix multiplication. The automorphism group of $\mathbb{C}^2$ is usually denoted by $\mathrm{GL}(2, \mathbb{C})$ and called *general linear group of degree* 2. Since $p : \mathbb{C}^2 \setminus \{0\} \to \mathbb{P}^1_{\mathbb{C}} = \mathbb{CP}^1$ is a projection, an invertible 2×2 matrix A with complex entries acts on the projective line $\mathbb{CP}^1$ via $f([z_0 : z_1]) = [z'_0 : z'_1]$, where

$$\begin{bmatrix} z'_0 \\ z'_1 \end{bmatrix} = M \cdot \begin{bmatrix} z_0 \\ z_1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} z_0 \\ z_1 \end{bmatrix}.$$

This is well defined, since $f([\lambda z_0 : \lambda z_1]) = [\lambda z'_0 : \lambda z'_1] = [z'_0 : z'_1] \quad$ for $\lambda \in \mathbb{C}^*$.

There are, however, the matrices in $\mathrm{GL}(2, \mathbb{C})$ that have no effect on points in the projective line: the diagonal matrix $M = \alpha I_{2\times 2}$ with $\alpha \in \mathbb{C}^*$ fixes every $[z_0 : z_1] \in \mathbb{CP}^1$. Also, the matrices $M \in \mathrm{GL}(2, \mathbb{C})$ and αM have the same effects on $\mathbb{CP}^1$ (in fact, $\alpha M = \alpha I \cdot M$).

The group of diagonal matrices with entry $\alpha \in \mathbb{C}^*$ is isomorphic to $\mathbb{C}^*$, and we can make the projective general linear group of order 2, $\mathrm{PGL}(2,\mathbb{C}) = \mathrm{GL}(2,\mathbb{C})/\mathbb{C}^*$, act on the projective line. Its elements are 2×2 complex matrices with nonzero determinant and two such matrices are considered to be equal if they differ by a nonzero factor $\alpha \in \mathbb{C}^*$. In addition, $\dim \mathrm{PGL}(2,\mathbb{C}) = 3$.

Let us identify the point $[z:1]$ with z, choose the frame 0, 1 and $\infty := [1:0]$. Set $\infty/\infty = 1$, $k/0 = \infty$ for $k \neq 0$, and so on, for convenience, and remember the fact $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$. $\mathrm{PGL}(2,\mathbb{C}) = \mathrm{Aut}(\mathbb{CP}^1)$ can also be considered as the group of all biholomorphic linear fractional transformations, namely *Möbius transformations*,

$$f : z \in \mathbb{CP}^1 \to \frac{az+b}{cz+d} \in \mathbb{CP}^1, \quad ad-bc \neq 0. \tag{2.2.2}$$

Note that, in the case of $ad-bc=0$, the rational function f takes constant value.

Proposition 2.2.1. *Let z_1, z_2 and z_3 be three points on the Riemann sphere $\mathbb{CP}^1$. Then there is a unique Möbius transformation such that $f(z_1)=\infty$, $f(z_2)=0$ and $f(z_3)=1$.*

Proof. The equations $f(z_1)=\infty$, $f(z_2)=0$ and $f(z_3)=1$ implies $cz_1+d=0$, $az_2+b=0$ and $az_3+b=cz_3+d$, respectively. Then $c\neq 0$, otherwise all of a, b, c and d will be zero. Since the Möbius transformation is a rational linear transformation, we can choose $c=1$. Therefore, we have $d=-z_1$, $a=\frac{z_3-z_1}{z_3-z_2}$ and $b=-z_2\frac{z_3-z_1}{z_3-z_2}$. Hence, the required Möbius transformation is

$$f(z) = \frac{(z_3-z_1)(z-z_2)}{(z_3-z_2)(z-z_1)}. \tag{2.2.3}$$

□

Corollary 2.2.2. *A three-point set in $\mathbb{CP}^1$ is projectively rigid, i.e., given any pair of distinct three points $\{z_1,z_2,z_3\}$ and $\{z_1',z_2',z_3'\}$ on the Riemann sphere $\mathbb{CP}^1$, there is a unique Möbius transformation f such that $f(z_i)=f(z_i')$, $i=1,2,3$.*

Proof. Let g and h be the Möbius transformations sending the frames $\{z_1, z_2, z_3\}$ and $\{z'_1, z'_2, z'_3\}$ to the standard frame $\{\infty, 0, 1\}$, respectively. Then $f = h^{-1} \circ g$ is the required transformation. □

Definition 2.2.3 (Cross-ratio)**.** The cross-ratio of a quadruple of distinct points on the projective line with coordinates $[\alpha_i : \beta_i]$, $i = 1, 2, 3, 4$, is the point of $\mathbb{CP}^1$ defined by

$$\left[\frac{\det \begin{pmatrix} \alpha_1 & \alpha_3 \\ \beta_1 & \beta_3 \end{pmatrix}}{\det \begin{pmatrix} \alpha_1 & \alpha_4 \\ \beta_1 & \beta_4 \end{pmatrix}} : \frac{\det \begin{pmatrix} \alpha_2 & \alpha_3 \\ \beta_2 & \beta_3 \end{pmatrix}}{\det \begin{pmatrix} \alpha_2 & \alpha_4 \\ \beta_2 & \beta_4 \end{pmatrix}} \right] \tag{2.2.4}$$

If $\beta_i \neq 0$ for all $i = 1, 2, 3, 4$, then we can identify each point $[\alpha_i : \beta_i] = [\frac{\alpha_i}{\beta_i} : 1]$ with non-zero complex number $\frac{\alpha_i}{\beta_i}$, for simplicity say z_i, then the cross ratio of z_1, z_2, z_3, z_4 is a non-zero number given by the formula

$$(z_1, z_2\,; z_3, z_4) = \frac{z_1 - z_3}{z_2 - z_3} : \frac{z_1 - z_4}{z_2 - z_4} = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_2 - z_3)(z_1 - z_4)} \tag{2.2.5}$$

If one of $\beta_i = 0$, say $\beta_1 = 0$, then $z_1 = \infty$ and $(\infty, z_2\,; z_3, z_4) = \frac{z_2 - z_4}{z_2 - z_3}$.

Note that the cross ratio $(z_1, z_2\,; z_3, z_4)$ of distinct four points z_1, z_2, z_3, z_4 on the projective line is the image of z_4 under the Möbius transformation sending the points z_1, z_2, z_3 to the points $\infty, 0, 1$ respectively (See equation (2.2.3)).

There are different definitions of the cross-ratio used in the literature. However, they all differ from each other by some possible permutation of the coordinates. In general, there are six possible different values the cross-ratio can take depending on the order in which the points z_i are given. Since there are 24 possible permutations of the four coordinates, some permutations must leave the cross-ratio unaltered. In fact, exchanging any two pairs of coordinates preserves the cross-ratio:

$$(z_1, z_2\,; z_3, z_4) = (z_2, z_1\,; z_4, z_3) = (z_3, z_4\,; z_1, z_2) = (z_4, z_3\,; z_2, z_1) \tag{2.2.6}$$

Using these symmetries, there can then be 6 possible values of the cross-ratio, depending on the order in which the points are given. These are:

$$\begin{array}{lll} (z_1,z_2\,;z_3,z_4)=\lambda, & (z_1,z_3\,;z_2,z_4)=1-\lambda, & (z_1,z_4\,;z_2,z_3)=\frac{\lambda-1}{\lambda} \\ (z_1,z_2\,;z_4,z_3)=\frac{1}{\lambda}, & (z_1,z_3\,;z_4,z_2)=\frac{1}{1-\lambda}, & (z_1,z_4\,;z_3,z_2)=\frac{\lambda}{\lambda-1}. \end{array} \tag{2.2.7}$$

Proposition 2.2.4. *Cross-ratios are invariant under Möbius transformations.*

Proof. Let z_1, z_2, z_3 and z_4 be four distinct points on $\mathbb{CP}^1$ and g the Möbius transformation sending z_1,z_2,z_3 to $\infty,0,1$, respectively, so that $(z_1,z_2\,;z_3,z_4)=g(z_4)$. Then for any Möbius transformation f, $g\circ f^{-1}$ is the Möbius transformation sending $f(z_1)$, $f(z_2)$, $f(z_3)$, $f(z_4)$ to $\infty,0,1,g(z_4)$, i.e., $\big(f(z_1),f(z_2)\,;f(z_3),f(z_4)\big)=g(z_4)$. $\square$

Now, let us go one step further and choose $V=V'=\mathbb{C}^3$ in the diagram (2.2.1), then the automorphisms of $\mathbb{C}^3$ are just the 3×3 invertible matrices with complex entries, and these automorphisms forms a group under ordinary matrix multiplication. The automorphism group of $\mathbb{C}^3$ is usually denoted by $\mathrm{GL}(3,\mathbb{C})$ and called *General Linear group of order* 3. Since $p:\mathbb{C}^3\setminus\{0\}\to\mathbb{P}^2_{\mathbb{C}}=\mathbb{CP}^2$ is a projection, then an invertible 3×3 matrix A with complex entries acts on the projective plane $\mathbb{CP}^2$ via $f([x:y:z])=[x':y':z']$, where

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = M\cdot\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\cdot\begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

This is well defined, since $f([\lambda x:\lambda y:\lambda z])=[\lambda x':\lambda y':\lambda z']=[x':y':z']$ for $\lambda\in\mathbb{C}^*$.

There are, however, the matrices in $\mathrm{GL}(3,\mathbb{C})$ have no effect on points in the projective plane: the diagonal matrix $M=\alpha I_{3\times 3}$ with $\alpha\in\mathbb{C}^*$ fixes every $[x:y:z]\in\mathbb{CP}^2$. Also, the matrices $M\in\mathrm{GL}(3,\mathbb{C})$ and αM have the same effects on $\mathbb{CP}^2$ (in fact, $\alpha M=\alpha I\cdot M$). The group of diagonal matrices with entries $\alpha\in\mathbb{C}^*$ is isomorphic to $\mathbb{C}^*$, and we can make the projective general linear group of order three, $\mathrm{PGL}(3,\mathbb{C}) = \mathrm{GL}(3,\mathbb{C})/\mathbb{C}^*$, act on the projective plane. Its elements are 3×3 complex matrices with nonzero determinant, and two such matrices are considered to be equal if they differ by a nonzero factor $\alpha \in \mathbb{C}^*$. In addition, $\dim \mathrm{PGL}(3,\mathbb{C}) = 8$.

Proposition 2.2.5. *Let $P_i = [x_i : y_i : z_i]$, $i = 1,2,3,4$ be four points in $\mathbb{CP}^2$, no three of which are collinear. Then there is a unique projective transformation sending the standard frame, namely $[1:0:0]$, $[0:1:0]$, $[0:0:1]$ and $[1:1:1]$, to the points P_1, P_2, P_3 and P_4, respectively.*

Proof. The transformation defined by $A \in \mathrm{PGL}(3,\mathbb{C})$ will map $[1:0:0]$ to P_1, if and only if there is $\alpha_1 \in \mathbb{C}^*$ with

$$\alpha_1 \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = M \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} a_{11} \\ a_{12} \\ a_{13} \end{bmatrix}.$$

Similarly the second and the third rows are determined up to nonzero factors $\alpha_2, \alpha_3 \in \mathbb{C}^*$. Thus,

$$M = \begin{bmatrix} \alpha_1 x_1 & \alpha_1 y_1 & \alpha_1 z_1 \\ \alpha_2 x_2 & \alpha_2 y_2 & \alpha_2 z_2 \\ \alpha_3 x_3 & \alpha_3 y_3 & \alpha_3 z_3 \end{bmatrix}.$$

Now, P_4 will be the image of $[1:1:1]$ if and only if

$$\alpha_4 \begin{bmatrix} x_4 \\ y_4 \\ z_4 \end{bmatrix} = M \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \alpha_1 \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + \alpha_2 \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} + \alpha_3 \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix}$$

Rescaling allows us to assume $\alpha_4 = 1$. Thus, the vector (x_4, y_4, z_4) is a linear combi- nation of (x_i, y_i, z_i), $\quad i = 1,2,3$. Since the vectors (x_i, y_i, z_i) are linearly independent, there is a unique solution $(\alpha_1, \alpha_2, \alpha_3)$, and since no three of the points P_i are collinear then $\alpha_i \neq 0$. This implies that M is an invertible matrix and defines

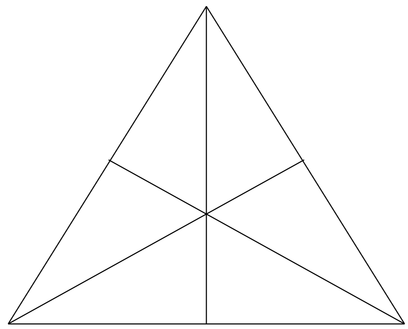


Figure 2.1 Complete quadrilateral.

a unique projective transformation f given by a matrix $M \in \mathrm{PGL}(3,\mathbb{C})$. □

Corollary 2.2.6. *Let $\{P_i\}$ and $\{Q_i\}$ denote the sets of four points in the projective plane such that no three of P_i and no three of Q_i are collinear. Then there is a unique projective transformation sending P_i to Q_i for $i = 1,2,3,4$.*

Proof. Let f denote the projective transformation given by a matrix M that sends the standard frame to the P_i's; let g denote the projective transformation given by a matrix N that does the same with Q_i's. Then the transformation $g \circ f^{-1}$ defined by the matrix $N \cdot M^{-1}$ is the projective transformation we are looking for. □

Corollary 2.2.7. *Complete quadrilateral, configuration of six lines with four simple triple points and three nodes, is projectively rigid.*

Proof. As it is seen from the Figure 2.1 that the complete quadrilateral is completely determined by four triple points. Then by Corollary 2.2.6, one can transform this four points to any four points for which none of three is collinear. Hence, the complete quadrilateral is projectively unique. □

An ordered quadruple of distinct points z_1, z_2, z_3, z_4 of $\mathbb{CP}^1$ is called a *harmonic quadruple* if $(z_1,z_2\,;z_3,z_4) = -1$. Let us assume that these four points lie on a complex line L in $\mathbb{CP}^2$. By choosing a frame on L, one can identify L with $\mathbb{CP}^1$ and extend this definition for arbitrary complex line in $\mathbb{CP}^2$.

Proposition 2.2.8. *The quadruple of distinct points p_1, p_2, p_3, p_4 of $L \subset \mathbb{CP}^2$ is harmonic if and only if there are points $a,b,c,d \in \mathbb{CP}^2 \setminus L$ such that the intersection*

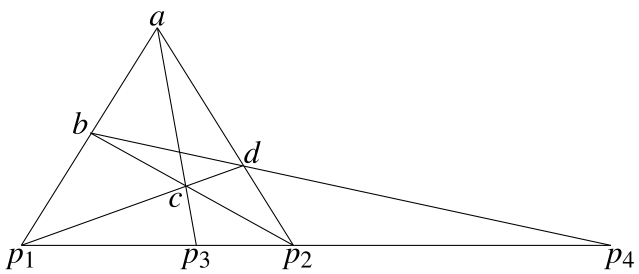


Figure 2.2 Harmonic configuration.

points of the complete quadrilateral, having the points a,b,c,d as triple points, with ℓ are the points p_1, p_2, p_3, p_4. Such configuration is known as harmonic configuration (See Figure 2.2).

Proof. First, let us show the necessary part. Corollaries 2.2.2 and 2.2.6 impliy that one may choose a homogeneous coordinate system on $\mathbb{CP}^2$ such that $a = [0:0:1]$, $p_1 = [1:0:0]$, $p_2 = [1:1:0]$, $p_4 = [0:1:0]$ and $d = [1:1:1]$. Then $b = [1:0:1]$, $c = [2:1:1]$, $p_3 = [2:1:0]$ and ℓ is the line $Z = 0$. Hence by omitting the third coordinates one can identify L with $\mathbb{CP}^1$ and obtains $(p_1, p_2; p_3, p_4) = \frac{1-2}{1-0} = -1$.

Conversely, we can draw a configuration from the points p_1, p_2 and p_4 as in Figure 2.2. Put $p_3' = L \cap \overline{ac}$. Here $\overline{ac}$ denotes the line through a and c. Then by Proposition 2.2.4, $(p_1, p_2; p_3, p_4) = -1 = (p_1, p_2; p_3', p_4)$ implies $p_3 = p_3'$. $\square$

A Projective transformation f given by a matrix A act on the projective plane and therefore on a plane algebraic curve $\mathcal{C}_F : F(X,Y,Z) = 0$; the image of $\mathcal{C}_F$ under f is some curve $\mathcal{C}_G : G(U,V,W) = 0$. How can be computed G from F? Let us first look at simple example. Take $F(X,Y,Z) = X^2 - YZ$ and the transformation $[U:V:W] = f([X:Y:Z]) = [X:Y+Z:Y-Z]$. For getting G, we solve X, Y, Z and then plug the result $(X,Y,Z) = (U, \frac{V+W}{2}, \frac{V-W}{2})$ into F, hence $G(U,V,W) = F(U, \frac{V+W}{2}, \frac{V-W}{2}) = U^2 - \frac{V^2}{4} + \frac{W^2}{4}$. It has been seen from this example that we get G by evaluating F at $f^{-1}([X:Y:Z])$, that is, $G = F \circ f^{-1}$. This ensures that a point $[X:Y:Z]$ on $\mathcal{C}_F$ will get mapped by f to a point $[U:V:W]$ on $\mathcal{C}_G$.

Proposition 2.2.9. *Projective transformations preserve the degree of curves.*

Proof. Projective transformations map a monomial $X^iY^jZ^k$ of degree $m = i+j+k$ either to 0 or to another homogeneous polynomial of degree m. If $F(X,Y,Z)$ is transformed by some transformations f into the zero polynomial, then inverse transformation maps the zero polynomial into F, which is nonsense. □

Definition 2.2.10. A point $[X_0 : Y_0 : Z_0] \in \mathbb{CP}^2$ is called the singular point of the curve $\mathcal{C}_F :\ F(X,Y,Z) = 0$ if

$$\frac{\partial F}{\partial X}(X_0,Y_0,Z_0) = \frac{\partial F}{\partial Y}(X_0,Y_0,Z_0) = \frac{\partial F}{\partial Z}(X_0,Y_0,Z_0) = 0. \tag{2.2.8}$$

Proposition 2.2.11. *Projective transformations preserve singularities.*

Proof. Suppose a projective curve $\mathcal{C}_F : F(X,Y,Z) = 0$ is mapped to a projective curve $\mathcal{C}_G : G(U,V,W) = 0$ via a projective transformation f given by a matrix M. Then, we have $F = G \circ f$ and $[U \quad V \quad W]^T = [X \quad Y \quad Z]^T \cdot M^T$. Hence the chain rule implies

$$\begin{bmatrix} \frac{\partial F}{\partial X} \\ \frac{\partial F}{\partial Y} \\ \frac{\partial F}{\partial Z} \end{bmatrix} = M \cdot \begin{bmatrix} \frac{\partial G}{\partial U} \\ \frac{\partial G}{\partial V} \\ \frac{\partial G}{\partial W} \end{bmatrix} \tag{2.2.9}$$

Therefore a point $P_0 = [X_0 : Y_0 : Z_0]$ on $\mathcal{C}_F$ is singular if and only if all three derivatives of F vanish at P_0. Since $M \in \mathrm{PGL}(3,\mathbb{C})$ then it is nonsingular and the equation (2.2.9) implies that the point $[U_0 : V_0 : W_0] = f([X_0 : Y_0 : Z_0])$ is a singular point of the curve $\mathcal{C}_G$. □

Similarly, after some calculations one can also show that projective transformations preserve the multiplicities, tangents, flexes ,etc.

2.3 Projective Conics

A conic in the complex plane is given by a quadric equation $a_1x^2+a_2xy+a_3y^2+a_4x+a_5y+a_6=0$, where at least one of the complex coefficients a_i is non zero. By using homogeneous coordinates and reindexing the coefficients, a conic in $\mathbb{CP}^2$ is given by homogenous ternary quadric equation

$$a_1X^2+a_2Y^2+a_3Z^2+a_4XY+a_5YZ+a_6ZX=0, \tag{2.3.1}$$

where at least one of the complex coefficients a_i is non zero. In matrix notation, the equation (2.3.1) can be written as

$$\begin{bmatrix} X & Y & Z \end{bmatrix} \cdot M \cdot \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} X & Y & Z \end{bmatrix} \cdot \begin{bmatrix} a_1 & \frac{a_4}{2} & \frac{a_6}{2} \\ \frac{a_4}{2} & a_2 & \frac{a_5}{2} \\ \frac{a_6}{2} & \frac{a_5}{2} & a_3 \end{bmatrix} \cdot \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = 0. \tag{2.3.2}$$

If $\det M = 0$, then the conic is said to be *reducible* (or *degenerate*), this means that the conic is either a double line or a union of two lines, otherwise it is called *irreducible* (or *non degenerate*).

Note that, at least one of the coefficients of a conic in $\mathbb{CP}^2$ is non zero. This means that it is enough to know five points which conic passes or five independent info about conic, to determine a conic in $\mathbb{CP}^2$. On the other hand, there is a bijection between the conics in $\mathbb{CP}^2$ and the points $[a_1 : a_2 : a_3 : a_4 : a_5 : a_6]$ of $\mathbb{CP}^5$. Then one may prefer to analyse configuration of points in $\mathbb{CP}^5$, instead of configuration of conics in $\mathbb{CP}^2$.

Projective transformations preserve the degree of curves, thus they map lines into lines and conics into conics. Affine transformations preserve the line at infinity; hence can not a (real) circle (no point at infinity) into a hyperbola (two points at infinity). Projective transformations can do this: the projective circle has equation $X^2+Y^2-Z^2=0$, the projective transformation $U=Z$, $V=X$, $W=Y$ transform this equation into $V^2 - U^2 + W^2 = 0$, which after dehomogenizing with respect to W, is just the hyperbola $u^2 - v^2 = 1$. What happened here is that $Y = W$ has moved to the two points with $Y = 0$ to infinity.

Similarly, the hyperbola $XY - Z^2 = 0$ can be transformed into a parabola via $U = Z$, $V = X$, $W = Y$: after dehomogenizing we get $v = u^2$. The hyperbola had two points $[1:0:0]$ and $[0:1:0]$ at infinity; the first one was moved to the point $[0:1:0]$ at infinity, the second one to $[0:0:1]$ which is the origin in the affine plane. As a matter of fact it can be proved that, over the complex numbers, there is only one class of non degenerate conics up to projective transformations (See Proposition 2.3.2).

Anymore, since a conic in $\mathbb{CP}^2$ is given by a homogeneous ternary quadric equation in three variables, the term *quadric* will be used instead of the term *conic*.

Definition 2.3.1. Two quadrics are called projectively equivalent if there is a projective transformation, mapping one to the other.

Proposition 2.3.2. *Any non degenerate projective quadric defined over $\mathbb{C}$ is projectively equivalent to the quadric $XY + YZ + ZX = 0$. More exactly, given a non degenerate quadric Q and three points on Q, there is a unique projective transformation which maps Q to a quadric and three points to* $[1:0:0]$, $[0:1:0]$ *and* $[0:0:1]$, *respectively.*

Proof. Take any three points on a quadric. Then by corollary 2.2.6, there is a projective transformation, mapping them into $[1:0:0]$, $[0:1:0]$ and $[0:0:1]$, respectively (note that the three points on a quadric are not collinear since the quadric is non degenerate). If the transformed quadric has the equation

$$a_1U^2 + a_2V^2 + a_3W^2 + a_4UV + a_5VW + a_6WU = 0 \tag{2.3.3}$$

then we immediately see that $a_1 = a_2 = a_3 = 0$. Moreover, $a_4a_5a_6 \neq 0$ since otherwise the quadric is degenerate. Using the transformation $U = a_5X$, $V = a_6Y$, $W =$

a_4Z, this becomes $XY+YZ+ZX=0$. If there are two such maps f and g, then $g\circ f^{-1}$ maps the standard quadric onto itself and preserves the three points of the standard frame. It is then easily seen that the corresponding matrix to $g\circ f^{-1}$ must be the identity map in $\mathrm{PGL}(3,\mathbb{C})$. $\square$

2.4 Duality

Given any vector space V over a field $\Bbbk$, the dual space $V^\star$ is defined to be the set of all linear functionals on V, i.e., scalar valued linear transformations on V (in this context, a "scalar" is a member of the base field $\Bbbk$). $V^\star$ itself becomes a vector space over $\Bbbk$ under the following definition of addition and scalar multiplication:

$$(\phi+\psi)(x)=\phi(x)+\psi(x) \qquad \text{and} \qquad (\lambda\phi)(x)=\lambda\phi(x)$$

for all ϕ and ψ in $V^\star$, λ in $\Bbbk$ and x in V. If the dimension of V is finite, then $V^\star$ has the same dimension as V; if $\{e_1,\cdots,e_n\}$ is a basis for V, then the associated dual basis $\{e^1,\cdots,e^n\}$ of $V^\star$ is given by

$$e^i(e_j)=\delta_{ij}=\begin{cases}1, & \text{if } i\neq j\\ 0, & \text{if } i=j.\end{cases}$$

Concretely, if we interpret $\mathbb{C}^3$ as the space of columns of three complex numbers, then its dual space is typically written as the space of rows of there complex numbers. Such a row acts on $\mathbb{C}^3$ as a linear functional by ordinary matrix multiplication. In addition, the elements of $(\mathbb{C}^3)^\star$ can be intuitively represented as collections of parallel planes.

If $[x:y:z]\in\mathbb{CP}^2$ then $(x,y,z)\sim(\lambda x,\lambda y,\lambda z)$ for any nonzero complex number λ. Let us consider the set of functionals $\phi\in(\mathbb{C}^3)^\star$ so that $\phi(x,y,z)=\phi(\lambda x,\lambda y,\lambda z)=\lambda\phi(x,y,z)$ for any λ in $\mathbb{C}^*$. It is clear that, these functionals vanish on $\mathbb{C}^3$ and $\phi([x:$

$y:z]) = 0$ for any $[x:y:z] \in \mathbb{CP}^2$. Thus, dual of the projective plane contains the linear functionals vanishing on $\mathbb{CP}^2$. Also, one can view such kind of functionals as lines in $\mathbb{CP}^2$.

$$[A:B:C] \in \mathbb{CP}^2 \quad \rightleftarrows \quad L:\ AX+BY+CZ = 0 \subset \mathbb{CP}^2 \tag{2.4.1}$$

Duality for the projective plane $\mathbb{CP}^2$ concerns the interchangeability between points and lines which preserves incidence properties (More generally, duality for $\mathbb{CP}^n$ interchanges *dimension*+1 to *codimension*). We now extend this property for projective, algebraic curves. For any projective curve $\mathcal{C} \subset \mathbb{CP}^2$, consider the subset

$$\mathcal{C}^\star = \{L^\star \mid L \text{ is a line of tangency to } \mathcal{C}\} \tag{2.4.2}$$

and refer to it as the *dual curve of* $\mathcal{C}$. Indeed, it turns out that this subset of $\mathbb{CP}^2$ is actually a projective curve, in $\mathbb{CP}^2$, except for the case when $\mathcal{C}$ is a projective line, in which case $\mathcal{C}^\star$ consists of just one point.

Proposition 2.4.1. *The dual curve of a non degenerate quadric in* $\mathbb{CP}^2$ *is again a quadric in* $\mathbb{CP}^2$.

Proof. In Proposition 2.3.2, it is shown that all non degenerate quadrics are projectively equivalent. It is enough to prove that, dual curve of the quadric Q given by the equation $F(X,Y,Z) = X^2 - YZ = 0$ is again a non degenerate quadric. We have

$$\frac{\partial F}{\partial X} = 2X,\ \frac{\partial F}{\partial Y} = -Z,\ \frac{\partial F}{\partial Z} = -Y,$$

then by eliminating X, Y and Z between the equations

$$2X = U, \quad -Z = V, \quad -Y = W \quad \text{and} \quad X^2 - YZ = 0$$

we obtain the equation of the dual curve $\mathcal{C}^\star$ as $U^2 - 4VW = 0$ which defines a non degenerate quadric in $\mathbb{CP}^2$. □

Corollary 2.4.2. $(Q^{\star})^{\star} = Q$.

2.5 Intersection Behaviour of Quadrics

Definition 2.5.1. Let $(f,0)$ and $(g,0)$ be two smooth germs of algebraic curves in $\mathbb{C}^2$ and let $\varphi : \Delta_t \to \mathbb{C}^2$ be the parametrization of $(f,0)$. The vanishing degree of $g \circ \varphi$ at the origin is called the *intersection number* or *intersection multiplicity* of the algebraic curves at the origin.

Example 2.5.2. The non degenerate quadrics $Q_1 :\ X^2 - YZ = 0$ and $Q_2 :\ X^2 + XY - YZ = 0$ intersect each other at the points $[0:0:1]$ and $[0:1:0]$. Let us find their intersection multiplicities. For the point $[0:0:1]$, dehomogenizing the equations of quadrics we get $f :\ x^2 - y = 0$ and $g :\ x^2 + xy - y = 0$. The germ $(f,0)$ can be parameterized as $\varphi : \Delta_t \to \mathbb{C}^2$, $\varphi(t) = (t,t^2)$, then $(g \circ \varphi)(t) = t^3$ and its vanishing degree at the origin is 3 ,i.e. the intersection multiplicity of the quadrics Q_1 and Q_2 at the point $[0:0:1]$ is 3. In addition, after some calculations it can be easily seen that the intersection multiplicity of the quadrics Q_1 and Q_2 at the point $[0:1:0]$ is 1.

The well known *Bézout's theorem* was originally stated by French mathematician *Etienne Bézout* in 1779 as "*The degree of the final equation resulting from any number of complete equations in the same number of unknowns, and of any degrees, is equal to the product of the degrees of the equations*" to solve the system of equations.

Theorem 2.5.3 (Weak Bézout's Theorem)**.** *If two curves of degree m and n have more then mn distinct points in common then they have a common component.*

Even for the weak form of Bézout's theorem, it has many important consequences:

Theorem 2.5.4. *If two curves of order n intersect at n^2 distinct points, and if mn*

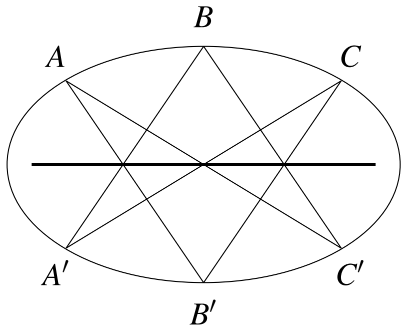


Figure 2.3 Pascal's theorem.

of this points lie on an irreducible curve of degree m, then the remaining $n^2 - mn$ *points lie on a curve of degree* $n - m$*.*

Theorem 2.5.5 (Pascal's Theorem)**.** *If one is given six points on a non degenerate quadric and makes a hexagon out of them in an arbitrary order, then the points of intersection of opposite sides of this hexagon will all lie on a single line.*

Proof. Let $ABCA'B'C'$ be a hexagon on an irreducible quadric. Let AB' and $A'B$, AC' and $A'C$, BC' and $B'C$ be the opposite sides of the hexagon. The triples of lines AC', BA', CB' and AB',BC', CA' define two cubics. They intersect at 9 points, and six of them lie on an irreducible quadric. Thus the remaining three lie on a curve of degree $3 - 2 = 1$, i.e, the remaining 3 points are collinear. □

The Pascal's Theorem was discovered by *Blaise Pascal* when he was only 16 years old. It is the generalization of the "*Pappus's hexagon theorem*". The original proof of Pascal's theorem has been lost and it is supposed to be he proved his theorem via Menelaus' theorem. We used the consequence of Bézout's Theorem to prove it.

The Pascal's theorem was generalized by *Möbius* in 1847 as follows: suppose a polygon with $4n + 2$ sides is inscribed in a quadric, and opposite pairs of sides are extended until they meet in $2n + 1$ points. Then if $2n$ of those points lie on a common line, the last point will be on that line, too.

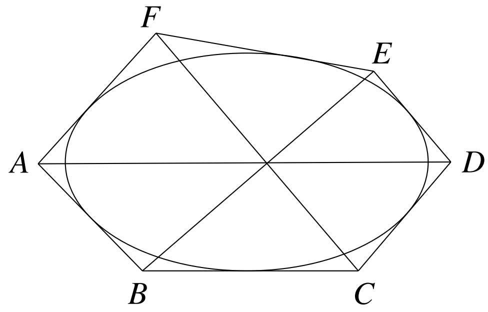


Figure 2.4 Brianchon's theorem.

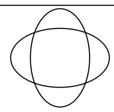
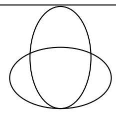
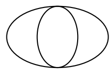
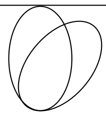
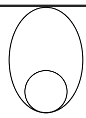
Theorem 2.5.6 (Brianchon's Theorem)**.** *Let $ABCDEF$ be a hexagon formed by six tangent lines of a non degenerate quadric. Then the lines AD, BE, CF intersect at a single point.*

Proof. Since, duality for $\mathbb{CP}^2$ interchanges the roles of points and lines and preserves the incidence relations meanwhile the dual of a quadric is again a quadric in $\mathbb{CP}^2$, then the dual of the Brianchon's Theorem is just the Pascal's Theorem. □

Theorem 2.5.7 (Strong Bézout's Theorem)**.** *Let C_1 and C_2 be plane projective algebraic curves of degree m and n without common component over an algebraically closed field $\mathbb{k}$. Then they intersect in exactly mn points counting multiplicities.*

As a result of Theorem 2.5.7 over the algebraically closed field $\mathbb{C}$, two quadric have only four intersection points counting multiplicities. Thus, there are five (=the number of positive integer partitions of 4) situations for the intersection behavior of two non degenerate quadrics. To describe these non degenerate cases, we will investigate a graph whose vertices denotes the quadrics and edges denote the intersection behavior of non degenerate quadrics (See Table 2.1). In addition, we will describe the degenerate cases in the Table 2.2.

Table 2.1 Intersection behavior of two non degenerate quadrics.

Graph	Configuration	Meaning
Q_1 Q_2		Two quadrics Q_1 and Q_2 intersect each other at four distinct points, i.e, they are in general position.
Q_1 Q_2		Two quadrics Q_1 and Q_2 intersect each other at three distinct points with multiplicities 2, 1 and 1, i.e, they have a tacnode.
Q_1 Q_2		Two quadrics Q_1 and Q_2 intersect each other at two distinct points with multiplicities 2 and 2, i.e., they tangent to each other at two distinct points or they have two tacnodes.
Q_1 Q_2		Two quadrics Q_1 and Q_2 intersect each other at two distinct points with multiplicities 3 and 1.
Q_1 Q_2		Two quadrics Q_1 and Q_2 tangent each other at a point with multiplicity 4.

2.6 Parametrization of Quadrics

Let Q be a quadric given by the equation,

$$a_1X^2 + a_2Y^2 + a_3Z^2 + a_4XY + a_5YZ + a_6ZX = 0, \tag{2.6.1}$$

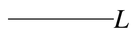
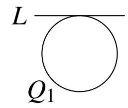
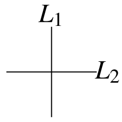
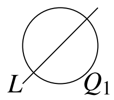
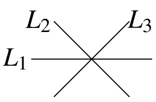
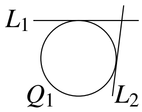
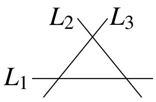
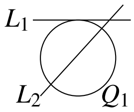
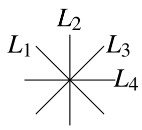
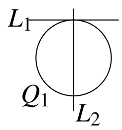
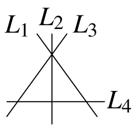
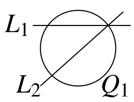
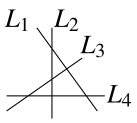
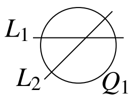
in $\mathbb{CP}^2$ and $[X_0 : Y_0 : Z_0]$ a point on it. The equation of the lines through this point are in the form

$$s(YZ_0 - Y_0Z) = t(Z_0X - X_0Z). \tag{2.6.2}$$

According to *Bézout's Theorem* there are two intersection points of this line and the quadric Q. These intersection points can be found by substituting the equation (2.6.2) into the equation (2.6.1) and solving it. After some calculations one can get these solutions as $[X_0 : Y_0 : Z_0]$ and $[p_1(s,t) : p_2(s,t) : p_3(s,t)]$, where $p_i(s,t) \in \mathbb{C}[s,t]$ are homogeneous of degree 2. Therefore a quadric can be parametrized by as:

$$X = p_1(s,t), \quad Y = p_2(s,t), \quad Z = p_3(s,t). \tag{2.6.3}$$

Table 2.2 Intersection behavior of two quadrics in degenerate cases.

Configuration	Configuration
 $Q_1 = Q_2 = L \cdot L$	 $Q_2 = L \cdot L$
 $Q_1 = L_1 \cdot L_1,\, Q_2 = L_2 \cdot L_2$	 $Q_2 = L \cdot L$
 $Q_1 = L_1 \cdot L_1,\, Q_2 = L_2 \cdot L_3$	 $Q_2 = L_1 \cdot L_2$
 $Q_1 = L_1 \cdot L_1,\, Q_2 = L_2 \cdot L_3$	 $Q_2 = L_1 \cdot L_2$
 $Q_1 = L_1 \cdot L_2,\, Q_2 = L_3 \cdot L_4$	 $Q_2 = L_1 \cdot L_2$
 $Q_1 = L_1 \cdot L_2,\, Q_2 = L_3 \cdot L_4$	 $Q_2 = L_1 \cdot L_2$
 $Q_1 = L_1 \cdot L_2,\, Q_2 = L_3 \cdot L_4$	 $Q_2 = L_1 \cdot L_2$

2.7 Cubic Curves

A cubic curve in the projective plane is given by a third degree homogeneous equation

$$\begin{aligned} \mathcal{C}:\ F(X,Y,Z) &= a_1X^3 + a_2X^2Y + a_3XY^2 + a_4Y^3 + a_5X^2Z + a_6XYZ + a_7Y^2Z \\ &\quad + a_8XZ^2 + a_9YZ^2 + a_{10}Z^3 = 0 \qquad (2.7.1) \end{aligned}$$

Note that the equation (2.7.1) has 10 coefficients, since at least one of these coefficients is non-zero, it is enough to know 9 info about cubic to determine it explicitly. Unfortunately, projective transformations may not determine cubics uniquely as in the case quadrics, since $\dim \mathrm{PGL}(3,\mathbb{C}) = 8$.

In case of the quadrics the words "singular quadric" and "reducible quadric" are the same. But this is not true in general for cubics. A cubic is called an *irreducible* (resp. *reducible*) if $F(X,Y,Z)$ is an irreducible (resp. reducible) polynomial. In reducible case, it consists of either three lines (lines may not need to be distinct) or a quadric and a line. Since we are in projective space, every curve must meet at some points. So, as we have defined in Definition 2.2.10, these intersection points are the singular points of reducible cubic. Therefore one may consider that every reducible cubic is singular. But the converse is not true, e.g. the curve $X^3 - Y^2Z = 0$ is irreducible but have a singularity at $[0:0:1]$.

A *flex* of a curve $\mathcal{C}$ is a point p of $\mathcal{C}$ such that $\mathcal{C}$ is non singular at this point and tangent of $\mathcal{C}$ at p intersects with the curve at least 3 times. Flex points are the intersection points of $\mathcal{C}$ with its *Hessian curve*

$$\det \begin{bmatrix} F_{XX} & F_{XY} & F_{XZ} \\ F_{YX} & F_{YY} & F_{YZ} \\ F_{ZX} & F_{ZY} & F_{ZZ} \end{bmatrix} = 0. \tag{2.7.2}$$

Since the projective transformations preserves tangents and intersection multiplicities, then clearly preserves flexes.

Proposition 2.7.1. *Every irreducible cubic curve can be represented in Weierstraß form*

$$Y^2Z = 4X^3 - aXZ^2 - bZ^3. \tag{2.7.3}$$

Proof. Assume we have an irreducible cubic. Then it has a flex point and flex tangent. Let us consider a projective transformation moving this flex point to $[0 : 0 : 1]$ and tangent to the line $Y = 0$. Also, assume that the new equation of cubic is in the form (2.7.1). Clearly, $a_8 = a_{10} = 0$ and $a_9 \neq 0$. Since we assume cubic is irreducible then a_1 and a_5 are not both zero. In addition, since $Y = 0$ is the flex tangent with intersection multiplicity 3, then so $a_1 \neq 0$, $a_5 = 0$. Since at least one of the coefficients is non zero and we have already know $_9 \neq$, by rescaling the equation we can also assume $a_9 = 1$. If we apply the projective transformation $[X : Y : Z] \to [X : Z : Y]$, then cubic curve will reduce to the cubic curve $a_1X^3 + a_2X^2Z + a_3XZ^2 + a_4Z^3 + a_6XYZ + a_7YZ^2 + Y^2Z = 0$ with flex point $[0 : 1 : 0]$ and flex tangent $Z = 0$. By completing the square some terms, this equation can be written as $(Y + \frac{a_6}{2}X + \frac{a_7}{2}Z)^2Z + a_1X^3 + (a_2 - \frac{a_6^2}{4})X^2Z + (a_3 - \frac{a_6a_7}{2})XZ^2 + (a_4 - \frac{a_7^2}{4})Z^3 = 0$. Then by using the transformation

$$[X : Y : Z] \to \left[\left(-\frac{a_1}{4}\right)^{\frac{1}{3}} X : Y + \frac{a_6}{2}X + \frac{a_7}{2}Z : Z\right]$$

and renaming the coefficients we obtain $Y^2Z - 4X^3 + g_2X^2Z + g_1XZ^2 + g_0Z^3 = 0$. If one use the transformation $[X : Y : Z] \to [X + \frac{g_2}{2} : Y : Z]$ and rename the coefficients once again, then reaches the desired equation. $\square$

Corollary 2.7.2. *The cubic curve $Y^2Z = 4X^3 - aXZ^2 - bZ^3$ is non-singular if and only if $\Delta := a^3 - 27b^2 \neq 0$.*

Proof. Let $F := Y^2Z - 4X^3 + aXZ^2 + bZ^3$. Then the partial derivatives $F_X = -12X^2 + aZ^2$, $F_Y = 2YZ$ and $F_Z = Y^2 + 2aXZ + 3bZ^2$ all vanishes if and only if $a^3 - 27b^2 = 0$. $\square$

If a and b are both zero, the singular cubic is called *cuspidal cubic*. If $\Delta = 0$ but not both of a, b is zero then singular cubic is called *nodal cubic*.

Remark 2.7.3. Every nonsingular cubic curve in projective plane is also projectively equivalent to a nonsingular cubic defined by the $X^3 + Y^3 + Z^3 - 3\alpha XYZ =$, where $a^3 \neq 1$ and $a \neq \infty$.

In the literature, nonsingular irreducible cubic curves are also known as *elliptic curves*. The name “elliptic” comes from the Weierstraßelliptic $\wp$ function. Because, the real curve

$$y^2 = 4x^3 - ax - b\,, \quad \Delta = a^3 - 27b^2 \neq 0, \tag{2.7.4}$$

may be parametrized by $x = \wp(u)$, $y = \frac{d\wp}{du}(u)$, where $\wp(u)$ is the Weierstraßelliptic function defined by

$$u = \int_{\wp(u)}^{\infty} \frac{dx}{(4x^3 - ax - b)^{\frac{1}{2}}}.$$

The Weierstraßelliptic function $\wp(u)$ is not only defined on the real plane, it can also be defined over the complex plane $\mathbb{C}$. Let Λ be a lattice generated by 1 and a point τ of the upper half plane. Meromorphic functions on $T = \mathbb{C}/\Lambda$ correspond precisely to doubly periodic meromorphic functions on $\mathbb{C}$ with periods 1 and τ. The Weierstrass $\wp$-function on T explicitly defined as

$$\wp(u) := \frac{1}{u^2} + \sum_{\omega \in \Lambda \setminus \{0\}} \left(\frac{1}{(u-\omega)^2} - \frac{1}{u^2} \right). \tag{2.7.5}$$

This series converges uniformly on compact subsets of T. The derivative

$$\wp'(u) = -\sum_{\omega \in \Lambda} \frac{2}{(u-w)^3}$$

of $\wp(u)$ is also meromorphic function on T, and satisfies the equation

$$\wp'(u)^2 = 4\wp(u)^3 - a\wp(u) - b \tag{2.7.6}$$

with $a = 60 \sum_{\omega \in \Lambda \setminus \{0\}} \omega^{-4}$ and $b = 140 \sum_{\omega \in \Lambda \setminus \{0\}} \omega^{-6}$. So, the map

$$u \to [\wp(u) : \wp'(u) : 1] \tag{2.7.7}$$

is an embedding of the torus $T = \mathbb{C}/\Lambda$ into $\mathbb{CP}^2$. In homogeneous coordinates, the image is clearly the elliptic curve $Y^2Z - 4X^3 + aXZ^2 + bZ^3 = 0$. Because of this reason, topologically an elliptic curve is a torus, so their genus is $g = 1$, and Euler characteristic is $e = 0$.

Elliptic curves are not only geometric or topological objects, but also arithmetical objects. Choosing a fixed point O on an elliptic curve $\mathcal{C} \in \mathbb{CP}^2$, one can make the following construction: for any points $A, B \in \mathcal{C}$, let $A * B$ be the third point of intersection of $\mathcal{C}$ with the line $\overline{AB}$, then define an operation "$+$" over $\mathcal{C}$ so that $A + B := O * (A * B)$. Then, the he set of all points of $\mathcal{C}$ forms a group under the operation "$+$" with identity O, and inverse $-A = (O * O) * A$ for any given point A (Silverman & Tate, 1992, p. 18-22).

CHAPTER THREE
CONFIGURATION OF LINES

In this chapter, we will study the line arrangements, mainly the combinatorics of simplicial line arrangements. Simplicial arrangements are not only related with incidence problems, polytopes, graphs, and complexes but also important objects of Geometry and Topology. Since all faces are triangular, every member of the arrangement meets with other lines in a special position, possibly the configuration will be rigid. Rigid arrangements plays an important role for the algebraic surface geography. It is known that, if an algebraic surface associated to arrangement has $\mathbf{B}_2$ as universal cover, then underlying arrangement have to be rigid, i.e only the rigid arrangements may be uniformized by a complex ball. For this reason, in the light of the facts in (Grünbaum, 1967, 1971, 1972, 2009), we will first deal with the isomorphism types of line arrangements.

Secondly, we will introduce the Füredi & Palásti (1984)'s method to construct an arrangement of lines with maximum number of triangles. Then by using the group law of Elliptic curves we generalize their result and discuss the Orchard problem.

3.1 Isomorphism Type of Simplicial Line Arrangements

An arrangement of lines $\mathcal{A}$ is a finite collection of $n = n(\mathcal{A})$ lines $L_1, L_2, \cdots, L_n$. If there exists a point common to all lines L_i, then $\mathcal{A}$ is called *trivial*. Unless the opposite is explicitly stated we shall in the sequel assume that all arrangements we are dealing with are *non-trivial*, therefore also $n \geq 3$. An arrangement is called *simple* if no point belongs to more than two of the lines L_i, i.e., L_i's are in general position.

With a real arrangement $\mathcal{A}$ there is an associated 2-dimensional cell complex into which the lines of $\mathcal{A}$ decompose $\mathbb{RP}^2$. The vertices are the intersection points of two or more lines, the edges are the segments into which the lines are partitioned by the vertices and the faces are the connected components of the complement of the set of lines generating the arrangement. The number of vertices, edges and faces are denoted by $f_0 = f_0(\mathcal{A})$, $f_1 = f_1(\mathcal{A})$ and $f_2 = f_2(\mathcal{A})$, respectively. It is clear that $n \leq f_0 \leq \binom{n}{2}$, with equality on the left only if $n-1$ of the lines all pass through one point, and on the right only if the arrangement is simple.

If all faces are triangles, arrangement is called *simplicial*, and simplicial arrangements first introduced by Melchior (1942) and extensively appeared in (Grünbaum, 1971, 1972). It is not hard to see that simplicial arrangements satisfy the equality $2f_1 = 3f_2$ (Use the equalities (3.1.1), (3.1.2) and (3.1.3)).

Two arrangements are said to be *isomorphic* provided that the associated cell complexes are isomorphic; that is, if and only if there exist an incidence preserving one to one correspondence between the vertices, edges and faces of one arrangement and those of the other. The totality of all mutually isomorphic arrangements forms an *isomorphism type* of arrangements.

For limited number of lines, one can easily determine the isomorphism types of arrangements by drawing figures (see Figure 3.1). But, if the number of lines increases then the number of isomorphism types of an arrangement of n lines, which is bounded by 2^{an^2} for a positive constant a (Edelsbrunner, 1987, Theorem 1.4), groves rapidly. So, we will only deal with the special case, simplicial arrangements. To determine two arrangements are whether isomorphic, one may need to know some extra information about the number of lines, vertices, edges, faces, etc.

One of the simplest and best known such results is the *Euler's relation*; though it holds more generally for arbitrary cell decomposition of the projective plane, in the case of arrangements it becomes particularly elementary. As is established by induction, the numbers f_i $(i = 0, 1, 2)$ of vertices, edges, and faces of each arrangement $\mathcal{A}$ satisfy *Euler's relation*:

$$f_0 - f_1 + f_2 = e(\mathbb{RP}^2) = 1. \tag{3.1.1}$$

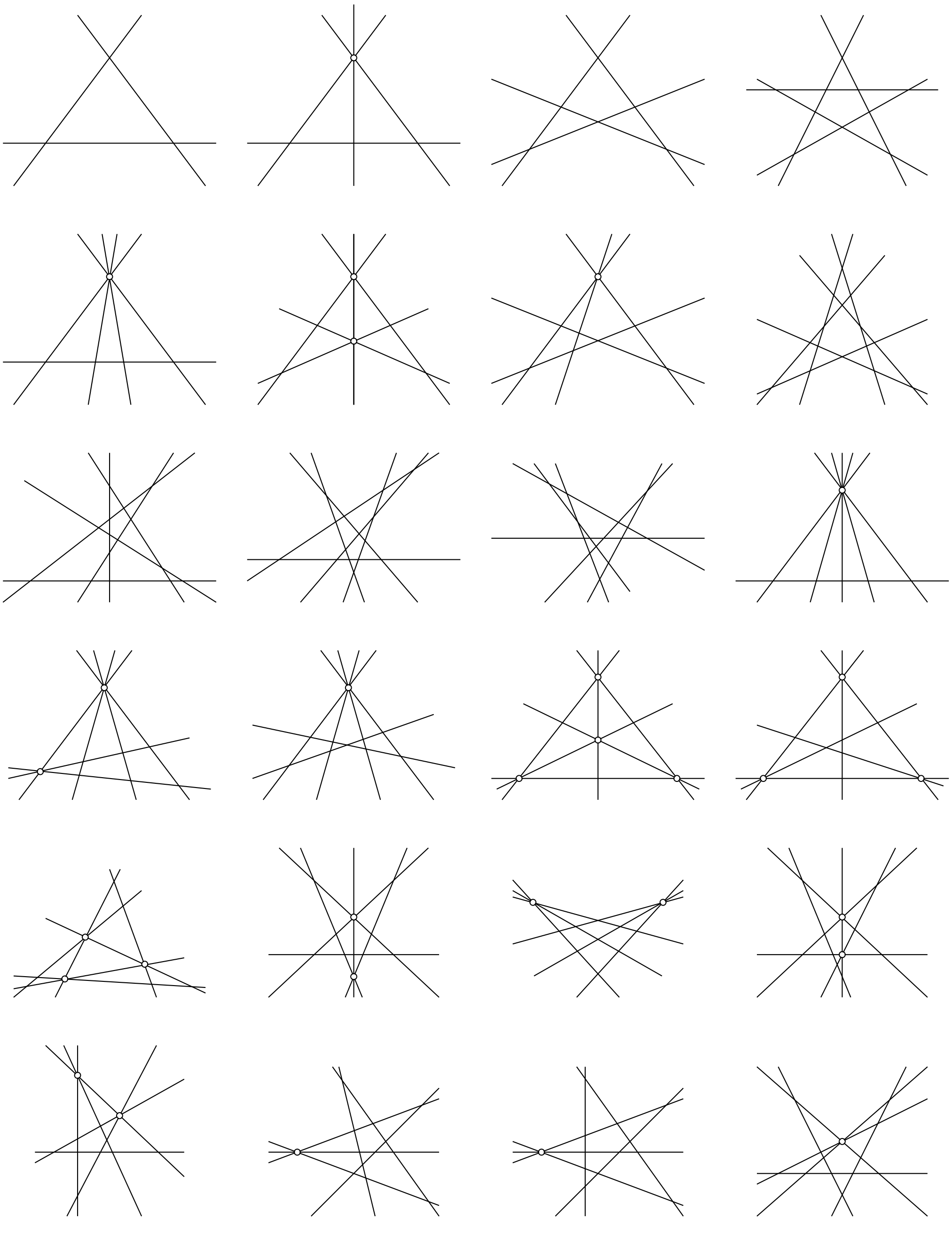


Figure 3.1 The different isomorphism types of non-trivial arrangements of 3, 4, 5 and 6 lines (Figure 2.1 Grünbaum, 1972, p. 5).

Let us denote the number of s-fold points of $\mathcal{A}$ by t_s $(s \geq 2)$, the number of lines each of which is incident with precisely $j \geq 2$ of the vertices of $\mathcal{A}$ by r_j and the number of k-gons among the cells of $\mathcal{A}$ by p_k. Then, one can easily discover the following equalities:

$$
\begin{aligned}
f_0 &= \sum_{s\geq 2} t_s, && (3.1.2)\\
f_1 &= \sum_{s\geq 2} st_s = \sum_{j\geq 2} jr_j = \frac{1}{2}\sum_{k\geq 3} kp_k, && (3.1.3)\\
f_2 &= 1 - f_0 + f_1 = 1 + \sum_{s\geq 2}(s-1)t_s, && (3.1.4)\\
\binom{n}{2} &= \sum_{s\geq 2}\binom{s}{2} t_s, && (3.1.5)\\
n &= \sum_{j\geq 2} r_j, && (3.1.6)
\end{aligned}
$$

Melchior (1942) has showed that if arrangement $\mathcal{A}$ has at least three non collinear points, then

$$t_2 \geq 3 + t_4 + 2t_5 + 3t_6 + \cdots \qquad (3.1.7)$$

This inequality shows that $2f_1 - 3f_2 \geq 0$. Then by using Euler's relation (3.1.1), one can easily obtain the linear inequality

$$1 + f_0 \leq f_2 \leq 2f_0 - 2. \qquad (3.1.8)$$

Indeed, the inequalities (3.1.8) determine the convex hull of the set of pairs (f_0, f_2) for all arrangements $\mathcal{A}$. The equality on the left holds in (3.1.8) if and only if $\mathcal{A}$ is a simple arrangement, while equality on the right is characteristic for simplicial arrangements (Grünbaum, 1967, pp.401–402). In addition, one gets the following inequality:

$$2n - 2 \leq f_2 \leq 1 + \binom{n}{2} \qquad (3.1.9)$$

Indeed, the upper bound follows from the observation that the number of faces does not decrease if the lines of an arrangement are subjected to sufficiently small perturbations which change the given arrangement into a simple one. For simple arrangements (and only for such arrangements) $f_2 = 1 + \binom{n}{2}$. The lower bound $f_2 \geq 2n-2$ is also established using induction on n. The equality at right holds in (3.1.9) if only if $\mathcal{A}$ is a simple arrangement; and equality on the left holds if and only if $\mathcal{A}$ is near pencil. Unfortunately, there is no hope of completely characterizing the sets of pairs (f_0, f_2) and (n, f_2). However, Grünbaum (1971, 1972) has some partial results. For example, $f_2 \geq 3n-6$ if $\mathcal{A}$ is not a near pencil and $n \geq 6$. It is also known that $t_2(n) \geq \frac{3}{7}n$ and $t_3(n) \geq \frac{(n-1)^2+4}{8}$ for all n.

Three infinite families $\mathcal{R}(0)$, $\mathcal{R}(1)$ and $\mathcal{R}(2)$ of isomorphism classes of are known.

Family $\mathcal{R}(0)$ consists of all *near pencils*. A near pencil denoted by $\mathcal{A}(n,0)$, $n \geq 3$, consists of $n-1$ lines that have a point in common, the last line does not belong to a pencil. The isomorphism invariants of this family is $(f_0, f_1, f_2) = (n, 3n-3, 2n-2)$, $(t_2, t_3, \cdots, t_{n-1}) = (n-1, 0^{n-4}, 1)$ and $(r_2, r_3, \cdots, r_{n-1}) = (n-1, 0^{n-4}, 1)$, where $0^{n-4} := \underbrace{0, \cdots, 0}_{n-4 \text{ times}}$.

Family $\mathcal{R}(1)$ consists of simplicial arrangements $\mathcal{A}(2n, 1)$, which consists of the sides of regular convex n-gon, $n \geq 3$, and its n symmetry axes.

Family $\mathcal{R}(2)$ consists of simplicial arrangements $\mathcal{A}(4n+1, 1)$, which is obtained from $\mathcal{A}(4n, 1)$ in the family $\mathcal{R}(1)$ by adjoining the line at infinity.

Beside this three infinite families of simplicial arrangements only 91 other types were known (Grünbaum, 1971). But, as it is reported in (Hirzebruch, 1983) and (Barthel et al., 1987, p. 64), the arrangements $\mathcal{A}_2(17)$ and $\mathcal{A}_7(17)$ are isomorphic. In addition, the arrangement $\mathcal{A}(16, 7)$ discovered later by (Grünbaum, 1972, p. 7). Recently, Grünbaum (2009) have been updated his catalogue. By cheating from Grünbaum's recent paper, we will give this catalogue in Table 3.1, and illustrate some figures.

In this table, we denote the sequence $\overbrace{a,a,\cdots,a}^{b \text{ times}}$ by a^b, the non maximal sporadic simplicial arrangements by M and the pseudo-minimal sporadic simplicial arrangements by m, that is, arrangements that do not contain as sub-arrangement any sporadic arrangements. This table contains simplicial arrangements up to 37 lines, because of the following conjecture:

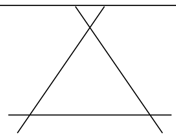
Conjecture 3.1.1. *(Grünbaum, 1972, Conjecture 2.1) For $n \geq 38$, the number of isomorphism types of simplicial arrangement of n lines is*

$$c^{\triangle}(n) = \begin{cases} 2 & \textit{if } n \equiv 0,1,2 \pmod 4 \\ 1 & \textit{if } n \equiv 3 \pmod 4. \end{cases} \tag{3.1.10}$$

This conjecture is still open. If one proves it, then he will prove the conjecture that the Table 3.1 in page33 is the complete enumeration of isomorphism classes of sporadic arrangements with $n \leq 37$; and for $n \geq 38$ they are either $\mathcal{R}(0)$, or $\mathcal{R}(1)$, or $\mathcal{R}(2)$.

In addition, the Figure 3.2 in page 63 is the Hasse diagram of the simplicial arrangements in Table 3.1. In the diagram, the maximal arrangements are indicated by bold framed numerals. The numerals with shaded backgrounds indicate pseudo minimal sporadic simplicial arrangements. Note that, non of the arrangements in the families $\mathcal{R}(0)$, $\mathcal{R}(1)$ and $\mathcal{R}(2)$ is maximal, while the diagram shows there are only ten sporadic ones.

Table 3.1 Isomorphism types of simplicial arrangements in $\mathbb{RP}^2$.

$\mathcal{A}(n,k)$	**f** **t** **r**	**Figures**	**Notes**
$\mathcal{A}(3,0)$	$\mathbf{f} = (3,6,4)$ $\mathbf{t} = (3)$ $\mathbf{r} = (3)$		$\mathcal{R}(0)$

Continued on next page

Table 3.1 Isomorphism types of simplicial arrangements in $\mathbb{RP}^2$. *– continued from previous page*

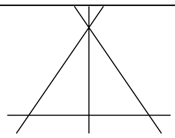
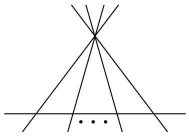
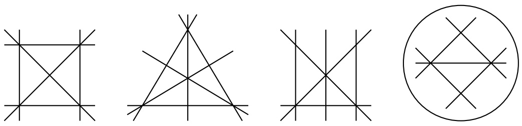
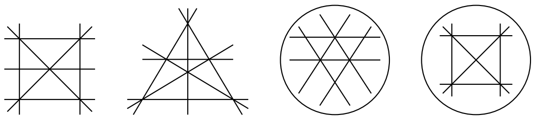
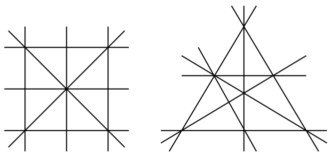
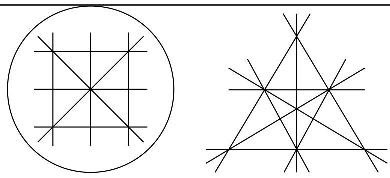
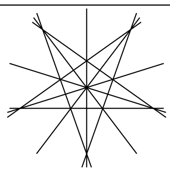
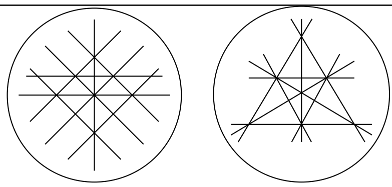
$\mathcal{A}(n,k)$	**f** **t** **r**	Figures	Notes
$\mathcal{A}(4,0)$	$\mathbf{f}=(4,9,6)$ $\mathbf{t}=(3,1)$ $\mathbf{r}=(3,1)$		$\mathcal{R}(0)$
$\mathcal{A}(n,0),\ n>4$	$\mathbf{f}=(n,3n-3,2n-2)$ $\mathbf{t}=(n-1,0^4n-4,1)$ $\mathbf{r}=(n-1,0^{n-4},1)$		$\mathcal{R}(0)$
$\mathcal{A}(6,1)$	$\mathbf{f}=(7,18,12)$ $\mathbf{t}=(3,4)$ $\mathbf{r}=(0,6)$		$\mathcal{R}(1)$
$\mathcal{A}(7,1)$	$\mathbf{f}=(9,24,16)$ $\mathbf{t}=(3,6)$ $\mathbf{r}=(0,4,3)$		m
$\mathcal{A}(8,1)$	$\mathbf{f}=(11,30,20)$ $\mathbf{t}=(4,6,1)$ $\mathbf{r}=(0,2,6)$		$\mathcal{R}(1)$
$\mathcal{A}(9,1)$	$\mathbf{f}=(13,36,24)$ $\mathbf{t}=(6,4,3)$ $\mathbf{r}=(0^2,9)$		$\mathcal{R}(2)$
$\mathcal{A}(10,1)$	$\mathbf{f}=(16,45,30)$ $\mathbf{t}=(5,10,0,1)$ $\mathbf{r}=(0^2,5^2)$		$\mathcal{R}(1)$
$\mathcal{A}(10,2)$	$\mathbf{f}=(16,45,30)$ $\mathbf{t}=(6,7,3)$ $\mathbf{r}=(0^2,6,3,1)$		

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Table 3.1 Isomorphism types of simplicial arrangements in $\mathbb{RP}^2$. *– continued from previous page*

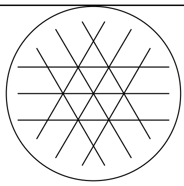
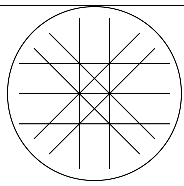
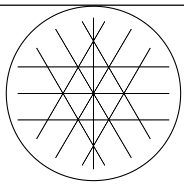
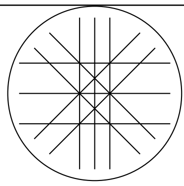
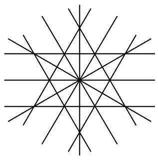
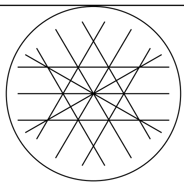
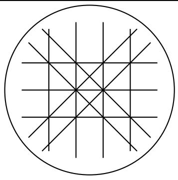
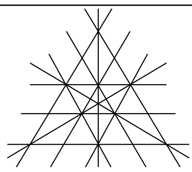
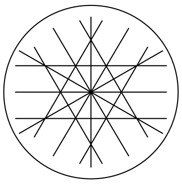
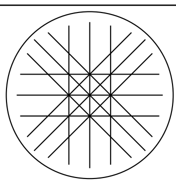
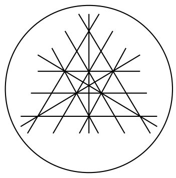
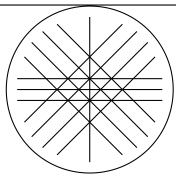
$\mathcal{A}(n,k)$	**f** **t** **r**	Figures	Notes
$\mathcal{A}(10,3)$	$\mathbf{f}=(16,35,30)$ $\mathbf{t}=(6,7,3)$ $\mathbf{r}=(0,1,3,6)$	 	
$\mathcal{A}(11,1)$	$\mathbf{f}=(19,54,36)$ $\mathbf{t}=(7,8,4)$ $\mathbf{r}=(0^2,4^2,3)$	 	
$\mathcal{A}(12,1)$	$\mathbf{f}=(22,63,42)$ $\mathbf{t}=(6,15,0^2,1)$ $\mathbf{r}=(0^2,3^2,6)$		$\mathcal{R}(1)$
$\mathcal{A}(12,2)$	$\mathbf{f}=(22,63,42)$ $\mathbf{t}=(8,10,3,1)$ $\mathbf{r}=(0^2,3^2,6)$	 	
$\mathcal{A}(12,3)$	$\mathbf{f}=(22,63,42)$ $\mathbf{t}=(9,7,6)$ $\mathbf{r}=(0^2,3^2,6)$		$\mathcal{R}(1)$
$\mathcal{A}(13,1)$	$\mathbf{f}=(25,72,48)$ $\mathbf{t}=(9,12,3,0,1)$ $\mathbf{r}=(0^2,3,0,10)$		$\mathcal{R}(2)$
$\mathcal{A}(13,2)$	$\mathbf{f}=(25,72,48)$ $\mathbf{t}=(12,4,9)$ $\mathbf{r}=(0^2,3,0,10)$	 	
$\mathcal{A}(13,3)$	$\mathbf{f}=(25,72,48)$ $\mathbf{t}=(10^2,3,2)$ $\mathbf{r}=(0^2,1,4,8)$		

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Table 3.1 Isomorphism types of simplicial arrangements in $\mathbb{RP}^2$. *– continued from previous page*

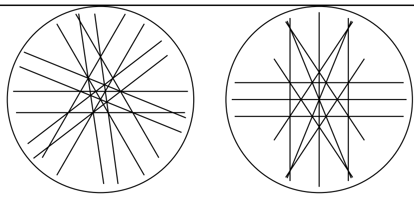
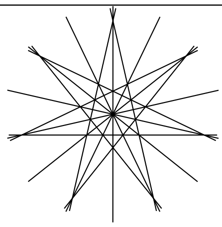
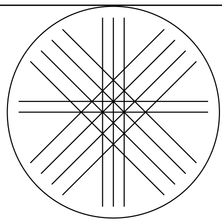
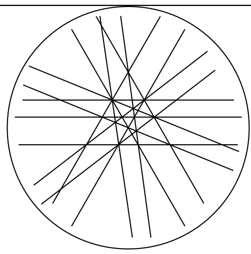
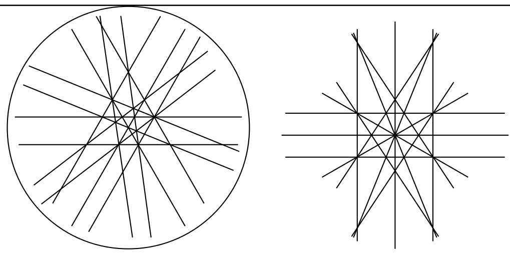
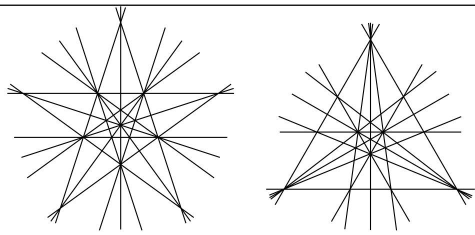
$\mathcal{A}(n,k)$	**f** **t** **r**	Figures	Notes
$\mathcal{A}(13,4)$	$\mathbf{f}=(27,78,52)$ $\mathbf{t}=(6,18,3)$ $\mathbf{r}=(0^4,13)$		*m*
$\mathcal{A}(14,1)$	$\mathbf{f}=(29,84,56)$ $\mathbf{t}=(7,21,0^3,1)$ $\mathbf{r}=(0^3,7,0,7)$		$\mathcal{R}(1)$
$\mathcal{A}(14,2)$	$\mathbf{f}=(29,84,56)$ $\mathbf{t}=(11,12,4,2)$ $\mathbf{r}=(0^2,1,4^3,1)$		
$\mathcal{A}(14,3)$	$\mathbf{f}=(30,87,58)$ $\mathbf{t}=(9,16,4,1)$ $\mathbf{r}=(0^4,11,3)$		
$\mathcal{A}(14,4)$	$\mathbf{f}=(29,84,56)$ $\mathbf{t}=(10,14,4,0,1)$ $\mathbf{r}=(0^3,4,6,4)$		*m*
$\mathcal{A}(15,1)$	$\mathbf{f}=(31,90,60)$ $\mathbf{t}=(15,10,0,6)$ $\mathbf{r}=(0^4,15)$		*m*

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Table 3.1 Isomorphism types of simplicial arrangements in $\mathbb{RP}^2$. *– continued from previous page*

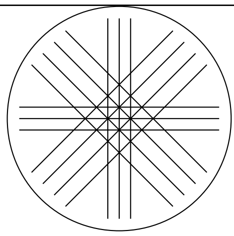
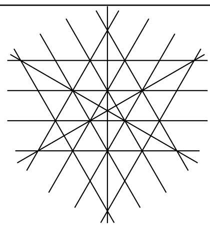
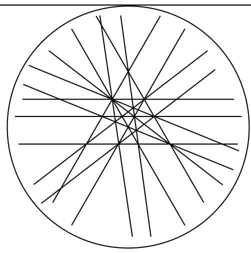
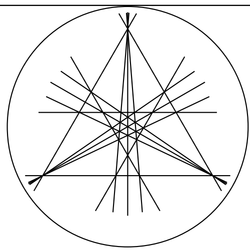
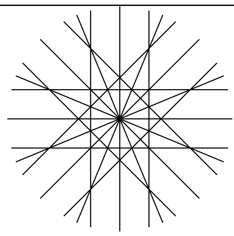
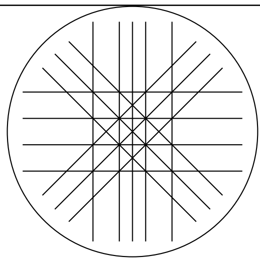
$\mathcal{A}(n,k)$	**f** **t** **r**	Figures	Notes
$\mathcal{A}(15,2)$	$\mathbf{f} = (33,96,64)$ $\mathbf{t} = (13,12,6,2)$ $\mathbf{r} = (0^2,1,4,2,4^2)$		
$\mathcal{A}(15,3)$	$\mathbf{f} = (34,99,66)$ $\mathbf{t} = (12,13,9)$ $\mathbf{r} = (0^4,9,3^2)$		
$\mathcal{A}(15,4)$	$\mathbf{f} = (33,96,64)$ $\mathbf{t} = (12,14,6,0,1)$ $\mathbf{r} = (0^4,10,4,1)$		
$\mathcal{A}(15,5)$	$\mathbf{f} = (34,99,66)$ $\mathbf{t} = (9,22,0,3)$ $\mathbf{r} = (0^4,9,3^2)$		*m*
$\mathcal{A}(16,1)$	$\mathbf{f} = (37,108,72)$ $\mathbf{t} = (8,28,0^4,1)$ $\mathbf{r} = (0^3,4^2,0,8)$		$\mathcal{R}(1)$
$\mathcal{A}(16,2)$	$\mathbf{f} = (37,108,72)$ $\mathbf{t} = (14,15,6,1^2)$ $\mathbf{r} = (0^2,1,2,4,2,7)$		

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Table 3.1 Isomorphism types of simplicial arrangements in $\mathbb{RP}^2$. – *continued from previous page*

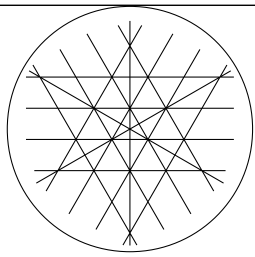
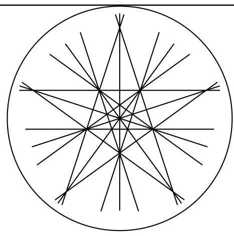
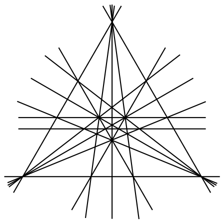
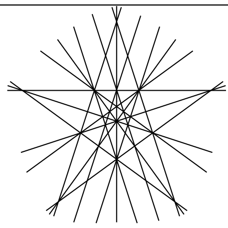
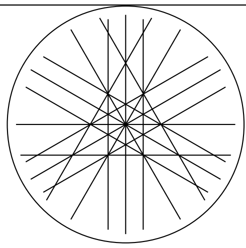
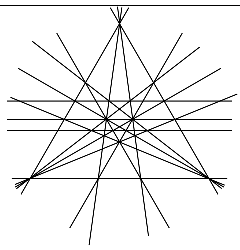
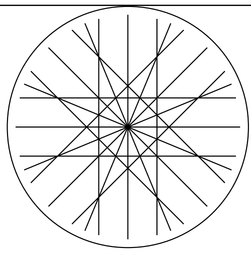
$\mathcal{A}(n,k)$	**f** **t** **r**	Figures	Notes
$\mathcal{A}(16,3)$	$\mathbf{f}=(37,108,72)$ $\mathbf{t}=(15,13,6,3)$ $\mathbf{r}=(0^4,10,0,6)$		
$\mathcal{A}(16,4)$	$\mathbf{f}=(36,105,70)$ $\mathbf{t}=(15^2,0,6)$ $\mathbf{r}=(0^4,10,5,0^2,1)$	 	
$\mathcal{A}(16,5)$	$\mathbf{f}=(37,108,72)$ $\mathbf{t}=(14,16,3,4)$ $\mathbf{r}=(0^3,2,4,8,0,2)$		*m*
$\mathcal{A}(16,6)$	$\mathbf{f}=(37,108,72)$ $\mathbf{t}=(15,12,9,0,1)$ $\mathbf{r}=(0^4,7,6,3)$		
$\mathcal{A}(16,7)$	$\mathbf{f}=(38,111,74)$ $\mathbf{t}=(12,19,6,0,1)$ $\mathbf{r}=(0^3,3^2,2,8)$		*m*
$\mathcal{A}(17,1)$	$\mathbf{f}=(41,120,80)$ $\mathbf{t}=(12,24,4,0^3,1)$ $\mathbf{r}=(0^4,8,0,9)$		$\mathcal{R}(2)$

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Table 3.1 Isomorphism types of simplicial arrangements in $\mathbb{RP}^2$. *– continued from previous page*

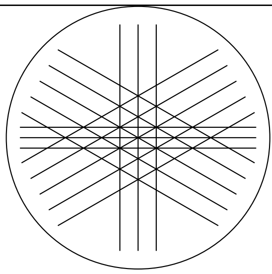
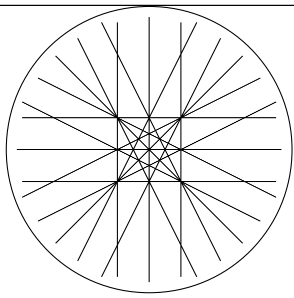
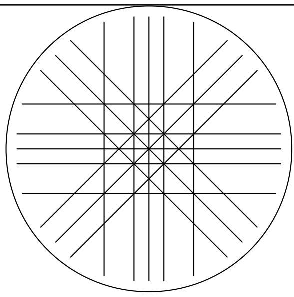
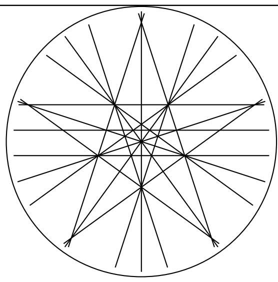
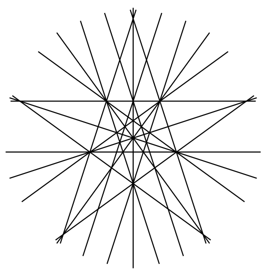
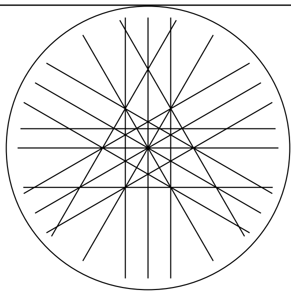
$\mathcal{A}(n,k)$	**f** **t** **r**	Figures	Notes
$\mathcal{A}(17,2)$	$\mathbf{f}=(41,120,80)$ $\mathbf{t}=(16,16,7,0,2)$ $\mathbf{r}=(0^2,1,0,6,0,10)$		
$\mathcal{A}(17,3)$	$\mathbf{f}=(41,120,80)$ $\mathbf{t}=(18,12,7,4)$ $\mathbf{r}=(0^4,8,0,9)$		
$\mathcal{A}(17,4)$	$\mathbf{f}=(41,120,80)$ $\mathbf{t}=(16^2\ 7,0,2)$ $\mathbf{r}=(16^2,7,0,2)$		$\mathcal{A}(17,4)$ has two lines with four quadruple points on each, while $\mathcal{A}(17,2)$ has no such line.
$\mathcal{A}(17,5)$	$\mathbf{f}=(41,120,80)$ $\mathbf{t}=(16,18,1,6)$ $\mathbf{r}=(0^4,6,8,1,0,2)$	 	
$\mathcal{A}(17,6)$	$\mathbf{f}=(42,123,82)$ $\mathbf{t}=(16,15,10,0,1)$ $\mathbf{r}=(0^4,6,3,7,0,1)$		

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Table 3.1 Isomorphism types of simplicial arrangements in $\mathbb{RP}^2$. *– continued from previous page*

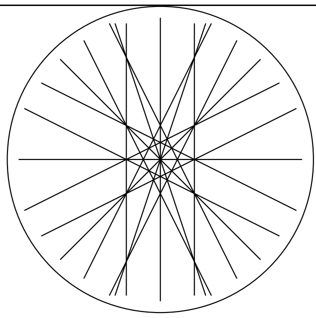
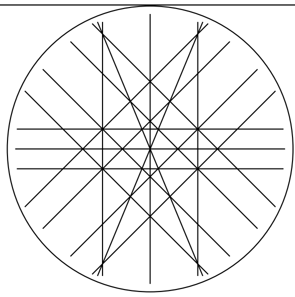
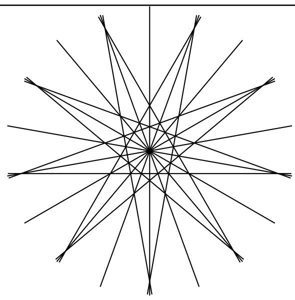
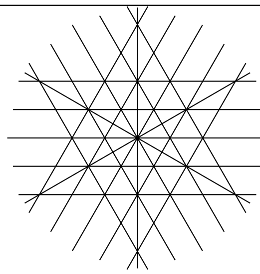
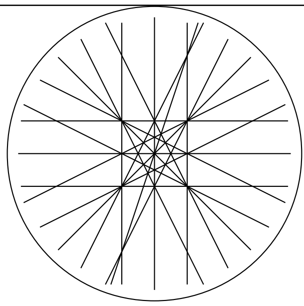
$\mathcal{A}(n,k)$	**f** **t** **r**	Figures	Notes
$\mathcal{A}(17,7)$	$\mathbf{f}=(43,126,84)$ $\mathbf{t}=(13,22,7,0,1)$ $\mathbf{r}=(0^4,6,0,10,0,1)$		
$\mathcal{A}(17,8)$	$\mathbf{f}=(43,126,84)$ $\mathbf{t}=(14,20,7,2)$ $\mathbf{r}=(0^4,1,8^2)$		*m*
$\mathcal{A}(18,1)$	$\mathbf{f}=(46,135,90)$ $\mathbf{t}=(9,36,0^5,1)$ $\mathbf{r}=(0^4,9,0^2,9)$		$\mathcal{R}(1)$
$\mathcal{A}(18,2)$	$\mathbf{f}=(46,135,90)$ $\mathbf{t}=(18^2,6,3,1)$ $\mathbf{r}=(0^4,3^2,12)$		*m*
$\mathcal{A}(18,3)$	$\mathbf{f}=(46,135,90)$ $\mathbf{t}=(19,16,6,5)$ $\mathbf{r}=(0^4,6,2,6,3,1)$		

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Table 3.1 Isomorphism types of simplicial arrangements in $\mathbb{RP}^2$. *– continued from previous page*

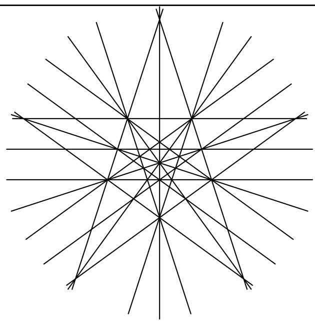
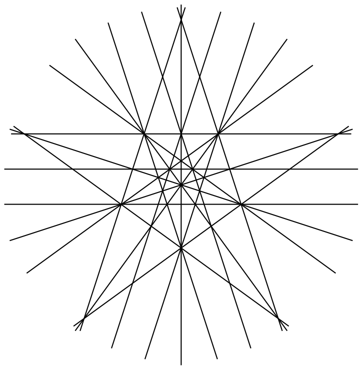
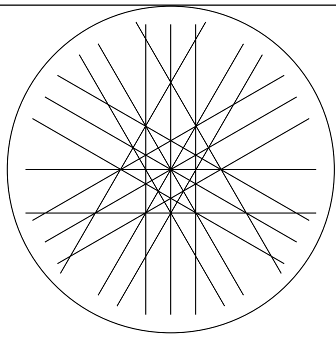
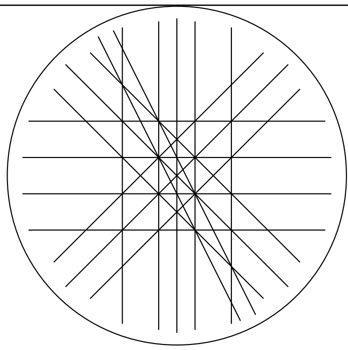
$\mathcal{A}(n,k)$	**f** **t** **r**	Figures	Notes
$\mathcal{A}(18,4)$	$\mathbf{f}=(46,135,90)$ $\mathbf{t}=(18,19,3,6)$ $\mathbf{r}=(0^4,3,9,3,0,3)$		
$\mathcal{A}(18,5)$	$\mathbf{f}=(46,135,90)$ $\mathbf{t}=(18,19,3,6)$ $\mathbf{r}=(0^4,3,9,3,0,3)$		Each of $\mathcal{A}(18,4)$ and $\mathcal{A}(18,5)$ contains three quadruple points that determine three lines. These lines determine 4 triangles. In $\mathcal{A}(18,4)$ there is a triangle that contains three of the quintuple points, while no such triangle exists in $\mathcal{A}(18,5)$.
$\mathcal{A}(18,6)$	$\mathbf{f}=(47,138,92)$ $\mathbf{t}=(18,16,12,0,1)$ $\mathbf{r}=(0^4,5,2,7,2^2)$		
$\mathcal{A}(18,7)$	$\mathbf{f}=(46,135,90)$ $\mathbf{t}=(18^2,6,3,1)$ $\mathbf{r}=(0^4,3,3,0,6^2)$		

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Table 3.1 Isomorphism types of simplicial arrangements in $\mathbb{RP}^2$. *– continued from previous page*

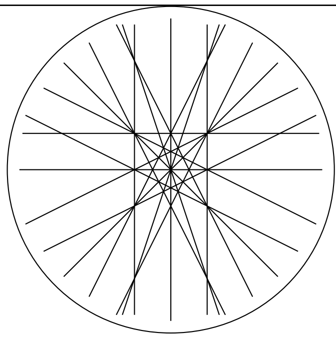
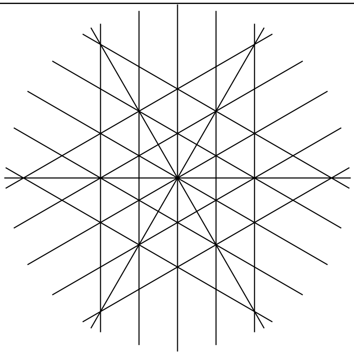
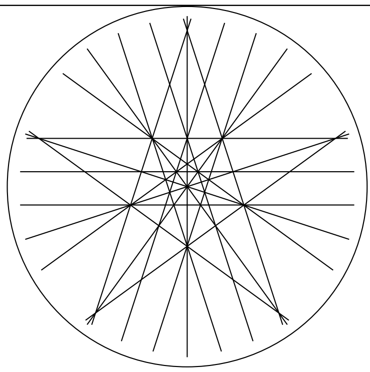
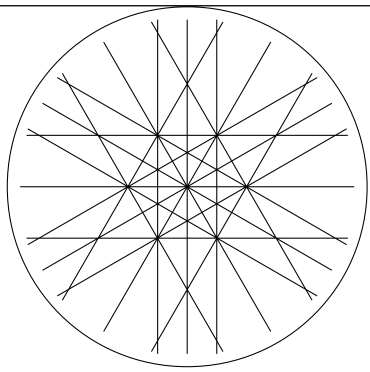
$\mathcal{A}(n,k)$	**f** **t** **r**	Figures	Notes
$\mathcal{A}(18,8)$	$\mathbf{f}=(47,138,92)$ $\mathbf{t}=(16,22,6,2,1)$ $\mathbf{r}=(0^4,6,0,7,4,1)$		
$\mathcal{A}(19,1)$	$\mathbf{f}=(49,144,96)$ $\mathbf{t}=(21,18,6,0,4)$ $\mathbf{r}=(0^4,4,0,15)$		
$\mathcal{A}(19,2)$	$\mathbf{f}=(51,150,100)$ $\mathbf{t}=(21,18,6^2)$ $\mathbf{r}=(0^4,1,8,6,0,4)$		
$\mathcal{A}(19,3)$	$\mathbf{f}=(49,144,96)$ $\mathbf{t}=(24,12,6^2,1)$ $\mathbf{r}=(0^4,4,0,15)$		

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Table 3.1 Isomorphism types of simplicial arrangements in $\mathbb{RP}^2$. *– continued from previous page*

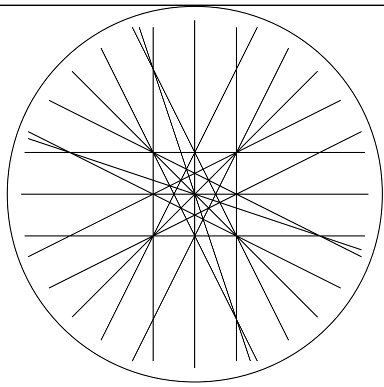
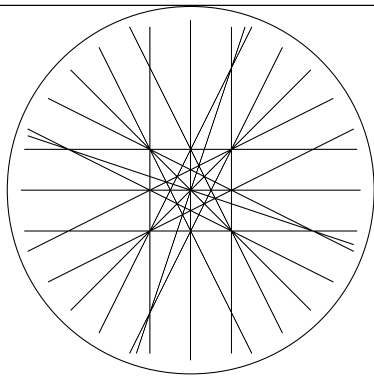
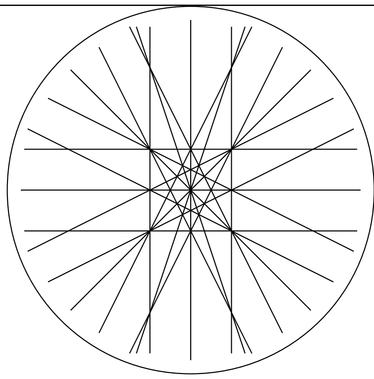
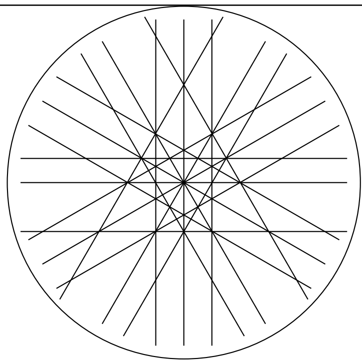
$\mathcal{A}(n,k)$	**f** **t** **r**	Figures	Notes
$\mathcal{A}(19,4)$	$\mathbf{f} = (51,150,100)$ $\mathbf{t} = (20^2,6,4,1)$ $\mathbf{r} = (0^4,4^4,3)$		
$\mathcal{A}(19,5)$	$\mathbf{f} = (51,150,100)$ $\mathbf{t} = (20^2,6,4,1)$ $\mathbf{r} = (0^4,4^4,3)$		$\mathcal{A}(19,4)$ and $\mathcal{A}(19,5)$ differ by the order of the points at infinity of different multiplicities.
$\mathcal{A}(19,6)$	$\mathbf{f} = (51,150,100)$ $\mathbf{t} = (20^2,6,4,1)$ $\mathbf{r} = (0^4,6,0,6,4,3)$		
$\mathcal{A}(19,7)$	$\mathbf{f} = (52,153,102)$ $\mathbf{t} = (21,15^2,0,1)$ $\mathbf{r} = (0^4,4,3^2,6,3)$		

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Table 3.1 Isomorphism types of simplicial arrangements in $\mathbb{RP}^2$. *– continued from previous page*

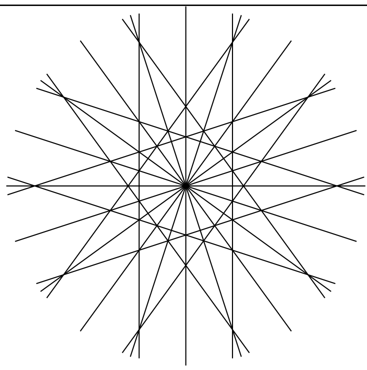
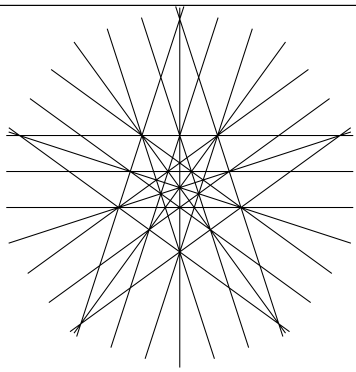
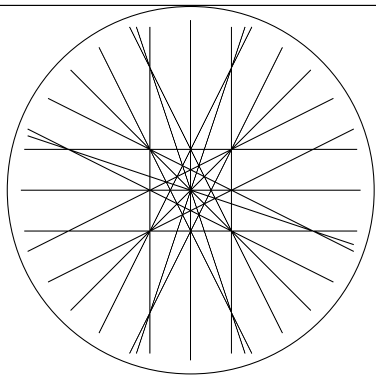
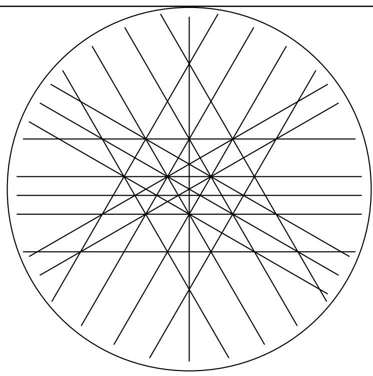
$\mathcal{A}(n,k)$	**f** **t** **r**	Figures	Notes
$\mathcal{A}(20,1)$	$\mathbf{f}=(56,165,110)$ $\mathbf{t}=(10,45,0^6,1)$ $\mathbf{r}=(0^4,5^2,0^2,10)$		$\mathcal{R}(1)$
$\mathcal{A}(20,2)$	$\mathbf{f}=(56,165,110)$ $\mathbf{t}=(25,15,10,6)$ $\mathbf{r}=(0^5,5,10,0,5)$		
$\mathcal{A}(20,3)$	$\mathbf{f}=(56,165,110)$ $\mathbf{t}=(21,24,6,4,0,1)$ $\mathbf{r}=(0^4,4,2,4,6,3,1)$		
$\mathcal{A}(20,4)$	$\mathbf{f}=(56,165,110)$ $\mathbf{t}=(23,20,7,5,1)$ $\mathbf{r}=(0^4,5,1,4^2,6)$		

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Table 3.1 Isomorphism types of simplicial arrangements in $\mathbb{RP}^2$. *– continued from previous page*

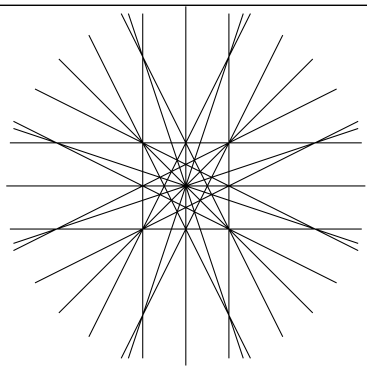
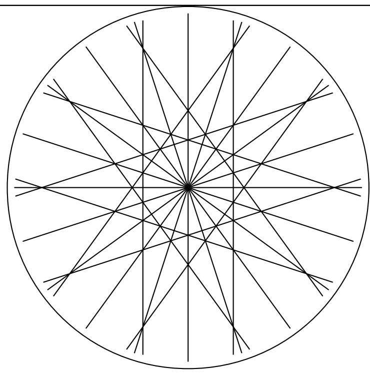
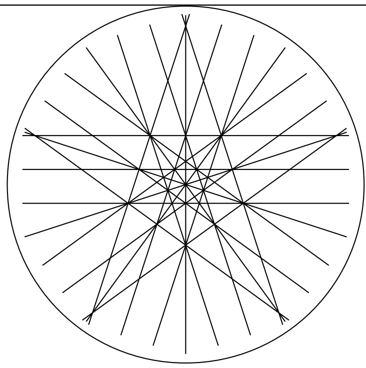
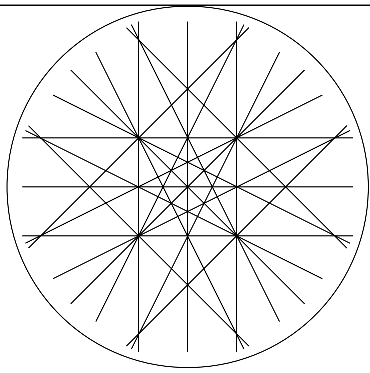
$\mathcal{A}(n,k)$	**f** **t** **r**	Figures	Notes
$\mathcal{A}(20,5)$	$\mathbf{f} = (55,162,108)$ $\mathbf{t} = (20,26,4^2,0^2,1)$ $\mathbf{r} = (0^3,2^2,0,4,12)$		
$\mathcal{A}(21,1)$	$\mathbf{f} = (61,180,120)$ $\mathbf{t} = (15,40,5,0^5,1)$ $\mathbf{r} = (0^3,5,0,5,0,11)$		$\mathcal{R}(2)$
$\mathcal{A}(21,2)$	$\mathbf{f} = (61,180,120)$ $\mathbf{t} = (30,10,15,6)$ $\mathbf{r} = (0^6,15,0,6)$		
$\mathcal{A}(21,3)$	$\mathbf{f} = (61,180,120)$ $\mathbf{t} = (24^2,9,0,4)$ $\mathbf{r} = (0^4,6,0,3,0,12)$		

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Table 3.1 Isomorphism types of simplicial arrangements in $\mathbb{RP}^2$. *– continued from previous page*

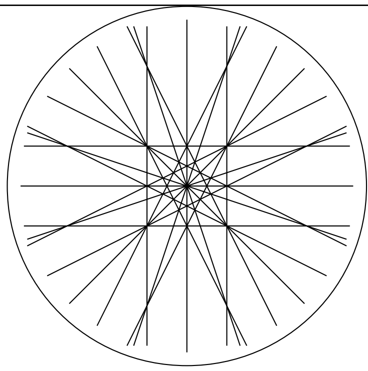
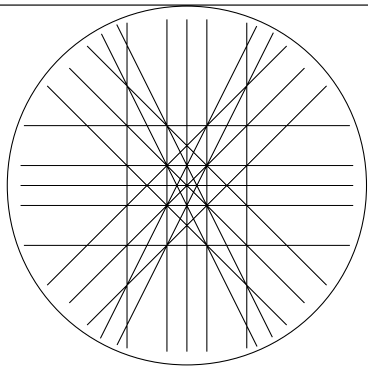
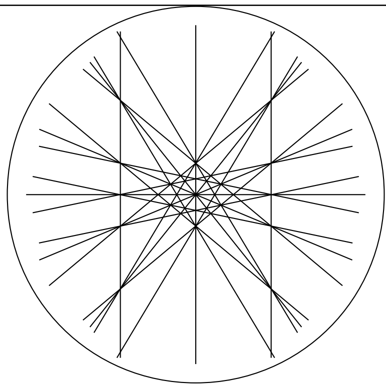
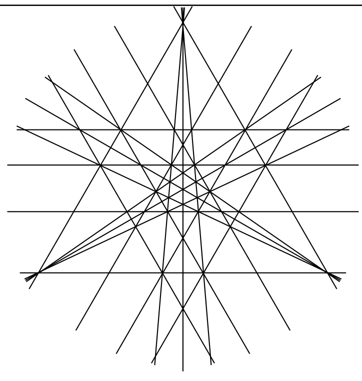
$\mathcal{A}(n,k)$	**f** **t** **r**	Figures	Notes
$\mathcal{A}(21,4)$	$\mathbf{f} = (61,180,120)$ $\mathbf{t} = (22,28,6,4,0^2,1)$ $\mathbf{r} = (0^4,4,0,4,8,4,0,1)$		M
$\mathcal{A}(21,5)$	$\mathbf{f} = (61,180,120)$ $\mathbf{t} = (26,20,9,4,2)$ $\mathbf{r} = (0^4,5,0,3,4,9)$		
$\mathcal{A}(21,6)$	$\mathbf{f} = (63,186,124)$ $\mathbf{t} = (25,20,15,2,1)$ $\mathbf{r} = (0^4,1,0,11,0,8,0,1)$		M
$\mathcal{A}(21,7)$	$\mathbf{f} = (64,189,126)$ $\mathbf{t} = (24,22,15,3)$ $\mathbf{r} = (0^6,12,0,6,3)$		M

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Table 3.1 Isomorphism types of simplicial arrangements in $\mathbb{RP}^2$. *– continued from previous page*

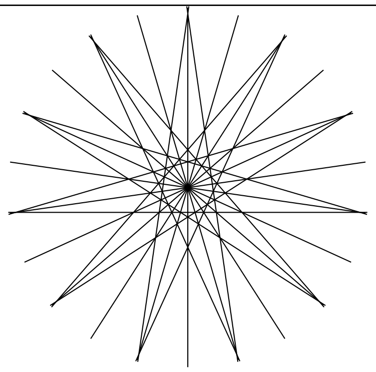
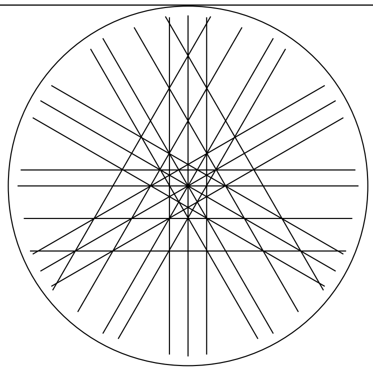
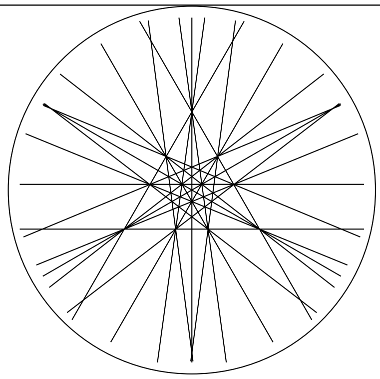
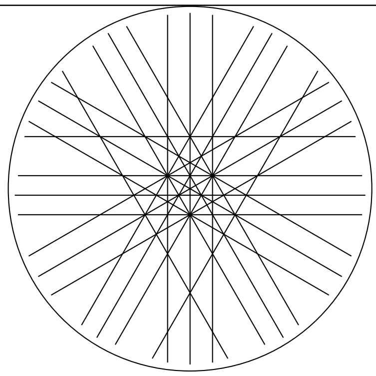
$\mathcal{A}(n,k)$	**f** **t** **r**	Figures	Notes
$\mathcal{A}(22,1)$	$\mathbf{f} = (67,198,132)$ $\mathbf{t} = (11,55,0^7,1)$ $\mathbf{r} = (0^5,11,0^3,11)$		$\mathcal{R}(1)$
$\mathcal{A}(22,2)$	$\mathbf{f} = (70,207,138)$ $\mathbf{t} = (24,30,12,3,1)$ $\mathbf{r} = (0^4,1,0,6,3,9,0,3)$		
$\mathcal{A}(22,3)$	$\mathbf{f} = (67,198,132)$ $\mathbf{t} = (27,28,0,12)$ $\mathbf{r} = (0^6,12,0,9,0,1)$		
$\mathcal{A}(22,4)$	$\mathbf{f} = (67,198,132)$ $\mathbf{t} = (27,25,9,3^2)$ $\mathbf{r} = (0^4,4,0,6,0,6^2)$		

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Table 3.1 Isomorphism types of simplicial arrangements in $\mathbb{RP}^2$. *– continued from previous page*

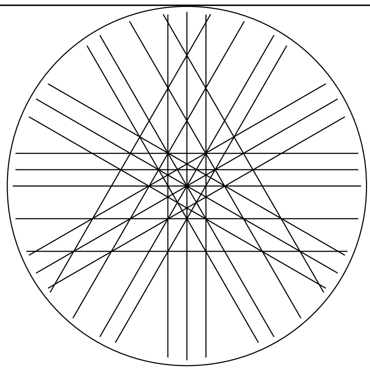
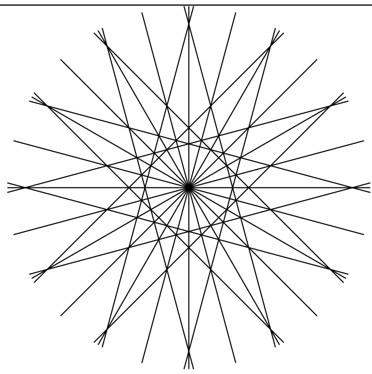
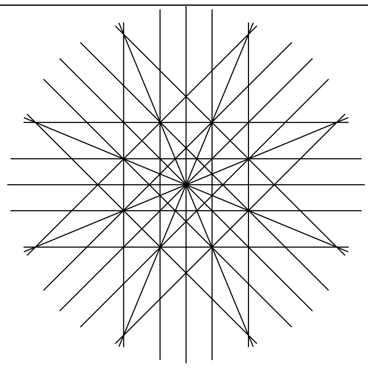
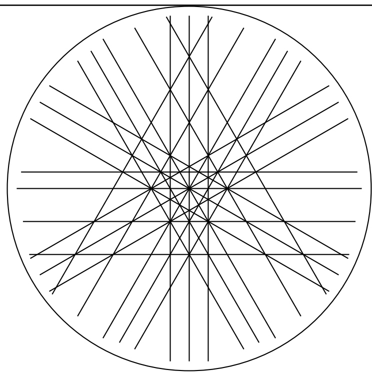
$\mathcal{A}(n,k)$	**f** **t** **r**	Figures	Notes
$\mathcal{A}(23,1)$	$\mathbf{f}=(75,222,148)$ $\mathbf{t}=(27,32,10,4,2)$ $\mathbf{r}=(0^4,1,0,6,2,7,4,3)$		
$\mathcal{A}(24,1)$	$\mathbf{f}=(79,234,156)$ $\mathbf{t}=(12,66,0^8,1)$ $\mathbf{r}=(0^5,6^2,0^3,12)$		$\mathcal{R}(1)$
$\mathcal{A}(24,2)$	$\mathbf{f}=(77,228,152)$ $\mathbf{t}=(32^2,0,12,0^2,1)$ $\mathbf{r}=(0^5,4,0^2,20)$		m
$\mathcal{A}(24,3)$	$\mathbf{f}=(80,237,158)$ $\mathbf{t}=(31,32,9,5,3)$ $\mathbf{r}=(0^4,1,0,6,1,6^2,4)$		

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Table 3.1 Isomorphism types of simplicial arrangements in $\mathbb{RP}^2$. *– continued from previous page*

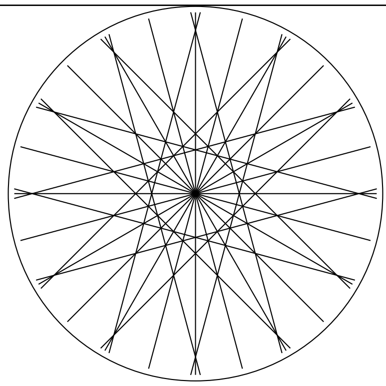
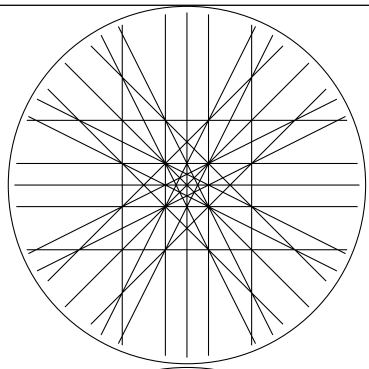
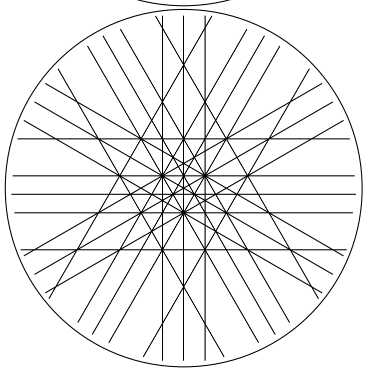
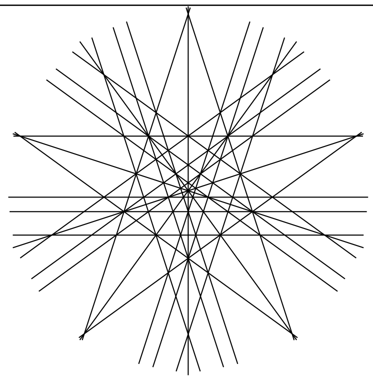
$\mathcal{A}(n,k)$	**f** **t** **r**	Figures	Notes
$\mathcal{A}(25,1)$	$\mathbf{f}=(85,252,168)$ $\mathbf{t}=(18,60,6,0^7,1)$ $\mathbf{r}=(0^6,12,0^3,13)$		$\mathcal{R}(2)$
$\mathcal{A}(25,2)$	$\mathbf{f}=(85,252,168)$ $\mathbf{t}=(36,28,15,0,6)$ $\mathbf{r}=(0^4,4,0,3,0,6,0,12)$	 	M
$\mathcal{A}(25,3)$	$\mathbf{f}=(91,270,180)$ $\mathbf{t}=(30,40,15,6)$ $\mathbf{r}=(0^8,15,0,10)$		

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Table 3.1 Isomorphism types of simplicial arrangements in $\mathbb{RP}^2$. *– continued from previous page*

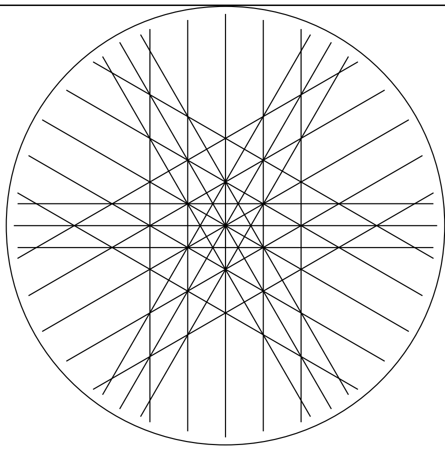
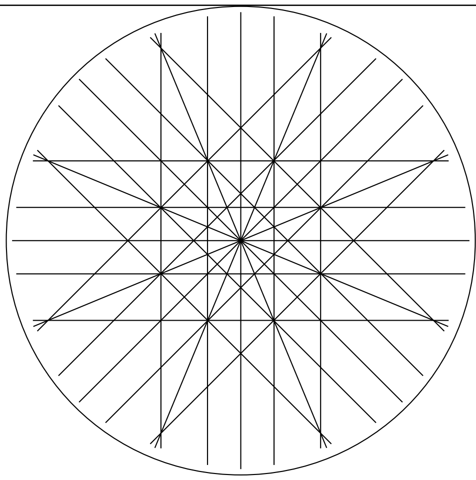
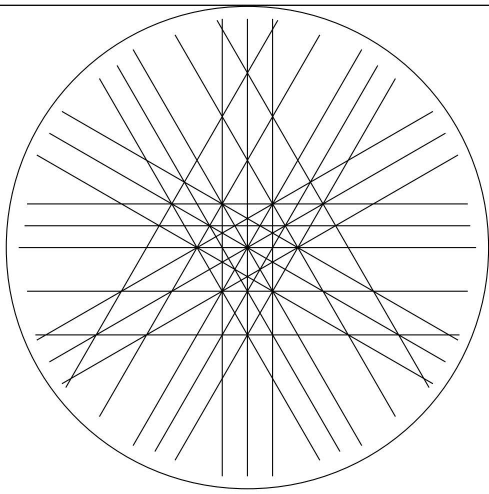
$\mathcal{A}(n,k)$	**f** **t** **r**	Figures	Notes
$\mathcal{A}(25,4)$	$\mathbf{f} = (85,252,168)$ $\mathbf{t} = (36,30,9,6,4)$ $\mathbf{r} = (0^4,1,0,9,0,3,0,12)$		
$\mathcal{A}(25,5)$	$\mathbf{f} = (81,240,160)$ $\mathbf{t} = (36,32,0,8,4,0,1)$ $\mathbf{r} = (0^6,5,0,20)$		*M*
$\mathcal{A}(25,6)$	$\mathbf{f} = (85,252,168)$ $\mathbf{t} = (36,30,9,6,4)$ $\mathbf{r} = (0^4,1,0,6,0,6^3)$		

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Table 3.1 Isomorphism types of simplicial arrangements in $\mathbb{RP}^2$. *– continued from previous page*

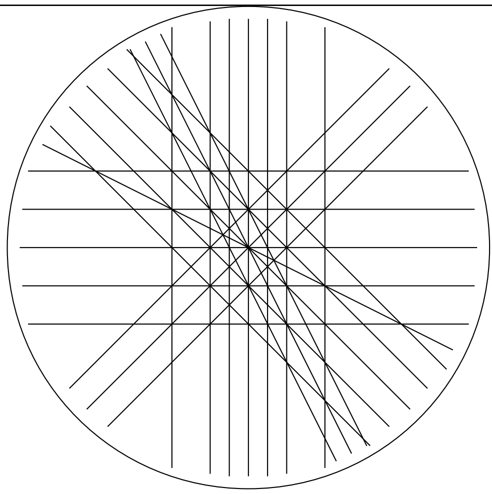
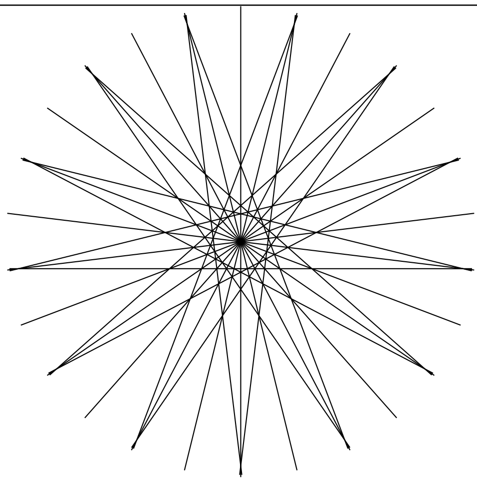
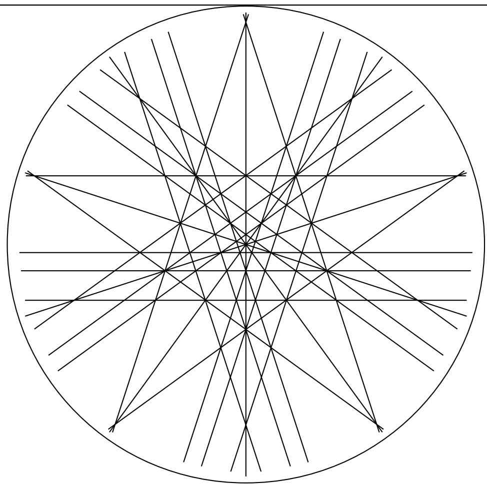
$\mathcal{A}(n,k)$	**f** **t** **r**	Figures	Notes
$\mathcal{A}(25,7)$	$\mathbf{f} = (85, 252, 168)$ $\mathbf{t} = (33, 34, 12, 2, 3, 0, 1)$ $\mathbf{r} = (0^4, 2, 0, 4^3, 0, 1)$		
$\mathcal{A}(26,1)$	$\mathbf{f} = (92, 273, 182)$ $\mathbf{t} = (13, 78, 0^9, 1)$ $\mathbf{r} = (0^6, 13, 0^4, 13)$		$\mathcal{R}(1)$
$\mathcal{A}(26,2)$	$\mathbf{f} = (96, 285, 190)$ $\mathbf{t} = (35, 40, 10, 11)$ $\mathbf{r} = (0^8, 11, 5, 10)$		

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Table 3.1 Isomorphism types of simplicial arrangements in $\mathbb{RP}^2$. *– continued from previous page*

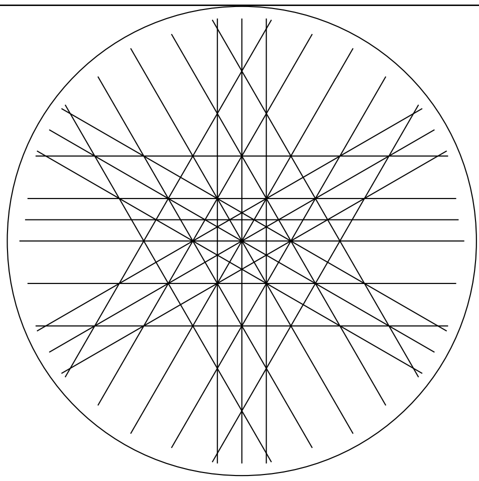
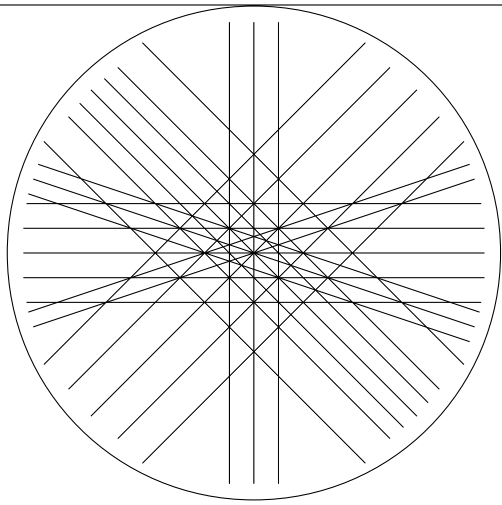
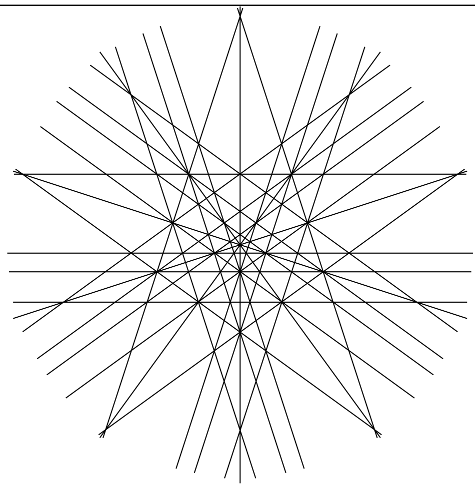
$\mathcal{A}(n,k)$	**f** **t** **r**	Figures	Notes
$\mathcal{A}(26,3)$	$\mathbf{f}=(92,273,182)$ $\mathbf{t}=(37,36,9,6,3,1)$ $\mathbf{r}=(0^4,1,0,7,2^2,1,8,4,1)$		
$\mathcal{A}(26,4)$	$\mathbf{f}=(92,273,182)$ $\mathbf{t}=(35,39,10,4,3,0,1)$ $\mathbf{r}=(0^4,1^2,4^2,2^2,7,4,1)$		
$\mathcal{A}(27,1)$	$\mathbf{f}=(101,300,200)$ $\mathbf{t}=(40^2\,6,14,1)$ $\mathbf{r}=(0^8,8^2,11)$		

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Table 3.1 Isomorphism types of simplicial arrangements in $\mathbb{RP}^2$. *– continued from previous page*

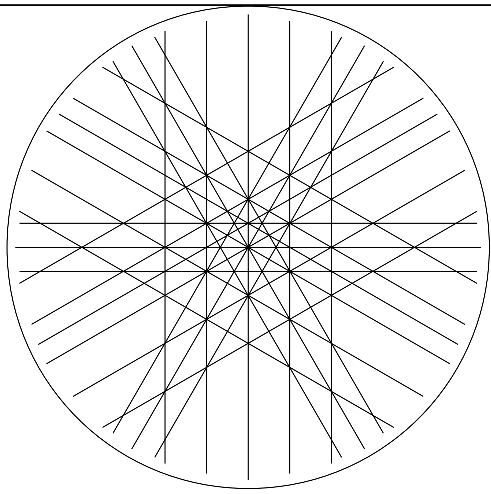
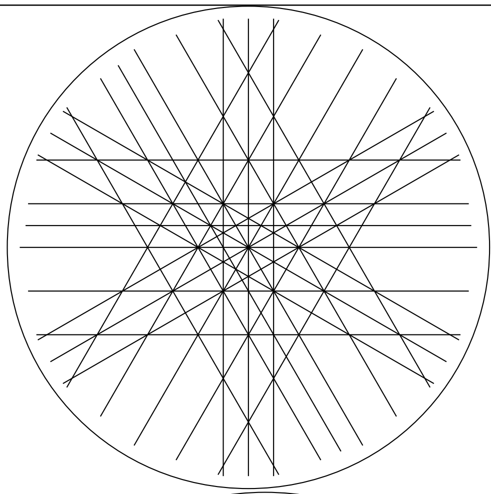
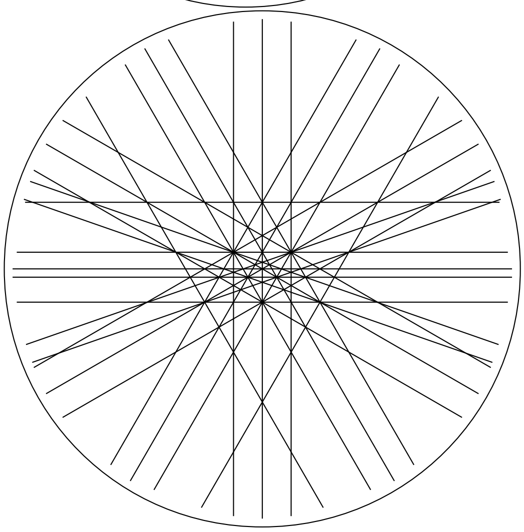
$\mathcal{A}(n,k)$	**f** **t** **r**	Figures	Notes
$\mathcal{A}(27,2)$	$\mathbf{f} = (99,294,196)$ $\mathbf{t} = (39,40,10,6,2^2)$ $\mathbf{r} = (0^4,1,0,5,4,1,2,4,8,2)$		
$\mathcal{A}(27,3)$	$\mathbf{f} = (99,294,196)$ $\mathbf{t} = (39,40,10,6,2^2)$ $\mathbf{r} = (0^4,1,0,6,2^3,5,6,3)$	 	

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Table 3.1 Isomorphism types of simplicial arrangements in $\mathbb{RP}^2$. *– continued from previous page*

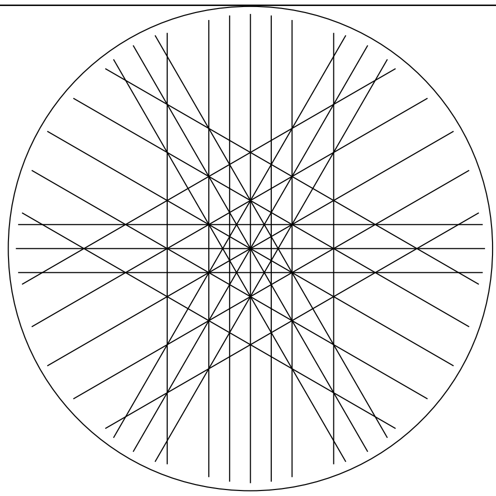
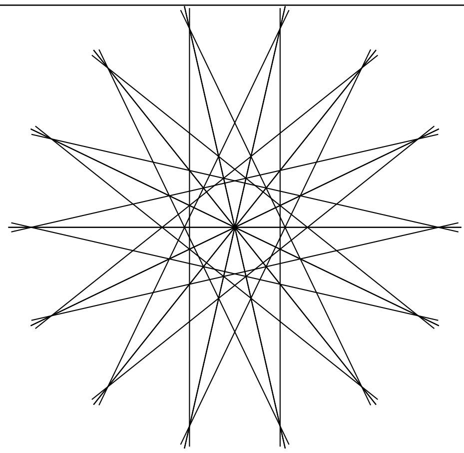
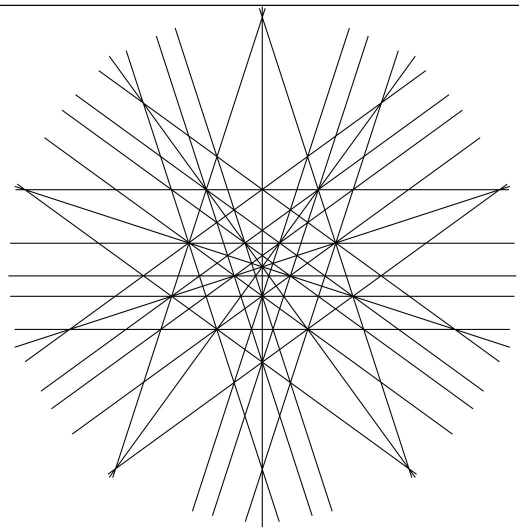
$\mathcal{A}(n,k)$	**f** **t** **r**	Figures	Notes
$\mathcal{A}(27,4)$	$\mathbf{f} = (99,294,196)$ $\mathbf{t} = (38,42,9,6,3,0,1)$ $\mathbf{r} = (0^4,1,0,5,4,2,0,7,4^2)$		
$\mathcal{A}(28,1)$	$\mathbf{f} = (106,315,210)$ $\mathbf{t} = (14,91,0^{10},1)$ $\mathbf{r} = (0^6,7^2,0^4,14)$		$\mathcal{R}(1)$
$\mathcal{A}(28,2)$	$\mathbf{f} = (106,315,210)$ $\mathbf{t} = (45,40,3,15,3)$ $\mathbf{r} = (0^8,6,9,13)$		

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Table 3.1 Isomorphism types of simplicial arrangements in $\mathbb{RP}^2$. *– continued from previous page*

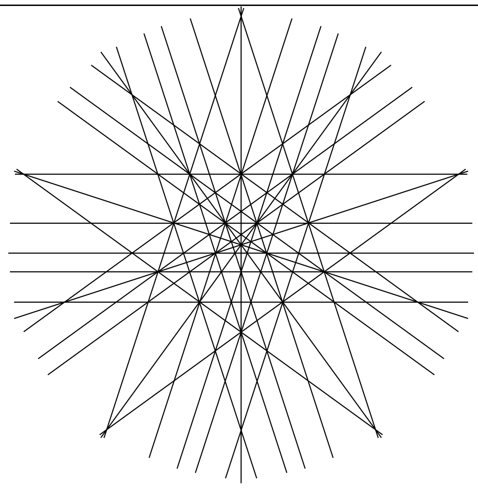
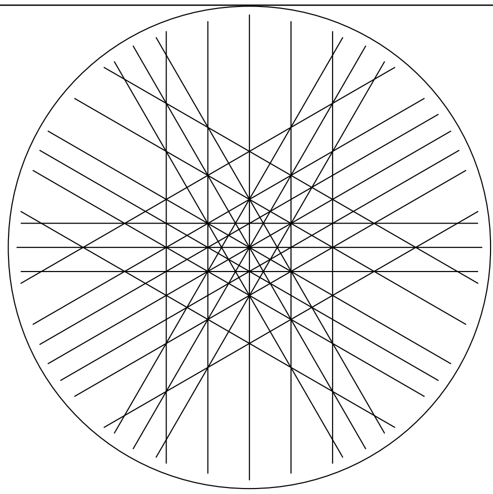
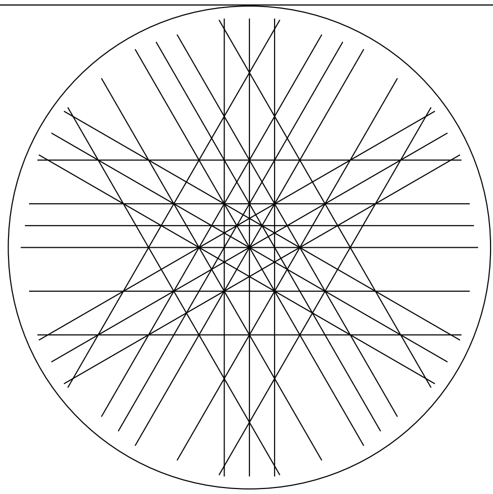
$\mathcal{A}(n,k)$	**f** **t** **r**	Figures	Notes
$\mathcal{A}(28,3)$	$\mathbf{f} = (106,315,210)$ $\mathbf{t} = (45,40,3,15,3)$ $\mathbf{r} = (0^8,6,9,13)$		In $\mathcal{A}(28,3)$ one of the triangles determined by the 3 sextuple points contains no quintuple point. In $\mathcal{A}(28,2)$ there is no such triangle.
$\mathcal{A}(28,4)$	$\mathbf{f} = (106,315,210)$ $\mathbf{t} = (41,44,9,11,6,2,1^2)$ $\mathbf{r} = (0^4,1,0,4^2,2,1,4,6^2)$		
$\mathcal{A}(28,5)$	$\mathbf{f} = (106,315,210)$ $\mathbf{t} = (42^2,12,6,1,3)$ $\mathbf{r} = (0^4,1,0,4^2,1,3,1,10,4)$		

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Table 3.1 Isomorphism types of simplicial arrangements in $\mathbb{RP}^2$. *– continued from previous page*

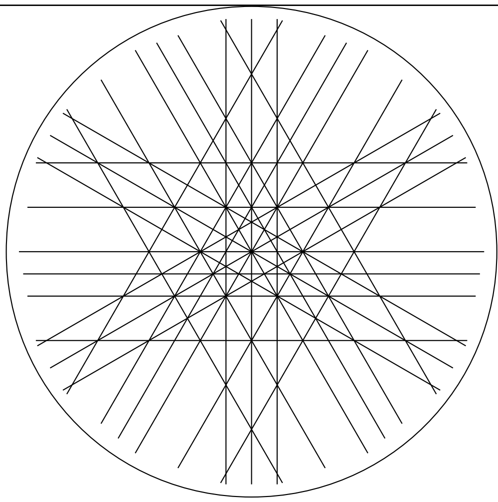
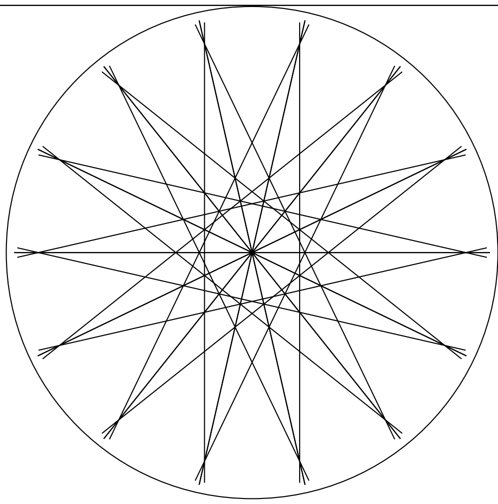
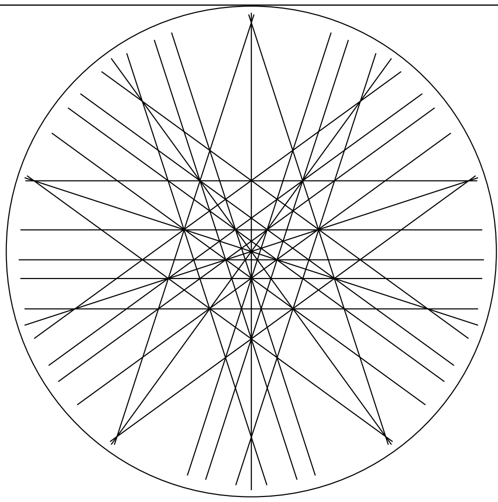
$\mathcal{A}(n,k)$	**f** **t** **r**	Figures	Notes
$\mathcal{A}(28,6)$	$\mathbf{f} = (106,315,210)$ $\mathbf{t} = (42^2,12,6,1,3)$ $\mathbf{r} = (0^4,1,0,6,0,3^3,6^2)$		
$\mathcal{A}(29,1)$	$\mathbf{f} = (113,336,224)$ $\mathbf{t} = (21,84,7,0^9,1)$ $\mathbf{r} = (0^6,7,0,7,0^3,15)$		$\mathcal{R}(2)$
$\mathcal{A}(29,2)$	$\mathbf{f} = (113,336,224)$ $\mathbf{t} = (50,40,1,14,6)$ $\mathbf{r} = (0^8,5,8,16)$		

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Table 3.1 Isomorphism types of simplicial arrangements in $\mathbb{RP}^2$. *– continued from previous page*

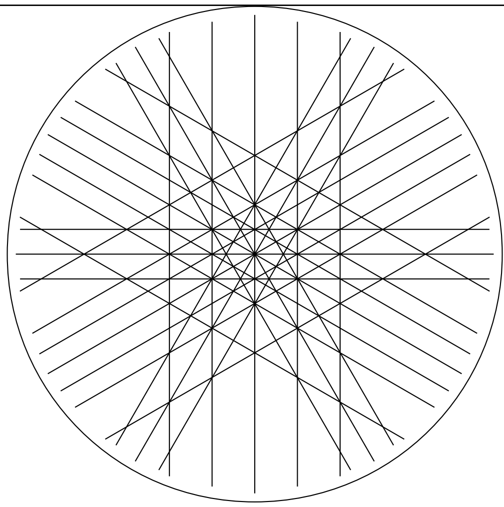
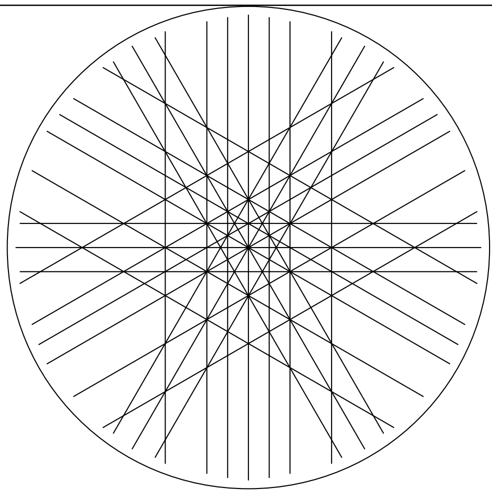
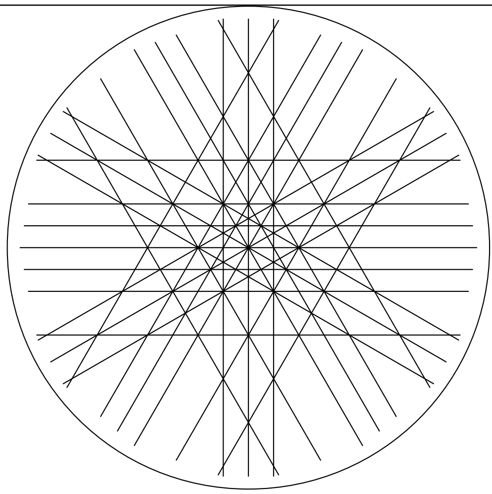
$\mathcal{A}(n,k)$	**f** **t** **r**	Figures	Notes
$\mathcal{A}(29,3)$	$\mathbf{f} = (113,336,224)$ $\mathbf{t} = (44,46,13,6,2,0,2)$ $\mathbf{r} = (0^4,1,0,3,4,3,0,4^2,10)$		
$\mathcal{A}(29,4)$	$\mathbf{f} = (113,336,224)$ $\mathbf{t} = (45,44,14,6,1,2,1)$ $\mathbf{r} = (0^4,1,0,3,4,2^2,1,8^2)$		
$\mathcal{A}(29,5)$	$\mathbf{f} = (113,336,224)$ $\mathbf{t} = (45,44,14,6,1,2,1)$ $\mathbf{r} = (0^4,1,0,4,2,3,2^2,6,9)$		

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Table 3.1 Isomorphism types of simplicial arrangements in $\mathbb{RP}^2$. *– continued from previous page*

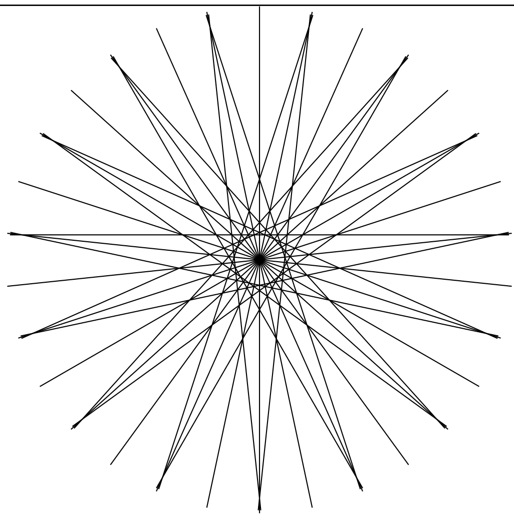
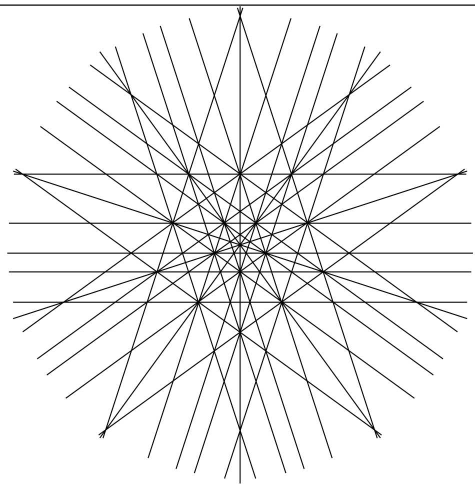
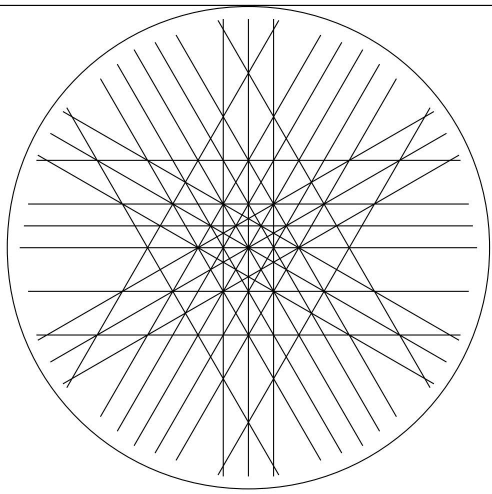
$\mathcal{A}(n,k)$	**f** **t** **r**	Figures	Notes
$\mathcal{A}(30,1)$	$\mathbf{f} = (121, 360, 240)$ $\mathbf{t} = (15, 105, 0^{11}, 1)$ $\mathbf{r} = (0^7, 15, 0^5, 15)$		$\mathcal{R}(1)$
$\mathcal{A}(30,2)$	$\mathbf{f} = (116, 345, 230)$ $\mathbf{t} = (55, 40, 0, 11, 10)$ $\mathbf{r} = (0^8, 5^2, 20)$		
$\mathcal{A}(30,3)$	$\mathbf{f} = (120, 357, 238)$ $\mathbf{t} = (49, 44, 17, 6, 1^2, 2)$ $\mathbf{r} = (0^4, 1, 0, 3, 2, 4, 1, 2, 4, 13)$		

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Table 3.1 Isomorphism types of simplicial arrangements in $\mathbb{RP}^2$. *– continued from previous page*

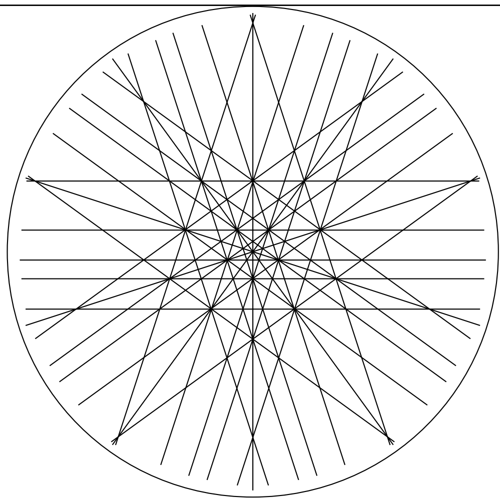
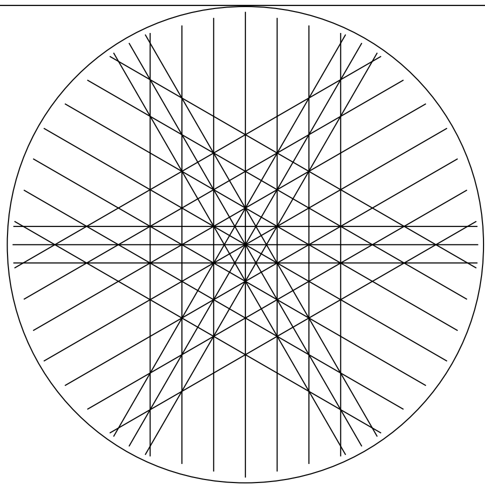
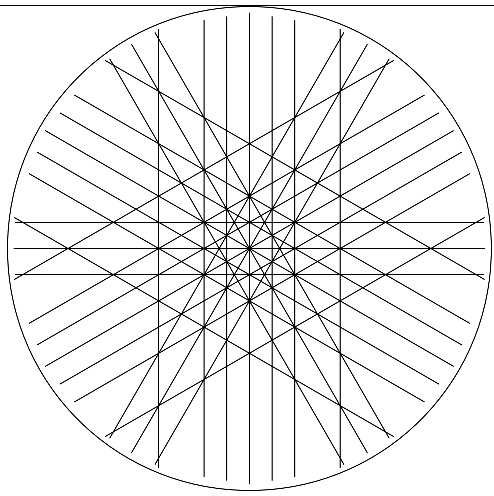
$\mathcal{A}(n,k)$	**f** **t** **r**	Figures	Notes
$\mathcal{A}(31,1)$	$\mathbf{f} = (121,360,240)$ $\mathbf{t} = (60,40,0,6,15)$ $\mathbf{r} = (0^8,6,0,25)$		M
$\mathcal{A}(31,2)$	$\mathbf{f} = (127,378,252)$ $\mathbf{t} = (54,42,21,6,1,0,3)$ $\mathbf{r} = (0^4,1,0^3,9,0,6,0,15)$		M
$\mathcal{A}(31,3)$	$\mathbf{f} = (127,378,252)$ $\mathbf{t} = (54,42,21,6,1,0,3)$ $\mathbf{r} = (0^4,1,0,3,0,6,0,3,0,18)$		M

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Table 3.1 Isomorphism types of simplicial arrangements in $\mathbb{RP}^2$. *– continued from previous page*

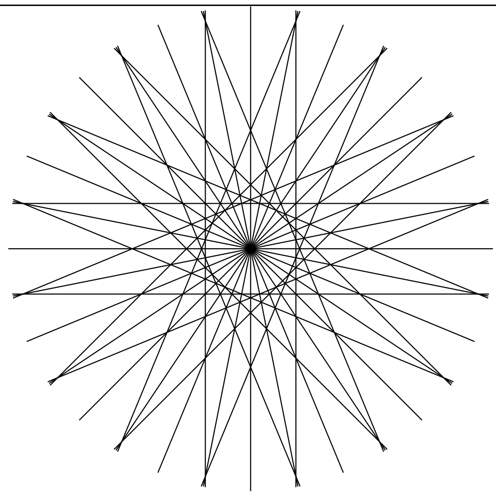
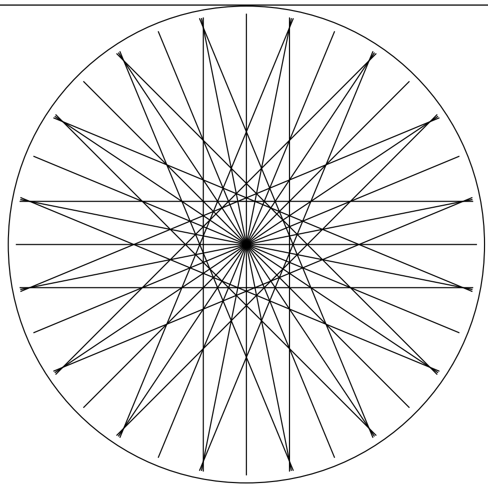
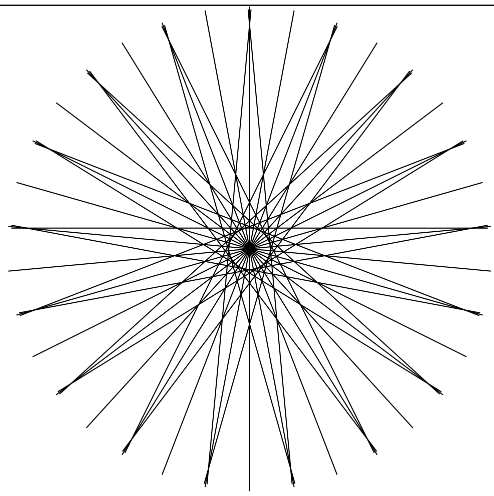
$\mathcal{A}(n,k)$	**f** **t** **r**	Figures	Notes
$\mathcal{A}(32,1)$	$\mathbf{f} = (137,408,272)$ $\mathbf{t} = (16,120,0^{12},1)$ $\mathbf{r} = (0^7,8^2,0^5,16)$		$\mathcal{R}(1)$
$\mathcal{A}(33,1)$	$\mathbf{f} = (145,432,288)$ $\mathbf{t} = (24,112,8,0^{11},1)$ $\mathbf{r} = (0^8,16,0^5,17)$		$\mathcal{R}(2)$
$\mathcal{A}(34,1)$	$\mathbf{f} = (154,459,306)$ $\mathbf{t} = (17,136,0^{13},1)$ $\mathbf{r} = (0^8,17,0^6,17)$		$\mathcal{R}(1)$

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Table 3.1 Isomorphism types of simplicial arrangements in $\mathbb{RP}^2$. *– continued from previous page*

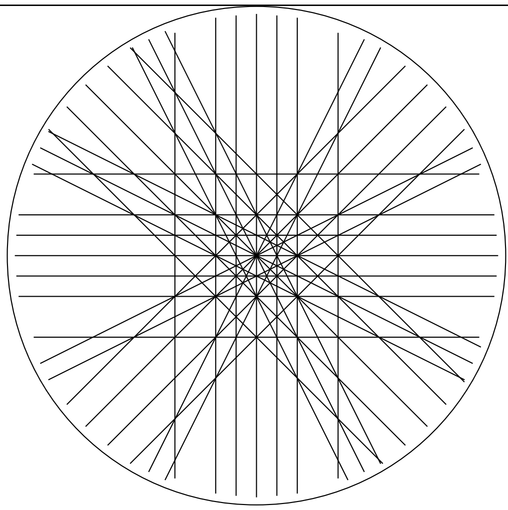
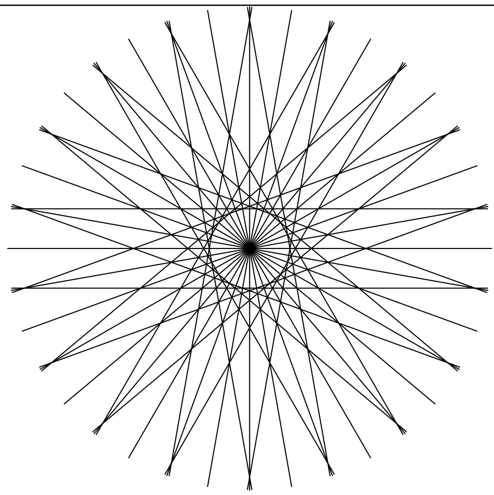
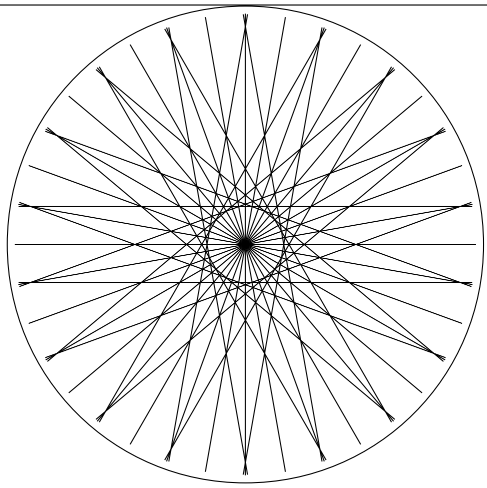
$\mathcal{A}(n,k)$	**f** **t** **r**	Figures	Notes
$\mathcal{A}(34,2)$	$\mathbf{f} = (154, 459, 306)$ $\mathbf{t} = (60, 63, 18, 6, 4, 0, 3)$ $\mathbf{r} = (0^6, 3^3, 0, 4, 0, 6, 0, 9, 6)$		$\mathcal{R}(1)$
$\mathcal{A}(36,1)$	$\mathbf{f} = (172, 513, 342)$ $\mathbf{t} = (18, 153, 0^{14}, 1)$ $\mathbf{r} = (0^8, 9^2, 0^6, 18)$		$\mathcal{R}(1)$
$\mathcal{A}(37,1)$	$\mathbf{f} = (181, 540, 360)$ $\mathbf{t} = (0^8, 9, 0, 9, 0^5, 19)$ $\mathbf{r} = (27, 144, 9, 0^{13}, 1)$		$\mathcal{R}(2)$

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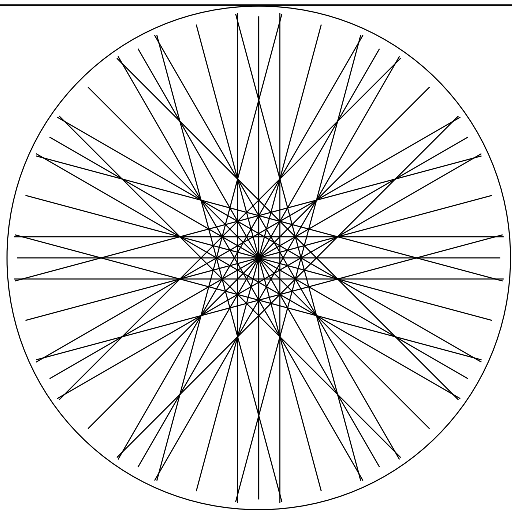
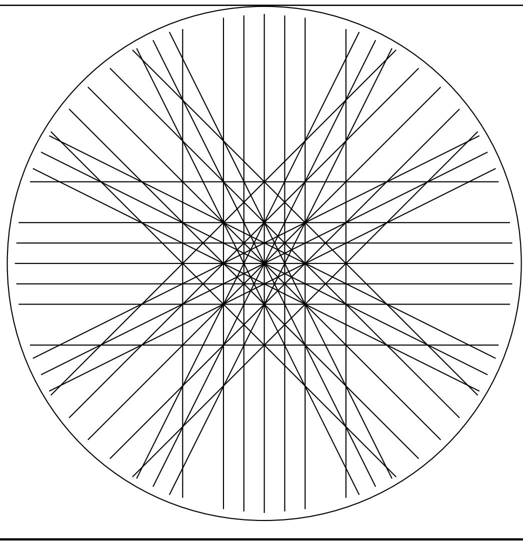
Table 3.1 Isomorphism types of simplicial arrangements in $\mathbb{RP}^2$. *– continued from previous page*

$\mathcal{A}(n,k)$	**f** **t** **r**	Figures	Notes
$\mathcal{A}(37,2)$	$\mathbf{f} = (181, 540, 360)$ $\mathbf{t} = (72^2, 12, 24, 0^6, 1)$ $\mathbf{r} = (0^{10}, 13, 0^3, 24)$		m, M
$\mathcal{A}(37,3)$	$\mathbf{f} = (181, 540, 360)$ $\mathbf{t} = (72^2, 24, 0, 10, 0, 3)$ $\mathbf{r} = (0^6, 3, 0, 6, 0, 4, 0^3, 12, 0, 12)$		M

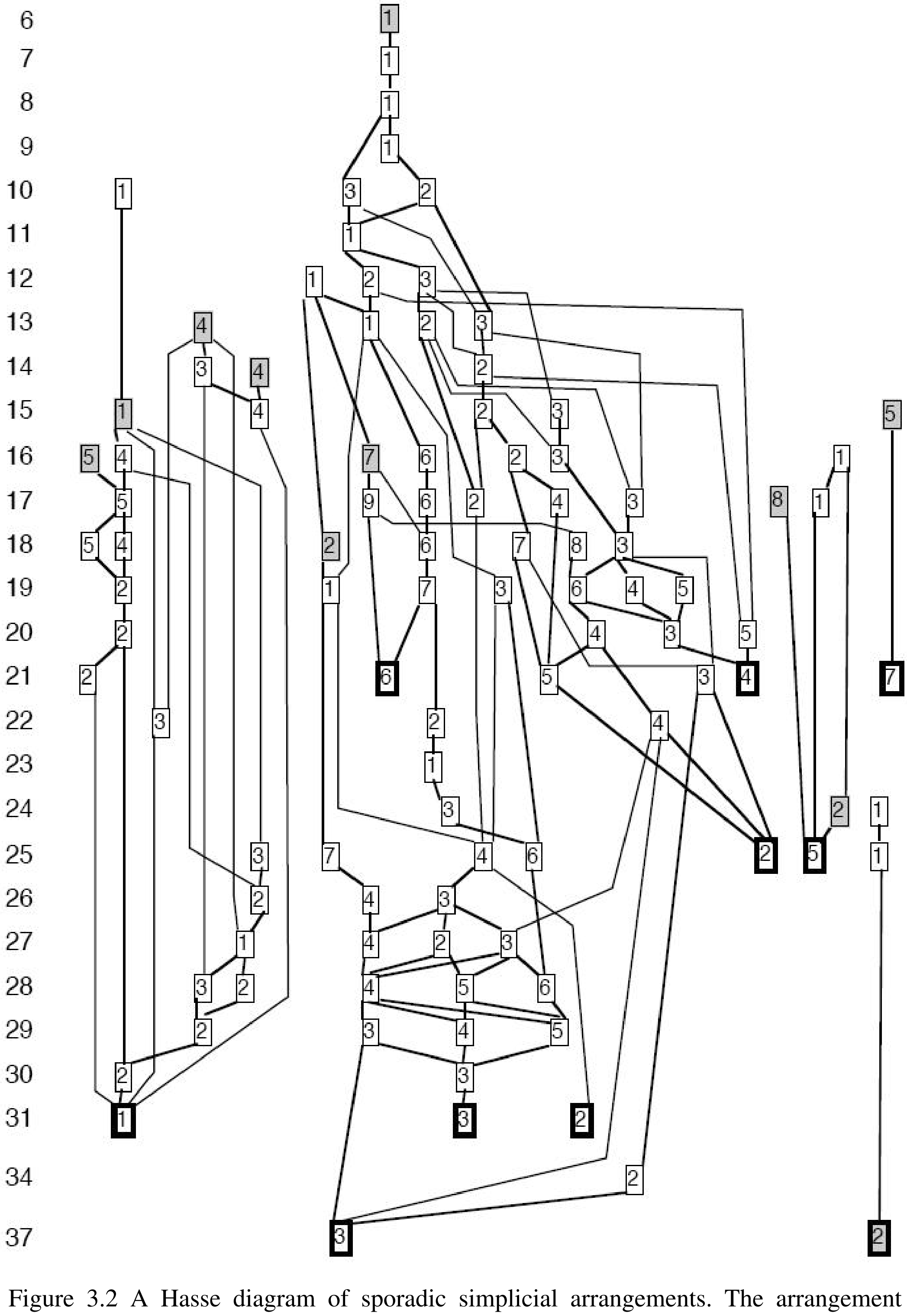


Figure 3.2 A Hasse diagram of sporadic simplicial arrangements. The arrangement $\mathcal{A}(n,k)$ is the indicated by the entry k in row n (Grünbaum, 2009, p. 5).

3.2 Füredi and Palasti's Method, and Triangles in Arrangements of Lines

Grünbaum (1972) pointed out that the maximal number of triangles in a simple arrangement p_3^s can be estimated by $p_3^s(n) \leq \frac{n(n-1)}{3}$ for even n, and $p_3^s(n) \leq \frac{n(n-2)}{3}$ if n is odd. Moreover, he conjectured that this latter inequality holds for all n, $n \not\equiv 4 \pmod 6$. The exact value of $p_3^s(n)$ is known only for some small values of n (e.g., (Simmons, 1972) for the case $n = 15$, (Grünbaum, 1972) for $n = 20$). To find best lower bounds for $p_3^s(n)$, Füredi & Palásti (1984) construct two arrangements by using the facts of Euclidean geometry in an intelligent way. First, let us explain their method.

Consider a circle $\mathcal{C}$ of radius 1 with center O, and chose a fixed point $P(0)$ on it. For any real α, let $P(\alpha)$ be the point obtained by rotating $P(0)$ around O, with angle α. Further denote by $L(\alpha)$ the straight line through the points $P(\alpha)$ and $P(\pi - 2\alpha)$. In case $\alpha \equiv \pi - 2\alpha \pmod{2\pi}$, $L(\alpha)$ is the line tangent to $\mathcal{C}$ at $P(\alpha)$.

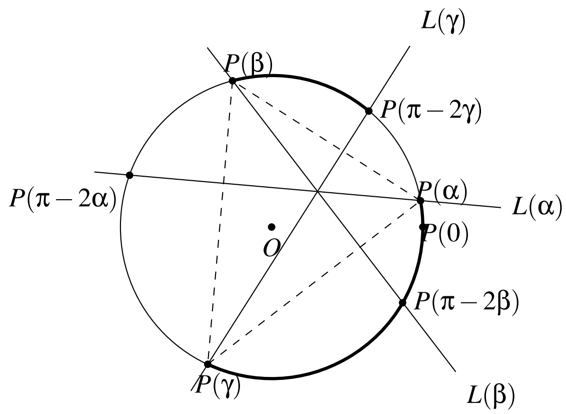


Figure 3.3 Concurrent lines $L(\alpha)$, $L(\beta)$ and $L(\gamma)$.

Lemma 3.2.1 (Füredi & Palásti (1984))**.** *The lines* $L(\alpha)$*,* $L(\beta)$ *and* $L(\gamma)$ *are concurrent if and only if* $\alpha+\beta+\gamma \equiv 0 \pmod{2\pi}$.

Proof. If $\alpha+\beta+\gamma \equiv 0 \pmod{2\pi}$, then sum of the lengths of directed arcs $(P(\alpha), P(\gamma))$ and $(P(\beta), P(\pi-2\gamma))$ is equal to π. This implies that $L(\gamma)$ is perpendicular to the line $\overline{P(\alpha)P(\beta)}$. In a similar way, one can easily see that the lines $L(\alpha)$, $L(\beta)$ and

$L(\gamma)$ are altitudes of the triangle $P(\alpha)P(\beta)P(\gamma)$ (see Figure3.3), consequently they meet at one point.

Conversely, assume the lines $L(\alpha)$, $L(\beta)$ and $L(\gamma)$ are concurrent. Then the sum of the lengths of directed arcs $(P(\alpha), P(\pi-2\gamma))$ and $(P(\beta), P(\gamma))$ is equal to π, since the sum of length of the remaining directed arcs is π. This implies that $\alpha+\beta+\gamma \equiv 0 \pmod{2\pi}$. □

Remark 3.2.2. The set of lines $\{L(\alpha) \mid 0 \leq \alpha < 2\pi\}$ may be regarded as a set of tangents to the arcs of a hypocycloid of third order (which is also known as three cuspidal quartic curve), drawn in a circle of center O and radius 3.

Remark 3.2.3. In the case of $\alpha+\beta+\gamma \equiv 0 \pmod{2\pi}$, if one takes dual of the concurrent lines $L(\alpha)$, $L(\beta)$ and $L(\gamma)$, the corresponding dual points $L^{\star}(\alpha)$, $L^{\star}(\beta)$ and $L^{\star}(\gamma)$ lie on a line, dual to the meeting point $L(\alpha) \cap L(\beta) \cap L(\gamma)$. So, Lemma 3.2.1 plays an important role for the solution of *Orchard problem*.

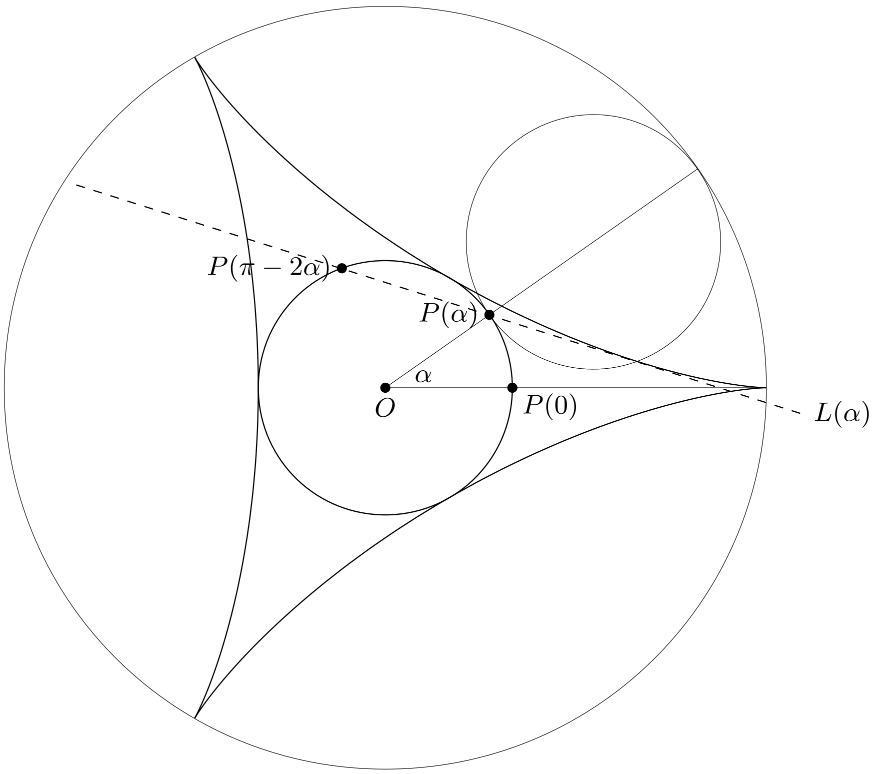


Figure 3.4 The line $L(\alpha)$ as a tangent to hypocycloid.

Füredi & Palásti (1984) considered the following arrangements of lines for $n \geq 3$:

$$\mathcal{A}_n = \left\{ L_i = L\left(\frac{(2i+1)\pi}{n}\right) \mid i = 0,1,\cdots,n-1 \right\}, \tag{3.2.1}$$

$$\mathcal{B}_n = \left\{ L_i = L\left(\frac{2i\pi}{n}\right) \mid i = 0,1,\cdots,n-1 \right\}. \tag{3.2.2}$$

See Figures 3.5 and 3.6.

The arrangement $\mathcal{A}_n$ is arrangement of n diagonals of a regular $2n$-gon. Lemma 3.2.1 implies that the line $L\left(\frac{(2n-2i-2j-2)\pi}{n}\right) \notin \mathcal{A}_n$ is concurrent to L_i and L_j of $\mathcal{A}_n$. Therefore, the lines $L_i, L_j, L_{n-i-j-1}$ and $L_i, L_j, L_{n-i-j-2}$ of $\mathcal{A}_n$ respectively form triangular cells, which tells us that $\mathcal{A}_n$ is a simple arrangement. As it is seen from the Figure 3.5 that its cells are k-gons, $3 \leq k \leq 6$. By considering the values of n relative to $\pmod 6$, they obtained the results in Table 3.2 for $p_k(\mathcal{A}_n)$. These results tell us that $p_3(\mathcal{A}_n) \geq \frac{n(n-3)}{3}$, hence $p_3^s(n) = \frac{n^2}{3} + O(n)$. On the other hand, the arrangement $\mathcal{A}_n$ is an example of two coloring arrangements. They calculated the number of black regions as $b(\mathcal{A}_n) = \frac{n^2+\varepsilon}{3}$ and the number of white regions as $w(\mathcal{A}_n) = \frac{n^2+3n-2\varepsilon+6}{6}$, where $\varepsilon = 0,2,2$ if $n \equiv 0,1,2 \pmod 3$, respectively. Hence, $b(\mathcal{A}_n) = 2w(\mathcal{A}_n) - (n+2-\varepsilon)$.

Table 3.2 The number of k-gons of the arrangement $\mathcal{A}_n$.

$n \geq 5$	$p_3(\mathcal{A}_n)$	$p_4(\mathcal{A}_n)$	$p_5(\mathcal{A}_n)$	$p_6(\mathcal{A}_n)$
$n \equiv 0 \pmod 6$	$\frac{n^2-3n}{3}$	$\frac{n}{2}+6$	$n-6$	$\frac{n^2-6n+6}{6}$
$n \equiv \mp 1 \pmod 6$	$\frac{n^2-3n+5}{3}$	5	$2n-9$	$\frac{n^2-9n+20}{6}$
$n \equiv \mp 2 \pmod 6$	$\frac{n^2-3n+8}{3}$	$\frac{n}{2}$	$n-2$	$\frac{n^2-6n+2}{6}$
$n \equiv 3 \pmod 6$	$\frac{n^2-3n+9}{3}$	3	$2n-9$	$\frac{n^2-9n+24}{6}$

The arrangement $\mathcal{B}_n$ also consists of n diagonals of a regular $2n$-gon. Lemma 3.2.1 implies that the line $L_{n-i-j}\left(\frac{2(n-i-j)\pi}{n}\right) \in \mathcal{B}_n$ is concurrent to the lines L_i and L_j of $\mathcal{B}_n$. Therefore, all cells in $\mathcal{B}_n$ either is a triangle or rectangle (See Figure 3.5). By considering the values of n relative to $\pmod 6$, they obtained the results $p_3(\mathcal{B}_n) \geq \frac{n(n-3)-2\varepsilon}{3} + 6$ and $p_4(\mathcal{B}_n) = n - 6 + \varepsilon$, where $\varepsilon = 0,2,2$ according to whether $n \equiv 0,1,2 \pmod 3$. Then it is clear that $p_3(\mathcal{B}_n) \geq \frac{n(n-3)}{3} + 4$.

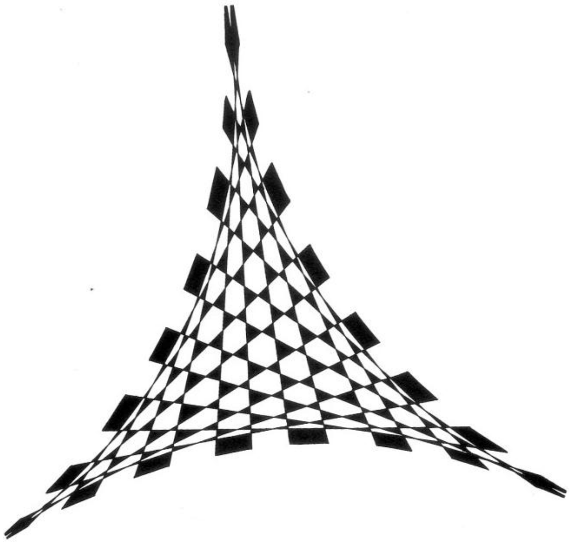


Figure 3.5 The arrangement $\mathcal{A}_n$ (Füredi & Palásti, 1984, Figure 2).

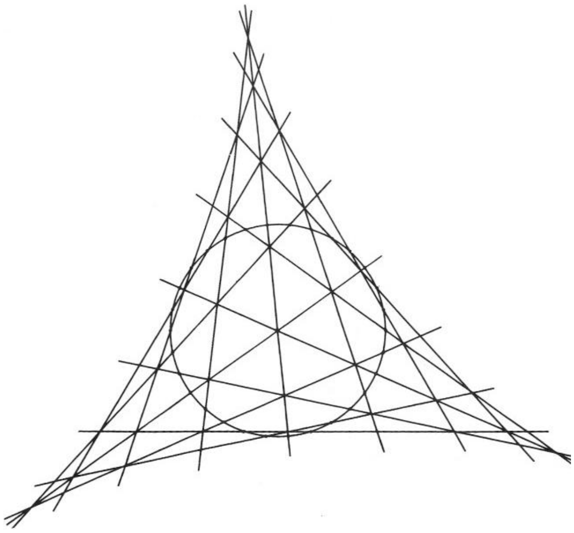


Figure 3.6 The arrangement $\mathcal{B}_n$ (Füredi & Palásti, 1984, Figure 3).

In fact, first important results for Grünbaum's conjecture $p_3(n) \leq \frac{n(n-1)}{3}$ were obtained by Purdy (1979, 1980), who in 1979 proved $p_3(n) \leq \frac{5}{12}n(n-1)$ and in 1980 he improved this to $p_3(n) \leq \frac{7}{18}n(n-1)+\frac{1}{3}$ for $n > 6$. Further, Gu (1999) extended Purdy's result and proved that $p_3(n) \leq \frac{n(n-1)}{3}$ if $t_3 = 0$, which was a generalization of the known result: $p_3(n) \leq \frac{n(n-1)}{3}$ for $t_s = 0$, $s \geq 3$. Also, he proved that $p_3(n) \leq \frac{8}{21}n(n-1)+\frac{2}{7}$ if $n \geq 7$.

3.3 Orchard Problem

The orchard problem is a tree planting problem asks that n trees be planted so that there will be $\sigma(n,k)$ straight rows with k trees in each row. The problem is to find an arrangement with the greatest $\sigma(n,k)$ for each given value of n. This very old problem is formulated by Sylvester (1867) as asking how to plant n trees in an orchard so as to maximize the number of rows, $\sigma(n)$, containing exactly 3 trees (i.e., $\sigma(n) := \sigma(n,3)$). Figure 3.7 shows examples of optimal arrangements with $n \leq 10$ points. Sylvester (1867) construct some arrangements and first showed that $\sigma(n) \geq \lfloor \frac{(n-1)^2}{8} \rfloor$, and several year later he found a better lower bound: $\sigma(n) \geq \lfloor \frac{(n-1)(n-2)}{6} \rfloor$. This was known as the best lower bound till 1974. Burr et al. (1974) considered a real cubic real cubic curve $\mathcal{C}: \ y^2 = 4x^3 - 1$ with one flex point at infinity. By using the parametrization $P(u) = (\wp(u), \wp'(u))$ of elliptic curves by Weierstrass $\wp$ function, they applied the group law of elliptic curves to orchard problem. The collinearity condition is as follows: three points $P(u_1), P(u_2)$ and $P(u_3)$ are collinear if and only if

$$u_1 + u_2 + u_3 \equiv 0 \pmod{2\omega}, \tag{3.3.1}$$

where ω is the period of $\wp(u)$.

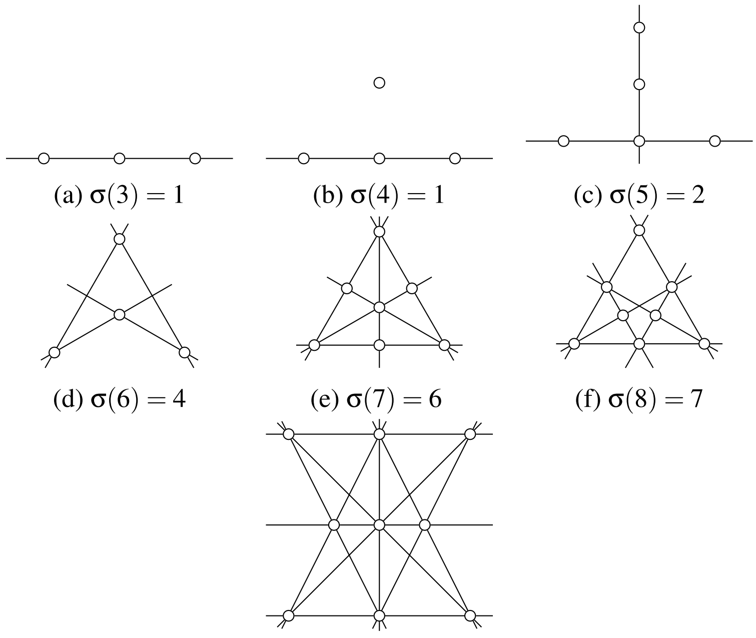
Then, they considered the n real points $P(u_s)$ of $\mathcal{C}$, where $u_s = \frac{2s}{n}\omega$, $s \in \mathbb{Z}_n$. So the collinearity condition (3.3.1) reduces to

$$s_1 + s_2 + s_3 \equiv 0 \pmod{n}. \tag{3.3.2}$$

By solving this equation in $\mathbb{Z}_n$, they found a lower bound

$$\sigma(n) \geq 1 + \frac{n(n-3)}{6}, \quad n \geq 3. \tag{3.3.3}$$

Indeed, if we denote the unordered triples (s_1, s_2, s_3) satisfying the equation (3.3.2) by σ, then σ is one-sixth of the number of ordered triples (s_1, s_2, s_3) of $\mathbb{Z}_n$. This



(a) $\sigma(3) = 1$ (b) $\sigma(4) = 1$ (c) $\sigma(5) = 2$

(d) $\sigma(6) = 4$ (e) $\sigma(7) = 6$ (f) $\sigma(8) = 7$

(g) $\sigma(9) = 10$

Figure 3.7 Orchards for $\sigma \leq 10$.

number is equal to the number σ_3 of all solutions of (3.3.2) decreased by 3 times the number σ_2 of all solutions of (3.3.2) in the case of two of s_i coincides, and increased by twice the number σ_1 of all solutions of (3.3.2) for the case $s_1 = s_2 = s_3$. Clearly, $\sigma_3 = n^2$, $\sigma_2 = n$ and $\sigma_1 = 3$ or 1 depending on whether $3 \mid n$. Combining these results, one obtains

$$\sigma(n) = \sigma_3 - 3\sigma_2 + 2\sigma_1 = 1 + \frac{n(n-3)}{6}.$$

The lower bound (3.3.3) can also be obtained by using the Füredi & Palásti's arrangements $\mathcal{B}_n$ in the page 66. This arrangement contains n diagonals of regular $2n$-gon. Three lines L_i, L_j, L_k of $\mathcal{B}_n$ meets at a point if and only if $i + j + k \equiv 0 \pmod{n}$. We have already found the number of solutions of this equation. So, configuration consists of $1 + \frac{n(n-3)}{6}$ triple points. If we take the duals of those points and lines, then we obtain exactly the n points and $1 + \frac{n(n-3)}{6}$ lines, each of which consists of 3 points. As it can be easily seen that, in the real case these two methods are dual. If one consider the complex line arrangements, the lower bound (3.3.3) is not so good. For example $\sigma(9) = 12$. This can be complex realizable by Hessian arrangement.

Hessian arrangement consists of 12 lines passing through the 9 flex points of Fermat cubic $X^3+Y^3+Z^3=0$. To find a best lower bound for $\sigma(n)$ one can use the group law of (complex) elliptic curves.

Let E_n denotes the n-torsion points of an irreducible elliptic curve $\mathcal{C}: Y^2Z = 4X^3 - aXZ^2 - bZ^3$ with $\Delta = a^3 - 27b^2 \neq 0$. This elliptic curve consists of nine flex points, and only one of them, $[0:1:0]$, is at infinity. By fixing this point as zero, define the group law. Then set of n-torsion points $E_n(\mathbb{C}) := \{P \in \mathcal{C} : nP = 0\}$ is clearly a subgroup of $\mathcal{C}$, and $E_3(\mathbb{C})$ consists of only nine flex points.

Elliptic curves can be parametrized by using the Weierstraß $\wp$ function. The collinearity condition $P(u_1)+P(u_2)+P(u_3) = O$ is equivalent to the condition $u_1 + u_2 + u_3 = 0$ for $u_i \in \Lambda$, where Λ is the underlying lattice of the cubic curve $\mathcal{C}$. If λ is generated by ω_1, ω_2, then for given positive integer n, the points $u = (\lambda_1\omega_1 + \lambda_2\omega_2)$ for $0 \geq \lambda_1, \lambda_2 \geq n-1$ all have $nu \equiv 0 \mod \Lambda$, and these are the n^2 points with order dividing n in the group $\mathbb{C}/\Lambda$. The images of these points corresponds to n-torsion points of elliptic curve, and $E_n(\mathbb{C}) \cong \mathbb{Z}_n \oplus \mathbb{Z}_n$. Thus, subgroups of $E_n(\mathbb{C})$ consisting of the collinear points solves the orchard problem, and best upper bounds can be obtained in this way. If one takes the dual of points in these subgroup and lines so that collinear points lie on it, then he get an arrangement of lines having only triple points. This kind of arrangements are important for the uniformization problem (See Theorem 6.1.4).

CHAPTER FOUR
CONFIGURATION OF QUADRICS

In this chapter, we will be interested in combinatorics of quadric arrangements, and so investigate the some possible configurations of non-degenerate quadrics with contact order ≥ 2 and derive their equations. We will also mention some impossible graphs. To describe the intersection behavior of non-degenerate quadrics for such configurations, we will use the dual graphs explained in the section 2.5 (See Table 2.1 on page 22), and unless otherwise indicated we assume that all quadrics are distinct and non-degenerate, and any three of them have no common point.

To derive equations for quadrics, we will need the parametrization of the quadrics as explained in Section 2.6. If one parametrizes one of the quadrics and substitute them into the equation of the second quadric, then he gets a polynomial equation $q(t) = 0$ of degree at most 4. The number of roots and the vanishing orders of the roots determines the number of intersection points, and contact order of them at these points, respectively. Note that, if the degree of $q(t)$ less than four, then it has a root at ∞.

4.1 Configuration of Quadrics with Contact Order Four

Proposition 4.1.1. *Any configuration of two quadrics with graph* •—• *is projectively equivalent to the quadrics*

$$\begin{aligned} Q_1 &: \ X^2 - YZ = 0 \\ Q_2 &: \ X^2 + aZ^2 - YZ = 0, \quad a \in \mathbb{C}^*. \end{aligned} \tag{4.1.1}$$

Proof. The fact of $\dim \mathrm{PGL}(3,\mathbb{C}) = 8$ allows us to fix one of the quadrics and their contact point. So, assume that Q_1 is the quadric given by equation $X^2 - YZ = 0$, and it has contact with Q_2 of order 4 at the point $[0:1:0]$. Also, assume that the equation of the second quadric Q_2 is of the form $a_1X^2 + a_2Y^2 + a_3Z^2 + a_4XY + a_5YZ +$

$a_6ZX = 0$. Since $[0:1:0] \in Q_2$, then one knows that $a_2 = 0$. By dehomogenizing their equations with respect to the variable Y, we get $Q_1 : x^2 - z = 0$ and $Q_2 : a_1x^2 + a_3z^2 + a_4x + a_5z + a_6xz = 0$. If we substitute the parametrization $(x,z) = (t,t^2)$ of the affine part of Q_1 into the equation of the affine part of Q_2, we get the polynomial $f(t) = a_3t^4 + a_6t^3 + (a_1 + a_5)t^2 + a_4t$. This polynomial has 4-fold root at $t = 0$ if and only if $a_4 = a_6 = a_1 + a_5 = 0$ and $a_3 \neq 0$. Then, the equation of Q_2 must be of the form $a_1X^2 + a_3Z^2 - a_1YZ = 0$. Since Q_2 is non-degenerate, then $a_1 \neq 0$. So, by dividing both sides by a_1, and renaming the nonzero coefficient $\frac{a_3}{a_1}$ as a we obtain that the quadric $Q_2 : X^2 + aZ^2 - YZ = 0$, where $a \in \mathbb{C}^*$. $\square$

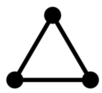
One can easily discover that the quadrics Q_1 and Q_2 have the following parametrizations:

$$Q_1 = \left\{[uv : v^2 : u^2] \mid [u:v] \in \mathbb{CP}^1\right\}, \tag{4.1.2}$$

$$Q_2 = \left\{[st : as^2 + t^2 : s^2] \mid [s:t] \in \mathbb{CP}^1\right\}. \tag{4.1.3}$$

and their common tangent line is the line $Z = 0$.

Proposition 4.1.2. *The graph*



can not be (complex) realized, i.e., there are no three distinct quadrics, pairwise tangent to each other of order 4 *at distinct points.*

Proof. Let Q_1 and Q_2 be the quadrics in Proposition 4.1.1, and suppose that there exist a quadric Q_3 such that Q_1 and Q_3 has a contact of order 4. Also, assume $Q_3 : a_1X^2 + a_2Y^2 + a_3Z^2 + a_4XY + a_5YZ + a_6XZ = 0$. By substituting parametrization (4.1.2) of Q_1 into the equation of Q_3 one gets $f_{13}(u,v) = a_3u^4 + a_6u^3v + (a_1 + a_5)u^2v^2 + a_4uv^3 + a_2v^4 = 0$. On the other hand, the contact point of Q_1 and Q_3 must be in the form of $[\alpha : \alpha^2 : 1]$, where $\alpha = \frac{v}{u} \in \mathbb{C}$, since the point $[0:1:0]$ does not lie on Q_3. Therefore $f_{13}(u,v) = A(\alpha u - v)^4$ for some $A \in \mathbb{C}^*$. Hence, by comparing the coefficients of these two equations for $f_{13}(u,v)$, one gets the equation of Q_3 in the form of

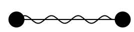
$$\beta X^2+Y^2+\alpha^4Z^2-4\alpha XY+(6\alpha^2-\beta)YZ-4\alpha^3XZ=0 \tag{4.1.4}$$

for some $\alpha,\beta\in\mathbb{C}$. Let us substitute the parametrization (4.1.3) of Q_2 into the equation (4.1.4) of Q_3. Then, we have

$$\begin{aligned}
f_{23}(s,t)&=(a^2+\alpha^4+6a\alpha^2-a\beta)s^4-(4a\alpha+4\alpha^3)s^3t+(2a+6\alpha^2)s^2t^2-4\alpha st^3+t^4\\
&=(\alpha s-t)^4+as[(a+6\alpha^2-\beta)s^3-4\alpha s^2t+2st^2].
\end{aligned} \tag{4.1.5}$$

Since the point $[0:1:0]$ does not lie on Q_3, the contact points of Q_2 and Q_3 must be in the form of $[\gamma:a+\gamma^2:1]$, where $\gamma=\frac{t}{s}\in\mathbb{C}$. Therefore, $f_{23}(s,t)$ contains the factor $(\gamma s-t)^{m_\gamma}$, where m_γ is the contact order of Q_2 and Q_3 at the point $[\gamma:a+\gamma^2:1]$. Clearly $f_{23}(s,t)=(\gamma s-t)^4$ if and only if $a=0$ and $\gamma=\alpha$. This is not the case since the quadrics Q_1 and Q_2 are distinct. Hence, the configuration of three distinct quadrics having contact orders 4 at distinct points is not possible. $\square$

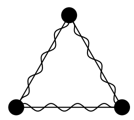
4.2 Configuration of Quadrics with Contact Order Three

Proposition 4.2.1. *Any configuration of two quadrics with graph*  *is projectively equivalent to the configuration of the quadrics*

$$\begin{aligned}
&Q_1: X^2-YZ=0\\
&Q_2: X^2+bY^2+cXY-YZ=0\,,\quad b,c\in\mathbb{C},\ c\neq 0.
\end{aligned} \tag{4.2.1}$$

Proof. Projective transformations allows us to choose the quadric $Q_1: X^2-YZ=0$ and the contact point $[0:0:1]$ of order three. Now assume that $Q_2: a_1X^2+a_2Y^2+a_3Z^2+a_4XY+a_5YZ+a_6ZX=0$. Since $[0:0:1]\in Q_2$, then one knows that $a_3=0$. By dehomogenizing their equations with respect to the variable Z, we get $Q_1 : x^2 - y = 0$ and $Q_2 : a_1x^2 + a_2y^2 + a_4xy + a_5y + a_6x = 0$. If we substitute the parametrization $(x, y) = (t, t^2)$ of the affine part of Q_1 into the equation of the affine part of Q_2, we get the polynomial $q(t) = a_2t^4 + a_4t^3 + (a_1 + a_5)t^2 + a_6t$. This polynomial has 3-fold root at 0 if and only if $a_6 = 0$, $a_5 = -a_1$ and $a_4 \neq 0$. In addition, $a_1 \neq 0$, because if it were then Q_2 would be degenerate. Then Q_2 has the equation $a_1X^2 + a_2Y^2 + a_4XY - a_1YZ = 0$. Dividing each side of this equation by a_1, and setting $\frac{a_2}{a_1} = b$ and $\frac{a_4}{a_1} = c$ we obtain the required equation for Q_2. Note that for each $b \in \mathbb{C}$ and $c \in \mathbb{C}^*$, Q_2 is non-degenerate. $\square$

Proposition 4.2.2. *Three quadrics in the graph*



are projectively equivalent to the quadrics

$$Q_1 : -(1 + a_{14} + a_{16})X^2 + a_{14}XY + YZ + a_{16}ZX = 0,$$
$$Q_2 : -(1 + a_{24} + a_{25})Y^2 + a_{24}XY + a_{25}YZ + ZX = 0,$$
$$Q_3 : a_{33}Z^2 + XY + a_{35}YZ + a_{36}ZX = 0,$$

where either $a_{16} = a_{25} = 1$, $a_{14} = a_{24} = \alpha$, $a_{35} = a_{36} = \frac{1}{\alpha}$, $a_{33} = -\frac{\alpha+2}{\alpha^3}$, $\alpha \in \mathbb{C} \setminus \{0, \mp 1, -2\}$*; or*

$$\begin{aligned}
a_{16} &= \beta, \quad a_{14} = \frac{(\beta-1)^2(\beta+1)}{\beta^2}, \quad a_{24} = \frac{(\beta-1)^3(\beta+1)}{\beta^2}, \quad a_{25} = \frac{1}{\beta}, \\
a_{33} &= \frac{\beta(\beta^3-\beta^2+1)(2\beta^2-2\beta+1)}{(\beta-1)^8(\beta+1)^2}, \quad a_{35} = \frac{\beta^2}{(\beta-1)^2(\beta+1)}, \\
a_{36} &= \frac{\beta^2}{(\beta-1)^3(\beta+1)}, \quad \beta^4 - 2\beta^3 + 2\beta^2 - \beta + 1 = 0.
\end{aligned}$$

Proof. Let $Q_i : a_{i1}X^2 + a_{i2}Y^2 + a_{i3}Z^2 + a_{i4}XY + a_{i5}YZ + a_{i6}ZX = 0, i = 1, 2, 3$. Projective transformations allow us to choose four points. Let Q_1 and Q_2 have contact of order 3 at $[0 : 0 : 1]$ and transverse at $[1 : 1 : 1]$. Assume $[0 : 1 : 0]$ and $[1 : 0 : 0]$ are the third order contact points of Q_3 with Q_1 and Q_2, respectively. Then

$a_{12} = a_{13} = a_{21} = a_{23} = a_{31} = a_{32} = a_{11} + a_{14} + a_{15} + a_{16} = a_{22} + a_{24} + a_{25} + a_{26} = 0$. In addition, the coefficients a_{15}, a_{26}, a_{34} are non zero, otherwise quadrics will be degenerate. Rescaling the equations of quadrics, we can assume that $a_{15} = a_{26} = a_{34} = 1$. Since each quadric is non-degenerate, then the determinants of corresponding symmetric matrices must be nonzero. This condition gives $a_{14}, a_{16}, a_{24}, a_{25} \neq -1$ and $a_{33} \neq a_{35}a_{36}$. Then equations of quadrics Q_i will be

$$
\begin{aligned}
&Q_1 : -(1 + a_{14} + a_{16})X^2 + a_{14}XY + YZ + a_{16}ZX = 0,\\
&Q_2 : -(1 + a_{24} + a_{25})Y^2 + a_{24}XY + a_{25}YZ + ZX = 0,\\
&Q_3 : a_{33}Z^2 + XY + a_{35}YZ + a_{36}ZX = 0,
\end{aligned}
$$

with conditions $a_{14}, a_{16}, a_{24}, a_{25} \neq -1$ and $a_{33} \neq a_{35}a_{36}$. On the other hand, the quadrics Q_1 and Q_2 can be parametrized as

$$
\begin{aligned}
Q_1 &= \left\{ [st + a_{16}s^2 : t^2 + a_{16}st : (1 + a_{14} + a_{16})s^2 - a_{14}st] \mid [t : s] \in \mathbb{CP}^1 \right\},\\
Q_2 &= \left\{ [v^2 + a_{25}uv : uv + a_{25}u^2 : (1 + a_{24} + a_{25})u^2 - a_{24}uv] \mid [u : v] \in \mathbb{CP}^1 \right\}.
\end{aligned}
$$

By substituting the parametrization of Q_1 into the equations of Q_2 and Q_3, and the parametrization of Q_2 into the equation of Q_3, respectively we obtain

$$
\begin{aligned}
f_{12}(s,t) =&(s - t)(a_{16}s + t)[(1 + a_{14} + a_{16})s^2\\
&+ ((1 + a_{24} + a_{25})a_{16} + a_{14}a_{25} + a_{25} + 1)st\\
&+ (1 + a_{24} + a_{25})t^2] = 0,
\end{aligned}
$$

$$
\begin{aligned}
f_{13}(s,t) =&(1 + a_{14} + a_{16})((1 + a_{14} + a_{16})a_{33} + a_{16}a_{36})s^4\\
&- [2a_{14}a_{33}(1 + a_{14} + a_{16}) + (1 + a_{14})a_{36}\\
&- (1 + a_{35})(a_{16}a_{36} + a_{16}^2) - a_{14}a_{16}a_{35}]s^3t\\
&+ [(1 + a_{14} + a_{16})a_{35} + a_{14}^2a_{33} - a_{14}a_{16}a_{35} - a_{14}a_{36} + 2a_{16}]s^2t^2\\
&+ (1 - a_{14}a_{35})st^3 = 0,
\end{aligned}
$$

and

$$\begin{aligned}f_{23}(u,v) =&(1+a_{24}+a_{25})((1+a_{24}+a_{25})a_{33}+a_{25}a_{35})u^4\\&-[(2a_{24}a_{33}-a_{25}a_{36}-a_{35})(1+a_{24}+a_{25})-a_{25}^2+a_{24}a_{25}a_{35}]u^3v\\&+[(1+a_{24}+a_{25})a_{36}+a_{24}^2a_{33}-a_{24}a_{25}a_{36}-a_{24}a_{35}+2a_{25}]u^2v^2\\&+(1-a_{24}a_{36})uv^3=0.\end{aligned}$$

Q_1 has a third order contact with Q_2 at $[0:0:1]$ if and only if

$$f_{12}(s,t)=A(s-t)(a_{16}s+t)^3$$

for a non-zero constant A. This is possible only when

$$a_{16}^2(1+a_{24}+a_{25})=1+a_{14}+a_{16} \quad \text{and} \quad a_{16}(1+a_{24}+a_{25})=1+a_{25}+a_{14}a_{25}, \tag{4.2.2}$$

or equivalently

$$a_{25}=\frac{1}{a_{16}} \quad \text{and} \quad a_{24}=\frac{1+a_{14}}{a_{16}^2}-1, \quad (a_{16}\neq 0). \tag{4.2.3}$$

Second, Q_1 has a third order contact with Q_3 at $[0:1:0]$ if and only if the coefficients of the terms s^2t^2 and st^3 in $f_{13}(s,t)$ are zero while the coefficients of s^4 and s^3t are non-zero. Then we have

$$a_{35}=\frac{1}{a_{14}}, \quad \text{and} \quad a_{36}=\frac{(1+a_{14})(1+a_{16})}{a_{14}^2}+a_{14}a_{33}, \quad (a_{14}\neq 0). \tag{4.2.4}$$

Last, Q_2 has a third order contact with Q_3 at $[0:1:0]$ if and only if the coefficients of the terms u^2v^2 and uv^3 in $f_{23}(u,v)$ are zero while the coefficients of u^4 and u^3v are non-zero. Then we have

$$a_{36}=\frac{1}{a_{24}}, \quad \text{and} \quad a_{35}=\frac{(1+a_{24})(1+a_{25})}{a_{24}^2}+a_{24}a_{33}, \quad (a_{24}\neq 0). \tag{4.2.5}$$

From the equations (4.2.3), we have $(1+a_{14})(1+a_{16}) = a_{16}^3(1+a_{24})(1+a_{25})$. On the other hand, the equations (4.2.4) and (4.2.5) implies that $a_{24}^3(1+a_{14})(1+a_{16}) = a_{14}^3(1+a_{24})(1+a_{25})$. Then we get $a_{16}^3 a_{24}^3 = a_{14}^3$. If ω is a third root of unity, then clearly $a_{14} = a_{16}a_{24}\omega$. Substituting it into the equation (4.2.3) we get

$$a_{16}a_{24}(a_{16}-\omega) = 1 - a_{16}^2, \tag{4.2.6}$$

which implies that $\omega = 1$ if $a_{16} = \omega$, since $a_{16}, a_{24} \neq 0$.

Now suppose, $a_{16} = \omega = 1$, then the equations (4.2.3), (4.2.4) and (4.2.5) tell us that

$$a_{16} = a_{25} = 1, \quad a_{24} = a_{14}, \quad a_{35} = a_{36} = \frac{1}{a_{14}} \quad \text{and} \quad a_{33} = -\frac{2+a_{14}}{a_{14}^3}. \tag{4.2.7}$$

Note that $1 + a_{33} + a_{35} + a_{36} = \frac{(a_{14}+2)(a_{14}^2-1)}{a_{14}^3} = 0$ if and only if $a_{14} = \mp 1$ or $a_{14} = -2$. In addition, quadrics are degenerate if $a_{14} = -1$; $Q_1 = Q_2 = Q_3$ if $a_{14} = -2$; and quadrics are non-degenerate but meet at $[1:1:1]$ if $a_{14} = 1$. So, these are not cases.

Smilarly, if $a_{16} \neq \omega$, then $a_{16} \neq \mp 1$ by the equation (4.2.6), and therefore

$$\begin{aligned}
&a_{24} = \frac{1-a_{16}^2}{a_{16}(a_{16}-\omega)}, \quad a_{14} = \frac{(1-a_{16}^2)\omega}{a_{16}-\omega}, \quad a_{25} = \frac{1}{a_{16}}, \quad a_{36} = \frac{a_{16}(a_{16}-\omega)}{1-a_{16}^2} \\
&a_{35} = \frac{a_{16}-\omega}{(1-a_{16}^2)\omega}, \quad a_{33} = \frac{a_{16}(a_{16}-\omega)(1-\omega)(1-a_{16}-a_{16}\omega)}{\omega(1-a_{16})^3(1+a_{16})^2}, \\
&a_{36} = \frac{a_{16}\omega(a_{16}-\omega)(1+\omega)+a_{16}(1-a_{16}^2\omega^2)}{(1-a_{16})^2(1+a_{16})}
\end{aligned} \tag{4.2.8}$$

by the equations (4.2.3), (4.2.4) and (4.2.5). In addition, two equalities for a_{36} in (4.2.8) imply that $(1-\omega)[a_{16}(1-a_{16})(1+\omega)-\omega] = 0$, so either $\omega = 1$ or $\omega = \frac{a_{16}-a_{16}^2}{1-a_{16}+a_{16}^2}$.

If $\omega = 1$, then $a_{14} = a_{16}a_{24}$, and therefore by the equation (4.2.6) one has either

$a_{16}=1$ or $1+a_{14}+a_{16}=0$. We have already studied the case $a_{16}=1$. If $1+a_{14}+a_{16}=0$ then the coefficient of s^4 in $f_{13}(s,t)$ will be zero, so this is not the case.

If $\omega=\frac{a_{16}-a_{16}^2}{1-a_{16}+a_{16}^2}=1$, then $2a_{16}^2-2a_{16}+1=0$, i.e., $a_{16}=\frac{1\mp i}{2}$. By using the equations (4.2.8), one can easily calculate that $1+a_{14}+a_{16}=0$. This is also not the case.

Now, suppose $\omega=\frac{a_{16}-a_{16}^2}{1-a_{16}+a_{16}^2}\neq 1$. Then ω satisfy the equation $\omega^2+\omega+1=0$. So, one gets $a_{16}^4-2a_{16}^3+2a_{16}^2-a_{16}+1=0$, i.e.,

$$a_{16}=\frac{1}{2}\mp\frac{\sqrt{2\sqrt{13}-2}}{4}\mp\frac{\sqrt{2\sqrt{13}+2}}{4}i.$$

Then, the equations in (4.2.8) reduces to

$$\begin{aligned}
a_{14}&=\frac{(a_{16}-1)^2(a_{16}+1)}{a_{16}^2}, \quad a_{24}=\frac{(a_{16}-1)^3(a_{16}+1)}{a_{16}^2}, \quad a_{25}=\frac{1}{a_{16}},\\
a_{33}&=\frac{a_{16}(a_{16}^3-a_{16}^2+1)(2a_{16}^2-2a_{16}+1)}{(a_{16}-1)^8(a_{16}+1)^2},\\
a_{35}&=\frac{a_{16}^2}{(a_{16}-1)^2(a_{16}+1)}, \quad a_{36}=\frac{a_{16}^2}{(a_{16}-1)^3(a_{16}+1)}.
\end{aligned}$$

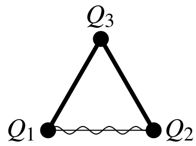
For such coefficients, quadrics neither degenerate nor meet at a point.

□

Remark 4.2.3. If one allows that quadrics in Proposition 4.2.2 has one simple triple point, then their equations are projectively equivalent to

$$\begin{aligned}
Q_1 &: -3X^2+XY+YZ+ZX=0,\\
Q_2 &: -3Y^2+XY+YZ+ZX=0, \qquad (4.2.9)\\
Q_3 &: -3Z^2+XY+YZ+ZX=0.
\end{aligned}$$

Proposition 4.2.4. *The graph*



can not be (complex) realized.

Proof. By the Proposition 4.2.1, we may assume that $Q_1 : X^2 - YZ = 0$ and $Q_2 : X^2 + bY^2 + cXY - YZ = 0$, where $b, c \in \mathbb{C}$ and $c \neq 0$. The quadrics Q_1 and Q_2 meet at $[0 : 0 : 1]$ and $[-bc : c^2 : b^2]$ with multiplicities 3 and 1, respectively. In addition, these two quadrics have parametrizations $Q_1 = \{[uv : v^2 : u^2] \mid [u : v] \in \mathbb{CP}^1\}$ and $Q_2 = \{[st : s^2 : t^2 + bs^2 + cst] \mid [s : t] \in \mathbb{CP}^1\}$. Assume that there exists a quadric $Q_3 : a_1X^2 + a_2Y^2 + a_3Z^2 + a_4XY + a_5YZ + a_6XZ = 0$ which meet with Q_1 at $[p : 1 : p^2]$ and Q_2 at $[q : 1 : q^2 + cq + b]$ with multiplicities 4. Then, both $f_{13}(p) = a_3p^4 + a_6t^3 + (a_1 + a_5)t^2 + a_4t + a_2$ and $f_{23}(q) = a_3q^4 + (2ca_3 + a_6)q^3 + (a_1 + c^2a_3 + 2ba_3 + a_5 + ca_6)q^2 + (2bca_3 + a_4 + ca_5 + ba_6)q + (a_2 + b^2a_3 + ba_5)$ are fourth power of linear polynomials. Suppose $f_{13}(p) = (\gamma p + \delta)^4 = 0$, then clearly $a_3 = \gamma^4 \neq 0, a_6 = 4\gamma^3\delta, a_5 = -a_1 + 6\gamma^2\delta^2, a_4 = 4\gamma\delta^3, a_2 = \delta^4$ and $p = -\frac{\delta}{\gamma}$. Moreover,

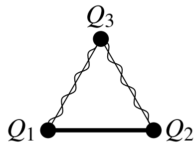
$$\begin{aligned} f_{23}(q) =& \gamma^4q^4 + 2\gamma^3(c\gamma + 2\delta)q^3 + \gamma^2((c^2 + 2b)\gamma^2 + 6\delta^2 + 4c\gamma\delta)q^2 \\ &+ (2bc\gamma^4 + 4b\gamma^3\delta + 6c\gamma^2\delta^2 + 4\gamma\delta^3 - a_1c)q + (-a_1b + b^2\gamma^4 + 6b\gamma^2\delta^2 + \delta^4) \\ =& (\gamma q + \eta)^4 = 0 \end{aligned}$$

if and only if $\delta = \frac{4b-c^2}{4c}\gamma$, $\eta = \frac{c\gamma+2\delta}{2} = \frac{4b+c^2}{4c}\gamma$, $a_1 = 4\delta^2\gamma^2 = \frac{(4b-c^2)^2}{4c^2}\gamma^4$ and $q = -\frac{4b+c^2}{4c}$. Hence the equation of the quadric Q_3 must be in the form of

$$4\delta^2\gamma^2X^2 + \delta^4Y^2 + \gamma^4Z^2 + 4\gamma\delta^3XY + 2\gamma^2\delta^2YZ + 4\gamma^3\delta XZ = (2\delta\gamma X + \delta^2Y + \gamma^2Z)^2 = 0.$$

This means, such a quadric Q_3 must be degenerate. $\square$

Proposition 4.2.5. *Three quadrics in the graph*



are projectively equivalent to the quadrics

$$
\begin{aligned}
Q_1 :& X^2 - YZ = 0,\\
Q_2 :& X^2 + aZ^2 - YZ = 0,\\
Q_3 :& -\left(\frac{2}{3}(m^2 - 8mp + p^2) + \frac{2(m-p)^3(m+p)}{a}\right)X^2 - Y^2\\
&+\left(\frac{2ap^3}{3(m-p)} + p^3(2m-p)\right)Z^2 + \left(-\frac{2a}{3(m-p)} + (m+p)\right)XY\\
&+\left(\frac{2ap}{3(m-p)} + 2mp\right)YZ + \left(-\frac{2ap^2}{m-p} + 3m - p\right)XZ = 0,
\end{aligned}
$$

where $m, p \in \mathbb{C}$, $m \neq p$ *and* $a^2 = -3(m-p)^4$.

Proof. By the Proposition 4.2.1, we may assume that $Q_1 : X^2 - YZ = 0$ and $Q_2 : X^2 + aZ^2 - YZ = 0$, where $a \in \mathbb{C}^*$. These two quadrics meet at the point $[0:1:0]$ with multiplicity 4 and they have parametrizations $Q_1 = \{[uv : v^2 : u^2] \mid [u:v] \in \mathbb{CP}^1\}$ and $Q_2 = \{[st : t^2 + as^2 : s^2] \mid [s:t] \in \mathbb{CP}^1\}$. Suppose, such a quadric Q_3 exist. Since $[0:1:0] \notin Q_1 \cap Q_3$ and $[0:1:0] \notin Q_2 \cap Q_3$, then Q_3 will meet with Q_1 at the points $[p : p^2 : 1]$ and $[q : q^2 : 1]$ with multiplicities 3 and 1, respectively, where $p \neq q$. Similarly, Q_3 will meet with Q_2 at the points $[m : m^2 + a : 1]$ and $[n : n^2 + a : 1]$ with multiplicities 3 and 1, respectively, where $m \neq n$. In addition, the line $\ell_1 :\ 2pX - Y - p^2Z = 0$ is tangent to Q_1 at $[p : p^2 : 1]$ and the line $\ell_2 : (p+q)X - Y - pqZ = 0$ pass through the intersection points $[p : p^2 : 1]$ and $[q : q^2 : 1]$ of Q_1 and Q_3. Therefore, the equation of Q_3 must be in the form of

$$
\begin{aligned}
\lambda Q_1 + \ell_1\ell_2 : \quad & (\lambda + 2p(p+q))X^2 + Y^2 + p^3qZ^2 - (3p+q)XY + p(p+q)YZ\\
& - p^2(p+3q)XZ = 0 \qquad (4.2.10)
\end{aligned}
$$

for some $\lambda \in \mathbb{C}^*$. Substituting the affine parametrization $x = t$, $y = t^2 + a$, $z = 1$ of Q_2 into the equation (4.2.10), we obtain

$$\begin{aligned} f_{23}(t) &= t^4 - (3p+q)t^3 + (3p(p+q)+2a)t^2 - (p^2(p+3q)+a(3p+q))t \\ &\quad +(p^3q + a^2 + ap(p+q) - a\lambda) = 0. \end{aligned}$$

On the other hand, by the intersection behavior of Q_2 and Q_3, $f_{23}(t)$ must be in the form of

$$f_{23}(t) = (t-m)^3(t-n) = t^4 - (3m+n)t^3 + 3m(m+n)t^2 - m^2(m+n)t + m^3 n = 0.$$

Comparing these two equations for $f_{23}(t)$ term by term we will get the following equations:

$$3m+n = 3p+q \tag{4.2.11}$$

$$3m(m+n) = 3p(p+q)+2a \tag{4.2.12}$$

$$m^2(m+3n) = p^2(p+3q)+a(3p+q) \tag{4.2.13}$$

$$m^3 n = p^3q + a^2 + ap(p+q) - a\lambda \tag{4.2.14}$$

Note that $m \neq p$ and consequently $n \neq q$, otherwise a would be zero but this is not the case. From the equations (4.2.11), (4.2.12) and (4.2.14) one obtains $n = -m + 2p + \frac{2a}{3(m-p)}$, $q = 2m - p + \frac{2a}{3(m-p)}$, $\lambda = \frac{a^2(3m-p) - 2a(m-p)^3 + 3(m-p)^4(m+p)}{3a(m-p)}$, and substituting them into the equation (4.2.13) one gets $a^2 = -3(m-p)^4$. Therefore, the equation (4.2.10) reduces to

$$\begin{aligned} &-\left(\frac{2}{3}(m^2 - 8mp + p^2) + \frac{2(m-p)^3(m+p)}{a}\right)X^2 - Y^2 \\ &+\left(\frac{2ap^3}{3(m-p)} + p^3(2m-p)\right)Z^2 + \left(-\frac{2a}{3(m-p)} + (m+p)\right)XY \\ &+\left(\frac{2ap}{3(m-p)} + 2mp\right)YZ + \left(-\frac{2ap^2}{m-p} + 3m - p\right)XZ = 0. \end{aligned}$$

□

4.3 Configuration of Quadrics with Many Tacnodes

The problem Naruki interested in determines when two quadrics are tangent to each other at one point or two points. For this aim he used the singular members of pencil introduced some invariants. First let us explain these invariants.

Let Q_1 and Q_2 be two quadrics given by the ternary quadric equations $F_1(X,Y,Z) = 0$ and $F_2(X,Y,Z) = 0$, corresponding to 3×3 symmetric matrices are M_1 and M_2, respectively. Assume that they are in general position. Then there are four distinct intersection points. Denote further the intersection points by p_0, p_1, p_2, p_3; and the $(2,2)-$partitions $\{p_0, p_1; p_2, p_3\}$, $\{p_0, p_2; p_1, p_3\}$, $\{p_0, p_3; p_1, p_2\}$ by l_1, l_2, l_3. The partitions are called the *references* of the pair $\{Q_1, Q_2\}$. They are in a one to one correspondence with the singular members of the pencil $\underset{\sim}{Q} = \{Q(t)\}$ generated by Q_1 and Q_2 (See Figure4.1). Indeed, the equations for members of family $\underset{\sim}{Q}$ is of the form:

$$Q(t):\ tF_1 + F_2 = 0 \tag{4.3.1}$$

Note that $Q(\infty) = Q_1$ and $Q(0) = Q_2$.

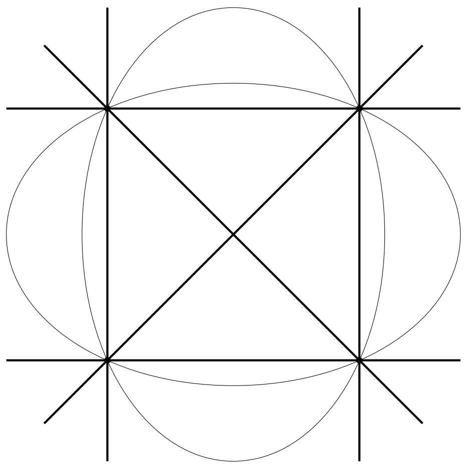


Figure 4.1 Singular members of the family of two quadrics in general position.

The symmetric matrix corresponding to the quadric $Q(t)$ is $tA_1 + A_2$, and the singular members of the family Q corresponds to roots t_1, t_2, t_3 of the cubic equation

$$\det(tM_1 + M_2) = 0. \tag{4.3.2}$$

By changing the indices in a suitable way, it can be assumed that

$$Q(t_1) = \overline{p_0p_1} \cup \overline{p_2p_3}\,, \quad Q(t_2) = \overline{p_0p_2} \cup \overline{p_1p_3} \quad \text{and} \quad Q(t_3) = \overline{p_0p_3} \cup \overline{p_1p_2},$$

where $\overline{p_ip_j}$ denotes the line passing through the points p_i and p_j for each (i, j). Thus, the references l_1, l_2, l_3 correspond to $Q(t_1)$, $Q(t_2)$, $Q(t_3)$.

The first invariant is defined by Naruki (1983) for ordered pairs of two quadrics and their references by setting

$$[Q_2/Q_1;\, l_1] = \frac{t_1^2}{t_2t_3}\,, \quad [Q_2/Q_1;\, l_2] = \frac{t_2^2}{t_1t_3}\,, \quad [Q_2/Q_1;\, l_3] = \frac{t_3^2}{t_1t_3} \tag{4.3.3}$$

which give some obvious properties:

$$[Q_2/Q_1;\, l_1] \cdot [Q_2/Q_1;\, l_2] \cdot [Q_2/Q_1;\, l_3] = 1, \tag{4.3.4}$$

and

$$[Q_2/Q_1;\, l_i] \cdot [Q_1/Q_2;\, l_i] = 1, \quad i = 1, 2, 3. \tag{4.3.5}$$

Projective invariance of these quantities follows from the fact of the change of the coordinate t of the family Q. Indeed, one can choose the coordinate τ of Q such that $\tau = \infty, 0, 1$ correspond to singular members $Q(t_1)$, $Q(t_2)$ and $Q(t_3)$, and $\tau = \alpha, \beta$ correspond to the quadrics Q_1, Q_2; explicitly $\tau = \frac{(t_1-t_3)(t_2-t)}{(t_2-t_3)(t_1-t)}$, which is the cross ratio $(t_1, t_2; t_3, t)$. Then,

$$\alpha = (t_1, t_2; t_3, \infty) = \frac{t_1 - t_3}{t_1 - t_2}\,, \quad \beta = (t_1, t_2; t_3, 0) = \frac{t_2(t_1 - t_3)}{t_3(t_1 - t_2)}\,, \quad \frac{\alpha}{\beta} = \frac{t_1}{t_2}\,, \quad \frac{\alpha - 1}{\beta - 1} = \frac{t_1}{t_3},$$

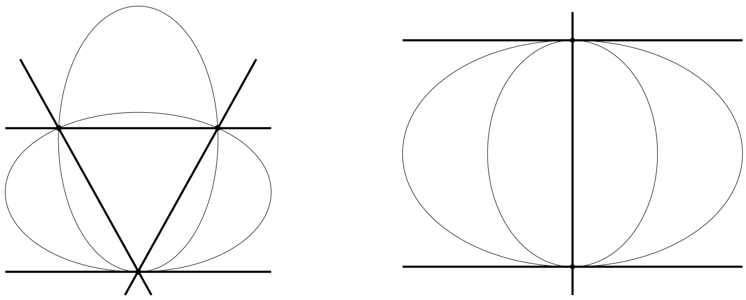


Figure 4.2 Singular members of the family of tangent quadrics.

and therefore

$$[Q_2/Q_1;\, l_1] = \frac{\alpha(\alpha-1)}{\beta(\beta-1)}\,, \quad [Q_2/Q_1;\, l_2] = \frac{\beta^2(\alpha-1)}{\alpha^2(\beta-1)}\,, \quad [Q_2/Q_1;\, l_3] = \frac{\alpha(\beta-1)^2}{\beta(\alpha-1)^2}.$$

Since both α and β are cross ratios, then by Proposition 2.2.4, these quantities remain invariant under coordinate changes.

Now consider the case that the quadrics are in a *special position*, i.e., they are tangent to each other at least at one point (contact of order 3 and 4 are excluded). Then the equation (4.3.2) has one simple root t' and one double root t''. The singular member $Q(t')$ contains common tangent (or tangents) while $Q(t'')$ contains the contact point (or points) in its singular locus (See Figure 4.2). In addition, there are only two references l', l'' of the pair $\{Q_1, Q_2\}$ corresponding to t' and t''.

Second invariant for quadrics in a special position is also defined by Naruki (1983) by setting

$$[Q_2/Q_1] = \frac{t'}{t''}. \tag{4.3.6}$$

Thus, it gives some obvious properties

$$[Q_2/Q_1;\, l'] = [Q_2/Q_1]^2 \tag{4.3.7}$$

and

$$[Q_2/Q_1;\, l''] = [Q_2/Q_1]^{-1} = [Q_1/Q_2]. \tag{4.3.8}$$

The invariant $[Q_2/Q_1]$ can also be defined without the use of coordinates. The (possibly singular) quadrics passing through given points and having given tangent lines at those points form a pencil, so they correspond to points on the projective line. Let Q_0 and Q_∞ be the union of two tangent lines, and twice the line connecting the two given points, respectively; and $\bar{Q}_1, \bar{Q}_2, \bar{Q}_0, \bar{Q}_\infty$ are the corresponding points of these quadrics on the projective line, then $[Q_1/Q_2]$ is nothing but short of the cross ratio $(\bar{Q}_0, \bar{Q}_\infty; \bar{Q}_1, \bar{Q}_2)$.

Finding the roots of the equation (4.3.2), gives some clues about the intersection of quadrics as follows: if there are three simple roots then quadrics are in general position; if there are one simple and one double root then quadrics are tangent at least at one point; and if there is 3-fold root then quadrics have contact of order ≥ 3. But less suitable for when two quadrics are tangent to each other at a point or at two distinct points. Similarly it also does not distinguish the contact orders 3 and 4. Distinguish these cases, we need parametrization of quadrics. First, we parameterize one of the quadrics as explained in Section 2.6, and then substitute them into the equation of the second quadric. This will give us a polynomial equation $q(t) = 0$ of degree at most 4. The number of roots and the vanishing orders of the roots determines the number of intersection points, and contact order of them at these points, respectively. Note that, if the degree of $q(t)$ less than four, then it has a root at ∞.

4.3.1 Two Quadrics with Two Tacnodes

Proposition 4.3.1. *Any configuration of two quadrics with graph* $\bullet\!=\!=\!\bullet$*, i.e, quadrics have two tacnodes, is projectively equivalent to the quadrics*

$$
\begin{aligned}
Q_1 &: X^2 + Y^2 - Z^2 + 2pXY = 0, \\
Q_2 &: \frac{1}{q^2}X^2 + Y^2 - Z^2 + 2pXY = 0,
\end{aligned}
\tag{4.3.9}
$$

where $p,q \in \mathbb{C}$, $q \neq 0$, $p,q \neq \pm 1$, and $p^2q^2 \neq 1$. In addition, $[Q_1/Q_2] = \frac{q^2(p^2-1)}{p^2q^2-1}$ is the Naruki invariant.

Proof. Since $\dim Aut(\mathbb{CP}^2) = \dim \mathrm{PGL}(3,\mathbb{C}) = 8$, we can choose homogeneous coordinates on $\mathbb{CP}^2$ such that the points $[0:\mp 1:1], [\mp 1:0:1]$ lie on Q_1,and $[0:\mp 1:1]$ are the tangency points of Q_2 with Q_1. The conditions $[0:\mp 1:1], [\mp 1:0:1] \in Q_1$ implies that $Q_1 : X^2+Y^2-Z^2+2pXY = 0$. For non-degeneracy, one should add the condition $p \neq \mp 1$. Then, the lines $L_\pm :\ pX+Y \mp Z = 0$ are the tangents of Q_1 at the points $[0:\pm 1:1]$, respectively.

Let $Q_2 :\ a_1X^2+a_2Y^2+a_3Z^2+a_4XY+a_5YZ+a_6ZX = 0$. Then the conditions $[0:\mp 1:1] \in Q_2$ implies $a_3 = -a_2$ and $a_5 = 0$. Since the lines $L_\pm :\ pX+Y \mp Z = 0$ are tangent to Q_2 at the points $[0:\pm 1:1]$, respectively, then one has the conditions $a_6 = 0$ and $a_4 = 2pa_2$. Therefore, the equation of Q_2 reduces to $a_1X^2+a_2Y^2-a_2Z^2+2pa_2XY = 0$. Note that a_2 must be non zero, otherwise Q_2 will be a double line. By dividing each side of the equation of Q_2 by a_2 and setting $\frac{a_1}{a_2} = \frac{1}{q^2}$ we obtain the required equation. Non-degeneracy condition of Q_2 is $p^2q^2 \neq 1$. In addition, $q \neq \mp 1$ since the quadrics are distinct.

Last, the cubic equation (4.3.2) for these quadrics Q_1 and Q_2 has simple root $t' = -\frac{q^2(p^2-1)}{p^2q^2-1}$ and double root $t'' = -1$. Hence the Naruki invariant is $[Q_1/Q_2] = \frac{q^2(p^2-1)}{p^2q^2-1}$. □

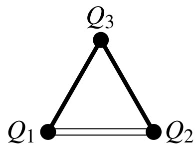
Remark 4.3.2. Megyesi (2000) proved this proposition for the case $p = 0$. Indeed, he said that any smooth quadric with two tacnodes was projectively equivalent to the pair defined by the equations $X^2+Y^2-Z^2 = 0$ and $\frac{1}{q^2}X^2+Y^2-Z^2 = 0$ with conditions $q \in \mathbb{C} \setminus \{0, \mp 1\}$. But this is just a special case.

The quadrics in (4.3.9) have parametrizations

$$\begin{aligned} Q_1 &= \left\{ [2st+2ps^2 : t^2-s^2 : t^2+s^2+2pst] \mid [s:t] \in \mathbb{CP}^1 \right\}, \\ Q_2 &= \left\{ [2qst+2pq^2s^2 : t^2-s^2 : t^2+s^2+2pqst] \mid [s:t] \in \mathbb{CP}^1 \right\}. \end{aligned} \tag{4.3.10}$$

and we shall use without writing them later again explicitly.

Proposition 4.3.3. *The graph*



can not be (complex) realized.

Proof. Suppose that such configuration of non-degenerate quadrics exist. By the Proposition 4.3.1, we may assume that $Q_1 : X^2+Y^2-Z^2+2pXY=0$ and Q_2 : $\frac{1}{q^2}X^2+Y^2-Z^2+2pXY=0$, where $p,q\in\mathbb{C}$, $q\neq 0$, $p,q\neq\pm 1$, and $p^2q^2\neq 1$. Let us assume that $Q_3 : a_1X^2+a_2Y^2+a_3Z^2+a_4XY+a_5YZ+a_6XZ=0$. Since $Q_1\cap Q_2=\{[0:\mp 1:1]\}$, then by the parametrizations (4.3.10) of Q_1 and Q_2, the contact points of Q_3 with Q_1 and Q_2 must be in the form of $[2(u+p):u^2-1:u^2+2pu+1]$ and $[2q(v+pq):v^2-1:v^2+2pqv+1]$, respectively. By substituting these points into the equation of Q_3 we will obtain the following equations:

$$
\begin{aligned}
f_{13}(u) =&(a_2+a_3+a_5)u^4+(4a_3p+2a_4+2a_5p+2a_6)u^3\\
&+(4a_1-2a_2+a_3(4p^2+2)+2a_4p+6a_6p)u^2\\
&+(8a_1p+4a_3p-2a_4-2a5p+2a_6(2p^2+1))u\\
&+4a_1p^2+a_2+a_3-2a_4p-a_5+2a_6p=0
\end{aligned}
$$

and

$$
\begin{aligned}
f_{23}(v) =&(a_2+a_3+a_5)v^4+(4a_3pq+2a_4q+2a_5pq+2a_6q)v^3\\
&+(4a_1q^2-2a_2+a_3(4p^2q^2+2)+2a_4pq^2+6a_6pq^2)v^2\\
&+(8a_1pq^3+4a_3pq-2a_4q-2a5pq+2a_6q(2p^2q^2+1))v\\
&+4a_1p^2q^4+a_2+a_3-2a_4pq^2-a_5+2a_6pq^2=0
\end{aligned}
$$

By the intersection behavior of Q_3 with Q_1 and Q_2, both $f_{13}(u)$ and $f_{23}(v)$ must be fourth power of some linear polynomials. Assume $f_{13}(u)=A(u-\lambda)^4$ and $f_{23}(v)=$

$B(v-\mu)^4$ for some non-zero constants A and B. Comparing the coefficients of two polynomials for $f_{13}(u)$ and also for $f_{23}(v)$ term by term we get

$$A = a_2 + a_3 + a_5 \tag{4.3.11}$$
$$-4A\lambda = 4a_3 p + 2a_4 + 2a_5 p + 2a_6 \tag{4.3.12}$$
$$6A\lambda^2 = 4a_1 q^2 - 2a_2 + a_3(4p^2q^2+2) + 2a_4 pq^2 + 6a_6 pq^2 \tag{4.3.13}$$
$$-4A\lambda^3 = 8a_1 pq^3 + 4a_3 pq - 2a_4 q - 2a5pq + 2a_6 q(2p^2q^2+1) \tag{4.3.14}$$
$$A\lambda^4 = 4a_1 p^2 q^4 + a_2 + a_3 - 2a_4 pq^2 - a_5 + 2a_6 pq^2 \tag{4.3.15}$$

and

$$B = a_2 + a_3 + a_5 \tag{4.3.16}$$
$$-4B\mu = 4a_3 pq + 2a_4 q + 2a_5 pq + 2a_6 q \tag{4.3.17}$$
$$6B\mu^2 = 4a_1 q^2 - 2a_2 + a_3(4p^2q^2+2) + 2a_4 pq^2 + 6a_6 pq^2 \tag{4.3.18}$$
$$-4B\mu^3 = 8a_1 pq^3 + 4a_3 pq - 2a_4 q - 2a5pq + 2a_6 q(2p^2q^2+1) \tag{4.3.19}$$
$$B\mu^4 = 4a_1 p^2 q^4 + a_2 + a_3 - 2a_4 pq^2 - a_5 + 2a_6 pq^2 \tag{4.3.20}$$

It is clear from the equations (4.3.11) and (4.3.16) that $A = B$, and from the equations (4.3.12) and (4.3.17) that $\mu = \lambda q$. Similary we obtain $a_2 = a_3$ by comparing the equations (4.3.13) and (4.3.18), $(2a_3 - a_5)p + (a_6 - a_4) = 0$ by comparing the equations (4.3.14) and (4.3.19), $(a_2 + a_3 - a_5)(q^2+1) + 2pq^2(a_6 - a_4) = 0$ by comparing the equations (4.3.15) and (4.3.20). If $1 + q^2 \neq 2p^2q^2$, then $a_5 = 2a_2$ and $a_6 = a_4$. Hence we get $a_2 = a_3 = \frac{A}{4}$ and $a_5 = \frac{A}{2}$ by (4.3.11), $2a_4 = -A(p+2\lambda)$ by (4.3.12), $(4a_1 + 8a_4 p) = A(6\lambda^2 - p^2)$ by (4.3.13), $8a_1 p + 4a_4 p^2 = -4A\lambda^3$ by (4.3.14) and $4p^2 a_1 = A\lambda^4$ by (4.3.15). Then, either $p = \lambda = a_1 = a_4 = a_6 = 0$, $4a_2 = 4a_3 = 2a_5 = A$ or $\lambda = -p \neq 0$, $a_1 = \frac{Ap^2}{4}$, $a_2 = a_3 = \frac{A}{4}$, $a_4 = a_6 = -\frac{Ap}{2}$ and $a_5 = \frac{A}{2}$. The last solution is also true when $1 + q^2 = 2p^2q^2$. In all cases, the quadric Q_3 will be degenerate. So, such configuration of three on degenerate quadrics can not be realized. □

4.3.2 Two Quadrics with a Tacnode

Proposition 4.3.4. *Any configuration of two quadrics with a tacnode, i.e with graph* $\bullet\!\!-\!\!\!-\!\!\!-\!\!\bullet$*, are projectively equivalent to the quadrics*

$$\begin{aligned} Q_1 &: Y^2+Z^2-2XY=0, \\ Q_2 &: \alpha Y^2+\beta Z^2+2XY=0, \end{aligned} \tag{4.3.21}$$

where $\alpha,\beta \in \mathbb{C}\setminus\{-1\}$*,* $\beta \neq 0$*. In addition,* $[Q_1/Q_2] = -\frac{1}{\beta}$ *is the Naruki invariant.*

Proof. Projective transformations allows us to choose the coordinates such that Q_1 : $Y^2+Z^2-2XY=0$, Q_2 is tangent to Q_1 at $[1:0:0]$ and also one of the coefficient of the equation for Q_2 is fixed, for simplicity choose the coefficient of YZ as zero. Say $Q_2:\ a_1X^2+a_2Y^2+a_3Z^2+a_4XY+a_5XZ=0$. The condition $[1:0:0]\in Q_2$ implies $a_1=0$. In addition, the tangency condition at $[1:0:0]$ implies $a_4\neq 0$, and $a_5=0$. Then $Q_2:\ a_2Y^2+a_3Z^2+a_4XY=0$. Note that Q_2 is non-degenerate iff and only if $a_3\neq 0$. Dividing by $\frac{a_4}{2}$ each side of the equation of Q_2 and setting $\alpha := \frac{2a_2}{a_4}$ and $\beta := \frac{2a_3}{a_4}$ we obtain $Q_2:\ \alpha Y^2+\beta Z^2+2XY=0$, where $\alpha,\beta\in\mathbb{C}$, $\beta\neq 0$.

On the other hand, $\{[s^2+t^2 : 2st : 2s^2] \mid [s,t]\in\mathbb{CP}^1\}$ is a parametrization of Q_1. By substituting this parametrization into the equation of Q_2, we get the homogeneous equation

$$4t^2((1+\beta)s^2+(1+\alpha)t^2)=0.$$

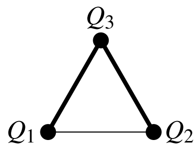
So, the configuration of the quadrics Q_1 and Q_2 given by the equations above has only one tacnode if and only if $\beta\neq -1$ and $\alpha\neq -1$. Otherwise, either quadrics have two tacnode when $\alpha=-1$ and $\beta\neq -1$, or a fourth order contact at $[1:0:0]$ when $\beta=-1$ and $\alpha\neq -1$, or they will coincide when $\alpha=\beta=-1$. In addition, Q_2 is non-degenerate if $\beta\neq 0$.

Last, the cubic equation (4.3.2) for these quadrics Q_1 and Q_2 have simple root $t' = -\frac{1}{beta}$ and double root $t''=1$. Hence the Naruki invariant is $[Q_1/Q_2] = -\frac{1}{\beta}$. $\square$

Remark 4.3.5. The quadrics given by the equations in (4.3.21) has the following parametrizations:

$$\begin{aligned} Q_1 &= \{[s^2+t^2 : 2st : 2s^2] \mid [s:t] \in \mathbb{CP}^1\} && (4.3.22) \\ Q_2 &= \{[\alpha t^2+\beta s^2 : -2t^2 : -2st] \mid [s:t] \in \mathbb{CP}^1\}. && (4.3.23) \end{aligned}$$

Proposition 4.3.6. *Any three quadrics with graph*



are projectively equivalent to the quadrics

$$\begin{aligned} &Q_1 : Y^2+Z^2-2XY = 0, \\ &Q_2 : Y^2+Z^2+2XY = 0, \qquad (4.3.24) \\ &Q_3 : 4X^2-Y^2-2Z^2 = 0. \end{aligned}$$

Proof. Let Q_1 and Q_2 be as in Proposition 4.3.4, and $Q_3 : a_1X^2+a_2Y^2+a_3Z^2+a_4XY+a_5YZ+a_6XZ = 0$. Then by the parametrizations of Q_1 and Q_2, we know that the contact points of Q_3 with Q_1 and Q_2 must be in the form of $[u^2+1 : 2 : 2u]$ and $[\alpha+\beta v^2 : -2 : -2v]$, respectively. By substituting these points into the equation of Q_3 we obtain

$$\begin{aligned} f_{13}(u) =& a_1u^4+2a_6u^3+2(a_1+2a_3+a_4)u^2+2(2a_5+a_6)u+(a_1+4a_2+2a_4) = 0 \\ f_{23}(v) =& a_1\beta^2v^4-2a_5\beta v^3+2(a_1\alpha\beta+2a_3-a_4\beta) \\ &+2(2a_5-a_6\alpha)v+(a_1\alpha^2+4a_2-2a_4\alpha) = 0 \end{aligned}$$

Notice that Q_3 has a contact of order 4 with Q_1 and Q_2 if $f_{13}(u) = a_1(u+\frac{a_6}{2a_1})^4$, $f_{22}(v) = a_1\beta^2(v-\frac{a_5}{2a_1\beta})^4$. Note that a_1 must be non-zero. By rescalling the equations of quadrics we may assume $a_1 = 1$. Comparing the coefficients of two equations for

$f_{13}(u)$ and $f_{23}(v)$, we get the following equations:

$$\begin{aligned}
4+4(2a_3+a_4)-3a_6^2 &= 0,\\
4(2a_5+a_6)-a_6^3 &= 0,\\
16+32(2a_2+a_4)-a_6^4 &= 0,\\
4\alpha\beta+4(2a_3-a_4\beta)-3a_6^2 &= 0,\\
4\beta(2a_5-a_6\alpha)+a_6^3 &= 0,\\
16\alpha^2\beta^2+32a_1^3(2a_2-a_4\alpha)-a_6^4 &= 0.
\end{aligned}$$

The first four of them give $a_2=\frac{a_6^4}{64}-\frac{2\alpha\beta+\beta-1}{4(\beta+1)}$, $a_3=\frac{3a_6^2}{8}-\frac{\beta(\alpha+1)}{2(\beta+1)}$, $a_4=\frac{\alpha\beta-1}{\beta+1}$, $a_5=\frac{a_6(a_6^2-4)}{8}$. Substituting these solutions into the last two equations we get

$$a_6(a_6^2(\beta+1)-4\beta(\alpha+1))=0 \quad \text{and} \quad (\beta-1)(a_6^4(\beta+1)^2-16\beta^2(\alpha+1)^2)=0.$$

These equations are valid if either $a_6=0$ and $\beta=1$, or $a_6^2=\frac{4\beta(\alpha+1)}{\beta+1}$. In case $a_6^2=\frac{4\beta(\alpha+1)}{\beta+1}$, the quadric Q_3 will be degenerate, so we have only the case $a_6=0$ and $\beta=1$ for which $a_2=-\frac{\alpha}{4}$, $a_3=-\frac{\alpha+1}{4}$, $a_4=\frac{\alpha-1}{2}$ and $a_5=0$. Hence $Q_3:\,4X^2-\alpha Y^2-(\alpha+1)Z^2+2(\alpha-1)XY=0$ and it is non-degenerate if $\alpha\in\mathbb{C}\setminus\{-1,3\mp2\sqrt{2}\}$. Notice that, such quadrics are projectively equivalent to $Q_1:Y^2+Z^2-2XY=0$, $Q_2:Y^2+Z^2+2XY=0$ and $Q_3:4X^2-Y^2-2Z^2=0$ via $[X:Y:Z]\mapsto[X+\frac{\alpha-1}{4}Y:\frac{\alpha+1}{2}Y:\sqrt{\frac{\alpha+1}{2}}Z]$. $\square$

4.3.3 *Three Quadrics with Six Tacnode*

First, let us introduce the following lemma, which is useful to determine when two quadrics are tangent to each other at one or two points, or to construct quadrics tangent to given ones.

Lemma 4.3.7 (Megyesi (2000), Lemma 2)**.** *Let Q_i, $i = 1,2$, be two quadrics given by the homogeneous ternary quadratic equations $F_i := F_i(X,Y,Z) = 0$, $i = 1,2$. If Q is a quadric which is tangent to both Q_1 and Q_2 at two points, then its equation can be written in the form*

$$F(X,Y,Z) = F_1 + L_1^2 = \lambda F_2 + L_2^2, \tag{4.3.25}$$

where $\lambda \in \mathbb{C}$, and $L_i := L_i(X,Y,Z) = 0$ define the line connecting the two points where Q is tangent to Q_i, $i = 1,2$. Furthermore, $F_1 - \lambda F_2 = L_2^2 - L_1^2 = 0$ defines a degenerate quadric, and L_1 and L_2 are linear combinations of the defining equations of the components of this quadric (if it is a double line, $L_1 = 0$ and $L_2 = 0$ define this line with the reduced structure). λ is uniquely determined by Q, while L_1 and L_2 determined up to sign.

Proof. Let $L_i = 0$ be the equations of the lines connecting the two points where the quadric Q is tangent to Q_i, $i = 1,2$. Then the quadric Q belongs to the families $\mathcal{P}_i :\ \lambda_i F_i + L_i^2 = 0$, $i = 1,2$. Therefore, for suitable λ_i's we have $F = \lambda_1 F_1 + L_1^2 = \lambda_2 F_2 + L_2^2 = 0$. Multiply F, L_1 and L_2 by suitable scalars so that the equation (4.3.25) holds. Then the quadric $F_1 - \lambda F_2 = L_2^2 - L_1^2 = 0$ belongs to one of the references of the pair $\{Q_1, Q_2\}$. Writing $(L_2 - L_1)(L_2 + L_1) = 0$ makes it obvious that L_1 and L_2 are linear combinations of the equations of the components of this degenerate quadric.

L_1 and L_2 are determined by Q up to multiplication by scalars. If they define the same line then $F_1 - \lambda F_2$ is a multiple of L_1^2 and λ is obviously unique. If they define different lines, let p be the point of intersection of these lines, then the degenerate quadric $F_1 - \lambda F_2 = 0$ is the union of two lines meeting at p. These lines must either pass through two points of intersection of Q_1 and Q_2, or be tangent to Q_1 and Q_2 at a point where the quadrics are tangent to each other, given p, this determines the two lines, hence λ uniquely. The uniqueness of λ implies that L_1 and L_2 are determined up to sign by Q. $\square$

By the above lemma, a quadric Q determines a singular member in the pencil spanned by Q_1 and Q_2, then there exist a corresponding partition of intersection points of Q_1 and Q_2 into two pairs. If the two points in a pair coincide then we take the line to be the common tangent line to Q_1 and Q_2 at that point. Following Naruki, this partition is called *reference* and said that Q belongs to a given reference.

Proposition 4.3.8 (Naruki (1983), Proposition 5.2)**.** *Suppose that Q, Q_1 and Q_2 are three quadrics such that Q is tangent to Q_1 and Q_2 at two points and that l is the reference of $\{Q_1, Q_2\}$ to which Q belongs. Then*

$$[Q_2/Q_1;\, l] = [Q_1/Q] \cdot [Q/Q_2]. \tag{4.3.26}$$

In particular if Q_1 and Q_2 are in a special position, then

$$[Q_2/Q_1]^2 = [Q_1/Q] \cdot [Q/Q_2]. \tag{4.3.27}$$

Now we apply Proposition 4.3.8 to the problem of obtaining necessary conditions for three or four quadrics to form some interesting configurations.

Proposition 4.3.9 (Naruki (1983), Proposition 6.1)**.** *If the quadrics Q_1, Q_2, Q_3 are pairwise tangent to each other at two distinct points, then*

$$[Q_3/Q_2] = [Q_2/Q_1] = [Q_1/Q_3] \tag{4.3.28}$$

Proof. Suppose that the quadrics Q_1, Q_2, Q_3 are pairwise tangent to each other at two distinct points. Then, by Proposition 4.3.8, we have $[Q_j/Q_i]^2 = [Q_i/Q_k] \cdot [Q_k/Q_j]$ for any permutation (i, j, k) of $(1,2,3)$. It follows that,

$$[Q_j/Q_i]^3 = [Q_j/Q_i] \cdot [Q_i/Q_k] \cdot [Q_k/Q_j] = [Q_k/Q_j]^3 = [Q_i/Q_k]^3. \tag{4.3.29}$$

Let ω be a third root of unity. Then by the equations (4.3.27) and (4.3.29), we have

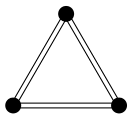
$$[Q_3/Q_2] = \omega[Q_1/Q_3] = \omega^2[Q_2/Q_1].$$

Therefore, $\omega = [Q_2/Q_1][Q_3/Q_1]$ and $\omega^2 = [Q_3/Q_2][Q_1/Q_2]$. On the other hand,

$$\begin{aligned}
\omega^2 &= [Q_3/Q_2][Q_1/Q_2] \\
&= [Q_3/Q_2][Q_1/Q_3][Q_3/Q_1][Q_1/Q_2] \\
&= [Q_2/Q_1]^2[Q_3/Q_1][Q_1/Q_2] \\
&= [Q_2/Q_1][Q_3/Q_1] = \omega,
\end{aligned}$$

i.e., $\omega = 1$. Thus $[Q_3/Q_2] = [Q_1/Q_3] = [Q_2/Q_1]$. □

Proposition 4.3.10 (Megyesi (2000), Proposition 4.)**.** *Any configuration of three quadrics with graph*



are projectively equivalent to the quadrics

$$\begin{aligned}
&Q_1 : X^2 + Y^2 - Z^2 = 0 \\
&Q_2 : \frac{1}{q^2}X^2 + Y^2 - Z^2 = 0 \qquad (4.3.30) \\
&Q_3 : X^2 + Y^2 - q^2Z^2 = 0,
\end{aligned}$$

where $q \in \mathbb{C} \setminus \{0, \mp 1\}$*. In addition* $[Q_1/Q_2] = [Q_2/Q_3] = [Q_3/Q_1] = q^2$.

Proof. By the section 4.3.1, it may be assumed that two of the quadrics Q_1 and Q_2 are given by the equations (4.3.9). Let L_1 and L_2 be as in Lemma 4.3.7. Since singular members of family generated by Q_1 and Q_2 are $(1 - \frac{1}{q^2})X^2 = 0$ and $(pX + Y - Z)(pX + Y + Z) = 0$, then for suitable choice of the constant α we may assume that $L_2 :\ \alpha(pX + Y) = 0$, then $L_1 :\ \mp\alpha Z = 0$. Then by Lemma 4.3.7, $\alpha^2 = 1 - \frac{1}{q^2}$, the equation of the quadric Q_3 is $X^2 + Y^2 - q^2Z^2 + 2pXY = 0$, and the quadric Q_3 tangents to quadrics Q_1 and Q_2 at the intersection points $L_1 \cap Q_1 = \{[p \mp \sqrt{p^2 - 1} : 1 : 0]\}$ and $L_2 \cap Q_2 = \{[1 : -p : \mp\frac{\sqrt{1-p^2q^2}}{q}]\}$,respectively. The conditions $[p \mp \sqrt{p^2 - 1} : 1 : 0], [1 : -p : \mp\frac{\sqrt{1-p^2q^2}}{q}] \in Q_3$ together with the conditions of Proposition 4.3.1 imply that p must be zero. In addition, the Propositions 4.3.1

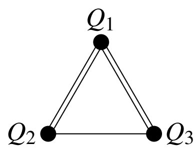
and 4.3.9 imply that $[Q_1/Q_2] = [Q_2/Q_3] = [Q_3/Q_1] = q^2$. $\square$

The quadrics in (4.3.30) have parametrizations:

$$\begin{aligned} Q_1 &= \left\{[2st : t^2 - s^2 : t^2 + s^2] \mid [s:t] \in \mathbb{CP}^1\right\}, \\ Q_2 &= \left\{[2qst : t^2 - s^2 : t^2 + s^2] \mid [s:t] \in \mathbb{CP}^1\right\}, \\ Q_3 &= \left\{[2qst : qt^2 - qs^2 : t^2 + s^2] \mid [s:t] \in \mathbb{CP}^1\right\}. \end{aligned} \tag{4.3.31}$$

4.3.4 Three Quadrics with Five Tacnodes

Proposition 4.3.11 (Megyesi (2000), Proposition 5)**.** *Any configuration of three quadrics with graph*



are projectively equivalent to the three quadrics

$$\begin{aligned} &Q_1 : X^2 + Y^2 - Z^2 = 0, \\ &Q_2 : p^2X^2 + (p^2+1)Y^2 - 2pYZ = 0, \\ &Q_3 : q^2X^2 + (q^2+1)Y^2 - 2qYZ = 0, \end{aligned} \tag{4.3.32}$$

where $p, q \in \mathbb{C} \setminus \{0, \mp 1\}$, $p \neq q$ and $pq \neq 1$; and $[Q_3/Q_2] = \frac{q}{p}$, $[Q_1/Q_2] = p^2$, $[Q_1/Q_3] = q^2$ are the Naruki invariants.

Proof. Let us assume that the quadrics Q_2 and Q_3 are tangent to Q_1 at two points, and to each other at one point. Projective transformations allow us to choose the homogeneous coordinates so that $Q_1 : \; X^2 + Y^2 - Z^2 = 0$ and that $[0:0:1]$ is the tangent point of Q_2 and Q_3 and their common tangent line is the line $Y = 0$.

Let the equation of Q_2 be $a_1X^2 + a_2Y^2 + a_3Z^2 + a_4XY + a_5YZ + a_6ZX = 0$. The conditions that $[0:0:1] \in Q_2$ and the tangent line to Q_2 at $[0:0:1]$ is the line $Y = 0$

imply that $a_3 = a_6 = 0$ and $a_5 \neq 0$. By substituting the standard parametrization (4.3.31) of Q_1 into the equation of Q_2, we obtain the quartic equation

$$f(t,s) = (a_2 - a_5)t^4 - 2a_4st^3 + (4a_1 - 2a_2)s^2t^2 + 2a_4s^3t + (a_2 + a_5)s^4 = 0.$$

Q_2 is tangent to Q_1 at two point if and only if $f(t,s)$ is a square of a quadric polynomial. Assume $f(t,s) = \sum_{k=0}^{4} f_k s^k t^{4-k}$ is a square, then

$$f_4f_1^2 - f_0f_3^2 = 4a_4^2(a_2 - a_5) + 4a_4^2(a_2 + a_5) = -8a_4^2a_5 = 0,$$

which gives either $a_4 = 0$ or $a_5 = 0$. But, Q_2 is degenerate if $a_5 = 0$, and also we know from tangency condition that $a_5 \neq 0$. Hence $a_4 = 0$. Therefore if Q_2 tangent to Q_1 at two points, these must be in the form of $[\mp\sqrt{1-p^2} : p : 1]$. Because, if $[\mp i : 1 : 0]$ were tangency points, then by comparing tangent lines at these points we would get $a_5 = 0$ and the quadric Q_2 would be degenerate. So, assume Q_2 is tangent to Q_1 at two points $[\mp\sqrt{1-p^2} : p : 1]$, where $p \neq 0, \mp 1$. Because, the points $[\mp 1 : 0 : 1]$ would be the tangency points of Q_1 and Q_2 if $p = 0$, and $[\mp 1 : 0 : 1] \in Q_2$ would imply that $a_1 = 0$ which means Q_2 is degenerate. In addition, $[0 : \mp 1 : 1]$ would be the tangency points of Q_1 and Q_2 if $p = \mp 1$, and $[0 : \mp 1 : 1] \in Q_2$ would imply that $a_2 + a_5 = 0$. Moreover, by comparing the tangent lines of Q_1 and Q_2 at these points, we would get $a_2 = a_5 = 0$, i.e., Q_2 is degenerate. So these are not the cases.

In addition to the condition $[\mp\sqrt{1-p^2} : p : 1] \in Q_2$, by comparing the equations of tangent lines to Q_1 and Q_2 at these points, we obtain $a_1 = p^2$, $a_2 = p^2 + 1$ and $a_5 = -2p$. So, the equation of Q_2 must be in the form of $p^2X^2 + (p^2+1)Y^2 - 2pYZ = 0$, where $p \in \mathbb{C} \setminus \{0, \mp 1\}$. Similarly, the quadric Q_3 is given by the equation $q^2X^2 + (q^2+1)Y^2 - 2qYZ = 0$ for some $q \in \mathbb{C} \setminus \{0, \mp 1\}$. We have the conditions $p \neq q$ since the quadrics are distinct, and $pq \neq 1$ since the quadrics Q_2 and Q_3 have only one tacnode, which is at $[0 : 0 : 1]$.

Let M_i be the symmetric matrix corresponding to Q_i, $i = 1,2,3$. Then the cubic equations $\lambda M_2 + M_3 = 0$, $\mu M_1 + M_2 = 0$ and $\eta M_3 + M_1 = 0$ have simple roots $\lambda' = -\frac{q^2}{p^2}$, $\mu' = -1$, $\eta' = -1$ and double roots $\lambda'' = -\frac{q}{p}$, $\mu'' = -p^2$, $\eta'' = -\frac{1}{q^2}$. So, $[Q_3/Q_2] = \frac{q}{p}$, $[Q_2/Q_1] = \frac{1}{p^2}$, $[Q_1/Q_3] = q^2$ and Proposition 4.3.9 is verified.

Last, $pq \neq 1$. If pq was equal to 1, then the singular member $p^2X^2 + (pq-1)Y^2 = 0$ of the family $\lambda Q_2 + Q_3$ corresponding to double root $\lambda'' = -\frac{q}{p}$ would be double line and so the quadrics Q_2 and Q_3 would be tangent at two points, but this is not the case. □

4.3.5 Four Quadrics with Twelve or Eleven Tacnodes

As a corollary of the Proposition 4.3.9 we have the following:

Proposition 4.3.12 (Naruki (1983), Proposition 6.1'). *Suppose that four quadrics Q_1, Q_2, Q_3 and Q_4 are pairwise tangent to each other at two distinct points. Then,*

$$[Q_i/Q_j] = -1 \quad \textit{for } 1 \le i \neq j \le 4. \tag{4.3.33}$$

Proof. By Proposition 4.3.9 we know that for any permutation (i,j,k), we have $[Q_k/Q_j] = [Q_j/Q_i] = [Q_i/Q_k]$. Therefore, we get $[Q_i/Q_j] = [Q_j/Q_i]$ for $1 \le i \neq j \le 4$. Moreover, the property (4.3.8) implies that $[Q_i/Q_j]$ is either 1 or -1. If it was 1 then the double and single roots of the equation (4.3.2) would coincide, and contact order at tangency point would be at least 3. But this contradicts the fact that quadrics are tangent to each other at two distinct points. □

Without giving a proof, Naruki (1983) pointed out that such four quadrics are given by the four choices of the signs in

$$\mp X^2 \mp Y^2 \mp Z^2 = 0 \tag{4.3.34}$$

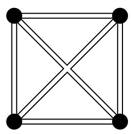
and they are projectively unique. Before proving this fact, let us remember the following fact of projective transformations acting on quadrics:

Consider the subgroup

$$G=\left\{M(\varphi,\theta)=\begin{bmatrix}1&0&0\\0&\varphi&\theta\\0&\theta&\varphi\end{bmatrix}\mid \det M(\varphi,\theta)=\varphi^2-\theta^2=1\right\}\cong\mathbb{C}^* \qquad (4.3.35)$$

of $\mathrm{PGL}(3,\mathbb{C})$. Any element $M(\varphi,\theta)$ of G fixes the quadric $Q_1:\ X^2+Y^2-Z^2=0$ and the points $[0:\mp1:1]$. Note that the quadrics in (4.3.30) are invariant under the action $[X:Y:Z]\to[X:-Y:-Z]$, in other words $M(-1,0)\in G$ act trivially on them. Let H be the quotient of G by the two element subgroup generated by $M(-1,0)$. Then H act on quadrics that are tangent to both Q_1 and Q_2 at two points. Moreover, any two quadrics both tangent to Q_1 and Q_2 are images of each other under the action of H.

Proposition 4.3.13. *The graph*



can not be realized but it is complex realizable and projectively unique equations for these quadrics are $\mp X^2\mp Y^2\mp Z^2=0$.

Proof. By the Proposition 4.3.10 we may first assume that three quadrics are in the form $Q_1:\ X^2+Y^2-Z^2=0$, $Q_2:\ \frac{1}{q^2}X^2+Y^2-Z^2=0$ and $Q_3:\ X^2+Y^2-q^2Z^2=0$ for some $q\in\mathbb{C}\setminus\{0,\mp1\}$. Q_4 must be the image of Q_3 under the action of some $M(\varphi,\theta)\in H$, so its equation is

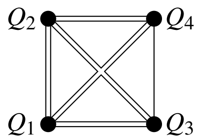
$$X^2+(\varphi Y-\theta Z)^2-q^2(-\theta Y+\varphi Z)^2=0 \qquad (4.3.36)$$

. On the other hand, Lemma 4.3.7 implies that the singular members of the family generated by Q_2 and Q_3 are $(1-\frac{1}{q^2})Y=0$ and $(\frac{1}{q}X-Z)(\frac{1}{q}X+Z)=0$. Assume

$L_2 : X = 0$, and $L_1 : Z = 0$. Then Q_4 must be tangent to Q_2 at $Q_2 \cap L_1 = \{[\mp iq : 1 : 0]\}$ and to Q_3 at $Q_3 \cap L_2 = \{[0 : \mp q : 1]\}$. These conditions together with the condition $\varphi^2 - \theta^2 = 1$ implies that $(\varphi, \theta) = (0, \mp i)$ and $q^4 = 1$. Since $q \neq \mp 1$, then $q^2 = -1$. This also verifies the necessary condition $[Q_i/Q_j] = -1$ in Proposition 4.3.12.

Note that this configuration is projectively rigid since it does not depend on the choice of signs for θ. In addition, this configuration contains six imaginary intersection points and the imaginary smooth quadric $X^2 + Y^2 + Z^2 = 0$, so it can not be realized in $\mathbb{RP}^2$. □

Proposition 4.3.14. *Any configuration of four quadrics with graph*



is projectively equivalent to the quadrics

$$\begin{aligned}
&Q_1 : X^2 + Y^2 - Z^2 = 0,\\
&Q_2 : \frac{1}{q^2}X^2 + Y^2 - Z^2 = 0,\\
&Q_3 : X^2 + Y^2 - q^2 Z^2 = 0,\\
&Q_4 :\ (1 - q^2)X^2 + (3q^2 + 1)Y^2 - q^2(q^2 + 3)Z^2 - 4q(q^2 + 1)YZ = 0.
\end{aligned} \tag{4.3.37}$$

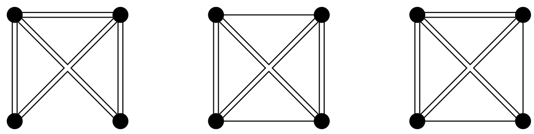
for some $q \in \mathbb{C} \setminus \{0, \mp 1, \mp i\}$. *Alternatively, one can take last two quadrics as* $X^2 + (q^2 + 1)Y^2 \mp 2qYZ = 0$.

Proof. Assume that the quadrics Q_1, Q_2, Q_3 are given as in Proposition 4.3.10, and we use the idea of the proof of Proposition 4.3.13 and assume that the quadrics Q_3 and Q_4 have only one tacnode. Then Q_4 must be the image of Q_3, given in (4.3.36), under the action of some $M(\varphi, \theta) \in H$ with the additional condition $q^2 \neq -1$. Then one can get Q_4 is tangent to Q_3 if and only if $M(\varphi, \theta) = M(\frac{q^2+1}{q^2-1}, \mp \frac{2q}{q^2-1})$. If we choose the sign "+", the contact point will be $[0 : \mp q : 1]$. In addition, if the element

$M(\frac{1}{\sqrt{1-q^2}}, -\frac{q}{\sqrt{1-q^2}})$, which is a square root of $M^{-1}(\frac{q^2+1}{q^2-1}, \mp\frac{2q}{q^2-1})$, acts on $\mathbb{CP}^2$ then it transform the quadrics Q_3 and Q_4 to $X^2+(q^2+1)y^2 \mp 2qYZ = 0$ while fixing Q_1 and Q_2. □

4.3.6 *Four Quadrics with Ten Tacnodes*

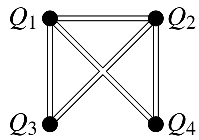
There are three possible graphs for the configuration of four quadrics with ten tacnodes, and these are the graphs in Figure 4.3.



(a) (b) (c)

Figure 4.3 Three graphs on 4 vertices and 10 edges.

Proposition 4.3.15 (Megyesi (2000), Proposition 9)**.** *Any configuration of four quadrics with graph*



is projectively equivalent to the quadrics

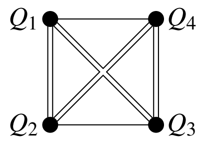
$$\begin{aligned} Q_1 &: X^2+Y^2-Z^2 = 0, \\ Q_2 &: \frac{1}{q^2}X^2+Y^2-Z^2 = 0, \\ Q_3, Q_4 &: X^2+(\varphi^2-q^2\theta^2)Y^2+(\theta^2-q^2\varphi^2)Z^2 \mp 2\varphi\theta(q^2-1)YZ = 0. \end{aligned} \tag{4.3.38}$$

for some $q \in \mathbb{C}\setminus\{0,\mp 1\}$*,* $\varphi,\theta \in \mathbb{C}$*,* $\varphi^2-\theta^2 = 1$*.*

Proof. Assume that the quadrics are labeled as in the graph, and Q_1, Q_2 are the quadrics in Proposition 4.3.1. As in the proof of Propositions 4.3.13 and 4.3.14, Q_3 and Q_4 are images of $X^2+Y^2-q^2Z^2 = 0$ under suitable elements $M(\varphi_1,\theta_1)$ and

$M(\varphi_2, \theta_2)$ of H. Acting on them by a square root of $M(\varphi_1, \theta_1) \cdot M^{-1}(\varphi_2, \theta_2)$, we can transport them into such a position that they are the images of $X^2 + Y^2 - r^2 Z^2 = 0$ under $M(\varphi, \theta)$ and $M^{-1}(\varphi, \theta)$ and then their equations will be as stated. $\square$

Proposition 4.3.16 (Megyesi (2000), Proposition 12)**.** *Any configuration of four quadrics with graph*



is projectively equivalent to the quadrics

$$
\begin{aligned}
&Q_1 : X^2 + Y^2 - Z^2 = 0, \\
&Q_2 : p^2 X^2 + (p^2 + 1) Y^2 - 2pYZ, \\
&Q_3 : q^2 X^2 + (q^2 + 1) Y^2 - 2qYZ, \\
&Q_4 : (2pq - p - q)^2 X^2 + [(p+q)(4pq - 3p - 3q + 4) - 4pq] Y^2 \\
&\qquad - (p-q)^2 Z^2 - 4(p-1)(q-1)(p+q) YZ = 0.
\end{aligned} \tag{4.3.39}
$$

for some $p, q \in \mathbb{C} \setminus \{0, \mp 1\}$*,* $p \neq \mp q$*,* $pq \neq 1$*,* $p + q \neq 2$*,* $p + q \neq 2pq$*.*

Proof. Assume that the quadrics are labeled as in the graph, and Q_1, Q_2 and Q_3 are the quadrics in Proposition 4.3.11. Since Q_4 is tangent to Q_2 and Q_3 at two points, then its equation never contains the terms XY and XZ. Indeed, by Proposition 4.3.11 we know that for any quadric Q_4 which is tangent to both Q_2 and Q_3 at two points, the triple (Q_4, Q_2, Q_3) is projectively equivalent to the triple (Q_1, Q_2, Q_3). Since, the quadrics Q_1, Q_2, Q_3 remain fixed under the involution $[X : Y : Z] \to [-X : Y : Z]$, then Q_4 must be fixed under this involution. Q_4 must be tangent to Q_1 at one of the points $[0 : \mp 1 : 1]$, by changing the sign of Y (and of p, q) we may assume that it is $[0 : 1 : 1]$. So the equation of Q_4 must be of the form $X^2 + aY^2 + bZ^2 - (a+b)YZ = 0$. By substituting a parametrization $[2pst : 2pt^2 : p^2 s^2 + (1 + p^2) t^2]$ of Q_2 into the equation of Q_4 we get

$$\begin{aligned} f(t,s) &= f_4t^4 + f_2s^2t^2 + f_0s^4 \\ &= [4ap^2 + b(1+p^2)^2 - 2(a+b)p(1+p^2)]t^4 \\ &\quad +[4p^2 + 2bp^2(1+p^2) - 2(a+b)p^3]s^2t^2 + bp^4s^4. \end{aligned}$$

Since the polynomial $f(t,s)$ is the square of a reducible polynomial, then $f_2^2 - 4f_0f_4 = 0$. Therefore we have the condition $4p^4[((a-b)^2+4b)p^2 - 4(a+b)p + 4(b+1)] = 0$. By the same argument with Q_3 instead of Q_2, we obtain the same equation with q instead of p. The equation

$$\left[(a-b)^2 + 4b\right]u^2 - 4(a+b)u + 4(b+1) = 0$$

has two distinct non zero roots $u = p$ and $u = q$, if $b \neq -1$, $(a-b) \neq 2$ and $(a-b)^2 + 4b \neq 0$. By taking suitable linear combinations of the relations between the roots and coefficients of the quadric equation, we obtain

$$(2pq - p - q)^2 b + (p-q)^2 = 0 \quad \text{and} \quad (pq - p - q)b + pqa - p - q = 0.$$

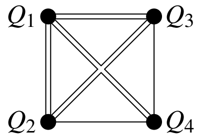
Hence, we have the solutions

$$b = -\frac{(p-q)^2}{(2pq-p-q)^2} \quad \text{and} \quad a = \frac{(p+q)(4pq-3p-3q+4) - 4pq}{(2pq-p-q)^2},$$

if $p+q \neq 2pq$.

In addition to conditions on p, q imposed in Proposition 4.3.11, we must also require that $p+q \neq 2pq$ to avoid division by zero, $p+q \neq 2$ and $p \neq \mp q$ to ensure that Q_4 is not singular and $Q_1 \neq Q_4$. Hence the equation of Q_4 is found as stated. $\square$

Proposition 4.3.17 (Megyesi (2000), Proposition 9)**.** *Any configuration of four quadrics with graph*



is projectively equivalent to quadrics given by the equations

$$\begin{aligned}
&Q_1 : X^2+Y^2-Z^2=0,\\
&Q_2 : \frac{1}{\rho^4}X^2+Y^2-Z^2=0,\\
&Q_3 : X^2+Y^2-\rho^4Z^2=0,\\
&Q_4 : X^2+Y^2-Z^2+\frac{[(1-\rho^2\sigma^2)X+2\rho\sigma Y+(\rho^2+\sigma^2)Z]^2}{\sigma^4(1-\rho^4)}=0,
\end{aligned} \tag{4.3.40}$$

for some $\rho,\sigma\in\mathbb{C}\setminus\{0,\mp1,\mp i\}$, $\rho^4\neq\sigma^4$, *and* $\rho^4\sigma^4\neq 1$.

Proof. Assume that the quadrics Q_1, Q_2 and Q_3 are given as in Proposition 4.3.10, and Q_4 as in graph. Let $[Q_1/Q_4]=\tau^2$, then by Lemma 4.3.7 the equation of Q_4 can be written as

$$X^2+Y^2+Z^2+\frac{1-\tau^2}{\tau^2(\alpha^2+\beta^2-\gamma^2)}(\alpha X+\beta Y+\gamma Z)^2=0,$$

where $\alpha X+\beta Y+\gamma Z=0$ is the equation of the line L_4, which is the line connecting two tangency points of Q_1 and Q_4. α, β and γ are only determined up to scalars. From the condition that Q_4 is tangent to Q_2 and Q_3, we get two equations for α, β and γ. After discarding the solutions corresponding to the cases when Q_4 is tangent to Q_2 and Q_3 at two points or when it passes through the contact points of some of the other quadrics, the only solutions are $[\alpha:\beta:\gamma]=[(1-q\tau):\mp2\sqrt{q\tau}:\mp(q+\tau)]$ and $[\alpha:\beta:\gamma]=[(1+q\tau):\mp2i\sqrt{q\tau}:\mp(q-\tau)]$. These can all be obtained from one another by changing the sign of r, s or one of the coordinates. They can all be written in the form

$$[\alpha:\beta:\gamma]=[1-\rho^2\sigma^2:\mp2\rho\sigma:\mp(\rho^2+\sigma^2)],$$

where ρ and σ are suitable fourth roots of q^2 and τ^2, respectively. The pairs (ρ,σ) and $(-\rho,-\sigma)$ determines the same quadrics. The contact point of Q_2 and Q_4 is $[\rho^2(\rho^2\sigma^2-1) : -2\rho^3\sigma : \rho^2\sigma^2+1]$. If $q = \mp\tau$ or $q\tau = \mp 1$ then one of these contact point is the contact point of Q_1 with Q_2 or Q_3, which we have to exclude. Thus the equations of four quadrics are obtained as stated. $\square$

4.3.7 Five Quadrics with Seventeen Tacnodes

Proposition 4.3.13 implies that the complete graph K_4 with double edges can be complex realized and this configuration is unique up to projective equivalence, and these quadrics are given by the equations in (4.3.34). Concordantly, one can wonder that whether the complete graph K_5 with double edges can be (complex) realized or not. The answer is "No". Because, if there was such a quadric Q, then the configurations of quadrics Q_1,Q_2,Q_3,Q_4 and Q_1,Q_2,Q_3,Q would be projectively equivalent, which implies $Q_4 = Q$. Next question is "What is the maximum number of tacnodes $t(5)$ for configuration of five quadrics?". Normally, one can expect $t(5) = 5\cdot 4 = 20$. But this is false, since the complete graph K_5 with double edges can not be (complex) realized.

By considering the double cover of $\mathbb{CP}^2$ branched along the union of quadrics, and applying the Miyoka-Yau inequality to the double cover, Hirzebruch (1986) gave the inequality $t(n) \leq \frac{4}{9}n(n+3)$ for the number of tacnodes in configuration of n-quadrics. This inequality implies the Miyaoka-Yau bound for $t(5)$ is 17. Due to their combinatorics, the candidates for the configuration of five quadrics with $t(5) = 17$ are given by the graphs in Figure 4.4.

First, let us consider the graph in Figure 4.4a. Then we may assume that the quadrics Q_1,Q_2,Q_3,Q_4 are as in Proposition 4.3.13. Since these quadrics are projectively unique, the only quadric which is tangent to Q_1 and Q_2 at two points and also must be tangent to Q_3, Q_4. So, this graph is impossible.

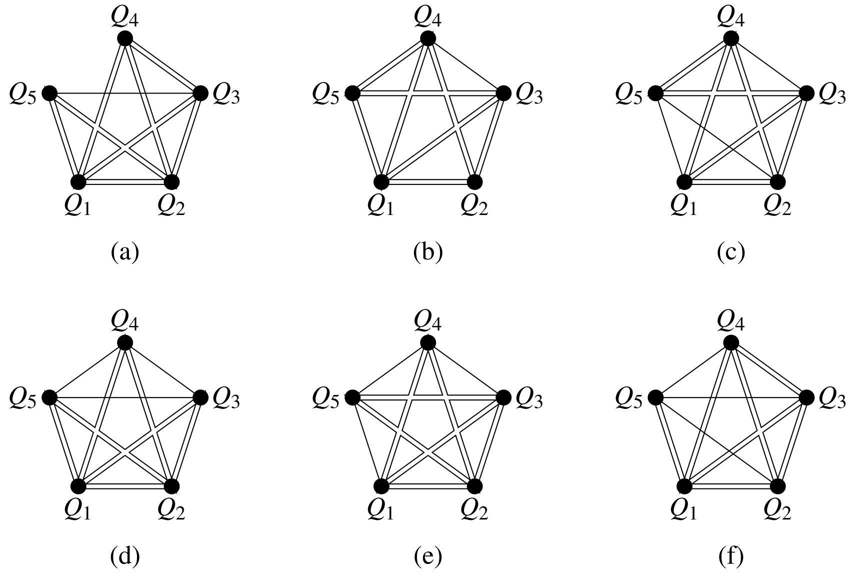


Figure 4.4 The six graph on 5 vertices and 17 edges.

Second, consider the graph in Figure 4.4b. Then by Proposition 4.3.14 we may assume that $Q_1 : X^2+Y^2-Z^2=0$, $Q_2 : \frac{1}{q^2}X^2+Y^2-Z^2=0$, $Q_3 : X^2+(q^2+1)Y^2+2qYZ=0$ and $Q_4 : X^2+(q^2+1)Y^2-2qYZ=0$ for some $q \in \mathbb{C}\setminus\{0,\mp 1,\mp i\}$. Suppose that there is a quadric Q_5, which is in general position with Q_2, such that quadrics Q_1, Q_3, Q_4, Q_5 form a configuration of 11 tacnodes. The involutions $[X : Y : Z] \to [X : -Y : Z]$ and $[X : Y : Z] \to [X : Y : -Z]$, exchanges Q_3 and Q_4, Q_5 and Q_4 while fixing Q_1 and Q_3. Then $Q_5 = Q_3$, and hence, this graph is impossible. For the same reason, the graph in Figure 4.4c is also impossible.

Next, consider the graph in Figure 4.4d. By Proposition 4.3.10, we may assume that $Q_1 : X^2+Y^2-Z^2=0$, $Q_2 : \frac{1}{q^2}X^2-Y^2+Z^2=0$, $Q_3 : X^2+Y^2-q^2Z^2=0$ for some $r \in \mathbb{C}\setminus\{0,\mp 1\}$. Then $Q_4 = M(Q_3)$ and $Q_4 = M^{-1}(Q_3)$, where $M = M(\frac{q^2+1}{q^2-1}, \frac{2q}{q^2-1}) \in H$. In general, two quadrics which are both tangent to Q_1 and Q_2 are tangent to each other if and only if one of them is the image of the other under M, so Q_4 and Q_5 are tangent to each other if and only if $M^3 = 1$, which happens if and only if $q^2 = -1/3$ or $q^2 = -3$. These are reciprocals of each other and give projectively equivalent configurations. If we take $q^2 = -1/3$, we obtain the quadrics $X^2+Y^2-Z^2=0$, $-3X^2+Y^2-Z^2=0$, $3X^2+3Y^2+Z^2=0$ and $3X^2-2Z^2\mp$

$i\sqrt{3}YZ = 0$. Note that this configuration is unique up to projective equivalence.

Fourth, consider the graph in Figure 4.4e. By Proposition 4.3.14 we may assume that $Q_1 : X^2+Y^2-Z^2 = 0$, $Q_2 : \frac{1}{q^2}X^2+Y^2-Z^2 = 0$, $Q_3 : X^2+(q^2+1)Y^2+2qYZ = 0$ and $Q_4 : X^2+(q^2+1)Y^2-2qYZ = 0$ for some $q \in \mathbb{C} \setminus \{0, \mp 1, \mp i\}$. By applying the argument of the proof of Proposition 4.3.14 with the roles of Q_1 and Q_3 reversed, Q_5 must be the image of Q_1 under the action of an element of the subgroup G' of $\mathrm{PGL}(3,\mathbb{C})$, fixing Q_2, Q_3 and the points $[\mp q : 0 : 1]$. The subgroup G' is the group

$$G = \left\{ N(\varphi,\theta) = \begin{bmatrix} \varphi & 0 & q^2\theta \\ 0 & 1 & 0 \\ \theta & 0 & \varphi \end{bmatrix} \mid \det N(\varphi,\theta) = \varphi^2 - q^2\theta^2 = 1 \right\} \cong \mathbb{C}^*$$

Q_1 and Q_5 must be tangent to each other at one of the points $[\mp 1 : 0 : 1]$, we may assume it is $[1 : 0 : 1]$, then Q_5 is the image of Q_1 under a group element which maps $[-1 : 0 : 1]$ to $[1 : 0 : 1]$, which is $N(0,-1)$. Hence the equation of Q_5 is $(q^2+3)X^2 + (q^2-1)Y^2 + (3q^2+1)Z^2 - 4(q^2+1)XZ = 0$. The discriminant expressing condition that Q_4 and Q_5 are tangent to each other, is $2^{18}(q^2+1)^6(q^2-1)^{10}q^2(q^4-6q^2+1)^2$. The only feasible solutions are the roots of $q^4-6q^2+1 = 0$, $q = \mp 1 \mp \sqrt{2}$, but then Q_2, Q_4 and Q_5 are tangent to each other at the same point, for example if $q = \sqrt{2}-1$, then this is the common point is $[\sqrt{2}-1 : 1\sqrt{2}]$. Thus, this graph is also impossible.

Finally, let us consider the graph in Figure 4.4f. By Proposition 4.3.13 we may assume that $Q_1 : X^2+Y^2+Z^2 = 0$, $Q_2 : X^2+Y^2-Z^2 = 0$, $Q_3 : X^2-Y^2+Z^2 = 0$ and $Q_4 : -X^2+Y^2+Z^2 = 0$. Let $\alpha X + \beta Y + \gamma Z = 0$ be the equation of the line connecting the tangency points of Q_1 and and Q_5. Then by Lemma 4.3.7, the equation of Q_5 is $\lambda(X^2+Y^2+Z^2) + (\alpha X + \beta Y + \gamma Z)^2 = 0$ for some suitable $\lambda \in \mathbb{C}$. By substituting parametrization $[s^2-t^2 : 2st : s^2+t^2]$ into the equation of Q_5, we obtain the equation

$$\begin{aligned} f(s,t) &= (2\lambda+(\gamma-\alpha)^2)t^4+4\beta(\gamma-\alpha)st^3+(4\lambda-2\alpha^2+4\beta^2+2\gamma^2)s^2t^2 \\ &\quad +4\beta(\gamma+\alpha)s^3t+(2\lambda+(\gamma+\alpha)^2)s^4=0. \end{aligned}$$

Q_1 and Q_5 are tangent to each other at two points if and only if $f(s,t)$ is a square of reducible polynomial. So, either $\alpha^2+\beta^2=0$ or $\lambda=-\frac{(\alpha^2+\beta^2+\gamma^2)}{2}\mp\gamma\sqrt{\alpha^2+\beta^2}$. But, if $\alpha^2+\beta^2=0$, then Q_5 passes through one of the contact points of Q_1 and Q_2, $[1:\mp i:0]$, so we must have the second possibility.

By doing the same calculations with Q_3 and Q_4, and comparing the expressions for λ, one can obtain $\alpha^2(\beta^2+\gamma^2)=\beta^2(\alpha^2+\gamma^2)=\gamma^2(\alpha^2+\beta^2)$. So, α,β,γ can only differ from each other by a sign. By changing the sign of some of the coordinates in a suitable way, we may assume that $\alpha=\beta=\gamma=1$. Then, $\lambda=-\frac{3}{2}+\mp\sqrt{2}$, and the equations for Q_5 are

$$(\mp 2\sqrt{2}-1)(X^2+Y^2+Z^2)+4(XY+YZ+ZX)=0. \qquad (4.3.41)$$

Let Q_5^+ and Q_5^- be the quadrics obtained by choosing "+" and "−" sign in (4.3.41), respectively. Two configurations of quadrics $Q_1,Q_2,Q_3,Q_4,Q_5^{\mp}$ are not projectively equivalent. Indeed, if there were such a projective transformation ϕ, then Q_1 would remain invariant, Q_3,Q_4,Q_5 might be permuted among each other and Q_5^+ must be mapped to Q_5^-. But, such a map ϕ only permutes X,Y,Z and leaves Q_5^+, Q_5^- invariant.

Theorem 4.3.18. *There exist exactly three configuration of conics of five quadrics with seventeen tacnodes up to projective equivalence. First configuration corre-*

sponds to graph in Figure 4.4d, and equations of quadrics are

$$\begin{aligned}
Q_1 &: X^2+Y^2-Z^2=0,\\
Q_2 &: -3X^2+Y^2-Z^2=0,\\
Q_3 &: 3X^2+3Y^2-Z^2=0, \qquad (4.3.42)\\
Q_4 &: 3X^2-2Z^2-i\sqrt{3}YZ=0,\\
Q_5 &: 3X^2-2Z^2+i\sqrt{3}YZ=0.
\end{aligned}$$

Last two configurations corresponds to graph in Figure 4.4d, and equations of quadrics are

$$\begin{aligned}
Q_1 &: X^2+Y^2+Z^2=0,\\
Q_2 &: X^2+Y^2-Z^2=0,\\
Q_3 &: X^2-Y^2+Z^2=0, \qquad (4.3.43)\\
Q_4 &: -X^2+Y^2+Z^2=0,\\
Q_5^{\mp} &: (\mp 2\sqrt{2}-1)(X^2+Y^2+Z^2)+4(XY+YZ+ZX)=0.
\end{aligned}$$

Question 4.3.19. Are the quintuplets (Q_1,Q_2,Q_3,Q_4,Q_5^-) and (Q_1,Q_2,Q_3,Q_4,Q_5^+) Zariski pairs?

4.3.8 Six Quadrics with Twenty Four Tacnodes

The inequality $t(n)\leq \frac{4}{9}n(n+3)$ implies that the maximum number of tacnodes for six quadrics may be 24. Suppose such configuration exist. Then, each vertices of possible graphs must have degree eight (Megyesi & Szabó, 1996, Theorem 6). Therefore, the possible graphs for such configurations are as in Figure 4.5.

Theorem 4.3.20. *There is no six nondegenerate quadrics with twenty four tacnodes, i.e., non of the graphs in Figure 4.5 is (complex) realizable.*

Proof. First, let us consider the graph in Figure 4.5a. We may assume that the

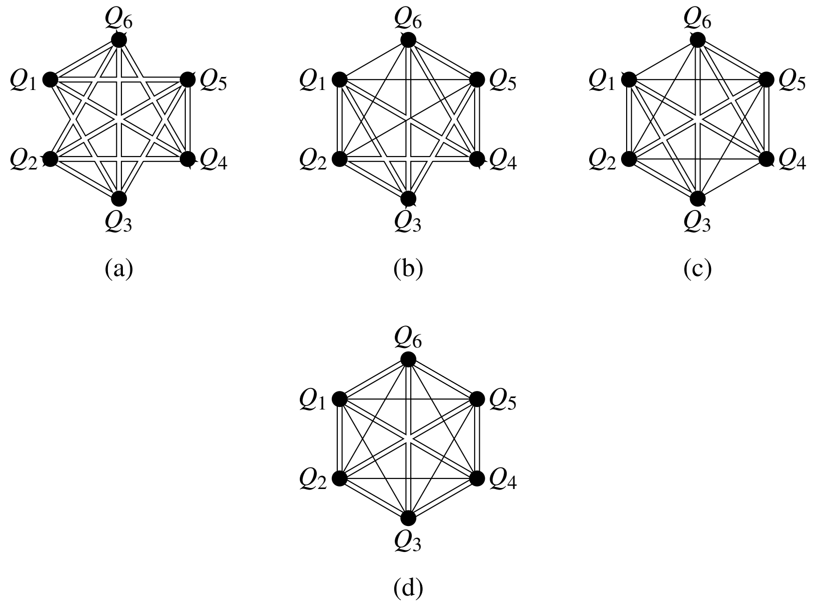


Figure 4.5 The four graph on 6 vertices and 24 edges.

quadrics Q_1, Q_2, Q_4 and Q_5 are as in Proposition 4.3.15. Since the configurations of quadruples (Q_1, Q_2, Q_4, Q_5) and (Q_3, Q_2, Q_6, Q_5) are projectively equivalent, then Q_3 and Q_6 must be respectively the images of Q_1 and Q_4 under a projective transformation fixing Q_2, Q_5 and their intersection points. This implies that Q_5 must be tangent to Q_6 at two distinct points. But this contradicts the fact that the quadrics Q_5 and Q_6 are in general position. So this graph can not be realized.

Second, consider the graph in Figure 4.5b. We may assume that the quadrics Q_1, Q_2, Q_3 and Q_4 are as in Proposition 4.3.15, then Q_1, Q_2 and Q_5 must have the same reference with respect to the quadrics Q_3 and Q_4. This tell us that either $Q_5 = Q_1$ or $Q_5 = Q_2$. Then, such configuration of the quadrics Q_1, Q_2, Q_3, Q_4 and Q_5 is impossible. Therefore, Figure 4.5b consist of an impossible configuration as a subgraph, then it is also impossible.

Next, consider the graph in the Figure 4.5c. By the Proposition 4.3.16, we may assume that

$$
\begin{aligned}
Q_1 &: X^2+Y^2-Z^2=0, \\
Q_2 &: p^2X^2+(p^2+1)Y^2-2pYZ=0, \\
Q_4 &: q^2X^2+(q^2+1)Y^2-2qYZ=0, \\
Q_5 &: (2pq-p-q)^2X^2+[(p+q)(4pq-3p-3q+4)-4pq]Y^2 \\
&\quad -(p-q)^2Z^2-4(p-1)(q-1)(p+q)YZ=0,
\end{aligned}
$$

where $p,q \in \mathbb{C}\setminus\{0,\mp 1\}$, $p \neq \mp q$, $p+q \neq 2$ and $p+q \neq 2pq$. The degenerate quadric

$$-(1-p^2)X^2+(pY-Z)^2 = (\sqrt{1-p^2}X+pY-Z)(-\sqrt{1-p^2}X+pY-Z)=0$$

consists of the common tangent lines of the quadrics Q_1 and Q_2. Then by Lemma 4.3.7, the equation of the line connecting the tangency points of Q_1 and Q_3 is $\frac{1}{2}[\alpha(\sqrt{1-p^2}X+pY-Z)+\frac{1}{\alpha}(-\sqrt{1-p^2}X+pY-Z)]=0$, and therefore the equation of Q_3 is

$$X^2+Y^2-Z^2+\frac{1}{4}[(\alpha-\frac{1}{\alpha})\sqrt{1-p^2}X+(\alpha+\frac{1}{\alpha})(pY-Z)]^2=0,$$

where $\alpha \in \mathbb{C}\setminus\{0,\mp 1\}$.

Let us substitute the parametrization $\{[2pst:2ps^2:(p^2+1)s^2+t^2] \mid [s:t]\in\mathbb{CP}^1\}$ of Q_2 into the equation of Q_3. Then we have obtained

$$
\begin{aligned}
f_{23}(s,t) =& \sum_{i=0}^{4} a_i s^i t^{n-i} \\
=& \frac{p^4(1-\alpha^2)^2}{4\alpha^2}t^4+\frac{p^3(1-\alpha^2)(1-\alpha^2)}{\alpha^2}st^3 \\
& -\frac{p^2(p^2(\alpha^4+6\alpha^2+1)-(3\alpha^4+2\alpha^2+3))}{2\alpha^2}s^2t^2 \\
& +\frac{p(1-p^2)(1-\alpha^2)(1+\alpha^2)}{\alpha^2}s^3t+\frac{(1-p^2)^2(1-\alpha^2)^2}{4\alpha^2}=0.
\end{aligned}
$$

If $f(s,t)$ is a square of a reducible polynomial then

$$f_3^3+8f_1f_4^2-4f_2f_3f_4=\frac{4p^{11}(1+\alpha^2)(1-\alpha^2)^3}{\alpha^4}=0.$$

This is possible only when $\alpha^2=-1$. But, $f_{23}(s,t)=-p^4t^4-2p^2(1+p^2)s^2t^2-(1-p^2)^2s^4$ will never be a square for $\alpha^2=-1$, i.e Q_2 and Q_3 are in general position which contradicts to fact that the quadrics Q_2 and Q_3 have two tacnodes. Thus, this graph can not be realized.

Last, consider the graph in Figure 4.5d. By the Proposition 4.3.16, we may assume that $Q_1: X^2+Y^2-Z^2=0$, $Q_2: p^2X^2+(p^2+1)Y^2-2pYZ=0$, $Q_4: q^2X^2+(q^2+1)Y^2-2qYZ=0$ and $Q_3: (2pq-p-q)^2X^2+[(p+q)(4pq-3p-3q+4)-4pq]Y^2-(p-q)^2Z^2-4(p-1)(q-1)(p+q)YZ=0$, where $p,q\in\mathbb{C}\setminus\{0,\mp1\}$, $p\neq\mp q$, $p+q\neq 2$, $p+q\neq 2pq$. By taking $[0:-1:1]$ instead of $[0:1:1]$ as tangency points of Q_1 and Q_4 in the proof of the Proposition 4.3.16, we will get the equation $(2pq+p+q)^2X^2+[(p+q)(4pq+3p+3q+4)-4pq]Y^2-(p-q)^2Z^2-4(p+1)(q+1)(p+q)YZ=0$ for Q_5, where $p,q\in\mathbb{C}\setminus\{0,\mp1\}$, $p\neq\mp q$, $pq\neq 1$, $p+q\neq\mp 2$, $p+q\neq\mp 2pq$. Now assume that such quadric Q_6 exist. Then the configuration of quadrics Q_3, Q_5 and Q_6 has five tacnodes and they are projectively equivalent to quadrics in (4.3.32). Note that the quadrics in (4.3.32) are invariant under the involution $[X:Y:Z]\to[-X:Y:Z]$, therefore the quadric Q_6 must be invariant under this involution since both Q_3 and Q_5 are invariant. Hence, Q_6 is tangent to Q_2 at $[0:2p:p^2+1]$, and to Q_4 at $[0:2q:q^2+1]$, so its equation must be in the form $aX^2+((p^2+1)Y-2pZ)((q^2+1)Y-2qZ)=0$ for some $a\in\mathbb{C}^*$. Then by substituting the parametrization $\{[2st:s^2-t^2:s^2+t^2]\mid[s:t]\in\mathbb{CP}^1\}$ of Q_1 into the equation of Q_6, we will get

$$\begin{aligned} f_{16}(s,t) &= (p+1)^2(q+1)^2t^2+(4a+8pq-2(1+p^2)(1+q^2))s^2t^2 \\ &\quad +(p-1)^2(q-1)^2s^4=0, \end{aligned}$$

which is a square of a reducible polynomial if $a=(p-q)^2$ or $a=(pq-1)^2$.

On the other hand since the point $[0:1:1]$ lies on Q_3, we can parametrize it by using the line $-sX+t(Y-Z)=0$, and its parametrization is

$$\{[2(p+q-2pq)(2-p-q)st : -(p-q)^2s^2+(2pq-p-q)^2t^2 :$$
$$(4pq(p+q-1)-(p+q)(3p+3q-4))s^2+(2pq-p-q)^2t^2] \mid [s:t] \in \mathbb{CP}^1\}.$$

By substituting the parametrization of Q_3 into the equation of Q_6 we will get

$$\begin{aligned} f_{36}(s,t) =& (p-1)^2(q-1)^2(2pq-p-q)^4t^4 \\ & +2(2pq-p-q)^2[(p+q-2)^2-p^4(5q^2-4q+1) \\ & -2p^3(3q^3-14q^2+9q-2)-p^2(5q^4-28q^3+58q^2-28q+5) \\ & +2pq(2q^3-9q^2+14q-3)-q^2(q^2-4q+5)]s^2t^2 \\ & +(p^2+3pq-3p-q)^2(q^2+3pq-p-3q)s^4=0. \end{aligned}$$

$f_{3,6}(s,t)$ is a square of a reducible polynomial if and only if either $a=(2pq-p-q)^2$ or $a=\frac{(p-q)^2(pq-1)^2}{(p+q-2)^2}$.

Similarly, by parametrizing the quadric Q_5 and substituting into the equation of Q_6, and taking into account the tangency conditions we will see that either $a=(2pq+p+q)^2$ or $a=\frac{(p-q)^2(pq-1)^2}{(p+q+2)^2}$.

Hence $a=(p-q)^2$ if $pq \mp (p+q)=3$, or $a=(pq-1)^2$ if $3pq \mp (p+q)=1$. In both cases $p+q=0$ and we have already excluded this case. So, the graph in Figure 4.5d can not be realized.

$\square$

CHAPTER FIVE
ZARISKI VAN-KAMPEN THEOREM: AN OVERVIEW

Zariski van-Kampen theorem is a tool for computing fundamental groups of complements to curves (germs of curve singularities, affine or projective plane curves). It gives us the fundamental groups in terms of generators and relations. Roughly speaking, the generators can be taken in a generic line and the relations consist of identifying these generators with their images by some monodromies. Before introducing this theorem we will overview definitions of homotopy between continuous map, fundamental group, and give the statement of the classical van-Kampen theorem. Then we will investigate the braid monodromy and give the statement of the Zariski van-Kampent heorem based on the lecture notes of Shimada (2007). In addition, we will also compute the local fundamental groups of the germs in Figure 6.1, and fundamental groups of some quadric arrangements related to line arrangements.

5.1 Homotopy Between Continuous Maps

Let us denote by I the closed interval $[0,1]$ of $\mathbb{R}$. Let X and Y be two topological spaces, and let $f_i : X \to Y$, $i = 1,2$, be two continuous maps. A continuous map $F : X \times I \to Y$ is called a *homotopy* from f_0 to f_1 if it satisfies $F(x,0) = f_0(x)$, $F(x,1) = f_1(x)$ for all $x \in X$. We say that f_0 and f_1 are *homotopic* and write $f_0 \simeq f_1$ if there exists a homotopy from f_0 to f_1. The relation $\simeq$ is an equivalence relation, and the equivalence class under the relation $\simeq$ is called the *homotopy class*.

If there are continuous maps $f : X \to Y$ and $g : Y \to X$ such that $g \circ f$ is homotopic to the identity of X, and $f \circ g$ is homotopic to the identity of Y , then X and Y are said to be *homotopically equivalent*.

Let A be a subspace of X. A homotopy $F : X \times I \to Y$ from f_0 to f_1 is said to be *stationary on A* if $F(a,s) = f_0(a)$ for all $(a,s) \in A \times I$. If there exists a homotopy stationary on A from f_0 to f_1, the maps f_0 and f_1 are called *homotopic relative to* A and it is written as $f_0 \simeq_A f_1$. It is clear that $\simeq_A$ is an equivalence relation.

5.2 Definition of the Fundamental Group

Let x_0 and x_1 be points of a topological space X. A continuous map $\alpha : I \to X$ satisfying $\alpha(0) = x_0$ and $\alpha(1) = x_1$ is called a *path* from x_0 to x_1. Denote by $[\alpha]$ the homotopy class relative to $\partial I = \{0,1\}$ containing α. We define a path $\bar{\alpha} : I \to X$ from x_1 to x_0 by $\bar{\alpha}(t) := \alpha(1-t)$ and call $\bar{\alpha}$ *the inverse path of* α. A constant map to the point x_0 is a path with both of the initial point and the terminal point being x_0. This path is denoted by e_{x_0}.

Given two paths $\alpha, \beta : I \to X$ such that $\alpha(1) = \beta(0)$, there is a *composition* or *product path* $\alpha \cdot \beta$ that traverses first α and then β, defined by the formula

$$\alpha \cdot \beta(t) = \begin{cases} \alpha(2t), & 0 \le t \le 1/2 \\ \beta(2t-1), & 1/2 \le t \le 1. \end{cases}$$

This product operation respects homotopy classes since if $\alpha_0 \simeq \alpha_1$ and $\beta_0 \simeq \beta_1$ via homotopies $F(s,t)$ and $G(s,t)$, respectively, and if $\alpha_0(1) = \beta_1(0)$ so that $\alpha_0 \cdot \beta_0$ is defined, then the continuous map

$$H(s,t) = \begin{cases} F(s,2t), & 0 \le t \le 1/2 \\ G(s,2t-1), & 1/2 \le t \le 1. \end{cases}$$

provides a homotopy $\alpha_0 \cdot \beta_0 \simeq \alpha_1 \cdot \beta_1$.

In particular, suppose we restrict attention to paths $\alpha : I \to X$ with the same starting and ending point $\alpha(0) = \alpha(1) = x_0 \in X$. Such paths are called *loops*, and the common starting and ending point x_0 is referred as the *basepoint*. The set of all homotopy classes $[\alpha]$ of loops $\alpha : I \to X$ at the base point x_0 is denoted by $\pi_1(X,x_0)$.

Proposition 5.2.1. $\pi_1(X,x_0)$ *is a group with respect to the product* $[\alpha][\beta]=[\alpha\cdot\beta]$.

This group is called the *fundamental group* of X at the base point x_0. If X is path connected, then for any two base points x_0 and x_1 the fundamental groups $\pi_1(X,x_0)$ and $\pi_1(X,x_1)$ are isomorphic. Indeed, if δ is a path from x_0 to x_1, then the isomorphism $\Phi_\delta : \pi_1(X,x_1)\to\pi_1(X,x_0)$ is given by $\Phi_\delta([\alpha])=[\delta\cdot\alpha\cdot\bar{\delta}]$. The inverse is given by $\Phi_{\bar{\delta}}$. Thus if X is path connected, the group $\pi_1(X,x_0)$ is, up to isomorphism, independent of the choice of base point x_0. In this case the notation $\pi_1(X,x_0)$ is often abbreviated to $\pi_1(X)$.

In general, a space X is called *simply connected* if it is path connected and has trivial fundamental group. For example, if $n\geq 2$, then S^n is simply connected; the circle S^1 is path connected, but $\pi_1(S^1)\simeq\mathbb{Z}$.

Theorem 5.2.2. *If X is path connected, then the abelianization* $\pi_1/[\pi_1,\pi_1]$ *of* $\pi_1 := \pi_1(X)$ *is isomorphic to* $H_1(X,\mathbb{Z})$.

5.3 Van Kampen Theorem

The van Kampen theorem gives a method for computing the fundamental groups of spaces that can be decomposed into simpler spaces whose fundamental groups are already known. By systematic use of the van Kampen theorem one can compute the fundamental groups of a very large number of spaces.

Theorem 5.3.1 (van Kampen)**.** *If X is a union of path connected open sets U_i each containing the base point $x_0\in X$ and if each intersection $U_i\cap U_j$ is path connected, then the homomorphism $\Psi : *_i\pi_1(U_i)\to\pi_1(X)$ is surjective. If in addition each intersection $U_i\cap U_j\cap U_k$ is path connected, then the kernel of ψ is the normal subgroup N generated by all elements of the form $\iota_{ij}(\mu)\iota_{ji}(\mu)^{-1}$, where $\iota_{ij} : \pi_1(U_i\cap U_j)\to\pi_1(U_i)$ is the homomorphism induced by the inclusion $U_i\cap U_j\hookrightarrow U_i$, and so ψ induces an isomorphism $\pi_1(X)\simeq *_i\pi_1(U_i)/N$.*

Example 5.3.2. Let X_n be the bouquet of n circles: $X_n = S^1 \vee S^1 \vee \cdots \vee S^1$. Then $\pi_1(X_n)$ is isomorphic to the free group F_n of n letters. Let A be the set of distinct n points on the complex plane $\mathbb{C}$. Then $\mathbb{C} \setminus A$ has homotopy type X_n, and therefore $\pi_1(\mathbb{C} \setminus A)$ is also isomorphic to the free group F_n.

Example 5.3.3. Let A be the set of distinct n points on the complex projective line $\mathbb{CP}^1$. Then $\pi_1(\mathbb{CP}^1 \setminus A)$ is isomorphic to the free group F_{n-1}.

5.4 Braid Group

Let $\mathcal{R} := \{R_\lambda\}_{\lambda \in \Lambda}$ be a subset of $F_n := \langle a_1, a_2, \cdots, a_n \rangle$, and let $N(\mathcal{R})$ be the smallest normal subgroup of F_n containing $\mathcal{R}$. The group generated by $a_1, a_2, \cdots, a_n$ with defining relations R_λ $(\lambda \in \Lambda)$ is denoted by

$$F_n/N(\mathcal{R}) = \langle a_1, a_2, \cdots, a_n \mid R_\lambda = e, \lambda \in \Lambda \rangle$$

Example 5.4.1. The group $\langle a \mid a^n = e \rangle$ is isomorphic to $\mathbb{Z}_n$, and the group $\langle a, b \mid aba^{-1}b^{-1} = e \rangle$ is isomorphic to $\mathbb{Z} \times \mathbb{Z}$.

Example 5.4.2. Let n be an integer ≥ 2. Then the group generated by $a_1, a_2, \cdots, a_{n-1}$ with defining relations

$$\begin{aligned}
&a_i^2 = e \quad \text{for } i = 1, 2, \cdots, n-1, \\
&a_i a_j = a_j a_i \quad \text{if } |i - j| > 1, \\
&a_i a_{i+1} a_i = a_{i+1} a_i a_{i+1} \quad \text{for } i = 1, 2, \cdots, n-1,
\end{aligned}$$

is isomorphic to the full symmetric group $\mathfrak{S}_n$ via $a_i \mapsto (i, i+1)$.

Put $M_n := \mathbb{C}^n \setminus \{\text{the big diagonal}\} = \{(z_1, z_2, \cdots, z_n) \in \mathbb{C}^n \mid z_i \neq z_j \text{ if } i \neq j\}$. The symmetric group $\mathfrak{S}_n$ acts on M_n by interchanging the coordinates. We then put $\overline{M}_n := M_n/\mathfrak{S}_n$. This space $\overline{M}_n$ is the space parametrizing non-ordered sets of distinct n points on the complex plane $\mathbb{C}$ (sometimes it is called the *configuration space* of

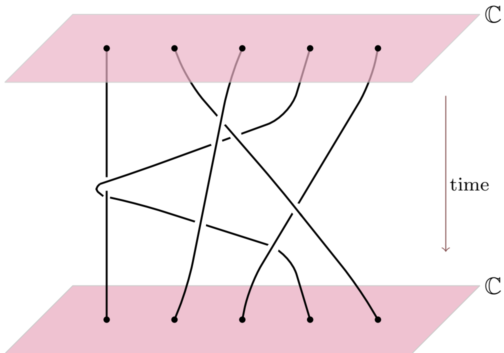


Figure 5.1 A braid.

non-ordered sets of distinct n points on the complex plane $\mathbb{C}$). By associating to a non-ordered set of distinct n points $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ the coefficients $\lambda_1, \lambda_2, \cdots, \lambda_n$ of $z^n + \lambda_1 z^{n-1} + \lambda_2 z^{n-2} + \cdots + \lambda_{n-1} z + \lambda_n = (z - \alpha_1)(z - \alpha_2) \cdots (z - \alpha_n)$, we obtain an isomorphism from $\overline{M}_n$ to the complement to the discriminant hypersurface of monic polynomials of degree n in $\mathbb{C}^n$. We put $P_n := \pi_1(M_n)$ and $B_n := \pi_1(\overline{M}_n)$. The group P_n is called the *pure braid group* on n strings, and the group B_n is called the *braid group* on n strings. By definition, we have a short exact sequence

$$1 \to P_n \to B_n \to \mathfrak{S}_n \to 1 \tag{5.4.1}$$

corresponding to the Galois covering $M_n \to \overline{M}_n$ with Galois group $\mathfrak{S}_n$. The point of the configuration space $\overline{M}_n$ is a set of distinct n points on the complex plane $\mathbb{C}$. Hence a loop in $\overline{M}_n$ is a movement of these distinct points on $\mathbb{C}$, which can be express by a braid as in Figure 5.1, whence the name the braid group.

The product in B_n is defined by the conjunction of of the braids. In particular, the inverse is represented by the braid upside-down. For $i = 1, 2, \cdots, n-1$, let σ_i be the element of B_n represented by the braid given in Figure 5.2

Theorem 5.4.3 (Artin)**.** *The braid group B_n is generated by $\sigma_1, \sigma_2, \cdots, \sigma_{n-1}$, and*

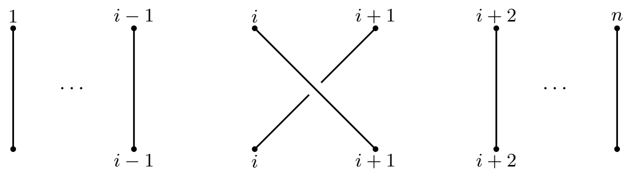


Figure 5.2 The element σ_i.

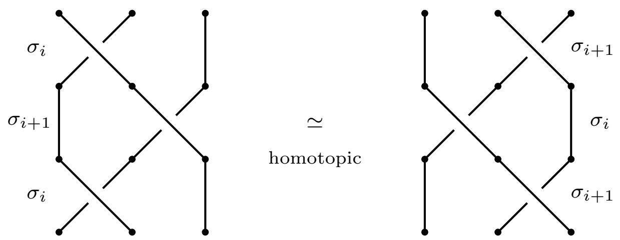


Figure 5.3 The relation $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$.

defined by the following relations:

$$\begin{aligned} &\sigma_i\sigma_j = \sigma_j\sigma_i \quad \textit{if } |i-j| > 1, \\ &\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \quad \textit{for } i = 1, 2, \cdots, n-1. \end{aligned} \tag{5.4.2}$$

The fact that B_n is generated by $\sigma_1, \sigma_2, \cdots, \sigma_{n-1}$ is easy to see. The relations actually hold can be checked easily by drawing braids. See Figure 5.3. The difficult part is that any other relations among the generators can be derived from these relations. See Birman (1974) for the proof.

We can define an action from right of the braid group B_n on the free group F_n by the following

$$a_j^{\sigma_i} := \begin{cases} a_j, & j \neq i, i+1, \\ a_i a_{i+1} a_i^{-1}, & j = i, \\ a_i, j = i+1. \end{cases} \tag{5.4.3}$$

This definition is compatible with the defining relation of the braid group.

5.5 Monodromy on Fundamental Groups

Let $p : \tilde{X} \to X$ be a locally trivial fiber space. Suppose that $p : \tilde{X} \to X$ has a section $s : X \to \tilde{X}$, that is, s is a continuous map satisfying $p \circ s = \mathrm{id}_X$. We chose a base point $\tilde{x}_0$ of $\tilde{X}$ and x_0 of X in such a way that $\tilde{x}_0 = s(x_0)$ holds. We then put $F_{x_0} := p^{-1}(x_0)$. We can regard $\tilde{x}_0$ as a base point of the fiber F_{x_0}. Then $\pi_1(X, x_0)$ acts on $\pi_1(F_{x_0}, \tilde{x}_0)$ from right. This action is called the *monodromy action* on the fundamental group of the fiber.

Indeed, suppose that we are given a loop $\gamma : I \to X$ with the base point x_0, and a loop $\mu : I \to F_{x_0}$ with the base point $\tilde{x}_0 = s(\gamma(0))$. The the fibers $\big(p^{-1}(\gamma(t)), s(\gamma(t))\big)$, $t \in I$ form a trivial fiber space over I. We can deform the loop μ into a loop $\mu_t : I \to p^{-1}(\gamma(t))$ with the base point $s(\gamma(t))$ continuously. The loop $\mu_1 : I \to p^{-1}(\gamma(1))$ with the base point $s(\gamma(1)) = \tilde{x}_0$ represents $[\mu]^{[\gamma]} \in \pi_1(F_{x_0}, \tilde{x}_0)$. Serre's lifting property of locally trivial fiber space implies that $[\mu]^{[\gamma]} = [\mu_1]$ is independent of the choice of the representing loops $\gamma : I \to X$ and $\mu : I \to F_{x_0}$.

Suppose we have a trivial fibration $p : \tilde{X} \to X$, where $\tilde{X} = X \times F$. For a point $y_0 \in F$, the map $x \mapsto (x, y_0)$ defines a section of $p : \tilde{X} \to X$. In this case $\pi_1(X, x_0)$ acts on $\pi_1(F, y_0)$ trivially. On the other hand, for any continuous map $\eta : X \to F$, the map $x \mapsto (x, \eta(x))$ defines a section of $p : \tilde{X} \to X$. In this case the pointed fibers are $(F, \eta(\gamma(t)))$. Let $\eta_t : [0,t] \to F$ be the path defined on F from $\eta(x_0)$ to $\eta(\gamma(t))$ by $\eta_t := \eta(\gamma(s))$. Then $\mu_t := \eta_t^{-1} \mu \eta_t$ is a deformation of μ. Hence $\pi_1(X, x_0)$ acts on $\pi_1(F, \eta(x_0))$ by $[\mu]^{[\gamma]} = (\eta_*[\gamma])^{-1}[\mu](\eta_*[\gamma])$.

Definition 5.5.1. A good set of loops $\mu_0, \mu_1, \cdots \mu_d$ based at $z \in \mathbb{C} \setminus \{z_0, z_1, \cdots, z_d\}$ is constructed in the following manner. Let Δ_i be closed discs around z_i mutually disjoint and not containing z. For each $i \in \{0, 1, \cdots, d\}$, let ω_i be a path connecting z to a point of the boundary of $\partial\Delta_i$ of Δ_i, and $\partial\Delta_i$ runs once in counter clockwise direction. The paths ω_i are required not to meet together except at their origin. For $0 \le i \le d$, take the loops $\mu_i = \omega_i \partial\Delta_i \omega_i^{-1}$. Such kind of good loops μ_i are called *meridians* of z_i in $\mathbb{C} \setminus \{z_0, \cdots, z_d\}$. Note that any two meridians of z_i are conjugate in $\pi_1(\mathbb{C} \setminus \{z_0, \cdots, z_d\})$. From now on, for the sake of simplicity we will denote by μ_i the homotopy class $[\mu_i]$.

5.6 Monodromy around a Curve Singularity

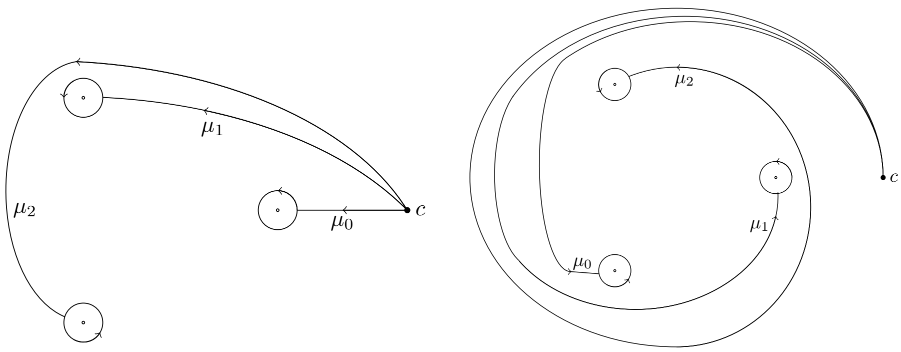
Let Δ_ρ denote the open disc $\{z \in \mathbb{C} \mid |z| < \rho\}$. Consider the curve C on $\Delta_{2\varepsilon} \times \Delta_{2\rho}$ defined by $x^m - y^d = 0$, where $m, d \in \mathbb{Z}_{\geq 2}$. Let $\bar{p} : \Delta_{2\varepsilon} \times \Delta_{2\rho} \to \Delta_{2\varepsilon}$ be the first projection $(x, y) \mapsto x$. We assume ρ is large enough compared with ε. We put

$$\Delta_{2\varepsilon}^* := \Delta_{2\varepsilon} \setminus \{0\} \quad \text{and} \quad \mathcal{Y} := \bar{p}^{-1}(\Delta_{2\varepsilon}^*) \cap ((\Delta_{2\varepsilon} \times \Delta_{2\rho}) \setminus C).$$

Then the restriction $p : \mathcal{Y} \to \Delta_{2\varepsilon}^*$ of $\bar{p}$ is locally trivial. The fiber over $x \in \Delta_{2\varepsilon}^*$ is $\Delta_{2\rho}$ minus the d-th roots of x^m. Choose the base point of $\Delta_{2\varepsilon}^*$ at $x_0 := \varepsilon$. Let c be a positive real number such that $|2\varepsilon|^{m/d} < c < \rho$. Then the map $x \mapsto (x, c)$ gives us a section of $p : \mathcal{Y} \to \Delta_{2\varepsilon}^*$. Put $F_{x_0} := p^{-1}(x_0)$, and $\tilde{x}_0 := s(x_0) = (\varepsilon, c)$.

The group $\pi_1(\Delta_{2\varepsilon}^*, x_0)$ is an infinite cyclic group generated by the homotopy class $\gamma = [g]$ of the loop $g(t) = \varepsilon \exp(2\pi i t)$. On the other hand, the fiber F_{x_0} is homotopic to the bouquet of d circles, and hence its fundamental group $\pi_1(F_{x_0}, \tilde{x}_0)$ is a free group generated by d elements $\mu_0, \mu_1, \cdots, \mu_{d-1}$ which are represented by the meridians given in Figure 5.5 (It is drawn for the case $(m, d) = (2, 3)$).

Main idea of the Braid monodromy technique is analyse the deformation of the fiber $p^{-1}(g(t))$ while t goes from 0 to 1 with the base point $s(g(t))$. The base point is constant at c. The deleted points moves around the origin with angular speed $2\pi m/d$, since $g(t)^d$ moves around the origin with angular speed $2\pi m$. Therefore, the meridians around the deleted points are dragged around the origin, and when $g(t)$ comes back to the starting point , the meridian μ_i is deformed into the meridian $\tilde{\mu}_i$. Therefore the monodromy action of $\pi_1(\Delta_{2\varepsilon}^*, x_0) = \langle \gamma \rangle$ on the free group $\pi_1(F_{x_0}, \tilde{x}_0) =$



(a) (b)

Figure 5.4 The monodromy action on $\pi_1(F_{x_0},\tilde{x}_0)$ when $C: x^2-y^3=0$.

$\langle\mu_0,\mu_1,\cdots,\mu_{d-1}\rangle$ is given by $\mu_i^\gamma=\tilde{\mu}_i$. Note that, the big loop around the origin is represented by the homotopy class $\delta:=\mu_{d-1}\mu_{d-2}\cdots\mu_1\mu_0$. Let $j\in\mathbb{Z}_{\geq 0}$, and r is the remainder of j divided by d. Set $\mu_j=\mu_{ad+r}:=\delta^a\mu_r\delta^{-a}$, then we have $\tilde{\mu}_i=\mu_{i+m}$. Hence the monodromy action of $\pi_1(\Delta^*_{2\varepsilon},x_0)$ on $\pi_1(F_{x_0},\tilde{x}_0)$ is given by $\mu_i^\gamma=\mu_{i+m}$. We will discuss local fundamental group of curve singularities in the Section 5.10.

5.7 The Fundamental Group of the Total Space

Suppose that a group H acts on a group N from right, i.e., $n\mapsto n^h$ $(n\in N,\ h\in H)$. Define a product on the set $N\times H$ by $(n_1h_1)(n_2h_2)=(n_1n_2^{h_1^{-1}},h_1h_2)$. Under this product, $N\times H$ becomes a group, which is called the *Semi-direct product* of N and H, and denoted by $N\rtimes H$.

The map $n\mapsto(n,e_h)$ defines an injective homomorphism $\iota: N\to N\rtimes H$, whose image is a normal subgroup of $N\rtimes H$, so one can regard N as a normal subgroup of $N\rtimes H$. On the other hand, the map $(n,h)\mapsto h$ defines a surjective homomorphism $\vartheta: N\rtimes H\to H$ whose kernel is N. Hence H can be identified with $(N\rtimes H)/N$. In addition, the map $h\mapsto(e_N,h)$ defines an injective homomorphism $\sigma: H\to N\rtimes H$ such that $\vartheta\circ\sigma=\mathrm{id}_H$, and one can regard H as a subgroup of $N\rtimes H$. Thus we have a splitting short exact sequence

$$1 \longrightarrow N \xrightarrow{\iota} N \rtimes H \underset{\sigma}{\overset{\vartheta}{\rightleftarrows}} H \longrightarrow 1. \tag{5.7.1}$$

Proposition 5.7.1. *Let $p : \tilde{X} \to X$ be a locally trivial fiber space with a section $s : X \to \tilde{X}$. Suppose $\tilde{X}$ is path connected. Let x_0 be a base point of X, and put $\tilde{x}_0 := s(x_0)$, $F_{x_0} := p^{-1}(x_0)$. Then the fundamental group $\pi_1(\tilde{X}, \tilde{x}_0)$ of total space $\tilde{X}$ is isomorphic to the semi-direct product $\pi_1(F_{x_0}, \tilde{x}_0) \rtimes \pi_1(X, x_0)$ constructed from the monodromy action of $\pi_1(X, x_0)$ on the free group $\pi_1(F_{x_0}, \tilde{x}_0)$.*

Proof. Since $\tilde{X}$ is path connected, than both of the fiber F_{x_0} and the base space X are path connected, and there is a section $s : X \to \tilde{X}$. Let $i : F_{x_0} \hookrightarrow \tilde{X}$ be the inclusion. Then we have the homotopy exact sequence

$$\xrightarrow{i_*} \pi_2(\tilde{X}, \tilde{x}_0) \xrightarrow{p_*} \pi_2(X, x_0) \longrightarrow \pi_1(F_{x_0}, \tilde{x}_0) \xrightarrow{i_*} \pi_1(\tilde{X}, \tilde{x}_0) \xrightarrow{p_*} \pi_1(X, x_0) \to 1.$$

Moreover, the section s induces a homomorphism $s_* : \pi_2(X, x_0) \to \pi_2(\tilde{X}, \tilde{x}_0)$ such that the composition $\pi_2(X, x_0) \xrightarrow{s_*} \pi_2(\tilde{X}, \tilde{x}_0) \xrightarrow{p_*} \pi_2(X, x_0)$ is the identity. Therefore, $p_* : \pi_2(\tilde{X}, \tilde{x}_0 \to \pi_2(X, x_0)$ is surjective and hence we obtain a short exact sequence

$$1 \longrightarrow \pi_1(F_{x_0}, \tilde{x}_0) \xrightarrow{i_*} \pi_1(\tilde{X}, \tilde{x}_0) \xrightarrow{p_*} \pi_1(X, x_0) \longrightarrow 1. \tag{5.7.2}$$

There is a section $s_* : \pi_1(X, x_0) \to \pi_1(\tilde{X}, \tilde{x}_0)$ of $p_* : \pi_1(\tilde{X}, \tilde{x}_0) \to \pi_1(X, x_0)$, and one can regard $\pi_1(F_{x_0}, \tilde{x}_0)$ as a normal subgroup of $\pi_1(\tilde{X}, \tilde{x}_0)$ by i_*. Define an action of $\pi_1(X, x_0)$ on $\pi_1(F_{x_0}, \tilde{x}_0)$ by $\mu \mapsto s_*(\gamma)^{-1} \mu s_*(\gamma)$, where $\gamma \in \pi_1(F_{x_0}, \tilde{x}_0)$ and $\mu \in \pi_1(X, x_0)$. This group theoretic action coincides with the monodromy action $\mu \mapsto \mu^\gamma$ of $\pi_1(X, x_0)$ on $\pi_1(F_{x_0}, \tilde{x}_0)$. The short exact sequence (5.7.2) implies that $\pi_1(\tilde{X}, \tilde{x}_0)$ is isomorphic to the semi direct product $\pi_1(F_{x_0}, \tilde{x}_0) \rtimes \pi_1(X, x_0)$, and the isomorphism $\pi_1(F_{x_0}, \tilde{x}_0) \rtimes \pi_1(X, x_0) \to \pi_1(\tilde{X}, \tilde{x}_0)$ is given by $(\mu, \gamma) \mapsto i_*(\mu) s_*(\gamma)$. $\square$

5.8 Fundamental Groups of Complemets to Subvarieties

Let M be a connected complex manifold, and V a proper closed analytic subspace of M. Let $\iota : M \setminus V \hookrightarrow M$ be the inclusion. Chosen a base point $x_0 \in M \setminus V$, we have an epimorphism $\iota_* : \pi_1(M \setminus V, x_0) \to \pi_1(M, x_0)$. If the codimension of V in M is at least 2, then ι_* is an isomorphism. Indeed, if V is of codimension ≥ 2, than $M \setminus V$ is simply connected and the group $H_1(M \setminus V)$ is trivial.

The following well known theorem is the most famous result considering only the case $n = 2$.

Theorem 5.8.1 (Zariski-Lefschetz hyperplane section theorem (Zariski, 1937)). *Let V be a hypersurface in $\mathbb{CP}^n$. Then the inclusion homomorphism $\pi_1(\mathcal{H} \setminus V) \to \pi_1(\mathbb{CP}^n \setminus V)$ is an isomorphism for a generic plane $\mathcal{H} = \mathbb{CP}^2$ in $\mathbb{CP}^n$.*

Abelianizing the above isomorphism, we get $H_1(\mathbb{CP}^2 \setminus C) = H_1(\mathbb{CP}^n \setminus V)$, where $C := \mathcal{H} \cap V = \mathbb{CP}^2 \cap V$. Now, if C is reduced plane algebraic curve with the irreducible components C_i of d_i for $1 \leq i \leq k$, then the homology groups of $\mathbb{CP}^2 \setminus C$ are quite simple and do not give to much information. By the Lefschetz duality and by the exact sequence of the pair $(\mathbb{CP}^2, C)$, one has

$$H_1(\mathbb{CP}^2 \setminus C, \mathbb{Z}) \simeq \mathbb{Z}^{k-1} \oplus \mathbb{Z}_d, \quad d := \gcd(d_1, d_2, \cdots, d_k) \tag{5.8.1}$$

whereas the fundamental group $\pi_1(\mathbb{CP}^2 \setminus C)$ is much more informative. In particular if C is irreducible ($k = 1$), we have $H_1(\mathbb{CP}^2 \setminus C, \mathbb{Z}) \simeq \mathbb{Z}_{d_1}$.

5.9 Zariski Van-Kampen Theorem

Let $p : M \to C$ be a surjective homomorphic map from a connected complex manifold M to a 1-dimensional complex manifold C. Suppose that the following conditions are satisfied.

(a) The curve C is simply connected.

(b) There exists a holomorphic map $s : C \to M$ such that $p \circ s = id_C$.

(c) There exists a set $\mathcal{P}_m$ of m points of C such that the restriction $p_0 : M_0 \to C \setminus \mathcal{P}_m$ of p to $M_0 := p^{-1}(C \setminus \mathcal{P}_m)$ is a locally trivial fiber space.

Let z_0 and $\tilde{z}_0 := s(z_0)$ be base points of $C \setminus \mathcal{P}_m$ and M_0, respectively. Denote $F_{z_0} := p^{-1}(z_0)$ the fiber over z_0 and by $i : F_{z_0} \hookrightarrow M$ the inclusion map. As it is explained in the Section 5.5, the fundamental group $\pi_1(C \setminus \mathcal{P}_m)$ acts on $\pi_1(F_{z_0}, \tilde{z}_0)$ from the right via $\mu \mapsto \mu^\gamma$, where $\mu \in \pi_1(F_{z_0}, \tilde{z}_0)$ and $\gamma \in \pi_1(C \setminus \mathcal{P}_m, z_0)$. The following the theorem of Zarsiki van-Kampen in this general setting.

Theorem 5.9.1 (Zarsiki van-Kampen theorem)**.** *Suppose that the conditions (a), (b) and (c) are satisfied. Then $i_* : \pi_1(F_{z_0}, \tilde{z}_0) \to \pi_1(M, \tilde{z}_0)$ is surjective. Suppose moreover that the following condition is satisfied:*

(d) For each point $z_i \in \mathcal{P}_m$, the fiber $p^{-1}(z_i)$ is irreducible.

Then the kernel of i_* is the smallest subgroup of $\pi_1(F_{z_0}, \tilde{z}_0)$ containing the subset $\{\mu^{-1}\mu^\gamma \mid \mu \in \pi_1(F_{z_0}, \tilde{z}_0),\ \gamma \in \pi_1(C \setminus \mathcal{P}_m, z_0)\}$. In addition, $\pi_1(M_0, \tilde{z}_0)$ is isomorphic to the semi-direct product $\pi_1(F_{z_0}, \tilde{z}_0) \rtimes \pi_1(C \setminus \mathcal{P}_m, z_0)$ constructed from the monodromy action of $\pi_1(C \setminus \mathcal{P}_m, z_0)$ on $\pi_1(F_{z_0}, \tilde{z}_0)$.

As a consequence of the Theorem 5.9.1 we have the following corollary.

Corollary 5.9.2. *Suppose that $p : M \to C$ satisfies the conditions (a), (b), (c), (d) and $\pi_1(F_{z_0}, \tilde{z}_0)$ is a free group generated by $\mu_0, \mu_1, \cdots, \mu_{d-1}$. Suppose that the group $\pi_1(C \setminus \mathcal{P}_m, z_0)$ is generated by $\gamma_1, \gamma_2, \cdots, \gamma_m$. Then $\pi_1(M, \tilde{z}_0)$ is isomorphic to the group defined by the presentation*

$$\langle \mu_0, \mu_1, \cdots, \mu_{d-1} \mid \mu_j^{\gamma_i} = \mu_j, \quad i = 1, 2, \cdots, m, \quad j = 0, 1, \cdots, d-1 \rangle.$$

5.10 Local Fundamental Group of Curve Singularities

In the Section 5.6, we have discussed the monodromy action around the curve singularity for the affine curve $C:\ x^m - y^d = 0$, where $m,d \in \mathbb{Z}_{\geq 2}$. Assume that $\pi_1(F_{z_0},\tilde{z}_0)$ is a free group generated by $\mu_0,\mu_1,\cdots,\mu_{d-1}$. The monodromy relation was $\mu_{j+m} = \mu_j$. Then by Corollary 5.9.2, the fundamental group $\pi_1(\Delta_{2\varepsilon}\times\Delta_{2\rho}\setminus C)$ is isomorphic to $G_{m,d}$ defined by the presentation below:

$$G_{m,d} := \left\langle \delta,\mu_j \mid \delta = \mu_{d-1}\mu_{d-2}\cdots\mu_0,\ \mu_{j+d} = \delta\mu_j\delta^{-1},\ \mu_j = \mu_{j+m},\ j\in\mathbb{Z}\right\rangle \quad (5.10.1)$$

Theorem 5.10.1. *Assume that C is a curve given by the equation $x^m - y^m = 0$, which is a pencil of m lines. Then the local fundamental group of its complement is isomorphic to the group*

$$G_{m,m} = \langle \delta,\mu_j \mid \delta = \mu_{m-1}\mu_{m-2}\cdots\mu_0,\ [\delta,\mu_j] = 1,\ j = 0,1,\cdots,m-1\rangle. \quad (5.10.2)$$

Proof. Set $d = m$ in (5.10.1), then the relations $\mu_j = \mu_{j+m}$ and $\mu_{j+m} = \delta\mu_j\delta^{-1}$ imply that $\mu_j\delta = \delta\mu_j$, i.e, $[\delta,\mu_j] = 1,\ j = 0,1,,\cdots,m-1$. □

If $m = 2$, then $\delta = \mu_1\mu_0$. The relations $[\delta,\mu_j] = 1$ reduces to $[\mu_0,\mu_1] = 1$. Hence, $G_{2,2} = \langle \mu_0,\mu_1 \mid \mu_0\mu_1 = \mu_1\mu_0\rangle$ is isomorphic to the abelian group $\mathbb{Z}\times\mathbb{Z}$.

Theorem 5.10.2. *Suppose C is an affine curve given by the equation $x^2 - y^{2n} = 0$. Then the local fundamental group of its complement is isomorphic to the group $G_{2,2n}$ defined by the presentation $\langle \mu_0,\mu_1 \mid (\mu_0\mu_1)^n = (\mu_1\mu_0)^n\rangle$.*

Proof. Set $m = 2$ and $d = 2n$ in (5.10.1), then the relation $\mu_j = \mu_{j+2}$ imply that $\mu_{j+2n} = \mu_j$ for any $j\in\mathbb{Z}$, and $\mu_{2k-1} = \mu_1$, $\mu_{2k} = \mu_0$ for any $k\in\mathbb{Z}$. Then we have $\delta = \mu_{2n-1}\mu_{2n-2}\cdots\mu_0 = (\mu_1\mu_0)^n$ and $[\delta,\mu_j] = 1$ by the relation $\mu_{j+2n} = \mu_j = \delta\mu_j\delta^{-1}$. Note that, $\mu_0\delta = (\mu_0\mu_1)^n\mu_0$ and $\delta\mu_1 = \mu_1(\mu_0\mu_1)^n$. Therefore we have $(\mu_1\mu_0)^n = (\mu_0\mu_1)^n$ from the relations $\delta\mu_j = \mu_j\delta,\ j = 1,2$. □

Theorem 5.10.3 (Oka, 1975)**.** *Suppose C is the affine curve given by the equation $x^m - y^d = 0$, where m and d are co-prime integers. Then the local fundamental group $G_{m,d}$ of its complement is isomorphic to the group G' defined by the presentation $\langle \alpha, \beta \mid \alpha^m = \beta^d \rangle$.*

Proof. For any $j \in \mathbb{Z}$, let (a_j, b_j) be a pair of integers satisfying $a_j d + b_j m = j$. From the relations $\mu_{j+d} = \delta \mu_j \delta^{-1}$ and $\mu_j = \mu_{j+m}$, we have

$$\mu_{j+k} = \mu_{a_j d + b_j m + k} = \mu_{a_j d + k} = \delta^{a_j} \mu_k \delta^{-a_j}$$

for all $k \in \mathbb{Z}$. Therefore $\mu_{j+d-1}\mu_{j+d-2}\cdots\mu_j = \delta^{a_j}(\mu_{d-1}\mu_{d-2}\cdots\mu_0)\delta^{-a_j} = \delta$. Define an element $\tau \in G_{m,d}$ by $\tau = \mu_{m-1}\mu_{m-2}\cdots\mu_0$. Then we have

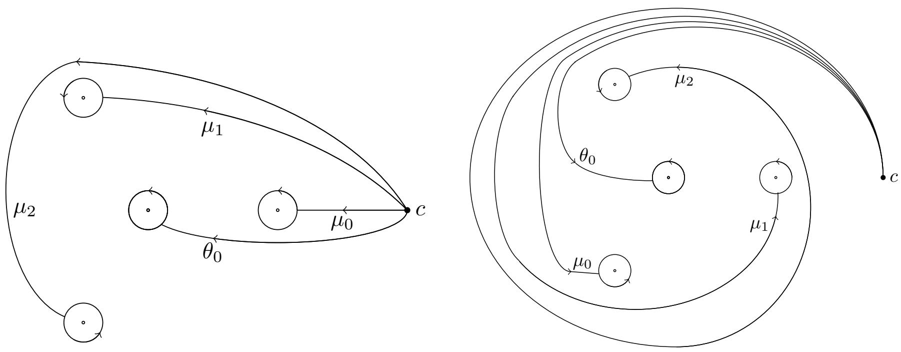
$$\delta^m = \mu_{md-1}\mu_{md-2}\cdots\mu_0 = \tau^d.$$

In addition, since $\delta^{a_1+km}\tau^{b_1-kd} = \delta^{a_1}\tau^{b_1}$ for any integer k, we can assume $b_1 > 0$ and $a_1 < 0$. Then we have

$$\begin{aligned} \delta^{a_1}\tau^{b_1} &= (\mu_{|a_1|d}\mu_{|a_1|d-1}\cdots\mu_1)^{-1}(\mu_{b_1 m-1}\mu_{b_1 m-2}\cdots\mu_0) \\ &= (\mu_1^{-1}\cdots\mu_{|a_1|d-1}^{-1}\mu_{|a_1|d}^{-1})(\mu_{b_1 m-1}\mu_{b_1 m-2}\cdots\mu_0) \\ &= \mu_0, \end{aligned}$$

because $b_1 m - 1 = |a_1|d$. This means, every element of $G_{m,d}$ can be written in terms of δ and τ, Explicitly $\mu_j = \delta^{a_j}\mu_0\delta^{-a_j} = \delta^{a_j}(\delta^{a_1}\tau^{b_1})\delta^{-a_j}$. Hence we can define a surjective homomorphism $\varphi : G' \to G_{m,d}$ by $\alpha \mapsto \delta, \beta \mapsto \tau$. It's inverse homomorphism $\varphi^{-1} : G_{m,d} \to G'$ is $\delta \mapsto \alpha$, $\mu_j \mapsto \alpha^{a_j}(\alpha^{a_1}\beta^{b_1})\alpha^{-a_j}$. Note that, from $\alpha^m = \beta^d$, the right hand side does not depend on the choice of the pair (a_j, b_j). Thus φ is an isomorphism. $\square$

Under the notations in Section 5.6 let us now consider the curve C on $\Delta_{2\varepsilon} \times \Delta_{2\rho}$ defined by $y(x^m - y^d) = 0$. The fiber over $x \in \Delta_{2\varepsilon}^*$ is $\Delta_{2\rho}$ minus 0 and the d-th roots



(a) (b)

Figure 5.5 The monodromy action on $\pi_1(F_{x_0}, \tilde{x}_0)$ when $C : y(x^2 - y^3) = 0$.

of x^m. Choose the base point of $\Delta^*_{2\varepsilon}$ at $x_0 := \varepsilon$. The group $\pi_1(\Delta^*_{2\varepsilon}, x_0)$ is an infinite cyclic group generated by the homotopy class $\gamma = [g]$ of the loop $g(t) = \varepsilon \exp(2\pi i t)$.

On the other hand, denote by θ_0 the meridian around 0 in F_{x_0} , and by μ_j the meridians around d-th roots of x^m in F_{x_0}. The fiber F_{x_0} is homotopic to the bouquet of $d+1$ circles, and hence its fundamental group $\pi_1(F_{x_0}, \tilde{x}_0)$ is a free group generated by $d+1$ elements $\theta_0, \mu_0, \mu_1, \cdots, \mu_{d-1}$ which are represented by the meridians given in Figure 5.5a (It is drawn for the case $(m,d) = (2,3)$).

The monodromy action $\pi_1(\Delta^*_{2\varepsilon}, x_0)$ on $\pi_1(F_{x_0}, \tilde{x}_0)$ rotates the d-th roots of x^m around the origin with angular speed $2\pi m/d$ while fixing the point 0. Therefore the meridians θ_0 and μ_i are deformed to the meridians $\tilde{\theta}_0$ and $\tilde{\mu}_i$, respectively (See Figure 5.5b). Therefore the monodromy action of $\pi_1(\Delta^*_{2\varepsilon}, x_0) = \langle \gamma \rangle$ on the free group $\pi_1(F_{x_0}, \tilde{x}_0) = \langle \theta_0, \mu_0, \mu_1, \cdots, \mu_{d-1} \rangle$ is given by $\theta_0^\gamma = \tilde{\theta}_0$ and $\mu_i^\gamma = \tilde{\mu}_i$. Set $\delta := \mu_{d-1}\mu_{d-2}\cdots\mu_0$ and $\delta_0 = \delta\theta_0$. The homotopy class δ_0 is represented by the big loop around all deleted points. Let$j \in \mathbb{Z}_{\geq 0}$, $j = ad + r$ and r is the remainder of j divided by d. Set $\mu_j := \delta_0^a \mu_r \delta_0^{-a}$, then we have the relation $\tilde{\mu}_i = \mu_{i+m}$. In addition, set $\theta_k = \delta_0^k \theta_0 \delta_0^{-k}$ and $\tau := \mu_{m-1}\mu_{m-2}\cdots\mu_0$, then we have $\tilde{\theta}_k = \tau\theta_k\tau^{-1}$. Hence the monodromy action of $\pi_1(\Delta^*_{2\varepsilon}, x_0)$ on $\pi_1(F_{x_0}, \tilde{x}_0)$ is given by the relations $\theta_k^\gamma = \tau\theta_k\tau^{-1}$ and $\mu_i^\gamma = \mu_{i+m}$. Then by Corollary 5.9.2, the fundamental group $\pi_1(\Delta_{2\varepsilon} \times \Delta_{2\rho} \setminus C)$ is isomorphic to $G_{m,d,0}$ defined by the presentation below:

$$G_{m,d,0} := \left\langle \theta_k, \delta_0, \tau, \mu_j \;\middle|\; \begin{array}{l} \delta_0 = \mu_{d-1}\mu_{d-2}\cdots\mu_0\theta_0,\ \tau = \mu_{m-1}\mu_{m-2}\cdots\mu_0,\ \tau\theta_k = \theta_k\tau, \\ \theta_k = \delta_0^k\theta_0\delta_0^{-k},\ \mu_{j+d} = \delta_0\mu_j\delta_0^{-1},\ \mu_j = \mu_{j+m},\ j,k \in \mathbb{Z} \end{array} \right\rangle \tag{5.10.3}$$

First of all, let us discuss the basic cases $y(x-y^n) = 0$ and $y(x^n - y) = 0$.

- If $(m,d) = (1,n)$, then by the notations of (5.10.3), the monodromy relations for μ_j's are $\mu_j = \mu_{j+1}$. Therefore $\delta_0 = \mu_0^n\theta_0$, and the relation $\mu_j = \mu_{j+n} = \delta_0\mu_j\delta_0^{-1}$ implies $\mu_0\theta_0 = \theta_0\mu_0$ which is the monodromy relation for θ_0. Since $\mu_j = \mu_0$, $\delta_0 = \mu_0^n\theta_0$ and $\theta_k = \delta_0^k\theta_0\theta_{-k}$, then every element of $G_{1,n,0}$ can be written in terms of μ_0 and θ_0. Therefore, the group $G_{1,n,0}$ has presentation $\langle \mu_0, \theta_0 \mid \mu_0\theta_0 = \theta_0\mu_0 \rangle \simeq \mathbb{Z}\times\mathbb{Z}$. Note that this is isomorphic to the group $G_{2,2}$ in (5.10.2). Because, the line $y=0$ and the curve $x-y^n=0$ meet transversally at the origin.

- If $(m,d) = (n,1)$, the fiber over $x \in \Delta^*_{2\varepsilon}$ is $\Delta_{2\rho}$ minus 0 and the point x^n, denote the loops around them by θ_0 and μ_0, respectively. The loop $\delta_0 := \mu_0\theta_0$ is the big loop surrounding these two deleted points. The monodromy action $\pi_1(\Delta^*_{2\varepsilon}, x_0)$ on $\pi_1(F_{x_0}, \tilde{x}_0)$ rotates n times the the point x^n around the origin while fixing the point 0. Thus the monodromy relations are $\mu_0 = \delta_0^n\mu_0\delta_0^{-n}$ and $\theta_0 = \delta_0^n\theta_0\delta_0^{-n}$. Taking into account the relation $\delta_0 = \mu_0\theta_0$, one can easily show that these two relations are equivalent to the relation $(\mu_0\theta_0)^n = (\theta_0\mu_0)^n$. Thus, $G_{n,1,0} = \langle \theta_0, \mu_0 \mid (\mu_0\theta_0)^n = (\theta_0\mu_0)^n \rangle$ which is isomorphic to the group $G_{2,2n}$ in Theorem 5.10.2.

Now, we will study the cases, $(m,d) \in \{(m,m),(2,2n),(2n,2)\}$ and the case m and d are co-prime. These cases are stated explicitly in the following theorems.

Theorem 5.10.4. *Assume that C is a curve given by the equation $y(x^m - y^m) = 0$, which is a pencil of $m+1$ lines. Then the local fundamental group of its complement is isomorphic to the group $G_{m+1,m+1}$ in* (5.10.1).

Proof. Set $d = m$ in (5.10.3), then the relations $\mu_j = \mu_{j+m}$ and $\mu_{j+m} = \delta_0\mu_j\delta_0^{-1}$ imply that $\mu_j\delta_0 = \delta_0\mu_j$, i.e, $[\delta_0, \mu_j] = 1$, $j = 0, 1, , \cdots, m-1$. In addition,

$$\theta_k = \delta_0^k\theta_0\delta_0^{-k} = (\tau\theta_0)^k\theta_0(\tau\theta_0)^{-k} = (\theta_0\tau)^k\theta_0(\tau\theta_0)^{-k} = \theta_0(\tau\theta_0)^k(\tau\theta_0)^{-k} = \theta_0$$

for all $k \in \mathbb{Z}$, and the relation $\delta_0 = \tau\theta_0 = \theta_0\tau$ is equivalent to $\delta_0\theta_0 = \theta_0\tau\theta_0 = \theta_0\delta_0$. Then, it is clear that $G_{m,m,0}$ is isomorphic to group

$$\langle \theta_0, \delta_0, \mu_j \mid \delta_0 = \mu_{m-1}\mu_{m-2}\cdots\mu_0\theta_0,\ [\delta_0, \mu_j] = [\delta_0, \theta_0] = 1,\ j = 0, 1, \cdots, m-1\rangle.$$

This group is also isomorphic to the group $G_{m+1,m+1}$ via $\theta_0 \mapsto \mu_0$, $\delta_0 \mapsto \delta$, $\mu_j \mapsto \mu_{j+1}$, $j = 0, 1, \cdots, m-1$. $\square$

Theorem 5.10.5. *Suppose C is an affine curve given by the equation $y(x^2 - y^{2n}) = 0$. Then the local fundamental group of its complement is isomorphic to the group $G_{2,2n,0}$ defined by the presentation*

$$\left\langle \theta_0, \mu_0, \mu_1 \ \middle|\ \mu_1\mu_0\theta_0 = \theta_0\mu_1\mu_0, \quad (\mu_1\mu_0)^n\theta_0 = \mu_0\theta_0\mu_1(\mu_0\mu_1)^{n-1} \right\rangle \tag{5.10.4}$$

Proof. Set $m = 2$ and $d = 2n$ in (5.10.3), then the relation $\mu_j = \mu_{j+2}$ imply that $\mu_{j+2n} = \mu_j$ for any $j \in \mathbb{Z}$, and $\mu_{2k-1} = \mu_1$, $\mu_{2k} = \mu_0$ for any $k \in \mathbb{Z}$. Then we have $\tau = \mu_1\mu_0$, $\delta_0 = \mu_{2n-1}\mu_{2n-2}\cdots\mu_0\theta_0 = (\mu_1\mu_0)^n\theta_0$, $[\mu_1\mu_0, \theta_0] = 1$; and $[\delta_0, \mu_j] = 1$. Note that,

$$\begin{aligned}
&\mu_0\delta_0 = \mu_0(\mu_1\mu_0)^n\theta_0 = \mu_0\theta_0(\mu_1\mu_0)^n = \mu_0\theta_0\mu_1(\mu_0\mu_1)^{n-1}\mu_0,\\
&\delta_0\mu_0 = (\mu_1\mu_0)^n\theta_0\mu_0,\\
&\mu_1\delta_0 = \mu_1(\mu_1\mu_0)^n\theta_0,\\
&\delta_0\mu_1 = (\mu_1\mu_0)^n\theta_0\mu_1 = \mu_1(\mu_0\mu_1)^{n-1}\mu_0\theta_0\mu_1.
\end{aligned}$$

Therefore the relations $[\delta_0, \mu_j] = 1$ imply that $(\mu_1\mu_0)^n\theta_0 = \mu_0\theta_0\mu_1(\mu_0\mu_1)^{n-1}$. To complete proof it is enough to show that $\theta_k = \theta_0$ for all $k \in \mathbb{Z}$. This comes from the relations $\tau\theta_k = \theta_k\tau$ and $\theta_k = \delta_0^k\theta_0\delta_0^{-k}$, $k \in \mathbb{Z}$. Indeed,

$$\theta_k = (\tau^n\theta_0)^k\theta_0(\tau^n\theta_0)^{-k} = \tau^{nk}\theta_0^k\theta_0\theta_0^{-k}\tau^{-nk} = \theta_0 \quad \text{for all } k \in \mathbb{Z}.$$

Thus, the group $G_{2,2n,0}$ has presentation (5.10.5). □

Theorem 5.10.6. *Suppose C is an affine curve given by the equation $y(x^{2n} - y^2) = 0$. Then the local fundamental group of its complement is isomorphic to the group $G_{2n,2,0}$ defined by the presentation*

$$\langle \theta_0, \mu_0, \mu_1 \ | \ (\mu_1\mu_0\theta_0)^n = (\mu_0\theta_0\mu_1)^n = (\theta_0\mu_1\mu_0)^n \rangle \tag{5.10.5}$$

Proof. Set $m = 2n$ and $d = 2$ in (5.10.3), then clearly $\delta_0 = \mu_1\mu_0\theta_0$, and the relation $\mu_{j+2} = \delta_0\mu_j\delta_0^{-1}$ imply that

$$\mu_j = \begin{cases} \delta_0^k\mu_1\delta_0^{-k} & j = 2k+1 \\ \delta_0^k\mu_0\delta_0^{-k} & j = 2k. \end{cases} \tag{5.10.6}$$

Therefore, we have $\delta_0^n = \tau\theta_0^n$. Indeed,

$$\begin{aligned} \tau &= \mu_{2n-1}\mu_{2n-2}\cdots\mu_1\mu_0 \\ &= (\delta_0^{n-1}\mu_1\delta_0^{-n+1})(\delta_0^{n-1}\mu_0\delta_0^{-n+1})\cdots(\delta_0\mu_1\delta^{-1})(\delta_0\mu_0\delta^{-1})\mu_1\mu_0 \\ &= \delta_0^n(\delta_0^{-1}\mu_1\mu_0)^n \\ &= \delta_0^n\theta_0^{-n} \end{aligned}$$

Then by using the relation $\tau\theta_0 = \theta_0\tau$, we get $\delta_0^n\theta_0 = \theta_0\delta_0^n$ which implies

$$(\mu_1\mu_0\theta_0)^n = (\theta_0\mu_1\mu_0)^n. \tag{5.10.7}$$

On the other hand the relations $\mu_{j+2n} = \mu_j$ and $\mu_{j+2} = \delta_0\mu_j\delta_0^{-1}$ implies

$$\mu_j = \delta_0^n\mu_j\delta_0^{-n}, \quad \text{i.e.,} \quad (\mu_1\mu_0\theta_0)^n\mu_j = \mu_j(\mu_1\mu_0\theta_0)^n, \quad j = 1,2. \tag{5.10.8}$$

These two relations (5.10.7) and (5.10.8) equivalent to the relation

$$(\mu_1\mu_0\theta_0)^n = (\mu_0\theta_0\mu_1)^n = (\theta_0\mu_1\mu_0)^n. \tag{5.10.9}$$

Finally, since the equality (5.10.6) is valid together with the equalities $\theta_k = \delta_0^k\theta_0\delta_0^{-k}$, $\delta_0 = \mu_1\mu_0\theta_0$ and $\tau = \delta_0^n\theta_0^{-n}$ for all $j,k \in \mathbb{Z}$, then any element of $G_{2n,2,0}$ can be written as a word of the letters μ_0, μ_1, θ_0 and their inverses. Thus, $G_{2n,2,0}$ is the group generated by μ_0, μ_1, θ_0 with relations (5.10.9). $\square$

Theorem 5.10.7. *Suppose C is the affine curve given by the equation $y(x^m - y^d) = 0$, where m and d are co-prime integers. Then the local fundamental group $G_{m,d,0}$ of its complement is isomorphic to the group G_0' defined by the presentation*

$$\langle \alpha, \beta, \theta \mid \alpha^m\theta^{-m} = \beta^d,\ \beta\theta = \theta\beta \rangle. \tag{5.10.10}$$

Proof. Proof is similar to the proof of Theorem 5.10.3. For any $j \in \mathbb{Z}$, let (a_j, b_j) be a pair of integers satisfying $a_jd + b_jm = j$. In particular, since $(m,d) = 1$ then $a_{kd} = k$, $b_{km} = k$ while $a_{km} = b_{kd} = 0$. From the relations $\mu_{j+d} = \delta_0\mu_j\delta_0^{-1}$ and $\mu_j = \mu_{j+m}$, we get

$$\mu_{j+k} = \mu_{a_jd+b_jm+k} = \mu_{a_jd+k} = \delta_0^{a_j}\mu_k\delta_0^{-a_j}$$

for all $k \in \mathbb{Z}$. Then we have

$$\mu_{j+d-1}\mu_{j+d-2}\cdots\mu_j\theta_{a_j} = \delta_0^{a_j}(\mu_{d-1}\mu_{d-2}\cdots\mu_0\theta_0)\delta_0^{-a_j} = \delta_0^{a_j}\delta_0\delta_0^{-a_j} = \delta_0,$$

which implies $\mu_{j+d-1}\mu_{j+d-2}\cdots\mu_j = \delta_0\theta_{a_j}^{-1}$. Therefore we have the relation

$$\begin{aligned}
\tau^d &= \mu_{md-1}\mu_{md-2}\cdots\mu_0 \\
&= \delta_0\theta_{m-1}^{-1}\delta_0\theta_{m-2}^{-1}\cdots\delta_0\theta_1^{-1}\delta_0\theta_0^{-1} \\
&= \delta_0(\delta_0^{(m-1)}\theta_0^{-1}\delta_0^{-(m-1)})\delta_0(\delta_0^{(m-2)}\theta_0^{-1}\delta_0^{-(m-2)})\cdots\delta_0(\delta_0\theta_0^{-1}\delta_0^{-1})\delta_0\theta_0^{-1} \\
&= \delta_0^m\theta_0^{-m}
\end{aligned}$$

Since $(\delta_0\theta^{-1})^{a_1+km}\tau^{b_1-kd} = (\delta_0\theta^{-1})^{a_1}\tau^{b_1}$ for any integer k, we can assume $b_1 > 0$ and $a_1 < 0$. Then we have

$$\begin{aligned}
(\delta_0\theta_0^{-1})^{a_1}\tau^{b_1} &= (\mu_{|a_1|d}\mu_{|a_1|d-1}\cdots\mu_1)^{-1}(\mu_{b_1m-1}\mu_{b_1m-2}\cdots\mu_0) \\
&= (\mu_1^{-1}\cdots\mu_{|a_1|d-1}^{-1}\mu_{|a_1|d}^{-1})(\mu_{b_1m-1}\mu_{b_1m-2}\cdots\mu_0) \\
&= \mu_0,
\end{aligned}$$

because $b_1m-1 = |a_1|d$. Therefore $\mu_j = \delta_0^{a_j}\mu_0\delta_0^{-a_j} = \delta_0^{a_j}(\delta_0\theta_0^{-1})^{a_1}\tau^{b_1}\delta_0^{-a_j}$. We also know that $\theta_k = \delta_0^k\theta_0\delta_0^{-k}$. Hence, every element of $G_{m,d,0}$ can be written in terms of δ_0, θ_0 and τ.

Thus, we can define a surjective homomorphism $\varphi : G_0' \to G_{m,d,0}$ by $\alpha \mapsto \delta_0$, $\beta \mapsto \tau$ and $\theta \mapsto \theta_0$. It's inverse homomorphism $\varphi^{-1} : G_{m,d,0} \to G_0'$ is given by $\delta_0 \mapsto \alpha$, $\mu_j \mapsto \alpha^{a_j}(\alpha\theta^{-1})^{a_1}\beta^{b_1}\alpha^{-a_j}$ and $\theta_0 \mapsto \theta$. Note that, from $\alpha^m\theta^{-m} = \beta^d$, the right hand side does not depend on the choice of the pair (a_j, b_j). Thus φ is an isomorphism.

$\square$

5.11 Zariski Van-Kampen Theorem for Projective Plane Curves

Let $C \subset \mathbb{CP}^2$ be a complex projective plane curve defined by a homogeneous equation $\Phi(X,Y,Z) = 0$ of degree d. Suppose that C is reduced, that is Φ does not have any multiple factor. The complement $\mathbb{CP}^2 \setminus C$ is path-connected. We consider the fundamental group $\pi_1(\mathbb{CP}^2 \setminus C)$. Choose a base point $a \in \mathbb{CP}^2 \setminus C$. By a linear coordinate transformations, we can assume that $a := [0:1:0]$. Since $a \notin C$, the coefficient of Y^d in Φ is not zero. Let $L \subset \mathbb{CP}^2$ be the line defined by the equation $Y = 0$. For a point $P \in L$, let $\overline{pa} \subset \mathbb{CP}^2$ be the line connecting p and a. Put

$$\widetilde{X} := \{(p,q) \in L \times \mathbb{CP}^2 \mid Q \in \overline{pa}\}, \tag{5.11.1}$$

and let $\tilde{f} : \widetilde{X} \to L$ and $\rho : \widetilde{X} \to \mathbb{CP}^2$ be the projections onto each factors. If $q \neq a$, then $\tilde{f}^{-1}(q)$ consists of a single point, while $E := \rho^{-1}(a)$ is isomorphic to L by $\tilde{f}$. The morphism $\rho : \widetilde{X} \to \mathbb{CP}^2$ is called the *blowing up* of $\mathbb{CP}^2$ at a, and E is called the exceptional divisor (See Figure 5.6).

Put $X := \widetilde{X} \setminus \rho^{-1}(C)$, and let $f : X \to L$ be the restriction of $\tilde{f}$. Since the lifting $\rho^{-1}(C)$ and the exceptional divisor E has no common point , then ρ induces an isomorphism from $X \setminus E$ to $\mathbb{CP}^2 \setminus (C \cup \{a\})$, and we the following commutative diagram:

$$\begin{array}{ccc} \pi_1(X \setminus E) & \xrightarrow{\sim} & \pi_1(\mathbb{CP}^2 \setminus (C \cup \{a\})) \\ \downarrow & & \downarrow \wr \\ \pi_1(X) & \xrightarrow[(\rho|_X)_*]{} & \pi_1(\mathbb{CP}^2 \setminus C), \end{array} \tag{5.11.2}$$

where the vertical arrows are induced from inclusions. The left vertical arrow is surjective because E is a proper subvariety of X, and the right vertical arrow is an isomorphism because $\{a\}$ is a proper subvariety of $\mathbb{CP}^2 \setminus C$ with codimension 2. Hence $\rho\,|_X$ induces an isomorphism. Therefore, we will calculate $\pi_1(X)$.

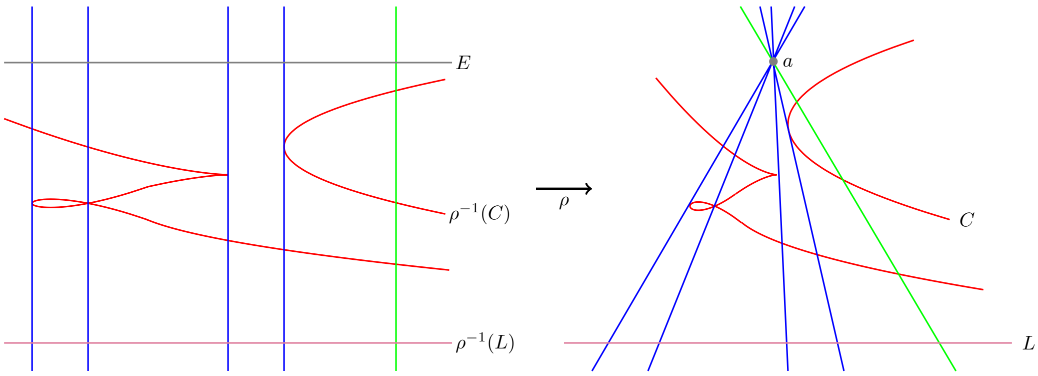


Figure 5.6 Blowing up at a.

For any point $p \in L$, the blow-up map ρ maps the intersection points of $\tilde{f}^{-1}(p)$ and $\rho^{-1}(C)$ to the intersection points of $\overline{pa}$ and C bijectively. Suppose that p is the point $[\xi : 0 : \eta]$, then $\overline{pa}$ is the line $\{[\xi : t : \eta] \mid t \in \mathbb{C} \cup \{\infty\}\}$, which correspond to a if $t = \infty$. Hence the intersection points of $\overline{pa}$ and C correspond to the roots of $\Phi(\xi, t, \eta) = 0$ bijectively. Let $D_\Phi(\xi, \eta)$ be the discriminant of $\Phi(\xi, t, \eta)$ regarded as a polynomial of t. Since we assumed Φ has no multiple factors, $D_\Phi(\xi, \eta)$ is not zero. It is a homogeneous polynomial of degree $d(d-1)$ in ξ and η. Put

$$\mathcal{P} := \{[\xi : 0 : \eta] \mid D_\Phi(\xi, \eta) = 0\}.$$

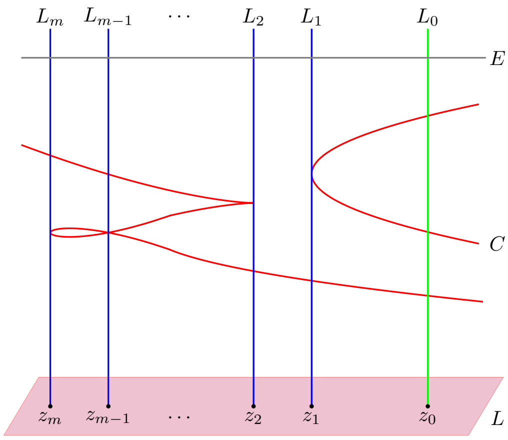
If $p \in L \setminus \mathcal{P}$, then $f^{-1}(p)$ is the line $\overline{pa}$ minus d distinct points. Hence the restriction of f to $f^{-1}(L \setminus \mathcal{P})$ is a locally trivial fiber space over $L \setminus \mathcal{P}$.

Choose a base point of $\mathcal{X}$ at $\tilde{z}_0 \in E \setminus (E \cap f^{-1}(\mathcal{P}))$, and let $z_0 := f(\tilde{z}_0)$ be the base point of L and $F_{z_0} := f^{-1}(z_0)$ be the fiber of f at z_0. The map $p \mapsto (p, a)$ is the holomorphic section $s : L \to \mathcal{X}$ of $f : \mathcal{X} \to L$ that passes through $\tilde{z}_0$. The image of s id E. Hence $\pi_1(L \setminus \mathcal{P})$ acts on $\pi_1(F_{z_0})$ from right. The projective line L is simply connected, and every fiber of f is irreducible since it is a projective line minus some points. Moreover, $\pi_1(F_{z_0})$ is the free group generated by homotopy classes $\mu_1, \mu_2, \cdots, \mu_{d-1}$ of $d-1$ meridians around $d-1$ points of $F_{z_0} \cap \rho^{-1}(C)$. Remember if one choose one of the points as the point at infinity, then a complex projective line minus d points is homotopic to complex plane $\mathbb{C}$ minus $d-1$ points, which has homotopy type of bouquet of $d-1$ circles. So, $\pi_1(F_{z_0}, \tilde{z}_0)$ is a free group of $d-1$ generators. But one may add μ_d as a generator with the relation $\mu_d \mu_{d-1} \cdots \mu_1 = 1$. Now we can apply the Corollary 5.9.2. Suppose that $\mathcal{P} := \mathcal{P}_m = \{z_1, z_2, \cdots, z_m\} \subset L$. Then $\pi_1(L \setminus P_m, z_0)$ is the free group generated by homotopy classes $\gamma_1, \gamma_2, \cdots \gamma_{m-1}$ of $m-1$ meridians around $m-1$ points of $\mathcal{P}_m$. One may add γ_m to $\pi_1(L \setminus P_m, z_0)$ as generator with the relation $\gamma_m \gamma_{m-1} \cdots \gamma_1 = 1$ (See Figure 5.7).

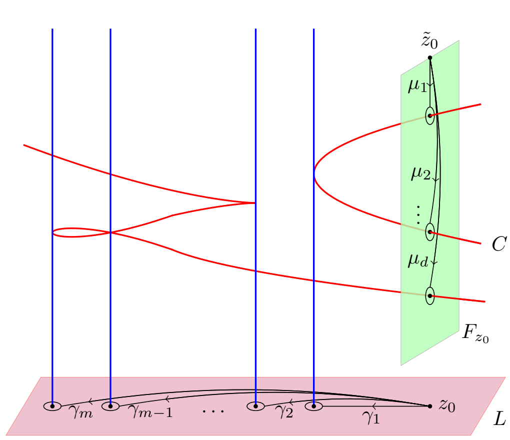
Theorem 5.11.1 (Zariski (1929), van Kampen (1933))**.** *Under the notations above, the fundamental group $\pi_1(\mathbb{CP}^2 \setminus C)$ is isomorphic to the group*

$$\langle \mu_1, \mu_2, \cdots, \mu_{d-1} \mid \mu_j^{\gamma_i} = \mu_i, \quad i = 1, 2, \cdots, m-1, \quad j = 1, 2, \cdots, d-1 \rangle.$$

Before considering the fundamental groups of complement of quadric line arrangement, a first insight on the fundamental groups of complements of some simple line arrangements.



(a) Projection.



(b) Merideans in L and F_{z_0}.

Figure 5.7 The generators of $\pi_1(L \setminus P_m, z_0)$ and $\pi_1(F_{z_0}, \tilde{z}_0)$.

1. If $C = L$, a single line, then $\mathbb{CP}^2 \setminus C = \mathbb{C}^2$ which is simply connected and therefore the fundamental group $\pi_1(\mathbb{CP}^2 \setminus C)$ is trivial.

2. If $C = L_1 \cup L_2$ consists of two lines, then considering one of the lines to be the line at infinity, say L_2, one obtains $\mathbb{CP}^2 \setminus C = \mathbb{C}^2 \setminus L_1 = \mathbb{C} \times \mathbb{C}^*$, so that $\pi_1(\mathbb{CP}^2 \setminus C) = \mathbb{Z}$.

3. If $C = L_0 \cup L_1 \cup \cdots L_m$ is a pencil of $m+1$ lines, considering L_m to be the line at infinity one obtains m parallel lines in $\mathbb{C}^2$, and the complement can be identified with $\mathbb{C} \setminus \{m \text{ points}\} \times \mathbb{C}$. Hence, in this case one has $\pi_1(\mathbb{CP}^2 \setminus C) = F_m$, the free group of rank n.

4. If $C = L_0 \cup L_1 \cup \cdots L_m$ is a near-pencil, i.e, the lines $L_0, L_1, \cdots, L_{m-1}$ meet at a single point while L_m transverse to them, considering L_m to be the line at infinity one obtains a pencil of m lines in $\mathbb{C}^2$. By using local braid monodromy, we computed its fundamental group in Theorem 5.10.1. Hence, in this case one has $\pi_1(\mathbb{CP}^2 \setminus C) = \langle \delta, \mu_j \mid \delta = \mu_{m-1}\mu_{m-2}\cdots\mu_0,\ [\delta, \mu_j] = 1,\ 0 \leq j \leq m-1 \rangle$. In particular, if $m = 2$ then $\pi_1(\mathbb{CP}^2 \setminus C)$ is isomorphic to $\mathbb{Z} \times \mathbb{Z}$.

5. If $C = L_0 \cup L_1 \cup \cdots L_m$ is a generic line arrangement, considering L_m to be the line at infinity one obtains m lines in general position in $\mathbb{C}^2$ which has $m(m-1)/2$ nodes. Let μ_i be a meridian around the line L_i. At each nodal

point $L_i \cap L_j$, take a projection $\mathbb{C}^2 \to \mathbb{C}$. The local braid monodromy gives only the condition $\mu_i\mu_j = \mu_j\mu_i$, does not effect the other meridians. Therefore, $\pi_1(\mathbb{CP}^2 \setminus C)$ is the abelian group $\langle \mu_i | \mu_i\mu_j = \mu_j\mu_i,\ \mu_m\mu_{m-1}\cdots\mu_0 = 1\rangle$.

6. Now suppose p and q be two points in $\mathbb{CP}^2$ and N be the line through p and q. Assume the pencils through p and q has $m+1$ and $n+1$ lines, respectively. Denote by C the union of this $m+n+1$ lines. Then $\widehat{\mathbb{CP}^2}$ is obtained from $\mathbb{CP}^2$ by blowing up the points p and q. As is well known, if one blows down the proper transform of the line N then obtains $\mathbb{CP}^1 \times \mathbb{CP}^1$ (See Figure 5.8). Then we have a birational morphisms $\mathbb{CP}^2 \leftarrow \widehat{\mathbb{CP}^2} \to \mathbb{CP}^1 \times \mathbb{CP}^1$. The primage $\hat{C}$ of C in $\widehat{\mathbb{CP}^2}$ equals the union of the proper transform of the lines in C and two exceptional divisors E_p and E_q. The image of $\hat{C}$ in $\mathbb{CP}^1 \times \mathbb{CP}^1$ equals $m+1$ in one ruling and $n+1$ lines in the other ruling. This birational morphism induces an isomorphism of complements. Therefore,

$$\pi_1(\mathbb{CP}^2 \setminus C) \simeq \pi_1(\mathbb{CP}^1 - \{n+1 points\}) \times \pi_1(\mathbb{CP}^1 - \{m+1 points\}) \simeq \mathbb{Z}^n \times \mathbb{Z}^m.$$

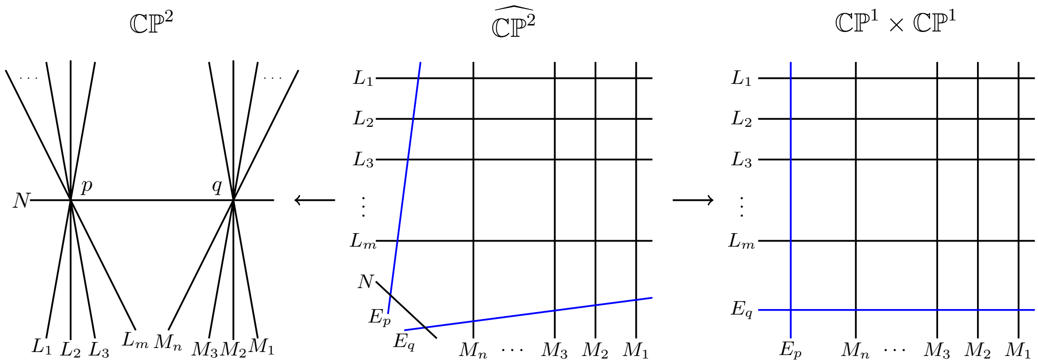


Figure 5.8 Birational morphism.

7. *Oka & Sakamato (1978)'s theorem:* Let C_1 and C_2 be two plane curves in $\mathbb{C}^2$ of degrees d_1 and d_2, respectively. If C_1 and C_2 meets at d_1d_2 distinct points, then $\pi_1(\mathbb{C}^2 \setminus (C_1 \cup C_2)) \simeq \pi_1(\mathbb{C}^2 \setminus C_1) \times \pi_1(\mathbb{C}^2 \setminus C_2)$.If these curves are projective algebraic curves in $\mathbb{CP}^2$, assuming L_∞ is a line at infinity in general position to C_1 and C_2, then $\pi_1(\mathbb{CP}^2 \setminus (C_1 \cup C_2))$ is decided by the following central

extension:

$$1 \to \mathbb{Z} \to \pi_1(\mathbb{C}^2 \setminus (C_1 \cup C_2)) \to \pi_1(\mathbb{CP}^2 \setminus (C_1 \cup C_2)) \to 1.$$

Some quadric arrangements can be obtained from line arrangements by using birational morphisms. Assume that $\mathcal{A}$ is a line arrangement, and φ be the involution $\varphi : \mathbb{CP}^2 \to \mathbb{CP}^2$ defined by $[X : Y : Z] \to [YZ : XZ : XY]$. Suppose that the lines H_1, H_2 and H_3 are respectively given by the equations $X = 0$, $Y = 0$ and $Z = 0$. If $\mathcal{A}$ is in general position with respect to $H_1 \cup H_2 \cup H_3$, then $\varphi(\mathcal{A})$ is an arrangement of smooth quadrics. In addition to those of $\mathcal{A}$, this arrangement has three more singular points where all irreducible components of $\varphi(\mathcal{A})$ meet transversally. In this case the group $\pi_1(\mathbb{CP}^2 \setminus \varphi(\mathcal{A}))$ can easily be found in terms of $\pi_1(\mathbb{CP}^2 \setminus \mathcal{A})$ as follows: Let $\mathcal{A} \cup_{i=1}^n L_i$, and μ_i be a meridian of L_i in $\mathbb{CP}^2 \setminus \mathcal{A}$. Let

$$\pi_1(\mathbb{CP}^2 \setminus \mathcal{A}) \simeq \langle \mu_1, \cdots, \mu_n | w_1 = w_2 = \cdots = w_m = \mu_n \cdots \mu_1 = 1 \rangle \tag{5.11.3}$$

be a presentation obtained by Zarsiki-van Kampen theorem. Set $\mathcal{A}' := \mathcal{A} \cup H_1 \cup H_2 \cup H_3$ and assume σ_i is a meridian around H_i. Since $\mathcal{A}$ is in general position to $H_1 \cup H_2 \cup H_3$, then one has

$$\pi_1(\mathbb{CP}^2 \setminus \mathcal{A}') = \left\langle \begin{array}{c|l} \mu_1, \cdots, \mu_n & [\mu_i, \sigma_j] = [\sigma_j, \sigma_k] = 1 \\ \sigma_1, \sigma_2, \sigma_3 & w_1 = \cdots = w_n = \mu_n \cdots \mu_1 \sigma_1 \sigma_2 \sigma_3 = 1 \end{array} \right\rangle. \tag{5.11.4}$$

Notice that $\sigma_j \sigma_k$ is a meridian of $\mathcal{A}'$ at $H_j \cap H_k$. Hence the group $\pi_1(\mathbb{CP}^2 \setminus \varphi(\mathcal{A}))$ can be obtained by setting $\sigma_1\sigma_2 = \sigma_1\sigma_3 = \sigma_2\sigma_3 = 1$ in the presentation of $\pi_1(\mathbb{CP}^2 \setminus \mathcal{A}')$. But these relations imply $\sigma := \sigma_1 = \sigma_2 = \sigma_3$ and $\sigma^2 = 1$. In addition, the relations $[\mu_i, \sigma_j] = 1$ and $\mu_n \cdots \mu_1 \sigma_1 \sigma_2 \sigma_3 = 1$ implies $(\mu_n \cdots \mu_1)^2 = 1$. Hence

$$\pi_1(\mathbb{CP}^2 \setminus \varphi(\mathcal{A})) = \left\langle \mu_1, \cdots, \mu_n \;\middle|\; \begin{array}{l} [\mu_i, \mu_n \cdots \mu_1] = 1 \\ w_1 = \cdots = w_n = (\mu_n \cdots \mu_1)^2 = 1 \end{array} \right\rangle. \tag{5.11.5}$$

Since σ is a central element of this group,

$$1 \to \mathbb{Z}_2 \to \pi_1(\mathbb{CP}^2 \setminus \varphi(\mathcal{A})) \to \pi_1(\mathbb{CP}^2 \setminus \mathcal{A}) \to 1$$

is an exact sequence.

For example, let $\mathcal{A}$ be the pencil of n lines $L_m : mX + Y - Z = 0$, $m = 1, \cdots, n$. Then $\varphi(\mathcal{A})$ is a pencil of n smooth quadrics $mYZ + XZ - XY = 0$ which are tangent to each other at $[1:0:0]$ and transverse at $[0:1:0]$ and $[0:0:1]$ (This intersection behavior of quadrics are independent of the choice of singular point of the pencil of lines whenever $\mathcal{A}$ is in general position with respect to $H_1 \cup H_2 \cup H_3$). Either using Zariski van Kampen theorem or assuming one of the lines L_m as a line at infinity one will see that the fundamental group $\pi_1(\mathbb{CP}^2 \setminus \mathcal{A})$ is a free group F_{n-1} of rank $n-1$, which has a presentation $\langle \mu_1, \mu_2, \cdots, \mu_n \,|\, \mu_n \cdots \mu_1 = 1 \rangle$. Then by equation (5.11.5), $\pi_1(\mathbb{CP}^2 \setminus \varphi(\mathcal{A}))$ has a presentation

$$\left\langle \mu_1, \cdots, \mu_n \right| [\mu_i, \mu_n \cdots \mu_1] = (\mu_n \cdots \mu_1)^2 = 1 \left. \right\rangle.$$

Next suppose, $\mathcal{A}$ has n lines in general position such that $\mathcal{A} \cup H_1 \cup H_2 \cup H_3$ is an arrangement of $n+3$ lines in general position. Then $\varphi(\mathcal{A})$ consists of n smooth conics in general position. Since $\pi_1(\mathbb{CP}^2 \setminus \mathcal{A}) < F_n$ is abelian, then $\pi_1(\mathbb{CP}^2 \setminus \varphi(\mathcal{A}))$ is an abelian group having a presentation

$$\left\langle \mu_1, \cdots, \mu_n \right| [\mu_i, \mu_j] = [\mu_i, \mu_n \cdots \mu_1] = (\mu_n \cdots \mu_1)^2 = 1 \left. \right\rangle.$$

Another method to get quadric arrangements are branched coverings. Assume $\mathcal{A} = \cup_{i=1}^n L_i$ and $\phi : \mathbb{CP}^2 \to \mathbb{CP}^2$ be the branched covering defined by $[X : Y : Z] \mapsto [X^2 : Y^2 : Z^2]$. Suppose the lines H_1, H_2 and H_3 are respectively given by the equations $X = 0, Y = 0$ and $Z = 0$, and set $\mathcal{A}' := \mathcal{A} \cup H_1 \cup H_2 \cup H_3$. If $\mathcal{A}$ is in general position to $H_1 \cup H_2 \cup H_3$, then $\phi^{-1}(\mathcal{A})$ is an arrangement of smooth quadrics. Any singular point of $\mathcal{A}$ lie four singular points of $\phi^{-1}(\mathcal{A})$ of the same type. In this case the group $\pi_1(\mathbb{CP}^2 \setminus \phi^{-1}(\mathcal{A}))$ can easily be found in terms of $\pi_1(\mathbb{CP}^2 \setminus \mathcal{A})$. Assume $\pi_1(\mathbb{CP}^2 \setminus \mathcal{A})$ has a presentation (5.11.3), then the presentation (5.11.5) is valid and there is an exact sequence

$$1 \to \pi_1(\mathbb{CP}^2 \setminus \phi^{-1}(\mathcal{A}')) \to \pi_1(\mathbb{CP}^2 \setminus \mathcal{A}') \to \mathbb{Z}_2 \times \mathbb{Z}_2 \to 1.$$

The group $\pi_1(\mathbb{CP}^2 \setminus \phi^{-1}(\mathcal{A}))$ is the quotient $\pi_1(\mathbb{CP}^2 \setminus \phi^{-1}(\mathcal{A}'))$ by the sub-group generated by the meridians of $\phi^{-1}(H_1)$, $\phi^{-1}(H_2)$ and $\phi^{-1}(H_3)$.

Suppose that $\mathcal{A}$ is a pencil of n lines $L_i : m_iX - Y + (b - m_ia)Z = 0$, $i = 1, \cdots n$. The singular point of $\mathcal{A}$ is $[a : b : 1]$. Assume $b \neq m_ia$ and $m_i \neq 0$ for each i, otherwise $\mathcal{A}$ will not be in general position with respect to $H_1 \cup H_2 \cup H_3$. Then $\varphi^{-1}(\mathcal{A})$ is an arrangement of n smooth quadrics $Q_i := \varphi^{-1}(L_i) : m_iX^2 - Y^2 + (b - m_ia)Z^2 = 0$. These n quadrics form a pencil through $[\mp\sqrt{a} : \mp b : 1]$. If $ab \neq 0$ there are four singular point but if one of a, b is zero while other is not, there are two singular points and the quadrics Q_i tangent to each other at these points. Before computing $\pi_1(\mathbb{CP}^2 \setminus \phi^{-1}\mathcal{A})$, first notice that $\pi_1(\mathbb{CP}^2 \setminus \mathcal{A}) = \{\mu_1, \mu_2, \cdots, \mu_n |\ \mu_n\mu_{n-1} \cdots \mu_1 = 1\}$ is a free group of rank $n - 1$.

First assume $ab \neq 0$ and take a projection onto a suitable line. Here the suitable means that the singular fibers does not contain no more than one multiple points. Therefore singular fibers either tangent to quadrics, or goes through singular points of $\phi^{-1}(\mathcal{A})$. Each smooth fiber F meets with each quadric Q_i at two points. In these smooth fibers, denote by μ_{i1} and μ_{i2} the meridians around $F \cap Q_i$. Around the tangency point of F with Q_i, braid monodromy gives the relations $(\mu_{i1}\mu_{i2})^2 = (\mu_{i2}\mu_{i1})^2$. Around the singular points of $\phi^{-1}(\mathcal{A})$, braid monodromies gives relations $[\mu_{1i}, \mu_{1n} \cdots \mu_{11}] = [\mu_{2i}, \mu_{2n} \cdots \mu_{21}] = 1$. Since any two meridians of Q_i are homotopic, then $\phi^{-1}(\mathcal{A})$ has a presentation $\langle \mu_1, \mu_2, \cdots, \mu_n |\ \mu_n\mu_{n-1} \cdots \mu_1 = 1\rangle$.

Incase $ab = 0$ (assume $a = 0$, $b \neq 0$), quadrics are tangent each other at two points. Around the tangency point of F with Q_i, braid monodromy gives the relations $(\mu_{i1}\mu_{i2})^2 = (\mu_{i2}\mu_{i1})^2$. Around the tangency points of $\phi^{-1}(\mathcal{A})$, braid monodromies gives relations $[\mu_{i1}, (\mu_{n1}\cdots\mu_{11})^2] = [\mu_{i2}, (\mu_{n2}\cdots\mu_{12})^2] = 1$. Since any two meridians of Q_i are homotopic, then $\phi^{-1}(\mathcal{A})$ has a presentation

$$\langle \mu_1, \mu_2, \cdots, \mu_n | \ (\mu_n\mu_{n-1}\cdots\mu_1)^2 = 1\rangle.$$

Next consider the arrangement $\mathcal{A}$ of lines $X \mp Y \mp Z = 0$ in general position. These lines together with the coordinate lines $X = 0$, $Y = 0$ and $Z = 0$, form an arrangement in Figure 6.17 and branched cover of this arrangement is the Naruki arrangement $Q_i : X^2 \mp Y^2 \mp Z^2 = 0$ which has twelve tacnodes as singularities. Take a projection onto a suitable line. Around the tangency point of the fiber F with Q_i, braid monodromy gives the relations $(\mu_{i1}\mu_{i2})^2 = (\mu_{i2}\mu_{i1})^2$. Around the tangency points of $\phi^{-1}(\mathcal{A})$, braid monodromies gives relations $(\mu_{i1}\mu_{j1})^2 = (\mu_{j1}\mu_{i1})^2$ and $(\mu_{i2}\mu_{j2})^2 = (\mu_{j2}\mu_{i2})^2$. Since any two meridians of Q_i are homotopic, then $\phi^{-1}(\mathcal{A})$ has a presentation $\langle \mu_1, \mu_2, \mu_3, \mu_4 | \ (\mu_i\mu_j)^2 = (\mu_j\mu_i)^2,\ 1 \leq, i < j \leq 4\rangle$.

CHAPTER SIX
BRANCHED COVERINGS AND ORBIFOLDS

In the Section 6.1, first we give some facts of branched covering due to references (Uludağ, 2007) and (Namba, 1987) and study the branched Galois coverings of complex manifolds, in particular the branched coverings of $\mathbb{CP}^1$ as motivation, and introduce some partial results by several authors to Fenchel's problem. We will introduce the notions of orbifold and sub-orbifold in the Section 6.2, by using the reference (Uludağ, 2007) and (Namba, 1987). Due to Yoshida (1987), orbifold germs are related via covering maps. We will discuss in details of such covering relations of orbifold germs and exhibit them by drawing pictures in the Section 6.2.3. Section 6.3 is a survey on Chern classes and Chern numbers. Orbifold version of Chern numbers will be introduced in the Section 6.4. Kobayashi et al. (1989)'s Theorem 6.4.2 plays an important role to determine the uniformization of orbifolds. In the Sections 6.5 and 6.6, by applying this theorem to quadric-line arrangements we have obtained some new ball-quotient orbifolds. As in the covering relation among orbifold germs, these ball-quotient orbifolds are also related each other via covering maps. We have exhibited such covering relations in the Section 6.7.

6.1 Branched Coverings

Let X be an n-dimensional (connected) complex manifold. A surjective finite (proper) holomorphic mapping $\varphi : M \to X$, where M is an irreducible normal complex space, is called a *branched covering* of X. A topological finite covering map is a very special kind of branched covering. Any non-constant map between compact Riemann surfaces and the covering map

$$\varphi_m : (z_1, z_2, \cdots, z_n) \in \mathbb{C}^n \to (z_1^m, z_2^m, \cdots, z_n^m) \in \mathbb{C}^n \qquad (6.1.1)$$

are the most well known examples of branched coverings.

A *morphism* between branched coverings $\varphi : M \to X$ and $\psi : M \to X$ is a surjective holomorphic map $\vartheta : M \to N$ such that $\varphi(p) = \psi(\vartheta(p))$ for all $p \in M$. If ϑ is a biholomorphism then it is an isomorphism. The group G_φ of all automorphisms of φ is finite and acts on every fiber of φ. If G_φ acts transitively on every fiber of φ, then the covering map $\varphi : M \to X$ is called *branched Galois covering*. In this case, the orbit space M/G_φ is biholomorphic to X. The covering map $\varphi : M \to X$ is called an *abelian* (resp. *cyclic*) if φ is a Galois covering and G_φ is an abelian (resp. cyclic) group.

The *ramification locus* R_φ of a finite branched covering $\varphi : M \to X$ is the set of points p of M such that φ is not biholomorphic around p. The image $B_\varphi := \varphi(R_\varphi)$ is called the *branch locus* of φ and the map φ is said to be branched along B_φ. Both of the ramification locus and the branch locus are hypersurfaces (i.e. codimension 1 at every point) of M and X, respectively. In case φ is a topological covering then both R_φ and B_φ are empty, such φ is said to be *unbranched*. For a given branched covering map $\varphi : M \longrightarrow X$, the restriction $\varphi' :\ M \setminus R_\varphi \longrightarrow X \setminus B_\varphi$ is an unbranched covering. By a property of normal complex spaces we have the following properties (Namba, 1987):

1. $G_\varphi = G_{\varphi'}$ naturally,
2. φ is a Galois covering if and only if φ' is a Galois covering,
3. $|G_\varphi| \leq \deg \varphi$, where $|G_\varphi|$ is the order of the group G_φ, and $\deg \varphi$ is the mapping degree of φ. The equality holds if and only if φ is a Galois covering.

Conversely, the Grauert & Remmert (1958) theorem says that *"Given a topological unbranched finite covering $\varphi' : M' \longrightarrow X \setminus B$ with M' being connected, where X is a normal variety and B is a finite union of proper subvarieties of codimension 1; there exist an irreducible normal variety M with a finite branched covering $\varphi : M \longrightarrow X$ and a homeomorphism $s : M' \longrightarrow \varphi^{-1}(X \setminus B)$ such that $\varphi(x) = \varphi'\big(s(x)\big)$ for all $x \in M'$"* (Serre, 1960). So, there is a correspondence between subgroups of

$\pi_1(X\setminus B)$ of finite index and finite coverings of X branched along B. If φ' is Galois, then so is φ and therefore the covering φ is Galois if and only if the corresponding subgroup is normal (Namba, 1987, Theorem 1.1.17).

The *ramification divisor* of a finite branched covering $\varphi : M \to X$ of smooth spaces is the divisor of its jacobian; for singular spaces it can be defined for the restriction of φ to smooth parts of M and X (If φ is ramified only along a singular part then the ramification divisor is empty). If $\varphi : M \to X$ is Galois, it is possible to define the *branch divisor* on X as follows: Let $H_1, H_2, \cdots H_k$ be the irreducible components of the branch locus B_φ. Let $p \in H_i$ be a smooth point of B_φ, U be a small neighborhood of p and V be a connected component of $\varphi^{-1}(U)$. The degree m_i of $\varphi\,|_V$ does not depend on p and is called the *branching index* of φ along H_i. Then the branch divisor is defined as $D_\varphi := \sum_{i=1}^k m_i H_i$.

Definition 6.1.1. Let X be a complex manifold and $D = \sum_{i=1}^k m_i H_i$ be a divisor with coefficients in $m_i \in \mathbb{Z}_{>0}$. A Galois covering $\varphi : M \to X$ is said to be *branched at* D if $D_\varphi = D$.

Let X be a normal variety and $B = \cup_{i=1}^k H_i$ be a hypersurface with irreducible components H_i and $D = \sum_{i=1}^k m_i H_i$ be a divisor. Then the *orbifold fundamental group* of the pair (X, D) is defined as

$$\pi_1^{orb}(X, D) := \pi_1(X\setminus B, \star)/\langle\langle \mu_1^{m_1}, \cdots, \mu_k^{m_k} \rangle\rangle\ , \tag{6.1.2}$$

where $\star \in X\setminus B$ is a base point, μ_i is a meridian of H_i in $X\setminus B$, and $\langle\langle\rangle\rangle$ denotes the normal closure. Let N be a normal subgroup of finite index in $\pi_1(X\setminus B)$. The Galois covering corresponding to N is branched at D if and only if $\mu_i^{m_i} \in N$ and $\mu^m \notin N$ for $m < m_i$ and $i = 1, 2, \cdots, k$ (this condition is known as *branching condition* in the sequel). The condition $\mu_i^{m_i} \in N$ amounts the existence of the factorization

$$\begin{tikzcd}
\pi_1(X\setminus B) \arrow[rr, two heads, "\varphi"] \arrow[dr, two heads] & & \pi_1(X\setminus B)/N =: G \\
& \pi_1^{orb}(X,D) \arrow[ur, two heads, "\psi"'] &
\end{tikzcd}$$

whereas the branching condition $\mu_i^m \notin N$ for $m < m_i$ means that $\varphi(\mu_i) \in G$ is strictly of order m_i. Thus, the coverings of X branched along D are really controlled by the group $\pi_1^{orb}(X,D)$, and there is a Galois correspondence between the Galois covering of X branched along D and normal subgroups of $\pi_1^{orb}(X,D)$ satisfying the branching condition. In particular, a covering of X branched at D is simply connected if and only if it is universal,i.e, the Galois group is the full group $\pi_1^{orb}(X,D)$.

Lemma 6.1.2 (Fox, 1957, §7)**.** *Let $M \to X$ be a Galois covering branched at D and with Galois group G. We have the exact sequence*

$$0 \to \pi_1(M) \to \pi_1^{orb}(X,D) \to G \to 0.$$

6.1.1 Branched Coverings of $\mathbb{CP}^1$

Let $X = \mathbb{CP}^1$, take distinct points $p_0, p_1, \cdots, p_k \in \mathbb{CP}^1$ and let $m_0, m_1, \cdots, m_k \in \mathbb{Z}_{>1}$. Put, $D := \sum_{i=1}^k m_i p_i$. Then, one has presentation

$$\pi_1(\mathbb{CP}^1 \setminus \{p_0, p_1, \cdots, p_k\}) \simeq \langle \mu_0, \mu_1, \cdots, \mu_k \mid \mu_0\mu_1\cdots\mu_k = 1\rangle$$

which is a free group of rank k. Then

$$\pi_1^{orb}(\mathbb{CP}^1, D) =< \mu_0, \mu_1, \cdots, \mu_k \mid \mu_0^{m_0} = \mu_1^{m_1} = \cdots = \mu_k^{m_k} = \mu_0\mu_1\cdots\mu_k = 1 >$$

Let $M \to \mathbb{CP}^1$ be a covering branched at D with Galois group G. By the Riemann-Hurwitz formula, the *Euler number* $e(M)$ of M equals

$$e(M) = |G| \left[e(\mathbb{CP}^1 \setminus \{p_0, p_1, \cdots, p_k\}) + \sum_{i=0}^{k} \frac{1}{m_i} \right] = |G| \left[1 - k + \sum_{i=0}^{k} \frac{1}{m_i} \right] \quad (6.1.3)$$

On the other hand, by the Koebe-Poincaré theorem, up to biholomorphism there are only three simply connected Riemann surfaces: the Riemann sphere $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$, the affine plane $\mathbb{C}$, and the Poincaré disc $\mathbf{B}_1 = \{z \in \mathbb{C} \mid |z| < 1\}$. If M is a compact Riemann surface, either $e(M) > 0$ and $M \simeq \mathbb{CP}^1$ (and therefore $e(M) = 2$), or $e(M) = 0$ and the universal cover of M is $\mathbb{C}$, or $e(M) < 0$ and the universal cover of M is $\mathbf{B}_1$. Note that the signature of $e(M)$ is completely determined by the data $(\mathbb{CP}^1, D)$ and no information on G is needed. Accordingly, the *orbifold Euler number* of $(\mathbb{CP}^1, D)$ is defined as

$$e^{orb}(\mathbb{CP}^1, D) := 1 - k + \sum_{i=0}^{k} \frac{1}{m_i} \quad \Rightarrow \quad e(M) = |G| e^{orb}(\mathbb{CP}^1, D). \quad (6.1.4)$$

Then, if $M \to \mathbb{CP}^1$ is a covering branched at D with G as Galois group, then

$$|G| = \frac{e(M)}{e^{orb}(\mathbb{CP}^1, D)}. \quad (6.1.5)$$

For $k = 0$, one has $e^{orb}(\mathbb{CP}^1, D) = 1 + 1/m_0 > 0$. Hence, if $M \to \mathbb{CP}^1$ is a covering branched at D, then $e(M) > 0$, which implies $M \simeq \mathbb{CP}^1$, and by the equation (6.1.5) one has $|G| = 2/(1 + 1/m_0)$, which is not positive integer unless $m_0 = 1$. Hence for $k = 0$ there are no coverings branched at D unless $m_0 = 1$. This also can be seen from the fact the group $\pi_1^{orb}(\mathbb{CP}^1, D)$ is trivial for $k = 0$.

For $k = 1$, one has $e^{orb}(\mathbb{CP}^1, D) = 1/m_0 + 1/m_1 > 0$. Hence, if a covering $M \to \mathbb{CP}^1$ branched at D exists, then $M \simeq \mathbb{CP}^1$, and by the equation (6.1.5) one has $|G| = 2m_0 m_1/(m_0 + m_1)$, which is a positive integer if and only if $m_0 = m_1 = m$. In this case such covering is the power map $[X : Y] \in \mathbb{CP}^1 \to [X^m : Y^m]$, and $\pi_1^{orb}(\mathbb{CP}^1, D) =$

$\langle \mu_0, \mu_1 \mid \mu_0^m = \mu_1^m = \mu_0\mu_1 = 1 \rangle \simeq \mathbb{Z}_m.$

Now, let us consider the case $k = 2$. Observe that the set $B = \{p_0, p_1, p_2\}$ is projectively rigid (See Corollary 2.2.2). Assume $m_0 \leq m_1 \leq m_2$ and put $\rho := 1/m_0 + 1/m_1 + 1/m_2 - 1$. Then, $e^{orb}(\mathbb{CP}^1, D) = \rho$. If $\rho > 0$ then the covering must be $\mathbb{CP}^1$ and $|G| = 2\rho^{-1}$. In this case (m_0, m_1, m_2) is one of the following: $(2,2,m)$, $(2,3,3)$, $(2,3,4)$ or $(2,3,5)$; the corresponding Galois groups must be of orders $2m$, 12, 24 and 60, respectively. Then the group

$$\pi_1^{orb}(\mathbb{CP}^1, D) \simeq \langle \mu_0, \mu_1, \mu_2 \mid \mu_0^{m_0} = \mu_1^{m_1} = \mu_2^{m_2} = \mu_0\mu_1\mu_2 = 1 \rangle \tag{6.1.6}$$

is called a *triangle group*, and it is finite of order $2\rho^{-1}$ if $\rho > 0$ and the branching condition is satisfied. Hence there exist a Galois coverings $\mathbb{CP}^1 \to \mathbb{CP}^1$ branched at D. Historically this follows from Klein's classification of finite subgroups of $\mathrm{PGL}(2;\mathbb{C}) \simeq \mathrm{Aut}(\mathbb{CP}^1)$. Each group is the symmetry group of one of the platonic solids inscribed in a sphere and they correspond to symmetry groups.

If $\rho = 0$, then $e^{orb}(\mathbb{CP}^1, D)$ vanishes and (m_0, m_1, m_2) is one of $(2,3,6)$, $(2,4,4)$, $(3,3,3)$ and $(2,2,\infty)$. In these cases the abelianizations of orbifold fundamental group are finite and satisfy the branching condition. Hence they are covered by a Riemann surfaces of genus 1 (an elliptic curve), and their universal covering is $\mathbb{C}$. The groups $\pi_1^{orb}(\mathbb{CP}^1, D)$ are infinite solvable. Similarly, Galois coverings of $\mathbb{CP}^1$ branched at four points with branching indices 2 are also elliptic curves. Each one of these coverings corresponds to a regular tessellation of the plane.

Any pair $(\mathbb{CP}^1, D)$ not considered above has negative orbifold Euler characteristic. The question of existence of finite coverings branched at D is known as *Fenchel's problem*. Fenchel's problem has been solved by Bundgaard & Nielsen (1951) and was generalized by Fox (1952) to branched coverings of Riemann surfaces.

Theorem 6.1.3. *Let $k \geq 2$ and $D := \sum_{i=0}^{k} m_i p_i$ be a divisor on $\mathbb{CP}^1$. Then there exists* *a finite Galois covering $M \to \mathbb{CP}^1$ branched at D; and M is*

i. ***(elliptic case)*** $\mathbb{CP}^1$ *if* $k = 1$ *and* $m_0 = m_1$ *or* $k = 2$ *and* $\frac{1}{m_0} + \frac{1}{m_1} + \frac{1}{m_2} > 1$,

ii. ***(parabolic case)*** *a Riemann surface of genus* 1 *if* $k = 2$ *and* $\frac{1}{m_0} + \frac{1}{m_1} + \frac{1}{m_2} = 1$, *or* $k = 3$ *and* $m_0 = m_1 = m_2 = m_3 = 2$,

iii. ***(hyperbolic case)*** *a Riemann surface of genus* > 1, *otherwise.*

6.1.2 Fenchel's Problem

A natural generalization of Fenchel's problem to higher dimensions is: given a complex manifold X and a divisor with coefficients in $\mathbb{Z}_{>1}$ on X, decide whether there exists a Galois covering $M \to X$ branched at D, regardless of the question of desingularization. There is no hope for a complete solution of generalized Fenchel's problem as in Theorem 6.1.3, since the group $\pi_1(X \setminus supp(D))$ does not admit a simple presentation, and it can be trivial, abelian, finite non-abelian or infinite. However, there are some partial results obtained by several authors. But the most important one related with line arrangements was proved by Kato (1987).

For a divisor $D = \sum_{i=1}^{n} m_i C_i$ on $\mathbb{CP}^2$, let us define the group of the divisor D as $Gr_n(D) := \pi_1(\mathbb{CP}^2 \setminus B)/\langle\langle \mu_1^{m_1}, \mu_2^{m_2}, \cdots, \mu_n^{m_n} \rangle\rangle$, where $B = \cup_{i=1}^{n} C_i$ is the support of D and μ_i is a meridian of C_i in $\mathbb{CP}^2 \setminus B$ and each of C_i is of degree d_i. First consider the basic case: $n = 1$ and C_1 is smooth. Then it is clear that $\pi_1(\mathbb{CP}^2 \setminus C_1) = \mathbb{Z}_{d_1}$ and $Gr_1(D) = \mathbb{Z}_{k_1}$, where $k_1 := \gcd(m_1, d_1)$. Thus, Fenchel's problem for $D = m_1 C_1$ has a positive solution if and only if $m_1 \mid d_1$, and the solution is given by an abelian covering. Obviously this still gives a solution if $\pi_1(\mathbb{CP}^2 \setminus C_1)$ is non-abelian, since the abelianization of $\pi_1(\mathbb{CP}^2 \setminus C_1)$ is $\mathbb{Z}_{d_1}$. Similarly, if $n > 1$, then the abelianization $H_1(\mathbb{CP}^2 \setminus C, \mathbb{Z})$ of $\pi_1(\mathbb{CP}^2 \setminus B)$ is the abelian group

$$H_1(\mathbb{CP}^2 \setminus B, \mathbb{Z}) = \langle \mu_1, \mu_2, \cdots, \mu_n \mid \mu_1^{d_1} \mu_2^{d_2} \cdots \mu_n^{d_n} = 1 \rangle.$$

Hence, the abelianization of $Gr_n(D)$ has the presentation

$$\langle \mu_1, \mu_2, \cdots, \mu_n \mid \mu_1^{d_1}\mu_2^{d_2}\cdots\mu_n^{d_n} = \mu_1^{m_1} = \mu_2^{m_2} = \cdots = \mu_n^{m_n} = 1 \rangle.$$

Put $\kappa_i := m_i/\gcd(m_i, d_i)$, and let ρ_i be the smallest common multiple of $\{\kappa_j \mid i \neq j\}$. Then an abelian covering solves the Fenchel's problem provided that κ_i divides ρ_i for $1 \leq i \leq n$.

However, abelian coverings give a solution to Fenchel's problem only for very restricted cases. If one assume the divisor $D = \sum_{i=1}^{n} m_i L_i$, whose support B is a line arrangement, the coefficients m_i being prime. Then the condition $\kappa_i \mid \rho_i$ is never satisfied. But, $\pi_1(\mathbb{CP}^2 \setminus B)$ is big if it is not abelian. Hence, some non-abelian covers must give a solution to Fenchel's problem. Indeed, Kato proved the following theorem:

Theorem 6.1.4 (Kato, 1987). *Let $\mathcal{A} = \{H_i : i = 0, 1, \cdots, k\}$ be an arrangement of lines in $\mathbb{CP}^2$ such that any line contains a point of multiplicity at least* 3*. Let $m_i \in \mathbb{Z}_{>1}$ and put $D := \sum_{i=0}^{k} m_i H_i$. Then there exists a finite Galois covering of $\mathbb{CP}^2$ branched D.*

Kato also describes the resolution of singularities of the covering surfaces, and this resolution is compatible with the blowing-up of points of multiplicity > 2 of the branch locus. There is a generalization of the Kato's theorem to quadric arrangements given by Namba (1987).

Theorem 6.1.5 (Theorem 1.5.8, Namba (1987)). *Let $k \geq 2$ and $Q_1, Q_2, \cdots Q_k$ be irreducible quadrics in $\mathbb{CP}^2$. Assume that, for every Q_i there is another Q_j such that they have two tacnodes. Then for any positive integers $m_1, m_2, \cdots, m_k$ greater than* 1*, there is a finite Galois covering $\varphi : M \to \mathbb{CP}^2$ which branches at $D = \sum_{i=1}^{k} m_i Q_i$.*

Another extreme example is the Oka curve. For co-prime integers p and q, Oka

(1975) constructed the following irreducible curves

$$\begin{aligned} &\mathcal{C}^1_{p,q} : x^p - y^q = 0 \quad \subset \mathbb{C}^2 \\ &\mathcal{C}^2_{p,q} : (X^p + Y^q)^q + (Y^q + Z^q)^p = 0 \quad \subset \mathbb{CP}^2 \end{aligned} \tag{6.1.7}$$

and observed that

$$\pi_1(\mathbb{C}^2 \setminus \mathcal{C}^1_{p,q}) = \langle a, b \mid \ a^p = b^q \rangle \tag{6.1.8}$$

and

$$\pi_1(\mathbb{CP}^2 \setminus \mathcal{C}^2_{p,q}) = \langle a, b \mid \ a^p = b^q = 1 \rangle \simeq \mathbb{Z}_p * \mathbb{Z}_q \tag{6.1.9}$$

with free commutator subgroup $\mathbb{F}_{(p-1)(q-1)}$ of rank $(p-1)(q-1)$. In his Ph.D. thesis, Uludağ (2000) proved the following theorem.

Theorem 6.1.6 (Corollary 6.1.1, Uludağ (2000))**.** *If $\mathcal{C}_{p,q}$ is an Oka curve, then for any $m \geq 1$, there exist a finite Galois covering of $\mathbb{CP}^2$ branched at $m\mathcal{C}_{p,q}$.*

Given a projective manifold X, which groups can appear as the Galois group of a branched covering of X? This question has the following solution.

Theorem 6.1.7 (Namba (1991))**.** *(i)For any projective manifold X and any finite group G, there is a finite branched Galois covering $M \to X$ with G as the Galois group. (ii) For $n \geq 2$ there exists a covering of the germ $(\mathbb{C}^n, \mathbf{0})$ with a given finite Galois group.*

6.2 Orbifolds

6.2.1 Transformation Groups

An *action* of a topological group G on a space M is a (continuous) map $G \times M \to M$, denoted by $(g, z) \mapsto gz$, so that $g(hz) = (gh)z$ and $1z = z$ for all $g, h \in G$ and $z \in M$. In the sequel, it is written $G \curvearrowright M$ to mean that G acts on M. Given $z \in M$, $G_z := \{g \in G \mid \ gz = z\}$ is the *isotropy subgroup* (or *stabilizer subgroup*) of G and

$G(z) := \{gz \in M \mid g \in G\}$ is the *orbit* of z. The action is *free* if $G_z = \{1\}$, for all $z \in M$, and it is *transitive* if there is only one orbit. Given $z \in M$, the natural map $G/G_z \to G(z)$ defined by $gG_z \to gz$ is a continuous bijection. The orbit space M/G is the set of orbits in M endowed with the quotient topology. A *slice* at a point $z \in M$ is a G_z-stable subset U_z so that the map $G \times_{G_z} U_z \to M$ is an equivariant homeomorphism onto a neighborhood of $G(z)$.

Suppose, G is a discrete group, M a Hausdorff space and $G \curvearrowright M$. The G-action is *proper* if given any two points $z_1, z_2 \in M$, there are open neighborhoods U of z_1 and V of z_2 so that $gU \cap V \neq \emptyset$ for only finitely many g.

Lemma 6.2.1. *A G-action on M is proper if and only if M/G is Hausdorff, each isotropy subgroup is finite, and each point $z \in M$ has a slice, i.e., there is a G_z-stable open neighborhood U_z so that $gU_z \cap U_z = \emptyset$ for all $g \in G \setminus G_z$.*

If G is a discrete group acting on a topological space M, the action is *properly discontinuous* if for any point $z \in M$, there is an open neighborhood U of z in M, such that the set of all $g \in G$ for which $gU \cap U \neq \emptyset$ consists of the identity only.

Let M be a connected complex manifold. By a *transformation group*, we shall mean a pair (G, M), where and G is a group of holomorphic automorphisms of M acting properly discontinuously, in particular for any $z \in M$ the isotropy group G_z is finite. The most important example of a transformation group is (G, M) , where M is a symmetric space such as the n-ball $\mathbf{B}_n$. Let (G, M) be a transformation group and X its orbit space with the projection $\varphi : M \to X$. The orbit space X is an irreducible normal analytic space endowed with a β-*map* defined as

$$\beta_\varphi : x \in X \to |G_z| \in \mathbb{Z}_{>0},$$

where $z \in \varphi^{-1}(x)$. In dimension 1, the orbit space is always smooth. In higher dimensions, X may have singularities of quotient type.

Let (G,M) be transformation group with the orbit space X and orbit map $\varphi : M \to X$, and put

$$R_\varphi := \{z \in M \mid |G_z| > 1\} \quad \text{and} \quad B_\varphi := \{x \in X \mid \beta_\varphi(x) > 1\}.$$

Let $\overline{X} := X \setminus Sing(X)$ be the smooth part of X, $x \in \overline{X}$ and $z \in \varphi^{-1}(x)$. Let M_z be the germ of M at z and X_x the germ of X at x. Then G_z acts on M_z, and the orbit space X_x. Since $|G_z|$ is finite and X_x is smooth, then the orbit map of germs $\varphi_z : M_z \to X_x$ is a finite Galois covering branched along $B_{\varphi,x}$. Therefore, one can define the local branch divisor $D_{\varphi,x}$. The local branch divisors patch yield a global branch divisor $D_\varphi := \sum_i m_i H_i$ supported by B_φ, where H_i are the irreducible components of B_φ.

On the other hand, since M_z is a smooth germ, it is simply connected. Hence φ_z must be the universal covering branched at $D_{\varphi,x}$ in other words the Galois group of φ_z is $G_z \simeq \pi_1^{orb}(X,D_\varphi)_x$. In particular one has

$$\beta(x) = |G_z| = |\pi_1^{orb}(X,D_\varphi)_x| \tag{6.2.1}$$

What is said above is in fact true for a singular point $x \in X$. For simplicity, assume that $x \notin B_\varphi$. Since M_z is a smooth germ it is simply connected and thus φ_z must be universal.

6.2.2 β-Spaces and Orbifolds

Recall that a transformation group (G,M) induces a β-map on its orbit space X. Conversely, let X be a normal complex space and β a map $X \to \mathbb{Z}_{>0}$. The pair (X,β) is called a β-*space*. The basic question related to a β-space is the *uniformization problem*: Under what conditions on a β-space (X,β), does there exist a (finite) transformation group (G,M) equipped with the orbit space X and the orbit map $\varphi : M \to X$ such that $\beta = \beta_\varphi$? In case such a transformation group (G,M) exist, it is called a *uniformization* of (X,β) and (X,β) is said to be uniformizable. Moreover, if G is abelian then (G,M) is called an abelian uniformization. Observe that these definitions can be localizable.

Definition 6.2.2. A locally finite uniformizable β-space (X,β) is called an *orbifold*. The space X is said to be the *base space* of (X,β), and (X,β) is said to be an orbifold over X. The set, $\{x \in X \mid \beta(x) > 1\}$ is called the *locus* of the orbifold.

Orbifolds (X,β) and (X',β') are said to be equivalent if there is a biholomorphism $\varepsilon : X \to X'$ such that the following diagram commutes.

$$\begin{tikzcd} X \arrow[rr, "\varepsilon"] \arrow[dr, "\beta"'] & & X' \arrow[dl, "\beta'"] \\ & \mathbb{Z}_{>0} & \end{tikzcd}$$

The product of β-spaces (X_1,β_1) and (X_2,β_2) is the β-space $(X_1 \times X_2,\beta)$, where $\beta(x,y) := \beta_1(x)\beta_2(y)$. If (X_i,β_i) is uniformized by (G_i,M_i) for $i = 1,2$, then the product orbifold is uniformized by $(G_1,M_1) \times (G_2,M_2)$.

Let (X,β) be an orbifold. Then by locally finite uniformizability, its locus $B_\beta = \{x \in X \mid \beta(x) > 1\}$ is a locally finite union of hypersurfaces $H_1,H_2,\cdots$, and β must be constant along $H_i \setminus \big(Sing(B) \cup Sing(X)\big)$. Let m_i be this number and put $D_\beta := \sum_i m_i H_i$. The orbifold fundamental group of (X,β) is defined that of the pair (X,D_β), that is the group

$$\pi_1^{orb}(X,\beta) := \pi_1(X \setminus B_\beta)\Big/ \ll \mu_1^{m_1},\mu_2^{m_2},\cdots,\mu_k^{m_k} \gg, \qquad (6.2.2)$$

where μ_i is a meridian of H_i and "$\ll\gg$" denotes the normal closure.

Lemma 6.2.3 (Uludağ, 2007)**.** *If (X,β) is an orbifold, then $\beta(x) = \left|\pi_1^{orb}(X,\beta)_x\right|$ for any $x \in X$.*

Proof. Let $x \in X$. Since (X,β) is an orbifold, the germ $(X,\beta)_x$ admits a finite uniformization. Hence there is a unique transformation group (G_z,M_z) with $(X,\beta)_x$

as the orbit space such that $\beta_{\varphi_z} = b_x$, where $\varphi_z : M_z \to (X,\beta)_x$ is the quotient map and $\varphi_z^{-1}(x) = \{z\}$. By Lemma 6.1.2, one has the exact sequence

$$0 \to \pi_1(M_z) \to \pi_1^{orb}(X,\beta)_x \to G_z \to 0$$

Since M_z is smooth, it is simply connected, so that $G_z \simeq \pi_1^{orb}(X,\beta)_x$. Hence $\beta(x) = |G_z| = |\pi_1^{orb}(X,\beta)_x|$ for any $x \in X$. $\square$

Let (X,β) is an orbifold and $D_\beta = \sum_{i=1}^{k} m_i H_i$ be the associated divisor. By Lemma 6.2.3, β function is completely determined by D_β. In other words, the pair (X, D_β) determines the pair (X,β). On the other hand in dim≥ 2 most pairs (X,D) do not come from an orbifold. The local uniformizability condition puts an important restriction on the possible pairs (X,D), in particular local orbifold fundamental group of (X,D) must be finite. In dimension 2, this later condition is sufficient for local uniformizability, since by a theorem of Mumford (1961), a simply connected germ is smooth in dimension 2. This is no longer true in dimension ≥ 3 (see Brieskorn (1966) for counter examples).

Theorem 6.2.4 (Uludağ, 2007)**.** *In dimension 2, $(X,\beta)_x$ is an orbifold germ if and only if $\pi_1^{orb}(X,\beta)_x$ is finite.*

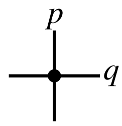
Proof. $(X,\beta)_x$ is an orbifold germ then by the definition of orbifold germ, clearly $\pi_1^{orb}(X,\beta)_x$ is finite. Conversely, if $\pi_1^{orb}(X,\beta)_x$ is finite then its universal covering is a finite covering by a simply connected germ. In dimension two, a simply connected germ is smooth by Mumford (1961)'s theorem. $\square$

To understand uniformization problem, let us consider the following examples:

Example 6.2.5. Let $p_0, p_1, \cdots, p_k$ be $k+1$ distinct points in $\mathbb{CP}^1$ and let $m_0, m_1, \cdots, m_k$ be positive integers. Let $\beta : \mathbb{CP}^1 \to \mathbb{Z}_{>0}$ be the function with $\beta(p_i) = m_i$ for $i = 0, 1, \cdots, k$ and $\beta(p) = 1$ otherwise. Around the point p_i, the β-space $(\mathbb{CP}^1, \beta)$ is uniformized by the transformation group $(\mathbb{Z}_{m_i}, \mathbb{C})$. Hence, $(\mathbb{CP}^1, \beta)$ is an orbifold. Theorem 6.1.3, completely answers the question of uniformizability of these orbifolds.

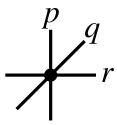
Example 6.2.6. Let p,q be two positive integers and consider the germ $(\mathbb{C}^2,\beta)_0$, where

$$\beta(x,y)=\begin{cases} pq & (x,y)=(0,0) \\ p & x=0, y\neq 0 \\ q & x\neq 0, y=0 \end{cases}$$



Put $H_1=\{x=0\}$ and $H_2=\{y=0\}$. The group $\pi_1(\mathbb{C}^2\setminus(H_1\cup H_2))_0$ is the free abelian group generated by the meridians of H_1 and H_2 so that $\pi_1^{orb}(\mathbb{C}^2,\beta)_0\simeq\mathbb{Z}_p\oplus\mathbb{Z}_q$ is finite. This is indeed an orbifold germ, the map $(\mathbb{C}^2\to\mathbb{C}^2)$ defined by $(x,y)\mapsto(x^p,y^q)$ is its uniformization.

Example 6.2.7. Let p,q,r be three positive integers and consider the germ of the pair $(\mathbb{C}^2,D)_0$, where $D=pH_1+qH_2+rH_3$, $H_1=\{x=0\}$, $H_2=\{y=0\}$ and $H_3=\{x-y=0\}$.



One has $\pi_1(\mathbb{C}^2\setminus(H_1\cup H_2\cup H_3))\simeq\langle\mu_1,\mu_2,\mu_3 \mid [\mu_i,\mu_1\mu_2\mu_3]=1,\ i=1,2,3\rangle$, where μ_i is a meridian of H_i for $i=1,2,3$ (See Theorem 5.10.1). Therefore, the local orbifold fundamental group admits the presentation

$$\pi_1^{orb}(C^2,D)\simeq\langle\mu_1,\mu_2,\mu_3 \mid [\mu_i,\mu_1\mu_2\mu_3]=\mu_1^p=\mu_2^q=\mu_3^r=1,\ i=1,2,3\rangle.$$

This group is a central extension of the triangle group and is finite of order $4\rho^{-2}$ if $\rho:=1/p+1/q+1/r-1>0$, infinite solvable when $\rho=0$ and "big" otherwise. Hence $(\mathbb{C}^2,D)_0$ do not come from an orbifold germ if $\rho<0$. For $\rho>0$ it comes from an orbifold germ and it is uniformizable. In this case the triple (p,q,r) is one of $(1,m,m)$, $(2,2,m)$, $(2,3,3)$, $(2,3,4)$, $(2,3,5)$ and the order of corresponding orbifold fundamental groups are m^2, m^2, 144, 576, 3600, respectively.

Let (X,β) be an orbifold and let D_β be the associated divisor. Recall that the group $\pi_1^{orb}(X,\beta)$ is the group $\pi_1^{orb}(X,D_\beta)$. If $\xi:\pi_1^{orb}(X,\beta)\twoheadrightarrow G$ is a surjection onto a finite group G with $\mathrm{Ker}(\varphi)$ satisfying the branching condition, then there exist a Galois covering $\varphi : M \to X$ branched at D_β, where M is a possibly singular normal space.

Lemma 6.2.8 (Uludağ, 2007, Lemma 2.3). *Let (X,β) be an orbifold, and $\varphi : M \to X$ a Galois covering branched at D_β. Then M is smooth if and only if $\beta_\varphi \equiv \beta$.*

Proof. For any $x \in X$ there is the induced branched covering of germs $\varphi_x : M_z \to X_x$, where $z \in \varphi^{-1}(x)$. The germ M_z is smooth if and only if φ_x is the uniformization map of the germ $(X,\beta)_x$, which is the universal branched covering and has $\pi_1^{orb}(X,\beta)_x$ as its Galois group. In other words, M_z is smooth if and only if $G_z \simeq \pi_1^{orb}(X,\beta)_x$, if and only if $\beta_\varphi(x) = |G_z| = |\pi_1^{orb}(X,\beta)_x| = \beta(x)$. □

For a point $x \in X$, there is a natural map $\iota_x : \pi_1^{orb}(X,\beta)_x \to \pi_1^{orb}(X,\beta)$ induced by the inclusion $\pi_1^{orb}(X,D_\beta)_x \hookrightarrow \pi_1^{orb}(X,D_\beta)$. The group G_z is the image of the composition map

$$\xi \circ \iota : \pi_1^{orb}(X,\beta)_x \to \pi_1^{orb}(X,\beta) \to G.$$

Theorem 6.2.9 (Uludağ, 2007, Theorem 2.4). *Let $\xi : \pi_1^{orb}(X,\beta) \twoheadrightarrow G$ be a surjection and let $\varphi : M \to X$ be the corresponding Galois covering of X branched along D_β. The pair (G,M) is a uniformization of the orbifold (X,β) if and only if for any $x \in X$, the map $\xi \circ \iota_x : \pi_1^{orb}(X,\beta)_x \to G$ is an injection*

Proof. One has $\beta_\varphi \equiv \beta$ if and only if for any $x \in X$ and $z \in \varphi^{-1}(x)$ the image G_z of $\xi \circ \iota_x$ is the full group $\pi_1^{orb}(X,\beta)_x$. The result follows from Lemma 6.2.3. □

The Theorem 6.2.9 may fail in higher dimensions (see Brieskorn (1966) for counter examples). So, we will mostly consider orbifolds in dimension 2.

Recall that an orbifold germ $(X,\beta)_x$ is a germ that admits a finite uniformization by a transformation group (G_z, M_z), where M_z is a smooth germ and G_z is finite group acting on M_z and fixes z. According to a classical result of **?**, any orbifold germ $(X,\beta)_x$ is equivalent to the quotient of the germ $\mathbb{C}^n_0$ by finite subgroup of

$\mathrm{GL}(n,\mathbb{C})$. In other words, any orbifold germ $(X,\beta)_x$. In dimension 2, Yoshida (1987) observed the following fact: If $H \subset \mathrm{GL}(2,\mathbb{C})$ is a reflection group with a non-abelian PG, then among the reflection groups with the same projectivization there is a maximal one G containing H. Every reflection group K with $PK = PG$ is a normal subgroup of this maximal reflection group. This means, the germ $\mathbb{C}^2/K$ is a Galois covering of $\mathbb{C}^2/G$. If G is maximal reflection group, then the quotient $\mathbb{C}^2/G$ is the orbifold $(\mathbb{C}^2, pX+qY+rZ)$ for some (p,q,r) with $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}-1>0$, where X,Y,Z are the lines meeting at the origin. Hence, any orbifold germ with a smooth base is a covering of the germ $(\mathbb{C}^2, pX+qY+rZ)$. The following result characterizes the germs with a smooth base.

Theorem 6.2.10 (Kato, 1987)**.** *In dimension 2, all orbifold germs with a smooth base are given in the in the Figure 6.1 and Table 6.1.*

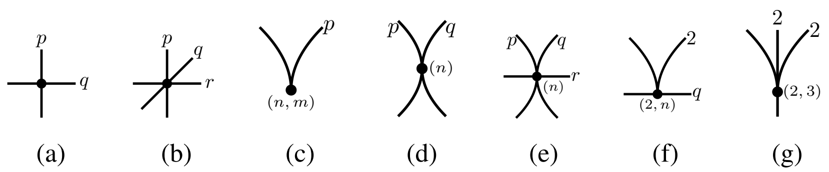


Figure 6.1 Orbifold germs.

Table 6.1 Orbifold germs and corresponding branching conditions and the order of corresponding orbifold fundamental groups.

	Equation	Condition	Order
Figure 6.1a	xy	—-	pq
Figure 6.1b	$xy(x+y)$	$0<\rho := \frac{1}{p}+\frac{1}{q}+\frac{1}{r}-1$	$4\rho^{-2}$
Figure 6.1c	$x^n - y^m$	$\gcd(n,m)=1, \quad 0<\rho := \frac{1}{p}+\frac{1}{n}+\frac{1}{m}-1$	$\frac{4\rho^{-2}}{nm}$
Figure 6.1d	x^2-y^{2n}	$0<\rho := \frac{1}{p}+\frac{1}{q}+\frac{1}{n}-1$	$\frac{4\rho^{-2}}{n}$
Figure 6.1e	$y(x^2-y^{2n})$	$0<\rho := \frac{1}{p}+\frac{1}{q}+\frac{1}{nr}-1$	$\frac{4\rho^{-2}}{n}$
Figure 6.1f	$y(x^2-y^n)$	n is odd	$2nq^2$
Figure 6.1g	$x(x^2-y^3)$	—-	96

Solutions to Conditions in Table 6.1 (including the equality) are as in Table 6.2. In case of $\rho=0$, we will obtain the orbifold germs with cusp points and the orbit space M/G admits a compactification by considering pairs (X,β) with extended β

Table 6.2 Solutions to the conditions in Table 6.1 together with the case equality.

Condition	Solution		Condition	Solution	
$\rho := \frac{1}{p}+\frac{1}{q}+\frac{1}{r}-1>0$	(p,q,r)	Order	$\rho=0$	(p,q,r)	Order
	$(2,2,n),\ n\in\mathbb{Z}_{>1}$	$4n^2$		$(2,3,6)$	∞
	$(2,3,3)$	144		$(2,4,4)$	∞
	$(2,3,4)$	576		$(3,3,3)$	∞
	$(2,3,5)$	3600			
$\rho := \frac{1}{p}+\frac{1}{n}+\frac{1}{m}-1>0$ $\gcd(n,m)=1$	(p,n,m)	Order	$\rho=0$ $\gcd(n,m)=1$	(p,n,m)	Order
	$(2,2,a)$, $a\in\mathbb{Z}_{>1}$ is odd	$2a$		$(6,2,3)$	∞
	$(2,3,4)$	48			
	$(2,3,5)$	240			
	$(3,2,3)$	24			
	$(3,2,5)$	360			
	$(4,2,3)$	96			
	$(5,2,3)$	600			
$\rho := \frac{1}{p}+\frac{1}{q}+\frac{1}{n}-1>0$	(p,q,n)	Order	$\rho=0$	(p,q,n)	Order
	$(2,2,a),\ a\in\mathbb{Z}_{>1}$	$4a$		$(2,3,6)$	∞
	$(2,a,2),\ a\in\mathbb{Z}_{>1}$	$2a^2$		$(2,4,4)$	∞
	$(2,3,3)$	48		$(2,6,3)$	∞
	$(2,3,4)$	144		$(3,3,3)$	∞
	$(2,3,5)$	720		$(3,6,2)$	∞
	$(2,4,3)$	192		$(4,4,2)$	∞
	$(2,5,3)$	1200			
	$(3,3,2)$	72			
	$(3,4,2)$	288			
	$(3,5,2)$	1800			
$\rho := \frac{1}{p}+\frac{1}{q}+\frac{1}{nr}-1>0$	(p,q,n,r)	Order	$\rho=0$	(p,q,n,r)	Order
	$(2,3,2,2)$	288		$(2,3,2,3)$	∞
	$(2,2,a,b)$, $a,b\in\mathbb{Z}_{>1}$	$4ab^2$		$(2,3,3,2)$	∞
				$(2,4,2,2)$	∞

functions with values in $\mathbb{N}\cup\{\infty\}$. In case $M=\mathbf{B}_2$, and G is a finite volume discrete subgroup of $\mathrm{Aut}(\mathbf{B}_2)$, for smooth X, a classification of ball cusp points was given in (Yoshida, 1987). Any such germ is a covering of one of the germs $(\mathbb{C}^2, pX+qY+rZ)_0$ with $\rho=0$ and $(\mathbb{C}^2, 2H_1+2H_2+2H_3+2H_4)_0$, where H_i's are smooth branches meting transversally at the origin. We will study the covering relations among orbifold germs in Section 6.2.3.

6.2.3 Sub-orbifolds and Orbifold Coverings

Let (X,β) be an orbifold. An orbifold (X,β') is said to be suborbifold of (X,β) if $\beta'(x)$ divides $\beta(x)$ for any $x \in X$.

Let $\varphi : Y \to X$ be a uniformization of (X,β). Define the function $\alpha : Y \to \mathbb{N}$ by

$$\alpha(y) := \frac{\beta(\varphi(y))}{\beta'(\varphi(y))}.$$

Then $\varphi : (Y,\alpha) \to (X,\beta)$ is called an orbifold covering, and (Y,α) is called the lifting of (X,β) to the uniformization of (X,β'). The exact sequence of Lemma 6.1.2 can be generalized to the following commutative diagram:

$$\begin{tikzcd}
 & 0 & 0 & & \\
0 \arrow[r] & \pi_1(Y) \arrow[r] \arrow[u] & \pi_1^{orb}(X,\beta') \arrow[r] \arrow[u] & G \arrow[r] & 0 \\
0 \arrow[r] & \pi_1^{orb}(Y,\alpha) \arrow[r] \arrow[u] & \pi_1^{orb}(X,\beta) \arrow[r] \arrow[u] & G \arrow[r] \arrow[u, "\simeq"'] & 0
\end{tikzcd}$$

Remark 6.2.11. The branching conditions in Table 6.1 of orbifold germs are related with covering relations among orbifold germs. For example, suppose we have the germ $A := (\mathbb{C}^2,\beta)_0$ associated with the divisor $D = pH_1 + qH_2 + nH_3$, where $H_1 = x+y=0$, $H_2 = x-y=0$ and $H_3 = y=0$. $M := (\mathbb{C}^2,\beta')_0 = (\mathbb{C}^2,nH_3)_0$ is a sub-orbifold of A and its uniformizer is $\varphi_{1,n} : (x,y) \mapsto (x,y^n)$. Denote by H_1' the lifting $\varphi_{1,n}^{-1}(H_1) = \{x+y^n = 0\}$ and by H_2' the lifting $\varphi_{1,n}^{-1}(H_2) = \{x-y^n = 0\}$. If one denotes $B := (\mathbb{C}^2,\alpha)_0 = (\mathbb{C}^2, pH_1' + qH_2')_0$, which is the germ in Figure 6.1d, then he has a covering $\varphi_{1,n} : B \to A$ and the exact sequence

$$1 \to \pi_1^{orb}(B) \to \pi_1^{orb}(A) \to \mathbb{Z}_n \to 1.$$

Therefore $|\pi_1^{orb}(B)| = \frac{1}{n}|\pi_1^{orb}(A)| = \frac{4}{n}(\frac{1}{p} + \frac{1}{q} + \frac{1}{n} - 1)^{-2}$ and the uniformizability condition of B is $\rho := \frac{1}{p} + \frac{1}{q} + \frac{1}{n} - 1 > 0$ (and $\rho \geq 1$ for singular base).

6.2.4 Covering Relations among Orbifold Germs

6.2.4.1 Coverings of the Abelian Germs

The local orbifold fundamental group of the germs $(\mathbb{C}^2, pX+qY)_0$ is isomorphic to the the abelian group $\mathbb{Z}_p \oplus \mathbb{Z}_q$, where $X=\{x=0\}$ and $Y=\{y=0\}$. Any smooth sub-orbifold of this orbifold is of the form $(\mathbb{C}^2, rX+sY)_0$, where $r|p$, $s|q$ and $r,s \in \mathbb{Z}_{\geq 1}$. This latter orbifold germ is uniformized by $\mathbb{C}_0$ via the map $\varphi_{r,s} : (x,y) \in \mathbb{C}^2 \to (x^r, y^s) \in \mathbb{C}^2$ with $\mathbb{Z}_r \oplus \mathbb{Z}_s$ as its Galois group. The lifting of $(\mathbb{C}^2, pX+qY)_0$ to this uniformization is the orbifold $(\mathbb{C}^2, \frac{p}{r}X + \frac{q}{s}Y)_0$.

6.2.4.2 Coverings of the Dihedral Germs

Consider the dihedral germ $(\mathbb{C}^2, 2X+2Y+mZ)_0$ in Figure 6.1b, where $X=\{x=0\}$, $Y=\{y=0\}$ and $Z=\{x-y=0\}$. In the Theorem 5.10.1 we have computed the local fundamental group of complement to pencil of m-lines in $\mathbb{C}^2$. By using the presentation of $G_{3,3}$ we get the triangle group

$$\langle \mu_1, \mu_2, \mu_3 \mid [\mu_i, \mu_3\mu_2\mu_1] = \mu_1^2 = \mu_2^2 = \mu_3^m = 1,\ i=1,2,3 \rangle$$

of order $4m^2$ as the orbifold fundamental group of the germ $(\mathbb{C}^2, 2X+2Y+mZ)_0$. This group acts on $\mathbb{C}^2$ and the branch divisor is the dihedral germ. Now we will discus the coverings of the dihedral germ. Due to oddness or evenness of m we have two cases:

1. If m is an odd number, then $(\mathbb{C}^2, 2X)_0$, $(\mathbb{C}^2, 2Y)_0$, $(\mathbb{C}^2, mZ)_0$, $(\mathbb{C}^2, 2X+2Y)_0$, $(\mathbb{C}^2, 2X+mZ)_0$ and $(\mathbb{C}^2, 2Y+mZ)_0$ are its sub-orbifolds. Each one of these sub-orbifolds is uniformized by $\mathbb{C}_0^2$ via a cyclic map $\varphi_{p,q} : (x,y) \to (x^p, y^q)$ and note that $\varphi_{r,s} \circ \varphi_{p,q} = \varphi_{rp,sq}$.

   a. Consider the sub-orbifold $(\mathbb{C}^2, 2X)_0$ whose uniformizer is the map $\varphi_{2,1}$. If we denote the branch $\varphi_{2,1}^{-1}(Y) = \{y = 0\}$ by Y and the branch $\varphi_{2,1}^{-1}(Z) = \{x^2 - y = 0\}$ by W, then

$$\varphi_{2,1} : (\mathbb{C}^2, 2Y + mW)_0 \to (\mathbb{C}^2, 2X + 2Y + mZ)_0$$

is an orbifold covering. Note that, $\varphi_{1,2}$ is a covering map of $(\mathbb{C}^2, 2Y + mW)_0$ and one has $Z' := \varphi_{1,2}^{-1}(W) = \{x^2 - y^2 = 0\}$. By setting $Z'_1 = \{x + y = 0\}$ and $Z'_2 = \{x - y = 0\}$ one gets the covering

$$\varphi_{2,2} = \varphi_{2,1} \circ \varphi_{1,2} : (\mathbb{C}^2, mZ')_0 = (\mathbb{C}^2, mZ'_1 + mZ'_2)_0 \to (\mathbb{C}^2, 2X + 2Y + mZ)_0,$$

which is related to covering of the dihedral germ by the uniformizer of its sub-orbifold $(\mathbb{C}^2, 2X + 2Y)_0$.

On the other hand, if one would have changed the coordinates by the map $\sigma :\ (x,z) = (x, x - y)$, then $\sigma^{-1} \circ \varphi_{2,1}$ would be the uniformizer of the sub-orbifold $(\mathbb{C}^2, 2X)_0$. In this case, denote by Z the branch $\varphi_{2,1}^{-1}(Z) = \{z = 0\}$ and by V the branch $\varphi_{2,1}^{-1}(Y) = \{x^2 - z = 0\}$. Then

$$\sigma^{-1} \circ \varphi_{2,1} : (\mathbb{C}^2, 2V + 2Z)_0 \to (\mathbb{C}^2, 2X + 2Y + mZ)_0$$

is an orbifold covering. Note that, $\varphi_{1,m}$ is a covering map of $(\mathbb{C}^2, 2V + mZ)_0$ via its sub orbifold $(\mathbb{C}^2, mZ)_0$. Denote by Y' the lifting $\varphi_{1,m}^{-1}(V) = \{x^2 - z^m = 0\}$ of V. Then one has the covering,

$$\sigma^{-1} \circ \varphi_{2,m} = \tau^{-1} \circ \varphi_{2,1} \circ \varphi_{1,m} : (\mathbb{C}^2, 2Y')_0 \to (\mathbb{C}^2, 2X + 2Y + mZ)_0,$$

which is related to covering of the dihedral germ by the uniformizer of its sub-orbifold $(\mathbb{C}^2, 2X + mZ)_0$.

b. Coverings of the dihedral germ $(\mathbb{C}^2, 2X + 2Y + mZ)_0$ via its sub-orbifold $(\mathbb{C}^2, 2Y)_0$ is similar to the case 1.a. It is enough to interchange the roles of X and Y to see such coverings.

c. Consider the sub-orbifold $(\mathbb{C}^2, mZ)_0$ and change coordinates by the map $\sigma: (x,z) = (x, x-y)$. Then it is clear that the sub-orbifold $(\mathbb{C}^2, mZ)_0$ is uniformized by $\sigma^{-1} \circ \varphi_{1,m}$. Denote the branch $\varphi_{1,m}^{-1}(X) = \{x = 0\}$ by X and the branch $\varphi_{1,m}^{-1}(Y) = \{x - z^m = 0\}$ by V. Then

$$\sigma^{-1} \circ \varphi_{1,m} : (\mathbb{C}^2, 2X + 2V)_0 \to (\mathbb{C}^2, 2X + 2Y + mZ)_0$$

is an orbifold covering. Note that, $\varphi_{2,1}$ is a covering map of $(\mathbb{C}^2, 2X + mV)_0$ via its sub orbifold $(\mathbb{C}^2, 2X)_0$. Denote by Y' the lifting $\varphi_{2,1}^{-1}(V) = \{x^2 - z^m = 0\}$ of V. Then one has the covering,

$$\sigma^{-1} \circ \varphi_{2,m} = \sigma^{-1} \circ \varphi_{1,m} \circ \varphi_{2,1} : (\mathbb{C}^2, 2Y')_0 \to (\mathbb{C}^2, 2X + 2Y + mZ)_0,$$

which is related to covering of the dihedral germ by the uniformizer of its sub-orbifold $(\mathbb{C}^2, 2X + mZ)_0$. On the other hand, if one would have changed the coordinates by the map $\tau: (z,y) = (x-y, y)$, then $\tau^{-1} \circ \varphi_{m,1}$ would be the uniformizer of the sub-orbifold $(\mathbb{C}^2, mZ)_0$. In this case, denote the branch $\varphi_{m,1}^{-1}(Y) = \{y = 0\}$ by Y and the branch $\varphi_{m,1}^{-1}(X) = \{y + z^m = 0\}$ by U. Then

$$\tau^{-1} \circ \varphi_{m,1} : (\mathbb{C}^2, 2U + 2Y)_0 \to (\mathbb{C}^2, 2X + 2Y + mZ)_0$$

is an orbifold covering. Note that, $\varphi_{1,2}$ is a covering map of $(\mathbb{C}^2, 2U + 2Y)_0$ via its sub orbifold $(\mathbb{C}^2, 2Y)_0$. Denote by X' the lifting $\varphi_{1,2}^{-1}(U) = \{x^2 - z^m = 0\}$ of V. Then one has the covering,

$$\tau^{-1} \circ \varphi_{m,2} = \tau^{-1} \circ \varphi_{m,1} \circ \varphi_{1,2} : (\mathbb{C}^2, 2X')_0 \to (\mathbb{C}^2, 2X + 2Y + mZ)_0,$$

which is related to covering of the dihedral germ by the uniformizer of its sub-orbifold $(\mathbb{C}^2, 2Y + mZ)_0$.

d. The uniformizer of the sub-orbifold $(\mathbb{C}^2, 2X + 2Y)_0$ is the map $\varphi_{2,2}$ and one has the covering

$$\varphi_{2,2} : (\mathbb{C}^2, Z') \to (\mathbb{C}^2, 2X + 2Y + mZ)_0,$$

where the branch $Z' = \varphi_{2,2}^{-1}(Z) = \{x^2 - y^2 = 0\}$ is the lifting of the divisor Z by $\varphi_{2,2}$.

e. Consider the sub-orbifold $(\mathbb{C}^2, 2X + mZ)_0$ and change the coordinates by the map $\sigma :\ (x,z) = (x, x-y)$. Then $X = \{x = 0\}$, $Y = \{x - z = 0\}$, $Z = \{z = 0\}$, and the map $\varphi_{2,m} : (x,z) \mapsto (x^2, z^m)$ is the uniformizer of $(\mathbb{C}^2, 2X + mZ)_0$. Then one has the covering

$$\sigma^{-1} \circ \varphi_{2,m} : (\mathbb{C}^2, 2Y')_0 \to (\mathbb{C}^2, 2X + 2Y + mZ)_0,$$

where $Y' = \varphi_{2,m}^{-1}(Y) = \{x^2 - z^m = 0\}$ is a lifting of Y by $\varphi_{2,m}$.

f. Covering of the dihedral germ $(\mathbb{C}^2, 2X + 2Y + mZ)_0$ via its sub-orbifold $(\mathbb{C}^2, 2Y + mZ)_0$ is similar to the case 1.e. It is enough to interchange the roles of X and Y to see this covering.

In case of m is an odd prime, to see all of covering relations above see Figure 6.2. If m is odd but not prime, then it has prime factorization which induces factorization of covering relations of dihedral germ. We have omitted to explain such factorizations but exhibited in Figure 6.3. In both cases we have omitted the change of coordinate maps in these figures.

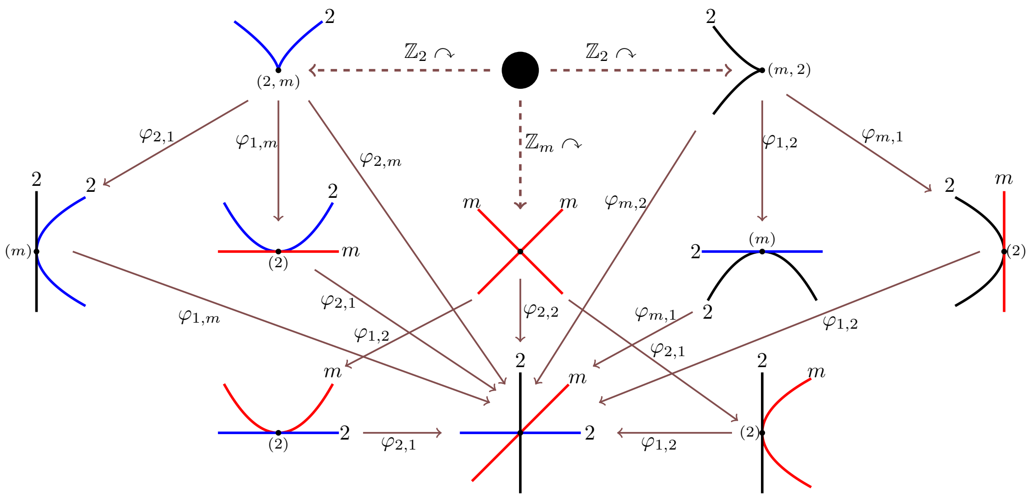


Figure 6.2 Coverings of the dihedral germ $(\mathbb{C}^2, 2X+2Y+mZ)_0$, where m is an odd prime.

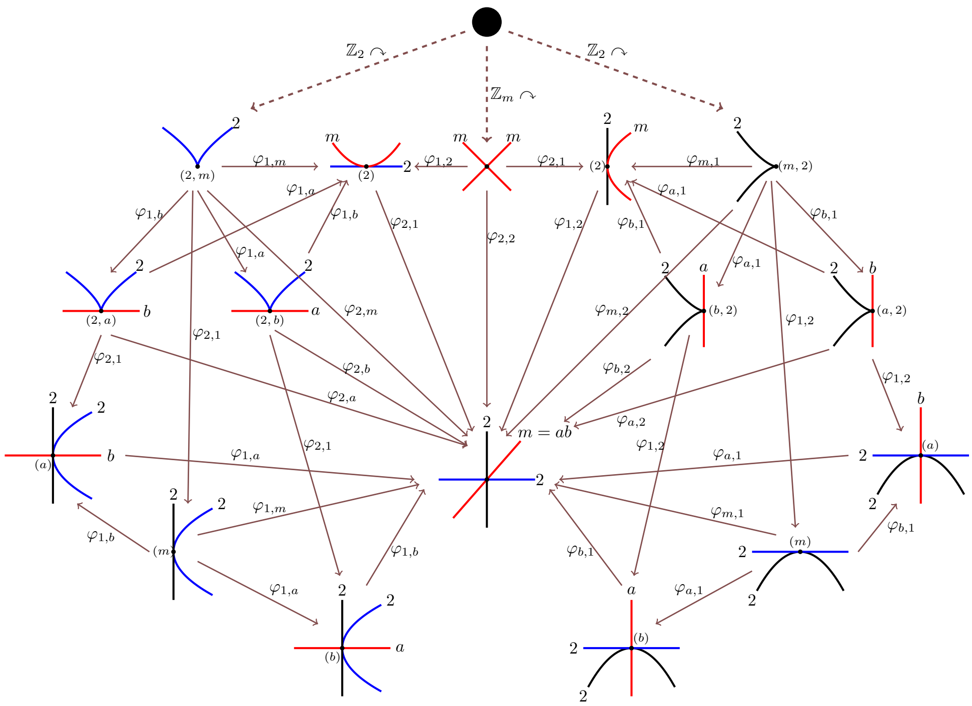


Figure 6.3 Coverings of the dihedral germ $(\mathbb{C}^2, 2X+2Y+mZ)_0$, where $m = ab$ and a, b are odd primes.

2. If m is even, say $m = 2^k n$, where n is odd. Then $|\pi_1^{orb}(\mathbb{C}^2, D)_0| = 2^{2k+2}n^2$ and the sub-orbifolds are $(\mathbb{C}^2, 2X)_0$, $(\mathbb{C}^2, 2Y)_0$, $(\mathbb{C}^2, 2X+2Y)_0$, $(\mathbb{C}^2, 2X+nZ)_0$, $(\mathbb{C}^2, 2Y+nZ)_0$, $(\mathbb{C}^2, 2X+2^sZ)_0$, $(\mathbb{C}^2, 2Y+2^sZ)_0$, $(\mathbb{C}^2, 2X+2^snZ)_0$, $(\mathbb{C}^2, 2Y+2^snZ)_0$, where $s = 1, \cdots, k$.

a. For the sub-orbifolds $(\mathbb{C}^2,2X)_0$, $(\mathbb{C}^2,2Y)_0$, $(\mathbb{C}^2,mZ)_0$ and $(\mathbb{C}^2,2X+2Y)_0$, the lifting and uniformization of the dihedral germ are the same as in cases 1.a., 1.b., 1.c. and 1.d., respectively.

b. Consider the sub-orbifold $(\mathbb{C}^2,2X+nZ)_0$. For simplicity, let us first change the coordinates by $\sigma:\ (x,z)=(x,x-y)$. Then $X=\{x=0\}$, $Y=\{x-z=0\}$ and $Z=\{z=0\}$, and the map $\varphi_{2,n}:(x,z)\mapsto(x^2,z^n)$ is the uniformizer of $(\mathbb{C}^2,2X+nZ)_0$. Denote the branch $\varphi_{2,n}^{-1}(Y)=\{x^2-z^n=0\}$ by V , and the branch $\varphi_{2,n}^{-1}(Z)=\{z=0\}$ by Z. Then,

$$\sigma^{-1}\circ\varphi_{2,n}:\ (\mathbb{C}^2,2Y'+2^kZ)_0\to(\mathbb{C}^2,2X+2Y+mZ)_0$$

is an orbifold covering. Note that the branch Y' is smooth for $n=1$, and it is a cusp of $(2,n)$-type for other odd n's. Denote by Y' the lifting $\varphi_{1,2^k}^{-1}(V)=\{x^2-z^m=0,\ m=2^kn\}$ of V via the uniformizer of $(\mathbb{C}^2,2^kZ)_0$. Then one has a covering

$$\sigma^{-1}\circ\varphi_{2,m}=\sigma^{-1}\circ\varphi_{2,n}\circ\varphi_{1,2^k}:(\mathbb{C}^2,2Y')_0\to(\mathbb{C}^2,2X+2Y+mZ)_0.$$

c. Coverings of the dihedral germ $(\mathbb{C}^2,2X+2Y+mZ)_0$ via the uniformizer of its the sub-orbifold $(\mathbb{C}^2,2Y+nZ)_0$ is similar to the case 2.b. It is enough to change the roles of X and Y to see such coverings explicitly.

d. Consider the sub-orbifold $(\mathbb{C}^2,2X+2^sZ)_0$, $s=1,2,\dots,k$. For simplicity, let us first change the coordinates by the map $\sigma:\ (x,z)=(x,x-y)$. Then $X=\{x=0\}$, $Y=\{x-z=0\}$ and $Z=\{z=0\}$ and the map $\varphi_{2,2^s}:(x,z)\mapsto(x^2,z^{2^s})$ is the uniformizer of $(\mathbb{C}^2,2X+2^sZ)_0$. Therefore one has the covering

$$\sigma^{-1}\circ\varphi_{2,2^s}:(\mathbb{C}^2,2V+2^{k-s}nZ)_0\to(\mathbb{C}^2,2X+2Y+mZ)_0,$$

where V is the lifting $\varphi_{2,2^s}^{-1}(Y)=\{x^2-z^{2^s}=0\}$ of Y by $\varphi_{2,2^s}$. Since the uniformizer of $(\mathbb{C}^2, 2^{k-s}nZ)_0$ is $\varphi_{1,2^{k-s}n}$ then one has the covering

$$\sigma^{-1}\circ\varphi_{2,m} = \sigma^{-1}\circ\varphi_{2,2^s}\circ\varphi_{1,2^{k-s}n} : (\mathbb{C}^2, 2Y')_0 \to (\mathbb{C}^2, 2X+2Y+mZ)_0,$$

where $Y' := \varphi_{1,2^{k-s}n}(V) = \{x^2 - z^m = 0,\ m = 2^k n\}$ is the lifting of V by $\varphi_{1,2^{k-s}n}$.

e. Coverings of $(\mathbb{C}^2, 2X+2Y+mZ)_0$ by the uniformizer $\varphi_{2,2^s}$ of the sub-orbifold $(\mathbb{C}^2, 2Y+2^sZ)_0$ is similar to the case 2.d.

f. Consider the sub-orbifold $(\mathbb{C}^2, 2X+2^snZ)_0$ and change the coordinates by $\sigma : (x,z) = (x, x-y)$, then $X = \{x = 0\}$, $Z = \{z = 0\}$, $Y = \{x - z = 0\}$ and the map $\varphi_{2,2^sn}$ is the uniformizer of $(\mathbb{C}^2, 2X+2^snZ)_0$. Denote by Z the lifting $\varphi_{2,2^sn}^{-1}(Z)$ and by V the lifting $\varphi_{2,2^sn}^{-1}(Y) = \{x^2 - z^{2^sn} = 0\}$. Then we have the covering

$$\sigma^{-1}\circ\varphi_{2,2^sn} : (\mathbb{C}^2, 2V+2^{k-s}Z)_0 \to (\mathbb{C}^2, 2X+2Y+mZ)_0.$$

Since the uniformizer of $(\mathbb{C}^2, 2^{k-s}Z)_0$ is $\varphi_{1,2^{k-s}}$ then one has the covering

$$\sigma^{-1}\circ\varphi_{2,m} = \sigma^{-1}\circ\varphi_{2,2^sn}\circ\varphi_{1,2^{k-s}} : (\mathbb{C}^2, 2Y')_0 \to (\mathbb{C}^2, 2X+2Y+mZ)_0,$$

where $Y' := \varphi_{1,2^{k-s}}(V) = \{x^2 - z^m = 0,\ m = 2^k n\}$ is the lifting of V by $\varphi_{1,2^{k-s}n}$.

g. Coverings of the germ $(\mathbb{C}^2, 2X+2Y+mZ)_0$ by the uniformizer of the sub-orbifold $(\mathbb{C}^2, 2Y+2^snZ)_0$ is similar to the case 2.f.

h. Note that, Y' has two components and they are normal crossing two lines if $k = n = 1$. Otherwise, set $V_{0,1}^0 := \{x + z^{2^{k-1}n} = 0\}$, $V_{0,2}^0 := \{x - z^{2^{k-1}n} = 0\}$, and change the coordinates by $\alpha_1 : (x_1, z) = (\frac{x+z^{2^{k-1}n}}{2}, z)$, then $V_{0,1}^0 = \{x_1 = 0\}$ and $V_{0,2}^0 = \{x_1 - z^{2^{k-1}n} = 0\}$. Denote by V_2^1 by the lifting $\varphi_{2,1}^{-1}(V_{0,2}^0) = \{x_1^2 - z^{2^{k-1}n} = 0\}$, then we have a covering

$$\alpha_1\circ\varphi_{2,1} : (\mathbb{C}, 2V_2^1)_0 \to (\mathbb{C}, 2Y')_0.$$

If $k = 1$ and $n \neq 1$, then clearly V_2^1 is a cusp of $(2,n)$-type. Now suppose $k > 1$ and set $V_{1,1}^1 := \{x_1 + z^{2^{k-2}n} = 0\}$, $V_{1,2}^1 := \{x_1 - z^{2^{k-2}n} = 0\}$, and change the coordinates by $\alpha_2 : (x_2, z) = (\frac{x_1 + z^{2^{k-2}n}}{2}, z)$, then $V_{1,1}^1 = \{x_2 = 0\}$ and $V_{1,2}^1 = \{x_1 - z^{2^{k-2}n} = 0\}$. Denote by V_2^2 by the lifting $\varphi_{2,1}^{-1}(V_{1,2}^1) = \{x_2^2 - z^{2^{k-2}n} = 0\}$, then we have a covering $\alpha_2 \circ \varphi_{2,1} : (\mathbb{C}, 2V_2^2)_0 \to (\mathbb{C}, 2V_2^1)_0$. If $k = 2$ and $n \neq 1$, then clearly V_2^2 is a cusp of $(2,n)$-type. Apply this procedure $k-1$ times. If $n = 1$ then V_2^{k-1} consists of normal crossing lines. Otherwise applying the procedure above once again we obtain V_2^k as cusp of $(2,n)$ type. Thus we have a covering

$$\alpha_1 \circ \varphi_{2,1} \circ \alpha_2 \circ \varphi_{2,1} \circ \cdots \circ \alpha_k \circ \varphi_{2,1} : (\mathbb{C}, 2V_2^k)_0 \to (\mathbb{C}, 2Y')_0.$$

A similar covering relation is also valid for the orbifold $(\mathbb{C}, 2X')_0$.

To see coverings of the dihedral germ $(\mathbb{C}^2, 2X + 2Y + mZ)_0$, where m is even, see Figures 6.4, 6.5 and 6.6. We have omitted the change of coordinate maps.

Remark 6.2.12. The black dot on top of the Figures 6.2, 6.3, 6.4, 6.5, 6.6 represents the isolated surface (Du Val) singularity of type A_{m-1}, given by the equation

$$\mathcal{S}_m := \{(x,y,z) \in \mathbb{C}^3 \mid -x^2 + y^2 + z^m = 0\}, \quad m \geq 2.$$

It is clear that the projection $(x,y,z) \to (x,y)$ defines a $\mathbb{Z}_m$ orbifold covering by this singularity of the orbifold $(\mathbb{C}^2, mZ')_0$. Other coordinate projections define $\mathbb{Z}_2$ coverings by the same singularity of the orbifolds $(\mathbb{C}^2, 2X')_0$ and $(\mathbb{C}^2, 2Y')_0$.

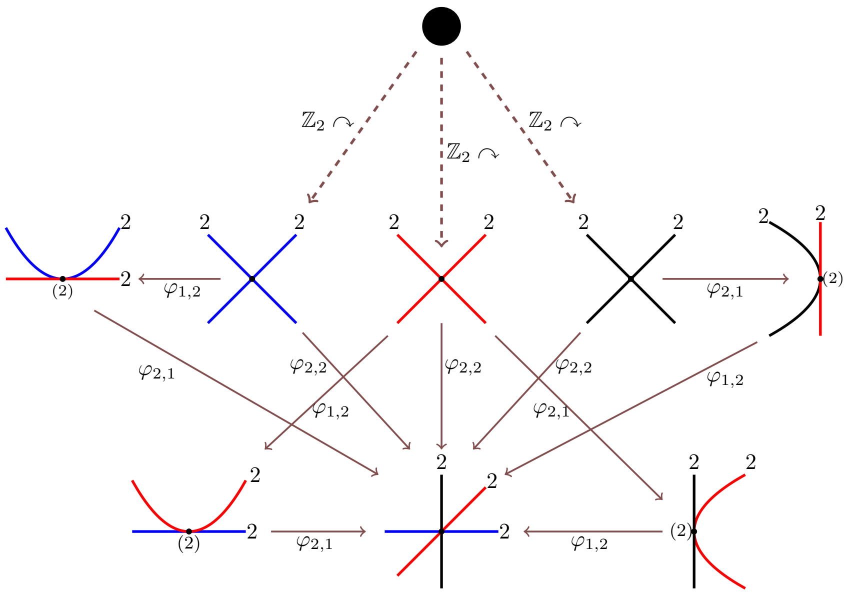


Figure 6.4 Coverings of the dihedral germ $(\mathbb{C}^2, 2X+2Y+2Z)_0$.

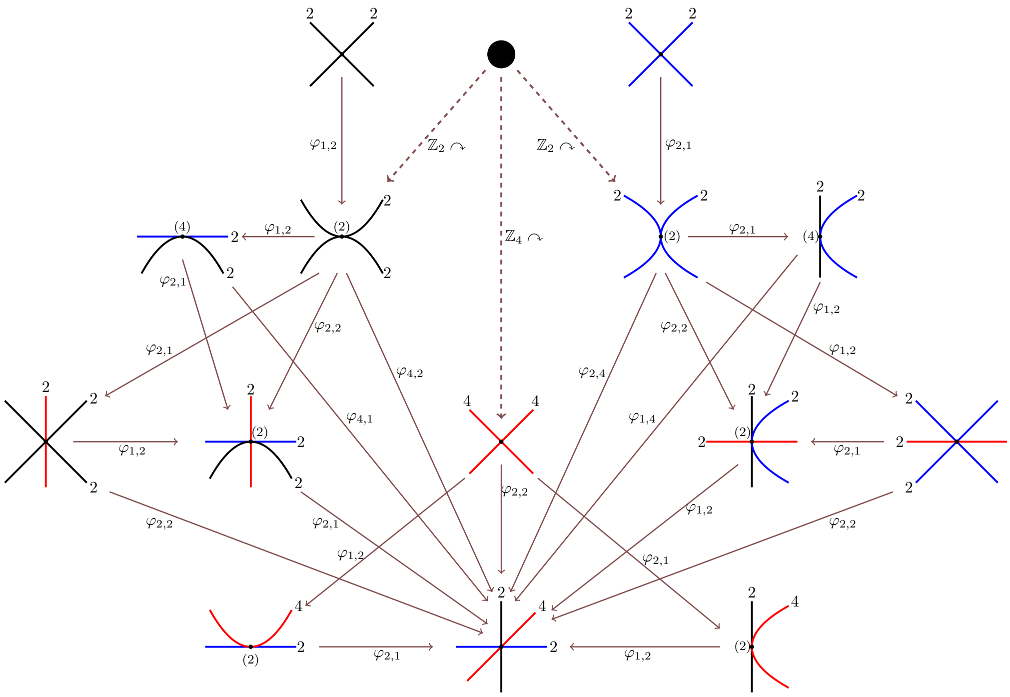


Figure 6.5 Coverings of the dihedral germ $(\mathbb{C}^2, 2X+2Y+4Z)_0$.

6.2.4.3 *Coverings of the Tetrahedral Germ*

Consider the tetrahedral germ $(\mathbb{C}^2, 2X+3Y+3Z)_0$ in Figure 6.1b, where $X = \{x=0\}$, $Y = \{y=0\}$ and $Z = \{x-y=0\}$. In the Theorem 5.10.1 we have computed

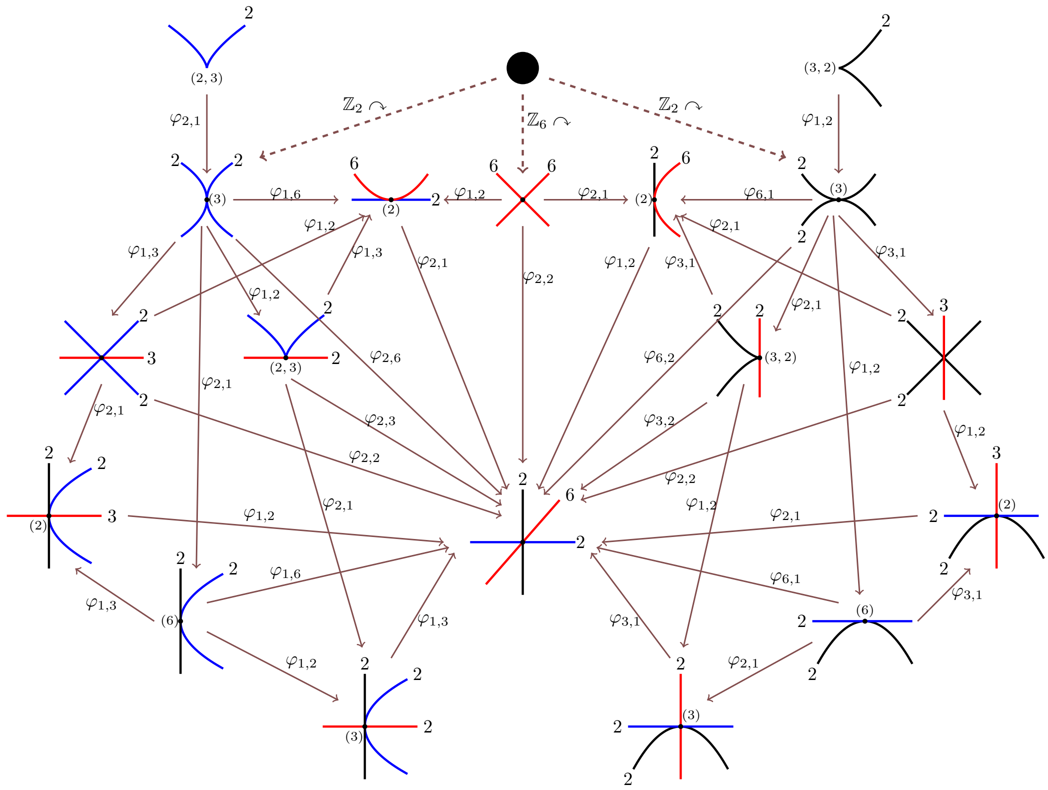


Figure 6.6 Coverings of the dihedral germ $(\mathbb{C}^2, 2X+2Y+6Z)_0$.

the local fundamental group of complement to pencil of m-lines in $\mathbb{C}^2$. By using the presentation of $G_{3,3}$, we get the triangle group

$$\langle \mu_1, \mu_2, \mu_3 \mid [\mu_i, \mu_3\mu_2\mu_1] = \mu_1^2 = \mu_2^3 = \mu_3^3 = 1,\ i = 1,2,3 \rangle$$

of order 144 as the orbifold fundamental group of the germ $(\mathbb{C}^2, 2X+3Y+3Z)_0$. This group acts on $\mathbb{C}^2$ and the branch divisor of this action is the tetrahedral germ. The sub-orbifolds of $(\mathbb{C}^2, 2X+3Y+3Z)_0$ are $(\mathbb{C}^2, 2X)_0$, $(\mathbb{C}^2, 3Y)_0$, $(\mathbb{C}^2, 3Z)_0$, $(\mathbb{C}^2, 2X+3Y)_0$, $(\mathbb{C}^2, 2X+3Z)_0$ and $(\mathbb{C}^2, 2Y+3Z)_0$. Now we will discus the coverings of the tetrahedral germ via uniformizers of its sub orbifolds.

a. The uniformizer of $(\mathbb{C}^2, 2X)_0$ is the map $\varphi_{2,1} : (x,y) \to (x^2, y)$. If we denote the branch $\varphi_{2,1}^{-1}(Y) = \{y = 0\}$ by Y and the branch $\varphi_{2,1}^{-1}(Z) = \{x^2 - y = 0\}$ by W, then $\varphi_{2,1} : (\mathbb{C}^2, 3Y+3W)_0 \to (\mathbb{C}^2, 2X+3Y+3Z)_0$ is an orbifold covering. Now, $\varphi_{1,3}$ is a covering of $(\mathbb{C}^2, 3Y+3W)_0$ and one has $Z' = \varphi_{1,3}^{-1}(W) = \{x^2 -$

$y^3 = 0\}$. Then we have the covering

$$\varphi_{2,3} = \varphi_{2,1} \circ \varphi_{1,3} : (\mathbb{C}^2, 3Z')_0 \to (\mathbb{C}^2, 2X + 3Y + 3Z)_0,$$

which is related to covering of the tetrahedral germ by the uniformizer of its sub-orbifold $(\mathbb{C}^2, 2X + 3Y)_0$. On the other hand, if one would have changed the coordinates by the map $\sigma : \ (x,z) = (x, x - y)$, then $\sigma^{-1} \circ \varphi_{2,1}$ would be the uniformizer of the sub-orbifold $(\mathbb{C}^2, 2X)_0$. In this case, denote the branch $\varphi_{2,1}^{-1}(Z) = \{z = 0\}$ by Z and the branch $\varphi_{2,1}^{-1}(Y) = \{x^2 - z = 0\}$ by V. Then

$$\sigma^{-1} \circ \varphi_{2,1} : (\mathbb{C}^2, 3V + 3Z)_0 \to (\mathbb{C}^2, 2X + 3Y + 3Z)_0$$

is an orbifold covering. Note that, $\varphi_{1,3}$ is a covering map of $(\mathbb{C}^2, 3V + 3Z)_0$ via its sub orbifold $(\mathbb{C}^2, 3Z)_0$. Denote by Y' the lifting $\varphi_{1,3}^{-1}(V) = \{x^2 - z^3 = 0\}$ of V. Then one has another covering,

$$\sigma^{-1} \circ \varphi_{2,3} = \sigma^{-1} \circ \varphi_{2,1} \circ \varphi_{1,3} : (\mathbb{C}^2, 3Y')_0 \to (\mathbb{C}^2, 2X + 3Y + 3Z)_0,$$

which is related to covering of the tetrahedral germ by the uniformizer of its sub-orbifold $(\mathbb{C}^2, 2X + 3Z)_0$.

b. The uniformizer of $(\mathbb{C}^2, 3Y)_0$ is the map $\varphi_{1,3} : (x,y) \to (x, y^3)$. If we denote by X the branch $\varphi_{1,3}^{-1}(X) = \{x = 0\}$ and by W the branch $\varphi_{1,3}^{-1}(Z) = \{x - y^3 = 0\}$, then $(\mathbb{C}^2, 2X + 3W)_0$ is a lifting of $(\mathbb{C}^2, 2X + 3Y + 3Z)_0$ via $\varphi_{1,3}$. Now, $\varphi_{2,1}$ is a covering map of $(\mathbb{C}^2, 2X + 3W)_0$ and one has $Z' = \varphi_{2,1}^{-1}(W) = \{x^2 - y^3 = 0\}$. Then we have the covering

$$\varphi_{2,3} = \varphi_{1,3} \circ \varphi_{2,1} : (\mathbb{C}^2, 2X + 3W)_0 \to (\mathbb{C}^2, 2X + 3Y + 3Z)_0,$$

which is related to covering of the tetrahedral germ by the uniformizer of its sub-orbifold $(\mathbb{C}^2, 2X + 3Y)_0$. On the other hand, if one would have changed the coordinates by the map $\tau : \ (z,y) = (x - y, y)$, then $\tau^{-1} \circ \varphi_{1,3}$ would be the uniformizer of the sub-orbifold $(\mathbb{C}^2,3Y)_0$. In this case, denote by Z the branch $\varphi_{1,3}^{-1}(Z)=\{z=0\}$ and by U the branch $\varphi_{1,3}^{-1}(X)=\{z+y^3=0\}$. Then

$$\tau^{-1}\circ\varphi_{1,3}:(\mathbb{C}^2,3U+3Z)_0\rightarrow(\mathbb{C}^2,2X+3Y+3Z)_0$$

is an orbifold covering. Note that, $(\mathbb{C}^2,3Z)_0$ is uniformized by $\mathbb{C}_0^2$ via $\varphi_{3,1}$. Denote by X' the lifting $\varphi_{3,1}^{-1}(U)=\{z^3+y^3=0\}$ of U. Then one has another covering,

$$\tau^{-1}\circ\varphi_{3,3}=\tau^{-1}\circ\varphi_{1,3}\circ\varphi_{3,1}:(\mathbb{C}^2,2X')_0\rightarrow(\mathbb{C}^2,2X+3Y+3Z)_0,$$

which is related to covering of the tetrahedral germ by the uniformizer of its sub-orbifold $(\mathbb{C}^2,3Y+3Z)_0$.

c. Coverings of the germ $(\mathbb{C}^2,2X+3Y+3Z)_0$ by the uniformizer of the sub-orbifold $(\mathbb{C}^2,3Z)_0$ is similar to the case b. It is enough to change the roles of Y and Z to see such coverings.

d. We know that the abelian germ $(\mathbb{C}^2,2X+3Y)_0$ is uniformized by $\mathbb{C}_0^2$ via the map $\varphi_{2,3}:(x,y)\rightarrow(x^2,y^3)$. If we denote by Z' the branch $\varphi_{2,3}^{-1}(Z)=\{x^2-y^3=0\}$, then we have the covering

$$\varphi_{2,3}:(\mathbb{C}^2,3Z')_0\rightarrow(\mathbb{C}^2,2X+3Y+3Z)_0.$$

e. After change of coordinates in a suitable way, one can easily see that the uniformization of the tetrahedral germ due to its sub-orbifold $(\mathbb{C}^2,2X+3Z)_0$ is similar to the case d.

f. First let us change the coordinates by a map $\tau:\ (z,y)=(x-y,y)$, then $X=\{z+y=0\}$, $Y=\{y=0\}$ and $Z=\{z=0\}$. We know that the sub-orbifold $(\mathbb{C}^2,3Y+3Z)_0$ is uniformized by $\mathbb{C}_0^2$ via the map $\varphi_{3,3}:(z,y)\rightarrow(z^3,y^3)$. If we denote by X' the branch $\varphi_{3,3}^{-1}(X)=\{z^3+y^3=0\}$ by X' then we have the

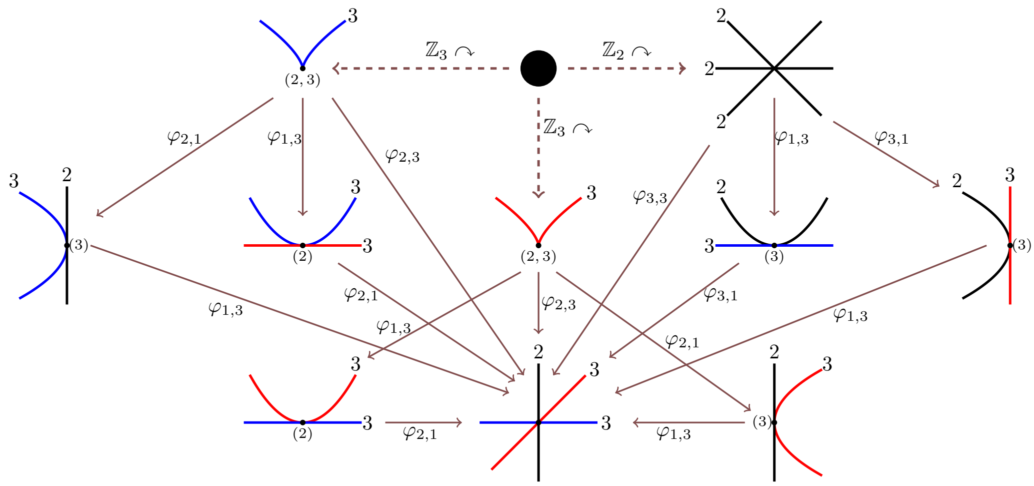


Figure 6.7 Coverings of the tetrahedral germ$(\mathbb{C}^2, 2X+3Y+3Z)_0$.

covering

$$\varphi_{3,3} : (\mathbb{C}^2, 2X')_0 \to (\mathbb{C}^2, 2X+3Y+3Z)_0.$$

Note that X' consists of three lines. Set $X_1 = \{z+y=0\}$, $X_2 = \{z+\omega y=0\}$ and $X_3 = \{z+\omega^2 y=0\}$, where ω is a third root of unity, then $X' = X_1 \cup X_2 \cup X_3$ and $(\mathbb{C}^2, 2X')_0$ is the dihedral germ $(\mathbb{C}^2, 2X_1+2X_2+2X_3)_0$. This tell us that, dihedral germ appears as a covering of the tetrahedral germ. The coverings of the dihedral germ has already been explained in Section 6.2.4.2.

Remark 6.2.13. The black dot on top of Figure 6.7 represents the surface

$$\mathcal{S} = \{(x,y,z) \in \mathbb{C}^3 \mid -x^2+y^3+z^3=0\}.$$

It is clear that the projection $(x,y,z) \to (y,z)$ defines a $\mathbb{Z}_2$ orbifold covering of the orbifold $(\mathbb{C}^2, 2X')_0$. Similarly, the coordinate projections $(x,y,z) \to (x,z)$ and $(x,y,z) \to (x,y)$ define $\mathbb{Z}_3$ coverings of the orbifolds $(\mathbb{C}^2, 3Y')_0$ and $(\mathbb{C}^2, 3Z')_0$, respectively. The surface $\mathcal{S}$ has a D_4 singularity at the origin. Indeed, the blow-up $\mathcal{S}' \to \mathcal{S}$ is covered by 3 affine pieces, of which I only write down one: consider $\mathbb{C}^3$ with coordinates x_1, y_1, z, and the morphism $\psi : \mathbb{C}^3 \to \mathbb{C}^3$ defined by $x = x_1 z$, $y = y_1 z$ and $z = z$. The inverse image of $\mathcal{S}$ under ψ is defined by $f_1(x_1 z, x_2 z, z) = -x_1^2 z^2 + y_1^3 z^3 + z + 3 = z^2 f_1$, where $f_1(x_1, y_1, z) = -x_1^2 + (y_1^3+1)z$. Here the factor

z^2 vanishes on the exceptional (x_1,y_1)-plane $\mathbb{C}^2 = \psi^{-1}O : (z=0) \subset \mathbb{C}^3$, and the residual component $\mathcal{S}' : (f_1(x_1,y_1,z)=0) \subset \mathbb{C}^3$ is the birational transform of $\mathcal{S}$. Now clearly the inverse image of $O=(0,0,0)$ under ψ is the y_1-axis, and $\hat{\mathcal{S}}$: $-x_1^2+(y_1^3+1)z=0$ has ordinary double points at the 3 points where $x_1=z=0$ and $y_1^3+1=0$. One can check that the other affine pieces of the blowup have no further singular points. The resolution $\hat{\mathcal{S}} \to \mathcal{S}' \to \mathcal{S}$ is obtained on blowing up these three points, and the corresponding Dynkin diagram is D_4. Because of that $\mathcal{S}$ is the isolated surface (Du Val) singularity of type D_4.

6.2.4.4 Coverings of the Octahedral Germ

Consider the octahedral germ $(\mathbb{C}^2, 2X+3Y+4Z)_0$, where $X=\{x=0\}$, $Y=\{y=0\}$ and $Z=\{x-y=0\}$. Its orbifold fundamental group is the triangle group

$$\langle \mu_1,\mu_2,\mu_3 \mid [\mu_i,\mu_1\mu_2\mu_3] = \mu_1^2 = \mu_2^3 = \mu_3^4 = 1,\ i=1,2,3\rangle$$

of order 596. This group acts on $\mathbb{C}^2$ and the corresponding branch divisor is the octahedral germ. The orbifolds $(\mathbb{C}^2,2X)_0$, $(\mathbb{C}^2,3Y)_0$, $(\mathbb{C}^2,2Z)_0$, $(\mathbb{C}^2,4Z)_0$, $(\mathbb{C}^2,2X+3Y)_0$, $(\mathbb{C}^2,2X+2Z)_0$, $(\mathbb{C}^2,2X+4Z)_0$, $(\mathbb{C}^2,3Y+2Z)_0$ and $(\mathbb{C}^2,3Y+4Z)_0$ are its sub-orbifolds. Let us study the liftings of $(\mathbb{C}^2,2X+3Y+4Z)_0$ due to uniformizers of its sub-orbifolds.

a. The uniformizer of $(\mathbb{C}^2,2X)_0$ is the map $\varphi_{2,1} : (x,y) \to (x^2,y)$. If we denote by Y the lifting $\varphi_{2,1}^{-1}(Y) = \{y=0\}$ and by W the lifting $\varphi_{2,1}^{-1}(Z) = \{x^2-y=0\}$, then

$$\varphi_{2,1} : (\mathbb{C}^2,3Y+4W)_0 \to (\mathbb{C}^2,2X+3Y+4Z)_0$$

is an orbifold covering. Now, $\varphi_{1,3}$ is a covering of $(\mathbb{C}^2,3Y+4W)_0$ and one has $Z' = \varphi_{1,3}^{-1}(W) = \{x^2-y^3=0\}$. Therefore we have the covering

$$\varphi_{2,3} : (\mathbb{C}^2,4Z')_0 \to (\mathbb{C}^2,2X+3Y+4Z)_0,$$

which is related to covering of octahedral germ by the uniformizer of its sub orbifold $(\mathbb{C}^2, 2X+3Y)_0$.

On the other hand, if one would have changed the coordinates by the map $\sigma:\ (x,z)=(x,x-y)$, then $\sigma^{-1}\circ\varphi_{2,1}$ would be the uniformizer of the sub-orbifold $(\mathbb{C}^2,2X)_0$. In this case, denote by Z the branch $\varphi_{2,1}^{-1}(Z)=\{z=0\}$ and by V the branch $\varphi_{2,1}^{-1}(Y)=\{x^2-z=0\}$. Then

$$\sigma^{-1}\circ\varphi_{2,1}:(\mathbb{C}^2,3V+4Z)_0\to(\mathbb{C}^2,2X+3Y+3Z)_0$$

is an orbifold covering. The sub-orbifold $(\mathbb{C}^2,4Z)_0$ is uniformized by $\mathbb{C}^2_0$ via $\varphi_{1,4}$. Denote by Y' the lifting $\varphi_{1,4}^{-1}(V)=\{x^2-z^4=0\}$ of V. Then one has another covering,

$$\sigma^{-1}\circ\varphi_{2,4}=\sigma^{-1}\circ\varphi_{2,1}\circ\varphi_{1,4}:(\mathbb{C}^2,3Y')_0\to(\mathbb{C}^2,2X+3Y+4Z)_0,$$

which is related to covering of the octahedral germ by the uniformizer of its sub-orbifold $(\mathbb{C}^2,2X+4Z)_0$. Note that $(\mathbb{C}^2,2Z)_0$ is a sub orbifold of $(\mathbb{C}^2,4Z)_0$ and $\varphi_{1,4}=\varphi_{1,2}\circ\varphi_{1,2}$. By using this fact one may obtain the factorization $\sigma^{-1}\circ\varphi_{2,1}\circ\varphi_{1,2}\circ\varphi_{1,2}$ of the covering $\sigma^{-1}\circ\varphi_{2,1}\circ\varphi_{1,4}$. We will omit to explain this factorization but exhibit in the Figure 6.8.

b. The uniformizer of $(\mathbb{C}^2,3Y)_0$ is the map $\varphi_{1,3}:(x,y)\to(x,y^3)$. If we denote by X the lifting $\varphi_{1,3}^{-1}(X)=\{x=0\}$ and by W the lifting $\varphi_{1,3}^{-1}(Z)=\{x-y^3=0\}$, then one has the covering

$$\varphi_{1,3}:(\mathbb{C}^2,2X+4W)_0\to(\mathbb{C}^2,2X+3Y+4Z)_0.$$

Now, $\varphi_{2,1}$ is a covering of $(\mathbb{C}^2,2X+4W)_0$ and one has $Z'=\varphi_{2,1}^{-1}(W)=\{x^2-y^3=0\}$. Therefore, we have the covering

$$\varphi_{2,3}:(\mathbb{C}^2,4Z')_0\to(\mathbb{C}^2,2X+3Y+4Z)_0,$$

which is related to covering of octahedral germ by the uniformizer of the sub-orbifold $(\mathbb{C}^2, 2X+3Y)_0$.

On the other hand, if one would have changed the coordinates by the map $\tau: (z,y) = (x-y,y)$, then $\tau^{-1} \circ \varphi_{1,3}$ would be the uniformizer of the sub-orbifold $(\mathbb{C}^2, 3Y)_0$. In this case, denote by Z the branch $\varphi_{1,3}^{-1}(Z) = \{z=0\}$ and by U the branch $\varphi_{1,3}^{-1}(X) = \{z+y^3=0\}$. Then

$$\tau^{-1} \circ \varphi_{1,3} : (\mathbb{C}^2, 2U+4Z)_0 \to (\mathbb{C}^2, 2X+3Y+3Z)_0$$

is an orbifold covering. In addition, the sub-orbifold $(\mathbb{C}^2, 4Z)_0$ is uniformized by $\mathbb{C}^2_0$ via $\varphi_{4,1}$. Denote by X' the lifting $\varphi_{4,1}^{-1}(U) = \{z^4+y^3=0\}$ of V. Then one has another covering,

$$\tau^{-1} \circ \varphi_{4,3} = \tau^{-1} \circ \varphi_{1,3} \circ \varphi_{4,1} : (\mathbb{C}^2, 2X')_0 \to (\mathbb{C}^2, 2X+3Y+tZ)_0,$$

which is related to covering of the octahedral germ via its sub-orbifold $(\mathbb{C}^2, 3Y+4Z)_0$. Note that $(\mathbb{C}^2, 2Z)_0$ is a sub orbifold of $(\mathbb{C}^2, 4Z)_0$ and $\varphi_{4,1} = \varphi_{2,1} \circ \varphi_{2,1}$. By using this fact one may obtain the factorization $\tau^{-1} \circ \varphi_{1,3} \circ \varphi_{2,1} \circ \varphi_{2,1}$ of the covering $\tau^{-1} \circ \varphi_{1,3} \circ \varphi_{4,1}$. We will omit to explain this factorization but exhibit in the Figure 6.8.

c. Now consider the sub-orbifold $(\mathbb{C}^2, 2Z)_0$, and change the coordinates by a map $\sigma : (x,z) = (x, x-y)$, then $X = \{x=0\}$, $Y = \{x-z=0\}$ and $Z = \{z=0\}$. The orbifold $(\mathbb{C}^2, 2Z)_0$ is uniformized by $\mathbb{C}^2_0$ via $\varphi_{1,2} : (x,z) \to (x,z^2)$. If we denote by X, V' and Z the branches $\varphi_{1,2}^{-1}(X) = \{x=0\}$, $\varphi_{1,2}^{-1}(Y) = \{x-z^2=0\}$ and $\varphi_{1,2}^{-1}(Z) = \{z=0\}$, respectively, then we have the covering

$$\sigma^{-1} \circ \varphi_{1,2} : (\mathbb{C}^2, 2X+3V'+2Z)_0 \to (\mathbb{C}^2, 2X+3Y+4Z)_0.$$

Taking the lifting of $(\mathbb{C}^2, 2X+3V'+2Z)_0$ by $\varphi_{1,2}$, and setting $X := \varphi_{1,2}^{-1}(X) =$

$\{x=0\}$ and $V := \varphi_{1,2}^{-1}(V') = \{x - z^4\}$ we will obtain the covering

$$\sigma^{-1} \circ \varphi_{1,4} = \sigma^{-1} \circ \varphi_{1,2} \circ \varphi_{1,2} = (\mathbb{C}^2, 2X + 3V)_0 \to (\mathbb{C}^2, 2X + 3Y + 4Z)_0$$

which is related to covering of the octahedral germ by uniformizer of its sub-orbifold $(\mathbb{C}^2, 2X + 3Y + 4Z)_0$. So, we will explain further coverings of the orbifold $(\mathbb{C}^2, 2X + 3V)_0$ in the case d.

Beside this, one may consider the orbifold $(\mathbb{C}^2, 2X + 3V' + 2Z)_0$ which appeared as a cover of octahedral germ, above. Its sub-orbifold $(\mathbb{C}^2, 2X)_0$ is uniformized by $\mathbb{C}^2_0$ via $\varphi_{2,1} : (x,z) \mapsto (x^2, z)$. Setting $V'' := \varphi_{2,1}^{-1}(V') = \{x^2 - z^2 = 0\}$ and $Z := \varphi_{2,1}^{-1}(Z) = \{z = 0\}$, we have an orbifold covering

$$\sigma^{-1} \circ \varphi_{2,2} = \sigma^{-1} \circ \varphi_{1,2} \circ \varphi_{2,1} : (\mathbb{C}^2, 3V'' + 2Z)_0 \to (\mathbb{C}^2, 2X + 3Y + 4Z)_0,$$

which is related to covering of the octahedral germ by the uniformizer of its sub-orbifold $(\mathbb{C}^2, 2X + 2Z)_0$. Note that $(\mathbb{C}^2, 3V'' + 2Z)_0$ is a tetrahedral germ and it appeared as covering of the octahedral germ. We will explain further coverings of $(\mathbb{C}^2, 3V'' + 2Z)_0$ in the case f.

On the other hand, if one would have changed the coordinates by the map $\tau : (z,y) = (x - y, y)$, then $\tau^{-1} \circ \varphi_{2,1}$ would be the uniformizer of the sub-orbifold $(\mathbb{C}^2, 2Z)_0$. In this case, denote by U', Y and Z the branches $\varphi_{2,1}^{-1}(X) = \{z^2 + y = 0\}$, $\varphi_{2,1}^{-1}(Y) = \{y = 0\}$ and $\varphi_{2,1}^{-1}(Z) = \{z = 0\}$, respectively. Then

$$\tau^{-1} \circ \varphi_{2,1} : (\mathbb{C}^2, 2U' + 3Y + 2Z)_0 \to (\mathbb{C}^2, 2X + 3Y + 3Z)_0$$

is an orbifold covering. Taking the lifting of $(\mathbb{C}^2, 2U' + 3Y + 2Z)_0$ by $\varphi_{2,1}$, and setting $U := \varphi_{2,1}^{-1}(U') = \{z + y^4 = 0\}$ and $Y := \varphi_{2,1}^{-1}(Y) = \{y = 0\}$ we will obtain the covering

$$\tau^{-1} \circ \varphi_{4,1} = \tau^{-1} \circ \varphi_{2,1} \circ \varphi_{2,1} = (\mathbb{C}^2, 2U + 3Y)_0 \to (\mathbb{C}^2, 2X + 3Y + 4Z)_0,$$

which is related to covering of the octahedral germ by uniformizer of its sub-orbifold $(\mathbb{C}^2, 2X+3Y+4Z)_0$. So, we will explain further coverings of the orbifold $(\mathbb{C}^2, 2U+3Y)_0$ in the case d.

Beside this, one may consider the orbifold $(\mathbb{C}^2, 2U'+3Y+2Z)_0$ which appeared as a cover of octahedral germ, above. Its sub-orbifold $(\mathbb{C}^2, 3Y)_0$ is uniformized by $\mathbb{C}^2_0$ via $\varphi_{1,3} : (z,y) \mapsto (z,y^3)$. Setting $U'' := \varphi_1^{-1}(U') = \{z^2+y^3=0\}$ and $Z := \varphi_{1,3}^{-1}(Z) = \{z=0\}$, we have an orbifold covering

$$\tau^{-1} \circ \varphi_{2,3} = \tau^{-1} \circ \varphi_{2,1} \circ \varphi_{1,3} : (\mathbb{C}^2, 2U''+2Z)_0 \to (\mathbb{C}^2, 2X+3Y+4Z)_0,$$

which is related to covering of the octahedral germ by the uniformizer of its sub-orbifold $(\mathbb{C}^2, 3Y+2Z)_0$. We will explain further coverings of $(\mathbb{C}^2, 2U''+2Z)_0$ in the case h.

d. Consider the sub-orbifold $(\mathbb{C}^2, 4Z)_0$ and change the coordinates by a map $\sigma : (x,z) = (x, x-y)$, then $X = \{x=0\}$, $Y = \{x-z=0\}$ and $Z = \{z=0\}$. Then it is uniformized by $\mathbb{C}^2_0$ via $\varphi_{1,4}$. Denote by X the lifting $\varphi_{1,4}^{-1}(X) = \{x=0\}$ of X and by V the lifting $\varphi_{1,4}^{-1}(Y) = \{x-z^4=0\}$ of Y. Then one has the covering

$$\sigma^{-1} \circ \varphi_{1,4} : (\mathbb{C}^2, 2X+3V)_0 \to (\mathbb{C}^2, 2X+3Y+4Z)_0.$$

Since the uniformizer of $(\mathbb{C}^2, 2X)_0$ is $\varphi_{2,1} : (x,z) \mapsto (x^2, z)$, then by setting $Y' := \varphi_{2,1}^{-1}(V) = \{x^2-z^4=0\}$ we obtain the covering

$$\sigma^{-1} \circ \varphi_{2,4} = \sigma^{-1} \circ \varphi_{1,4} \circ \varphi_{2,1} : (\mathbb{C}^2, 3Y')_0 \to (\mathbb{C}^2, 2X+3Y+4Z)_0,$$

which is related to covering of the octahedral germ by the uniformizer of its sub-orbifold $(\mathbb{C}^2, 2X+4Z)_0$.

On the other hand, if one would have changed the coordinates by the map $\tau : (z,y) = (x-y, y)$, then $X = \{z+y=0\}$, $Y = y = 0$, $Z = \{z=0\}$ and $\tau^{-1} \circ \varphi_{4,1}$ would be the uniformizer of the sub-orbifold $(\mathbb{C}^2, 4Z)_0$. In this case, denote by

U, Y the branches $\varphi_{4,1}^{-1}(X) = \{z^4 + y = 0\}$ and $\varphi_{4,1}^{-1}(Y) = \{y = 0\}$, respectively. Then one has the covering

$$\tau^{-1}\varphi_{4,1} : (\mathbb{C}^2, 2U + 3Y)_0 \to (\mathbb{C}^2, 2X + 3Y + 4Z)_0.$$

The orbifold $(\mathbb{C}^2, 3Y)_0$ is uniformized by $\mathbb{C}_0^2$ via $\varphi_{1,3} : (z, y) \mapsto (z, y^3)$. Then we have the covering

$$\tau^{-1}\varphi_{4,3} = \tau^{-1}\varphi_{4,1} \circ \varphi_{1,3} : (\mathbb{C}^2, 2X')_0 \to (\mathbb{C}^2, 2X + 3Y + 4Z)_0,$$

which is related to covering of the octahedral germ by the uniformizer of its sub-orbifold $(\mathbb{C}^2, 3Y + 4Z)_0$.

e. The uniformizer of $(\mathbb{C}^2, 2X + 3Y)_0$ is the map $\varphi_{2,3} : (x, y) \to (x^2, y^3)$. If we denote by Z' the branch $\varphi_{2,3}^{-1}(Z) = \{x^2 - y^3 = 0\}$, then $(\mathbb{C}^2, 4Z')_0$ is a lifting of $(\mathbb{C}^2, 2X + 3Y + 4Z)_0$ via $\varphi_{2,3}$ and we have the covering

$$\varphi_{2,3} : (\mathbb{C}^2, 4Z')_0 \to (\mathbb{C}^2, 2X + 3Y + 4Z)_0.$$

f. Consider the sub orbifold $(\mathbb{C}^2, 2X + 2Z)_0$ and change the coordinates by a map $\sigma : (x, z) = (x, x - y)$. Then $X = \{x = 0\}$, $Y = \{x - z = 0\}$ and $Z = \{z = 0\}$, and the uniformizer of $(\mathbb{C}^2, 2X + 2Z)_0$ is the map $\varphi_{2,2} : (x, z) \to (x^2, z^2)$. If we denote V'' the branch $\varphi_{2,2}^{-1}(Y) = \{x^2 - z^2 = 0\}$ and Z the branch $\varphi_{2,2}^{-1}(Z) = \{z = 0\}$, then one has the covering

$$\sigma^{-1} \circ \varphi_{2,2} : (\mathbb{C}^2, 3V'' + 2Z)_0 \to (\mathbb{C}^2, 2X + 3Y + 4Z)_0.$$

Note that V'' consists of two lines through the origin and the germ $(\mathbb{C}^2, 3V'' + 2Z)$ is tetrahedral. We have already study the coverings of tetrahedral germ in the Section 6.2.4.3. On the other hand, the sub-orbifold $(\mathbb{C}^2, 2Z)_0$ of $(\mathbb{C}^2, 3V'' +$

$2Z)_0$ is uniformized by $\mathbb{C}^2_0$ via $\varphi_{1,2} : (x,z) \mapsto (x,z^2)$. Then we have the covering

$$\varphi_{1,2} : (\mathbb{C}^2, 3Y')_0 \to (\mathbb{C}^2, 3V'' + 2Z)_0,$$

which naturally induces the covering

$$\sigma^{-1} \circ \varphi_{2,4} = \sigma^{-1} \circ \varphi_{2,2} \circ \varphi_{1,2} : (\mathbb{C}^2, 3Y')_0 \to (\mathbb{C}^2, 2X + 3Y + 4Z)_0,$$

where $Y' = \varphi_{1,2}^{-1}(V'') = \{x^2 - z^4 = 0\}$.

g. Consider the sub orbifold $(\mathbb{C}^2, 2X + 4Z)_0$ change the coordinates by a map $\sigma : (x,z) = (x, x-y)$. Then $X = \{x = 0\}$, $Y = \{x - z = 0\}$ and $Z = \{z = 0\}$, and the uniformizer of $(\mathbb{C}^2, 2X + 4Z)_0$ is the map $\varphi_{2,4} : (x,z) \to (x^2, z^4)$. If we denote by Y' the branch $\varphi_{2,4}^{-1}(Y) = \{x^2 - z^4 = 0\}$ then one has the covering

$$\sigma^{-1} \circ \varphi_{2,4} : (\mathbb{C}^2, 3Y')_0 \to (\mathbb{C}^2, 2X + 3Y + 4Z)_0.$$

h. Consider the sub orbifold $(\mathbb{C}^2, 3Y + 2Z)_0$ and change the coordinates by a map $\tau : (z,y) = (x - y, y)$. Then $X = \{z + y = 0\}$, $Y = \{y = 0\}$ and $Z = \{z = 0\}$, and the uniformizer of $(\mathbb{C}^2, 3Y + 2Z)_0$ is the map $\varphi_{2,3} : (z,y) \to (z^2, y^3)$. If we denote by U'' the branch $\varphi_{2,3}^{-1}(X) = \{z^2 + y^3 = 0\}$ and by Z the branch $\varphi_{2,3}^{-1}(Z) = \{z = 0\}$, then one has the covering

$$\tau^{-1} \circ \varphi_{2,3} : (\mathbb{C}^2, 2U'' + 2Z)_0 \to (\mathbb{C}^2, 2X + 3Y + 4Z)_0.$$

The sub-orbifold $(\mathbb{C}^2, 2Z)_0$ of $(\mathbb{C}^2, 2U'' + 2Z)_0$ is uniformized by $\mathbb{C}^2_0$ via $\varphi_{2,1} : (z,y) \mapsto (z^2, y)$. Then we have the covering

$$\varphi_{2,1} : (\mathbb{C}^2, 2X')_0 \to (\mathbb{C}^2, 2U'' + 2Z)_0,$$

which naturally induces the covering

$$\tau^{-1}\circ\varphi_{4,3}=\tau^{-1}\circ\varphi_{2,3}\circ\varphi_{2,1}:(\mathbb{C}^2,2X')_0\to(\mathbb{C}^2,2X+3Y+4Z)_0,$$

where $X'=\varphi_{2,1}^{-1}(U'')=\{z^4+y^3=0\}$.

i. Consider the sub orbifold $(\mathbb{C}^2,3Y+4Z)_0$ change the coordinates by a map $\tau:(z,y)=(x-y,y)$. Then $X=\{z+y=0\}$, $Y=\{y=0\}$ and $Z=\{z=0\}$, and the uniformizer of $(\mathbb{C}^2,3Y+4Z)_0$ is the map $\varphi_{4,3}:(x,z)\to(z^4,y^3)$. If we denote X' the branch $\varphi_{4,3}^{-1}(X)=\{z^4+y^3=0\}$ then one has the covering

$$\tau^{-1}\circ\varphi_{4,3}:(\mathbb{C}^2,2X')_0\to(\mathbb{C}^2,2X+3Y+4Z)_0.$$

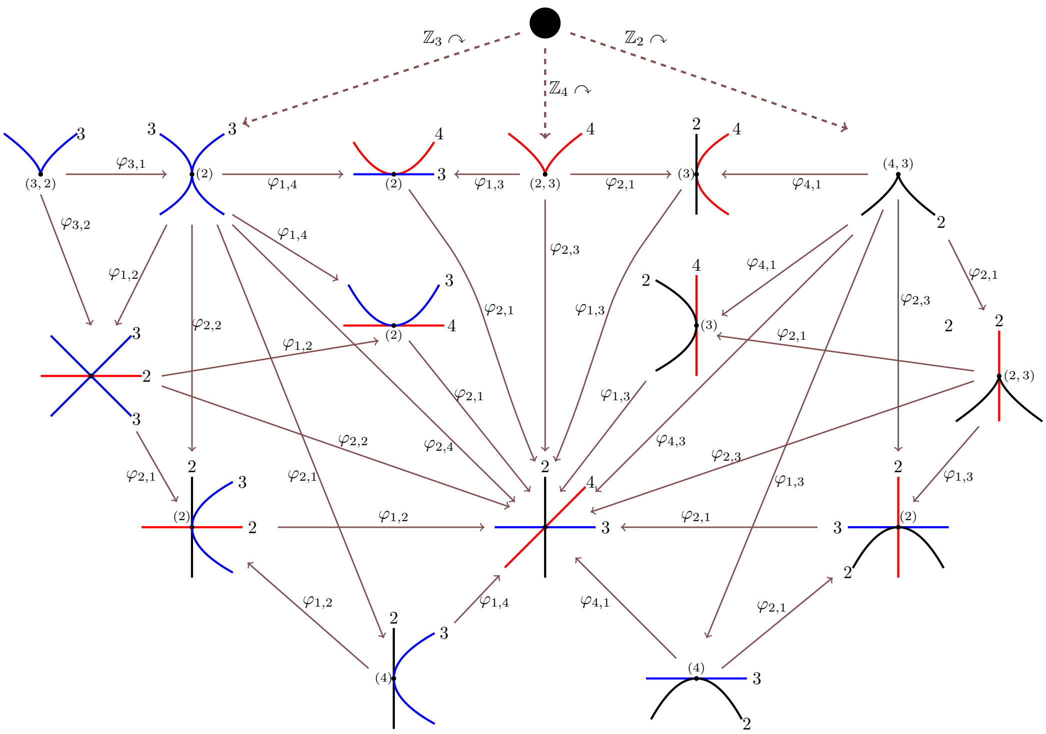


Figure 6.8 Coverings of the octahedral germ$(\mathbb{C}^2,2X+3Y+4Z)_0$.

Remark 6.2.14. The black dot on top of Figure 6.8 represents the isolated surface (Du Val) singularity of type E_6, given by the equation

$$\mathcal{E}_6 := \{(x,y,z) \in \mathbb{C}^3 \mid -x^2+y^3+z^4=0\}.$$

It is clear that the projection $(x,y,z) \to (x,y)$ defines a $\mathbb{Z}_4$ orbifold covering by this singularity of the orbifold $(\mathbb{C}^2, 4Z')_0$. Other coordinate projections $(x,y,z) \to (y,z)$ and $(x,y,z) \to (x,z)$ define respectively $\mathbb{Z}_2$ and $\mathbb{Z}_3$ coverings by the same singularity of the orbifolds $(\mathbb{C}^2, 2X')_0$ and $(\mathbb{C}^2, 3Y')_0$.

6.2.4.5 *Coverings of the Icosahedral Germ*

Consider the icosahedral germ $(\mathbb{C}^2, 2X+3Y+5Z)_0$, where $X = \{x=0\}$, $Y = \{y=0\}$ and $Z = \{x-y=0\}$. Its orbifold fundamental group is the triangle group

$$\langle \mu_1, \mu_2, \mu_3 \mid \; [\mu_i, \mu_1\mu_2\mu_3] = \mu_1^2 = \mu_2^3 = \mu_3^5 = 1, \; i = 1,2,3 \rangle$$

of order 3600. So, $(\mathbb{C}^2, 2X)_0$, $(\mathbb{C}^2, 3Y)_0$, $(\mathbb{C}^2, 5Z)_0$, $(\mathbb{C}^2, 2X+3Y)_0$, $(\mathbb{C}^2, 2X+5Z)_0$ and $(\mathbb{C}^2, 3Y+5Z)_0$ are its sub-orbifolds. Let us study the liftings of $(\mathbb{C}^2, 2X+3Y+5Z)_0$ due to uniformizer of its sub-orbifolds. Figure 6.9 exhibits all coverings of the icosahedral germ.

a. The uniformizer of $(\mathbb{C}^2, 2X)_0$ is the map $\varphi_{2,1} : (x,y) \to (x^2, y)$. If we denote by Y the branch $\varphi_{2,1}^{-1}(Y) = \{y=0\}$ and by W the branch $\varphi_{2,1}^{-1}(Z) = \{x^2 - y = 0\}$, then
$$\varphi_{2,1} : (\mathbb{C}^2, 3Y+5W)_0 \to (\mathbb{C}^2, 2X+3Y+5Z)_0$$
is an orbifold covering. Now, $\varphi_{1,3}$ is a covering of $(\mathbb{C}^2, 3Y+5W)_0$ and one has $Z' = \varphi_{1,3}^{-1}(W) = \{x^2 - y^3 = 0\}$. Then we get the covering
$$\varphi_{2,3} = \varphi_{2,1} \circ \varphi_{1,3} : (\mathbb{C}^2, 5Z')_0 \to (\mathbb{C}^2, 2X+3Y+5Z)_0,$$

which is related to covering of the icosahedral germ by the uniformizer of its sub-orbifold $(\mathbb{C}^2, 2X+3Y)_0$.

On the other hand, if one would have changed the coordinates by the map $\sigma: (x,z) = (x, x-y)$, then $X = \{x=0\}$, $Y = \{x-z=0\}$, $Z = \{z=0\}$ and $\sigma^{-1}\circ\varphi_{2,1}$ would be the uniformizer of the sub-orbifold $(\mathbb{C}^2, 2X)_0$. In this case, denote by Z the branch $\varphi_{2,1}^{-1}(Z) = \{z=0\}$ and by V the branch $\varphi_{2,1}^{-1}(Y) = \{x^2-z=0\}$. Then

$$\sigma^{-1}\circ\varphi_{2,1} : (\mathbb{C}^2, 3V+5Z)_0 \to (\mathbb{C}^2, 2X+3Y+5Z)_0$$

is an orbifold covering. The sub-orbifold $(\mathbb{C}^2, 5Z)_0$ is uniformized by $\mathbb{C}^2_0$ via $\varphi_{1,5}$. Denote by Y' the lifting $\varphi_{1,5}^{-1}(V) = \{x^2 - z^5 = 0\}$ of V. Then one has another covering,

$$\sigma^{-1}\circ\varphi_{2,5} = \sigma^{-1}\circ\varphi_{2,1}\circ\varphi_{1,5} : (\mathbb{C}^2, 3Y')_0 \to (\mathbb{C}^2, 2X+3Y+5Z)_0,$$

which is related to covering of the icosahedral germ by the uniformizer of its sub-orbifold $(\mathbb{C}^2, 2X+5Z)_0$.

b. The uniformizer of $(\mathbb{C}^2, 3Y)_0$ is the map $\varphi_{1,3} : (x,y) \to (x, y^3)$. If we denote by X the branch $\varphi_{1,3}^{-1}(X) = \{x=0\}$ and by W' the branch $\varphi_{1,3}^{-1}(Z) = \{x-y^3=0\}$, then then

$$\varphi_{1,3} : (\mathbb{C}^2, 2X+5W')_0 \to (\mathbb{C}^2, 2X+3Y+5Z)_0$$

is an orbifold covering. Now, $\varphi_{2,1}$ is a covering of $(\mathbb{C}^2, 2X+5W')_0$ and one has $Z' = \varphi_{2,1}^{-1}(W') = \{x^2-y^3=0\}$. Then we get the covering

$$\varphi_{2,3} = \varphi_{1,3}\circ\varphi_{2,1} : (\mathbb{C}^2, 5Z')_0 \to (\mathbb{C}^2, 2X+3Y+5Z)_0,$$

which is related to covering of the icosahedral germ by the uniformizer of its sub-orbifold $(\mathbb{C}^2, 2X+3Y)_0$.

On the other hand, if one would have changed the coordinates by the map

$\tau: (z,y) = (x-y,y)$, then $X = \{z+y=0\}$, $Y = \{y=0\}$, $Z = \{z=0\}$ and $\tau^{-1}\circ\varphi_{1,3}$ would be the uniformizer of the sub-orbifold $(\mathbb{C}^2,3Y)_0$. In this case, denote by Z the branch $\varphi_{1,3}^{-1}(Z) = \{z=0\}$ and by U the branch $\varphi_{1,3}^{-1}(X) = \{z+y^3=0\}$. Then

$$\tau^{-1}\circ\varphi_{1,3} : (\mathbb{C}^2,2U+5Z)_0 \to (\mathbb{C}^2,2X+3Y+5Z)_0$$

is an orbifold covering. The sub-orbifold $(\mathbb{C}^2,5Z)_0$ is uniformized by $\mathbb{C}^2_0$ via $\varphi_{5,1}$. Denote by X' the lifting $\varphi_{5,1}^{-1}(U) = \{z^5+y^3=0\}$ of V. Then one has another covering,

$$\tau^{-1}\circ\varphi_{5,3} = \tau^{-1}\circ\varphi_{1,3}\circ\varphi_{5,1} : (\mathbb{C}^2,2X')_0 \to (\mathbb{C}^2,2X+3Y+5Z)_0,$$

which is related to covering of the icosahedral germ by uniformizer of its sub-orbifold $(\mathbb{C}^2,3Y+5Z)_0$.

c. Consider the sub-orbifold $(\mathbb{C}^2,5Z)_0$ and change the coordinates by a map σ : $(x,z) = (x,x-y)$. Then $X = \{x=0\}$, $Y = \{x-z=0\}$ and $Z = \{z=0\}$. The uniformizer of $(\mathbb{C}^2,5Z)_0$ is the map $\varphi_{1,5} : (x,z) \to (x,z^5)$. If we denote by X the branch $\varphi_{1,5}^{-1}(X) = \{x=0\}$ and by V the branch $\varphi_{1,5}^{-1}(Y) = \{x-z^5=0\}$, then we have an orbifold covering

$$\sigma^{-1}\circ\varphi_{1,5} : (\mathbb{C}^2,2X+3V)_0 \to (\mathbb{C}^2,2X+3Y+5Z)_0.$$

Now, $\varphi_{2,1}$ is a covering of $(\mathbb{C}^2,2X+3V)_0$ and one has $Y' = \varphi_{2,1}^{-1}(V) = \{x^2-z^5=0\}$. Hence we have the covering

$$\sigma^{-1}\circ\varphi_{2,5} = \sigma^{-1}\circ\varphi_{1,5}\circ\varphi_{2,1} : (\mathbb{C}^2,3Y')_0 \to (\mathbb{C}^2,2X+3Y+5Z)_0,$$

which is related to covering of the icosahedral by the uniformizer of its sub-orbifold $(\mathbb{C}^2,2X+5Z)_0$.

On the other hand, if one would have changed the coordinates by the map

$\tau:\ (z,y) = (x-y,y)$, then $X = \{z+y=0\}$, $Y = \{y=0\}$, $Z = \{z=0\}$ and $\tau^{-1}\circ\varphi_{5,1}$ would be the uniformizer of the sub-orbifold $(\mathbb{C}^2,5Z)_0$. In this case, denote by U' the branch $\varphi_{5,1}^{-1}(X) = \{z^5+y=0\}$ and by Y the branch $\varphi_{5,1}^{-1}(Y) = \{y=0\}$. Then

$$\tau^{-1}\circ\varphi_{5,1} : (\mathbb{C}^2,2U'+3Y)_0 \to (\mathbb{C}^2,2X+3Y+5Z)_0$$

is an orbifold covering. The sub-orbifold $(\mathbb{C}^2,3Y)_0$ is uniformized by $\mathbb{C}^2_0$ via $\varphi_{1,3}$. Denote by X' the lifting $\varphi_{1,3}^{-1}(U') = \{z^5+y^3=0\}$ of U'. Then one has another covering

$$\tau^{-1}\circ\varphi_{5,3} = \tau^{-1}\circ\varphi_{5,1}\circ\varphi_{1,3} : (\mathbb{C}^2,2X')_0 \to (\mathbb{C}^2,2X+3Y+5Z)_0,$$

which is related to covering of the icosahedral germ by the uniformizer of its sub-orbifold $(\mathbb{C}^2,3Y+5Z)_0$.

d. The uniformizer of $(\mathbb{C}^2,2X+3Y)_0$ is the map $\varphi_{2,3} : (x,y) \to (x^2,y^3)$. If we denote by Z' the branch $\varphi_{2,3}^{-1}(Z) = \{x^2-y^3=0\}$, then one has the covering

$$\varphi_{2,3} : (\mathbb{C}^2,5Z')_0 \to (\mathbb{C}^2,2X+3Y+5Z)_0.$$

e. Consider the sub orbifold $(\mathbb{C}^2,2X+5Z)_0$ and change the coordinates by a map $\sigma : (x,z) = (x,x-y)$. Then the uniformizer of $(\mathbb{C}^2,2X+5Z)_0$ is the map $\varphi_{2,5} : (x,z) \to (x^2,z^5)$. If we denote by Y' the branch $\varphi_{2,5}^{-1}(Y) = \{x^2-z^5=0\}$, then one has the covering

$$\sigma^{-1}\circ\varphi_{2,5} : (\mathbb{C}^2,3Y')_0 \to (\mathbb{C}^2,2X+3Y+5Z)_0.$$

f. Consider the sub orbifold $(\mathbb{C}^2,3Y+5Z)_0$ and change the coordinates by a map $\tau : (z,y) = (x-y,y)$. Then the uniformizer of $(\mathbb{C}^2,3Y+5Z)_0$ is the map $\varphi_{5,3} : (z,y) \to (z^5,y^3)$. If we denote by X' the branch $\varphi_{5,3}^{-1}(Y) = \{z^5+x^3=0\}$, then one has the covering

$$\tau^{-1} \circ \varphi_{5,3} : (\mathbb{C}^2, 2X')_0 \to (\mathbb{C}^2, 2X + 3Y + 5Z)_0.$$

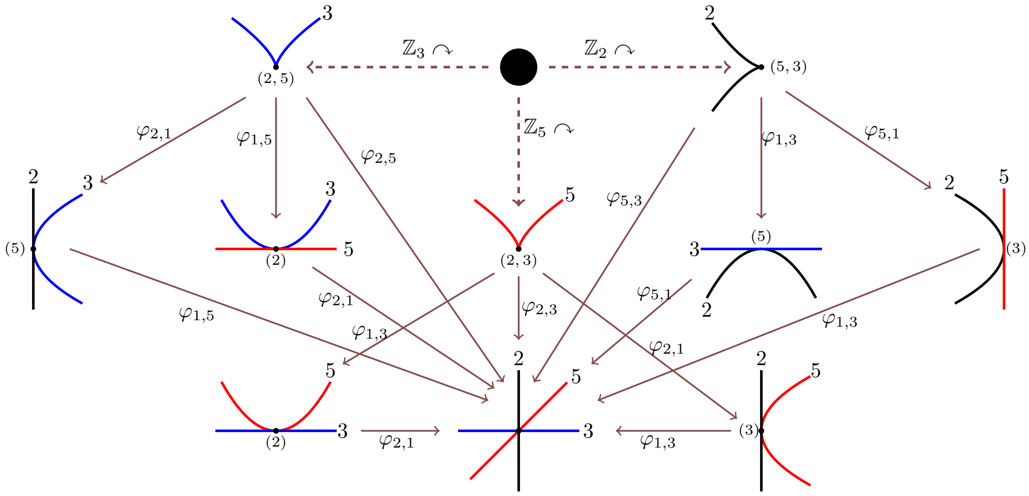


Figure 6.9 Coverings of the icosahedral germ$(\mathbb{C}^2, 2X + 3Y + 5Z)_0$.

Remark 6.2.15. The black dot on top of Figures 6.9 represents the isolated surface (Du Val) singularity of type E_8, given by the equation

$$\mathcal{E}_8 := \{(x,y,z) \in \mathbb{C}^3 \mid -x^2 + y^3 + z^5 = 0\}.$$

It is clear that the projection $(x,y,z) \to (x,y)$ defines a $\mathbb{Z}_5$ orbifold covering by this singularity of the orbifold $(\mathbb{C}^2, 5Z')_0$. Other coordinate projections $(x,y,z) \to (y,z)$ and $(x,y,z) \to (x,y)$ define $\mathbb{Z}_2$ and $\mathbb{Z}_3$ coverings by the same singularity of the orbifolds $(\mathbb{C}^2, 2X')_0$ and $(\mathbb{C}^2, 3Y')_0$, respectively.

6.2.4.6 *Coverings of the Other Orbifold Germs with Smooth Base*

In this section we will interested in coverings of the orbifold germs with smooth base and nonlinear branch loci in the Table 6.1. We omit drawing Figures since they appears as covers of orbifolds with linear branch loci. To see these coverings explicitly, see Figures 6.2, 6.3, 6.4, 6.5, 6.6, 6.7, 6.8 and 6.9.

(1) First consider the orbifold $(\mathbb{C}^2, pX)_0$, where $X = \{x^n - y^m = 0\}$ and $\rho := \frac{1}{p} + \frac{1}{n} + \frac{1}{m} - 1 > 0$ and $\gcd(n,m) = 1$. The possible triples (p,n,m) are listed in the Table 6.2. As we discussed in Remarks 6.2.12, 6.2.13, 6.2.14, 6.2.15, the uniformization of the orbifold $(\mathbb{C}^2, pX)_0$ is the surface

$$\mathcal{S} = \{(x,y,z) \in \mathbb{C}^3 \mid -x^n + y^m + z^p = 0\}$$

and the uniformizer is the $\mathbb{Z}_p$ covering corresponding to the projection $(x,y,z) \mapsto (x,y)$. Depending on the possible triples (p,n,m) listed in the Table 6.2, $\mathcal{S}$ is an isolated surface (Du Val) singularities of one of the types A_{p-1}, D_4, E_6, E_8.

(2) Second, consider the orbifold $(\mathbb{C}^2, pX + qY)_0$, where $X = \{x + y^n = 0\}$, $Y = \{x - y^n = 0\}$ and $\rho := \frac{1}{p} + \frac{1}{q} + \frac{1}{n} - 1 > 0$. The possible triples (p,q,n) are listed in the Table 6.2. Notice that $(\mathbb{C}^2, pX + qY)_0$ is a lifting of the orbifold germ $(\mathbb{C}^2, pH_1 + qH_2 + nH_3)_0$ via $\varphi_{1,n} : (x,y) \to (x,y^n)$, where $H_1 = \{x + y = 0\}$, $H_2 = \{x - y = 0\}$ and $H_3 = \{y = 0\}$. Fist of all let us change the coordinates by a map $\delta : (u,y) = (x+y,y)$, then we have $H_1 = \{u = 0\}$, $H_2 = \{u - 2y = 0\}$ and $H_3 = \{y = 0\}$. From the Sections 6.2.4.2, 6.2.4.3, 6.2.4.4 and 6.2.4.5, we know all coverings of $(\mathbb{C}^2, pH_1 + qH_2 + nH_3)_0$ and so all coverings of the germ $(\mathbb{C}^2, pX + qY)_0$. Since the uniformization of the germ $(\mathbb{C}^2, pH_1 + qH_2 + nH_3)_0$ is the surface $\mathcal{S}$ given by the equation $-u^p + 2y^n + z^q = 0$, returning back to original coordinates we get

$$\mathcal{S} = \{(x,y,z) \in \mathbb{C}^3 \mid -(x-y)^p + y^n + z^q = 0\}$$

as a universal cover of $(\mathbb{C}^2, pX + qY)_0$. Depending on the choice of possible triples (p,q,n), $\mathcal{S}$ is an isolated surface (Du Val) singularities of types A, D, E.

(3) Third, consider the orbifold $(\mathbb{C}^2, pX + qY + rZ)_0$, where $X = \{x - y^n = 0\}$, $Y = \{x + y^n = 0\}$, $Z = \{y = 0\}$ and $\rho := \frac{1}{p} + \frac{1}{q} + \frac{1}{nr} - 1 > 0$. The possible quadruples (p,q,r,n) are listed in the Table 6.2. Note that the lifting of this orbifold via the uniformizer $\varphi_{1,2}$ of its sub-orbifold $(\mathbb{C}^2, rZ)_0$ is the orbifold

$(\mathbb{C}^2, pX' + qY')_0$ with $X' = \{x - y^{nr}\}$ and $Y' = \{x + y^{nr}\}$. This is the orbifold in case (2) for which n is replaced by nr. Therefore, from the case (2) know its all coverings. Hence its uniformization is the surface

$$\mathcal{S} = \{(x,y,z) \in \mathbb{C}^3 \mid -(x-y)^p + y^{nr} + z^q = 0\}.$$

Depending on the choice of possible quadruples (p,q,r,n), $\mathcal{S}$ is an isolated surface (Du Val) singularities of types A, D, E.

(4) Next, consider the orbifold $(\mathbb{C}^2, 2X + qY)_0$, where $X = \{x^2 - y^n = 0\}$, $Y = \{y = 0\}$ and $n > 1$ is odd. The lifting of $(\mathbb{C}^2, 2X + qY)_0$ by the uniformizer $\varphi_{1,q}$ of $(\mathbb{C}^2, qY)_0$ is $(\mathbb{C}^2, 2X')_0$, where $X' = \{x^2 - y^{nq}\}$. Note that X' is a cusp $(2,nq)$-type if q is odd for which it corresponds to case (1), and reducible if q is even for which it corresponds to case (2). Therefore, from the cases (1) and (2), we know its all coverings. On the other hand, $(\mathbb{C}^2, 2X + qY)_0$ covers the orbifold $(\mathbb{C}^2, 2H_1 + nqH_2 + 2H_3)_0$ via $\varphi_{2,n} : (x,y) \mapsto (x^2, y^n)$, where $H_1 = \{x = 0\}$, $H_2 = \{y = 0\}$ and $H_3 = \{x - y = 0\}$. Since the uniformizations of $(\mathbb{C}^2, 2H_1 + nqH_2 + 2H_3)_0$ and $(\mathbb{C}^2, 2X + qY)_0$ are same, then the surface

$$\mathcal{S} = \{(x,y,z) \in \mathbb{C}^3 \mid -x^2 + y^{nq} + z^2 = 0\}$$

whose isolated surface singularity type is A, D, E, appears as the uniformization of $(\mathbb{C}^2, 2X + qY)_0$.

(5) Finally, consider the orbifold $(\mathbb{C}^2, 2X + 2Y)_0$, where $X = \{x = 0\}$, $Y = \{x^2 - y^3 = 0\}$. Since its orbifold fundamental group is of order 96, then $(\mathbb{C}^2, 2X)_0$ is its sub-orbifold and $\varphi_{2,1}$ is a uniformizer of $(\mathbb{C}^2, 2X)_0$. Denote by Y' the lifting $\varphi_{2,1}^{-1}(Y) = \{x^4 - y^3 = 0\}$. Then we have an orbifold covering

$$\varphi_{2,1} : (\mathbb{C}^2, 2Y')_0 \to (\mathbb{C}^2, 2X + 2Y)_0.$$

Note that the orbifold $(\mathbb{C}^2, 2Y')_0$ is uniformized by a surface of isolated surface

(Du Val) singularity of type E_6.

6.2.4.7 Coverings of the Other Orbifold Germs with Singular Base

In this section we will deal with only the covering relations between parabolic orbifolds with linear branch loci, but illustrate all covering relations containing parabolic orbifolds with non-linear branch loci in Figures 6.10, 6.10, 6.12 and 6.13. Note that the orbifold germs in this figures are consistent with the germs in Figure 6.1b, 6.1c, 6.1d and 6.1e. The solutions to condition $\rho = 0$ are given in the table 6.2.

1. First consider the orbifold $(\mathbb{C}^2, 2X+2Y+2Z+2W)_0$, where the lines X,Y,Z,W form a pencil at the origin. By the Theorem 5.10.1 and the equation (6.1.2), the orbifold fundamental group of this germ has the presentation

$$\langle \mu_1,\mu_2,\mu_3,\mu_4 \mid [\mu_4\mu_3\mu_2\mu_1,\mu_i] = \mu_i^2 = 1,\ i = 1,2,3,4\rangle.$$

   This group is infinite but solvable and isomorphic to a discrete subgroup Γ of $\mathrm{Aut}(\mathbb{C}^2)$ (Yoshida, 1987). This germ is uniformized by the transformation group $(\Gamma, \mathbb{C}^2)$. Since Γ is infinite but solvable, then many cusp points will appear in covers of $(\mathbb{C}^2, 2X+2Y+2Z+2W)_0$.

   Let us study the coverings of this orbifold. For the sake of simplicity we may choose the coordinates so that $X = \{x=0\}$, $Y = \{y=0\}$, $Z = \{x-y=0\}$ and $W = \{x+y=0\}$. The uniformizer the sub-orbifold $(\mathbb{C}^2, 2X)_0$ is $\varphi_{2,1}$. Denote by Y the lifting $\varphi_{2,1}^{-1}(Y) = \{y=0\}$, by Z' the lifting $\varphi_{2,1}^{-1}(Z) = \{x^2-y=0\}$, and by W' the lifting $\varphi_{2,1}^{-1}(W) = \{x^2+y=0\}$. Then we have an orbifold covering

$$\varphi_{2,1} : (\mathbb{C}^2, 2Y+2Z'+2W')_0 \to (\mathbb{C}^2, 2X+2Y+2Z+2W)_0.$$

   Consider the map $\varphi_{2,1}$ and denote by Z'' the lifting $\varphi_{1,2}^{-1}(Z')$ and by W'' the lifting $\varphi_{1,2}^{-1}(W')$ and set $Z_1'' = \{x+y=0\}$, $Z_2'' = \{x-y=0\}$, $W_1'' = \{x+iy=0\}$

and $W_2'' = \{x - iy = 0\}$. Then $Z'' = Z_1'' \cup Z_2''$, $W'' = W_1'' \cup W_2''$ and we have the covering

$$\varphi_{2,2} = \varphi_{2,1} \circ \varphi_{1,2} : (\mathbb{C}^2, 2Z_1'' + 2Z_2'' + 2W_1'' + 2W_2'')_0 \to (\mathbb{C}^2, 2X + 2Y + 2Z + 2W)_0,$$

which is related to cover of $(\mathbb{C}^2, 2X + 2Y + 2Z + 2W)$ by the uniformizer $\varphi_{2,2}$ of $(\mathbb{C}^2, 2X + 2Y)_0$. Let us now change the coordinates by $\delta : (u, v) = (x + y, x - y)$, then by rescaling the equations we have $Z_1'' = \{u = 0\}$, $Z_2'' = \{v = 0\}$, $W_1'' = \{u + iv = 0\}$ and $W_2'' = \{u - iv = 0\}$. Now, $\varphi_{2,2} : (u, v) \mapsto (u^2, v^2)$ is a uniformizer of $(\mathbb{C}^2, 2W_1'' + 2W_2'')_0$. Denote by W_1''' and W_2''' the branches $\varphi_{2,2}^{-1}(W_1'') = \{u^2 + iv^2 = 0\}$ and $\varphi_{2,2}^{-1}(W_2'') = \{u^2 - iv^2 = 0\}$, respectively. Set $W_{1,1} := \{u + \alpha iv = 0\}$, $W_{1,2} := \{u - \alpha iv = 0\}$, $W_{2,1} := \{u + \alpha v = 0\}$, $W_{2,2} := \{u - \alpha v = 0\}$ and $\overline{W} := W_{1,1} \cup W_{1,2} \cup W_{2,1} \cup W_{2,2}$, where $\alpha^2 = i$. Then $\overline{W} = \{u^4 + v^4 = 0\}$ and we have the coverings

$$\begin{aligned} \varphi_{2,2} : (\mathbb{C}^2, 2\overline{W})_0 &= (\mathbb{C}^2, W_{1,1} + 2W_{1,2} + 2W_{2,1} + 2W_{2,2})_0 \to \\ &\qquad (\mathbb{C}^2, 2Z_1'' + 2Z_2'' + 2W_1'' + 2W_2'')_0. \end{aligned}$$

and

$$\varphi_{2,2} \circ \delta^{-1} \circ \varphi_{2,2} : (\mathbb{C}^2, 2\overline{W})_0 \to (\mathbb{C}^2, 2X + 2Y + 2Z + 2W)_0.$$

Since the uniformization of $(\mathbb{C}^2, 2\overline{W})_0$ is the surface $\mathcal{S} = \{(u, v, z) \in \mathbb{C}^3 \mid u^4 + v^4 - 2z^2 = 0\}$, returning back to original coordinates we obtained the uniformization of the initial orbifold $(\mathbb{C}^2, 2X + 2Y + 2Z + 2W)_0$ as

$$\mathcal{S} = \{(x, y, z) \in \mathbb{C}^3 \mid x^4 + y^4 + 6x^2y^2 - z^2 = 0\}.$$

$\mathcal{S}$ is the surface of isolated surface singularity type X_9.

On the other hand, notice that the orbifolds $(\mathbb{C}^2, 2Z_1'' + 2Z_2'' + 2W_1'' + 2W_2'')_0$ and $(\mathbb{C}^2, W_{1,1} + 2W_{1,2} + 2W_{2,1} + 2W_{2,2})_0$ are similar to initial one (See Figure 6.10). By using this fact, one can construct an infinite tower of coverings, i.e, many ball-cusp points appears in covers. This is consistent with the solvability of its orbifold fundamental group.

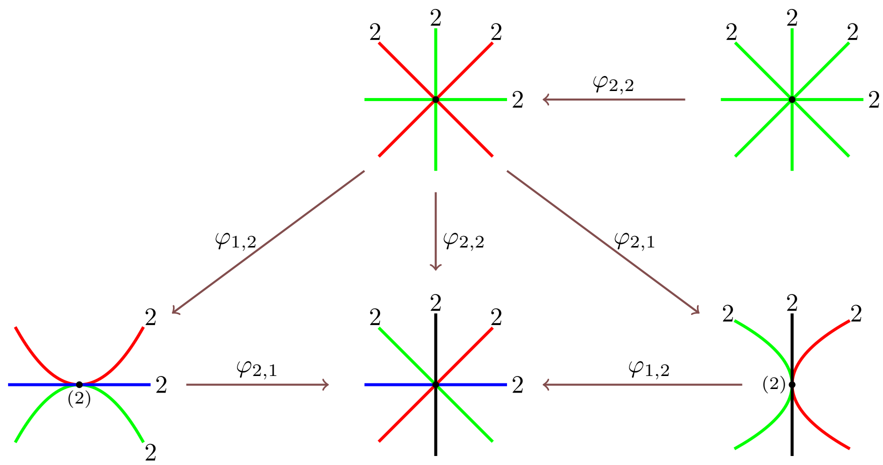


Figure 6.10 Coverings of the germ $(\mathbb{C}^2, 2X+2Y+2Z+2W)_0$.

2. Second, consider the orbifold $(\mathbb{C}^2, 3X+3Y+3Z)_0$, where the lines X, Y, Z form a pencil at the origin. The orbifold fundamental group of this germ is the triangle group

$$\langle \mu_1, \mu_2, \mu_3 \mid [\mu_3\mu_2\mu_1, \mu_i] = \mu_i^3 = 1,\ i = 1,2,3 \rangle.$$

This group is infinite but solvable and it is isomorphic to a discrete subgroup Γ of $\mathrm{Aut}(\mathbb{C}^2)$, and the transformation group $(\Gamma, \mathbb{C}^2)$ uniformizes this germ. Since Γ is infinite but solvable, then many cusp points will appear in covers of $(\mathbb{C}^2, 3X+3Y+3Z)_0$.

Now, let us study the covers of $(\mathbb{C}^2, 3X+3Y+3Z)_0$. For the sake of simplicity we may choose the coordinates so that $X = \{x=0\}$, $Y = \{y=0\}$, $Z = \{x-y=0\}$. The uniformizer the sub-orbifold $(\mathbb{C}^2, 3X)_0$ is $\varphi_{3,1}$. Denote by Y the lifting $\varphi_{3,1}^{-1}(Y) = \{y=0\}$ and by Z' the lifting $\varphi_{3,1}^{-1}(Z) = \{x^3-y=0\}$. Then we have an orbifold covering

$$\varphi_{3,1} : (\mathbb{C}^2, 3Y+3Z')_0 \to (\mathbb{C}^2, 3X+3Y+3Z)_0.$$

Taking the lifting of $(\mathbb{C}^2, 3Y+3Z')_0$ by $\varphi_{1,3}$, one can obtain the orbifold covering

$$\varphi_{3,3} = \varphi_{1,3} \circ \varphi_{3,1} : (\mathbb{C}^2, 3Z_1'' + 3Z_2'' + 3Z_3'')_0 \to (\mathbb{C}^2, 3X+3Y+3Z)_0,$$

where Z_i'' are linear components of $\varphi_{1,3}^{-1}(Z') = \{x^3 - y^3 = 0\}$ (See Figure 6.11). Hence the uniformization of $(\mathbb{C}^2, 3X+3Y+3Z)_0$ is the surface of isolated singularity of the type P_8 and it is given by the equation $T_{3,3,3} := \{(x,y,z) \in \mathbb{C}^3 \mid x^3+y^3+z^3 = 0\}$. This singularity type is also known as elliptic. Because, the germ at the origin of the isolated surface singularity $z^3 = xy(x-y)$ is a triple covering of the germ $(\mathbb{C}^2, 3X+3Y+3Z)_0$, it is resolved by a blow up which replace the origin by an elliptic curve.

Furthermore, note that the latest orbifold $(\mathbb{C}^2, 3Z_1''+3Z_2''+3Z_3'')_0$ is similar to $(\mathbb{C}^2, 3X+3Y+3Z)_0$. This means that there is an infinite tower of coverings and many ball-cusp points appears in covers.

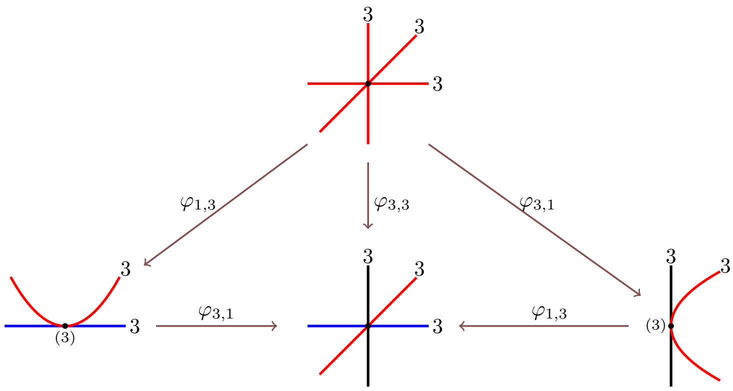


Figure 6.11 Coverings of the germ $(\mathbb{C}^2, 3X+3Y+3Z)_0$.

3. Next, consider the orbifold $(\mathbb{C}^2, 2X+4Y+4Z)_0$, where the lines X, Y, Z form a pencil at the origin. Its orbifold fundamental group has the presentation

$$\langle \mu_1, \mu_2, \mu_3 \mid [\mu_3\mu_2\mu_1, \mu_i] = \mu_1^2 = \mu_2^4 = \mu_3^4 = 1,\ i = 1,2,3 \rangle.$$

This group is infinite but solvable and it is isomorphic to a discrete subgroup Γ

of $\mathrm{Aut}(\mathbb{C}^2)$, and the transformation group $(\Gamma, \mathbb{C}^2)$ uniformizes this germ. Since Γ is infinite solvable, then many ball-cusp points appears in the covers of the germ $(\mathbb{C}^2, 2X+4Y+4Z)_0$. Let us now study its coverings. For the sake of simplicity we may choose the coordinates so that $X = \{x=0\}$, $Y = \{y=0\}$, $Z = \{x-y=0\}$. The uniformizer the sub-orbifold $(\mathbb{C}^2, 2X+2Y)_0$ is $\varphi_{2,2}$. Denote by Y the lifting $\varphi_{2,2}^{-1}(Y) = \{y=0\}$ and by W the lifting $\varphi_{2,2}^{-1}(Z) = \{x^2-y^2=0\}$, and set $W_1 = \{x-y=0\}$ and $W_2 = \{x+y=0\}$. Then we have an orbifold covering

$$\varphi_{2,2} : (\mathbb{C}^2, 2Y+4W_1+4W_2)_0 \to (\mathbb{C}^2, 2X+4Y+4Z)_0.$$

Note that the orbifold $(\mathbb{C}^2, 2Y+4W_1+4W_2)_0$ is same as the initial orbifold. (See Figure 6.12). This means that there is an infinite tower of coverings and many ball-cusp points appears in its covers.

Now, consider the sub orbifold $(\mathbb{C}^2, 2X+4Y)_0$ whose uniformizer is $\varphi_{2,4}$. Denote by Z' the branch $\varphi_{2,4}^{-1}(Z) = \{x^2-y^4=0\}$, then we have a covering

$$\varphi_{2,4} : (\mathbb{C}^2, 4Z')_0 \to (\mathbb{C}^2, 2X+4Y+4Z)_0.$$

On the other hand, consider the sub orbifold $(\mathbb{C}^2, 2Y+2Z)_0$ and change the coordinates by a map $\tau : (z,y) = (x-y, y)$, then $X = \{z+y=0\}$, $Y = \{y=0\}$ and $Z = \{z=0\}$. Clearly $\varphi_{2,2}$ is the uniformizer of $(\mathbb{C}^2, 2Y+2Z)_0$. Denote by Y the lifting $\varphi_{2,2}^{-1}(Y) = \{y=0\}$, by Z the lifting $\varphi_{2,2}^{-1}(Z) = \{z=0\}$ and by U the lifting $\varphi_{2,2}^{-1}(X) = \{z^2+y^2=0\}$, and set $X_1' = \{z+iy=0\}$ and $X_2' = \{z-iy=0\}$. Then we have an orbifold covering

$$\tau^{-1} \circ \varphi_{2,2} : (\mathbb{C}^2, 2X_1'+2X_2'+2Y+2Z)_0 \to (\mathbb{C}^2, 2X+4Y+4Z)_0.$$

Note that $(\mathbb{C}^2, 2X_1'+2X_2'+2Y+2Z)_0$ is the orbifold in the case 1. Take its lifting by $\varphi_{2,2}$ and set $X' = \varphi_{2,2}^{-1}(X_1' \cup X_2') = \{z^4 + y^4 = 0\}$. Then one has

$$\tau^{-1}\varphi_{4,4} = \tau^{-1} \circ \varphi_{2,2} \circ \varphi_{2,2} : (\mathbb{C}^2, 2X')_0 \to (\mathbb{C}^2, 2X + 4Y + 4Z)_0,$$

which is related the cover of $(\mathbb{C}^2, 2X + 4Y + 4Z)_0$ by the uniformizer $\varphi_{4,4}$ of the orbifold $(\mathbb{C}^2, 4Y + 4Z)_0$. Note that X' has four linear components and $(\mathbb{C}^2, 2X')_0$ is also the orbifold in the case 1.

As in these examples, there are many other coverings of the orbifold $(\mathbb{C}^2, 2X + 4Y + 4Z)_0$, which is is related with other orbifold germs with singular base via a power map $\varphi_{r,s} : (x, y) \to (x^r, y^s)$. We will omit to derive these covering relations but exhibit in the Figure 6.12. As it is seen from the coverings above, the uniformization of the germ $(\mathbb{C}^2, 2X + 4Y + 4Z)_0$ is the surface

$$T_{2,4,4} := \{(x, y, z) \in \mathbb{C}^3 \mid -x^2 + y^4 + z^4 = 0\}.$$

of isolated singularity of the type X_9.

4. Finally, consider the orbifold $(\mathbb{C}^2, 2X + 6Y + 3Z)_0$, where the lines X, Y, Z forms a pencil at the origin. Its orbifold fundamental group has the presentation

$$\langle \mu_1, \mu_2, \mu_3 \mid [\mu_3\mu_2\mu_1, \mu_i] = \mu_1^2 = \mu_2^3 = \mu_3^6 = 1,\ i = 1, 2, 3\rangle.$$

This group is infinite but solvable and it is isomorphic to a discrete subgroup Γ of $\mathrm{Aut}(\mathbb{C}^2)$, and the transformation group $(\Gamma, \mathbb{C}^2)$ uniformizes this germ. Since Γ is infinite solvable, then many cusp points appears in the covers. Let us study the coverings of $(\mathbb{C}^2, 2X + 3Y + 6Z)_0$. For the sake of simplicity, choose the coordinates so that $X = \{x = 0\}$, $Y = \{y = 0\}$, $Z = \{x - y = 0\}$.

The uniformizer the sub-orbifold $(\mathbb{C}^2, 2X + 2Y)_0$ is $\varphi_{2,2}$. Denote by Y the lifting $\varphi_{2,2}^{-1}(Y) = \{y = 0\}$ and by W the lifting $\varphi_{2,2}^{-1}(Z) = \{x^2 - y^2 = 0\}$, and set $W_1 = \{x + y = 0\}$ and $W_2 = \{x - y = 0\}$. Then we have an orbifold covering

$$\varphi_{2,2} : (\mathbb{C}^2, 3Y + 3W_1 + WZ_2)_0 \to (\mathbb{C}^2, 2X + 3Y + 6Z)_0.$$

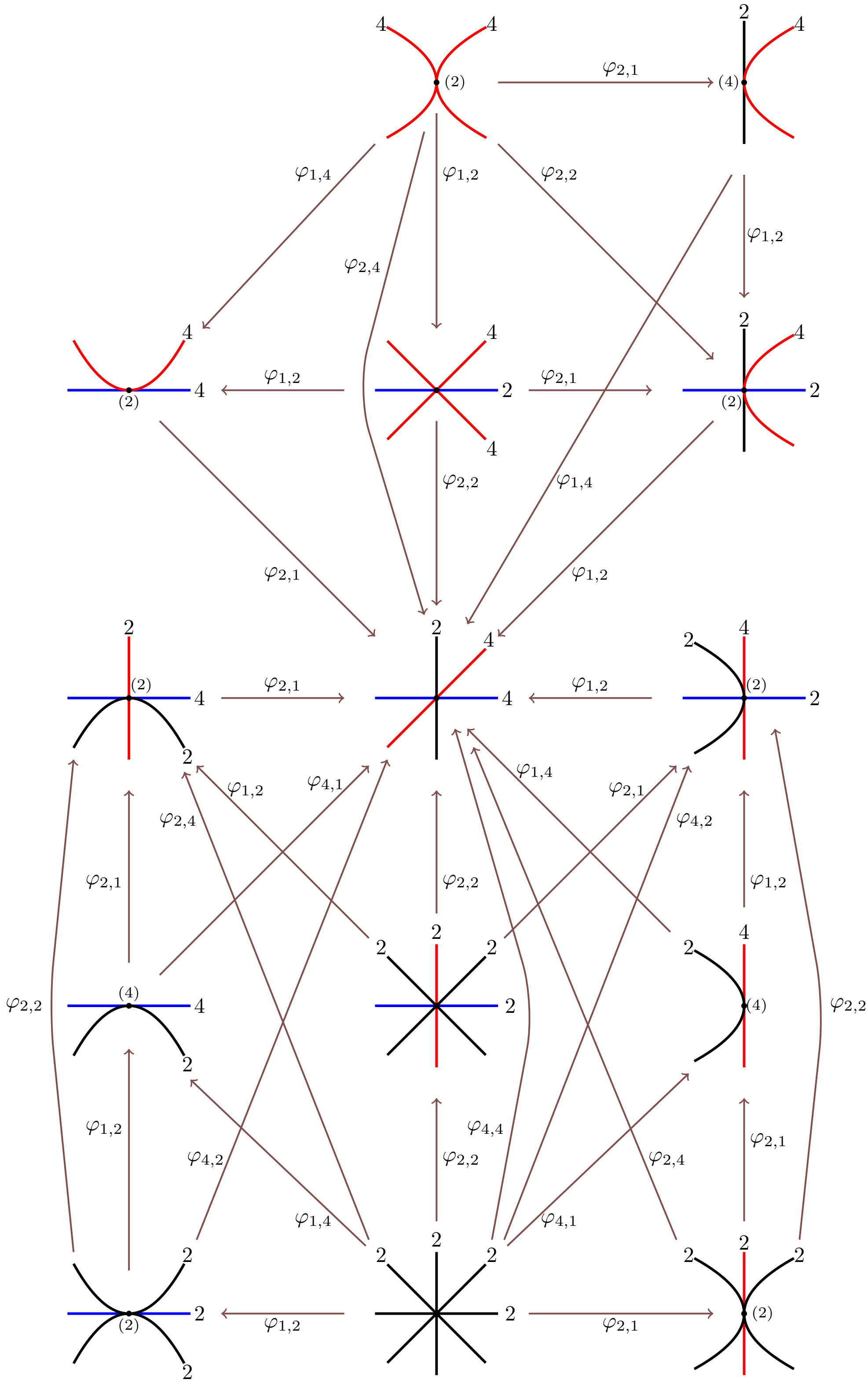


Figure 6.12 Coverings of the germ $(\mathbb{C}^2, 2X + 4Y + 4Z)_0$.

We know from the case the orbifold $(\mathbb{C}^2, 3Y+3W_1+3W_2)_0$ has an infinite tower of coverings and many ball-cusp points appears in the covers. Take the lifting of $(\mathbb{C}^2, 3Y+3W_1+WZ_2)_0$ by $\varphi_{1,3}$ and set $Z' := \varphi_{1,3}^{-1}(W_1 \cup W_2) = \{x^2 - y^6 = 0\}$. Then, one has the covering

$$\varphi_{2,6} = \varphi_{2,2} \circ \varphi_{1,3} : (\mathbb{C}^2, 3Z')_0 \to (\mathbb{C}^2, 2X+3Y+6Z)_0.$$

On the other hand, if one changes the coordinates by $\sigma : (x,z) = (x, x-y)$, then $\varphi_{2,3} : (x,z) \mapsto (x^2, z^3)$ is the uniformizer of the sub orbifold $(\mathbb{C}^2, 2X+3Z)_0$. Then we have the covering

$$\sigma^{-1}\varphi_{2,3} : (\mathbb{C}^2, 6Y')_0 \to (\mathbb{C}^2, 2X+3Y+6Z)_0,$$

where $Y' = \varphi_{2,3}^{-1}(Y) = \{x^2 - z^3 = 0\}$.

If one would have changed the coordinates by $\tau : (z,y) = (x-y, y)$, then $\varphi_{3,3} : (z,y) \mapsto (z^3, y^3)$ would be the uniformizer of the sub-orbifold $(\mathbb{C}^2, 3Y+3Z)_0$. Denote by Y the branch $\varphi_{-1}(Y) = \{y=0\}$ and by U the branch $\varphi_{3,3}^{-1}(X) = \{z^3+y^3=0\}$. Set $U_i := \{z+\omega^i y = 0\}$, where $\omega^3 = 1$ and $i = 0,1,2$. Then one has the covering

$$\tau^{-1} \circ \varphi_{2,3} : (\mathbb{C}^2, 2U+2Y)_0 = (\mathbb{C}^2, 2U_0+2U_1+2U_2+2Y)_0 \to (\mathbb{C}^2, 2X+3Y+6Z)_0.$$

Notice that $(\mathbb{C}^2, 2U_0+2U_1+2U_2+2Y)_0$ is the orbifold in the case 1 and it has an infinite tower of coverings. Now take the lifting of $(\mathbb{C}^2, 2U+2Y)_0$ by $\varphi_{1,2}$, and set $X' := \varphi_{1,2}^{-1}(U) = \{z^3+y^6 = 0\}$. Then we have the covering

$$\tau^{-1} \circ \varphi_{3,6} = \tau^{-1} \circ \varphi_{3,3} \circ \varphi_{1,2} : (\mathbb{C}^2, 2X')_0 \to (\mathbb{C}^2, 2X+3Y+6Z)_0.$$

As in these examples, there are many other coverings of the orbifold $(\mathbb{C}^2, 2X+6Y+3Z)_0$, which is is related with other orbifold germs with singular base via a power map $\varphi_{r,s} : (x,y) \to (x^r, y^s)$. We will omit to derive these covering relations but exhibit in the Figure 6.13.

As it is seen from the coverings above, the uniformization of the germ $(\mathbb{C}^2, 2X+6Y+3Z)_0$ is the surface $T_{2,6,3} := \{(x,y,z) \in \mathbb{C}^3 \mid -x^2+y^6+z^3 = 0\}$ of isolated singularity of the type J_{10}.

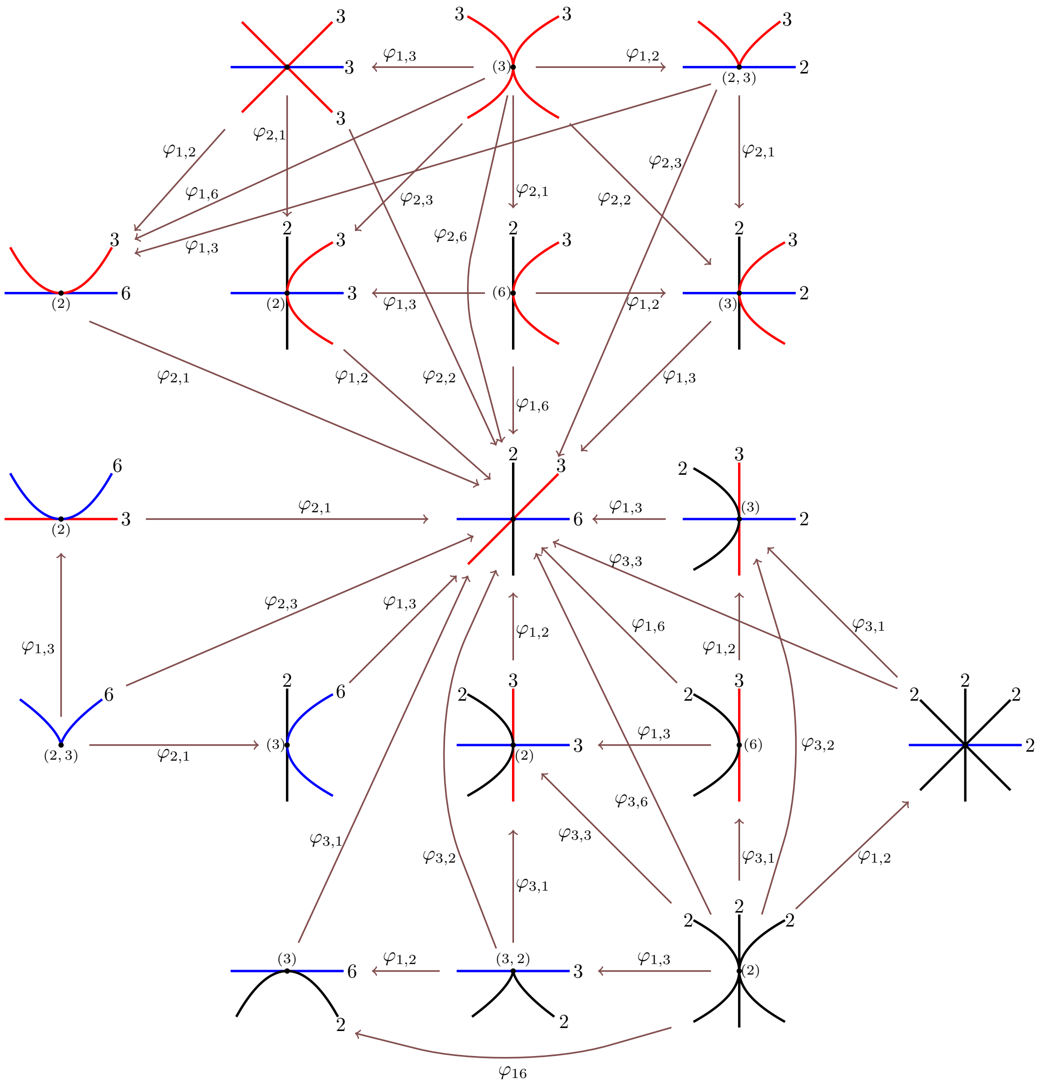


Figure 6.13 Coverings of the germ $(\mathbb{C}^2, 2X+3Z+6Y)_0$.

6.3 Chern Classes and Chern Numbers

Chern classes are characteristic classes. They are topological invariants associated to vector bundles on a smooth manifold. If you describe the same vector bundle on a manifold in two different ways, the Chern classes will be the same. Then, the Chern classes provide a simple test: if the Chern classes of a pair of vector bundles do not agree, then the vector bundles are different. (The converse is not true, though.)

Given a complex hermitian vector bundle V of complex rank n over a smooth manifold M, a representative of each *Chern class* $c_k[V]$ of V are given as the coefficients of the characteristic polynomial of the curvature form ω of V

$$c(t)[V] := \det\left(\frac{it}{2\pi}\omega + I_n\right) = \sum_k c_k[V]t^k. \tag{6.3.1}$$

Here the determinant is over the ring of $n \times n$ matrices whose entries are polynomials in t with coefficients in the commutative algebra of even complex differential forms on M. The *curvature form* ω of V is defined as

$$\omega = d\nabla + \frac{1}{2}[\nabla, \nabla] \tag{6.3.2}$$

with ∇ the hermitian *connection form* (with respect to a hermitian metric h) and d the *exterior derivative*. The scalar t is used here only as an indeterminate to generate the sum from the determinant, and I_n denotes the $n \times n$ identity matrix. More explicitly, the k-th Chern class of V is given by

$$c_k[V] = \mathrm{Tr}\left(\wedge^k \frac{i}{2\pi}\omega\right) \tag{6.3.3}$$

In addition, the *total Chern class* is defined as

$$c[V] = c_0[V] + c_1[V] + c_2[V] + \cdots . \tag{6.3.4}$$

To say that the expression (6.3.3) is a representative of the Chern class indicates that "class" here means up to addition of an exact differential form. That is, Chern classes are cohomology classes in the sense of de Rham cohomology, i.e., $c_k[V] \in H^{2k}(M,\mathbb{Z})$. The cohomology class of the Chern forms do not depend on the choice of connection in V (Kobayashi, 1983).

The Chern classes $c_k[V]$ satisfy the following properties (Hatcher, 2009):

(1) $c_0[V] = 1$ and $c_1[V] = \mathrm{Tr}\,(\frac{i}{2\pi}\omega)$ for all V,

(2) $c_k[V] = 0$ for all V, if $k > n$. Thus the total Chern class terminates,

(3) Functoriality: If $f : N \to M$ is continuous and f^*V is the vector bundle pullback of V, then $c_k[f^*V] = f^*c_k[V]$,

(4) Whitney sum formula: If one has complex vector bundles $p_i : V_i \to M$, $i = 1,2$, then the total Chern class and the Chern classes of the direct sum $V_1 \oplus V_2 = \{(\upsilon_1, \upsilon_2) \in V_1 \times V_2 \mid p_1(\upsilon_1) = p_2(\upsilon_2)\}$ are respectively given by

$$c[V_1 \oplus V_2] = c[V_1] \smile c[V_2] \quad \text{and} \quad c_k[V_1 \oplus V_2] = \sum_{i+j=k} c_i[V_1] \smile c_j[V_2],$$

(5) The top Chern class of V is always equal to the *Euler class* of the underlying real vector bundle, that is $c_n[V] = e[V]$.

(6) Additivity: If $0 \to V_1 \to V \to V_2 \to 0$ is an exact sequence of complex vector bundles, then V is isomorphic to $V_1 \oplus V_2$, and therefore $c[V] = c[V_1] \smile c[V_2]$.

Depending on the partition of n such that $\sum_{i=1}^{n} i a_i = n$, there are Chern forms $c_I[V] := c_1^{a_1}[V] c_2^{a_2}[V] \cdots c_n^{a_n}[V]$ in terms of wedge product of Chern classes, where $I := (a_1, a_2, \cdots a_n)$. The integral of these Chern forms on manifold M takes values in $\mathbb{Z}$ and they are called *Chern numbers* of V, and denoted by $c_I := c_1^{a_1} c_2^{a_2} \cdots c_n^{a_n}$. In case of $n = 1$, there is only one Chern number, c_1, that is the Euler number e. If $n = 2$, the Chern numbers are c_1^2 and c_2.

An important special case occurs when V is a line bundle L. Then the only nontrivial Chern class is the first Chern class, which is an element of the second cohomology group of M. As it is the top Chern class, it equals the Euler class of the bundle. If the vector bundle V is a direct sum of line bundles, i.e $V = L_1 \oplus L_2 \oplus \cdots \oplus L_n$, then $c(t)[V] = \prod_{i=1}^{n}(1 + c_1[L_i]t)$. This means that the first Chern class completely classify complex line bundles. That is, there is a bijection between the isomorphism classes of line bundles over M and the elements of $H^2(M,\mathbb{Z})$, which associates to a line bundle its first Chern class. Addition in the second dimensional cohomology group coincides with tensor product of complex line bundles. This classification of (isomorphism classes of) complex line bundles by the first Chern class is a crude approximation to the classification of (isomorphism classes of) holomorphic line bundles by linear equivalence classes of divisors.

Now, suppose V is the (holomorphic) tangent bundle TM of an n-dimensional complex manifold M. Assume, the coordinate patches $\{(U_\alpha, \mathbf{z}_\alpha)\}_{\alpha\in I}$ covers M and $\mathbf{z}_\alpha = (z_{1\alpha}, z_{2\alpha}, \cdots, z_{n\alpha})$ be the local affine coordinates on U_α. Then, the coordinate derivatives define a frame $\{\frac{\partial}{\partial z_{1\alpha}}, \frac{\partial}{\partial z_{2\alpha}}, \cdots, \frac{\partial}{\partial z_{n\alpha}}\}$ of TM. The complex structure of M defines an endomorphism J of TM such that $J(\frac{\partial}{\partial z_{j\alpha}}) = i\frac{\partial}{\partial z_{j\alpha}}$ and $J(\frac{\partial}{\partial \bar{z}_{j\alpha}}) = -i\frac{\partial}{\partial \bar{z}_{j\alpha}}$ on U_α for $j = 1, 2, \cdots, n$. Then clearly $J^2 = -id$. Beside this,

$$h = \sum_{i,j=1}^{n} h_{i\bar{j}}dz_{i\alpha}d\bar{z}_{j\alpha}, \quad \text{where} \quad h_{i\bar{j}} := h(\frac{\partial}{\partial z_{i\alpha}}, \frac{\partial}{\partial \bar{z}_{j\alpha}}) \tag{6.3.5}$$

is a hermitian metric on U_α. Let $\{\rho_\alpha\}_{\alpha\in I}$ be the partition of unity subordinating to the cover $\{(U_\alpha, \mathbf{z}_\alpha)\}_{\alpha\in I}$. Then,

$$h = \sum_{\alpha\in I}\sum_{i,j=1}^{n} \rho_\alpha h_{i\bar{j}}dz_{i\alpha}d\bar{z}_{j\alpha} \tag{6.3.6}$$

is a Hermitian metric on M. Moreover, associated curvature form ω is given by

$$\omega = \sum_{\alpha\in I}\sum_{i,j=1}^{n} \rho_\alpha h_{i\bar{j}}dz_{i\alpha} \wedge d\bar{z}_{j\alpha}. \tag{6.3.7}$$

Denote by H the determinant of $(h_{i\bar{j}})_{n\times n}$ and set $R_{i\bar{j}} := -\partial\bar{\partial}\log H = -\frac{\partial^2}{\partial z_{i\alpha}\partial \bar{z}_{j\alpha}}\log H$. Then, the *Ricci form* is given by

$$\Theta = \frac{i}{2\pi}\sum_{\alpha\in I}\sum_{i,j=1}^{n}\rho_\alpha R_{i\bar{j}} dz_{i\alpha}\wedge d\bar{z}_{j\alpha}. \tag{6.3.8}$$

On the other hand, the local functions $H_\alpha = (\det(h_{i\bar{j}}))^{-1} = H^{-1}$ provide a natural Hermitian metric of the canonical bundle K_M. The *canonical bundle* K_M is the holomorphic 1-vector bundle $\wedge^n T^*M$, where T^*M is cotangent bundle to TM. The cohomology class

$$-\frac{i}{2\pi}\partial\bar{\partial}\log H_\alpha = \frac{i}{2\pi}\partial\bar{\partial}\log H$$

is the first Chern class of the line bundle. Then we have the the following theorem.

Proposition 6.3.1 ((Yau, 1977), (Hwang, 1997))**.** *The Ricci form is closed, and represents $c_1(M)$. If the Ricci curvature R is viewed as a symmetric endomorphism of $\bigwedge^{1,1} TM$, then $\Theta = \frac{i}{2\pi}R(\omega)$. The Ricci form is the curvature of the canonical bundle K_M of M.*

For a complete, simply-connected Kähler manifold (M, J, h) of dimension n with complex structure J. The sectional curvature of a real two-plane $P \subset T_zM$ is the value $R(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_1, \mathbf{e}_2)$ of the curvature tensor on an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ of P. Geometrically, the sectional curvature is the Gaussian curvature at z of the surface in M obtained by exponentiating P. The sectional curvature function K is defined on the Grassmannian bundle of real two-planes in TM. If P is a complex line, then the sectional curvature is equal to $R(\mathbf{e}, \mathbf{e}, \mathbf{e}, \mathbf{e})$. The restriction of the sectional curvature function to the bundle of complex lines is called *the holomorphic sectional curvature* and denoted by K_{hol}.

If the sectional curvature function K is constant, then the curvature tensor has an explicit algebraic expression in terms of the metric h, in particular, for each $c \in \mathbb{R}$, there is a local model space with constant sectional curvature c. A similar fact is true when h is a Kähler metric with constant holomorphic sectional curvature. If h is (geodesically) complete, then simply connected spaces of constant curvature are classified. The following theorem locally classifies the metrics on a complex manifold M.

Theorem 6.3.2 ((Yau, 1977), (Hwang, 1997))**.** *Let (M,J,h) be a complete, simply-connected Kähler manifold of dimension n and constant holomorphic sectional curvature c. Then followings are true.*

- *If $c<0$, then h is isometric to a multiple of the Bergman metric and the canonical bundle K_M is ample.*
- *If $c=0$, then h is isometric to the flat metric on $\mathbb{C}^n$ and the canonical bundle K_M is trivial.*
- *If $c>0$, then h is isometric to a multiple of the Fubini-Study metric on $\mathbb{CP}^n$ and the anti-canonical bundle is ample.*
- *If ω is the curvature form (for Fubini-Study metric or flat metric or Bergman metric), then the Ricci form is $\Theta=\frac{i}{2\pi}c\omega$.*

Example 6.3.3. Consider the complex space $\mathbb{C}^n$ and its tangent bundle $T\mathbb{C}^n$ as the vector bundle V. The standard hermitian metric (flat metric) on $\mathbb{C}^n$ is $h=\sum_{i=0}^n dz_i d\bar{z}_i$. Consequently, the curvature form is $\omega=\sum_{i=0}^n dz_i\wedge d\bar{z}_i$, which is an exact form. Therefore, the Ricci form Θ is trivial. Hence by the Proposition 6.3.1 first Chern class $c_1[T\mathbb{C}^n]$ vanishes, so the first Chern number $c_1^n(\mathbb{C}^n)$ is zero . In addition, the top Chern class $c_n[T\mathbb{C}^n]$ is the Euler class $e[T\mathbb{C}^n]$, and the Euler number of $\mathbb{C}^n$ is $c_n(\mathbb{C}^n)=e(\mathbb{C}^n)=1$ since $\mathbb{C}^n$ is contractible.

Definition 6.3.4 (Line bundles $\mathscr{O}(k)$ over $\mathbb{CP}^n$)**.** Let W be a complex vector space of dimension $n+1$, $n>1$ and $\mathbb{P}W$ be the its projectivization, that is the quotient topological space $\mathbb{P}W=(W\setminus\{\mathbf{0}\})/\mathbb{C}^*$. It is clear that $\mathbb{P}W=\mathbb{CP}^n$ if $W=\mathbb{C}^{n+1}$. The trivial bundle is $\mathbb{P}W\times W$. Denote by $\mathscr{O}(-1)\subset\mathbb{P}W\times W$ the tautological line sub-bundle. Then the restriction $\mathscr{O}(-1)\mid_{U_j}$ of $\mathscr{O}(-1)$ to the local chart $U_j=\{[\mathbf{z}]\,|\,z_j\neq 0\}$ admits a non-vanishing local section $[\mathbf{z}]\to\varepsilon_j([\mathbf{z}])=(z_0,\cdots,z_{j-1},1,z_{j+1},\cdots,z_n)$.

In particular $\mathscr{O}(-1)$ is a holomorphic line bundle. For every $k \in \mathbb{Z}$, the line bundle $\mathscr{O}(k)$ is defined by

$$\begin{aligned} \mathscr{O}(1) &= \mathscr{O}(-1)^{\star}, \quad \mathscr{O}(0) = \mathbb{P}W \times \mathbb{C}, \\ \mathscr{O}(k) &= \mathscr{O}(1)^{\otimes k} = \mathscr{O}(1) \otimes \mathscr{O}(1) \otimes \cdots \otimes \mathscr{O}(1), \quad \text{for } k \geq 1, \\ \mathscr{O}(-k) &= \mathscr{O}(-1)^{\otimes k} = \mathscr{O}(-1) \otimes \mathscr{O}(-1) \otimes \cdots \otimes \mathscr{O}(-1), \quad \text{for } k \geq 1. \end{aligned} \tag{6.3.9}$$

Therefore, we have canonical exact sequences of vector bundles over $\mathbb{P}W$:

$$\begin{aligned} &0 \to \mathscr{O}(-1) \to \mathbb{P}W \times W \to \mathbb{P}W \times W/\mathscr{O}(-1) \to 0, \\ &0 \to (\mathbb{P}W \times W/\mathscr{O}(-1))^{\star} \to (\mathbb{P}W \times W)^{\star} \to \mathscr{O}(1) \to 0. \end{aligned} \tag{6.3.10}$$

The holomorphic map $\mu : \mathscr{O}(-1) \to W$ defined by $\mu : \mathscr{O}(-1) \hookrightarrow \mathbb{P}W \times W \to W$ send the zero section $\mathbb{P}W \times \{\mathbf{0}\}$ of $\mathscr{O}(-1)$ to the point $\{\mathbf{0}\}$ and induces a biholomorphism of $\mathscr{O}(-1) \setminus (\mathbb{P}W \times \{\mathbf{0}\})$ onto $W \setminus \{\mathbf{0}\}$. Moreover there is a canonical isomorphism $(\mathbb{P}W \times W)/\mathscr{O}(-1) \simeq T\mathbb{P}W \otimes \mathscr{O}(-1)$, i.e., $\big((\mathbb{P}W \times W)/\mathscr{O}(-1)\big) \otimes \mathscr{O}(1) \simeq T\mathbb{P}W$.

Example 6.3.5 (Barthel et al., 1987)**.** Consider the complex projective space $\mathbb{CP}^n$, which is a quotient of $= \mathbb{C}^{n+1}$ by $\mathbb{C}^*$. One may also think this quotient as $\mathbb{CP}^n = S^{2n+1}/S^1$. The standard hermitian metric on $\mathbb{C}^{n+1}$ is $ds^2 = d\mathbf{Z} \cdot d\overline{\mathbf{Z}} = \sum_{i=0}^{n} dZ_i d\overline{Z}_i$. It is invariant under the diagonal actions of S^1 (group of rotations), while it is not invariant under the diagonal action of $\mathbb{C}^*$. So, a hermitian metric on $\mathbb{CP}^n$ is the standard metric on S^{n+1} restricted to $\mathbb{C}^{n+1}$. This metric is known as the *Fubini-Study metric* and it is a Kähler metric. Let us write this metric explicitly.

The coordinate patches $U_i = \{[Z_0 : Z_1 : \cdots : Z_n] \mid Z_i \neq 0\}$ covers $\mathbb{CP}^n$ and it is possible to define the Fubini-Study metric on each local charts. Let $\mathbf{z} = (z_1, z_2, \cdots, z_n)$ be the local affine coordinates of $[Z_0 : Z_1 : \cdots : Z_n]$ in the coordinate patch U_0 provided $z_i := Z_i/Z_0$. Then the coordinate derivatives define a frame $\{\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \cdots, \frac{\partial}{\partial z_n}\}$ of the holomorphic tangent bundle $T\mathbb{CP}^n$ of $\mathbb{CP}^n$, in terms of which the Fubini-

Study metric has hermitian components

$$h_{i,\bar{j}} := h\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_j}\right) = \frac{\partial^2}{\partial z_i \partial \bar{z}_j} \log(1+|\mathbf{z}|^2) = \frac{(1+|\mathbf{z}|^2)\delta_{ij} - \bar{z}_i z_j}{(1+|\mathbf{z}|^2)^2}, \tag{6.3.11}$$

where $|\mathbf{z}|^2 = |z_1|^2 + |z_2|^2 + \cdots + |z_n|^2$ and δ_{ij} is the Kronecker delta. Thus, we get the hermitian metric h and the corresponding curvature form ω as

$$h = \sum h_{i\bar{j}} dz_i d\bar{z}_j \quad \text{and} \quad \omega = \sum h_{i\bar{j}} dz_i \wedge d\bar{z}_j. \tag{6.3.12}$$

In additon, $H = \det(h_{i\bar{j}}) = (1+|\mathbf{z}|^2)^{-(n+1)}$ and

$$R_{i\bar{j}} = -\frac{\partial^2}{\partial z_i \partial \bar{z}_j} \log H = (n+1)\frac{\partial^2}{\partial z_i \partial \bar{z}_j} \log(1+|\mathbf{z}|^2) = (n+1)h_{i\bar{j}}.$$

Thus, the Ricci form, the curvature of the canonical bundle, is

$$\Theta = \frac{i}{2\pi} \sum R_{i,\bar{j}} dz_i \wedge d\bar{z}_j = \frac{i}{2\pi}(n+1)\omega, \tag{6.3.13}$$

For the sake of simplicity, denote by ϖ the form $\frac{i}{2\pi}\omega$. Then we have $c_1[T\mathbb{CP}^n] = \Theta = (n+1)\varpi$. Note that $\varpi \in H^2(\mathbb{CP}^n, \mathbb{Z})$ is a positive generator and ϖ^n is a volume form, i.e., $\int_{\mathbb{CP}^n} \varpi^n = 1$. There is an exact sequence of vector bundles

$$0 \to \mathbb{C} \to \mathscr{O}_{\mathbb{CP}^n}(1) \otimes \mathbb{C}^{n+1} \to T\mathbb{CP}^n \to 0 \tag{6.3.14}$$

over $\mathbb{CP}^n$. From the Splitting principle and the Whitney sum formula we have

$$c(1)[T\mathbb{CP}^n] = c(1)[\mathscr{O}_{\mathbb{CP}^n}(1) \otimes \mathbb{C}^{n+1}] = (1+\varpi)^{n+1} \in H^*(\mathbb{CP}^n, \mathbb{Z}) \tag{6.3.15}$$

If $n = 1$, then $c(1)[T\mathbb{CP}^1] = (1+\varpi)^2 = 1+2\varpi$, i.e, $c_1[T\mathbb{CP}^1] = 2\varpi$. Therefore, $c_1(\mathbb{CP}^1) = \int_{CP^1} 2\varpi = 2$, which is the Euler number of $\mathbb{CP}^1$. In case $n = 2$, by the formula (6.3.15), we have $c(1)[T\mathbb{CP}^2] = (1+\varpi)^3 = 1+3\varpi+3\varpi^2$, i.e, $c_1[T\mathbb{CP}^2] = 3\varpi$ and $c_2[T\mathbb{CP}^2] = 3\varpi^2$. Therefore, $c_1^2(\mathbb{CP}^2) = \int_{CP^2} 9\varpi^2 = 9$ and $c_2(\mathbb{CP}^2) = \int_{CP^2} 3\varpi^2 = 3$. Finally, notice that, $T\mathbb{CP}^n$ is the line bundle $\mathscr{O}(n+1)$.

Example 6.3.6 (Barthel et al., 1987)**.** Consider the n-ball

$$\mathbf{B}_n = \{\mathbf{z} \in \mathbb{C}^n \mid |z_1|^2 + |z_2|^2 + \cdots + |z_n|^2 < 1\}.$$

It is homeomorphic to the embedded ball

$$\mathbf{B}_n(U_0) = \{[1 : z_1 : \cdots : z_n] \mid 1 - |\mathbf{z}|^2 = 1 - |z_1|^2 - |z_2|^2 - \cdots - |z_n|^2 > 0\} \subset \mathbb{CP}^n.$$

By considering the indefinite Hermitian form $F(z,w) = -z_0\bar{w}_0 + \sum_{i=1}^n z_i\bar{w}_i$ of $\mathbb{C}^{n+1}$, Bergman defined a metric and the corresponding curvature form on $\mathbf{B}_n \simeq \mathbf{B}_n(U_0)$ as

$$h = \sum h_{i\bar{j}} dz_i d\bar{z}_j \quad \text{and} \quad \omega = \sum h_{i\bar{j}} dz_i \wedge d\bar{z}_j, \tag{6.3.16}$$

where

$$h_{i\bar{j}} = -\partial\bar{\partial}\log N = -\frac{\partial^2}{\partial z_i \partial\bar{z}_j}\log N = \frac{N\delta_{ij} + \bar{z}_i z_j}{N^2} \quad \text{and} \quad N := 1 - |\mathbf{z}|^2 = 1 - \sum_{i=1}^n |z_i|^2. \tag{6.3.17}$$

In additon, $H = \det(h_{i\bar{j}}) = N^{-(n+1)}$ and

$$R_{i\bar{j}} = -\frac{\partial^2}{\partial z_i \partial\bar{z}_j}\log H = (n+1)\frac{\partial^2}{\partial z_i \partial\bar{z}_j}\log N = -(n+1)h_{i\bar{j}}.$$

Therefore, the Ricci form, the curvature of the canonical bundle, is

$$\Theta = \frac{i}{2\pi}\sum R_{i,\bar{j}} dz_i \wedge d\bar{z}_j = -\frac{i}{2\pi}(n+1)\omega, \tag{6.3.18}$$

For the sake of simplicity, denote by ϖ the form $\frac{i}{2\pi}\omega$. Then we have $c_1[T\mathbf{B}_n] = \Theta = -(n+1)\varpi$. Note that $\varpi \in H^2(\mathbf{B}_n, \mathbb{Z})$ is a positive generator and $\int_{\mathbf{B}_n(U_0)} \varpi^n = 1$. There is an exact sequence of vector bundles

$$0 \to \mathbb{C} \to \mathscr{O}_{\mathbf{B}_n}(-1) \otimes \mathbb{C}^{n+1} \to T\mathbf{B}_n \to 0 \tag{6.3.19}$$

over $\mathbf{B}_n \simeq \mathbf{B}_n(U_0)$. From the Splitting principle and the Whitney sum formula we have

$$c(1)[T\mathbf{B}_n] = c(1)[\mathscr{O}_{\mathbf{B}_n}(-1)\otimes\mathbb{C}^{n+1}] = (1-\varpi)^{n+1} \in H^*(\mathbf{B}_n,\mathbb{Z}) \tag{6.3.20}$$

If $n = 1$, then $c(1)[T\mathbf{B}_1] = (1-\varpi)^2 = 1-2\varpi$, i.e, $c_1[T\mathbf{B}_1] = -2\varpi$. Therefore, $c_1(\mathbf{B}_1) = \int_{\mathbf{B}_1(U_0)} -2\varpi = -2$, which is the Euler number of $\mathbf{B}_1$. In case $n = 2$, by the formula (6.3.20), we have $c(1)[T\mathbf{B}_2] = (1-\varpi)^3 = 1-3\varpi+3\varpi^2$, i.e, $c_1[T\mathbf{B}_2] = -3\varpi$ and $c_2[T\mathbf{B}_2] = 3\varpi^2$. Therefore, first and second Chern numbers of $\mathbf{B}_2$ are $c_1^2(\mathbf{B}_2) = \int_{\mathbf{B}_2(U_0)} 9\varpi^2 = 9$ and $c_2(\mathbf{B}_2) = \int_{\mathbf{B}_2(U_0)} 3\varpi^2 = 3$, respectively. Finally, notice that, $T\mathbf{B}_n$ is the line bundle $\mathscr{O}(-(n+1))$.

6.3.1 Divisors and Line Bundles

A *divisor* $D := \sum m_i H_i$ on a complex manifold M is a locally finite sum of closed, reduced, irreducible analytic hypersurfaces H_i (the components of D) with non-zero integer coefficients m_i. "Closed" means closed as subsets in the complex topology, "sum with integer coefficients" should be taken in the spirit of free Abelian groups, with the distinction that the sum here may be infinite, and "locally finite" means that every $z \in M$ has a neighborhood U which intersects only finitely many components. A divisor is *effective* or *positive* (notation: $D > 0$) if every component has positive coefficient. An effective divisor is locally cut out by a holomorphic function Φ, the function Φ vanishes along a union of irreducible analytic hypersurfaces, and the integer attached is the order of vanishing.

If M itself is compact, then a divisor is exactly an element of the free Abelian group on the set of closed, irreducible analytic hypersurfaces. The group of divisors is denoted by $\mathrm{Div}(M)$. The support of a divisor D is the union of the components of D. The degree of D is defined to be $\deg D = \sum m_i$.

Let $\mathcal{U} = \{U_\alpha\}$ be a locally finite open cover. A *meromorphic section* of a line bundle is defined to be a collection of local meromorphic functions $\{f_\alpha\}$ satisfying the compatibility condition $f_\alpha = \psi_{\alpha\beta} f_\beta$ on $U_\alpha \cap U_\beta$. If L is a line bundle on M, then every meromorphic section s, different from the zero section, determines a divisor (s) on M, namely its zero divisor minus its polar divisor, that is $(s) = (s)_0 - (s)_\infty$. In this case, $c_1[L]$ is Poincaré dual of $(s) = (s)_0 - (s)_\infty$. Conversely, if any divisor D is given, then there is (up to isomorphism) exactly one line bundle L_D with a meromorphic section s, such that $(s) = D$ (Barth et al., 2004). In this case, $\mathscr{O}_M(D)$ is used the denote the sheaf of germs of sections of L_D. In addition, if M is a Riemann surface, then $\deg D = \deg(s) = \int_M c_1[\mathscr{O}_M(D)]$ (Demailly, 2009).

Let D be a divisor on a compact complex manifold M, the cohomology class $c_1[D] := c_1[\mathscr{O}_M(D)] \in H^2(M, \mathbb{Z})$ depends only on the class of D up to linear equivalence ($D \sim D'$ if $D - D' = (f)$ for a meromorphic function f). If $C \subset M$ is a smooth irreducible curve, then the fundamental class $[C] \in H^{2n-2}(M, \mathbb{Z})$ due to Poincaré duality. Hence the intersection number $\langle c_1[D], [C] \rangle$ is well defined and depends only on the class of D up to linear equivalence. If D is an irreducible divisor, the number $\langle c_1[D], [C] \rangle$ coincides with the topological intersection number $D \cdot C$. If C intersects D transversally in at least one point, then this number is strictly positive. In particular, if D_1 and D_2 are two divisors on an algebraic surface M, then

$$D_1 \cdot D_2 = \int_M c_1[D_1] \smile c_1[D_2]. \tag{6.3.21}$$

In addition, assume both N and M are algebraic surfaces and $\varphi : N \to M$ is surjective holomorphic map. The canonical divisor K_N of N is related with the canonical divisor K_M of M via $K_N = \varphi^* K_M + J_\varphi$, where J_φ denotes the Jacobi divisor of φ. It is clear from the functoriality property of Chern classes, $c_1[\varphi^* D] = \varphi^* c_1[D]$ for a divisor D on M. If D_1 and D_2 are two divisors on M, then

$$\varphi^* D_1 \cdot \varphi^* D_2 = (\mathrm{grad}\varphi) \cdot (D_1 \cdot D_2). \tag{6.3.22}$$

The final is the relation between canonical class an the first Chern class. The Cohomology class corresponding to cananical bundle K_M is called the canonical class and often denoted also by K_M. It is the negative of the first Chern class $c_1[M] = c_1[K_M^{-1}]$, where K_M^{-1} is the anti-canonical bundle of K_M.

6.3.2 Algebraic Surfaces of General Type and Some Known Results

An *algebraic surface* is an algebraic variety of dimension two. In the case of geometry over the field of complex numbers, an algebraic surface is therefore of complex dimension two (as a complex manifold, when it is non-singular) and so of dimension four as a smooth manifold. Assume, M is an algebraic surface, and K_M be the canonical line bundle on M (i.e, the holomorphic 1-vector bundle $\wedge^2 TM^*$, where TM^* is cotangent bundle to the holomorphic tangent bundle TM). The *canonical class* is the divisor class of a Cartier divisor K on M giving rise to the canonical bundle $K_M = \mathscr{O}_M(K)$. It is an equivalence class for linear equivalence on M, and any divisor in it may be called a *canonical divisor*.

The *Kodaira dimension* $\kappa(M)$ of an algebraic surface M measures the size of the canonical model of M Indeed, it is a birational invariant of M and measures the dimensions of the spaces of global sections of $K_M^{\otimes r}$. As $r \to \infty$, these numbers either behave asymptotically like $C r^k$ for a unique integer k or are eventually zero. The Kodaira dimension κ to be this integer in the first case and $-\infty$ in the second case. Note that, since the complex dimension of M is 2, then $K_M^{\otimes r}$ is trivial when $r > 2$. Therefore, the Kodaira dimension $\kappa(M)$ takes values in $\{-\infty, 0, 1, 2\}$ for an algebraic surface M.

Due to Kodaira dimension, examples for the (coarse) classification of algebraic surfaces are as follows:

- $\kappa = -\infty$: The projective plane, quadrics in $\mathbb{CP}^3$, cubic surfaces, Veronese surface, del Pezzo surfaces, ruled surfaces,
- $\kappa = 0$: $K3$ surfaces, abelian surfaces, Enriques surfaces, hyperelliptic surfaces,

- $\kappa = 1$: Elliptic surfaces,
- $\kappa = 2$: Surfaces of general types.

The *algebraic dimension* of an algebraic surface M is the transcendental degree of $\mathbb{C}(M)$ over $\mathbb{C}$, and denoted by $a(M) := \mathrm{tran}_{\mathbb{C}}\mathbb{C}(M)$. Here, $\mathbb{C}(M)$ denotes the field of rational (meromorphic) functions on M. Its clear from the definitions of The Kodaira dimension $\kappa(M)$ and the algebraic dimension $a(M)$ that

$$\kappa(M) \leq a(M) \leq 2 = \dim_{\mathbb{C}} M.$$

If $\kappa(M) = 2$, then M is said to be of *general type*. If $a(M) = 2$, M is called *Moišhezon*. By the inequality above, any algebraic surface M of general type is *Moišhezon*. Due to Kodaira and Chow's theorem, If M is compact, complex analytic surface with $a(M) = 2$, then M is projective algebraic. Therefore, if M is of general type, then it is automatically projective algebraic. Since we are interested with surfaces of general type, from now on we will assume M is projective algebraic.

There are lots of invariants of algebraic surfaces: Hodge and Betti numbers, $\pi_1(M)$, signature, etc. The basic topological invariants for surfaces of general type however are just the Chern numbers c_1^2 and c_2. Recall that, the *first Chern number* c_1^2 of M is the self intersection number of the canonical class K, that is $c_1^2(M) = K \cdot K$, and the *second Chern number* c_2 of M is the Euler number of M, that is $c_2(M) = e(M)$. Due to Zariski, Every algebraic surface with $\kappa \geq 0$ has a unique minimal model (minimal model is a smooth surface is called minimal if there are no (-1) curves lying on it), i.e its canonical bundle is nef. Then one has a well defined map

$$\begin{aligned} \{\text{Minimal surfaces of general type}\} &\to \mathbb{Z} \oplus \mathbb{Z} \\ M &\mapsto (c_1^2(M), c_2(M)) \end{aligned} \tag{6.3.23}$$

Due to a theorem of Gieseker for Global moduli of surfaces of general types, for given c_1^2 and c_2 there are only finitely many diffeomorphism types of minimal

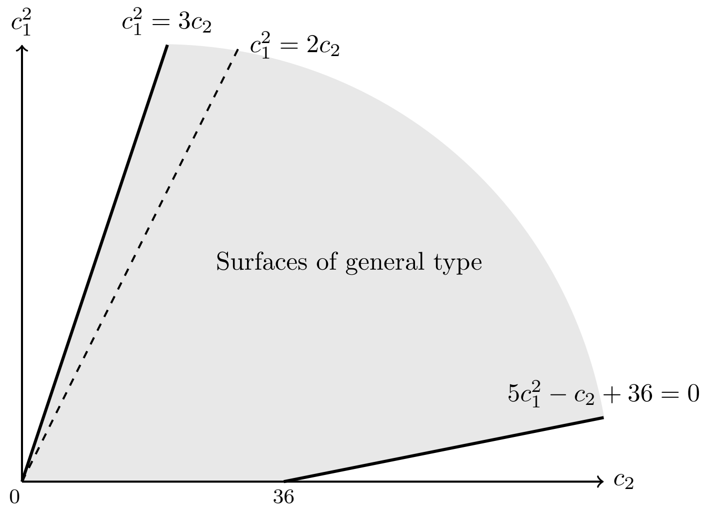


Figure 6.14 Surfaces of general type.

surfaces of general type. In addition, for a minimal surface of general types, the Chern numbers c_1^2 and c_2 are positive (See (Miyaoka, 1977) and (Yau, 1977)) and satisfy the following properties:

$$c_1^2 + c_2 \equiv 0 \pmod{12}, \tag{6.3.24}$$

$$5c_1^2 - c_2 + 36 \geq 12q \geq 0, \quad \text{(Noether inequality)} \tag{6.3.25}$$

where q is the irregularity of a surface M.

In 1956, Hirzebruch proved the following proportionality theorems:

Theorem 6.3.7 (Hirzebruch, 1956)**.**

(1) If M is a quotient of two ball $\mathbf{B}_2$, then one has the proportionality $c_1^2(M) = 3e(M)$.

(2) If M is a quotient of bidisc $\mathbf{B}_1 \times \mathbf{B}_1$, then the proportionality $c_1^2(M) = 2e(M)$ holds.

In 1977, Miyaoka and Yau proved the inequality

$$c_1^2(M) \leq 3e(M) \tag{6.3.26}$$

for an arbitrary algebraic surface and the following converse to Hirzebruch's proportionality theorem:

Theorem 6.3.8 (Miyaoka & Yau, 1977)**.** *If M satisfies the equality $c_1^2(M) = 3e(M) \geq 0$ then either M is $\mathbb{CP}^2$ or its universal covering is $\mathbf{B}_2$.*

The analogue of this result for surfaces with $c_1^2(M) = 2e(M) > 0$ is not correct! Kobayashi (1990) gave an example by using arrangement of five quadrics with 16 tacnodes such and 17 tacnodes. Assuming $\mathcal{A}_t$ as degeneration of these arrangement and M_t is a double plane branching over $\mathcal{A}_t$i obtained $c_1^2 - 2c_2 = \frac{3}{2}$ for singular members while generic double planes fulfill the proprotionality $c_1^2 = 2c_2$. Hence, he obtained that any general member close to a singular member is not uniformized by $\mathbf{B}_1 \times \mathbf{B}_1$.

6.4 Orbifold Chern Numbers

In the Section 6.3, we have introduced the Chern classes and Chern numbers of a complex manifolds M. As in fundamental group, the Chern numbers have also orbifold versions. Below we introduce the Chern numbers for orbifolds over the base $\mathbb{CP}^1$ and $\mathbb{CP}^2$, respectively. Let us first consider the base space $\mathbb{CP}^1$ and the divisor $D = \sum_{i=0}^{k} m_i p_i$.

Theorem 6.4.1 (Nevanlinna, 1970)**.** *Every entire function $f : \mathbb{C} \to \mathbb{CP}^1$ which is ramified over D is constant if $\sum_{i=0}^{k}(1 - \frac{1}{m_i}) > 2$.*

This degeneracy property corresponds to bigness of the canonical divisor

$$K_{\mathbb{CP}^1} + \sum_{i=0}^{k} \big(1 - \frac{1}{m_i}\big) p_i \tag{6.4.1}$$

of the pair $(\mathbb{CP}^1, D)$. Note that, this canonical divisor is an ample $\mathbb{Q}$-divisor on $\mathbb{CP}^1$. Assume we have an orbifold metric on $(\mathbb{CP}^1, D)$. Therefore, integrating the canonical class of (6.4.1), we can define the *Euler number* of $(\mathbb{CP}^1, D)$ as

$$e^{orb}(\mathbb{CP}^1, D) := e(\mathbb{CP}^1) - \sum_{i=0}^{k}(1 - \frac{1}{m_i}) = 1 - k + \sum_{i=0}^{k} \frac{1}{m_i}. \tag{6.4.2}$$

This is exactly the formula (6.1.4) on the page 145, and the Theorem 6.1.3 completely classifies the uniformization of the orbifold $(\mathbb{CP}^1, D)$ due to the sign of $e^{orb}(\mathbb{CP}^1, D)$. Now let us introduce this orbifold metric. First assume, $e^{orb}(\mathbb{CP}^1, D) < 0$, then by the Theorem 6.1.3, uniformization of the orbifold $(\mathbb{CP}^1, D)$ is $\mathbf{B}_1$ and we have introduced the Bergman metric on $\mathbf{B}_1$ in the Example 6.3.6. Since the divisor (6.4.1) is ample, there exists a volume form ϖ, a Hermitian metric $\|\cdot\|$ for $\mathscr{O}_{\mathbb{CP}^1}(\sum_{i=0}^{k} p_i)$, holomorphic sections s_i for $\mathscr{O}_{\mathbb{CP}^1}(p_i)$ with zeros at p_i, such that $\|s_i\| < 1$, and the minus of the Ricci-form of the singular volume form

$$\Theta = \frac{\varpi}{\prod_{i=0}^{k} m_i^2 \|s_i\|^{2(1-\frac{1}{m_i})}(1 - \|s_i\|^{\frac{2}{m_i}})^2}$$

defines a complete orbifold Kähler form $\omega = \partial\overline{\partial}\log\Theta$ on the orbifold $(\mathbb{CP}^1, D)$. In case $m_i = \infty$, $m_i(1 - \|s_i\|^{\frac{2}{m_i}}) = \log\frac{1}{\|s_i\|^2}$. This metric looks like an orbifold metric $\frac{|dz^{\frac{1}{n}}|}{(1-|z|^{\frac{2}{n}})^2}$ around a point with $m_i = n$, and like a Poincaré metric $\frac{|dz|^2}{|z|^2(\log\frac{1}{|z|^2})^2}$ of the punctured disk around a point of $m_i = \infty$ (Kobayashi, 1990). In a similar way, one can define orbifold metric for $(\mathbb{CP}^1, D)$ in cases of $e^{orb} = 0$, or $e^{orb} > 0$.

Note that, to compute the orbifold Euler number e^{orb}, it is enough to know the existence of orbifold metric. So the formula (6.4.2), which is also a consequence of Riemann-Hurwitz formula (See the Section 6.1.1), can be directly used to compute the orbifold Euler number e^{orb}.

Now, let us assume that the base space is $\mathbb{CP}^2$, $D = \sum_{i=1}^{k} m_i H_i$ is a divisor on $\mathbb{CP}^2$, the curves H_i being irreducible of degree d_i for $1 \le i \le k$. Denote by $(\mathbb{CP}^2, \beta)$ the orbifold associated with the divisor D. The canonical divisor

$$K^{orb} := K_{\mathbb{CP}^2} + \sum_{i=1}^{k}\big(1 - \frac{1}{m_i}\big)H_i \tag{6.4.3}$$

of $(\mathbb{CP}^2,\beta)$ is big if

$$\sum_{i=1}^{k}\left(1-\frac{1}{m_i}\right)>3 \tag{6.4.4}$$

Note that, this canonical divisor (6.4.3) together with the condition(6.4.4) is an ample $\mathbb{Q}$-divisor on $\mathbb{CP}^2$. Kobayashi (1990, Section 3, Theorem 1) proved that, there exists a canonical orbifold Kähler metric h and orbifold Kähler form ω obtained from the holomorphic sections s_i of the divisor D. Then we can integrate the Chern forms. By the definition of first Chern number,

$$\begin{aligned}
c_1^2(\mathbb{CP}^2,\beta) &= \int c_1[K^{orb}]\smile c_1[K^{orb}]\\
&= K^{orb}\cdot K^{orb}\\
&= \Big(K_{\mathbb{CP}^2}+\sum_{i=1}^{k}\left(1-\frac{1}{m_i}\right)H_i\Big)\cdot\Big(K_{\mathbb{CP}^2}+\sum_{i=1}^{k}\left(1-\frac{1}{m_i}\right)H_i\Big)\\
&= K_{\mathbb{CP}^2}\cdot K_{\mathbb{CP}^2}+2\sum_{i=1}^{k}\left(1-\frac{1}{m_i}\right)K_{\mathbb{CP}^2}\cdot H_i\\
&\quad+\Big(\sum_{i=1}^{k}\left(1-\frac{1}{m_i}\right)H_i\Big)\cdot\Big(\sum_{i=1}^{k}\left(1-\frac{1}{m_i}\right)H_i\Big)\\
&= (-3)^2+2(-3)\sum_{i=1}^{k}\left(1-\frac{1}{m_i}\right)d_i+\Big(\sum_{i=1}^{k}\left(1-\frac{1}{m_i}\right)d_i\Big)^2.
\end{aligned}$$

Hence, the *first orbifold Chern number* of $(\mathbb{CP}^2,\beta)$ is defined as

$$c_1^2(\mathbb{CP}^2,\beta):=\left(-3+\sum_{i=1}^{k}d_i\left(1-\frac{1}{m_i}\right)\right)^2. \tag{6.4.5}$$

Second Chern class of $(\mathbb{CP}^2,\beta)$ is the Euler class of $(\mathbb{CP}^2,\beta)$ and Kobayashi (1990, Section 3.2.3) obtained this class after resolving log-canonical singularities and compute the second Chern number of $(\mathbb{CP}^2,\beta)$ by integrating the Euler class and computing correction terms coming from singularities. Hence, the *second orbifold Chern number* or *orbifold Euler number* of $(\mathbb{CP}^2,\beta)$ is defined as

$$e(\mathbb{CP}^2,\beta):=3-\sum_{i=1}^{k}\left(1-\frac{1}{m_i}\right)e(H_i\setminus SingB)-\sum_{p\in Sing(B)}\left(1-\frac{1}{\beta(p)}\right), \tag{6.4.6}$$

where $\beta(p)$ denotes the order of the local orbifold fundamental group. If $(\mathbb{CP}^2, \beta)$ is an orbifold with cusp points, set $\frac{1}{\beta(p)} = 0$ whenever $\beta(p) = \infty$.

The orbifold Chern numbers have the following property: if $M \to (X, \beta)$ is a finite uniformization with G as its Galois group, then

$$e(M) = |G|e(X,\beta) \quad \text{and} \quad c_1^2(M) = |G|c_1^2(X,\beta). \tag{6.4.7}$$

The following orbifold analogue of the Miyaoka-Yau Theorem 6.3.8 was proved by Kobayashi & Nakamura & Sakai 1989 by constructing a metric on orbifolds.

Theorem 6.4.2 (Kobayashi-Nakamura-Sakai,1989)**.** *Let* $(\mathbb{CP}^2, \beta)$ *be an orbifold of general type, possibly with ball-cusp points. Then* $c_1^2(\mathbb{CP}^2, \beta) \leq 3e(\mathbb{CP}^2, \beta)$*. The equality holds if and only if* $(\mathbb{CP}^2, \beta)$ *is uniformized by* $\mathbf{B}_2$*.*

The following theorem determines whether the orbifold $(\mathbb{CP}^2, \beta)$ is of general type or not? Also, remember the ampleness condition (6.4.4).

Theorem 6.4.3 (Sakai, 1984)**.** *For a normal surface pair* $(\mathbb{CP}^2, \beta)$ *with at worst log-canonical singularities, the following conditions are equivalent*

(1) $\kappa(\mathbb{CP}^2, \beta) = 2$*,*

(2) K^{orb} *is numerically very ample,*

(3) K^{orb} *is ample.*

6.5 Orbifolds Supported by Line Arrangements

In Section 6.2.4, we have studied the covering relations among orbifold germs and we know that finiteness or infinite solvability of the local orbifold fundamental group is necessary for local uniformization. In addition, by Kato's Theorem 6.1.4 we know that the orbifolds, which is supported by an arrangement so that any line contains a point of multiplicity at least 3, are uniformizable. In addition, rigid arrangements are candidates observing a ball-quotient orbifold branched along them. So, we will mostly deal with rigid line arrangements.

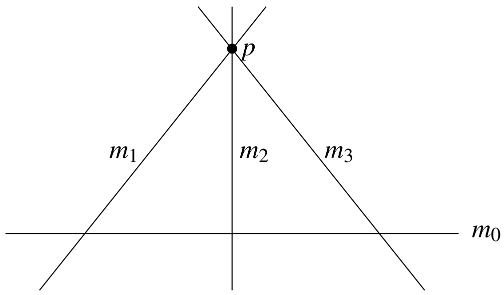


Figure 6.15 The orbifold $(\mathbb{CP}^2, \sum_{i=0}^{3} m_i H_i)$.

First, consider the orbifold $(\mathbb{CP}^2, \sum_{i=0}^{3} m_i H_i)$ in Figure 6.15, where $H_0 = \{Z = 0\}$, $H_1 = \{X = 0\}$, $H_2 = \{Y = 0\}$ and $H_3 = \{X - Y = 0\}$. The arrangement $\mathcal{A} = \{H_0, H_1, H_2, H_3\}$ is projectively rigid. Because one can maps $[0 : 0 : 1]$ to any point p and the line H_0 to any line which does not contain the point p, and projective transformations allow us to fix three points on the projective line. For simplicity, let us set $\kappa_i = 1/m_i$, $i = 0, 1, 2, 3$. The condition $\kappa_1 + \kappa_2 + \kappa_3 > 1$ is necessary for local uniformizability. Take a base point $\star \in \mathbb{CP}^2 \setminus \cup_{i=0}^{3} H_i$, and assume μ_i be the meridians around H_i and μ_p is meridian around p. Then $\mu_p \mu_0$ is contractible in $\mathbb{CP}^2 \setminus \cup_{i=0}^{3} H_i$, and hence $\mu_p = \mu_0^{-1}$. Therefore order m_0 of μ_0 in $\pi_1^{orb}(\mathbb{CP}^2, D)$ must equal the order of μ_p in $\pi_1^{orb}(\mathbb{CP}^2, D)_p$, .i.e, $m_0 = 2(\sum_{i=1}^{3} \frac{1}{m_i} - 1)$. Hence the quadruple $(m_0; m_1, m_2, m_3)$ must be one of $(2r; 2, 2, r)$, $(12; 3, 3, 2)$, $(24; 2, 4, 3)$ or $(60; 2, 3, 5)$.

The orbifold Chern numbers are

$$c_1^2(\mathbb{CP}^2, \sum_{i=0}^{3} m_i H_i) = (\kappa_0 + \kappa_1 + \kappa_2 + \kappa_3 - 1)^2$$

and

$$e(\mathbb{CP}^2, \sum_{i=0}^{3} m_i H_i) = \kappa_0(\kappa_1 + \kappa_2 + \kappa_3 - 1) + \frac{1}{4}(\kappa_1 + \kappa_2 + \kappa_3 - 1)^2.$$

Note that, they are not orbifolds of general type, since the condition (6.4.4) fails for all possible quadruples $(m_0; m_1, m_2, m_3)$. Although, $(c_1^2 - 3e)(\mathbb{CP}^2, \sum_{i=0}^{3} m_i H_i) = 0$ for such quadruples $(m_0; m_1, m_2, m_3)$, their uniformization is not $\mathbf{B}_2$. First three of them are uniformized by $\mathbb{CP}^2$ (Uludağ, 2007). Indeed,

- **Case** $(\mathbf{2r}; \mathbf{2}, \mathbf{2}, \mathbf{r})$**:** $(\mathbb{CP}^2, 2H_0 + 2H_1 + 2H_3)$ is a sub-orbifold of $(\mathbb{CP}^2, 2rH_0 + 2H_1 + 2H_2 + rH_3)$, and it is uniformized by $\mathbb{CP}^2$ via the bicyclic covering $\varphi_2 : [X : Y : Z] \to [X^2 : Y^2 : Z^2]$. The lifting $\varphi_2^{-1}(H_3)$ consists of two lines given by the equation $X^2 - Y^2 = 0$, which we denote by H_3^1 and H_3^2. Denote $\varphi^{-1}(H_0)$ by H_0 again. Hence $\varphi_2 : (\mathbb{CP}^2, rH_0 + rH_3^1 + rH_3^2) \to (CP^2, 2rH_0 + 2H_1 + 2H_2 + rH_3)$. Obviously, the covering orbifold is uniformized by $\mathbb{CP}^2$ via φ_r.

- **Case** $(\mathbf{24}; \mathbf{2}, \mathbf{4}, \mathbf{3})$**:** $(\mathbb{CP}^2, 2H_0 + 2H_1 + 2H_2)$ is a sub-orbifold of $(\mathbb{CP}^2, 24H_0 + 2H_1 + 4H_2 + 3H_3)$, and it is uniformized by $\mathbb{CP}^2$ via the bicyclic covering $\varphi_2 : [X : Y : Z] \to [X^2 : Y^2 : Z^2]$. Denote $\varphi_2^{-1}(H_2)$ by H_2 and $\varphi_2^{-1}(H_0)$ by H_0 again. The lifting $\varphi_2^{-1}(H_3)$ consists of two lines given by the equation $X^2 - Y^2 = 0$, which we denote by H_3^1 and H_3^2. Hence $\varphi_2 : (\mathbb{CP}^2, 12H_0 + 3H_3^1 + 3H_3^2 + 2H_2) \to (CP^2, 24H_0 + 2H_1 + 4H_2 + 3H_3)$. The covering orbifold is related with the Case $(12; 3, 3, 2)$.

- **Case** $(\mathbf{12}; \mathbf{3}, \mathbf{3}, \mathbf{2})$**:** $(\mathbb{CP}^2, 3H_0 + 3H_1 + 3H_2)$ is a sub-orbifold of $(\mathbb{CP}^2, 12H_0 + 3H_1 + 3H_2 + 2H_3)$, and it is uniformized by $\mathbb{CP}^2$ via the bicyclic covering $\varphi_3 : [X : Y : Z] \to [X^2 : Y^2 : Z^2]$. Denote $\varphi_2^{-1}(H_0)$ by H_0 again. The lifting $\varphi_3^{-1}(H_2)$ consists of three lines given by the equation $X^3 - Y^3 = 0$, which we denote by H_3^1, H_3^2 and H_3^3. Hence $\varphi_3 : (\mathbb{CP}^2, 4H_0 + 3H_3^1 + 3H_3^2 + 3H_3^3) \to (CP^2, 12H_0 + 3H_1 + 3H_2 + 2H_3)$. The covering orbifold appeared in the first case with $r = 2$ and is uniformized by $\mathbb{CP}^2$.

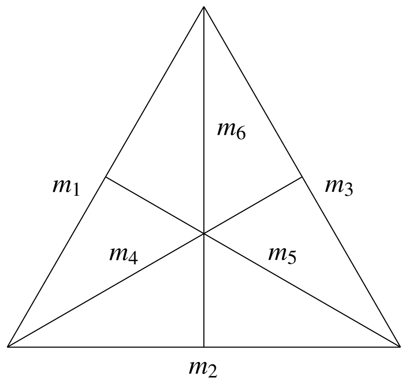


Figure 6.16 Complete quadrilateral.

Second, consider the orbifold $(\mathbb{CP}^2, \sum_{i=1}^{6} m_i H_i)$ in the Figure 6.16, where $H_1 = \{X = 0\}$, $H_2 = \{Y = 0\}$, $H_3 = \{Z = 0\}$, $H_4 = \{X - Y = 0\}$, $H_5 = \{Y - Z = 0\}$ and $H_6 = \{Z - X = 0\}$. The arrangement $\mathcal{A} = \{H_i \mid i = 1, \cdots, 6\}$ is projectively rigid. For simplicity, let us denote by D the divisor $\sum_{i=1}^{6} m_i H_i$, by κ_i the number $\frac{1}{m_i}$ and by $\rho_{i,j,k}$ the number $\kappa_i + \kappa_j + \kappa_k - 1$. The local uniformizability conditions of the orbifold $(\mathbb{CP}^2, D)$ are $\rho_{1,2,4} \geq 0$, $\rho_{1,3,6} \geq 0$, $\rho_{2,3,5} \geq 0$, $\rho_{4,5,6} \geq 0$ and the orbifold Chern numbers are

$$\begin{aligned} e(\mathbb{CP}^2, D) &= 2 - \sum_{i=1}^{6} \kappa_i + \kappa_1\kappa_5 + \kappa_2\kappa_6 + \kappa_3\kappa_4 + \frac{1}{4}(\rho_{1,2,4}^2 + \rho_{1,3,6}^2 + \rho_{2,3,5}^2 + \rho_{4,5,6}^2) \\ c_1^2(\mathbb{CP}^2, D) &= (3 - \sum_{i=1}^{6} \kappa_i)^2. \end{aligned} \tag{6.5.1}$$

Proposition 6.5.1. *Consider the orbifold $(\mathbb{CP}^2, D)$ supported by complete quadrilateral. Then*

i. $c_1^2(\mathbb{CP}^2, D) = e(\mathbb{CP}^2, D) = 0$ *if and only if* $m_i = 2$ *for all* $i = 1, \cdots, 6$.

ii. $(2e - c_1^2)(\mathbb{CP}^2, D) = 0$ *if and only if* $(m_1, m_2, m_3, m_4, m_5, m_6) = (m, 2, 2, 2, n, 2)$, *where* $m, n \in \mathbb{Z}^+$.

iii. $(3e-c_1^2)(\mathbb{CP}^2,D)=0$ *if and only if* $(m_1,m_2,m_3,m_4,m_5,m_6)=(m,m,m,n,n,n)$ *where* $m,n\in\mathbb{Z}^+$.

Proof. It is clear that $c_1^2(\mathbb{CP}^2,D)=0$ if and only if $m_i=2$ for all $i=1,\cdots,6$. Moreover, Euler orbifold number vanishes for such $m_i=2$.

If one use the equalities

$$\rho_{i,j,k}^2=(\kappa_i+\kappa_j+\kappa_k-1)^2=\kappa_i^2+\kappa_j^2+\kappa_k^2+2(\kappa_i\kappa_j+\kappa_j\kappa_k+\kappa_i\kappa_k)-2(\kappa_i+\kappa_j+\kappa_k)+1,$$

then orbifold Chern numbers (6.5.1) reduce to

$$c_1^2(\mathbb{CP}^2,D)=9-6\sum_{i=1}^{6}\kappa_i+(\sum_{i=1}^{6}\kappa_i)^2$$

and

$$e(\mathbb{CP}^2,D)=3-2\sum_{i=1}^{6}\kappa_i+\frac{1}{4}\sum_{i=1}^{6}\kappa_i^2+\frac{1}{4}(\sum_{i=1}^{6}\kappa_i)^2+\frac{1}{2}(\kappa_1\kappa_5+\kappa_2\kappa_6+\kappa_3\kappa_4).$$

Therefore,

$$\begin{aligned}2e-c_1^2&=-3+2\sum_{i=1}^{6}\kappa_i+\frac{1}{2}\sum_{i=1}^{6}\kappa_i^2-\frac{1}{2}(\sum_{i=1}^{6}\kappa_i)^2+\kappa_1\kappa_5+\kappa_2\kappa_6+\kappa_3\kappa_4\\&=-3+2\sum_{i=1}^{6}\kappa_i-\sum_{1\le i<j\le 6}\kappa_i\kappa_j+\kappa_1\kappa_5+\kappa_2\kappa_6+\kappa_3\kappa_4\\&=-3+2[(\kappa_1+\kappa_5)+(\kappa_2+\kappa_6)+(\kappa_3+\kappa_4)]-(\kappa_1+\kappa_5)(\kappa_2+\kappa_6)\\&\quad-(\kappa_1+\kappa_5)(\kappa_3+\kappa_4)-(\kappa_2+\kappa_6)(\kappa_3+\kappa_4)\\&=-3+2(a+b+c)-(ab+ac+bc),\end{aligned}$$

where $a=(\kappa_1+\kappa_5)$, $b=(\kappa_2+\kappa_6)$, $c=(\kappa_3+\kappa_4)$. The equation

$$2(a+b+c)=3+(ab+ac+bc)$$

has solutions in the interval $[0,1]$ if and only if two of a, b and c is 1 and the other one is free. Hence, any two of the tuples (m_1,m_5), (m_2,m_6) and (m_3,m_4) is $(2,2)$

and the third one is free, say (m,n). Since, complete quadrilateral is projectively rigid, and using the symmetries we may assume that $(m_1,m_2,m_3,m_4,m_5,m_6) = (m,2,2,2,n,2)$, where $m,n \in \mathbb{Z}^+$. For these weights, $(2e-c_1^2)(\mathbb{CP}^2,D)$ vanishes.

Finally,

$$\begin{aligned}
3e-c_1^2 =& \frac{3}{4}\sum_{i=1}^{6}\kappa_i^2 - \frac{1}{4}(\sum_{i=1}^{6}\kappa_i)^2 + \frac{3}{2}(\kappa_1\kappa_5+\kappa_2\kappa_6+\kappa_3\kappa_4)\\
=& \frac{1}{2}\sum_{i=1}^{6}\kappa_i^2 - \frac{1}{2}\sum_{1\le i<j\le 6}\kappa_i\kappa_j + \frac{3}{2}(\kappa_1\kappa_5+\kappa_2\kappa_6+\kappa_3\kappa_4)\\
=& \frac{1}{2}(\sum_{i=1}^{6}\kappa_i)^2 - \frac{3}{2}\sum_{1\le i<j\le 6}\kappa_i\kappa_j + \frac{3}{2}(\kappa_1\kappa_5+\kappa_2\kappa_6+\kappa_3\kappa_4)\\
=& \frac{1}{2}[(\kappa_1+\kappa_5)+(\kappa_2+\kappa_6)+(\kappa_3+\kappa_4)]^2 - \frac{3}{2}(\kappa_1+\kappa_5)(\kappa_2+\kappa_6)\\
&- \frac{3}{2}(\kappa_1+\kappa_5)(\kappa_3+\kappa_4) - \frac{3}{2}(\kappa_2+\kappa_6)(\kappa_3+\kappa_4)\\
=& \frac{1}{2}(a+b+c)^2 - \frac{3}{2}(ab+ac+bc)\\
=& \frac{1}{2}[(a^2+b^2+c^2)-(ab+ac+bc)]
\end{aligned}$$

The equation $(a^2+b^2+c^2)-(ab+ac+bc) = 0$ has solutions in the interval [0,1] if and only if $a=b=c$, which implies $(m_1,m_2,m_3,m_4,m_5,m_6) = (m,m,m,n,n,n)$, where $m,n \in \mathbb{Z}^+$. (m,n,n,m,n,m) is another solution but it is equivalent to previous one up to projective transformations. Hence $(3e-c_1^2)(\mathbb{CP}^2,D)$ vanishes if and only if $(m_1,m_2,m_3,m_4,m_5,m_6) = (m,m,m,n,n,n)$, where $m,n \in \mathbb{Z}_{>0}$. □

Theorem 6.5.2. *The orbifold $(\mathbb{CP}^2,D)$ branched along a complete quadrilateral is uniformized by complex 2-ball $\mathbf{B}_2$ if $(m_1,m_2,m_3,m_4,m_5,m_6)$ is one of $(2,2,2,3,3,3)$, $(3,3,3,2,2,2)$, $(3,3,3,3,3,3)$ and $(4,4,4,2,2,2)$ (Last two orbifolds consists of ball-cusp points).*

Proof. By the Proposition 6.5.1 we know all possibilities satisfying the orbifold version of Miyaoka-Yau equality. The ampleness condition (6.4.4) implies that $\frac{1}{m}+\frac{1}{n}<1$. Now, it is enough to check local uniformizability conditions. The inequalities $\frac{2}{m}+\frac{1}{n}\ge 1$ and $\frac{2}{n}+\frac{1}{m}\ge 1$ are valid if and only if (m,n) is either $(2,3)$ or $(3,2)$ or $(3,3)$ or $(4,2)$. Hence, the Theorem 6.4.2 completes the proof. □

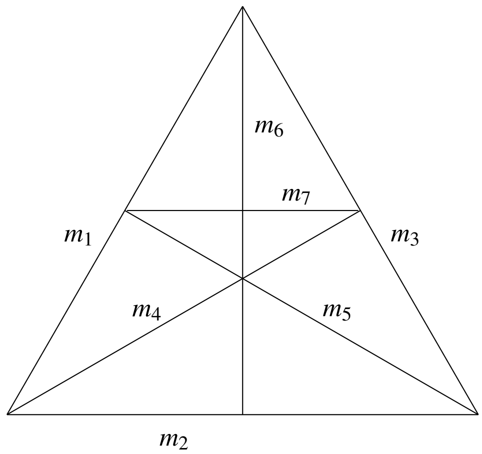


Figure 6.17 The orbifold $(\mathbb{CP}^2, \sum_{i=1}^{7} m_i H_i)$.

Third, consider the orbifold $(\mathbb{CP}^2, \sum_{i=1}^{7} m_i H_i)$ in the Figure 6.17, where $H_1 = \{X = 0\}$, $H_2 = \{Y = 0\}$, $H_3 = \{Z = 0\}$, $H_4 = \{X - Y = 0\}$, $H_5 = \{Y - Z = 0\}$, $H_6 = \{Z - X = 0\}$ and $H_7 = \{X - Y + Z = 0\}$. The arrangement $\mathcal{A} = \{H_i \mid i = 1, \cdots, 7\}$ is projectively rigid. For simplicity, let us set $D := \sum_{i=1}^{7} m_i H_i$, $\kappa_i := \frac{1}{m_i}$ and $\rho_{i,j,k} := \frac{1}{m_i} + \frac{1}{m_j} + \frac{1}{m_k} - 1$. The local uniformizability conditions of the orbifold $(\mathbb{CP}^2, D)$ are $\rho_{1,2,4} \geq 0$, $\rho_{1,3,6} \geq 0$, $\rho_{1,5,7} \geq 0$, $\rho_{2,3,5} \geq 0$, $\rho_{3,4,7} \geq 0$ and $\rho_{4,5,6} \geq 0$. In addition, the orbifold Chern numbers are

$$\begin{aligned} e(\mathbb{CP}^2, D) =& 4 - (\kappa_1 + 2\kappa_2 + \kappa_3 + \kappa_4 + \kappa_5 + 2\kappa_6 + 2\kappa_7) + (\kappa_2\kappa_6 + \kappa_2\kappa_7 + \kappa_6\kappa_7) \\ & + \frac{1}{4}(\rho_{1,2,4}^2 + \rho_{1,3,6}^2 + \rho_{1,5,7}^2 + \rho_{2,3,5}^2 + \rho_{3,4,7}^2 + \rho_{4,5,6}^2) \\ =& -1 + \frac{1}{4}(-5 + \sum_{i=1}^{7} \kappa_i)^2 + \frac{1}{2}(\kappa_1^2 + \kappa_3^2 + \kappa_4^2 + \kappa_5^2) + \frac{1}{4}(\kappa_2 + \kappa_6 + \kappa_7 - 1)^2 \end{aligned}$$

and

$$c_1^2(\mathbb{CP}^2, D) = (4 - \sum_{i=1}^{7} \kappa_i)^2.$$

In addition, the bigness condition (6.4.4) is satisfied for any branching indices m_i since $\sum_{i=1}^{7} \kappa_i < 4$.

Proposition 6.5.3. *Consider the orbifold $(\mathbb{CP}^2, D)$, where $D = \sum_{i=1}^{7} m_i H_i$ is divisor supported by the line arrangement in Figure 6.17. Then*

i. $(2e - c_1^2)(\mathbb{CP}^2, D) = 0$ *if and only if* $m_1 = m_3 = m_4 = m_5 = 2$.

ii. $(3e-c_1^2)(\mathbb{CP}^2,D)=0$ *if* $m_1=m_3=m_4=m_5=2k$ *and* (m_2,m_6,m_7) *is a permutation of* $(2,2,k)$ *for a positive integer* k*.*

Proof. For simplicity, set $a:=\kappa_1+\kappa_3+\kappa_4+\kappa_5$, $b:=\kappa_2+\kappa_6+\kappa_7-1$ and $c:=\frac{1}{2}(\kappa_1^2+\kappa_3^2+\kappa_4^2+\kappa_5^2)$. Note that $0<a\le 2$, $-1<b\le\frac{1}{2}$ and $0<c\le\frac{1}{2}$. Then, orbifold Chern numbers reduces to the forms

$$e(\mathbb{CP}^2,D)=-1+\frac{1}{4}(4-a-b)^2+c+\frac{1}{4}b^2 \quad\text{and}\quad c_1^2(\mathbb{CP}^2,D)=(3-a-b)^2.$$

Therefore,

$$\begin{aligned}(2e-c_1^2)(\mathbb{CP}^2,D)&=-\frac{1}{2}(a+b)^2+2(a+b)+2c+\frac{b^2}{2}-3\\&=-\frac{1}{2}(a+b-2)^2+2c+\frac{b^2}{2}-1=0\end{aligned}$$

Consider the function $f(x,y,z)=-\frac{1}{2}(x+y-2)^2+2z+\frac{1}{2}y^2-1$ in the domain

$$B=\{(x,y,z)\in\mathbb{R}^3\mid 0\le x\le 2,\ -1\le y\le\frac{1}{2},\ 0\le z\le\frac{1}{2}\}.$$

Since $\mathrm{grad}f=(-x-y+2,-x+2,2)\neq(0,0,0)$ for all $(x,y,z)\in B$, it takes its extremum values in the boundary ∂B of B. Except the boundary of B for which $x=2$, the function f takes negative values at ∂B. In case $x=2$, $f(2,y,z)=2z-1<0$ if $z<\frac{1}{2}$ and it vanishes for $z=\frac{1}{2}$. Thus f takes its maximum value, which is 0, on the edge of ∂B for which $x=0$ and $z=\frac{1}{2}$. Thus $(2e-c_1^2)(\mathbb{CP}^2,D)$ vanishes if and only if $a=2$ and $c=\frac{1}{2}$, i.e., $m_1=m_3=m_4=m_5=2$. Note that the integers m_2,m_6,m_7 are free.

On the other hand $(3e-c_1^2)(\mathbb{CP}^2,D)=-\frac{1}{4}((a+b)^2-3b^2)+3c=0$ if and only if $c=\frac{(a+b)^2-3b^2}{12}$ (i.e $b=\frac{a\pm\sqrt{3(a^2-8c)}}{2}$). Note that

$$\begin{aligned}a^2-8c=&(\kappa_1+\kappa_3+\kappa_4+\kappa_5)^2-4(\kappa_1^2+\kappa_3^2+\kappa_4^2+\kappa_5^2)\\=&-(\kappa_1-\kappa_3)^2-(\kappa_1-\kappa_4)^2-(\kappa_1-\kappa_5)^2-(\kappa_3-\kappa_4)^2-(\kappa_3-\kappa_5)^2\\&-(\kappa_4-\kappa_5)^2\le 0\end{aligned}$$

Thus, $(3e-c_1^2)=0$ if and only if $\kappa_1=\kappa_3=\kappa_4=\kappa_5$. Now suppose $m_1=m_3=m_4=m_5=m$, then $a=\frac{4}{m}$ and $c=\frac{2}{m^2}$. Thus $(3e-c_1^2)(\mathbb{CP}^2,D)=0$ if $b=\frac{2}{m}$, but $\frac{1}{m_2}+\frac{1}{m_6}+\frac{1}{m_7}=1+\frac{2}{m}$ has solutions only when m is even. Say $m=2k$, then (m_2,m_6,m_7) is a permutation of $(2,2,k)$. □

Proposition 6.5.4. *The orbifold $(\mathbb{CP}^2,D)$, where $D=4H_1+2H_2+4H_3+4H_4+4H_5+2H_6+2H_7$ is a divisor on $\mathbb{CP}^2$ supported by the line arrangement in Figure 6.17, is uniformized by $\mathbf{B}_2$.*

Proof. By the Proposition 6.5.3, we know that the Miyaoka-Yau equality satisfied for the orbifold $(\mathbb{CP}^2,D)$, where $D=\sum_{i=1}^{7}m_iH_i$, $m_1=m_3=m_4=m_5=2k$ and (m_2,m_6,m_7) is a permutation of $(2,2,k)$ for a positive integer k. The local uniformizability conditions $\rho_{1,2,4}\geq 0$, $\rho_{1,3,6}\geq 0$, $\rho_{1,5,7}\geq 0$, $\rho_{2,3,5}\geq 0$, $\rho_{3,4,7}\geq 0$, $\rho_{4,5,6}\geq 0$, which are equivalent to one of the conditions $\frac{1}{2k}+\frac{1}{2k}+\frac{1}{2}-1\geq 0$ or $\frac{1}{2k}+\frac{1}{2k}+\frac{1}{k}-1\geq 0$, implies $k=2$. Thus, the Theorem 6.3.8 completes the proof. □

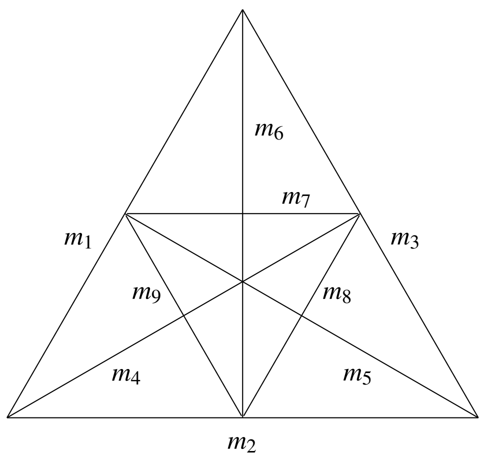


Figure 6.18 The orbifold $(\mathbb{CP}^2,\sum_{i=1}^{9}m_iH_i)$.

Fourth, consider the orbifold $(\mathbb{CP}^2,\sum_{i=1}^{9}m_iH_i)$ in the Figure 6.18, where $H_1=\{X=0\}$, $H_2=\{Y=0\}$, $H_3=\{Z=0\}$, $H_4=\{X-Y=0\}$, $H_5=\{Y-Z=0\}$, $H_6=\{Z-X=0\}$, $H_7=\{X-Y+Z=0\}$, $H_8=\{-X+Y+Z=0\}$ and $H_9=\{X+Y-Z=0\}$. The arrangement $\mathcal{A}=\{H_i\mid i=1,\cdots,9\}$ is projectively rigid. For simplicity, let us denote set $D:=\sum_{i=1}^{9}m_iH_i$, $\kappa_i:=\frac{1}{m_i}$ and $\rho_{i,j,k}:=\frac{1}{m_i}+\frac{1}{m_j}+\frac{1}{m_k}-1$. The local uniformizability conditions of the orbifold $(\mathbb{CP}^2,D)$ are $\rho_{1,2,4}\geq$

0, $\rho_{1,3,6} \geq 0$, $\rho_{2,3,5} \geq 0$, $\rho_{4,5,6} \geq 0$. In addition, the bigness condition (6.4.4) is satisfied for any branching indices m_i. This means, this orbifold is of general type. The orbifold Chern numbers are

$$\begin{aligned}
e(\mathbb{CP}^2, D) =& 8 - 2\sum_{i=1}^{9}\kappa_i + (\kappa_1+\kappa_5)\kappa_8 + (\kappa_3+\kappa_4)\kappa_9 + (\kappa_2+\kappa_6)\kappa_7 \\
& + \frac{1}{4}(\rho_{1,2,4}^2 + \rho_{1,3,6}^2 + \rho_{2,3,5}^2 + \rho_{4,5,6}^2) \\
=& 9 - 3\sum_{i=1}^{6}\kappa_i - 2\sum_{i=7}^{9}\kappa_i + \frac{1}{4}\sum_{i=1}^{6}\kappa_i^2 + \frac{1}{4}(\sum_{i=1}^{6}\kappa_i)^2 - \frac{1}{2}(\kappa_1\kappa_5 + \kappa_2\kappa_6 + \kappa_3\kappa_4) \\
& + (\kappa_1+\kappa_5)\kappa_8 + (\kappa_3+\kappa_4)\kappa_9 + (\kappa_2+\kappa_6)\kappa_7. \\
=& 9 - 3\sum_{i=1}^{6}\kappa_i - 2\sum_{i=7}^{9}\kappa_i + \frac{1}{4}(\sum_{i=1}^{6}\kappa_i)^2 + \frac{1}{2}\sum_{i=1}^{6}\kappa_i^2 + \sum_{i=7}^{9}\kappa_i^2 \\
& - \frac{1}{4}[(\kappa_1+\kappa_5-2\kappa_8)^2 + (\kappa_2+\kappa_6-2\kappa_7)^2 + (\kappa_3+\kappa_4-2\kappa_9)^2]
\end{aligned}$$

and

$$\begin{aligned}
c_1^2(\mathbb{CP}^2, D) =& (6 - \sum_{i=1}^{9}\kappa_i)^2 \\
=& 36 - 12\sum_{i=1}^{6}\kappa_i - 12\sum_{i=7}^{9}\kappa_i + (\sum_{i=1}^{6}\kappa_i)^2 + (\sum_{i=7}^{9}\kappa_i)^2 + (\sum_{i=1}^{6}\kappa_i)(\sum_{i=7}^{9}\kappa_i).
\end{aligned}$$

Therefore,

$$\begin{aligned}
(c_1^2 - 3e)(\mathbb{CP}^2, D) =& (3 - \frac{1}{2}\sum_{i=1}^{6}\kappa_i - \sum_{i=7}^{9}\kappa_i)^2 - \frac{3}{2}\sum_{i=1}^{6}\kappa_i^2 - 3\sum_{i=7}^{9}\kappa_i^2 + (\sum_{i=1}^{6}\kappa_i)(\sum_{i=7}^{9}\kappa_i) \\
& + \frac{3}{4}[(\kappa_1+\kappa_5-2\kappa_8)^2 + (\kappa_2+\kappa_6-2\kappa_7)^2 + (\kappa_3+\kappa_4-2\kappa_9)^2].
\end{aligned}$$

Up to projective equivalencies of the Figure 6.18, Maple gives solutions of ordered m_i's as $(n,n,n,2,2,2,2,2,2)$ or $(2,2,2,n,n,n,2,2,2)$, $n \in \mathbb{Z}_{\geq 2}$, for $(c_1^2 - 3e) = 0$. In these cases, the Chern numbers are $c_1^2 = (3-\frac{3}{n})^2$ and $e = \frac{1}{3}(3-\frac{3}{n})^2$. Since there are three fourfold point of the arrangement in Figure 6.18, at these points the β map takes infinite values. The local orbifold fundamental group at these points are infinite solvable if all the branching indices are 2, i.e $n = 2$. Then we have the following theorem:

Theorem 6.5.5. *The orbifold* $(\mathbb{CP}^2, D)$, *where* $D = \sum_{i=1}^{9} 2H_i$ *is the divisor supported by the line arrangement in Figure 6.18 is uniformized by* $\mathbf{B}_2$.

Another arrangement of nine line is the harmonic arrangement $\mathcal{A} = \{H_i \mid i = 1, \cdots, 9\}$ in Figure 6.19 defined by the equation

$$XYZ(X-Y)(Y-Z)(Z-X)(X-Y+Z)(Y-2Z)(2X-Y) = 0.$$

Harmonic arrangement is projectively rigid. Indeed, cross ratio of the singular points on H_8 and H_9 is -1. Note that, Harmonic arrangement is projectively equivalent to the arrangement in Figure 6.18 via $[X : Y : Z] \mapsto [X : X - Y + Z : Z]$. Thus, if we choose all branching indices are 2, then we get a ball quotient orbifold, but it is same as the previous one.

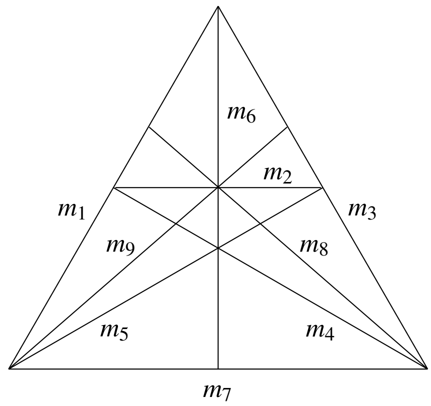


Figure 6.19 Harmonic arrangement.

Finally, consider the Ceva(n) arrangement, which is an arrangement $\mathcal{A}$ of $3n$ lines given by the equation $(X^n - Y^n)(Y^n - Z^n)(Z^n - X^n) = 0$. Let us divide it into three parts: $\mathcal{A}_1 = \{H_{1,i} \mid H_{1,i} : X - \omega^i Y = 0,\ i = 0, \cdots n-1\}$, $\mathcal{A}_2 = \{H_{2,i} \mid H_{2,i} : Y - \omega^i Z = 0,\ i = 0, \cdots n-1\}$ and $\mathcal{A}_3 = \{H_{3,i} \mid H_{3,i} : Z - \omega^i X = 0,\ i = 0, \cdots n-1\}$, where ω denotes the n-th root of unity. Each line has a point of order n, n triple points and no r-fold points if $r \neq 3, n$. Therefore, the arrangement $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3$ has 3 point of order n, n^2 triple points and no r-fold points if $r \neq 3, n$. Note that, triple points lies on the lines $H_{1,i}$, $H_{2,j}$ and $H_{3,k}$, where $i + j \equiv k \pmod{n}$. Let us denote by Γ the set $\{(i,j,k) \mid i+j \equiv k \pmod{n}\}$. Clearly, $|\Gamma| = n^2$. In addition, denote by $m_{s,i}$ the weights of $H_{s,i}$, set $\kappa_{s,i} := \frac{1}{m_{s,i}}$ and $D_n := \sum_{s=1}^{3}\sum_{i=0}^{n-1} m_{s,i}H_{s,i}$. The bigness condition (6.4.4) is satisfied for all $m_{s,i} \in \mathbb{Z}_{\geq 2}$, so the orbifold $(\mathbb{CP}^2, D_n)$ is of general type. Beside this, its orbifold Chern numbers are

$$c_1^2(\mathbb{CP}^2, D_n) = (3n - 3 - \sum_{s,i} \kappa_{s,i})^2 \tag{6.5.2}$$

and

$$e(\mathbb{CP}^2, D_n) = 2n^2 - 3n + (1-n)\sum_{s,i}\kappa_{s,i} + \frac{1}{4}\sum_{\Gamma}(\kappa_{1,i} + \kappa_{2,j} + \kappa_{3,k} - 1)^2 + P(n), \tag{6.5.3}$$

where

$$P(n) = \begin{cases} 0 & n \geq 4 \text{ or } n = 1, \\ \sum_{s=1}^{3} \kappa_{s,0}\kappa_{s,1} & n = 2, \\ \frac{1}{4}\sum_{s=1}^{3}(\kappa_{s,0} + \kappa_{s,1} + \kappa_{s,2} - 1)^2 & n = 3. \end{cases} \tag{6.5.4}$$

Then,

$$\begin{aligned}(c_1^2 - 3e)(\mathbb{CP}^2, D_n) =& (3n^2 - 9n + 9) - 3(n-1)\sum_{s,i}\kappa_{s,i} + (\sum_{s,i}\kappa_{s,i})^2 \\ & - \frac{3}{4}\sum_{\Gamma}(\kappa_{1,i} + \kappa_{2,j} + \kappa_{3,k} - 1)^2 - 3P(n).\end{aligned} \tag{6.5.5}$$

In case $n = 2$, the Ceva(2) arrangement is just the complete quadrilateral and we have already studied its uniformization (See Theorem 6.5.2).

If $n = 3$, the equation (6.5.5) reduces to

$$\begin{aligned} c_1^2 - 3e =& (3 - \sum_{s,i}\kappa_{s,i})^2 - \frac{3}{4}\sum_{\Gamma}(\kappa_{1,i} + \kappa_{2,j} + \kappa_{3,j} - 1)^2 - \frac{3}{4}\sum_{s=1}^{3}(\kappa_{s,0} + \kappa_{s,1} + \kappa_{s,2} - 1)^2 \\ =& -2(\sum_{s,i}\kappa_{s,i})^2 + \frac{9}{2}\sum_{1 \leq s < r \leq 3}\ \sum_{0 \leq i < j \leq 2} \kappa_{s,i}\kappa_{r,j} \end{aligned}$$

Therefore, $c_1^2 - 3e$ vanishes if and only if $\kappa_{s,i} = \frac{1}{m}$, i.e. $m_{s,i} = m$ for all s,i. Local uniformizability condition at triple points is $\frac{3}{m} - 1 \leq 0$ which implies m is either 2 or 3, respectively the orbifold Chern numbers are $c_1^2 = \frac{9}{4}$, $e = \frac{3}{4}$ or $c_1^2 = 9$, $e = 3$.

Now assume $n = 4$. The Ceva(4) arrangement $\mathcal{A}$ has three fourfold points and sixteen triple points, and each line has a fourfold point. The uniformizability condition at fourfold points implies $m_{s,i} = 2$. Indeed, if one assume $\kappa_{s,i} = \kappa$ for all s,i, then by (6.5.5) he get $c_1^2 - 3e = 9(2\kappa - 1)^2 = 0$ while $\kappa = \frac{1}{2}$. In fact, in general $c_1^2 - 3e = 0$ has many solutions $m_{s,i}$, but the uniformizability condition is satisfied only when $m_{s,i} = 2$ for all s,i.

Thus, by the Theorem 6.4.2 we have the following theorem:

Theorem 6.5.6. *The orbifold* $(\mathbb{CP}^2, D_n)$*, where* $D_n = \sum_{s,i} m_{s,i} H_{s,i}$ *is a divisor on* $\mathbb{CP}^2$ *supported by the Ceva(n) arrangement, is uniformized by* $\mathbf{B}_2$ *if*

i. $n = 2$ *and* $(m_{s,0}, m_{s,1})$ *is either* $(2,3)$ *or* $(2,4)$ *or* $(3,3)$ *for all* $s = 1,2,3$.

ii. $n = 3$ *and* $m_{s,i}$ *is either* 2 *or* 3 *for all* s,i.

iii. $n = 4$ *and* $m_{s,i} = 2$ *for all* s,i.

Remark 6.5.7. Note that, Ceva(n) arrangement is the degree n branch cover of the complete quadrilateral via $\varphi_n : [X : Y : Z] \mapsto [X^n : Y^n : Z^n]$. In the Proposition 6.5.1, we have showed that the Miyaoka-Yau equality $c_1^2 = 3c_2$ is satisfied for an orbifold associated with the divisor based on complete quadrilateral with weights (n,n,n,m,m,m). Thus, the branching indices of the orbifold branched along the Ceva(n) arrangement will be m.

6.6 Orbifolds Supported by Quadric-Line Arrangements

Let $A_n := Q \cup \bigcup_{i=1}^{n} T_i$ be an arrangement of a smooth conic with n-distinct tangent lines of Q, which is known as *Apollonius configuration*. Since the tangent lines are in general position, the configuration space A_n can be identified with the configuration space M_n of n-distinct points in $\mathbb{CP}^1$, via the contact points of T_i with $Q \simeq \mathbb{CP}^1$.

Let $(\mathbb{CP}^2, \beta)$ be an orbifold associated with the divisor $D = aQ + \sum_{i=1}^{n} m_i T_i$ supported by the Apollonius configuration. The orbifold Chern numbers are

$$c_1^2(\mathbb{CP}^2, \beta) = \left(-1 + n - \frac{2}{a} - \sum_{i=1}^{n} \frac{1}{m_i}\right)^2, \tag{6.6.1}$$

and

$$e(\mathbb{CP}^2, \beta) = \frac{(n-1)(n-2)}{2} + \frac{2-n}{a} + \sum_{i=1}^{n} \frac{2-n}{m_i} + \sum_{1 \leq i < j \leq n} \frac{1}{m_i m_j} + \frac{1}{2} \sum_{i=1}^{n} (\frac{1}{a} + \frac{1}{m_i} - \frac{1}{2})^2. \tag{6.6.2}$$

In addition, local uniformizability conditions are $\frac{1}{a} + \frac{1}{m_i} \geq \frac{1}{2}$ for all $i = 1, 2, \cdots, n$.

Proposition 6.6.1 (Uludağ, 2004). *Let $(\mathbb{CP}^2, \beta)$ be an orbifold associated with the divisor $D := aQ + \sum_{i=1}^{n} m_i T_i$ supported by the Apollonius configuration. Then*

i. $3e(\mathbb{CP}^2, \beta) = c_1^2(\mathbb{CP}^2, \beta) > 0$ *if and only if* $n = 3$ *and* $(a; m_1, m_2, m_3)$ *is one of* $(4;4,4,4)$, $(3;3,4,4)$, $(3;6,6,2)$ *or* $(3;6,3,3)$,

ii. $2e(\mathbb{CP}^2, \beta) = c_1^2(\mathbb{CP}^2, \beta) > 0$ *if and only if either* $a = 2$ *and* $\rho \neq n-2$ *or* $n = 2$ *and* $\frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{a} = \frac{1}{2}$, *or* $n = 3$ *and* $(a; m_1, m_2, m_3) = (3;2,3,4)$, *or* $n = 4$ *and* $(a; m_1, m_2, m_3, m_4) = (a;2,2,2,2)$.

iii. $e(\mathbb{CP}^2, \beta) = c_1^2(\mathbb{CP}^2, \beta) = 0$ *if and only if either* $n = 2$, $(a; m_1, m_2) = (2; \infty, \infty)$, *or* $n = 3$ *and* $(a; m_1, m_2, m_3)$ *is one of* $(2;2,2,\infty)$, $(2;3,3,3)$, $(2;2,4,4)$ *or* $(2;2,3,6)$*; or* $n = 4$ *and* $(a; m_1, m_2, m_3, m_4) = (2;2,2,2,2)$,

iv. $e(\mathbb{CP}^2, \beta) > 0$ *and* $c_1^2(\mathbb{CP}^2, \beta) = 0$ *if and only if either* $n = 2$, $(a; m_1, m_2)$ *is* *one of* $(4;4,4)$, $(3;6,6)$, $(6;3,3)$ *or* $n = 3$ *and* $m_1 = m_2 = \infty$, *or* $n = 3$ *and* $(a;m_1,m_2,m_3)$ *is one of* $(4;2,2,2)$ *or* $(3;3,2,2)$.

Proof. For simplicity, let us set $\rho := \sum_{i=1}^{n} \frac{1}{m_i}$ and $\kappa = \frac{1}{a}$. Then the equations (6.6.1) and (6.6.2) reduces to

$$c_1^2(\mathbb{CP}^2,\beta) = (-\rho - 2\kappa - 1 + n)^2 = (n-1)^2 - 2(n-1)(\rho + 2\kappa) + \rho^2 + 4\kappa\rho + 4\kappa^2$$

and

$$e(\mathbb{CP}^2,\beta) = \frac{(n-1)(n-2)}{2} - (n-2)(\kappa+\rho) + \frac{\rho^2 + n\kappa^2 + 2\rho\kappa - \rho - n\kappa}{2} + \frac{n}{8}.$$

Therefore $c_1^2(\mathbb{CP}^2,\beta) = 0$ if and only if $\rho + 2\kappa = n-1$. Note that the equality $\rho + 2\kappa = n-1$ is valid if $n \leq 4$, since $m_i, a \geq 2$. If $n = 2$, then the solution $(a;m_1,m_2)$ to the equation $\frac{2}{a} + \frac{1}{m_1} + \frac{1}{m_2} = 1$ is one of $(\infty;2,2)$, $(12;2,3)$, $(8;2,4)$, $(6;3,3)$, $(4;4,4)$, $(4;3,6)$, $(4;2,\infty)$, $(3;3,3)$, $(3;6,6)$ or $(2;\infty,\infty)$. In case $n = 3$, $\frac{2}{a} + \sum_{i=1}^{3} \frac{1}{m_i} = 2$ and $(a;m_1,m_2,m_3)$ is one of $(4;2,2,2)$, $(3;2,2,3)$ or $(2;m_1,m_2,m_3)$ satisfying $\rho = 1$. If $n = 4$, then $\frac{2}{a} + \sum_{i=1}^{4} \frac{1}{m_i} = 3$ and therefore $(a;m_1,m_2,m_3,m_4) = (2;2,2,2,2)$. It can be easily showed that only the possibilities stated in the case iii., both of the orbifold Chern numbers vanish. For the possibilities stated in the case iv., first Chern number vanishes while Euler number is always positive.

Furthermore,

$$\begin{aligned}(2e - c_1^2)(\mathbb{CP}^2,\beta) =& 1 - \frac{3n}{4} + \rho + n\kappa + (n-4)\kappa^2 - 2\rho\kappa \\ =& (\kappa - \frac{1}{2})\big((n-4)\kappa + \frac{3n-4}{2} - 2\rho\big)\end{aligned}$$

and

$$\begin{aligned}(3e-c_1^2)(\mathbb{CP}^2,\beta) =&\frac{n^2}{2}-\frac{17n}{8}+2+2\kappa+\frac{\rho^2}{2}-n\rho+\frac{5\rho}{2}-\frac{n\kappa}{2}-\rho\kappa-4\kappa^2\\ =&\frac{1}{2}(n^2+\rho^2+n\kappa-\kappa^2-2\rho\kappa-n\rho)+\frac{5}{2}(\rho-\kappa-n)+\frac{25}{8}\\ &+\frac{3}{8}(4n\kappa^2-12\kappa^2-4n\kappa+12\kappa+n-3)\\ =&\frac{1}{2}(\rho-\kappa-n+\frac{5}{2})^2+\frac{3}{8}(n-3)(2\kappa-1)^2.\end{aligned}$$

Thus, $(2e-c_1^2)(\mathbb{CP}^2,\beta)=0$ if and only if $\kappa=\frac{1}{2}$ or $(n-4)\kappa+\frac{3n-4}{2}-2\rho=0$. If $a=2$, then clearly $c_1^2(\mathbb{CP}^2,\beta)=2e(\mathbb{CP}^2,\beta)=(\rho+2-n)^2$ and it vanishes for $\rho=n-2$. The condition $(n-4)\kappa+\frac{3n-4}{2}-2\rho=0$ is valid only for $2\le n\le 4$, since $a,m_i\in\mathbb{Z}_{\ge 2}$. If $n=2$, then $\kappa+\rho=\frac{1}{2}$ which has infinitely many solutions, and $c_1^2=2e=\rho^2$. If $n=3$, then we have the equation $\kappa+2\rho=\frac{5}{2}$ whose solution is $(a;m_1,m_2,m_3)=(3;2,3,4)$ and orbifold Chern numbers are $c_1^2=2e=\frac{1}{16}$. In addition, if $n=4$ then the condition $(n-4)\kappa+\frac{3n-4}{2}-2\rho=0$ reduces to $\rho=2$ which implies $m_1=m_2=m_3=m_4=2$ and the orbifold Chern numbers are $c_1^2=2e=(1-\frac{2}{a})^2$.

On the other hand, $(3e-c_1^2)(\mathbb{CP}^2,\beta)=0$ if and only if $\rho-\kappa-n+\frac{5}{2}=0$ while either $n=3$ or $a=2$. Note that, if $a=2$ then the condition $\rho-\kappa-n+\frac{5}{2}=0$ reduces to $\rho=n-2$ which implies $2\le n\le 4$. But, the orbifold Chern numbers c_1^2 and e vanish. Now suppose $n=3$, then the condition $\rho-\kappa-n+\frac{5}{2}=0$ reduces to $\rho=\kappa+\frac{1}{2}$ for which a solution $(a;m_1,m_2,m_3)$ is one of $(4;4,4,4)$, $(3;3,4,4)$, $(3;6,6,2)$ or $(3;6,3,3)$. $\square$

Lemma 6.6.2 (Holzapfel & Vladov, 2001)**.** *There is an orbifold covering* $\big(\mathbb{CP}^1\times\mathbb{CP}^1,\ aQ+\sum_{i=1}^k m_i(T_i^v+T_i^h)\big)\to\big(\mathbb{CP}^2,\ 2aQ+\sum_{i=1}^k m_iT_i\big)$, *where* $T_i^v=\{p_i\}\times Q$, $T_i^h=Q\times\{p_i\}$ *and the covering map is* $\big([p:q],[r:s]\big)\to[ps+qr:qs:pr]$ *(See Figure 6.20).*

Proof. Consider the $\mathbb{Z}_2$-action defined by $(x,y)\in\mathbb{CP}^1\to(y,x)\in\mathbb{CP}^1$. The diagonal $Q=\{(x,x):x\in\mathbb{CP}^1\}$ is fixed under this action. Let $x=[p:q],y=[r:s]$, then the

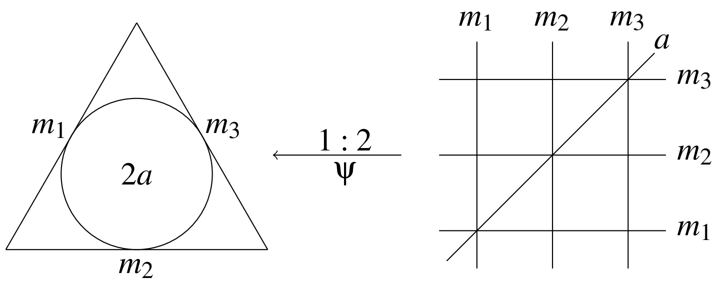


Figure 6.20 A covering of Apollonius configuration.

symmetric polynomials $\sigma_1(x,y) := ps+qr$, $\sigma_1(x,y) := qs$ and $\sigma_1(x,y) := pr$ are also invariant under this $\mathbb{Z}_2$-action. Consider the Viéte map

$$\psi : ([p:q],[r:s]) \in \mathbb{CP}^1 \times \mathbb{CP}^1 \to [ps+qr : qs : pr] \in \mathbb{CP}^2. \tag{6.6.3}$$

It is a branched covering of degree 2. The branching locus can be found as the image of the diagonal Q. Since $\psi|_Q$ is one-to-one, so we will denote $\psi(Q)$ by Q again. One has $\psi(Q) = \{[2pq : q^2 : p^2] \mid [p:q] \in \mathbb{CP}^1\}$, so that Q can be given by the equation $X^2 - 4YZ = 0$. One can identify $Q \times Q$ with $\mathbb{CP}^1 \times \mathbb{CP}^1$ via projections of diagonal. Let $P \in Q$, and put $T_P^h := Q \times \{P\}$ and $T_P^v = \{P\} \times Q$. Then $T_P := \psi(T_P^h) = \psi(T_p^v) = \{[rq+sp : sq : rp] \mid [r:s] \in \mathbb{CP}^1\} \subset \mathbb{CP}^2$ is the line $q^2Z + p^2Y - pqX = 0$ tangent to Q at the point $[2pq : q^2 : p^2]$. □

Remark 6.6.3. Consider the divisor $D = 2aQ + \sum_{i=1}^k m_i T_i$. Since, distinct tangent lines of a quadric meet transversally, then at these singular points local orbifold fundamental group is abelian and local uniformization always exist. In addition, at tangency points, orbifold germs have uniformization if and only if $\frac{1}{a} + \frac{1}{m_i} \geq \frac{1}{2}$ for each $i = 1,2,\cdots,k$. Therefore $(\mathbb{CP}^2, D)$ is an orbifold provided $\frac{1}{a} + \frac{1}{m_i} \geq \frac{1}{2}$ for each $i = 1,2,\cdots,k$.

Theorem 6.6.4. *The orbifolds in Figure 6.21 are uniformized by* $\mathbb{CP}^1 \times \mathbb{CP}^1$.

Proof. Consider the particular case of Lemma 6.6.2. If $a = 1$, then there is an orbifold covering

$$(\mathbb{CP}^1, \sum_{i=0}^k m_i p_i) \times (\mathbb{CP}^1, \sum_{i=0}^k m_i p_i) \to (\mathbb{CP}^2, 2Q + \sum_{i=0}^k m_i T_i). \tag{6.6.4}$$

So, by Theorem 6.1.3 the covering orbifold is uniformized by $\mathbb{CP}^1 \times \mathbb{CP}^1$ if $k = 1$ and $m_0 = m_1$, or $k = 2$ and $1/m_0 + 1/m_2 + 1/m_3 > 1$. Hence the orbifolds in Figure 6.21 are uniformized by $\mathbb{CP}^1 \times \mathbb{CP}^1$. It is also clear from the Proposition 6.6.1 that, these orbifolds satisfy the Hirzebruch's second proportionality theorem, i.e, $c_1^2 = 2e$. □

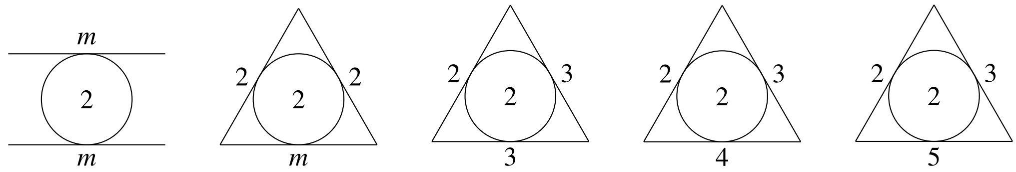


Figure 6.21 Orbifolds uniformized by $\mathbb{CP}^1 \times \mathbb{CP}^1$

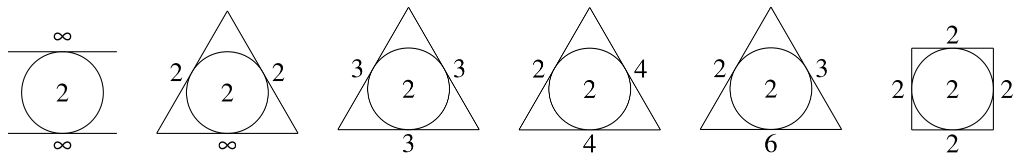


Figure 6.22 Orbifolds uniformized by $\mathbb{C} \times \mathbb{C}$.

Theorem 6.6.5. *The orbifolds in Figure 6.22 are uniformized by* $\mathbb{C} \times \mathbb{C}$.

Proof. Consider the covering given by (6.6.4). Then by the Theorem 6.1.3, the orbifold $(\mathbb{CP}^2, D)$ branched along Apollonious configuration is uniformized by $\mathbb{C} \times \mathbb{C}$ if $n = 2$ and $m_1 = m_1 = \infty$, or $n = 3$ and $1/m_1 + 1/m_2 + 1/m_3 = 1$, or $n = 4$ and $m_1 = m_2 = m_3 = m_4 = 2$. Note that by the Proposition 6.6.1, both of the orbifold Chern numbers vanish. □

Theorem 6.6.6. *The orbifolds in Figure 6.23 are uniformized by* $\mathbf{B}_2$.

Proof. Proof follows from the Theorem 6.4.2 and Proposition 6.6.1.i. □

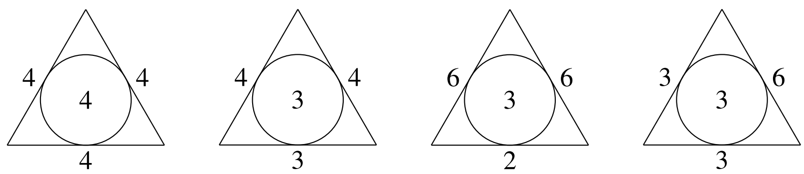


Figure 6.23 Orbifolds uniformized by $\mathbf{B}_2$

Lemma 6.6.7. *Let Q be a quadric in $\mathbb{CP}^2$ and T_1, T_2 and T_3 are its distinct tangent lines. Then the Apollonious configuration $\mathcal{A}_3 = Q \cup T_1 \cup T_2 \cup T_3$ is given by the equation*

$$XYZ[(X+Y-Z)^2 - 4XY] = 0 \tag{6.6.5}$$

up to projective transformations.

Proof. Since $\dim \mathrm{PGL}(3,\mathbb{C}) = 8$, we can choose homogeneous coordinates such that $T_1 = \{X = 0\}$, $T_2 = \{Y = 0\}$, $T_3 = \{Z = 0\}$ Suppose the quadric Q is given by the equation $F := aX^2 + bY^2 + Cz^2 + 2dXY + 2eYZ + 2fZX = 0$. For a given subgroup $\Sigma_3 < \mathrm{PGL}(3,\mathbb{C})$, isomorphic to the symmetric group S_3, the action of Σ_3 just permutes the coordinates. Thus, the Σ_3-invariant quadrics must satisfy simultaneously three equations

$$\begin{aligned} aY^2 + bX^2 + cZ^2 + 2dXY + 2eYZ + 2fZX &= 0 \\ aZ^2 + bY^2 + cX^2 + 2dYZ + 2eXZ + 2fXY &= 0 \\ aX^2 + bZ^2 + cY^2 + 2dXZ + 2eXY + 2fYZ &= 0, \end{aligned}$$

which have to be the same up to a factor. It follows that, $a = b = c = 1$ (without loss of generality) and $d = e = f = \lambda \in \mathbb{C}^*$. Therefore, Σ_3-invariant quadrics form a 1-parameter family $X^2 + Y^2 + Z^2 + 2\lambda XY + 2\lambda YZ + 2\lambda ZX = 0$. On the other hand, Q has contact of order 2 with $T_1 = \{X = 0\}$ at a point $[0 : 1 : t]$. Substituting the coordinates of this point in the quadric equation, we must have a unique solution for t of the equation $t^2 + 2\lambda t + 1 = 0$. Therefore, either $\lambda = 1$ or $\lambda = -1$, but Q is degenerate for $\lambda = 1$. Hence one gets a symmetric equation for the non-degenerate quadric Q as $X^2 + Y^2 + Z^2 - 2XY - 2YZ - 2ZX = (X+Y-Z)^2 - 4XY = 0$. $\square$

Now, let us add new lines to Apollonius configuration to discover new orbifolds uniformized by complex 2-ball $\mathbf{B}_2$. Consider the orbifold $(\mathbb{CP}^2, \beta)$ associated with the divisor $D = aQ + \sum_{i=1}^{3} m_i T_i + m_4 H_4$ supported by the arrangement in Figure 6.24 given by the homogeneous equation $XYZ(X-Z)[(X+Y-Z)^2 - 4XY] = 0$.

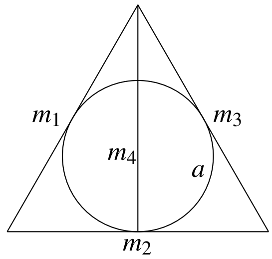


Figure 6.24

The orbifold Chern numbers are

$$c_1^2(\mathbb{CP}^2,\beta) = (3-\sum_{i=1}^{4}\kappa_i - 2\sigma)^2 = (3-\rho-2\sigma)^2 = 9-6(\rho+2\sigma)+(\rho+2\sigma)^2$$

and

$$\begin{aligned} e(\mathbb{CP}^2,\beta) =& 2-\sum_{i=1}^{4}\kappa_i - 2\sigma+\sigma\kappa_4+(\kappa_1+\kappa_3)\kappa_2+\frac{1}{4}(\kappa_1+\kappa_3+\kappa_4-1)^2 \\ &+\frac{1}{2}(\kappa_1+\sigma-\frac{1}{2})^2+\frac{1}{2}(\kappa_3+\sigma-\frac{1}{2})^2+\frac{1}{2}(\kappa_2+\sigma+\frac{\kappa_4}{2}-1)^2 \\ =& 3-2(\rho+2\sigma)+\frac{1}{4}(\rho+2\sigma)^2+\frac{1}{8}(2\sigma+\kappa_4)^2+\frac{1}{8}(2\kappa_1+\kappa_2)^2 \\ &+\frac{1}{8}(\kappa_2+2\kappa_3)^2 \end{aligned}$$

where $\kappa_i = \frac{1}{m_i}$, $\rho = \sum_{i=0}^{3}\kappa_i$ and $\sigma = \frac{1}{a}$. Therefore,

$$\begin{aligned} (3e-c_1^2)(\mathbb{CP}^2,\beta) =& -\frac{1}{4}(\rho+2\sigma)^2+\frac{3}{8}(2\sigma+\kappa_4)^2+\frac{3}{8}(2\kappa_1+\kappa_2)^2 \\ &+\frac{3}{8}(\kappa_2+2\kappa_3)^2 \end{aligned}$$

To find a solution to $3e-c_1^2=0$, set $a := \frac{1}{2}(2\kappa_1+\kappa_2)$, $b := \frac{1}{2}(2\kappa_3+\kappa_2)$, $c := \frac{1}{2}(2\sigma+\kappa_4)$, then clearly $a+b+c=\rho+2\sigma$ and $f(a,b,c) := 3e-c_1^2 = -\frac{1}{4}(a+b+c)+\frac{3}{2}a^2+$ $+\frac{3}{2}b^2+\frac{3}{8}c^2 \geq 0$. The function $f(a,b,c)$ takes its minimum value, 0, on the line $c=4a=4b$. The equation $a=b$ clearly implies $\kappa_1=\kappa_3$. In addition, the equation $c=4a$ implies $(a;m_1,m_2,m_3,m_4)$ is either $(p;4q,p,4q,q)$ or $(p,2p,2q,2p,q)$ for some $p,q\in\mathbb{Z}_{\geq 2}$.

At nodal points, the local orbifold fundamental group is abelian and it always admits a local uniformization at these points. At triple points and tangency points, there are local uniformizations if the local orbifold fundamental groups at these points are either finite or infinite solvable. The local uniformizability condition at these points are $\kappa_1+\kappa_3+\kappa_4 \geq 1$, $\kappa_1+\sigma \geq \frac{1}{2}$, $\kappa_3+\sigma \geq \frac{1}{2}$ and $\kappa_2+\sigma+\frac{\kappa_4}{2} \geq 1$. Checking these conditions for the quintuplets $(p;4q,p,4q,q)$ or $(p,2p,2q,2p,q)$ and taking into account the fact $p,q \in \mathbb{Z}_{\geq 2}$, we obtained the branching indices $(a;m_1,m_2,m_3,m_4)$ as $(2;4,4,4,2)$. Moreover, the first and second Chern numbers are $\frac{9}{16}$ and $\frac{3}{16}$, respectively. Notice that, $\kappa_1+\kappa_3+\kappa_4 = 1$ and $\kappa_2+\sigma+\frac{\kappa_4}{2} = 1$. These means, there are ball-cusp points at $T_1 \cap T_3 \cap H_4$ and $Q \cap T_2 \cap H_4$. As a result, we can state the following theorem:

Theorem 6.6.8. *Let $(\mathbb{CP}^2,\beta)$ be an orbifold associated with the divisor $D = 2Q + \sum_{i=1}^{3} 4T_i + 2H_4$ supported by the arrangement in Figure 6.24. Then, it is uniformized by $\mathbf{B}_2$.*

Third, let us add another tangent line to the quadric-line configuration whose uniformizability discussed above. Consider the orbifold $(\mathbb{CP}^2,\beta)$ associated with the divisor $D = aQ + \sum_{i=1}^{4} m_iT_i + m_5H_5$ supported by the arrangement in Figure6.25 given by the equation $XYZ(X-Z)(2X-Y+2Z)[(X+Y-Z)^2-4XY] = 0$. Its orbifold Chern numbers are

$$c_1^2(\mathbb{CP}^2,\beta) = (4-\sum_{i=1}^{5}\kappa_i - 2\sigma)^2$$

and

$$\begin{aligned} e(\mathbb{CP}^2,\beta) =& 2-(\kappa_1+\kappa_3+\kappa_5)-2(\kappa_2+\kappa_4+\sigma)+(\kappa_1+\kappa_3)(\kappa_2+\kappa_4)+\kappa_2\kappa_4 \\ &+\frac{1}{4}(\kappa_1+\kappa_3+\kappa_5-1)^2+\frac{1}{2}(\sigma+\kappa_1-\frac{1}{2})^2+\frac{1}{2}(\sigma+\kappa_3-\frac{1}{2})^2+ \\ &+\frac{1}{2}(\sigma+\kappa_2+\frac{\kappa_5}{2}-1)^2+\frac{1}{2}(\sigma+\kappa_4+\frac{\kappa_5}{2}-1)^2, \end{aligned}$$

where $\kappa_i = \frac{1}{m_i}$ and $\sigma = \frac{1}{a}$. Local orbifold fundamental groups at nodal points are abelian and admits local uniformization. Local uniformizability condition at triple and tangency points is related with the order of local orbifold fundamental group

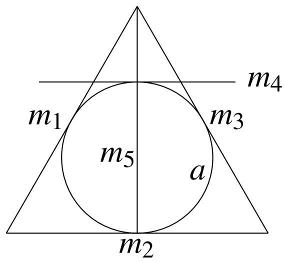


Figure 6.25

π_1^{orb}, and π_1^{orb} must be finite or at most infinite solvable. These correspond to the inequalities $\kappa_1+\kappa_3+\kappa_5 \geq 1$, $\kappa_i+\sigma \geq \frac{1}{2}$ and $\kappa_j+\sigma+\frac{\kappa_5}{2} \geq 1$, where $i=1,3$ and $j=2,4$. Equalities are valid if the orbifold has cusp points. The conditions $\kappa_j+\sigma+\frac{\kappa_5}{2} \geq 1$, $j=2,4$ tell us that a is either 2, 3 or 4.

First suppose $a=4$, then $\sigma=\frac{1}{4}$ and the inequality $\kappa_j+\sigma+\frac{\kappa_5}{2} \geq 1$ reduces to $\kappa_j+\frac{\kappa_5}{2} \geq \frac{3}{4}$, $j=2,4$, which imply $m_2=m_4=m_5=2$. In addition, the inequality $\kappa_i+\sigma \geq \frac{1}{2}$, $i=1,3$ implies that $m_1, m_3 \leq 4$. Under these conditions, the inequality $\kappa_1+\kappa_3+\kappa_5 \geq 1$ is automatically satisfied.

Next, assume that $a=3$, then $\sigma=\frac{1}{3}$ and the inequality $\kappa_j+\sigma+\frac{\kappa_5}{2} \geq 1$ reduces to $\kappa_j+\frac{\kappa_5}{2} \geq \frac{2}{3}$, $j=2,4$, which implies $m_2=m_4=2$ and m_5 is either 2 or 3. In addition, the conditions $\kappa_i+\sigma \geq \frac{1}{2}$, $i=1,3$ gives $m_1, m_3 \leq 6$. Under these conditions and depending on the choices of m_5, the inequality $\kappa_1+\kappa_3+\kappa_5 \geq 1$ has finite number of solutions.

Now suppose $a=2$, then $\kappa_j+\frac{\kappa_5}{2} \geq \frac{1}{2}$, $j=2,4$ and therefore (m_2,m_4,m_5) is one of $(2,2,k)$, $(2,3,2)$, $(2,3,3)$, $(2,4,2)$, $(3,2,2)$, $(3,3,2)$, $(3,4,2)$, $(3,2,3)$, $(3,3,3)$, $(4,2,2)$, $(4,3,2)$ and $(4,4,2)$. Beside these, the inequalities $\kappa_i+\sigma \geq \frac{1}{2}$, $i=1,3$ is always true. Depending on choice of (m_2,m_4,m_5) the inequality $\kappa_1+\kappa_3+\kappa_5 \geq 1$ has finite number of solutions.

By taking into account these restrictions on branching indices and using Maple we have obtained that $(3e-c_1^2)$ vanishes if $(a;m_1,m_2,m_3,m_4;m_5)$ is $(4;4,2,4,2;2)$ and first and second orbifold Chern numbers are $\frac{9}{4}$ and $\frac{3}{4}$, respectively. Note that,

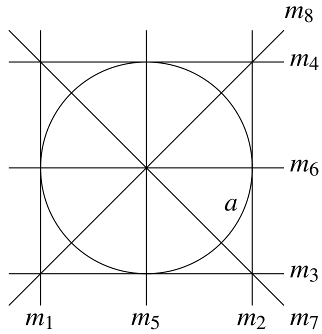


Figure 6.26

for such a and m_i's, $\kappa_1+\kappa_3+\kappa_5=1$, $\kappa_i+\sigma=\frac{1}{2}$ and $\kappa_j+\sigma+\frac{\kappa_5}{2}=1$, where $i=1,3$ and $j=2,4$. This means, at all multiple points except the nodal ones local orbifold fundamental groups are infinite solvable and these points are cusp points. In addition, this orbifold is an orbifold of general type. Then by Theorem 6.4.2, we can state the following theorem:

Theorem 6.6.9. *The orbifold* $(\mathbb{CP}^2,\beta)$ *associated with the divisor* $D=4Q+4T_1+2T_2+4T_3+2T_4+2H_5$ *supported by the arrangement in Figure 6.25 is uniformized by* $\mathbf{B}_2$.

Fourth, consider the orbifold $(\mathbb{CP}^2,\beta)$ associated with the divisor $D=aQ+\sum_{i=1}^{4}m_iT_i+\sum_{i=5}^{8}m_iH_i$ supported by the rigid arrangement in Figure 6.26 defined by the equation $XY(X-Y)(X+Y)(Y-Z)(Y+Z)(Z-X)(Z+X)(X^2+Y^2-Z^2)=0$. Its orbifold Chern numbers are

$$c_1^2(\mathbb{CP}^2,\beta)=(7-\sum_{i=1}^{8}\kappa_i-2\sigma)^2$$

and

$$\begin{aligned} e(\mathbb{CP}^2,\beta)=&12-2\sum_{i=1}^{6}\kappa_i-6\sigma+(2\sigma-3)(\kappa_7+\kappa_8)+\frac{1}{2}(\delta_{1,6}^2+\delta_{2,6}^2+\delta_{3,5}^2+\delta_{4,5}^2)\\ &+\frac{1}{4}(\rho_{1,2,5}^2+\rho_{1,3,8}^2+\rho_{1,4,7}^2+\rho_{2,3,7}^2+\rho_{2,4,8}^2+\rho_{3,4,6}^2),\end{aligned}$$

where $\kappa_i=\frac{1}{m_i}$, $\sigma=\frac{1}{a}$, $\rho_{i,j,k}=\kappa_i+\kappa_j+\kappa_k-1$ and $\delta_{r,s}=\sigma+\kappa_r+\frac{\kappa_s}{2}-1$.

Since $7-\sum_{i=1}^{8}\kappa_i-2\sigma>0$ for any $m_i,a\in\mathbb{Z}_{\geq 2}$, this orbifold is of general type. Note that, there is a fourfold point lies on the lines H_5, H_6, H_7 and H_8. The local orbifold fundamental group is infinite solvable if $m_5=m_6=m_7=m_8=2$ and big otherwise. Therefore, the weights $m_5=m_6=m_7=m_8=2$ admits the local uniformization at $H_5\cap H_6\cap H_7\cap H_8$. The other local uniformizability conditions are $\rho_{i,j,k}\geq 0$ and $\delta_{r,s}\geq 0$, where $(i,j,k)\in\{(1,2,5),(1,3,8),(1,4,7),(2,3,7),(2,4,8)\}$ and $(r,s)\in\{(1,6),(2,6),(3,5),(4,5)\}$. For any r in $\{1,2,3,4\}$, the conditions $\delta_{r,5}\geq 1$ and $\delta_{r,6}$ implies the inequality $\frac{1}{a}+\frac{1}{m_r}\geq\frac{3}{4}$, which is valid if either $a=4$ and $m_r=2$, or $a=3$ and $m_r=2$, or $a=2$ and $m_r\leq 4$. Notice that, in all cases the inequality $\rho_{i,j,k}=\frac{1}{m_i}+\frac{1}{m_j}+\frac{1}{2}\geq 1$ is satisfied, where $i,j\in\{1,2,3,4\}$ and $i\neq j$. Thus, we have candidates in the form of $(a;m_1,m_2,m_3,m_4;2,2,2,2)$, where either $a=4$ and $m_r=2$, or $a=3$ and $m_r=2$, or $a=2$ and $m_r\leq 4$. By taking into account these restrictions on branching indices and using Maple, we have obtained that the Miyaoka-Yau equality $(c_1^2-3e)(\mathbb{CP}^2,\beta)=0$ is satisfied if $(a;m_1,m_2,m_3,m_4;m_5,m_6,m_7,m_8)$ is $(2;4,4,4,4;2,2,2,2)$, and its first and second orbifold Chern numbers are 9 and 3, respectively. Notice that, all multiple points except the nodal ones are cusp points. Since this orbifold is of general type, then by Theorem 6.4.2, we can state the following theorem:

Theorem 6.6.10. *An orbifold $(\mathbb{CP}^2,\beta)$ associated with the divisor $D=2Q+\sum_{i=1}^{4}4T_i+\sum_{i=5}^{8}2H_i$ supported by the arrangement in Figure 6.26 is uniformized by $\mathbf{B}_2$.*

Fifth, consider the arrangement of a quadric Q and its four tangents T_i, $i=1,2,3,4$. Let H_5 be the line through $T_1\cap T_2$, $T_3\cap T_4$, and H_6 be the line through $Q\cap T_3$, $Q\cap T_4$. The line H_5 meets Q transversally. Arrangement of such quadric and lines are projectively rigid, and equations are $Q:=\{X^2-Y^2-Z^2=0\}$, $T_1=\{X+Z=0\}$, $T_2=\{X-Z=0\}$, $T_3=\{X+Y=0\}$, $T_4=\{X-Y=0\}$, $H_5=\{X=0\}$ and $H_6=\{Y=0\}$ (See Figure Figure 6.27). Consider the orbifold $(\mathbb{CP}^2,\beta)$ associated with the divisor $D=aQ+\sum_{i=1}^{6}m_iH_i$ supported by this arrangement. Its orbifold Chern numbers are

$$c_1^2(\mathbb{CP}^2,\beta)=(5-2\sigma-\sum_{i=1}^{6}\kappa_i)^2$$

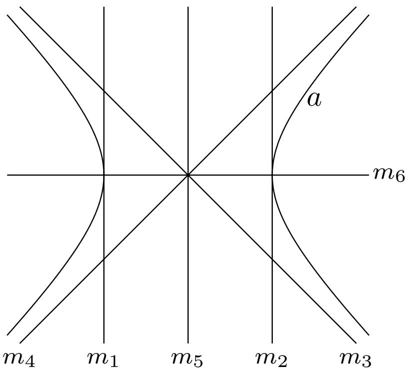


Figure 6.27

and

$$\begin{aligned} e(\mathbb{CP}^2,\beta) =& 6-\sigma-2\sum_{i=1}^{6}\kappa_i-\kappa_6+2\sigma\kappa_5+(\kappa_1+\kappa_2)(\kappa_3+\kappa_4) \\ &+\frac{1}{4}(\kappa_1+\kappa_2+\kappa_5-1)^2+\frac{1}{2}(\sigma+\kappa_1+\frac{1}{2}\kappa_6-1)^2 \\ &+\frac{1}{2}(\sigma+\kappa_2+\frac{1}{2}\kappa_6-1)^2+\frac{1}{2}(\sigma+\kappa_3-\frac{1}{2})^2+\frac{1}{2}(\sigma+\kappa_4-\frac{1}{2})^2, \end{aligned}$$

where $\sigma=\frac{1}{a}$ and $\kappa_i=\frac{1}{m_i}$. Notice that, there is a fourfold point. The local orbifold fundamental group at this point is infinite solvable if $m_3=m_4=m_5=m_6=2$, otherwise it is big. Such choice guarantees the local uniformization at tangency points $T_3\cap Q$ and $T_4\cap Q$. At nodal points, local orbifold fundamental group is abelian and local uniformization always exist at these points. For triple points, the local uniformizability conditions are $\kappa_1+\kappa_2\geq\frac{1}{2}$, $\kappa_1+\sigma\geq\frac{3}{4}$ and $\kappa_2+\sigma\geq\frac{3}{4}$. Therefore, this orbifold is locally uniformizable if (a,m_1,m_2) is one of the triples $(2,2,2)$, $(2,2,3)$, $(2,2,4)$, $(2,3,2)$, $(2,3,3)$, $(2,3,4)$, $(2,4,2)$, $(2,4,3)$, $(2,4,4)$, $(3,2,2)$ and $(4,2,2)$, while $m_3=m_4=m_5=m_6=2$. Taking into account this restrictions on branching indices and using Maple, we have obtained the Miyaoka-Yau equality $c_1^2(\mathbb{CP}^2,\beta)=3e(\mathbb{CP}^2,\beta)=9/4$. Since this orbifold is of general type, then by Theorem 6.4.2, we can state the following theorem.

Theorem 6.6.11. *Let $(\mathbb{CP}^2,\beta)$ be an orbifold associated with the divisor $D=2Q+\sum_{i=1}^{4}2T_i+\sum_{i=5}^{10}2H_i$ supported by the arrangement in Figure 6.27 is uniformized by* $\mathbf{B}_2$.

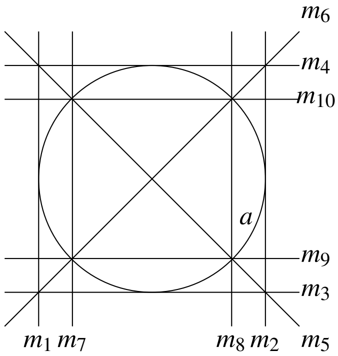


Figure 6.28

Sixth, consider the orbifold $(\mathbb{CP}^2,\beta)$ associated with divisor $D = aQ + \sum_{i=1}^{4} m_i T_i + \sum_{i=5}^{10} m_i H_i$ supported by the arrangement in Figure 6.28. Its orbifold Chern numbers are

$$c_1^2(\mathbb{CP}^2,\beta) = (9 - \sum_{i=1}^{10} \kappa_i - 2\sigma)^2$$

and

$$\begin{aligned} e(\mathbb{CP}^2,\beta) =& 20 - 4\sum_{i=1}^{10} \kappa_i + (\kappa_5+\kappa_6) - 6\sigma + (\kappa_1+\kappa_2)(\kappa_9+\kappa_{10}) \\ &+ (\kappa_3+\kappa_4)(\kappa_7+\kappa_8) + \frac{1}{2}(\eta_1^2+\eta_2^2+\eta_3^2+\eta_4^2) \\ &+ \frac{1}{4}(\rho_{1,3,6}^2+\rho_{1,4,5}^2+\rho_{2,3,5}^2+\rho_{2,4,6}^2), \end{aligned}$$

where $\kappa_i = \frac{1}{m_i}$, $\sigma = \frac{1}{a}$, $\rho_{i,j,k} = \kappa_i + \kappa_j + \kappa_k - 1$ and $\eta_r = \sigma + \kappa_r - \frac{1}{2}$. Notice that, there are four four-fold points and β map takes infinite values at these points. This means, the local orbifold fundamental groups at these points are infinite. Therefore, the local uniformizability at these points corresponds to solvability of local orbifold fundamental groups, which is possible if branching indices are 2, otherwise it will be too big. Then we may assume $a = m_5 = m_6 = m_7 = m_8 = m_9 = m_{10} = 2$. In this case, note that $\eta_r \geq 0$ for any $r \in \{1,2,3,4\}$ and therefore orbifold germs through tangency points are always locally uniformizable. In addition, the uniformizability conditions at triple points are $\rho_{1,3,6} \geq 0$, $\rho_{1,4,5} \geq 0$, $\rho_{2,3,5} \geq 0$ and $\rho_{2,4,6} \geq 0$ and they give us the relation $\frac{1}{m_i} + \frac{1}{m_j} \geq \frac{1}{2}$, where $(i,j) \in \{(1,3),(1,4),(2,3),(2,4)\}$. This is possible if for such (i,j), (m_i,m_j) is one of $(2,k)$, $(3,3)$, $(3,4)$, $(3,5)$, $(3,6)$,

$(4,3)$, $(4,4)$, $(5,3)$, $(6,3)$ and $(k,2)$, where $k \in \mathbb{Z}_{\geq 2}$. By taking into account these restrictions on branching indices and using Maple, we have obtained the Miyaoka-Yau equality $(3e - c_1^2)(\mathbb{CP}^2, \beta) = 0$ if all weights are 2. In this case, the first and second orbifold Chern numbers are 9 and 3, respectively. Since this orbifold is of general type, then by Theorem 6.4.2, we can state the following theorem:

Theorem 6.6.12. *Let $(\mathbb{CP}^2, \beta)$ be an orbifold associated with the divisor $D = 2Q + \sum_{i=1}^{4} 2T_i + \sum_{i=5}^{10} 2H_i$ supported by the arrangement in Figure 6.28 is uniformized by* $\mathbf{B}_2$.

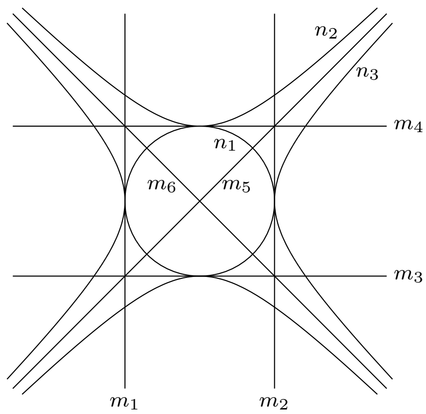


Figure 6.29

Seventh, consider the orbifold $(\mathbb{CP}^2, \beta)$ associated with the divisor $D = \sum_{j=1}^{3} n_j Q_j + \sum_{i=1}^{6} m_i H_i$ supported by the arrangement of three quadrics with six tacnodes and their pairwise six common tangents (See Figure 6.29). We know from the equation (4.3.30) that equations of three quadrics with six tacnodes is projectively equivalent to $(X^2+Y^2-Z^2)(\frac{1}{q^2}X^2+Y^2-Z^2)(X^2+Y^2-q^2Z^2) = 0$ and their pairwise common tangents are given by $(X-iY)(X+iY)(Y-Z)(Y+Z)(X+iqZ)(X-iqZ) = 0$. These six lines forms a complete quadrilateral if and only if $q^2 = -1$. Thus, considering fact $q^2 = -1$ and using the projective transformation $[X : Y : Z] \to [iX : Y : Z]$ one obtains the equation

$$(X^2-Y^2)(Y^2-Z^2)(Z^2-X^2)(X^2+Y^2-Z^2)(X^2-Y^2+Z^2)(-X^2+Y^2+Z^2) = 0$$

for the arrangement in Figure 6.29. In the Section 6.2.3 we have discussed the covering relations among orbifold germs and their uniformizations. A uniformizable germ consisting of two conics having a contact of order 2 and their common tangent line appeared as cover of four lines with branching indices 2 via $\varphi_{1,2}$ or $\varphi_{2,1}$ (See Figures 6.10 and 6.12). Therefore, such germs are uniformizable if the branching indices are 2. In this case, the β map takes infinite values and cusp points appears in covers of these points. Moreover, such choice of branching indices guarantees the local uniformization at triple points and nodal points. Omitting this fact, let us first compute its orbifold Chern numbers in terms of branching indices m_i and n_j. Orbifold Chern numbers of $(\mathbb{CP}^2, \beta)$ are

$$c_1^2(\mathbb{CP}^2, \beta) = (9 - \sum_{i=1}^{6} \kappa_i - 2\sum_{j=1}^{3} \sigma_j)^2$$

and

$$\begin{aligned} e(\mathbb{CP}^2, \beta) =& 20 - 4\sum_{i=1}^{6} \kappa_i - 6\sum_{j=1}^{3} \sigma_j + \kappa_1\kappa_2 + \kappa_3\kappa_4 + \kappa_5\kappa_6 + 2(\kappa_1 + \kappa_2)\sigma_2 \\ &+ 2(\kappa_3 + \kappa_4)\sigma_3 + 2(\kappa_5 + \kappa_6)\sigma_1 + \frac{1}{4}(\rho_{1,3,5}^2 + \rho_{1,4,6}^2 + \rho_{2,3,6}^2 + \rho_{2,4,5}^2), \end{aligned}$$

where $\kappa_i = \frac{1}{m_i}$, $\sigma_j = \frac{1}{n_j}$, $\rho_{i,j,k} = \kappa_i + \kappa_j + \kappa_k - 1$. Incase $m_i = n_j = 2$ for all i, j, then first and second orbifold Chern numbers are $c_1^2 = 9$ and $e = 3$, respectively. In addition, this orbifold is of general type. Therefore, as a consequence of the Theorem 6.4.2, we can state the following theorem:

Theorem 6.6.13. *The orbifold $(\mathbb{CP}^2, \beta)$ associated with the divisor $D = \sum_{j=1}^{3} 2Q_j + \sum_{i=1}^{6} 2H_i$ supported by the arrangement in Figure 6.29 is uniformized by $\mathbf{B}_2$.*

Eighth, consider an arrangement of three quadrics Q_j, such that the quadric Q_3 has a contact of order four with Q_1 and Q_2 while Q_1 and Q_2 has a tacnode. From the Proposition 4.3.6, we know that such quadrics are

$$Q_1 : Y^2 + Z^2 - 2XY = 0, \quad Q_2 : Y^2 + Z^2 + 2XY = 0, \quad Q_3 : 4X^2 - Y^2 - 2Z^2 = 0. \tag{6.6.6}$$

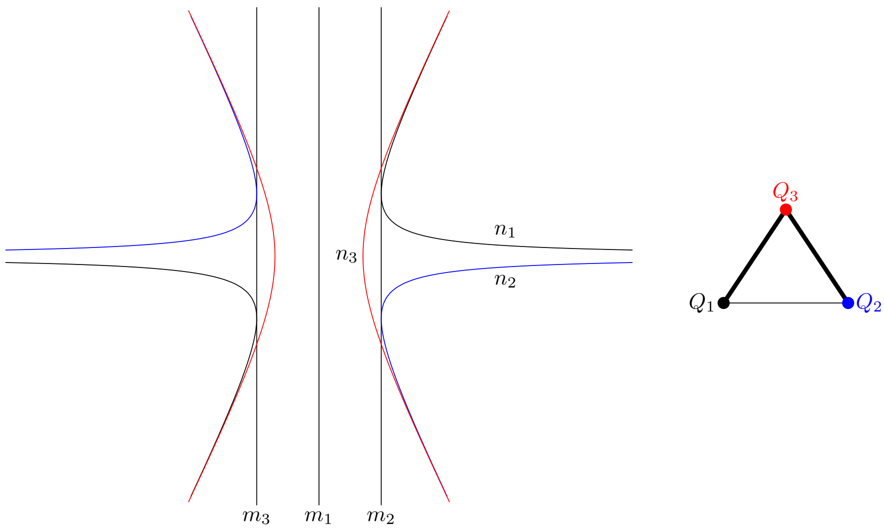


Figure 6.30 An orbifold $(\mathbb{CP}^2, \sum_{j=1}^{3} n_j Q_j + \sum_{i=1}^{3} m_i H_i)$.

Let H_1 be the the line through the nodal intersection points of Q_1 and Q_2, that is $H_1 : X = 0$. Let $H_2 : X + Z = 0$ and $H_3 : X - Z = 0$. They are common tangent lines of Q_1 and Q_2 at the points $[1 : 1 : 1]$ and $[-1 : 1 : 1]$, respectively. Also, the point $H_2 \cap H_3 = \{[\frac{1-\alpha}{4} : 1 : 0]\}$ lies on H_1. In addition, the line $H_4 : Y = 0$ is tangent to both of Q_1 and Q_2 at $[1 : 0 : 0]$. Configuration of these quadrics and lines are projectively rigid.

Consider the orbifold $(\mathbb{CP}^2, \beta)$ associated with the divisor $D = \sum_{j=1}^{3} n_j Q_j + \sum_{i=1}^{3} m_i H_i$ supported by the arrangement in Figure 6.30, where equations of quadrics Q_j and lines H_i are stated above. Its orbifold Chern numbers are

$$c_1^2(\mathbb{CP}^2, \beta) = (6 - \sum_{i=1}^{3} \kappa_i - 2\sum_{j=1}^{3} \sigma_j)^2,$$

$$\begin{aligned} e(\mathbb{CP}^2, \beta) = & 10 - 3\sum_{i=1}^{3} \kappa_i - 4(\sigma_1 + \sigma_2) - 6\sigma_3 + 2\sigma_3 \sum_{i=1}^{3} \kappa_i + \frac{1}{4}(\sum_{i=1}^{3} \kappa_i - 1)^2 \\ & + \frac{1}{2}(\sigma_1 + \sigma_2 + \kappa_1 - 1)^2 + (\sigma_1 + \sigma_3 - \frac{3}{4})^2 + (\sigma_2 + \sigma_3 - \frac{3}{4})^2 \\ & + \frac{1}{2}(\sigma_1 + \sigma_2 - \frac{1}{2})^2 + \frac{1}{2}(\sigma_1 + \kappa_2 - \frac{1}{2})^2 + \frac{1}{2}(\sigma_1 + \kappa_3 - \frac{1}{2})^2 \\ & + \frac{1}{2}(\sigma_2 + \kappa_2 - \frac{1}{2})^2 + \frac{1}{2}(\sigma_2 + \kappa_3 - \frac{1}{2})^2, \end{aligned}$$

where $\sigma_j = \frac{1}{n_j}$ and $\kappa_i = \frac{1}{m_i}$. Local orbifold fundamental group at nodal points are abelian and always admit local uniformization. In addition, uniformizability conditions at triple and tangency points are

$$\sum_{i=1}^{3} \kappa_i \geq 1, \quad \sigma_1 + \sigma_2 + \kappa_1 \geq 1, \quad \sigma_1 + \sigma_3 \geq \frac{3}{4}, \quad \sigma_2 + \sigma_3 \geq \frac{3}{4}, \quad \sigma_1 + \sigma_2 \geq \frac{1}{2}$$
$$\sigma_1 + \kappa_2 \geq \frac{1}{2}, \quad \sigma_1 + \kappa_3 \geq \frac{1}{2}, \quad \sigma_2 + \kappa_2 \geq \frac{1}{2}, \quad \sigma_2 + \kappa_3 \geq \frac{1}{2}.$$

Notice that, in Figure 6.30 the line H_2 is a reflection of H_3 and they have the same combinatorics. Similarly the quadric Q_1 is a reflection of Q_2 and they have same combinatorics. In addition, both orbifold Chern numbers and uniformizability conditions are symmetric w.r.t σ_1 and σ_2, and κ_2 and κ_3. Then, we can deduce $\sigma_1 = \sigma_2$ and $\kappa_2 = \kappa_3$. Therefore $(m_1, m_2, m_3, n_1, n_2, n_3)$ is in the form of (p, q, q, r, r, s), where (p, q, r, s) satisfy the inequalities

$$r \leq 4, \quad \frac{1}{r} + \frac{1}{s} \geq \frac{3}{4}, \quad \frac{1}{p} + \frac{2}{q} \geq 1, \quad \frac{2}{r} + \frac{1}{p} \geq 1 \quad \frac{1}{r} + \frac{1}{q} \geq \frac{1}{2},$$

which has solutions:

p	2	3	k	2	3	2
q	2, 3, 4	2, 3	2	3	3	4
r	3, 4	3	2	2	2	2
s	2	2	2, 3, 4	2, 3, 4	2, 3, 4	2, 3, 4

where $k \in \mathbb{Z}_{\geq 2}$. By using Maple, and taking into account the candidates above, we have obtained that $(3e - c_1^2)(\mathbb{CP}^2, \beta) = 0$ if $(m_1, m_2, m_3, n_1, n_2, n_3)$ is $(2, 4, 4, 4, 4, 2)$. In this case, notice that all multiple points admits cusp-points. Since it is an orbifold of general type, then by Theorem 6.4.2, we can state the following theorem:

Theorem 6.6.14. *The orbifold* $(\mathbb{CP}^2, \beta)$ *associated with the divisor* $D = 4Q_1 + 4Q_2 + 2Q_3 + 2H_1 + 4H_2 + 4H_3$ *supported by the arrangement in Figure 6.30 is uniformized by* $\mathbf{B}_2$.

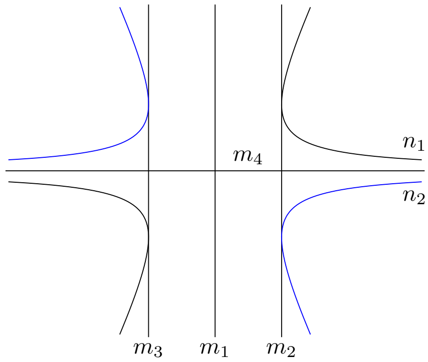


Figure 6.31

Ninth, consider the orbifold $(\mathbb{CP}^2, \beta)$ associated with the divisor $D = n_1Q_1 + n_2Q_2 + \sum_{i=1}^{4} m_iH_i$ supported by the arrangement in Figure 6.31. Here, $Q_1 : Y^2 + Z^2 - 2XY = 0$, $Q_2 : Y^2 + Z^2 + 2XY = 0$, $H_1 : X = 0$, $H_2 : X - Z = 0$, $H_3 : X + Z = 0$, $H_4 : Y = 0$. We have discussed the intersection behavior of this arrangement on page 239. Its orbifold Chern numbers are

$$c_1^2(\mathbb{CP}^2, \beta) = (5 - 2\sigma_1 - 2\sigma_2 - \sum_{i=1}^{4} \kappa_i)^2.$$

and

$$\begin{aligned} e(\mathbb{CP}^2, \beta) =& 6 - 3(\sigma_1 + \sigma_2) - 2\sum_{i=1}^{4} \kappa_i + \kappa_4(\kappa_1 + \kappa_2 + \kappa_3) \\ &+ \frac{1}{4}(\kappa_1 + \kappa_2 + \kappa_3 - 1)^2 + \frac{2}{4}(\sigma_1 + \sigma_2 + \kappa_1 - 1)^2 + \frac{1}{2}(\sigma_1 + \kappa_2 - \frac{1}{2})^2 \\ &+ \frac{1}{2}(\sigma_1 + \kappa_3 - \frac{1}{2})^2 + \frac{1}{2}(\sigma_2 + \kappa_2 - \frac{1}{2})^2 + \frac{1}{2}(\sigma_2 + \kappa_3 - \frac{1}{2})^2, \end{aligned}$$

where $\kappa_i = \frac{1}{m_i}$ and $\sigma_j = \frac{1}{n_j}$. Notice that, this orbifold is of general type. Local orbifold fundamental group at nodal points are abelian and it always admits local uniformization at nodal points. In addition, this orbifold is locally uniformizable at $H_4 \cap Q_1 \cap Q_2$ if $n_1 = n_2 = m_4 = 2$, which automatically verifies the local uniformizability conditions at each singular points on quadrics Q_j. Finally, there is a local uniformization at $[0:1:0]$ if $\kappa_1 + \kappa_2 + \kappa_3 \geq 1$, i.e, (m_1, m_2, m_3) is a permutation of $(2,2,k)$, $(2,3,3)$, $(2,3,4)$, $(2,3,5)$, $(2,3,6)$, $(2,4,4)$ and $(3,3,3)$, where $k \in \mathbb{Z}_{\geq 2}$. By using maple, and considering the candidates above we have obtained the Miyaoka-

Yau equality $(c_1^2 - 3e)(\mathbb{CP}^2, \beta) = 0$ for $(n_1, n_2; m_1, m_2, m_3, m_4) = (2,2;2,4,4,2)$. In this case orbifold Chern numbers are $c_1^2 = 3e = \frac{9}{4}$. Then by the Theorem 6.4.2, we can state the following theorem:

Theorem 6.6.15. *The orbifold* $(\mathbb{CP}^2, \beta)$ *associated with the divisor* $D = 2Q_1 + 2Q_2 + 2H_1 + 4H_2 + 4H_3 + 2H_4$ *supported by the arrangement in Figure 6.31 is uniformized by* $\mathbf{B}_2$.

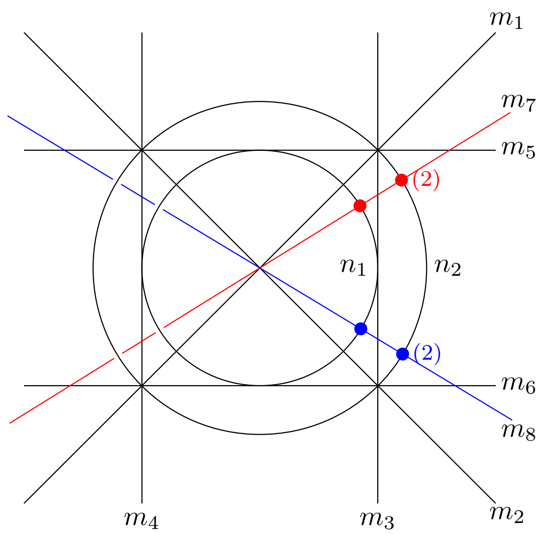


Figure 6.32

Tenth, consider the orbifold $(\mathbb{CP}^2, \beta)$ associated with the divisor $D = n_1 Q_1 + n_2 Q_2 + \sum_{i=1}^{8} m_i H_i$, supported by the arrangement of quadrics $Q_1 : X^2 + Y^2 - Z^2 = 0$, $Q_2 : X^2 + Y^2 - 2Z^2 = 0$ and the lines $H_1 : X - Y = 0$, $H_2 : X + Y = 0$, $H_3 : X - Z = 0$, $H_4 : X + Z = 0$, $H_5 : Y - Z = 0$, $H_6 : Y + Z = 0$, $H_7 : X - iY = 0$ and $H_8 : X + iY = 0$. Since this configuration can not be realized, we will draw an imaginary picture. An intersection behavior of the lines H_i, $i = 1, \cdots, 6$ and the quadrics Q_1 and Q_2 are as in Figure 6.32. The quadrics Q_1 and Q_2 has two tacnodes at $[\pm i : 1 : 0]$ (this points are labeled by red and blue colors on each quadric and to denote the intersection behavior at these points the intersection numbers are illustrated inside parenthesis). Common tangent lines of Q_1 and Q_2 at these points are the lines H_7 and H_8. In addition the lines H_7 and H_8 form a pencil together with H_1 and H_2 while they are transverse to other lines. In general settings of branching indices, one can compute its orbifold Chern numbers as

$$c_1^2(\mathbb{CP}^2,\beta) = (9 - 2\sigma_1 - 2\sigma_2 - \sum_{i=1}^{8} \kappa_i)^2$$

and

$$\begin{aligned} e(\mathbb{CP}^2,\beta) =& 19 - 6\sigma_1 - 4\sigma_2 + (2\sigma_1 - 3)(\kappa_1 + \kappa_2) - 4\sum_{i=3}^{8} \kappa_i + \kappa_3\kappa_4 + \kappa_5\kappa_6 \\ & + \frac{1}{2}\sum_{i=3}^{6}(\sigma_1 + \kappa_i - \frac{1}{2})^2 \end{aligned}$$

Notice that, there are four-fold points. At these points, the β map takes infinite values, i.e local orbifold fundamental group is infinite. Then solvability of local π_1^{orb} admits local uniformization at these points. Therefore, branching indices of curves through these points must be 2. Notice that, each line and quadric has at least one four-fold point. Thus, we can assume $n_j = m_i = 2$ for all i, j. In this case orbifold Chern numbers are $c_1^2 = 9$ and $e = 3$. Since this orbifold is of general type, then by Theorem 6.4.2 we have the following theorem:

Theorem 6.6.16. *The orbifold* $(\mathbb{CP}^2,\beta)$ *associated with the divisor* $D = 2Q_1 + 2Q_2 + \sum_{i=1}^{8} 2H_i$ *supported by the arrangement in Figure 6.32 is uniformized by the complex* 2*-ball* $\mathbf{B}_2$.

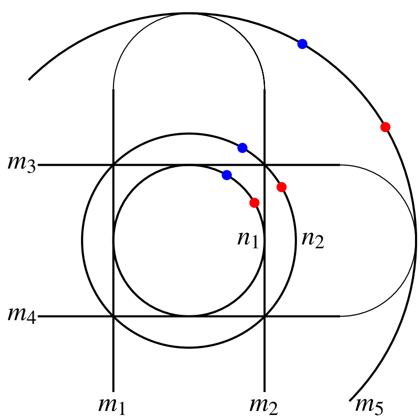


Figure 6.33

Eleventh, consider the arrangement of two quadrics Q_1, Q_2 and five lines H_i such that the quadrics Q_1 and Q_2 has two tacnodes and the line H_5 goes through these points. In addition, the lines H_1, H_2, H_3 and H_4 are distinct four tangent lines of Q_1 such that they pairwise meets on Q_2. Such configuration is projectively rigid and the equations of these quadrics and lines are $Q_1 : 2X^2+2Y^2-Z^2=0$, $Q_2 : X^2+Y^2-Z^2=0$, $H_1 : \sqrt{2}X+Z=0$, $H_2 : \sqrt{2}X-Z=0$, $H_3 : \sqrt{2}Y+Z=0$, $H_4 : \sqrt{2}Y-Z=0$ and $H_5 : Z=0$. Since this configuration can not be realized, we will draw a picture consisting its real part. The same colored points denote a tacnode of quadrics. The arc on the picture denotes the line $H_5=\{Z=0\}$, the line at infinity (See Figure 6.33). Now consider the orbifold $(\mathbb{CP}^2,\beta)$ associated with the divisor $D=n_1Q_1+n_2Q_2+\sum_{i=1}^5 m_iH_i$. Its orbifold Chern numbers are

$$c_1^2(\mathbb{CP}^2,\beta)=(6-2\sigma_1-2\sigma_2-\sum_{i=1}^{5}\kappa_i)^2$$

and

$$\begin{aligned}e(\mathbb{CP}^2,\beta)=&9-4(\sigma_1+\sigma_2)-2\sum_{i=1}^{5}\kappa_i+\frac{1}{4}(\kappa_1+\kappa_2+\kappa_5-1)^2\\&+\frac{1}{4}(\kappa_3+\kappa_4+\kappa_5-1)^2+\frac{1}{2}\sum_{i=1}^{4}(\sigma_1+\kappa_i-\frac{1}{2})^2\\&+\frac{1}{4}\sum_{i=1}^{2}\sum_{j=3}^{4}(\sigma_2+\kappa_i+\kappa_j-1)^2+(\sigma_1+\sigma_2+\frac{1}{2}\kappa_5-1)^2,\end{aligned}$$

where $\kappa_i=\frac{1}{m_i}$ and $\sigma_j=\frac{1}{n_j}$. Note that that both of the orbifold Chern numbers are symmetric in variables (κ_1,κ_2) and (κ_3,κ_4), i.e, $m_1=m_2=m_3=m_4=m$. Set $\kappa:=\frac{1}{m}$. The local uniformizability conditions at triple points and tangency points are

$$2\kappa+\kappa_5\geq 1,\quad \sigma_1+\kappa\geq\frac{1}{2},\quad \sigma_2+2\kappa\geq 1,\quad \sigma_1+\sigma_2+\frac{1}{2}\kappa_5\geq 1.$$

These conditions has solutions (m,m_5,n_1,n_2) for $m,n_1,n_2\leq 4$. Taking in to account these restriction on branching indices and using Maple we obtained the Miyaoka-Yau equality $c_1^2(\mathbb{CP}^2,\beta)-3e(\mathbb{CP}^2,\beta)=$ if $n_1=m=4$ and $n_2=m_5=2$. In this case, the orbifold Chern numbers are $c_1^2=3e=9$, and the β map vanishes at tangency points and triple points, i.e, local orbifold fundamental groups at these points are infinite and cusp points appears as cover of these points. Since this orbifold is of general type , then by Theorem 6.4.2, we can state the following theorem:

Theorem 6.6.17. *The orbifold* $(\mathbb{CP}^2, \beta)$ *associated with the divisor* $D = 4Q_1 + 2Q_2 + \sum_{i=1}^{4} 4H_i + 2H_5$ *supported by the arrangement in Figure 6.33 is uniformized by the complex* 2*-ball* $\mathbf{B}_2$.

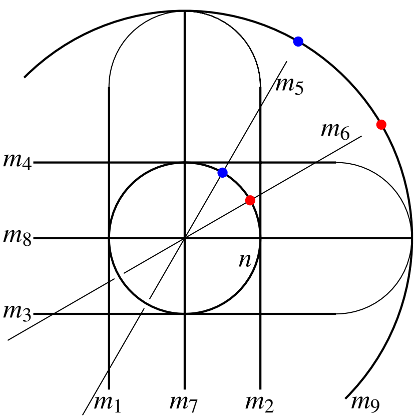


Figure 6.34

Twelfth, consider an arrangement of a quadrics Q, and nine lines H_i such that the lines H_1, H_2, H_3, H_4, H_5 and H_6 are distinct six tangents of Q. The line H_7 pass through the $H_3 \cap Q$, $H_4 \cap Q$, $H_1 \cap H_2 \cap H_9$ and $H_5 \cap H_6 \cap H_8$. The line H_8 pass through the $H_1 \cap Q$, $H_2 \cap Q$, $H_3 \cap H_4 \cap H_9$ and $H_5 \cap H_6 \cap H_7$. In addition, the line H_9 pass through the $H_5 \cap Q$, $H_6 \cap Q$, $H_3 \cap H_4 \cap H_8$ and $H_1 \cap H_2 \cap H_7$. This configuration is projectively rigid and complex realizable. The equations for these quadric and lines are $Q : X^2 + Y^2 - Z^2 = 0$, $H_1 : Z + X = 0$, $H_2 : Z - X = 0$, $H_3 : Y + Z = 0$, $H_4 : Y - Z = 0$, $H_5 : X + iY = 0\}$, $H_6 : X - iY = 0$, $H_7 : X = 0$, $H_8 : Y = 0$ and , $H_9 : Z = 0$. Since this configuration can not be realized, we will draw a picture consisting its real part, H_9 as the line at infinity. Wee will also draw imaginary lines H_5 and H_6 symbolically so that the colored points denote the tangency points of these lines to Q (See Figure 6.34). Consider the orbifold $(\mathbb{CP}^2, \beta)$ associated with the divisor $D = nQ + \sum_{i=1}^{9} m_i H_i$. Its orbifold Chern numbers are

$$c_1^2(\mathbb{CP}^2, \beta) = (8 - 2\sigma - \sum_{i=1}^{9} \kappa_i)^2$$

and

$$e(\mathbb{CP}^2,\beta) = 16-4\sigma-4\sum_{i=1}^{6}\kappa_i-2\sum_{i=7}^{9}\kappa_i+(\kappa_1+\kappa_2)(\kappa_3+\kappa_4)+(\kappa_5+\kappa_6)\sum_{i=1}^{4}\kappa_i$$
$$+\frac{1}{2}(\eta_{1,8}^2+\eta_{2,8}^2+\eta_{3,7}^2+\eta_{4,7}^2+\eta_{5,9}^2+\eta_{6,9}^2)$$

where $\sigma=\frac{1}{n}$, $\kappa_i=\frac{1}{m_i}$ and $\eta_{ij}=\sigma+\kappa_i+\frac{1}{2}\kappa_j-1$. Notice that each line in Figure 6.34 has a fourfold points. Local orbifold fundamental groups at these points are infinite solvable if $m_i=2$, otherwise they are big. Now assume $m_i=2$. Local orbifold fundamental group at nodal points are abelian and always admit local uniformization. To have local uniformization at tangency points on quadric Q, we must have $\eta_{ij}=\sigma-\frac{1}{4}\geq 0$, i.e, $2\leq n\leq 4$. The orbifold Chern numbers reduces to $c_1^2=(\frac{7}{2}-2\sigma)^2$ and $e=4-4\sigma+3(\sigma-\frac{1}{4})^2$. Therefore, $3e-c_1^2=5(\sigma-\frac{1}{4})^2=0$ if and only if $n=4$ which verifies uniformizability condition at singular points on Q. Notice that all multiple points except nodal ones, appears as cusp in covers. Since this orbifold is of general type, then by Theorem 6.4.2 we can state the following theorem:

Theorem 6.6.18. *The orbifold* $(\mathbb{CP}^2,\beta)$ *associated with the divisor* $D=4Q+\sum_{i=1}^{9}2H_i$ *supported by the arrangement in Figure 6.34 is uniformized by the complex* 2*-ball* $\mathbf{B}_2$.

Next, consider the configuration of n-quadrics, each has k tacnodes and do not allow the meetings of three or more quadrics at a point. Since the maximum number of tacnodes can not achieve the bound $\frac{4}{9}n(n+3)$ (Hirzebruch, 1986), then we have $k\leq\frac{8}{9}(n+3)$. Consider the orbifold $(\mathbb{CP}^2,\beta)$ associated with the divisor $D=\sum_{i=1}^{n}mQ_i$ supported by this configuration. Its orbifold Chern numbers are

$$c_1^2(\mathbb{CP}^2,\beta)=(-3+2n-\frac{2n}{m})^2.$$

and

$$
\begin{aligned}
e(\mathbb{CP}^2,\beta) =& 3-n(6+k-4n)-\frac{nk}{2}-n(2n-2-k)+(6+k-4n)\frac{n}{m} \\
&+(2n-2-k)\frac{n}{m^2}+\frac{nk}{4}(\frac{2}{m}-\frac{1}{2})^2
\end{aligned}
$$

Therefore

$$
3e-c_1^2(\mathbb{CP}^2,\beta)=\frac{32n(m^2n+m(3-2n)+n-3)-3kmn(7m-8)}{16m^2}.
$$

In addition, the uniformizability condition $\frac{2}{m}\geq\frac{1}{2}$ implies that $m\leq 4$. Incase $m=2$,

$$
3e-c_1^2(\mathbb{CP}^2,\beta)=\frac{n(8n+24-9k)}{16}
$$

and it vanishes if $k=\frac{8(n+3)}{9}$. But, the Theorem 4.3.20 tells us that there is no six non-degenerate quadrics with twenty four tacnodes. Thus, the claim $3e-c_1^2(\mathbb{CP}^2,\beta)=\frac{n(8n+24-9k)}{16}=0$ fails for $n=6$ and $k=8$ since such configuration does not exist.

If $m=3$, then

$$
3e-c_1^2(\mathbb{CP}^2,\beta)=\frac{64n(2n+3)-117nk}{144}
$$

and it vanishes when $k=\frac{64(2n+3)}{117}$, but this contradicts the fact $k\leq\frac{8(n+3)}{9}$ while $n>2$.

Now suppose $m=4$. Then

$$
3e-c_1^2(\mathbb{CP}^2,\beta)=\frac{3n}{16}(6(n+1)-5k)
$$

and it vanishes if $n=5\lambda-1$ and $k=6\lambda$. The number k achieves the bound $\frac{8(n+3)}{9}$ for $\lambda\geq 2$. So, one gets $\lambda=1$ which implies $n=4$ and $k=6$. This means, the arrangement supporting the divisor D is the Naruki arrangement given by equations $X^2\mp Y^2\mp Z^2=0$. Because of this reason, let us call this orbifold as *Naruki orbifold*. Noruki orbifold is an orbifold of general type and by the Theorem 6.4.2, we can state the following theorem:

Theorem 6.6.19. *The Naruki orbifold* $(\mathbb{CP}^2, \sum_{i=1}^4 4Q_i)$ *is uniformized by* $\mathbf{B}_2$.

Finally, consider the orbifold $(\mathbb{CP}^2, \beta) = (\mathbb{CP}^2, \sum_{i=1}^4 n_i Q_i + \sum_{j=1}^4 m_j H_j)$ supported by the arrangement containing Naruki arrangement $Q_i : X^2 \mp Y^2 \mp Z^2 = 0$ and four lines $H_j :\ X^4 - Y^4 = 0$. Note that the lines $H_1 : X - Y = 0$ and $H_2 : X + Y = 0$ are common tangent lines of the quadrics $Q_1 :\ -X^2 + Y^2 + Z^2 = 0$ and $Q_2 : X^2 - Y^2 + Z^2 = 0$; and the lines $H_3 : X - iY = 0$ and $H_4 : X + iY = 0$ are common tangent lines of the quadrics $Q_3 : X^2 + Y^2 - Z^2 = 0$ and $Q_4 : X^2 + Y^2 + Z^2 = 0$. These four common tangent lines meet at a single point. By local uniformizability condition at this point, weights m_j of the tangent lines H_j must be 2. In addition, at the contact of order 2 points of quadrics with these lines, orbifold germs are uniformizable if the weights of the quadrics are also 2, otherwise, local orbifold fundamental group will be big. Omitting fact the weights are all 2, first give formulas for its orbifold Chern number and then check for the weights 2. The orbifold Chern numbers are

$$c_1^2(\mathbb{CP}^2, \beta) = (9 - \sum_{i=1}^{4} \kappa_i - 2\sum_{j=1}^{4} \sigma_j)^2.$$

and

$$\begin{aligned} e(\mathbb{CP}^2, \beta) =& 22 - 4\sum_{i=1}^{3} \kappa_i - 8\sum_{j=1}^{4} \sigma_j + 2(\kappa_1 + \kappa_2)(\sigma_3 + \sigma_4) + 2(\kappa_3 + \kappa_4)(\sigma_1 + \sigma_2) \\ &+ (\sigma_1 + \sigma_3 - \frac{1}{2})^2 + (\sigma_1 + \sigma_4 - \frac{1}{2})^2 + (\sigma_2 + \sigma_3 - \frac{1}{2})^2 + (\sigma_2 + \sigma_4 - \frac{1}{2})^2, \end{aligned}$$

where $\kappa_i = \frac{1}{m_i}$ and $\sigma_j = \frac{1}{n_j}$. In case $m_i = n_j = 2$, the orbifold Chern numbers are $c_1^2 = 9$ and $e = 3$ and they satisfy the Miyaoka-Yau equality. Since this orbifold is of general type, as a consequence of the Theorem 6.4.2, we can state the following theorem:

Theorem 6.6.20. *An orbifold* $(\mathbb{CP}^2, \beta)$ *associated with the divisor* $D = \sum_{i=1}^4 2Q_i + \sum_{j=1}^4 2H_j$ *is uniformized by the complex 2-ball* $\mathbf{B}_2$*. Here the quadrics* Q_i *form a Naruki arrangement and the four lines* H_j *are common tangent lines of some of these quadrics so that the line* H_j *forms a pencil.*

6.7 Covering Relations among Ball-Quotient Arrangements

As a result of previous section, first we give a list of ball-quotient quadric-line arrangements in Table 6.3, and then study the covering relations among them.

Table 6.3 Ball-quotient quadric-line arrangements

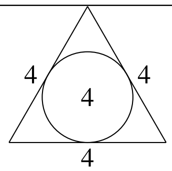
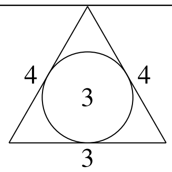
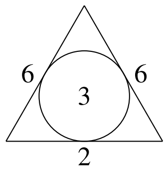
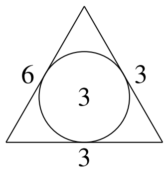
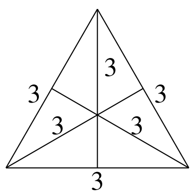
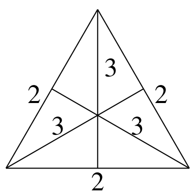
	Figure	Equations of quadrics and lines, c_1^2 and e
$\mathcal{A}_1$		$\mathcal{A}_1 := (\mathbb{CP}^2, D_1), \quad D_1 := 4Q + 4T_1 + 4T_2 + 4T_3,$ $Q : (X+Y-Z)^2 - 4XY = 0, T_1 : X = 0, T_2 : Y = 0, T_3 : Z = 0$ $c_1^2(\mathcal{A}_1) = 9/16, \quad e(\mathcal{A}_1) = 3/16$
$\mathcal{A}_2$		$\mathcal{A}_2 := (\mathbb{CP}^2, D_2), \quad D_2 := 3Q + 4T_1 + 3T_2 + 4T_3,$ $Q : (X+Y-Z)^2 - 4XY = 0, T_1 : X = 0, T_2 : Y = 0, T_3 : Z = 0$ $c_1^2(\mathcal{A}_2) = 1/4, \quad e(\mathcal{A}_2) = 1/12$
$\mathcal{A}_3$		$\mathcal{A}_3 := (\mathbb{CP}^2, D_3), \quad D_3 := 3Q + 6T_1 + 2T_2 + 6T_3,$ $Q : (X+Y-Z)^2 - 4XY = 0, T_1 : X = 0, T_2 : Y = 0, T_3 : Z = 0$ $c_1^2(\mathcal{A}_3) = 1/4, \quad e(\mathcal{A}_3) = 1/12$
$\mathcal{A}_4$		$\mathcal{A}_4 := (\mathbb{CP}^2, D_4), \quad D_4 := 3Q + 6T_1 + 3T_2 + 3T_3,$ $Q : (X+Y-Z)^2 - 4XY = 0, T_1 : X = 0, T_2 : Y = 0, T_3 : Z = 0$ $c_1^2(\mathcal{A}_4) = 1/4, \quad e(\mathcal{A}_4) = 1/12$
$\mathcal{A}_5$		$\mathcal{A}_5 := (\mathbb{CP}^2, D_5), \quad D_5 := \sum_{i=1}^{6} 3H_i,$ $H_1 : X = 0, H_2 : Y = 0, H_3 : Z = 0, H_4 : X - Y = 0, H_5 : Y - Z = 0, H_6 : Z - X = 0$ $c_1^2(\mathcal{A}_5) = 1, \quad e(\mathcal{A}_5) = 1/3$
$\mathcal{A}_6$		$\mathcal{A}_6 := (\mathbb{CP}^2, D_6), \quad D_6 := \sum_{i=1}^{3} 2H_i + \sum_{i=4}^{6} 3H_i,$ $H_1 : X = 0, H_2 : Y = 0, H_3 : Z = 0, H_4 : X - Y = 0, H_5 : Y - Z = 0, H_6 : Z - X = 0$ $c_1^2(\mathcal{A}_6) = 1/4, \quad e(\mathcal{A}_6) = 1/12$

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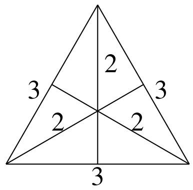
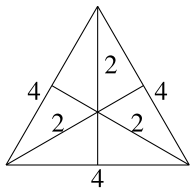
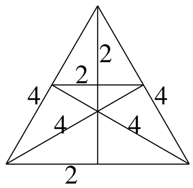
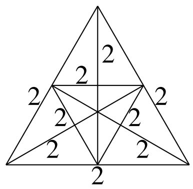
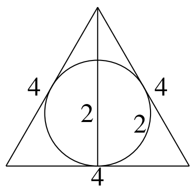
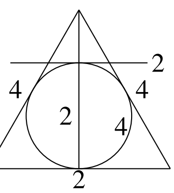
Table 6.3 Ball-quotient quadric-line arrangements. *(continued from previous page)*

	Figure	Equations of quadrics and lines, c_1^2 and e
$\mathcal{A}_7$		$\mathcal{A}_7 := (\mathbb{CP}^2, D_7)$, $D_7 := \sum_{i=1}^{3} 3H_i + \sum_{i=4}^{6} 2H_i$, $H_1 : X = 0, H_2 : Y = 0, H_3 : Z = 0, H_4 : X - Y = 0, H_5 : Y - Z = 0, H_6 : Z - X = 0$ $c_1^2(\mathcal{A}_7) = 1/4$, $e(\mathcal{A}_7) = 1/12$
$\mathcal{A}_8$		$\mathcal{A}_8 := (\mathbb{CP}^2, D_8)$, $D_8 := \sum_{i=1}^{3} 4H_i + \sum_{i=4}^{6} 2H_i$, $H_1 : X = 0, H_2 : Y = 0, H_3 : Z = 0, H_4 : X - Y = 0, H_5 : Y - Z = 0, H_6 : Z - X = 0$ $c_1^2(\mathcal{A}_8) = 9/16$, $e(\mathcal{A}_8) = 3/16$
$\mathcal{A}_9$		$\mathcal{A}_9 := (\mathbb{CP}^2, D_9)$, $D_9 := \sum_{i=1}^{3} 2H_i + \sum_{i=4}^{7} 4H_i$, $H_1 : X = 0, H_2 : Y = 0, H_3 : Z = 0, H_4 : X - Y + Z = 0$, $H_5 : -X + Y + Z = 0, H_6 : X + Y + Z = 0, H_7 : X + Y - Z = 0$, $c_1^2(\mathcal{A}_9) = 9/4$, $e(\mathcal{A}_9) = 3/4$
$\mathcal{A}_{10}$		$\mathcal{A}_{10} := (\mathbb{CP}^2, D_{10})$, $D_{10} := \sum_{i=1}^{9} 2H_i$, $H_1 : X = 0, H_2 : Y = 0, H_3 : Z = 0, H_4 : X - Y = 0, H_5 : Y - Z = 0$, $H_6 : Z - X = 0, H_7 : X - Y + Z = 0, H_8 : X + Y - Z = 0, H_9 : -X + Y + Z = 0$, $c_1^2(\mathcal{A}_{10}) = 9/4$, $e(\mathcal{A}_{10}) = 3/4$
$\mathcal{A}_{11}$		$\mathcal{A}_{11} := (\mathbb{CP}^2, D_{11})$, $D_{11} := 2Q + \sum_{i=1}^{3} 4T_i + 2H_4$, $T_1 : X = 0, T_2 : Y = 0, T_3 : Z = 0, H_4 : Z - X = 0, Q : (X + Y - Z)^2 - 4XY = 0$, $c_1^2(\mathcal{A}_{11}) = 9/16$, $e(\mathcal{A}_{11}) = 3/16$
$\mathcal{A}_{12}$		$\mathcal{A}_{12} := (\mathbb{CP}^2, D_{12})$, $D_{12} := 4Q + 4T_1 + 2T_2 + 4T_3 + 2T_4 + 2H_5$ $Q : (X + Y - Z)^2 - 4XY = 0, T_1 : X = 0, T_2 : Y = 0, T_3 : Z = 0$, $T_4 : 2X - Y + 2Z = 0, H_5 : Z - X = 0$ $c_1^2(\mathcal{A}_{11}) = 9/4$, $e(\mathcal{A}_{11}) = 3/4$

Continued on next page

Table 6.3 Ball-quotient quadric-line arrangements. *(continued from previous page)*

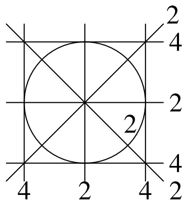
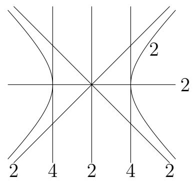
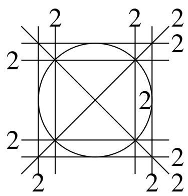
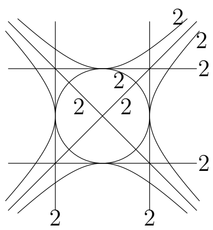
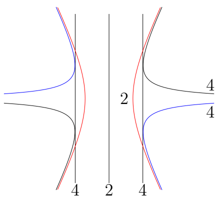
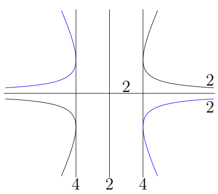
	Figure	Equations of quadrics and lines, c_1^2 and e
$\mathcal{A}_{13}$		$\mathcal{A}_{13} := (\mathbb{CP}^2, D_{13}), \quad D_{13} := 2Q + \sum_{i=1}^{4} 4T_i + \sum_{i=5}^{8} 2H_i$ $Q : X^2+Y^2-Z^2 = 0,\ T_1 : X+Z = 0,\ T_2 : X-Z = 0,\ T_3 : Y+Z = 0,$ $T_4 : Y-Z = 0,\ H_5 : X = 0,\ H_6 : Y = 0,\ H_7 : X+Y = 0,\ H_8 : X-Y = 0$ $c_1^2(\mathcal{A}_{13}) = 9, \quad e(\mathcal{A}_{13}) = 3$
$\mathcal{A}_{14}$		$\mathcal{A}_{14} := (\mathbb{CP}^2, D_{14}), \quad D_{14} := 2Q + 4H_1 + 4H_2 + \sum_{i=3}^{6} 2H_i$ $Q : X^2-Y^2-Z^2 = 0,\ H_1 : X+Z = 0,\ H_2 : X-Z = 0,\ H_3 : X+Y = 0,$ $H_4 : X-Y = 0,\ H_5 : X = 0,\ H_6 : Y = 0$ $c_1^2(\mathcal{A}_{14}) = 9/4, \quad e(\mathcal{A}_{14}) = 3/4$
$\mathcal{A}_{15}$		$\mathcal{A}_{15} := (\mathbb{CP}^2, D_{15}), \quad D_{15} := 2Q + \sum_{i=1}^{4} 4T_i + \sum_{i=5}^{8} 2H_i$ $Q : X^2+Y^2-Z^2 = 0,\ T_1 : X+Z = 0,\ T_2 : X-Z = 0,\ T_3 : Y+Z = 0,\ T_4 : Y-Z = 0,$ $H_5 : X+Y = 0,\ H_6 : X-Y = 0,\ H_7 : \sqrt{2}X+Z = 0,\ H_8 : \sqrt{2}X-Z = 0$ $H_9 : \sqrt{2}Y+Z = 0,\ H_{10} : \sqrt{2}Y-Z = 0$ $c_1^2(\mathcal{A}_{15}) = 9, \quad e(\mathcal{A}_{15}) = 3$
$\mathcal{A}_{16}$		$\mathcal{A}_{16} := (\mathbb{CP}^2, D_{16}), \quad D_{16} := \sum_{i=1}^{3} 2Q_i + \sum_{i=4}^{9} 2T_i$ $Q_1 : X^2+Y^2-Z^2 = 0,\ Q_2 : X^2-Y^2+Z^2 = 0,\ Q_3 : -X^2+Y^2+Z^2 = 0,$ $T_4 : X+Z = 0,\ T_5 : X-Z = 0,\ T_6 : Y+Z = 0,\ T_7 : Y-Z = 0,$ $T_8 : X+Y = 0,\ T_9 : X-Y = 0,$ $c_1^2(\mathcal{A}_{16}) = 9, \quad e(\mathcal{A}_{16}) = 3$
$\mathcal{A}_{17}$		$\mathcal{A}_{17} := (\mathbb{CP}^2, D_{17}), \quad D_{17} := 4Q_1 + 4Q_2 + 2Q_3 + 2H_1 + 4H_2 + 4H_3$ $Q_1 : Y^2+Z^2-2XY = 0,\ Q_2 : Y^2+Z^2+2XY = 0,\ Q_3 : 4X^2-Y^2-2Z^2 = 0,$ $H_1 : X = 0,\ H_2 : X-Z = 0,\ H_3 : X+Z = 0$ $c_1^2(\mathcal{A}_{17}) = 9, \quad e(\mathcal{A}_{17}) = 3$
$\mathcal{A}_{18}$		$\mathcal{A}_{18} := (\mathbb{CP}^2, D_{17}), \quad D_{18} := 2Q_1 + 2Q_2 + 2H_1 + 4H_2 + 4H_3 + 2H_4$ $Q_1 : Y^2+Z^2-2XY = 0,\ Q_2 : Y^2+Z^2+2XY = 0,$ $H_1 : X = 0,\ H_2 : X-Z = 0,\ H_3 : X+Z = 0,\ H_4 : Y = 0$ $c_1^2(\mathcal{A}_{18}) = 9/4, \quad e(\mathcal{A}_{18}) = 3/4$

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Table 6.3 Ball-quotient quadric-line arrangements. *(continued from previous page)*

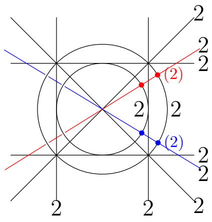
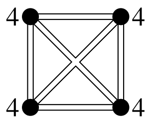
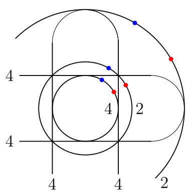
	Figure	Equations of quadrics and lines, c_1^2 and e
$\mathcal{A}_{19}$		$\mathcal{A}_{19} := (\mathbb{CP}^2, D_{18}),\quad D_{19} := 2Q_1 + 2Q_2 + 2\sum_{i=1}^{8} 2H_i$ $Q_1 : X^2+Y^2-Z^2=0,\ Q_2 : X^2+Y^2-2Z^2=0,$ $H_1 : X-Y=0,\ H_2 : X+Y=0,\ H_3 : X-Z=0,\ H_4 : X+Z=0$ $H_5 : Y-Z=0,\ H_6 : Y+Z=0,\ H_7 : X-iY=0,\ H_8 : X+iY=0$ $c_1^2(\mathcal{A}_{19})=9,\quad e(\mathcal{A}_{19})=3$
$\mathcal{A}_{20}$		The Naruki orbifold: $\mathcal{A}_{20} := (\mathbb{CP}^2, D_{20}),\quad D_{20} := \sum_{i=1}^{4} 4Q_i$ $Q_1 : X^2+Y^2-Z^2=0,\ Q_2 : X^2-Y^2+Z^2=0,$ $Q_3 : -X^2+Y^2+Z^2=0,\ Q_4 : X^2+Y^2+Z^2=0,$ $c_1^2(\mathcal{A}_{20})=9,\quad e(\mathcal{A}_{20})=3$
$\mathcal{A}_{21}$	Naruki arrangement plus four common tangents forming a pencil. Branching indices are all 2.	$\mathcal{A}_{21} := (\mathbb{CP}^2, D_{21}),\quad D_{21} := \sum_{i=1}^{4} 2Q_i + \sum_{j=1}^{4} 2H_j$ $Q_1 : X^2+Y^2-Z^2=0,\ Q_2 : X^2-Y^2+Z^2=0,\ Q_3 : -X^2+Y^2+Z^2=0,$ $Q_4 : X^2+Y^2+Z^2=0,\ H_1 : X-Y=0,\ H_2 : X+Y=0,\ H_3 : X-iY=0,$ $H_4 : X+iY=0,\quad c_1^2(\mathcal{A}_{21})=9,\quad e(\mathcal{A}_{21})=3$
$\mathcal{A}_{22}$	Ceva(3) arrangement. Branching indices are all 2.	$\mathcal{A}_{22} := (\mathbb{CP}^2, D_{22}),\quad D_{22} := \sum_{s=1}^{3}\sum_{i=0}^{2} 2H_{s,i}$ $H_{1,i} : X-\omega^i Y=0,\ H_{2,i} : Y-\omega^i Z=0,\ H_{3,i} : Z-\omega^i X=0,\ i=0,1,2,\ \omega^3=1$ $c_1^2(\mathcal{A}_{22})=9/4,\quad e(\mathcal{A}_{22})=3/4$
$\mathcal{A}_{23}$	Ceva(3) arrangement. Branching indices are all 3.	$\mathcal{A}_{23} := (\mathbb{CP}^2, D_{23}),\quad D_{23} := \sum_{s=1}^{3}\sum_{i=0}^{2} 3H_{s,i}$ $H_{1,i} : X-\omega^i Y=0,\ H_{2,i} : Y-\omega^i Z=0,\ H_{3,i} : Z-\omega^i X=0,\ i=0,1,2,\ \omega^3=1$ $c_1^2(\mathcal{A}_{23})=9,\quad e(\mathcal{A}_{23})=3$
$\mathcal{A}_{24}$	Ceva(4) arrangement. Branching indices are all 2.	$\mathcal{A}_{24} := (\mathbb{CP}^2, D_{24}),\quad D_{24} := \sum_{s=1}^{3}\sum_{i=0}^{3} 3H_{s,i}$ $H_{1,i} : X-\omega^i Y=0,\ H_{2,i} : Y-\omega^i Z=0,\ H_{3,i} : Z-\omega^i X=0,\ i=0,1,2,3,\ \omega^4=1$ $c_1^2(\mathcal{A}_{24})=9,\quad e(\mathcal{A}_{24})=3$
$\mathcal{A}_{25}$		$\mathcal{A}_{25} := (\mathbb{CP}^2, D_{25}),\quad D_{25} := 4Q_1 + 2Q_2 + \sum_{i=1}^{4} 4H_i + 2H_5$ $Q_1 : 2X^2+2Y^2-Z^2=0,\ Q_2 : X^2+Y^2-Z^2=0,\ H_1 : \sqrt{2}X+Z=0,$ $H_2 : \sqrt{2}X-Z=0,\ H_3 : \sqrt{2}Y-Z=0,\ H_4 : \sqrt{2}Y+Z=0,\ H_5 : Z=0$ $c_1^2(\mathcal{A}_{25})=9,\quad e(\mathcal{A}_{25})=3$

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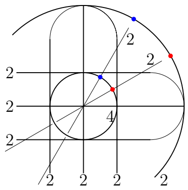
Table 6.3 Ball-quotient quadric-line arrangements. *(continued from previous page)*

	Figure	Equations of quadrics and lines, c_1^2 and e
$\mathcal{A}_{26}$		$\mathcal{A}_{26} := (\mathbb{CP}^2, D_{26}),\quad D_{26} := 4Q + \sum_{i=1}^{9} 2H_i,\quad Q : X^2+Y^2-Z^2=0,$ $H_1 : Z+X=0, H_2 : Z-X=0, H_3 : Z+Y=0, H_4 : Z-Y=0,$ $H_5 : X+iY=0, H_6 : X-iY=0, H_7 : X=0, H_8 : Y=0, H_9 : Z=0$ $c_1^2(\mathcal{A}_{26}) = 9,\quad e(\mathcal{A}_{26}) = 3$

The orbifolds listed in the Table 6.3 are related with eachother via covering maps. Under suitable choice of coordinates, the covering maps are the bicyclic maps $\varphi_n : \mathbb{CP}^2 \to \mathbb{CP}^2$ given by $[X:Y:Z] \to [X^n : Y^n : Z^n]$. Let us exhibit these covering relations among the orbifolds $\mathcal{A}_i$ in the Table 6.3. The diagram on page 265 in Figure 6.35 exhibits all covering relations among these orbifolds discussed below.

Coverings of $\mathcal{A}_1$:

Consider the orbifold $\mathcal{A}_1 = (\mathbb{CP}^2, 4Q + \sum_{i=1}^{3} 4T_i)$ in Table 6.3. Suppose without loss of generality that the lines T_1, T_2 and T_3 are defined by the equations $X=0$, $Y=0$ and $Z=0$, respectively. By the Lemma 6.6.7, a symmetric equation of Q is $(X+Y-Z)^2 - 4XY = 0$ which is tangent to the lines T_1, T_2 and T_3. If we consider the lifting of $\mathcal{A}_1$ due to the uniformization φ_2 of the sub-orbifold $(\mathbb{CP}^2, 2T_1+2T_2+2T_3)$, and denote by H_i the lifting $\varphi_2^{-1}(T_i)$ and by Q' the lifting $\varphi_2^{-1}(Q) = \{(X^2+Y^2-Z^2)^2 - 4X^2Y^2 = 0\}$, then $\varphi_2 : (\mathbb{CP}^2, 4Q' + 2T_1 + 2T_2 + 2T_3) \to \mathcal{A}_1$ is an orbifold covering. Note that Q' consists of the lines $X \mp Y \mp Z = 0$. If one denotes them by H_4, H_5, H_6 and H_7, then this covering orbifold will be the orbifold $\mathcal{A}_9$ in the Table 6.3. Hence one has $\varphi_2 : \mathcal{A}_9 \to \mathcal{A}_1$

If one takes φ_4 instead of φ_2, then he gets the covering orbifold $(\mathbb{CP}^2, 4Q'')$, where Q'' consists of four quadrics projectively equivalent to Naruki arrangement, and so the covering orbifold is the Naruki orbifold $\mathcal{A}_{20}$. Notice that this covering $\varphi_4 : \mathcal{A}_{20} \to \mathcal{A}_1$ is related with the orbifold covering $\varphi_2 : \mathcal{A}_{20} \to \mathcal{A}_9$.

Covering of $\mathcal{A}_3$:

Consider the orbifold $\mathcal{A}_3 = (\mathbb{CP}^2, 3Q + 6T_1 + 2T_2 + 6T_3)$ in the Table 6.3. Also assume that the equations of quadrics and lines are as stated in Table 6.3. If one consider the lifting of $\mathcal{A}_1$ due to the uniformization φ_2 of the sub-orbifold $(\mathbb{CP}^2, 2T_1 + 2T_2 + 2T_3)$, and denote by H_i the lifting $\varphi_2^{-1}(T_i)$, $i = 1,3$ and by Q' the lifting $\varphi_2^{-1}(Q)$, then he will get the orbifold covering $\varphi_2 : (\mathbb{CP}^2, 3Q' + 3T_1 + 3T_3) \to \mathcal{A}_3$. Notice that Q' consists of the lines $X \mp Y \mp Z = 0$, denote them by H_1, H_2, H_3 and H_4. The equation $XZ(X+Y-Z)(X-Y+Z)(-X+Y+Z)(X+Y+Z) = 0$ is an equation of complete quadrilateral. Therefore $(\mathbb{CP}^2, 3H_1 + 3H_2 + 3H_3 + 3H_4 + 3T_1 + 3T_3)$ is the orbifold $\mathcal{A}_5$ in the Table 6.3. Hence one has the covering $\varphi_2 : \mathcal{A}_5 \to \mathcal{A}_3$

Covering of $\mathcal{A}_4$:

Consider the orbifold $\mathcal{A}_4 = (\mathbb{CP}^2, 3Q + 6T_1 + 3T_2 + 3T_3)$ in the Table 6.3 and assume that the equations of quadrics and lines are as stated in Table 6.3. If we consider the lifting of $\mathcal{A}_4$ due to the uniformizer φ_3 of the sub-orbifold $(\mathbb{CP}^2, 3T_1 + 3T_2 + 3T_3)$, and denote by T_1 and $\overline{Q}$ the liftings $\varphi_3^{-1}(T_1)$, $\varphi_3^{-1}(Q)$, respectively, then $\varphi_3 : (\mathbb{CP}^2, 3\overline{Q} + 2T_1) \to \mathcal{A}_4$ is an orbifold covering. Note that $\overline{Q} : X^6 + Y^6 + Z^6 - 2X^2Y^2 - 2Y^2Z^2 - 2X^2Z^2 = 0$ is an irreducible sextic.

Covering of $\mathcal{A}_5$:

Consider the orbifold $\mathcal{A}_5 = (\mathbb{CP}^2, \sum_{s=1}^{6} 3H_s)$ in the Table 6.3 and assume that the equations of quadrics and lines are as stated in Table 6.3. Denote by H'_s the lifting $\varphi_3^{-1}(H_s)$, $s = 4,5,6$, of the lines H_4, H_5 and H_6 due to the uniformizer φ_3 of the sub-orbifold $(\mathbb{CP}^2, \sum_{s=1}^{3} 3H_s)$. Then one has the covering $\varphi_3 : (\mathbb{CP}^2, \sum_{s=4}^{6} 3H'_s) \to \mathcal{A}_5$. Notice that each H'_s consists of there lines $H_{s,i}$, $i = 0,1,2$. Here $H_{4,i} = \{X - \omega^i Y = 0\}$, $H_{5,i} = \{Y - \omega^i Z = 0\}$ and $H_{6,i} = \{Z - \omega^i X = 0\}$, $i = 0,1,2$, $\omega^4 = 1$. These lines form a Ceva(3) arrangement, and $(\mathbb{CP}^2, \sum_{s=4}^{6} 3H'_s) = (\mathbb{CP}^2, \sum_{s=4}^{6} \sum_{i=0}^{2} 3H_{s,i})$ is the orbifold $\mathcal{A}_{23}$. Thus we have the covering $\varphi_3 : \mathcal{A}_{23} \to \mathcal{A}_5$.

Covering of $\mathcal{A}_6$:

Consider the orbifold $\mathcal{A}_6 = (\mathbb{CP}^2, \sum_{s=1}^{3} 2H_s + \sum_{s=4}^{6} 3H_s)$ in the Table 6.3 and also and assume that the equations of quadrics and lines are as stated in Table 6.3. Denote by H'_s the liftings of the lines H_s, $s = 4,5,6$ due to the uniformizer φ_2 of the sub-orbifold $(\mathbb{CP}^2, \sum_{s=1}^{3} 2H_s)$. Each H'_s consists of two lines $H_{s,i}$, $i = 0,1$, $s = 4,5,6$. Set $H_{4,0} := \{X - Y = 0\}$, $H_{4,1} := \{X + Y = 0\}$, $H_{5,0} := \{Y - Z = 0\}$, $H_{5,1} := \{Y + Z = 0\}$, $H_{6,0} := \{Z - X = 0\}$ and $H_{6,1} := \{Z + X = 0\}$. Then $H'_s = H_{s,0} \cup H_{s,1}$, $s = 4,5,6$, and they form a complete quadrilateral. In addition, $(\mathbb{CP}^2, \sum_{s=4}^{6} 3H'_s) = (\mathbb{CP}^2, \sum_{s=4}^{6} \sum_{i=0}^{1} 3H_{s,i})$ is the orbifold $\mathcal{A}_5$ and one has the covering $\varphi_2 : \mathcal{A}_5 \to \mathcal{A}_6$.

Covering of $\mathcal{A}_7$:

Consider the orbifold $\mathcal{A}_7 = (\mathbb{CP}^2, \sum_{s=1}^{3} 3H_s + \sum_{s=4}^{6} 2H_s)$ in the Table 6.3 and assume that the equations of quadrics and lines are as stated in Table 6.3. As in covering of $\mathcal{A}_5$, liftings H'_s of the lines H_s, $s = 4,5,6$, due to the uniformizer φ_3 of the sub-orbifold $(\mathbb{CP}^2, \sum_{s=1}^{3} 3H_s)$, consists of three lines $H_{s,i}$, $i = 0,1,2$ and they form a Ceva(3) arrangement. Then $(\mathbb{CP}^2, \sum_{s=4}^{6} 2H'_s) = (\mathbb{CP}^2, \sum_{s=4}^{6} \sum_{i=0}^{2} 2H_{s,i})$ is the orbifold $\mathcal{A}_{22}$ and one has the covering $\varphi_3 : \mathcal{A}_{22} \to \mathcal{A}_7$.

Coverings of $\mathcal{A}_8$:

Consider the orbifold $\mathcal{A}_8 = (\mathbb{CP}^2, \sum_{i=1}^{3} 4H_i + \sum_{i=4}^{6} 2H_i)$ in the Table 6.3 and assume that the equations of quadrics and lines are as stated in Table 6.3. First consider the sub-orbifold $(\mathbb{CP}^2, \sum_{i=1}^{3} 2H_i)$ and its uniformizer φ_2. Denote by H_i the liftings $\varphi_2^{-1}(H_i)$, $i = 1,2,3$ and by H'_i the liftings $\varphi_2^{-1}(H_i)$, $i = 4,5,6$. Then we have the covering $\varphi_2 : (\mathbb{CP}^2, \sum_{i=1}^{3} 2H_i + \sum_{i=4}^{6} 2H'_i) \to \mathcal{A}_8$. Notice that each H'_i consists of two lines $H_{i,0}$ and $H_{i,1}$. Set $H_{4,0} := \{X - Y = 0\}$, $H_{4,1} := \{X + Y = 0\}$, $H_{5,0} := \{Y - Z = 0\}$, $H_{5,1} := \{Y + Z = 0\}$, $H_{6,0} := \{Z - X = 0\}$ and $H_{6,1} := \{Z + X = 0\}$. Then, together with the lines H_1, H_2 and H_3, they form an arrangement of 9 lines as in Figure 6.18. This means, up to projective equivalence, $(\mathbb{CP}^2, \sum_{i=1}^{3} 2H_i + \sum_{i=4}^{6} 2H'_i)$ is the orbifold $\mathcal{A}_{10}$ in the Table 6.3. Thus, we have the covering $\varphi_2 : \mathcal{A}_{10} \to \mathcal{A}_8$.

If one would consider the sub-orbifold $(\mathbb{CP}^2, \sum_{i=1}^{3} 4H_i)$ and its uniformizer φ_4, then he would get the covering $\varphi_4 : \mathcal{A}_{24} \to \mathcal{A}_8$. Indeed, we have the covering $\varphi_2 : (\mathbb{CP}^2, \sum_{i=4}^{6} 2H_i') \to \mathcal{A}_8$, where H_i' denotes the liftings $\varphi_2^{-1}(H_i)$, $i = 4,5,6$. Then . Notice that each H_i' consists of four lines $H_{i,j}$, $j = 0,1,2,3$. Set $H_{4,j} := \{X - \omega^j Y = 0\}$, $H_{5,j} := \{Y - \omega^j Z = 0\}$ and $H_{6,j} := \{Z - \omega^j X = 0\}$, where $j = 0,1,2,3$ and $\omega^4 = 1$. These twelve lines form a Ceva(4) arrangement. Therefore, $(\mathbb{CP}^2, \sum_{i=4}^{6} 2H_i') = (\mathbb{CP}^2, \sum_{i=4}^{6} \sum_{j=0}^{3} 2H_{i,j})$ is the orbifold $\mathcal{A}_{24}$ in the Table 6.3 and the covering is $\varphi_4 : \mathcal{A}_{24} \to \mathcal{A}_8$.

Now, let us consider the sub-orbifold $(\mathbb{CP}^2, 2H_1 + 2H_4 + 2H_6)$ and its uniformizer φ_2. By using projective transformations change the coordinates so that $H_1 = \{X = 0\}$, $H_2 = \{X - Y = 0\}$, $H_3 = \{X - Z = 0\}$, $H_4 = \{Y = 0\}$, $H_5 = \{Z - Y = 0\}$ and $H_6 = \{Z = 0\}$. Denote by H_1, H_2', H_3' and H_5' the liftings $\varphi_2^{-1}(H_1) = \{X = 0\}$, $\varphi_2^{-1}(H_2) = \{Y^2 - Z^2 = 0\}$, $\varphi_2^{-1}(H_3) = \{X^2 - Z^2 = 0\}$ and $\varphi_2^{-1}(H_5) = \{Z^2 - Y^2 = 0\}$, respectively. Then we have a covering $\varphi_2 : (\mathbb{CP}^2, 2H_1 + 4H_2' + 4H_3' + 2H_5') \to \mathcal{A}_8$. Notice that each of H_2', H_3', H_5' consists of two lines and they form a complete quadrilateral. If we add the line H_1 to this complete quadrilateral, we will get an arrangement of seven lines projectively equivalent to the arrangement in Figure 6.17. Thus, $(\mathbb{CP}^2, 2H_1 + 4H_2' + 4H_3' + 2H_5')$ is the orbifold $\mathcal{A}_9$ in the Table 6.3 and we have the covering $\varphi_2 : \mathcal{A}_9 \to \mathcal{A}_8$.

Next consider another sub-orbifold $(\mathbb{CP}^2, 2H_1 + 2H_4 + 2H_5)$ and its uniformizer φ_2. Projective transformations allow us to change coordinates, and we may chose them such that $H_1 = \{X = 0\}$, $H_2 = \{X - Z = 0\}$, $H_3 = \{X + Y - Z = 0\}$, $H_4 = \{Z = 0\}$, $H_5 = \{Y = 0\}$ and $H_6 = \{Y - Z = 0\}$. Denote by H_1, H_2', H_3' and H_6' the liftings $\varphi_2^{-1}(H_1) = \{X = 0\}$, $\varphi_2^{-1}(H_2) = \{X^2 - Z^2 = 0\}$, $\varphi_2^{-1}(H_3) = \{X^2 + Y^2 - Z^2 = 0\}$ and $\varphi_2^{-1}(H_6) = \{Y^2 - Z^2 = 0\}$, respectively. Then we have a covering $\varphi_2 : (\mathbb{CP}^2, 2H_1 + 4H_2' + 4H_3' + 2H_6') \to \mathcal{A}_8$. Notice that each of H_2' and H_6' consists of two lines tangent to H_3', and H_1 pass through the tangency points of $H_3' \cap H_6'$ and singular point of H_2'. Then H_1, H_2' H_3' and H_6' forms a rigid arrangement projectively equivalent to the Figure 6.24. Thus, $(\mathbb{CP}^2, 2H_1 + 4H_2' + 4H_3' + 2H_6')$ is the orbifold

$\mathcal{A}_{12}$ in the Table 6.3, and we have the covering $\varphi_2 : \mathcal{A}_{12} \to \mathcal{A}_8$.

Finally consider the sub-orbifold $(\mathbb{CP}^2, 2H_1+2H_3+2H_4)$ and its uniformizer φ_2. Change the coordinates so that $H_1 = \{X=0\}$, $H_2 = \{X-Z=0\}$, $H_3 = \{Y=0\}$, $H_4 = \{Z=0\}$, $H_5 = \{X-Y-Z=0\}$ and $H_6 = \{X-Y=0\}$. Denote, by H_1, H_3, H_2', H_5' and H_6' the liftings $\varphi_2^{-1}(H_1) = \{X=0\}$, $\varphi_2^{-1}(H_3) = \{Y=0\}$, $\varphi_2^{-1}(H_2) = \{X^2-Z^2=0\}$, $\varphi_2^{-1}(H_5) = \{X^2-Y^2-Z^2=0\}$ and $\varphi_2^{-1}(H_6) = \{X^2-Y^2=0\}$, respectively. Then we have a covering $\varphi_2 : (\mathbb{CP}^2, 2H_1+4H_2'+2H_3+2H_5'+2H_6') \to \mathcal{A}_8$. Notice that each of H_2' and H_6' consists of two lines, tangent to H_5'; and H_3 pass through the tangency points $H_5' \cap H_2'$ and singular point of H_6'. In addition, H_1 passes through the singular points of H_2' and H_6'. Therefore, $(\mathbb{CP}^2, 2H_1+4H_2'+2H_3+2H_5'+2H_6')$ is the orbifold $\mathcal{A}_{14}$ in the Table 6.3, and we have the covering $\varphi_2 : \mathcal{A}_{14} \to \mathcal{A}_8$.

Coverings of $\mathcal{A}_9$:

Consider the orbifold $\mathcal{A}_9 = (\mathbb{CP}^2, \sum_{i=1}^3 2H_i + \sum_{i=4}^7 4H_i)$ in the Table 6.3 and choose coordinates such that the equations of lines are as stated in the Table 6.3. First consider the sub-orbifold $(\mathbb{CP}^2, \sum_{i=1}^3 2H_i)$ and its uniformizer φ_2. Denote by H_i' the liftings $\varphi_2^{-1}(H_i)$, $i = 4,5,6,7$. H_i' are smooth quadrics and form a Naruki arrangement. Then, $(\mathbb{CP}^2, \sum_{i=4}^7 4H_i')$ is the Naruki orbifold $\mathcal{A}_{20}$ in the Table 6.3 and we have the covering $\varphi_2 : \mathcal{A}_{20} \to \mathcal{A}_9$.

Second consider the sub-orbifold $(\mathbb{CP}^2, 2H_3+2H_4+2H_7)$ and its uniformizer φ_2. Projective transformations allow us choose the coordinates such that $H_1 = \{Y-X=0\}$, $H_2 = \{X+Y-Z=0\}$, $H_3\{Z=0\}$, $H_4 = \{X=0\}$, $H_5 = \{X-Z=0\}$, $H_6 = \{Z-Y=0\}$ and $H_7 = \{Y=0\}$. Denote, by H_1', H_2', H_4, H_5', H_6' and H_7 the liftings $\varphi_2^{-1}(H_1) = \{Y^2-X^2=0\}$, $\varphi_2^{-1}(H_2) = \{X^2+Y^2-Z^2=0\}$, $\varphi_2^{-1}(H_4) = \{X=0\}$, $\varphi_2^{-1}(H_5) = \{X^2-Z^2=0\}$, $\varphi_2^{-1}(H_6) = \{Z^2-Y^2=0\}$ and $\varphi_2^{-1}(H_7) = \{Y=0\}$, respectively. Notice that, H_5' and H_6' each consist of two lines tangent to the quadric H_2'. Also, H_1' consists of two lines and these lines together with H_5' and H_6' form a complete quadrilateral. In addition, the line H_4 pass through the singular point of H_5' and the points $H_6' \cap H_2'$. Similarly, the line H_7 pass through the singular point of H_6' and the points $H_5' \cap H_2'$. This is exactly the arrangement in Figure 6.26. Therefore, $(\mathbb{CP}^2, 2H_1' + 2H_2' + 2H_4 + 4H_5' + 4H_6' + 2H_7)$ is the orbifold $\mathcal{A}_{13}$ in the Table 6.3 and we have the covering $\varphi_2 : \mathcal{A}_{13} \to \mathcal{A}_9$.

Next, consider the sub-orbifold $(\mathbb{CP}^2, 2H_1 + 2H_3 + 2H_4)$ and its uniformizer φ_2. Change the coordinates such that $H_1 = \{X = 0\}$, $H_2 = \{X + Y - Z = 0\}$, $H_3\{Y = 0\}$, $H_4 = \{Z = 0\}$, $H_5 = \{2Y - Z = 0\}$, $H_6 = \{2X + 2Y - Z = 0\}$ and $H_7 = \{2X - Z = 0\}$. Denote, by H_2', H_4, H_5', H_6', and H_7' the liftings $\varphi_2^{-1}(H_2) = \{X^2 + Y^2 - Z^2 = 0\}$, $\varphi_2^{-1}(H_4) = \{Z = 0\}$, $\varphi_2^{-1}(H_5) = \{2Y^2 - Z^2 = 0\}$, $\varphi_2^{-1}(H_6) = \{2X^2 + 2Y^2 - Z^2 = 0\}$, and $\varphi_2^{-1}(H_7) = \{2X^2 - Z^2 = 0\}$, respectively. Then we have the covering $(\mathbb{CP}^2, 2H_2' + 2H_4 + 4H_5' + 4H_6' + 4H_7') \to \mathcal{A}_9$. Notice that, H_5' and H_7' each consist of two lines tangent to the quadric H_6'. The quadric H_2' passes through the singular points of H_5' and H_7', and tangent to H_6' at two points on the line H_4. In addition, the singular points of H_5' and H_7' lies on the line H_4. Therefore, $(\mathbb{CP}^2, 2H_2' + 2H_4 + 4H_5' + 4H_6' + 4H_7')$ is the orbifold $\mathcal{A}_{25}$ in the Table 6.3 and the covering is $\varphi_2 : \mathcal{A}_{25} \to \mathcal{A}_9$.

Fourth, consider the sub-orbifold $(\mathbb{CP}^2, 2H_4 + 2H_5 + 2H_6)$ and its uniformizer φ_2. Change the coordinates such that $H_1 = \{Z - X = 0\}$, $H_2 = \{Z - Y = 0\}$, $H_3\{X + Y = 0\}$, $H_4 = \{X = 0\}$, $H_5 = \{Y = 0\}$, $H_6 = \{Z = 0\}$ and $H_7 = \{X + Y - Z = 0\}$. Denote, by H_1', H_2', H_3', H_4, H_5, H_6 and H_7' the liftings $\varphi_2^{-1}(H_1) = \{Z^2 - X^2 = 0\}$, $\varphi_2^{-1}(H_2) = \{Z^2 - Y^2 = 0\}$, $\varphi_2^{-1}(H_3) = \{X^2 + Y^2 = 0\}$, $\varphi_2^{-1}(H_4) = \{X = 0\}$, , $\varphi_2^{-1}(H_5) = \{Y = 0\}$, , $\varphi_2^{-1}(H_6) = \{Z = 0\}$, and $\varphi_2^{-1}(H_7) = \{X^2 + Y^2 - Z^2 = 0\}$, respectively. Then we have the covering $(\mathbb{CP}^2, 2H_1' + 2H_2' + 2H_3' + 2H_4 + 2H_5 + 2H_6 + 4H_7') \to \mathcal{A}_9$. Notice that, H_1', H_2' and H_3' each consist of distinct two lines tangent to the quadric H_7'. The line H_4 pass through the tangency points $H(7)' \cap H_2'$ and the point $H_1' \cap H_6$. The line H_5 pass through the tangency points $H(7)' \cap H_1'$ and the point $H_2' \cap H_6$. In addition the line H_6 goes through the tangency points $H(7)' \cap H_3'$. Therefore, $(\mathbb{CP}^2, 2H_1' + 2H_2' + 2H_3' + 2H_4 + 2H_5 + 2H_6 + 4H_7')$ is the orbifold $\mathcal{A}_{26}$ in the Table

6.3 and the covering is $\varphi_2 : \mathcal{A}_{26} \to \mathcal{A}_9$.

If one had considered the sub-orbifold $(\mathbb{CP}^2, 4H_4+4H_5+4H_6)$ and its uniformizer φ_4, the liftings would be $H_1'' := \varphi_4^{-1}(H_1) = \{Z^4 - X^4 = 0\}$, $H_2'' := \varphi_4^{-1}(H_2) = \{Z^4 - Y^4 = 0\}$, $H_3'' := \varphi_4^{-1}(H_3) = \{X^4 + Y^4 = 0\}$ and $H_7'' := \varphi_4^{-1}(H_7) = \{X^4 + Y^4 - Z^4 = 0\}$. Notice that H_7'' is the Fermat quartic and each of H_1'', H_2'' and H_3'' four lines which are flex tangents of H_7''. Then we have the orbifold covering $\varphi_4 : (\mathbb{CP}^2, 2H_1'' + 2H_2'' + 2H_3'' + 4H_7'') \to \mathcal{A}_9$.

Coverings of $\mathcal{A}_{10}$:

Consider the orbifold $\mathcal{A}_{10} = (\mathbb{CP}^2, 4Q + \sum_{i=1}^{9} 2H_i)$ in the Table 6.3 and choose coordinates such that the equations of lines are as stated in the Table 6.3. The uniformizer of the sub-orbifold $(\mathbb{CP}^2, \sum_{i=1}^{3} 2H_i)$ is φ_2. Denote by H_i', the liftings $\varphi_2^{-1}(H_i)$, $i = 4, \cdots, 9$. The liftings are $H_4' = \{X^2 - Y^2 = 0\}$, $H_5' = \{Y^2 - Z^2 = 0\}$, $H_6' = \{Z^2 - X^2 = 0\}$, $H_7' = \{X^2 - Y^2 + Z^2 = 0\}$, $H_8' = \{X^2 + Y^2 - Z^2 = 0\}$ and $H_9' = \{-X^2 + Y^2 + Z^2 = 0\}$. Notice that the quadrics H_7', H_8' and H_9' has six tacnodes and H_4', H_5', H_6' consists of pairwise common tangents of these quadrics. Therefore they form the arrangement in Figure 6.29 and the orbifold $(\mathbb{CP}^2, \sum_{i=4}^{9} 2H_i')$ is the orbifold $\mathcal{A}_{16}$ in the Table 6.3. Then e have the covering $\varphi_2 : \mathcal{A}_{16} \to \mathcal{A}_{10}$.

Second, consider the sub-orbifold $(\mathbb{CP}^2, \sum_{i=6}^{9} 2H_i)$, whose uniformizer is φ_2. Projective transformations allow us to change coordinates so that $H_1 : X + Z = 0$, $H_2 : X + Y = 0$, $H_3 : Y + Z = 0$, $H_4 : Y - Z = 0$, $H_5 : Z - X = 0$, $H_6 : X - Y = 0$, $H_7 : Z = 0$, $H_8 : X = 0$ and $H_9 : Y = 0$. The liftings H_i' of these lines, except the branch locus of φ_2, are $H_1' : X^2 + Z^2 = 0$, $H_2' : X^2 + Y^2 = 0$, $H_3' : Y^2 + Z^2 = 0$, $H_4' : Y^2 - Z^2 = 0$, $H_5' : Z^2 - X^2 = 0$, $H_6' : X^2 - Y^2 = 0$ and they form a Ceva(4) arrangement. Therefore the orbifold $(\mathbb{CP}^2, \sum_{i=1}^{6} 2H_i')$ is the orbifold $\mathcal{A}_{24}$ in the Table 6.3. Then we have the covering $\varphi_2 : \mathcal{A}_{24} \to \mathcal{A}_{10}$.

Third consider the sub-orbifold $(\mathbb{CP}^2, 2H_1 + 2H_5 + 2H_9)$, whose uniformizer is φ_2. Projective transformations allow us to change coordinates so that $H_1 : X = 0$,

$H_2 : X+Y+Z=0$, $H_3 : X+Y-Z=0$, $H_4 : -X+Y+Z=0$, $H_5 : Z=0$, $H_6 : X-Y+Z=0$, $H_7 : Z-X=0$, $H_8 : Z+X=0$ and $H_9 : Y=0$. The liftings H_i' of these lines, except the branch locus of φ_2, are $H_2' : X^2+Y^2+Z^2=0$, $H_3' : X^2+Y^2-Z^2=0$, $H_4' : -X^2+Y^2+Z^2=0$, $H_6' : X^2-Y^2+Z^2=0$, $H_7' : Z^2-X^2=0$, $H_9' : Z^2+X^2=0$. Notice that the quadrics H_2', H_3', H_4' and H_6' form a Naruki arrangement, and H_7', H_9' consists of four of the pairwise common tangents of these quadrics. In addition, H_7', H_9' form a pencil. Therefore the orbifold $(\mathbb{CP}^2, 2H_2'+2H_3'+2H_4'+2H_6'+2H_7'+2H_8')$ is the orbifold $\mathcal{A}_{21}$ in the Table 6.3. Then we have the covering $\varphi_2 : \mathcal{A}_{21} \to \mathcal{A}_{10}$.

Next, consider the sub-orbifold $(\mathbb{CP}^2, 2H_1+2H_3+2H_7)$, whose uniformizer is φ_2. Projective transformations allow us to change coordinates so that $H_1 : X=0$, $H_2 : X-Y+Z=0$, $H_3 : Z=0$, $H_4 : Y-Z=0$, $H_5 : X-Y=0$, $H_6 : Z-X=0$, $H_7 : Y=0$, $H_8 : 2X-Y=0$ and $H_9 : 2Z-Y=0$. The liftings H_i' of these lines, except the branch locus of φ_2, are $H_2' : X^2-Y^2+Z^2=0$, $H_4' : Y^2-Z^2=0$, $H_5' : X^2-Y^2=0$, $H_6' : Z^2-X^2=0$, $H_8' : 2X^2-Y^2=0$, $H_9' : 2Z^2-Y^2=0$ and they form an arrangement as in Figure 6.28. Therefore the orbifold $(\mathbb{CP}^2, 2H_2'+2H_4'+2H_5'+2H_6'+2H_8'+2H_9')$ is the orbifold $\mathcal{A}_{15}$ in the Table 6.3. Then e have the covering $\varphi_2 : \mathcal{A}_{15} \to \mathcal{A}_{10}$.

Last, consider the sub-orbifold $(\mathbb{CP}^2, 2H_2+2H_3+2H_6)$, whose uniformizer is φ_2. Projective transformations allow us to change coordinates so that $H_1 : Z-X=0$, $H_2 : Y=0$, $H_3 : Z=0$, $H_4 : X+Y-Z=0$, $H_5 : Y-Z=0$, $H_6 : X=0$, $H_7 : X+Y-2Z=0$, $H_8 : X-Y=0$ and $H_9 : X+Y=0$. The liftings H_i' of these lines, except the branch locus of φ_2, are $H_1' : Z^2-X^2=0$, $H_4' : X^2+Y^2-Z^2=0$, $H_5' : Y^2-Z^2=0$, $H_7' : X^2+Y^2-2Z^2=0$, $H_8' : X^2-Y^2=0$ and $H_9' : X^2+Y^2=0$. The quadrics H_4' and H_7' has two tacnodes and their common tangent lines are H_9'. Notice that H_1', H_5' and H_8' meets on H_7' while H_1' and H_5' are tangent to H_4'. In addition, H_8' and H_9' meets at a single point. Therefore, this is exactly the arrangement in the Figure 6.32 and the orbifold $(\mathbb{CP}^2, 2H_1'+2H_4'+2H_5'+2H_7'+2H_8'+2H_9')$ is the orbifold $\mathcal{A}_{19}$ in the Table 6.3. Then e have the covering $\varphi_2 : \mathcal{A}_{19} \to \mathcal{A}_{10}$.

Coverings of $\mathcal{A}_{11}$:

Consider the orbifold $\mathcal{A}_{11} = (\mathbb{CP}^2, 2Q + \sum_{i=1}^{3} 4T_i + 2H_4)$ in the Table 6.3 and choose coordinates such that the equations of lines and quadric are as stated in the Table 6.3. The uniformizer of the sub-orbifold $(\mathbb{CP}^2, \sum_{i=1}^{3} 2T_i)$ is φ_2. Denote by T_1, T_2, T_3, Q' and H_4', the liftings $\varphi_2^{-1}(T_1) = \{X = 0\}$, $\varphi_2^{-1}(T_2) = \{Y = 0\}$, $\varphi_2^{-1}(T_3) = \{Z = 0\}$, $\varphi_2^{-1}(Q) = \{(X^2+Y^2-Z^2)^2 - 4X^2Y^2 = 0\}$ and $\varphi_2^{-1}(H_4) = \{Z^2 - X^2 = 0\}$, respectively. Notice that Q' consists of four lines $X \mp Y \mp Z = 0$ and H_4' consists of two lines $Z \mp X = 0$. The configuration of these six lines forms a complete quadrilateral. If one add the lines T_1, T_2 and T_3 to complete quadrilateral, then he will get an arrangement of nine lines projectively equivalent to the arrangement in Figure 6.18. Therefore, $(\mathbb{CP}^2, 2Q' + \sum_{i=1}^{3} 2T_i + 2H_4')$ is the orbifold $\mathcal{A}_{10}$ in the Table 6.3 and we have an orbifold covering $\varphi_2 : \mathcal{A}_{10} \to \mathcal{A}_{11}$.

If one considers the sub-orbifold $(\mathbb{CP}^2, \sum_{i=1}^{3} 4T_i)$ whose uniformizer is φ_4, the liftings Q'' and H_4'' will consist of four quadrics $X^2 \mp Y^2 \mp Z^2 = 0$ and four lines $Z^4 - X^4 = 0$. Notice that Q'' is the Naruki arrangement and H_4'' consists of four pairwise common tangents of the quadrics in Q''. Thus, $(\mathbb{CP}^2, 2Q'' + 2H_4')$ is the the orbifold $\mathcal{A}_{21}$ in the Table 6.3 and we have an orbifold covering $\varphi_4 : \mathcal{A}_{21} \to \mathcal{A}_{11}$.

Next, consider the sub-orbifold $(\mathbb{CP}^2, 2T_1 + 2T_2 + 2H_4)$ whose uniformizer is φ_2. Choose coordinates so that $T_1 : X = 0$, $T_2 : Y = 0$, $T_3 : Z - X = 0$, $H_4 : Z = 0$ and $Q : (Y+Z)^2 - 4XY = 0$ and set $T_1 := \varphi_2^{-1}(T_1) = \{X = 0\}$, $T_2 := \varphi_2^{-1}(T_2) = \{Y = 0\}$, $T_3' := \varphi_2^{-1}(T_3) = \{Z^2 - X^2 = 0\}$ and $Q' = \varphi_2^{-1}(Q) = \{(Y^2+Z^2)^2 - 4X^2Y^2 = 0\}$. Notice that Q' consists of two quadrics $Y^2 + Z^2 \mp 2XY = 0$ with a tacnode, and T_3' consists of common tangent lines of these quadrics, while T_2 is a common tangent at tacnode. In addition, T_1 passes through the nodal intersection points of these quadrics and the singular point of T_3'. Then an arrangement of T_1, T_2, T_3' and Q' is exactly the arrangement in Figure 6.31. Therefore, $(\mathbb{CP}^2, Q' + 2T_1 + 2T_2 + 2H_4')$ is the the orbifold $\mathcal{A}_{18}$ in the Table 6.3 and we have an orbifold covering $\varphi_2 : \mathcal{A}_{18} \to \mathcal{A}_{11}$.

Coverings of $\mathcal{A}_{12}$:

Consider the orbifold $\mathcal{A}_{12} = (\mathbb{CP}^2, 4Q + 4T_1 + 2T_2 + 4T_3 + 2T_4 + 2H_5)$ in the Table 6.3 and choose coordinates such that the equations of lines and quadric are as stated in the Table 6.3. The uniformizer of the sub-orbifold $(\mathbb{CP}^2, \sum_{i=1}^{3} 2T_i)$ is φ_2. Denote by T_1, T_3, T_4', Q' and H_5', the liftings $\varphi_2^{-1}(T_1) = \{X = 0\}$, $\varphi_2^{-1}(T_3) = \{Z = 0\}$, $\varphi_2^{-1}(T_4) = \{2X^2 - Y^2 + Z^2 = 0\}$, $\varphi_2^{-1}(Q) = \{(X^2 + Y^2 - Z^2)^2 - 4X^2Y^2 = 0\}$ and $\varphi_2^{-1}(H_5) = \{Z^2 - X^2 = 0\}$, respectively. Notice that Q' consists of four lines $X \mp Y \mp Z = 0$ tangent to the quadric T_4', and H_5' consists of two lines $Z \mp X = 0$ through the tangency points $Q' \cap T_4'$. In addition, the lines T_1, T_2 and H_5' goes through the singular points of Q'. Configuration of such quadric and lines are projectively equivalent to the arrangement in Figure 6.26. Therefore, $(\mathbb{CP}^2, 4Q' + 2T_1 + 2T_3 + 2T_4' + 2H_5')$ is the orbifold $\mathcal{A}_{13}$ in the Table 6.3 and we have an orbifold covering $\varphi_2 : \mathcal{A}_{13} \to \mathcal{A}_{12}$.

Next, consider the sub-orbifold $(\mathbb{CP}^2, 2T_1 + 2T_2 + 2H_5)$ whose uniformizer is φ_2. Choose coordinates so that $T_1 : X = 0$, $T_2 : Y = 0$, $T_3 : X - Z = 0$, $T_4 : 4X - Y - 2Z = 0$, $H_5 : Z = 0$ and $Q : (Y + Z)^2 - 4XY = 0$, and set $T_1 := \varphi_2^{-1}(T_1) = \{X = 0\}$, $T_3' := \varphi_2^{-1}(T_3) = \{X^2 - Z^2 = 0\}$, $T_4' := \varphi_2^{-1}(T_4) = \{4X^2 - Y^2 - 2Z^2 = 0\}$ and $Q' = \varphi_2^{-1}(Q) = \{(Y^2 + Z^2)^2 - 4X^2Y^2 = 0\}$. Notice that Q' consists of two quadrics $Y^2 + Z^2 \mp 2XY = 0$ with a tacnode, and T_3' consists of common tangent lines of these quadrics, while T_1 passes through the nodal intersection points of these quadrics and the singular point of T_3'. The quadric T_4' has contacts of order four with the quadrics $Y^2 + Z^2 \mp 2XY = 0$. Then an arrangement of these quadrics and lines is exactly the arrangement in Figure 6.30. Therefore, $(\mathbb{CP}^2, 4Q' + 2T_1 + 4T_3' + 2T_4' + 2H_5')$ is the the orbifold $\mathcal{A}_{17}$ in the Table 6.3 and we have an orbifold covering $\varphi_2 : \mathcal{A}_{17} \to \mathcal{A}_{12}$.

Third, consider the sub-orbifold $(\mathbb{CP}^2, 2T_2 + 2T_4 + 2H_5)$ whose uniformizer is φ_2. Choose coordinates so that $T_1 : 2X + Y + Z = 0$, $T_2 : Y = 0$, $T_3 : -2X + Y + Z = 0$, $T_4 : Z = 0$, $H_5 : X = 0$ and $Q : X^2 - YZ = 0$, and set $T_1' := \varphi_2^{-1}(T_1) = \{2X^2 + Y^2 + Z^2 = 0\}$, $T_3' := \varphi_2^{-1}(T_3) = \{-2X^2 + Y^2 + Z^2 = 0\}$ and $Q' = \varphi_2^{-1}(Q) = \{X^4 - Y^2Z^2 = 0\}$. Notice that Q' consists of two quadrics $X^2 \mp YZ = 0$ with tacnode. If one check the intersections points of these quadrics, he will release that these four quadrics $X^2 \mp YZ = 0$ and $\mp 2X^2 + Y^2 + Z^2 = 0$ has twelve tacnodes and it is projectively equivalent to the Naruki arrangement. Therefore, $(\mathbb{CP}^2, 4Q' + 4T_1' + 4T_3')$ is the the orbifold $\mathcal{A}_{20}$ in the Table 6.3 and we have an orbifold covering $\varphi_2 : \mathcal{A}_{20} \to \mathcal{A}_{12}$.

Coverings of $\mathcal{A}_{14}$:

Consider the orbifold $\mathcal{A}_{14} = (\mathbb{CP}^2, 2Q + 4H_1 + 4H_2 + \sum_{i=3}^{6} 2H_i)$ in the Table 6.3. The uniformizer of the sub-orbifold $(\mathbb{CP}^2, 2H_1 + 2H_2 + 2H_6)$ is φ_2. For simplicity, let us choose homogeneous coordinates such that $H_1 : X = 0$, $H_2 : Z = 0$, $H_3 : X + 2Y + Z = 0$, $H_4 : X - 2Y + Z = 0$, $H_5 : X + Z = 0$, $H_6 : Y = 0$ and $Q : Y^2 - XZ = 0$. Let $H_1 : X = 0$, $H_2 : Z = 0$, $H_3' : X^2 + 2Y^2 + Z^2 = 0$, $H_4' : X^2 - 2Y^2 + Z^2 = 0$, $H_5' : X^2 + Z^2 = 0$ and $Q' : Y^4 - X^2Z^2 = 0$ be the liftings of the lines H_i and the quadric Q, respectively. Notice that Q' has two quadrics $Y^2 \mp XZ = 0$, and they form a Naruki arrangement together with the quadrics H_3' and H_4'. Also, the pencil $XZ(X^2 + Z^2) = 0$ consists of four pairwise common tangents of these quadrics. Hence $(\mathbb{CP}^2, 2H_1 + 2H_2 + 2H_3 + 2H_4' + 2H_5' + 2Q')$ is the orbifold $\mathcal{A}_{21}$ in the Table 6.3 and e have the covering $\varphi_2 : \mathcal{A}_{21} \to \mathcal{A}_{14}$.

Next consider the sub-orbifold $(\mathbb{CP}^2, 2H_2 + 2H_3 + 2H_4)$ whose uniformizer is φ_2. Projective transformations allow us to change coordinates such that $H_1 : X + Y - Z = 0$, $H_2 : Z = 0$, $H_3 : Y = 0$, $H_4 : X = 0$, $H_5 : X + Y = 0$, $H_6 : X - Y = 0$ and $Q : (X + Y - 2Z)^2 - 4XY = 0$. Let $H_1' : X^2 + Y^2 - Z^2 = 0$, $H_2 : Z = 0$, $H_5' : X^2 + Y^2 = 0$, $H_6' : X^2 - Y^2 = 0$ and $Q' : (X^2 + Y^2 - 2Z^2)^2 - 4X^2Y^2 = 0$ be the liftings of the lines H_1, H_2, H_5, H_6 and the quadric Q by φ_2, respectively. Notice that Q' consists of four lines $X \mp Y \mp \sqrt{2}Z = 0$ which are tangent to the quadric H_1', and components of H_6' goes through this tangency points $Q' \cap H_1'$. In addition H_5' consists of two imaginary lines tangent to Q_1 and the line H_2 at infinity pass through these tangency points. If one a picture of the arrangement of these lines and quadric, he will release that it is projectively equivalent to the arrangement in Figure 6.34. Hence, $(\mathbb{CP}^2, 2Q' + 4H_1' + 2H_2 + 2H_4 + 2H_5' + 2H_6')$ is the orbifold $\mathcal{A}_{26}$ in the Table

6.3 and we have the covering $\varphi_2 : \mathcal{A}_{26} \to \mathcal{A}_{14}$.

The following diagram in Figure 6.35 exhibits all covering relations among ball-quotient orbifolds discussed above.

$$
\begin{tikzcd}
 & & & \mathcal{A}_6 & & \\
\mathcal{A}_7 & \mathcal{A}_{22} \arrow[l, "\varphi_3"'] & \mathcal{A}_3 & \mathcal{A}_5 \arrow[l, "\varphi_2"'] \arrow[u, "\varphi_2"'] & \mathcal{A}_{23} \arrow[l, "\varphi_3"'] & \mathcal{A}_{18} \arrow[dl, "\varphi_2"] \\
\mathcal{A}_{25} \arrow[dr, "\varphi_2"] & \mathcal{A}_{26} \arrow[r, "\varphi_2"] \arrow[d, "\varphi_2"] & \mathcal{A}_{14} \arrow[d, "\varphi_2"] & \mathcal{A}_{21} \arrow[l, "\varphi_2"'] \arrow[r, "\varphi_4"] \arrow[dr, "\varphi_2"'] & \mathcal{A}_{11} & \mathcal{A}_{19} \arrow[dl, "\varphi_2"'] \\
\mathcal{A}_1 & \mathcal{A}_9 \arrow[l, "\varphi_2"'] \arrow[r, "\varphi_2"] & \mathcal{A}_8 & & \mathcal{A}_{10} \arrow[ll, "\varphi_2"'] \arrow[u, "\varphi_2"] & \mathcal{A}_{16} \arrow[l, "\varphi_2"'] \\
\mathcal{A}_{20} \arrow[u, "\varphi_4"] \arrow[ur, "\varphi_2"'] \arrow[rr, bend right, "\varphi_2"'] & \mathcal{A}_{13} \arrow[u, "\varphi_2"'] \arrow[r, "\varphi_2"] & \mathcal{A}_{12} \arrow[u, "\varphi_2"] & \mathcal{A}_{17} \arrow[l, "\varphi_2"] & \mathcal{A}_{24} \arrow[ull, "\varphi_4"] \arrow[u, "\varphi_2"] & \mathcal{A}_{15} \arrow[ul, "\varphi_2"]
\end{tikzcd}
$$

Figure 6.35 Covering relations among ball-quotient orbifolds in Table 6.3.

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