DOKUZ EYLÜL UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

MATHEMATICAL MODELS AND METHODS OF WAVE THEORY

by

Meltem ALTUNKAYNAK

August, 2010
İZMİR

MATHEMATICAL MODELS AND METHODS OF WAVE THEORY

A Thesis Submitted to the
Graduate School of Natural and Applied Sciences of Dokuz Eylül University
In Partial Fulfillment of the Requirements for the Degree
of Doctor of Philosophy in Mathematics

by

Meltem ALTUNKAYNAK

August, 2010
İZMİR

Ph.D. THESIS EXAMINATION RESULT FORM

We have read the thesis entitled **"MATHEMATICAL MODELS AND METHODS OF WAVE THEORY"** completed by **MELTEM ALTUNKAYNAK** under supervision of **PROF. DR. VALERY YAKHNO** and we certify that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Doctor of Philosophy.

. . . .

Prof. Dr. Valery YAKHNO

Supervisor

. . . .

Prof. Dr. Şennur SOMALI

Thesis Committee Member

. . . .

Yrd.Doç.Dr. Özlem Ege ORUÇ

Thesis Committee Member

. . . .

Examining Committee Member

. . . .

Examining Committee Member

Prof. Dr. Mustafa SABUNCU
Director
Graduate School of Natural and Applied Sciences

ACKNOWLEDGEMENTS

I would like to thanks first to my thesis advisor Prof.Dr. Valery Yakhno, for his invaluable guidance. He allowed this thesis to be my own work, but steered me in the right direction whenever he thought I needed it.

I would also want to thank the other members of my thesis committe: Prof.Dr. Şennur Somalı and Yrd.Doç.Dr. Özlem Ege Oruç for their helpful suggestions for my thesis and endless patience.

Last but not least, I am extremely grateful to my mother, my father, my brother and my husband for their endless patience, understanding, constant encouragement and confidence to me throughout my life.

Meltem ALTUNKAYNAK

MATHEMATICAL MODELS AND METHODS OF WAVE THEORY

ABSTRACT

In this thesis, initial value problems for the electromagnetic system and system of elasticity are studied. Properties of solutions for considered initial value problems are proved. Using these properties analytic methods are suggested for problems solving. These methods are based on symbolic transformations. As applications of these methods we construct the fundamental solutions for equations of anisotropic elasticity and derive electric fields, when the current density is presented in the polynomial form. Robustness of the methods are confirmed by computational examples. Simulations of elastic and electric fields are obtained in different anisotropic materials.

Keywords: Analytical Method, System of Crystal Optics, Elastic System, Initial Value Problem, Symbolic Computations, Fundamental Solutions

DALGA TEORİSİNİN MATEMATİKSEL METODLARI VE MODELLEMESİ

ÖZ

Bu tezde, electromanyetik ve elastik sistemler için başlangıç değer problemleri çalışılmıştır. Bu sistemlerin başlangıç değer problemlerinin çözümlerinin özellikleri ispatlanmıştır. Bu özellikler kullanılarak problemlerin çözümü için analitik metodlar önerilmiştir. Bu metodlar sembolik dönüşümlere dayanmaktadır. Bu metodların uygulaması olarak izotropik olmayan elastik sistemlerin temel çözümleri bulunmuş ve akım yoğunluğu polinom formunda olan sistemler için elektrik alan bulunmuştur. Metodların güvenilirliği örneklerle doğrulanmıştır. Elastik ve elektrik alanların izotropik olmayan farklı materyallerde simulasyonu yapılmıştır.

Anahtar sözcükler: Analitik Metod, Kristal Optik Sistemleri, Elastik Sistem, Başlangıç Değer Problemleri, Sembolik Hesaplamalar, Temel Çözümler

CONTENTS **Page**

CHAPTER ONE
INTRODUCTION

Search and development of new materials with specific properties are needed for different industries such as chemistry, microelectronics, etc. When new materials are created we must be able to have the possibility to model and study their properties. Mathematical models of physical processes can provide cutaway views that let you see aspects of something that would be invisible in the real artifact but computer models can also provide visualization tools.

The physical properties of a homogeneous isotropic medium do not depend on the direction and the position inside the medium. Physical properties of anisotropic media essentially depend on orientation and position. An anisotropic medium is called homogeneous when its physical properties depend on orientation and do not depend on position. The medium can be isotropic relative to some physical properties and anisotropic with respect to others. For example, anisotropic crystals and dielectrics are magnetically isotropic but electrically anisotropic. Some of materials are magnetically anisotropic but electrically isotropic and some of materials are electrically and magnetically anisotropic. Anisotropy of materials is related to their atomic lattice. A smallest block (three-dimensional array of atoms) of anisotropic materials is determined by repeated replication in three dimensions. Its symmetry tells how the constituent atoms are arranged in a regular repeating configuration. The structure of these three-dimensional unit cell of atoms in anisotropic materials may have one of seven basic shapes: cubic, hexagonal, tetragonal, trigonal, orthorhombic, monoclinic and triclinic (see, for example (Nye, 1967)). We need to note that anisotropy can be in the response to external fields (electric, magnetic, elastic fields, etc.) (Ramo & Whinnery & Duzer, 1994).

Thesis includes mathematical modeling and simulating the wave propagation in anisotropic solids and crystals.

Electromagnetic waves phenomenon is very well studied for different isotropic materials (Kong, 1986, Ramo & Whinnery & Duzer, 1994, Monk, 2003, Eom, 2004).

At the recent time the use and development of new anisotropic materials stimulates the growing interest for modelling electric and magnetic wave propagations inside these materials. This topic is an important interdisciplinary area of research with many cutting-edge scientific and technological applications.

Electric wave propagations in electrically and magnetically anisotropic materials is one of the objects of the thesis. Electric fields inside these materials are described by the time-dependent system (Cohen & Heikkola & Joly, 2003)

$$\mathcal{E}\frac{\partial^2 \mathbf{E}}{\partial t^2} \quad + \quad curl_x(\mathcal{M}^{-1} curl_x \mathbf{E}) = -\frac{\partial \mathbf{j}}{\partial t}, \tag{1.0.1}$$

where $x = (x_1, x_2, x_3)$ is a space variable from R^3, t is a time variable from R, $\mathbf{E} = (E_1, E_2, E_3)$ is the electric field, $E_k = E_k(x,t), \quad k = 1,2,3; \quad \mathbf{j} = (j_1, j_2, j_3)$ is the density of the electric current, $j_k = j_k(x,t), \quad k = 1,2,3$; $\mathcal{E} = (\varepsilon_{ij})_{3\times 3}$ is the permittivity matrix, $\mathcal{M}$ is the permeability matrix, $\mathcal{M}^{-1} = (\mu_{ij})_{3\times 3}$ is the matrix which is inverse to $\mathcal{M}$.

For a particular case, when $\mathcal{E} = diag(\varepsilon_{11}, \varepsilon_{11}, \varepsilon_{33})$, $\varepsilon_{jj} > 0, j = 1,2,3$; and $\mathcal{M} = diag(\mu, \mu, \mu)$, $\mu > 0$, the system (1.0.1) describes electric waves inside many crystals and is called the time-dependent system of crystal optics (Courant & Hilbert, 1979), pages 603-612). The Cauchy problem for this system with smooth data and the procedure of constructing an exact solution of this problem has been described by Courant and Hilbert in (Courant & Hilbert, 1979). Modelling and simulating electric waves in crystals by different procedures and explicit formulae for solutions of the initial value problems for the system of crystal optics are important issue of the modern research of material structures. Burridge and Qian in (Burridge, 2006) have used a plane wave approach to obtain an explicit formula for a fundamental solution of the same system of crystal optics. This formula has been used for modelling and simulating electric waves in anisotropic crystals with biaxial structures of anisotropy.

Different methods have been used to study problems for the system (1.0.1) in some particular cases. For example, decomposition method for the case of isotropic materials ($\mathcal{E}$ is a diagonal matrix of the form $\mathcal{E} = diag(\varepsilon_{11}, \varepsilon_{11}, \varepsilon_{11})$) has been suggested in (Linden, 1990). Analytic methods of Green's functions constructions have been studied for the case of isotropic materials in (Haba, 2004, Wijnands, 1997); for uniaxial anisotropic media ($\mathcal{E} = diag(\varepsilon_{11}, \varepsilon_{11}, \varepsilon_{33})$) in (Li, 2001, Gottis & Kondylis, 1995); for biaxial anisotropic crystals ($\mathcal{E} = diag(\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33})$) in (Ortner & Wagner, 2004); for arbitrary non-dispersive homogeneous anisotropic dielectrics ($\mathcal{E} = (\varepsilon_{ij})_{3\times 3}$ is a symmetric positive definite matrix) in (Ortner & Wagner, 2004, Yakhno, 2005, Yakhno & Kasap, 2006). Modelling lossy anisotropic dielectric waveguides with the method of lines has been made for inhomogeneous biaxial anisotropic media in (Berini & Wu, 1996).

Most of the studies and modelling electromagnetic waves had been made by numerical methods, in particular finite element method (Monk, 2003, Zienkiewicz, 2000, Cohen & Heikkola & Joly, 2003, Werner & Cary, 2007).

The propagation of elastic waves in a homogeneous solid is governed by a hyperbolic system of three linear second-order partial differential equations with constant coefficients. When the solid is also isotropic, the form of these equations is well known and provides the foundation of the conventional theory of elasticity (Love, 1944). The explicit solution of the initial value, or Cauchy, problem for the isotropic case was found by Poisson, and in a different way by (Stokes, 1883).

A mathematical model of wave propagations in anisotropic elastic materials is described by the dynamic system of anisotropic elasticity which usually has been studied by the plane wave approach (Fedorov, 1963) and (Ting, 1996).

The mathematical model of elastic wave propagation in a homogeneous, anisotropic medium is described by

$$\rho \frac{\partial^2 u_j}{\partial t^2} = \sum_{k=1}^{3} \frac{\partial \sigma_{jk}}{\partial x_k} + f_j, \quad j = 1, 2, 3, \tag{1.0.2}$$

where $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, $t > 0$, $u_j(x,t)$ are the components of the unknown displacement vector. The constant $\rho > 0$ is the density of the medium. Stress tensor

σ_{jk} are defined as

$$\sigma_{jk} = \sum_{l,m=1}^{3} c_{jklm}\epsilon_{lm}, \tag{1.0.3}$$

where

$$c_{jklm} = \left(\frac{\partial \sigma_{jk}}{\partial \epsilon_{lm}}\right)_{\epsilon_{lm}=0}, \tag{1.0.4}$$

and

$$\epsilon_{lm} = \frac{1}{2}\left(\frac{\partial u_l}{\partial x_m} + \frac{\partial u_m}{\partial x_l}\right). \tag{1.0.5}$$

$\{c_{jklm}\}_{j,k,l,m=1}^{3}$ are elastic moduli which is a forth-order positive definite constant tensor that satisfy the symmetry conditions $c_{jklm} = c_{kjlm} = c_{jkml}$. Due to the symmetry properties, it is convenient to represent the fourth-order tensor of elastic moduli in terms of a 6×6 matrix, which we denote by $\mathbf{C}$. This representation is realized by replacing the pairs (j,k) of indices $j,k = 1,2,3$ with a single index $\alpha = 1,\ldots,6$ according to the following rules:

$$\begin{aligned} &(1,1) \longleftrightarrow 1, \qquad (2,2) \longleftrightarrow 2, \qquad (3,3) \longleftrightarrow 3, \\ &(2,3),(3,2) \longleftrightarrow 4, \qquad (1,3),(3,1) \longleftrightarrow 5, \qquad (1,2),(2,1) \longleftrightarrow 6. \end{aligned} \tag{1.0.6}$$

Similarly, replacing the pairs (l,m) of indices $l,m = 1,2,3$ with index $\beta = 1,\ldots,6$ in accordance with in accordance with (1.0.6) gives

$$c_{\alpha\beta} = c_{jklm},$$

where $c_{\alpha\beta}$ are components of the matrix $\mathbf{C}$. From the property $c_{jklm} = c_{lmjk}$, we have the symmetry condition $c_{\alpha\beta} = c_{\beta\alpha}$, which implies that $\mathbf{C}$ is a symmetric matrix. The matrix $\mathbf{C}$ is positive-definite. As a result, the tensor of elastic moduli can be written as a 6×6 symmetric, positive- definite matrix

$$\mathbf{C} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} \\ c_{12} & c_{22} & c_{23} & c_{24} & c_{25} & c_{26} \\ c_{13} & c_{23} & c_{33} & c_{34} & c_{35} & c_{36} \\ c_{14} & c_{24} & c_{34} & c_{44} & c_{45} & c_{46} \\ c_{15} & c_{25} & c_{35} & c_{45} & c_{55} & c_{56} \\ c_{16} & c_{26} & c_{36} & c_{46} & c_{56} & c_{66} \end{bmatrix}, \tag{1.0.7}$$

with 21 independent components in general.

(Carcione & Kosloff & Kosloff, 1988) presented theoretical study for wave-propagation simulation in a transversely isotropic material. A pseudospectral time-integration technique to solve the equation of motion is used, where the propagation is done by a direct expansion of the evolution operator by a Chebycheff polynomial series.

However, nowadays there is a great interest to develop new methods for solving initial value problems (IVPs) and initial boundary value problems (IBVPs) for the dynamic system of anisotropic elasticity and simulate invisible elastic waves (Cohen, 2002), (Cohen & Heikkola & Joly, 2003) and (Yakhno & Akmaz, 2005). Most of the time the numerical methods, in particular the finite element method, are used for solving this kind of problems. Advantages and disadvantages of these methods are well known (Cohen & Heikkola & Joly, 2003) and (Zienkiewicz, 2000). Generally speaking, they are of a general purpose, rather labor-consuming, find approximate solutions, but do not always satisfy scientists and engineers at the needed scale and accuracy (Pavlovic, 2003). At the same time, analytic methods can provide the exact solution of the equations and also offer a fundamental understanding of the relevant physical phenomena. Unfortunately, the exact solutions cannot be found for all complex equations and systems. But when the exact solutions can be found it leads to a significant simplification of modeling and simulation. The modern methods of symbolic computations allow us to automate mathematical transformations on a very high level of complexity thanks to the truly remarkable achievements in computing power over the last decade (Pavlovic, 2003).

On the other hand nowadays computers can perform very complicated symbolic computations (in addition to numerical calculations) and this opens up new possibilities in modelling and simulation of wave propagation phenomena. Symbolic computations can be considered as a useful tool for analytical methods which can provide exact solutions of problems (Yakhno, 2005, Pavlovic, 2003, Pavlovic & Sapountzakis, 1986). Unfortunately the exact solutions can not be found for all complex equations and systems. But when the exact solution can be found it leads to the significant simplification of modelling and simulation. As it is mentioned in (Beltzer, 1990), 'the easiness with which these symbolic codes provide analytical results allows engineer to focus on the ideas rather than on overcoming calculational difficulties'. A successful application of an analytical approach based on symbolic computations of the initial value problem for the system (1.0.1) in the case when ($\mathcal{E} = (\varepsilon_{ij})_{3\times 3}$ is a symmetric positive definite matrix) and ($\mathcal{M} = diag(\mu, \mu, \mu),\ \mu > 0$ have been applied in (Yakhno & Kasap, 2006), (YakhnoV & YakhnoT, 2007).

The theory of generalized functions has exerted a strong influence on the development of the theory of linear differential equations. It is only in the setting of Laurent Schwartz' theory of distributions that fundamental solutions can be defined in general and can be applied -via the convolution of distributions- to the solution of linear partial differential equations with constant coefficients. In part, the relevant concepts were worked out by John Horvth, (Horváth, 1966)-(Horváth, 1977). The first use of a non-trivial fundamental solution can probably be ascribed to Jean dAlembert. In 1747, he considered the deflection u of a vibrating string. In 1789, Pierre Simon de Laplace used the fundamental solution ε of the elliptic operator $\triangle$, which bears his name, and thereby established the connexion of the Laplace operator with the Newtonian gravitational potential (Laplace, 1787). Laplace just recognized that $\triangle(\varepsilon * f) = 0$ outside the support of f , and it was Simon Denis Poisson, who obtained the equation $\triangle(\varepsilon * f) = f$ in 1813 (Poisson, 1813). In 1809, Laplace considered the first parabolic operator, namely the heat operator, and calculated its fundamental solution in the case $n = 1$, (Laplace, 1813). The generalization to higher n, in particular to $n = 2$, was found by Poisson in 1818 (Poisson, 1818). In 1818, Joseph Fourier was able to calculate the fundamental solution ε of the operator of the dynamic deflections of beams, an operator of fourth order (Fourier, 1818). As well in 1818, Poisson generalized d'Alembert's formula to three space dimensions by representing the solutions of the wave operator as convolution with the fundamental solution (Poisson, 1818). This notation, viz. the first use of Dirac's delta function, goes back to Gustav Kirchhof's paper of 1882 (Lützen, 1982). In 1849, George Stokes obtained -as the kernel of an integral representation- the fundamental matrix E of the system of partial differential operators which describes elastic waves in isotropic media (Stokes, 1883). This system can be found already in a memoir of 1829 by

Poisson (Poisson, 1829). The fundamental solution ε of the wave operator in two space dimensions was found as late as 1894 by Vito Volterra, (Volterra, 1894). Investigating the equations of static anisotropic elasticity, Ivar Fredholm found in 1900 (Fredholm, 1908) the fundamental matrix E of the elliptic 3 by 3 system of linear partial differential operators in three variables with constant coefficients and homogeneous of second order. In 1908, Fredholm succeeded in representing the fundamental solutions of elliptic homogeneous operators in 3 variables by Abelian integrals (Fredholm, 1908). In 1911, Nils Zeilon gave the first definition of a fundamental solution in case it is a locally integrable function (Zeilon, 1911). In 1913, Zeilon transferred Fredholm' fundamental solution results to non-elliptic operators (Zeilon, 1913). However, explicit formulae were found recently, (Wagner, 1999)-(Wagner, 2001). In three famous papers from 1926 to 1928 (Herglotz, 1926), Gustav Herglotz overcame the restriction to 2 or 3 independent variables and represented the fundamental solutions of elliptic and of strictly hyperbolic homogeneous operators of the degree m in n variables (with $n \leq m$) by $(n-1)$- fold and by $(n-2)$-fold integrals, respectively. Later, these formulae came to be known as the HerglotzûPetrovsky formulae. In 1945, Ivan Petrovsky represented -in the hyperbolic case- the fundamental solution ε by integrals over cycles in complex projective space and investigated the lacunas of ε by means of algebraic topology (Petrowsky, 1945). In 1950/51, Laurent Schwartz first published his Thorie des Distributions (Schwartz, 1966), in which framework he also gave the general definition of fundamental solutions. In 1952, Jean Leray stated a distributional version of the Herglotz-Petrovsky formulae for homogeneous hyperbolic operators, thereby also treating the case $m < n$ (Leray, 1953). The same goal was reached in 1959 by Vladimir A. Borovikov for operators of principal type (Borovikov, 1959) and presented in the textbook "Generalized Functions" by Israel M. Gel'fand and Georgi E. Shilov (Gel'fand, 1964). The first existence proofs for fundamental solutions $\varepsilon(x)$ in $\mathbb{D}'$os any linear differential operator $P(D) \neq 0$ with constant coefficients were given in 1953/54 by Bernard Malgrange and Leon Ehrenpreis (Ehrenpreis, 1960), (Malgrange, 1955). These proofs for fundamental solution were based on the Hahn-Banach theorem. In 1957, Lars Hormander showed that there always exist "regular" fundamental solutions (at that time called "proper" fundamental solutions) having "best" regularity properties (Hörmander, 1957). The existence of fundamental solutions depending C^{∞} or even holomorphic (in case of "constant strength") on the coefficients of $P(\partial)$ was proved by Francois Treves, (Trèves, 1962)-(Trèves, 1966); see also the survey paper (Ortner, 1997). A convenient tool to find a fundamental solution with the required properties of growth, of support, of smoothness, is the Fourier transform. The problem of seeking a fundamental solution of slow growth turns out to be a special case of the more general problem of ”dividing” a generalized function of slow growth by a polynomial. In 1957/58, Lars Hörmander and Stanislaw Lojasiewicz independently solved the ”division problem” and thereby proved the existence of temperate fundamental solutions (Hörmander, 1955), (Łojasiewicz, 1959). Different proofs for fundamental solution thereof were found later by Michael F. Atiyah (Atiyah, 1970) and Joseph N. Bernstein (Bernšteĭn, 1971). In 1970/73, Michael Atiyah, Raoul Bott, and Lars Garding extended and generalized Petrovsky's work, thereby developing a general theory of fundamental solutions of hyperbolic operators, (At′ya & Bott, 1984)). For general operators, this was established in the fundamental work of Lars Hörmander, (Hörmander, 1958), (Hörmander, 1963), (Hörmander, 1983). We also mention the first major table of fundamental solutions by Norbert Ortner in 1980 (Ortner, 1980) and the discovery of the connexion of lacunas of fundamental solutions with the existence of right inverses by Reinhold Meise, B. Alan Taylor, and Dietmar Vogt in 1990 (Meise, 1990).

In the middle of the nineteenth century, Lord Kelvin became the first to obtain the fundamental solutions or Greens function for 3D deformations of an infinite isotropic elastic solid subject to a point force. For materials that exhibit isotropic behaviour, expressions for the Greens function have been well established (see, e.g. (Mindlin, 1936); (Mindlin & Cheng, 1950); (Phan-Thien, 1983); (Huang & Wang, 1991)). Recently, (Ma & Lin, 2001) re-examined the Greens function for 2D plane stress and plane strain problems in an elastic half space with a free or rigidly fixed surface subject to line forces and line dislocations. For materials possessing anisotropic elasticity, the 3D Greens function for an infinite space has been investigated by (Fredholm, 1908), (Synge, 1957), (Barnett, 1972) and (Mura, 1987). Recently, (Ting & Lee, 1997) obtained the 3D elastostatic Greens function for a general anisotropic linear elastic solid. The novel feature of this work is that the Greens function is given explicitly in terms of the Stroh eigenvalues. In the case of an anisotropic half space, (Willis, 1966) obtained the Fourier integral representation of the surface Greens function due to the application of a point force on the surface. (Barnett & Lothe, 1975) obtained a line integral expression of the surface Greens function due to a point force applied on the free surface of an anisotropic half space based on Stroh formalism. (Walker, 1993) discusses the development of a Fourier integral representation of the Greens function for an anisotropic elastic half space. (Wu, 1998) employs the Stroh formalism in the Radon-transformed domain to derive the 3D Greens displacement function in an anisotropic half space due to a point force. (Suo, 1990) and (Qu & Li, 1991) obtained the Greens functions for an anisotropic bimaterial subjected to a line force and a line dislocation. (Pan & Yuan, 2000) studied the 3D Greens function for an anisotropic bimaterial using Stroh formalism and 2D Fourier transforms.

The plan of the thesis is as follows. In Chapter 1, we review the time dependent system of electric field, and linear system of elasticity. A brief historical background about development of these systems and the theory of generalized functions that has influence on the development of the theory of linear differential equations is given.

In Chapter 2, a new analytical method for computing a polynomial solution of the Cauchy problem for the system (1.0.1) is suggested. We suppose that initial data and inhomogeneous term have a polynomial presentation with respect to space variables and a solution of the initial value problem is found in the polynomial form with undetermined coefficients depending on the time variable. For these undetermined coefficients we find the recurrence relations and using these relations we obtain a procedure to recover the coefficients. The suggested method is based on this procedure and essentially uses symbolic computations. The implementation of our method is given in Maple 10. Stability estimates (energy inequalities) for solutions of (1.0.1) in a finite domain of the dependence (a finite domain containing characteristic cones) is described; using these stability estimates we show that polynomial solutions are approximate solutions of the initial value problems with non-polynomial smooth data. This theoretical result is confirmed by computational examples which compare an exact solution of the initial value problem corresponding to the given non-polynomial data and polynomial solutions which are found by polynomial approximations of given data and our method. The application of our method for computing electric fields and simulating their images in different anisotropic media (in particular, the sapphire) when initial data are polynomial approximations of Shannon's kernels is described. The Shennon kernels are not polynomials and they are widely used for modelling different processes and phenomena (Bonciu & Leger & Thiel, 1998), (Wei, 2001). This method gives an exact solution of IVP for any type of anisotropy and enables to create simulations of elastic waves.

In Chapter 3, generalized Cauchy problem for elastic system is considered. A new method is explained to find fundamental solution. This method is based on properties: fundamental solutions of the considered system have finite supports with respect to space variables for any fixed time variable; the Fourier images of solution components are analytic functions with respect to parameters of the Fourier transform and these Fourier images can be expanded in power series. The method consists of following. The system of equations of anisotropic elasticity is written for each cases. These equalities are written in the form of the Fourier images. Using power series presentations with unknown coefficients depending on t we construct the recurrence relations. These unknown coefficients are obtained using a procedure. Using these coefficients Fourier images of solution components can be obtained. Applying inverse Fourier transform to these images, fundamental solutions of the system of anisotropic elasticity can be constructed. Using mathematical tools (Maple 10) simulation of fundamental solutions in different anisotropic materials are presented. Computation examples confirm the robustness of our approach. In the chapter, applications of the fundamental solutions for solving the Initial Value Problems (IVP) for the system of anisotropic elasticity is described.

CHAPTER TWO
POLYNOMIAL SOLUTION METHOD
SOLVING CAUCHY PROBLEM

2.1 System of Electromagnetism

In this chapter, the time-dependent system of partial differential equations of the second order describing the electric wave propagation in electrically and magnetically anisotropic media is considered. A new analytical method for computing polynomial solutions of the initial value problem for the considered system is suggested. This method essentially uses symbolic computations and is implemented in Maple 10. The theoretical study and computational analysis of polynomial solutions and their comparison with non polynomial solutions corresponding to smooth data are given.

Let us consider the system describing electromagnetic wave propagation. The propagation of electromagnetic waves is described by the time-dependent Maxwell's system with a matrix of dielectric permittivity. Let $x = (x_1, x_2, x_3)$ be a space variable from $\mathbb{R}^3$ and t be a time variable from $\mathbb{R}$ then Maxwell's system is given by the following relations (see, Cohen & Heikkola & Joly, 2003):

$$curl_x \mathbf{H} = \varepsilon \frac{\partial \mathbf{E}}{\partial t} + \mathbf{j}, \tag{2.1.1}$$

$$curl_x \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t}, \tag{2.1.2}$$

$$div_x(\mu \mathbf{H}) = 0, \tag{2.1.3}$$

$$div_x(\varepsilon \mathbf{E}) = \rho, \tag{2.1.4}$$

where $\mathbf{E} = (E_1, E_2, E_3)$, $\mathbf{H} = (H_1, H_2, H_3)$ are electric and magnetic fields, $E_k = E_k(x,t)$, $H_k = H_k(x,t)$, $k = 1,2,3$; $\mathbf{j} = (j_1, j_2, j_3)$ is the density of the electric current, $j_k = j_k(x,t)$, $k = 1,2,3$; ε, μ are symmetric, positive-definite, dielectric permittivity and magnetic permeability matrices depending on space, ρ is the density of electric charges. The conservation law of charges is given by

$$\frac{\partial \rho}{\partial t} + div_x \mathbf{j} = 0. \tag{2.1.5}$$

Differentiating (2.1.1) with respect to t and using (2.1.2) we obtain following equation (Cohen & Heikkola & Joly, 2003)

$$\varepsilon(x)\frac{\partial^2 \mathbf{E}}{\partial t^2} + curl_x\Big(\mu^{-1} curl_x \mathbf{E}\Big) = \mathbf{f}_1, \tag{2.1.6}$$

and differentiating (2.1.2) with respect to t and using (2.1.1) we obtain

$$\mu(x)\frac{\partial^2 \mathbf{H}}{\partial t^2} + curl_x\Big(\varepsilon^{-1} curl_x \mathbf{H}\Big) = \mathbf{f}_2, \tag{2.1.7}$$

where $\mathbf{f}_1 = -\dfrac{\partial \mathbf{j}}{\partial t}.$ and $\mathbf{f}_2 = curl(\varepsilon^{-1} j)$

The system defining electromagnetic wave propagation is given by equations (2.1.1)-(2.1.4) is rewritten by the equations (2.1.6), (2.1.7).

2.2 A New Method for Computing a Solution of the Cauchy Problem with Polynomial Data for the System of Crystal Optics

In this Section, initial value problem (IVP) for the system of crystal optics with polynomial data and a polynomial inhomogeneous term is solved using a new analytical method. The found solution of IVP is a polynomial. Computational analysis of polynomial solutions and their comparison with non polynomial solutions corresponding to smooth data are given. Implementation of this method has been made by symbolic computations in Maple 10.

2.2.1 Problem Set-up

Let $x \in R^3$, $t \geq 0$, $\mathbf{e} = (e_1, e_2, e_3)$, $\mathbf{g} = (g_1, g_2, g_3)$ be vector functions with components depending on x; $\mathbf{f} = (f_1, f_2, f_3)$, $\mathbf{E} = (E_1, E_2, E_3)$ be vector functions with components depending on x and t. Let $\mathcal{E} = diag(\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33})$ be a given matrix with positive elements. Let us consider the Initial Value Problem (IVP) of finding $\mathbf{E}$ satisfying

$$\mathcal{E}\frac{\partial^2 \mathbf{E}}{\partial t^2} + curl_x(curl_x \mathbf{E}) = \mathbf{f}(x,t), \quad x \in R^3,\ t > 0, \tag{2.2.1}$$

$$\mathbf{E}(x,0) = \mathbf{e}(x), \quad \left.\frac{\partial \mathbf{E}(x,t)}{\partial t}\right|_{t=0} = \mathbf{g}(x), \quad x \in R^3, \tag{2.2.2}$$

where $\mathbf{e}(x), \mathbf{g}(x)$ are given vector functions for $x \in R^3$, $\mathbf{f}(x,t)$ is a given vector function for $x \in R^3, t \geq 0$.

This problem is the main object of our study. In this section we assume that components of initial data $\mathbf{e}(x), \mathbf{g}(x)$ and the inhomogeneous term $\mathbf{f}(x,t)$ have the following polynomial form

$$\begin{aligned}
e_j(x) &= \sum_{k=0}^{p}\sum_{m=0}^{p}\sum_{n=0}^{p} e_j^{k,m,n} x_1^k x_2^m x_3^n, &(2.2.3)\\
g_j(x) &= \sum_{k=0}^{p}\sum_{m=0}^{p}\sum_{n=0}^{p} g_j^{k,m,n} x_1^k x_2^m x_3^n, &(2.2.4)\\
f_j(x,t) &= \sum_{k=0}^{p}\sum_{m=0}^{p}\sum_{n=0}^{p} f_j^{k,m,n}(t) x_1^k x_2^m x_3^n, &(2.2.5)
\end{aligned}$$

where p is a given nonnegative integer; $e_j^{k,m,n}, g_j^{k,m,n}$ are given real numbers; $f_j^{k,m,n}(t)$ are given continuously differentiable functions of t; $j = 1, 2, 3$.

The main goal of the study is to derive components of a solution $\mathbf{E}$ of IVP (2.2.1), (2.2.2) in the form

$$E_j(x_1, x_2, x_3, t) = \sum_{k=0}^{p}\sum_{m=0}^{p}\sum_{n=0}^{p} E_j^{k,m,n}(t) x_1^k x_2^m x_3^n. \tag{2.2.6}$$

2.2.2 Method of Computing a Polynomial Solution of IVP for Crystal Optics

The method consists in two steps. On the first step some recurrence relations are obtained and on the second one these relations are used for finding successively all polynomial coefficients $E_j^{k,m,n}(t)$. Let us consider these steps in details.

Recurrence Relations for $E^{k,m,n}$

Substituting (2.2.3)-(2.2.6) into (2.2.1), (2.2.2) we obtain

$$\sum_{k=0}^{p}\sum_{m=0}^{p}\sum_{n=0}^{p}\Bigg(\varepsilon_{11}\frac{\partial^2 E_1^{k,m,n}}{\partial t^2} + (k+1)(m+1)E_2^{k+1,m+1,n} - (m+2)(m+1)E_1^{k,m+2,n}$$

$$-(n+2)(n+1)E_1^{k,m,n+2} + (k+1)(n+1)E_3^{k+1,m,n+1} - f_1^{k,m,n}\Bigg)x_1^k x_2^m x_3^n = 0, \quad (2.2.7)$$

$$\sum_{k=0}^{p}\sum_{m=0}^{p}\sum_{n=0}^{p}\Bigg(\varepsilon_{22}\frac{\partial^2 E_2^{k,m,n}}{\partial t^2} + (m+1)(n+1)E_3^{k,m+1,n+1} - (n+2)(n+1)E_2^{k,m,n+2}$$

$$-(k+2)(k+1)E_2^{k+2,m,n} + (k+1)(m+1)E_1^{k+1,m+1,n} - f_2^{k,m,n}\Bigg)x_1^k x_2^m x_3^n = 0, \quad (2.2.8)$$

$$\sum_{k=0}^{p}\sum_{m=0}^{p}\sum_{n=0}^{p}\Bigg(\varepsilon_{33}\frac{\partial^2 E_3^{k,m,n}}{\partial t^2} + (k+1)(n+1)E_1^{k+1,m,n+1} - (k+2)(k+1)E_3^{k+2,m,n}$$

$$-(m+2)(m+1)E_3^{k,m+2,n} + (m+1)(n+1)E_2^{k,m+1,n+1} - f_3^{k,m,n}\Bigg)x_1^k x_2^m x_3^n = 0, \quad (2.2.9)$$

$$\sum_{k=0}^{p}\sum_{m=0}^{p}\sum_{n=0}^{p}\Big(E_j^{k,m,n}(0) - e_j^{k,m,n}\Big)x_1^k x_2^m x_3^n = 0, \quad (2.2.10)$$

$$\sum_{k=0}^{p}\sum_{m=0}^{p}\sum_{n=0}^{p}\Big(\frac{\partial E_j^{k,m,n}}{\partial t}\Big|_{t=0} - g_j^{k,m,n}\Big)x_1^k x_2^m x_3^n = 0,\ j = 1,2,3. \tag{2.2.11}$$

where $j = 1,2,3$ and components of $\mathbf{E}^{p+2,m,n}$, $\mathbf{E}^{k,p+2,n}$, $\mathbf{E}^{k,m,p+2}$, $\mathbf{E}^{p+1,m,n}$, $\mathbf{E}^{k,p+1,n}$, $\mathbf{E}^{k,m,p+1}$, $\mathbf{E}^{p+1,p+1,n}$, $\mathbf{E}^{k,p+1,p+1}$, $\mathbf{E}^{p+1,m,p+1}$ are equal to zero for all $k,m,n = 0,1,2,\dots,p$.

Equations (2.2.7)-(2.2.11) are equivalent to relations

$$\varepsilon_{jj}\frac{\partial^2 E_j^{k,m,n}}{\partial t^2} = f_j^{k,m,n}(t)$$

$$+D_j^{k,m,n}[\mathbf{E}^{k+2,m,n}, \mathbf{E}^{k,m+2,n}, \mathbf{E}^{k,m,n+2}, \mathbf{E}^{k+1,m,n+1}, \mathbf{E}^{k+1,m+1,n}, \mathbf{E}^{k,m+1,n+1}], \tag{2.2.12}$$

$$t > 0,\ j = 1,2,3,$$

$$E_j^{k,m,n}(0) = e_j^{k,m,n}, \quad \frac{\partial E_j^{k,m,n}(t)}{\partial t}\bigg|_{t=0} = g_j^{k,m,n},\ j = 1,2,3. \tag{2.2.13}$$

where

$$D_1^{k,m,n}[\mathbf{E}^{k+2,m,n}, \mathbf{E}^{k,m+2,n}, \mathbf{E}^{k,m,n+2}, \mathbf{E}^{k+1,m,n+1}, \mathbf{E}^{k+1,m+1,n}, \mathbf{E}^{k,m+1,n+1}](t)$$

$$= -(k+1)(n+1)E_3^{k+1,m,n+1} - (k+1)(m+1)E_2^{k+1,m+1,n}$$

$$+(m+2)(m+1)E_1^{k,m+2,n} + (n+2)(n+1)E_1^{k,m,n+2}, \tag{2.2.14}$$

$$D_2^{k,m,n}[\mathbf{E}^{k+2,m,n}, \mathbf{E}^{k,m+2,n}, \mathbf{E}^{k,m,n+2}, \mathbf{E}^{k+1,m,n+1}, \mathbf{E}^{k+1,m+1,n}, \mathbf{E}^{k,m+1,n+1}](t)$$

$$= -(m+1)(n+1)E_3^{k,m+1,n+1} + (k+2)(k+1)E_2^{k+2,m,n}$$

$$-(k+1)(m+1)E_1^{k+1,m+1,n} + (n+2)(n+1)E_2^{k,m,n+2}, \tag{2.2.15}$$

$$D_3^{k,m,n}[\mathbf{E}^{k+2,m,n}, \mathbf{E}^{k,m+2,n}, \mathbf{E}^{k,m,n+2}, \mathbf{E}^{k+1,m,n+1}, \mathbf{E}^{k+1,m+1,n}, \mathbf{E}^{k,m+1,n+1}](t)$$

$$= -(k+1)(n+1)E_1^{k+1,m,n+1} + (k+2)(k+1)E_3^{k+2,m,n}$$

$$+(m+2)(m+1)E_3^{k,m+2,n} - (m+1)(n+1)E_2^{k,m+1,n+1}, \tag{2.2.16}$$

$$k,m,n = 0,1,\dots,p.$$

Equalities (2.2.12), (2.2.13) can be written equivalently as the following recurrence relations:

$$\mathbf{E}^{p+2,m,n} = 0,\ \mathbf{E}^{k,p+2,n} = 0,\ \mathbf{E}^{k,m,p+2} = 0,\ \mathbf{E}^{p+1,m,n} = 0,\ \mathbf{E}^{k,p+1,n} = 0,$$

$$\mathbf{E}^{k,m,p+1} = 0,\ \mathbf{E}^{p+1,m,p+1} = 0,\ \mathbf{E}^{k,p+1,p+1} = 0,\ \mathbf{E}^{p+1,p+1,n} = 0, \tag{2.2.17}$$

$$\mathbf{E}^{k,m,n}(t) = \mathbf{F}^{k,m,n}(t) + \frac{1}{\varepsilon_{jj}} \int_0^t (t-\tau)$$

$$\times \mathbf{D}^{k,m,n}[\mathbf{E}^{k+2,m,n}, \mathbf{E}^{k,m+2,n}, \mathbf{E}^{k,m,n+2}, \mathbf{E}^{k+1,m,n+1}, \mathbf{E}^{k+1,m+1,n}, \mathbf{E}^{k,m+1,n+1}](\tau) d\tau, \tag{2.2.18}$$

where the components of the vector operator $\mathbf{D}^{k,m,n} = (D_1^{k,m,n}, D_2^{k,m,n}, D_3^{k,m,n})$ are defined by (2.2.14)-(2.2.16) and the components of the vector functions $\mathbf{F}^{k,m,n}(t)$ are defined by the following relations

$$F_j^{k,m,n}(t) = g_j^{k,m,n} t + e_j^{k,m,n} + \frac{1}{\varepsilon_{jj}} \int_0^t (t-\tau) f_j^{k,m,n}(\tau) d\tau, \tag{2.2.19}$$

$j = 1,2,3;\ \ k = p, p-1, ..., 0;\ \ m = p, p-1, ..., 0;\ \ n = p, p-1, ..., 0.$

Procedure of finding $E^{k,m,n}$

We start this procedure with finding $\mathbf{E}^{p,p,p}$. Substituting $k = p,\ m = p,\ n = p$ into (2.2.18) and using (2.2.17), (2.2.14)-(2.2.16) we find

$$E_j^{p,p,p}(t) = \frac{1}{\varepsilon_{jj}} \int_0^t (t-\tau) f_j^{p,p,p}(\tau) d\tau + g_j^{p,p,p} t + e_j^{p,p,p},\ j = 1,2,3.$$

The main part of the procedure consists in the following. Let k, m, n be numbers from the set 0,1,2,...,p such that all components of $\mathbf{E}^{k+2,m,n}(t)$, $\mathbf{E}^{k,m+2,n}(t)$, $\mathbf{E}^{k,m,n+2}(t)$, $\mathbf{E}^{k+1,m,n+1}(t)$, $\mathbf{E}^{k+1,m+1,n}(t)$, $\mathbf{E}^{k,m+1,n+1}(t)$ have been given or constructed by previous steps. Using (2.2.14)-(2.2.18) we find $\mathbf{E}^{k,m,n}(t)$ successively for all $k = p, p-1, ..., 0;\ \ m = p, p-1, ..., 0;\ \ n = p, p-1, ..., 0.$

2.2.3 Implementation of the Method

For solving IVP (2.2.1), (2.2.2) with polynomial data and a polynomial inhomogeneous term we implement the procedure of section 2.2 by symbolic computations in Maple 10. For this, the explicit formulae for polynomial components of a vector function $\mathbf{E} = (E_1, E_2, E_3)$ are found by symbolic computations. By means of direct substitution of these components $E_j(x,t),\ j = 1,2,3$ into (2.2.1), (2.2.2) we can always check that $\mathbf{E}(x,t)$ is an exact solution. We note that when a degree of polynomials is greater than 10 the formulae for components of $\mathbf{E}$ are cumbersome and take several printed pages. The robustness of the method can be illustrated by the following example.

Example: Let $\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}$ be symbols describing the diagonal matrix $\mathcal{E} = diag(\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33})$; $\mathbf{f} = 0$; $\mathbf{e} = 0$; the components of $\mathbf{g} = (g_1, g_2, g_3)$ be defined by

$$g_1(x) = (x_1 + 2x_2 + 3x_3)^6 + (x_1 + x_2 + x_3) + 14,$$

$$g_2(x) = 0,\ g_3(x) = 0.$$

Applying our method we compute a vector function $\mathbf{E} = (E_1, E_2, E_3)$ where $E_2 = E_3 = 0$,

$$E_1 = \left(12\,\frac{t^6}{{\varepsilon_{11}}^2\varepsilon_{22}} + 507\,\frac{t^6}{{\varepsilon_{11}}^3} + 27\,\frac{t^6}{{\varepsilon_{11}}^2\varepsilon_{33}}\right)x_1\,x_3 + \frac{5}{7}\,\frac{t^8}{{\varepsilon_{11}}^2\varepsilon_{22}^2}$$

$$+\left(24\,\frac{t^6}{{\varepsilon_{11}}^2\varepsilon_{22}} + 1014\,\frac{t^6}{{\varepsilon_{11}}^3} + 54\,\frac{t^6}{{\varepsilon_{11}}^2\varepsilon_{33}}\right)x_2\,x_3 + 260\,\frac{t^4x_2^4}{{\varepsilon_{11}}^2} + 1/2\,\frac{t^2x_1}{\varepsilon_{11}}$$

$$+\frac{45}{56}\,\frac{t^8}{{\varepsilon_{11}}^2{\varepsilon_{33}}^2} + 32\,\frac{t^2{x_2}^6}{\varepsilon_{11}} + 1/2\,\frac{t^2x_3}{\varepsilon_{11}} + 1620\,\frac{t^2{x_1}^2x_2\,x_3^3}{\varepsilon_{11}} + \frac{117}{28}\,\frac{t^8}{{\varepsilon_{11}}^3\varepsilon_{33}}$$

$$+1/2\,\frac{t^2{x_1}^6}{\varepsilon_{11}} + \left(8\,\frac{t^6}{{\varepsilon_{11}}^2\varepsilon_{22}} + 338\,\frac{t^6}{\varepsilon_{11}^3} + 18\,\frac{t^6}{{\varepsilon_{11}}^2\varepsilon_{33}}\right)x_1\,x_2 + \frac{5265}{4}\,\frac{t^4x_3^4}{{\varepsilon_{11}}^2}$$

$$+\left(2\,\frac{t^6}{{\varepsilon_{11}}^2\varepsilon_{22}}+\frac{169}{2}\,\frac{t^6}{{\varepsilon_{11}}^3}+9/2\,\frac{t^6}{{\varepsilon_{11}}^2\varepsilon_{33}}\right){x_1}^2+\frac{729}{2}\,\frac{t^2{x_3}^6}{\varepsilon_{11}}+\frac{65}{4}\,\frac{t^4{x_1}^4}{{\varepsilon_{11}}^2}$$

$$+3240\,\frac{t^2x_1\,{x_2}^2{x_3}^3}{\varepsilon_{11}}+\left(18\,\frac{t^6}{{\varepsilon_{11}}^2\varepsilon_{22}}+\frac{1521}{2}\,\frac{t^6}{{\varepsilon_{11}}^3}+\frac{81}{2}\,\frac{t^6}{{\varepsilon_{11}}^2\varepsilon_{33}}\right){x_3}^2$$

$$+360\,\frac{t^2{x_1}^3{x_2}^2x_3}{\varepsilon_{11}}+\left(8\,\frac{t^6}{{\varepsilon_{11}}^2\varepsilon_{22}}+338\,\frac{t^6}{{\varepsilon_{11}}^3}+18\,\frac{t^6}{\varepsilon_{11}^2\varepsilon_{33}}\right){x_2}^2+\frac{2197}{56}\,\frac{t^8}{{\varepsilon_{11}}^4}$$

$$+7\,\frac{t^2}{\varepsilon_{11}}-\frac{9}{7}\,\frac{t^8}{{\varepsilon_{11}}^2\varepsilon_{22}\,\varepsilon_{33}}+130\,\frac{t^4{x_1}^3x_2}{{\varepsilon_{11}}^2}+6\,\frac{t^2{x_1}^5x_2}{\varepsilon_{11}}+390\,\frac{t^4{x_1}^2{x_2}^2}{{\varepsilon_{11}}^2}$$

$$+30\,\frac{t^2{x_1}^4{x_2}^2}{\varepsilon_{11}}+520\,\frac{t^4x_1\,{x_2}^3}{{\varepsilon_{11}}^2}+80\,\frac{t^2{x_1}^3{x_2}^3}{\varepsilon_{11}}+120\,\frac{t^2{x_1}^2{x_2}^4}{\varepsilon_{11}}+96\,\frac{t^2x_1\,{x_2}^5}{\varepsilon_{11}}$$

$$+1560\,\frac{t^4{x_2}^3x_3}{{\varepsilon_{11}}^2}+288\,\frac{t^2{x_2}^5x_3}{\varepsilon_{11}}+195\,\frac{t^4{x_1}^3x_3}{{\varepsilon_{11}}^2}+2430\,\frac{t^2{x_2}^2{x_3}^4}{\varepsilon_{11}}$$

$$+\frac{1215}{2}\,\frac{t^2{x_1}^2{x_3}^4}{\varepsilon_{11}}+729\,\frac{t^2x_1\,x_3^5}{\varepsilon_{11}}+1458\,\frac{t^2x_2\,{x_3}^5}{\varepsilon_{11}}+1755\,\frac{t^4x_1\,{x_3}^3}{{\varepsilon_{11}}^2}$$

$$+3510\,\frac{t^4x_2\,{x_3}^3}{{\varepsilon_{11}}^2}+2160\,\frac{t^2{x_2}^3{x_3}^3}{\varepsilon_{11}}+270\,\frac{t^2{x_1}^3{x_3}^3}{\varepsilon_{11}}+\frac{135}{2}\,\frac{t^2{x_1}^4{x_3}^2}{\varepsilon_{11}}$$

$$+9\,\frac{t^2{x_1}^5x_3}{\varepsilon_{11}}+3510\,\frac{t^4{x_2}^2{x_3}^2}{{\varepsilon_{11}}^2}+1080\,\frac{t^2{x_2}^4{x_3}^2}{\varepsilon_{11}}+\frac{1755}{2}\,\frac{t^4{x_1}^2{x_3}^2}{{\varepsilon_{11}}^2}$$

$$+1170\,\frac{t^4{x_1}^2x_2\,x_3}{{\varepsilon_{11}}^2}+90\,\frac{t^2{x_1}^4x_2\,x_3}{\varepsilon_{11}}+2340\,\frac{t^4x_1\,{x_2}^2x_3}{{\varepsilon_{11}}^2}+720\,\frac{t^2{x_1}^2{x_2}^3x_3}{\varepsilon_{11}}$$

$$+720\,\frac{t^2x_1\,{x_2}^4x_3}{\varepsilon_{11}}+3510\,\frac{t^4x_1\,x_2\,{x_3}^2}{{\varepsilon_{11}}^2}+540\,\frac{t^2{x_1}^3x_2\,{x_3}^2}{\varepsilon_{11}}+1620\,\frac{t^2{x_1}^2{x_2}^2{x_3}^2}{\varepsilon_{11}}$$

$$+2160\,\frac{t^2x_1\,{x_2}^3{x_3}^2}{\varepsilon_{11}}+2430\,\frac{t^2x_1\,x_2\,{x_3}^4}{\varepsilon_{11}}+\frac{13}{7}\,\frac{t^8}{{\varepsilon_{11}}^3\varepsilon_{22}}+\frac{1}{2}\,\frac{t^2x_2}{\varepsilon_{11}}$$

Substituting found explicit formulae for $E_j(x,t),\ j=1,2,3,$ into (2.2.1), (2.2.2), we check that the vector function $\mathbf{E}(x,t)$ is an exact solution of (2.2.1), (2.2.2).

2.2.4 Computational Analysis of Polynomial Solutions of IVP for Crystal Optics

Several cases of initial data for the Cauchy problem (2.2.1), (2.2.2) with zero inhomogeneous term are considered in this section. For these data the exact solutions of (2.2.1), (2.2.2) are given by explicit formulae. These data and solutions are differentiable but not polynomial. For each of considered cases we approximate initial data by polynomials and then, using the method, we compute a polynomial solution. The comparison of values of polynomial and original exact solutions is presented by tables.

Also, Shannon's kernels of the form

$$\frac{\sin \alpha x_3}{\pi x_3} \quad \text{and} \quad \frac{\sin \beta x_2}{\pi x_2}$$

appear in initial data. Here α, β are given numbers. Shannon's kernels are not polynomial and they are widely used for modeling different processes and phenomena. Initial data with Shannon's kernel are approximated by polynomial and then method we explained is applied to compute a polynomial solution of (2.2.1), (2.2.2). Graphs of the first component of the polynomial solution are presented on the figures.

Examples of exact solutions

Let $\mathbf{e} = 0,\ \mathbf{f} = 0$ and $\mathcal{E} = diag(1, 4, 9)$. The following three cases will be considered for components of $\mathbf{g} = (g_1, g_2, g_3)$

Case 1:

$$g_l = \sin x_3,\ l = 1, 2, 3; \tag{2.2.20}$$

Case 2:

$$g_1 = \sin(\frac{x_2}{3})\sin(\frac{x_3}{5}), g_2 = \cos(\frac{x_2}{3})\sin(\frac{x_3}{5}), \quad g_3 = \sin(\frac{x_2}{3})\cos(\frac{x_3}{5}); \tag{2.2.21}$$

Case 3:

$$g_1 = \cos x_1 \sin(\frac{x_2}{9}) \sin(\frac{x_3}{13}),\ g_2 = g_3 = 0. \tag{2.2.22}$$

The exact solution $\mathbf{E} = (E_1, E_2, E_3)$ of (2.2.1), (2.2.2) is given for each case by formulae:

Case 1:

$$E_1 = \sin x_3 \sin t, \quad E_2 = \frac{1}{2} \sin x_3 \sin(\frac{1}{2}t), \quad E_3 = \frac{1}{3} \sin x_3 \sin(\frac{1}{3}t),$$

Case 2:

$$E_1 = \sin(\frac{1}{3}x_2) \sin(\frac{1}{5}x_3) \sin t, \quad E_2 = \frac{1}{2} \cos(\frac{1}{3}x_2) \sin(\frac{1}{5}x_3) \sin(\frac{1}{2}t),$$

$$E_3 = \frac{1}{3} \sin(\frac{1}{3}x_2) \cos(\frac{1}{5}x_3) \sin(\frac{1}{3}t);$$

Case 3:

$$E_1 = \cos x_1 \sin(\frac{1}{9}x_2) \sin(\frac{1}{13}x_3) \sin t, \quad E_2 = 0, \quad E_3 = 0;$$

Polynomial Solutions

Let $\mathbf{e} = 0,\ \mathbf{f} = 0$ and $\mathbf{g} = \mathbf{g}^N$ where components of $\mathbf{g}^N = (g_1^N, g_2^N, g_3^N)$ be obtained from formulae (2.2.20), (2.2.21), (2.2.22) by finite Taylor series expansions of given functions g_l , i.e.

Case 1:

$$g_l^N = \sum_{n=0}^{N} g_l^{0,0,n} x_3^n, \tag{2.2.23}$$

Case 2:

$$g_l^N = \sum_{m=0}^{N} \sum_{n=0}^{N} g_l^{0,m,n} x_2^m x_3^n, \tag{2.2.24}$$

Case 3:

$$g_l^N = \sum_{k=0}^{N}\sum_{m=0}^{N}\sum_{n=0}^{N} g_l^{k,m,n} x_1^k x_2^m x_3^n, . \tag{2.2.25}$$

where $g_l^{k,m,n}$ are coefficients of Taylor series expansion of $g_l(x_1, x_2, x_3)$ at the point $x_1 = x_2 = x_3 = 0$.

Using the method we explained for approximated data (2.2.23)-(2.2.25) we compute polynomial solutions $\mathbf{E}^N = (E_1^N, E_2^N, E_3^N)$ for each case. The comparison of values of polynomial solutions $\mathbf{E}^N$ and original exact solutions E at some fixed points is listed in Table 2.1- Table 2.5.

Table 2.1 Values of E_1 and E_1^N for Case 1, $N = 40$.

t	x_1	x_2	x_3	E_1	E_1^N	$\lvert E_1 - E_1^N\rvert$
2	3	3	2	0.82682	0.82682	6.825×10^{-27}
2	4	4	3	0.12832	0.12832	7.985×10^{-23}
2	5	5	1.6	0.90890	0.90890	8.185×10^{-29}

Table 2.2 Values of E_2 and E_2^N for Case 1, $N = 40$.

t	x_1	x_2	x_3	E_2	E_2^N	$\lvert E_2 - E_2^N\rvert$
2	3	3	2	0.38257	0.38257	1.937×10^{-32}
2	4	4	3	0.05937	0.05937	3.412×10^{-27}
2	5	5	1.6	0.42055	0.42055	4.759×10^{-35}

Table 2.3 Values of E_3 and E_3^N for Case 1, $N = 40$.

t	x_1	x_2	x_3	E_3	E_3^N	$\lvert E_3 - E_3^N\rvert$
2	3	3	2	0.20206	0.20206	1.45756×10^{-38}
2	4	4	3	0.03136	0.03136	2.41084×10^{-31}
2	5	5	1.6	0.22212	0.22212	1.5511×10^{-42}

Table 2.4 Values of E_1 and E_1^N for Case 2, $N = 50$.

t	x_1	x_2	x_3	E_1	E_1^N	$\|E_1 - E_1^N\|$
1	3	2	2	0.23478	0.23478	$0.3 \cdot 10^{-9}$
14/10	4	2	1	0.16362	0.16362	$0.1 \cdot 10^{-9}$
2	5	3/2	3/2	0.25566	0.25566	$0.1 \cdot 10^{-9}$

Table 2.5 Values of E_1 and E_1^N for Case 3, $N = 30$.

t	x_1	x_2	x_3	E_1	E_1^N	$\|E_1 - E_1^N\|$
1	1	1	1	0.00473	0.00459	0.000145240705
1	2	2	2	-0.01445	-0.01401	0.00044336802
1.4	1	2	1	0.01353	0.01273	0.00079727107

Polynomial solutions for data with approximated Shannon's kernels

Let $\mathbf{e} = 0, \ \mathbf{f} = 0$ and for components of $\mathbf{g} = (g_1, g_2, g_3)$ the following two cases be taken

Case 4:

$$g_1(x_3) = \big(\frac{1}{x_3\pi}\big)\sin(4x_3), \quad g_2(x_3) = g_3(x_3) = 0, \tag{2.2.26}$$

Case 5:

$$g_1(x_2, x_3) = \big(\frac{1}{x_2\pi}\big)\sin(4x_2)\big(\frac{1}{x_3\pi}\big)\sin(4x_3), \quad g_2(x_2, x_3) = g_3(x_2, x_3) = 0. \tag{2.2.27}$$

The components of the vector function $\mathbf{g}^N = (g_1^N, g_2^N, g_3^N)$ are found from formulae (2.2.26), (2.2.27) by finite Taylor series expansions of given function g_1, i.e.

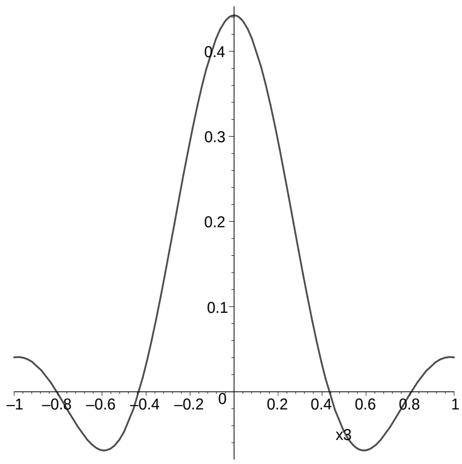
Case 4:

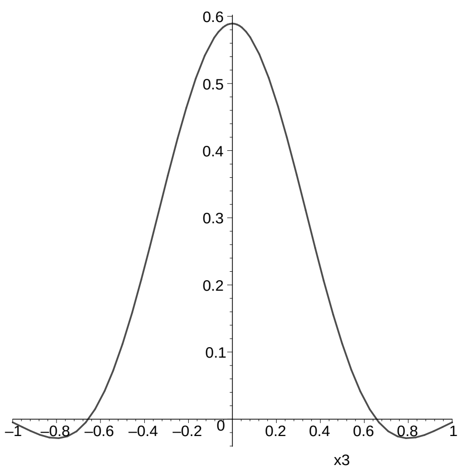
$$g_l^N = \sum_{n=0}^{N} g_l^{0,0,n} x_3^n, \tag{2.2.28}$$

Case 5:

$$g_l^N = \sum_{m=0}^{N}\sum_{n=0}^{N} g_l^{0,m,n} x_2^m x_3^n, \tag{2.2.29}$$

here $g_1^{k,m,n}$ are coefficients of Taylor series expansion of $g_1(x_1, x_2, x_3)$ at $x_1 = 0, \ x_2 = 0, \ x_3 = 0$. Using the method of the Section 2 for g_1^N given by (2.2.28)-(2.2.29) and $\mathbf{e} = 0, \ \mathbf{f} = 0$ we compute a polynomial solution $\mathbf{E}^N = (E_1^N, E_2^N, E_3^N)$ of (2.2.1), (2.2.2) for each case. The graphs of E_1^N for cases 4 and 5 are presented below.





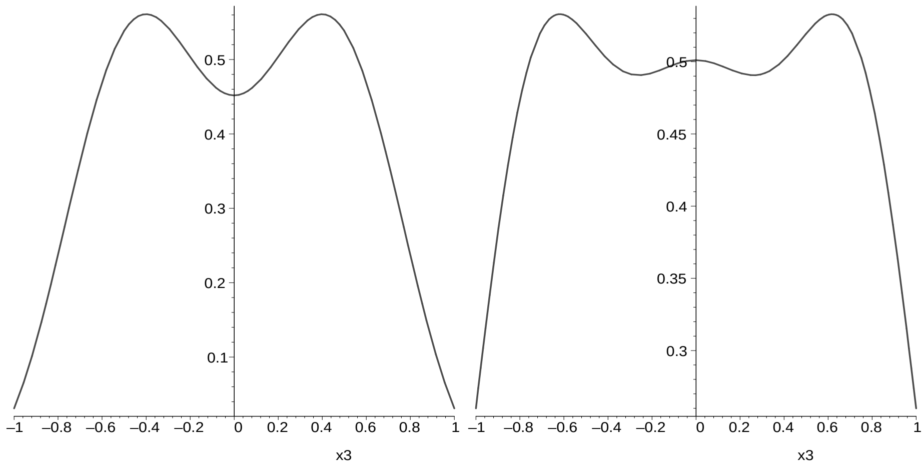


Figure 2.1 The first component of $\mathbf{E}^N$ for case 4 at $x_1 = 4, x_2 = 4, t = 0.2; 0.4; 0.8; 1, N = 40$.

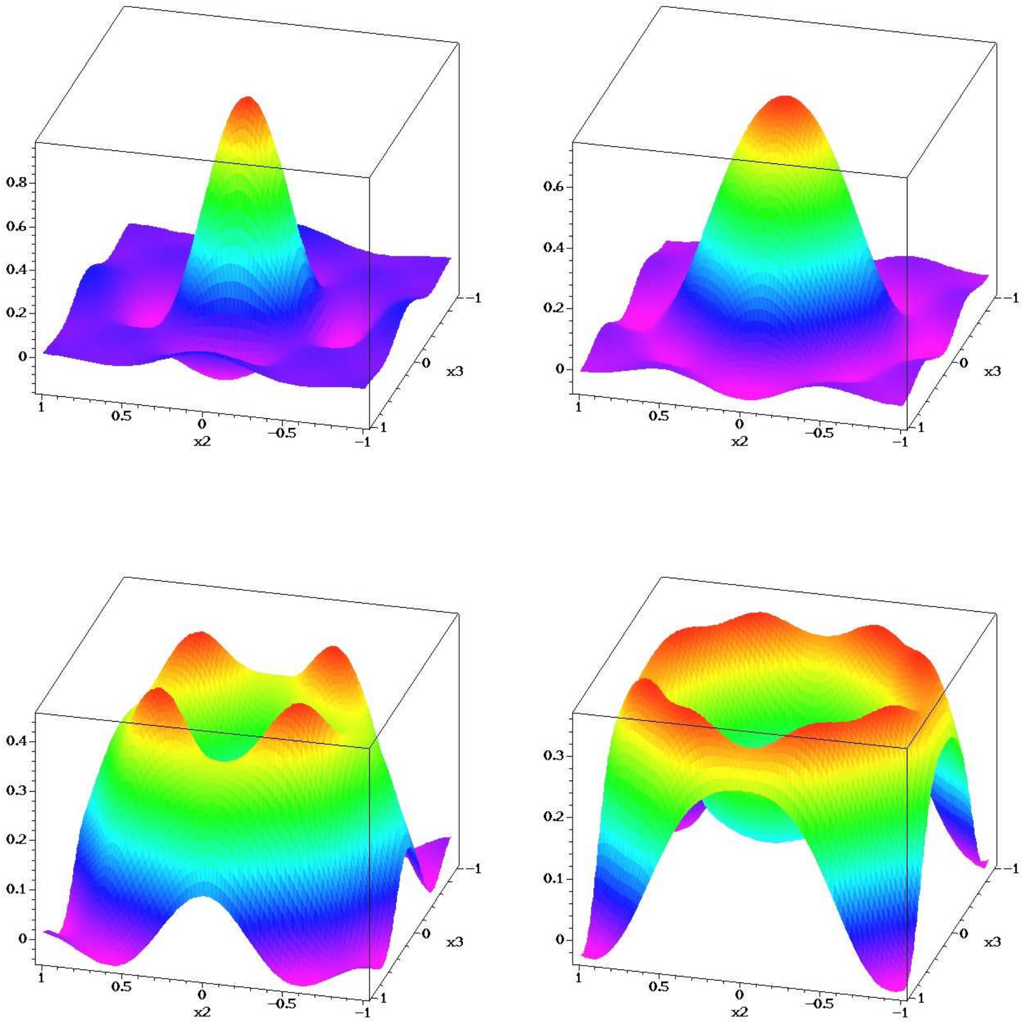


Figure 2.2 The first component of $\mathbf{E}^N$ for case 5 at $x_1 = 4, t = 0.2; 0.4; 0.8; 1, N = 50$.

2.3 Computing Polynomial Solutions of Electric Field Equations for Modelling Waves in Anisotropic Media

In this Section, the IVP describing the electric wave propagation in electrically and magnetically anisotropic media is considered. A new analytical method for computing polynomial solutions of the IVP for the considered system is suggested. Computational examples about a comparison of solutions of initial value problems corresponding to non-polynomial data and polynomial solutions which are found by polynomial approximations of given data and suggested method are described. The results of computations and simulations of electric fields in different electrically and magnetically anisotropic media (in particular, the sapphire) are presented by images.

2.3.1 Problem Set-up

Let us consider the initial value problem of finding a vector function $\mathbf{E}(x,t) = (E_1(x,t), E_2(x,t), E_3(x,t))$ that satisfies

$$\mathcal{E}\frac{\partial^2 \mathbf{E}}{\partial t^2} + curl_x(\mathcal{M}^{-1} curl_x \mathbf{E}) = \mathbf{f}(x,t), \quad x \in R^3, \; t > 0, \tag{2.3.1}$$

$$\mathbf{E}(x,0) = \mathbf{e}(x), \quad \left.\frac{\partial \mathbf{E}(x,t)}{\partial t}\right|_{t=0} = \mathbf{g}(x), \quad x \in R^3, \tag{2.3.2}$$

where $\mathbf{e} = (e_1, e_2, e_3)$, $\mathbf{g} = (g_1, g_2, g_3)$ are given vector functions with components depending on x; $\mathbf{f} = (f_1, f_2, f_3)$ is a given vector function with components depending on x and t. Let $\mathcal{E} = (\varepsilon_{ij})_{3\times 3}$ and $\mathcal{M}^{-1} = (\mu_{ij})_{3\times 3}$ be symmetric, positive definite matrices with constant elements.

We supposed that the components of vector functions $\mathbf{e}(x)$, $\mathbf{g}(x)$, $\mathbf{f}(x,t)$ are given in the following polynomial form

$$\begin{aligned} e_l(x) &= \sum_{k=0}^{p}\sum_{m=0}^{p}\sum_{n=0}^{p} e_l^{k,m,n} x_1^k x_2^m x_3^n, && (2.3.3)\\ g_l(x) &= \sum_{k=0}^{p}\sum_{m=0}^{p}\sum_{n=0}^{p} g_l^{k,m,n} x_1^k x_2^m x_3^n, && (2.3.4)\\ f_l(x,t) &= \sum_{k=0}^{p}\sum_{m=0}^{p}\sum_{n=0}^{p} f_l^{k,m,n}(t) x_1^k x_2^m x_3^n, && (2.3.5) \end{aligned}$$

where p is a given nonnegative integer; $e_l^{k,m,n}, g_l^{k,m,n}$ are given real numbers; $f_l^{k,m,n}(t)$ are given continuous functions of t; $l = 1, 2, 3$; k, m, n are running $0, 1, 2, ..., p$.

We find a solution $\mathbf{E}(x,t)$ of (2.3.1), (2.3.2) in the following polynomial form with undetermined coefficients depending on t:

$$E_l(x_1, x_2, x_3, t) = \sum_{k=0}^{p}\sum_{m=0}^{p}\sum_{n=0}^{p} E_l^{k,m,n}(t) x_1^k x_2^m x_3^n. \qquad (2.3.6)$$

2.3.2 Method of Computing a Polynomial Solution

In this Section we obtain the recurrence relations for undetermined coefficients $E_l^{k,m,n}(t)$ and then, we describe a procedure of their successive recovery.

Recurrence Relations for $\mathbf{E}^{k,m,n}$

We note that for the symmetric, positive definite matrix $\mathcal{E}$ there exists an orthogonal matrix $\mathcal{S} = (S_{ij})_{3\times 3}$ such that $\mathcal{S}^T\mathcal{E}\mathcal{S} = \mathcal{D}$, where $\mathcal{D}$ is a diagonal matrix with nonnegative diagonal entries that are eigenvalues of $\mathcal{E}$; $\mathcal{S}^T$ is transpose to $\mathcal{S}$.

Letting $\mathbf{E} = S\tilde{\mathbf{E}}$ and substituting this into (2.3.1), (2.3.2) and multiplying with $\mathcal{S}^T$

from left hand side we obtain

$$\mathcal{D}\frac{\partial^2 \tilde{\mathbf{E}}}{\partial t^2} + \mathcal{S}^T curl_x(\mathcal{M}^{-1} curl_x(\mathcal{S}\tilde{\mathbf{E}})) = \mathcal{S}^T \mathbf{f}(x,t), \quad x \in R^3,\ t > 0, \tag{2.3.7}$$

$$\tilde{\mathbf{E}}(x,0) = \mathcal{S}^T \mathbf{e}(x), \quad \left.\frac{\partial \tilde{\mathbf{E}}(x,t)}{\partial t}\right|_{t=0} = \mathcal{S}^T \mathbf{g}(x), \quad x \in R^3. \tag{2.3.8}$$

Using $\tilde{\mathbf{E}} = \mathcal{S}^T \mathbf{E}$ equation (2.3.6) may be written as follows

$$\tilde{\mathbf{E}}(x_1,x_2,x_3,t) = \sum_{k=0}^{p}\sum_{m=0}^{p}\sum_{n=0}^{p} \tilde{\mathbf{E}}^{k,m,n}(t) x_1^k x_2^m x_3^n, \tag{2.3.9}$$

where $\tilde{\mathbf{E}}^{k,m,n}(t) = \mathcal{S}^T \mathbf{E}^{k,m,n}(t)$, $\mathbf{E}^{k,m,n}(t) = (E_1^{k,m,n}(t), E_2^{k,m,n}(t), E_3^{k,m,n}(t))$.

Substituting (2.3.3), (2.3.4), (2.3.5) and (2.3.9) into (2.3.7), (2.3.8) we obtain

$$\mathcal{D}\frac{\partial^2 \tilde{\mathbf{E}}^{k,m,n}}{\partial t^2} = \mathcal{S}^T \mathbf{f}^{k,m,n}(t) - \boldsymbol{\Upsilon}^{k,m,n}\Big[\tilde{\mathbf{E}}^{k+2,m,n},$$

$$\tilde{\mathbf{E}}^{k,m+2,n}, \tilde{\mathbf{E}}^{k,m,n+2}, \tilde{\mathbf{E}}^{k+1,m+1,n}, \tilde{\mathbf{E}}^{k+1,m,n+1}, \tilde{\mathbf{E}}^{k,m+1,n+1}\Big](t), \tag{2.3.10}$$

$$\tilde{\mathbf{E}}^{k,m,n}(0) = \mathcal{S}^T \mathbf{e}^{k,m,n}, \tag{2.3.11}$$

$$\left.\frac{\partial \tilde{\mathbf{E}}^{k,m,n}}{\partial t}\right|_{t=0} = \mathcal{S}^T \mathbf{g}^{k,m,n}. \tag{2.3.12}$$

Here $\mathbf{e}^{k,m,n} = (e_1^{k,m,n}, e_2^{k,m,n}, e_3^{k,m,n})$, $\mathbf{g}^{k,m,n} = (g_1^{k,m,n}, g_2^{k,m,n}, g_3^{k,m,n})$, $\mathbf{f}^{k,m,n}(t) = (f_1^{k,m,n}(t), f_2^{k,m,n}(t), f_3^{k,m,n}(t))$ are known vector functions; the vector operators $\boldsymbol{\Upsilon}^{k,m,n}$ are defined by the following formulae

$$\boldsymbol{\Upsilon}^{k,m,n}\Big[\tilde{\mathbf{E}}^{k+2,m,n}, \tilde{\mathbf{E}}^{k,m+2,n}, \tilde{\mathbf{E}}^{k,m,n+2},$$

$$\tilde{\mathbf{E}}^{k+1,m+1,n}, \tilde{\mathbf{E}}^{k+1,m,n+1}, \tilde{\mathbf{E}}^{k,m+1,n+1}\Big](t) = \mathcal{S}^T \mathbf{B}^{k,m,n}(t),$$

here the components of the vector functions $\mathbf{B}^{k,m,n}(t)$ are defined by

$$B_1^{k,m,n} = \mu_{31} T_1^{k,m,n} + \mu_{32} T_2^{k,m,n} + \mu_{33} T_3^{k,m,n}$$
$$-\mu_{21} T_4^{k,m,n} - \mu_{22} T_5^{k,m,n} - \mu_{23} T_6^{k,m,n},$$

$$B_2^{k,m,n} = \mu_{11} T_4^{k,m,n} + \mu_{12} T_5^{k,m,n} + \mu_{13} T_6^{k,m,n}$$
$$-\mu_{31} T_7^{k,m,n} - \mu_{32} T_8^{k,m,n} - \mu_{33} T_9^{k,m,n},$$

$$B_3^{k,m,n} = \mu_{21} T_7^{k,m,n} + \mu_{22} T_8^{k,m,n} + \mu_{23} T_9^{k,m,n}$$
$$-\mu_{11} T_1^{k,m,n} - \mu_{12} T_2^{k,m,n} - \mu_{13} T_3^{k,m,n},$$

where

$$T_1^{k,m,n} = (m+2)(m+1)(S_{31}\tilde{E}_1^{k,m+2,n} + S_{32}\tilde{E}_2^{k,m+2,n} + S_{33}\tilde{E}_3^{k,m+2,n})$$
$$-(m+1)(n+1)(S_{21}\tilde{E}_1^{k,m+1,n+1} + S_{22}\tilde{E}_2^{k,m+1,n+1} + S_{23}\tilde{E}_3^{k,m+1,n+1});$$
$$T_2^{k,m,n} = (m+1)(n+1)(S_{11}\tilde{E}_1^{k,m+1,n+1} + S_{12}\tilde{E}_2^{k,m+1,n+1} + S_{13}\tilde{E}_3^{k,m+1,n+1})$$
$$-(k+1)(m+1)(S_{31}\tilde{E}_1^{k+1,m+1,n} + S_{32}\tilde{E}_2^{k+1,m+1,n} + S_{33}\tilde{E}_3^{k+1,m+1,n});$$
$$T_3^{k,m,n} = (k+1)(m+1)(S_{21}\tilde{E}_1^{k+1,m+1,n} + S_{22}\tilde{E}_2^{k+1,m+1,n} + S_{23}\tilde{E}_3^{k+1,m+1,n})$$
$$-(m+2)(m+1)(S_{11}\tilde{E}_1^{k,m+2,n} + S_{12}\tilde{E}_2^{k,m+2,n} + S_{13}\tilde{E}_3^{k,m+2,n});$$
$$T_4^{k,m,n} = (m+1)(n+1)(S_{31}\tilde{E}_1^{k,m+1,n+1} + S_{32}\tilde{E}_2^{k,m+1,n+1} + S_{33}\tilde{E}_3^{k,m+1,n+1})$$
$$-(n+2)(n+1)(S_{21}\tilde{E}_1^{k,m,n+2} + S_{22}\tilde{E}_2^{k,m,n+2} + S_{23}\tilde{E}_3^{k,m,n+2});$$
$$T_5^{k,m,n} = (n+2)(n+1)(S_{11}\tilde{E}_1^{k,m,n+2} + S_{12}\tilde{E}_2^{k,m,n+2} + S_{13}\tilde{E}_3^{k,m,n+2})$$
$$-(k+1)(n+1)(S_{31}\tilde{E}_1^{k+1,m,n+1} + S_{32}\tilde{E}_2^{k+1,m,n+1} + S_{33}\tilde{E}_3^{k+1,m,n+1});$$
$$T_6^{k,m,n} = (k+1)(n+1)(S_{21}\tilde{E}_1^{k+1,m,n+1} + S_{22}\tilde{E}_2^{k+1,m,n+1} + S_{23}\tilde{E}_3^{k+1,m,n+1})$$
$$-(m+1)(n+1)(S_{11}\tilde{E}_1^{k,m+1,n+1} + S_{12}\tilde{E}_2^{k,m+1,n+1} + S_{13}\tilde{E}_3^{k,m+1,n+1});$$
$$T_7^{k,m,n} = (k+1)(m+1)(S_{31}\tilde{E}_1^{k+1,m+1,n} + S_{32}\tilde{E}_2^{k+1,m+1,n} + S_{33}\tilde{E}_3^{k+1,m+1,n})$$
$$-(k+1)(n+1)(S_{21}\tilde{E}_1^{k+1,m,n+1} + S_{22}\tilde{E}_2^{k+1,m,n+1} + S_{23}\tilde{E}_3^{k+1,m,n+1});$$
$$T_8^{k,m,n} = (k+1)(n+1)(S_{11}\tilde{E}_1^{k+1,m,n+1} + S_{12}\tilde{E}_2^{k+1,m,n+1} + S_{13}\tilde{E}_3^{k+1,m,n+1})$$
$$-(k+2)(k+1)(S_{31}\tilde{E}_1^{k+2,m,n} + S_{32}\tilde{E}_2^{k+2,m,n} + S_{33}\tilde{E}_3^{k+2,m,n});$$
$$T_9^{k,m,n} = (k+2)(k+1)(S_{21}\tilde{E}_1^{k+2,m,n} + S_{22}\tilde{E}_2^{k+2,m,n} + S_{23}\tilde{E}_3^{k+2,m,n})$$
$$-(k+1)(m+1)(S_{11}\tilde{E}_1^{k+1,m+1,n} + S_{12}\tilde{E}_2^{k+1,m+1,n} + S_{13}\tilde{E}_3^{k+1,m+1,n}).$$

In these expressions we assume that the components of the vector functions

$$\tilde{\mathbf{E}}^{p+2,m,n}, \quad \tilde{\mathbf{E}}^{k,p+2,n}, \quad \tilde{\mathbf{E}}^{k,m,p+2}, \quad \tilde{\mathbf{E}}^{p+1,p+1,n}, \quad \tilde{\mathbf{E}}^{p+1,m,p+1}, \quad \tilde{\mathbf{E}}^{k,p+1,p+1}$$

are equal to zero.

Equalities (2.3.10)-(2.3.12) are equivalent to the following relations:

$$\tilde{\mathbf{E}}^{k,m,n} = \tilde{\mathbf{F}}^{k,m,n}(t) - \int_0^t (t-\tau)\mathcal{D}^{-1}\mathbf{\Upsilon}^{k,m,n}\Big[$$

$$\tilde{\mathbf{E}}^{k+2,m,n}, \tilde{\mathbf{E}}^{k,m+2,n}, \tilde{\mathbf{E}}^{k,m,n+2}, \tilde{\mathbf{E}}^{k+1,m+1,n}, \tilde{\mathbf{E}}^{k+1,m,n+1}, \tilde{\mathbf{E}}^{k,m+1,n+1}\Big](\tau)d\tau, \quad (2.3.13)$$

where the components of the vector functions $\tilde{\mathbf{F}}^{k,m,n}(t)$ are defined by

$$F_l^{k,m,n}(t) = S_{l1}^T e_1^{k,m,n} + S_{l2}^T e_2^{k,m,n} + S_{l3}^T e_3^{k,m,n} + t\Big(S_{l1}^T g_1^{k,m,n} + S_{l2}^T g_2^{k,m,n} + S_{l3}^T g_3^{k,m,n}\Big)$$

$$+\frac{1}{d_l}\int_0^t (t-\tau)\Big(S_{l1}^T f_1^{k,m,n}(\tau) + S_{l2}^T f_2^{k,m,n}(\tau) + S_{l3}^T f_3^{k,m,n}(\tau)\Big)d\tau,$$

where $k = p, p-1, ..., 0; \quad m = p, p-1, ..., 0; \quad n = p, p-1, ..., 0$ and $\frac{1}{d_l}, \; l = 1, 2, 3$ are diagonal elements of $\mathcal{D}^{-1}$ that is inverse of $\mathcal{D}$.

Procedure of finding $\mathbf{E}^{k,m,n}$

We suppose that the components of the vector functions

$$\mathbf{e}^{k,m,n} = (e_1^{k,m,n}, e_2^{k,m,n}, e_3^{k,m,n}), \quad \mathbf{g}^{k,m,n} = (g_1^{k,m,n}, g_2^{k,m,n}, g_3^{k,m,n}),$$

$$\mathbf{f}^{k,m,n}(t) = (f_1^{k,m,n}(t), f_2^{k,m,n}(t), f_3^{k,m,n}(t))$$

are known for all $k = p, p-1, ..., 0; \quad m = p, p-1, ..., 0; \quad n = p, p-1, ..., 0.$ The procedure of finding $\mathbf{E}^{k,m,n}$ consists of the sequence of the following iterative steps of constructing some formulae from the others using the relation (2.3.13).

Step 0:

$$\tilde{\mathbf{E}}^{p+2,m,n} = \tilde{\mathbf{E}}^{k,p+2,n} = \tilde{\mathbf{E}}^{k,m,p+2} = \tilde{\mathbf{E}}^{p+1,m,n} = \tilde{\mathbf{E}}^{k,p+1,n} = \tilde{\mathbf{E}}^{k,m,p+1} = 0$$

when $k = p+2,\ p+1,\ ...,\ 0$; $m = p+2,\ p+1,\ ...,\ 0$; $n = p+2,\ p+1,\ ...,\ 0$. This fact follows from (2.3.9).

Step 1: using zero values from step 0 we compute formulae for

$$\tilde{\mathbf{E}}^{p,m,n},\ \tilde{\mathbf{E}}^{k,p,n},\ \tilde{\mathbf{E}}^{k,m,p},\ k = p,\ p-1,\ ...,\ 0;\ \ m = p,\ p-1,\ ...,\ 0;\ \ n = p,\ p-1,\ ...,\ 0.$$

Step 2: from the relations obtained on previous steps we compute

$$\tilde{\mathbf{E}}^{p-1,m,n},\ \tilde{\mathbf{E}}^{k,p-1,n},\ \tilde{\mathbf{E}}^{k,m,p-1},\quad k = p-1,\ ...,\ 0;\ \ m = p-1,\ ...,\ 0;\ \ n = p-1,\ ...,\ 0.$$

... ...

Step p: from the relations obtained on previous steps we compute

$$\tilde{\mathbf{E}}^{1,m,n},\ \tilde{\mathbf{E}}^{k,1,n},\ \tilde{\mathbf{E}}^{k,m,1},\quad \text{for}\quad k = 1,\ 0;\ \ m = 1,\ 0;\ \ n = 1,\ 0;$$

and $\quad \tilde{\mathbf{E}}^{0,0,0}$.

Finally, components of $\mathbf{E}^{k,m,n}$ are found by $\mathbf{E}^{k,m,n} = \mathcal{S}\tilde{\mathbf{E}}^{k,m,n}$ for all $k = p, p-1, ..., 0;\ \ m = p, p-1, ..., 0;\ \ n = p, p-1, ..., 0$.

Constructing an explicit formula for a solution of (2.3.1), (2.3.2) with polynomial data

Using the procedure described in Section 3.2 and symbolic computations, a solution $\mathbf{E} = (E_1, E_2, E_3)$ of the IVP (2.3.1), (2.3.2) is constructed. The implementation of this method has been made in Maple 10. The explicit formulae for the components of $\mathbf{E} = (E_1, E_2, E_3)$ have been constructed for arbitrary polynomial initial data and polynomial inhomogeneous terms. Using the direct substitution we have checked that constructed formulae give exact solutions of the IVP. We note that when a degree of polynomials is greater than 10 the formulae for components of $\mathbf{E}$ are cumbersome and take several printed pages. The robustness of the method is illustrated by the following example.

Example: Let $\mathcal{E}$ and $\mathcal{M}$ be arbitrary matrices defined as follows:

$$\mathcal{E} = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{pmatrix}, \quad \mathcal{M} = \begin{pmatrix} 25 & 9 & 0 \\ 9 & 25 & 0 \\ 0 & 0 & 5 \end{pmatrix}.$$

The polynomial data and polynomial inhomogeneous term are given by $\mathbf{f} = 0;\ \mathbf{e} = 0;$ $\mathbf{g} = (g_1, g_2, g_3)$, where

$$g_1(x) = (x_1 + 5x_2 + 7x_3)^6 + (6x_1^3 + 2)^3 + (x_2^3 + 7)^3 + (6x_3^2 + 5)^2,$$

$$g_2(x) = 0,\ g_3(x) = 0.$$

Applying our method we compute the explicit formulae for the components of $\mathbf{E} = (E_1, E_2, E_3)$ and then we verify that $\mathbf{E}(x,t)$ is an exact solution of (2.3.1), (2.3.2). The formula for the first component of $\mathbf{E}$ is

$$E_1 = -\left(-\frac{12890969}{2720}\,t^4 - 1/4\,t^2\right) {x_3}^4 - \left(-\frac{16648236747}{157216000000000}\,t^{10} - \frac{9}{50}\,t^4\right) x_1 +$$
$$\frac{42}{5}\,t^2{x_1}^5x_3 + \frac{102255408409}{739840000}\,t^6x_2\,x_3 + 26250\,t^2{x_2}^5x_3 + \frac{353}{65280}\,t^4 + \frac{82}{5}\,t^2$$
$$+ 26250\,t^2x_1\,{x_2}^4x_3 + 10500\,t^2{x_1}^2{x_2}^3x_3 + \frac{7146489}{2176}\,t^4x_1\,{x_2}^2x_3$$
$$+ 2100\,t^2{x_1}^3{x_2}^2x_3 + 210\,t^2{x_1}^4x_2\,x_3 + \frac{7146489}{10880}\,t^4{x_1}^2x_2\,x_3 + 73500\,t^2x_1\,{x_2}^3{x_3}^2$$
$$+22050\,t^2{x_1}^2{x_2}^2{x_3}^2 + \frac{50025423}{10880}\,t^4x_1\,x_2\,{x_3}^2 + 2940\,t^2{x_1}^3x_2\,{x_3}^2 + 102900\,t^2x_1\,{x_2}^2{x_3}^3 +$$
$$20580\,t^2{x_1}^2x_2\,{x_3}^3 + 72030\,t^2x_1\,x_2\,{x_3}^4 + \left(\frac{2785143977}{23120000}\,t^6 + \frac{1}{272}\,t^4 + 5/2\,t^2\right){x_3}^2 +$$
$$\frac{15803}{4439040000}\,t^6 + \frac{89659509633399}{402472960000000}\,t^8 + \frac{50025423}{4352}\,t^4{x_2}^2{x_3}^2 + 91875\,t^2{x_2}^4{x_3}^2$$
$$+\frac{340309}{10880}\,t^4{x_1}^3x_2 + \frac{14607915487}{739840000}\,t^6x_1\,x_2 + 6\,t^2{x_1}^5x_2 + 75\,t^2{x_1}^4{x_2}^2$$
$$+ \frac{1020927}{4352}\,t^4{x_1}^2{x_2}^2 + 500\,t^2{x_1}^3{x_2}^3 + \frac{1701545}{2176}\,t^4x_1\,{x_2}^3 + \frac{50025423}{108800}\,t^4{x_1}^2{x_3}^2$$

$$
\begin{aligned}
&+147\,t^2{x_1}^4{x_3}^2+\frac{2382163}{54400}\,t^4{x_1}^3x_3+\frac{102255408409}{3699200000}\,t^6x_1\,x_3+3750\,t^2x_1\,{x_2}^5\\
&+1875\,t^2{x_1}^2{x_2}^4+\frac{100842}{5}\,t^2x_1\,{x_3}^5+100842\,t^2x_2\,{x_3}^5+\left(\frac{90486639}{4624000000}\,t^8+18\,t^2\right){x_1}^3\\
&+1/3\,\left(\frac{20713427}{13600}\,t^4+1/10\,t^2\right){x_3}^4+1/3\,\left(\frac{18258382989}{462400000}\,t^6+\frac{3}{1600}\,t^4+t^2\right){x_3}^2\\
&+1/3\,\left(-\frac{11586204279}{1572160000000}\,t^8+\frac{36}{5}\,t^2\right){x_1}^3+1/3\,\left(-\frac{2105971682967}{53453440000000000}\,t^{10}\right.\\
&\left.-\frac{6021}{68000}\,t^4\right)x_1-1/3\,\left(\frac{25332344433}{628864000000}\,t^8+18\,t^2\right){x_1}^3+\frac{365207660377}{7398400000}\,t^6{x_2}^2\\
&+\frac{61803}{544000}\,t^4x_2+\frac{106350977}{108800}\,t^4{x_2}^4+\frac{157377087}{1156000000}\,t^6{x_1}^5+\frac{15626}{5}\,t^2{x_2}^6\\
&+\frac{217}{5}\,t^2{x_1}^6+\frac{54189}{34000}\,t^4{x_1}^7+\frac{216}{5}\,t^2{x_1}^9+\frac{14727821839}{7398400000}\,t^6{x_1}^2+\frac{484813}{217600}\,t^4{x_1}^4\\
&-1/3\,\left(\frac{4686508909809}{21381376000000000}\,t^{10}+\frac{8667}{27200}\,t^4\right)x_1+\frac{11910815}{2176}\,t^4{x_2}^3x_3\\
&-1/3\,\left(\frac{163701753831}{1479680000}\,t^6+\frac{69}{21760}\,t^4+5/2\,t^2\right){x_3}^2+\frac{14}{5}\,t^2{x_2}^3+\frac{117649}{5}\,t^2{x_3}^6\\
&-1/3\,\left(\frac{194800333}{43520}\,t^4+1/4\,t^2\right){x_3}^4+7203\,t^2{x_1}^2{x_3}^4+180075\,t^2{x_2}^2{x_3}^4\\
&+171500\,t^2{x_2}^3{x_3}^3+1372\,t^2{x_1}^3{x_3}^3+\frac{116725987}{54400}\,t^4x_1\,{x_3}^3+\frac{116725987}{10880}\,t^4x_2\,{x_3}^3.
\end{aligned}
$$

Computational Comparison of Polynomial and Non-polynomial Solutions

In this Section we consider one example of an exact solution of (2.3.1), (2.3.2), corresponding to non-polynomial smooth data. This exact solution is presented by an explicit formula. Using our method we compute the polynomial solutions for data which are polynomial approximation of given smooth data. The numerical values of the computed polynomial solution and the numerical values of the non-polynomial solution are compared at the same fixed points. The results of this comparison are presented in the tables.

Example: Let $\mathcal{E} = diag(9,16,25)$, $\mathcal{M} = diag(16,25,36)$, $\mathbf{e} = 0$, $\mathbf{f} = 0$, $\mathbf{g} = (g_1, g_2, g_3)$, where

$$g_1 = \sin(\frac{x_2}{7})\sin(\frac{x_3}{9}), \qquad g_2 = g_3 = 0. \tag{2.3.14}$$

By the direct calculation we can check that the vector function $\mathbf{E} = (E_1, E_2, E_3)$, where

$$E_1 = \frac{1890}{\sqrt{421}}\sin(\frac{1}{7}x_2)\sin(\frac{1}{9}x_3)\sin(\frac{\sqrt{421}}{1890}t), \quad E_2 = E_3 = 0,$$

is an exact solution of (2.3.1), (2.3.2) corresponding to the given data. To find a polynomial solution we approximate g_1 by polynomial. Here we use the formula

$$\sin(az) = \sum_{m=0}^{N} \frac{(-1)^m (az)^{2m+1}}{(2m+1)!}.$$

After approximation of g_1 we have an IVP with polynomial data. Applying our method we have found an explicit formula for a polynomial solution $\mathbf{E}^N = (E_1^N, E_2^N, E_3^N)$. Taking points (x_1, x_2, x_3) as $(1,1,1)$, $(3,1,2)$, $(2,2,2)$, $(3,2,3)$, $(4,4,4)$, $(5,5,5)$ we find the numerical values of the exact and polynomial solutions. These values and results of the comparison of the first components of exact and polynomial solutions for $N = 5$ and $N = 30$ are listed in Table 2.6 and Table 2.7.

Table 2.6 Numerical values of E_1 and E_1^N for $N = 5$ at fixed points.

t	x_1	x_2	x_3	$E_1(x,t)$	$E_1^N(x,t)$	$\lvert E_1 - E_1^N \rvert$
7/5	1	1	1	0.02210	0.02210	$0.5 \cdot 10^{-9}$
1	3	1	2	0.03137	0.03137	$0.82 \cdot 10^{-9}$
2	2	2	2	0.12422	0.12422	$0.170 \cdot 10^{-7}$
7/5	3	2	3	0.12909	0.12909	$0.503 \cdot 10^{-7}$
2	4	4	4	0.46503	0.46503	$0.41361 \cdot 10^{-5}$
2	5	5	5	0.81163	0.81168	0.0000507747
90	0.1	0.1	0.1	0.01211	0.01211	$0.60647 \cdot 10^{-6}$

Table 2.7 Numerical values of E_1 and E_1^N for $N = 30$ at fixed points.

t	x_1	x_2	x_3	$E_1(x,t)$	$E_1^N(x,t)$	$\lvert E_1 - E_1^N \rvert$
7/5	1	1	1	0.02210	0.02210	0.
1	3	1	2	0.03137	0.03137	$0.1 \cdot 10^{-10}$
2	2	2	2	0.12422	0.12422	$0.1 \cdot 10^{-9}$
7/5	3	2	3	0.12909	0.12909	$0.1 \cdot 10^{-9}$
2	4	4	4	0.46503	0.46503	$0.2 \cdot 10^{-9}$
2	5	5	5	0.81163	0.81163	$0.1 \cdot 10^{-10}$
90	0.1	0.1	0.1	0.01211	0.01211	$0.1 \cdot 10^{-10}$

2.3.3 Computing and Simulating Electric Fields in Electrically and Magnetically Anisotropic Media

In this Section we describe an application of our method for computing electric fields and simulating their images in four different anisotropic media. The initial data are given by Shannon's kernels. Shannon's kernels are not polynomials and they are widely used for modelling data in real processes and phenomena (see, Bonciu & Leger & Thiel, 1998), (see, Wei, 2001). We take data as follows $\mathbf{e} = 0, \ \mathbf{f} = 0$, $\mathbf{g} = (g_1, g_2, g_3)$, where

$$g_1(x_1, x_2, x_3) = \Big(\frac{1}{x_2\pi}\Big)\sin\Big(\frac{x_2}{2}\Big)\Big(\frac{1}{x_3\pi}\Big)\sin\Big(\frac{x_3}{2}\Big), \tag{2.3.15}$$

$$g_2(x_1, x_2, x_3) = g_3(x_1, x_2, x_3) = 0. \tag{2.3.16}$$

We approximate $g_1(x_1, x_2, x_3)$ by the finite Taylor series

$$g_1^N = \sum_{k=0}^{N}\sum_{m=0}^{N}\sum_{n=0}^{N} g_1^{k,m,n} x_1^k x_2^m x_3^n, \tag{2.3.17}$$

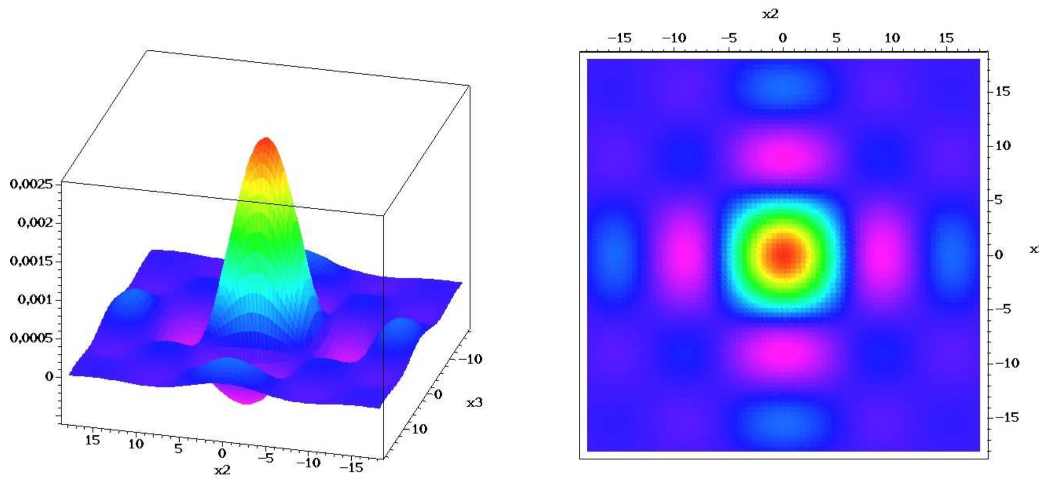
where $N = 30$.

The electric field in the sapphire

Let us consider the sapphire (this is a positive uniaxial crystal) in its principal axes (see, Werner & Cary, 2007, Nye, 1967). In the axes obtained from principal by rotating on $30°$ about x_1-axis and then on $45°$ about x_3-axis the magnetic permeability $\overline{\overline{\mu}}$ and the dielectric permittivity $\overline{\overline{\varepsilon}}$ can be written in the form (see, Werner & Cary, 2007) $\overline{\overline{\mu}} = \mu_0\mathcal{M}$, $\overline{\overline{\varepsilon}} = \varepsilon_0\mathcal{E}$. Here

$$\varepsilon_0 = \frac{1}{c^2\mu_0}, \quad \mathcal{M} = \mathbf{I}, \quad \mathcal{E} = \begin{pmatrix} 10.225 & -0.825 & -0.55\sqrt{\frac{3}{2}} \\ -0.825 & 10.225 & 0.55\sqrt{\frac{3}{2}} \\ -0.55\sqrt{\frac{3}{2}} & 0.55\sqrt{\frac{3}{2}} & 9.95 \end{pmatrix}, \tag{2.3.18}$$

where c is the speed of light, μ_0 is the magnetic permeability of vacuum, $\mathbf{I}$ is the identity 3×3 matrix. For matrices $\mathcal{E}$, $\mathcal{M}$ defined by (2.3.18) in the rotated axes, the system (2.3.1) is a mathematical model describing the electric wave propagation in the sapphire in a special time scale. In this Section we consider the system (2.3.1) subject to data (2.3.2), where $\mathbf{e} = 0$, $\mathbf{f} = 0$, $\mathbf{g} = (g_1, g_2, g_3)$, $g_1(x_1, x_2, x_3) = g_1^N(x_1, x_2, x_3)$, $g_2(x_1, x_2, x_3) = g_3(x_1, x_2, x_3) = 0$. Here $g_1^N(x_1, x_2, x_3)$ is defined by (2.3.17) for $N = 30$. Using our method we compute an explicit formula for the electric field (a polynomial solution $\mathbf{E}^N = (E_1^N, E_2^N, E_3^N)$ of (2.3.1), (2.3.2)). The graphs of the first component of $\mathbf{E}^N$ at $x_1 = 0$, $t = 1$ are presented in Fig.1(a)-1(b).



(a) 3-D plot of $E_1(0, x_2, x_3, 1)$

(b) 2-D plot of $E_1(0, x_2, x_3, 1)$

Figure 2.3 The first component of the electric field in sapphire: $x_1 = 0$, $t = 1$.

Fig. 1(a) is a $3 - D$ plot of $E_1(0, x_2, x_3, 1)$, where horizontal axes are x_2 and x_3, respectively. The vertical axis is the magnitude of $E_1^N(0, x_2, x_3, 1)$. The different colors correspond to different values of $E_1^N(0, x_2, x_3, 1)$. Fig.1(b) is a screen shot of 2D level plot of the same surface $E_1^N(0, x_2, x_3, 1)$, i.e. a view on the surface $z = E_1^N(0, x_2, x_3, 1)$ presented in Fig.1(a) from the top of z-axis. Fig.2(a)-(b) contain 2-D plots of $E_1^N(0, x_2, x_3, t)$ for $t = 3.4, 3.6, 4, 4.2, 4.4, 4.6$.

Electric fields in electrically and magnetically anisotropic media

In this Section we consider three anisotropic media with different material properties. Two of them are electrically anisotropic but magnetically isotropic. The third medium is electrically and magnetically anisotropic. The matrices of dielectric permittivity $\mathcal{E}$ and magnetically permeability $\mathcal{M}$ are listed below and do not correspond to real materials. We have taken these data for clarity in the graphical illustrations of the behavior of electric fields in the different anisotropic media.

Electrically anisotropic medium 1. The permittivity and permeability of this medium are defined by

$$\mathcal{E} = \begin{pmatrix} 0.00937 & 0.01776 & 0.01477 \\ 0.01776 & 0.005327 & 0.3453 \\ 0.01477 & 0.3453 & 0.08101 \end{pmatrix}; \quad \mathcal{M} = I.$$

Electrically anisotropic medium 2. This medium is characterized by the permittivity $\mathcal{E}$ and the permeability $\mathcal{M}$ defined by

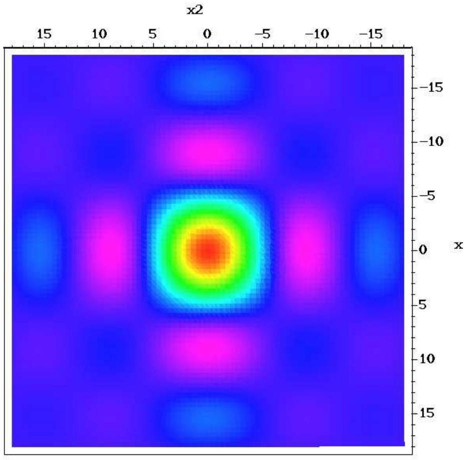
$$\mathcal{E} = \begin{pmatrix} 17.1598 & 13.0178 & 0 \\ 13.0178 & 23.6686 & 0 \\ 0 & 0 & 44.4444 \end{pmatrix}; \quad \mathcal{M} = \mathbf{I}.$$

Electrically and magnetically anisotropic medium 3. Here we consider a medium which is both electrically and magnetically anisotropic. The characteristics of this medium are given by

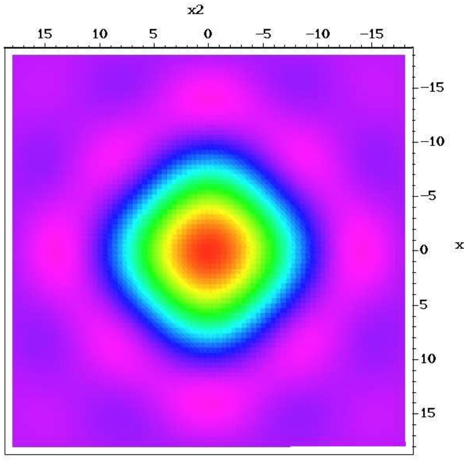
$$\mathcal{E} = \begin{pmatrix} 17.1598 & 13.0178 & 0 \\ 13.0178 & 23.6686 & 0 \\ 0 & 0 & 44.4444 \end{pmatrix}; \quad \mathcal{M} = \begin{pmatrix} 25 & 4 & 0 \\ 4 & 25 & 0 \\ 0 & 0 & 9 \end{pmatrix}.$$

For each of the mentioned media we take the same data: $\mathbf{e} = 0$, $\mathbf{f} = 0$, $\mathbf{g} = (g_1, g_2, g_3)$, $g_2 = g_3 = 0$, $g_1 = g_1^N$, where g_1^N is defined by (2.3.17) for $N = 30$.

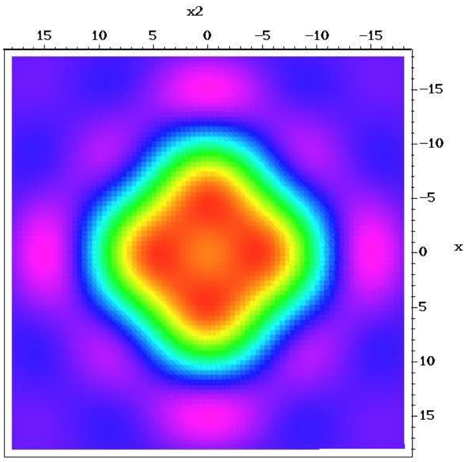
The results of the computation and simulation of electric fields for different anisotropic media 1-3 are presented on Fig.3-5, which are 2-D plots of the first component of the electric fields $E_1^N(0, x_2, x_3, t)$ of anisotropic media 1-3 for the different values t and $N = 30$.



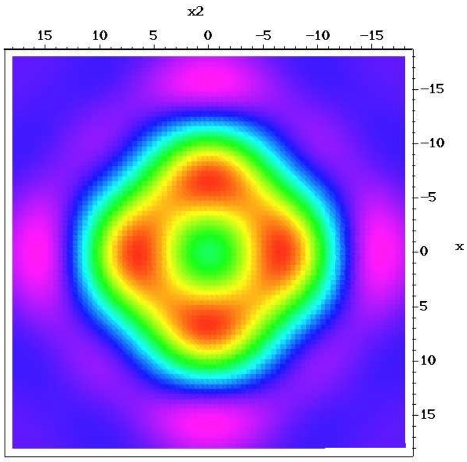
(a) 2-D plot of $E_1(0, x_2, x_3, 1)$



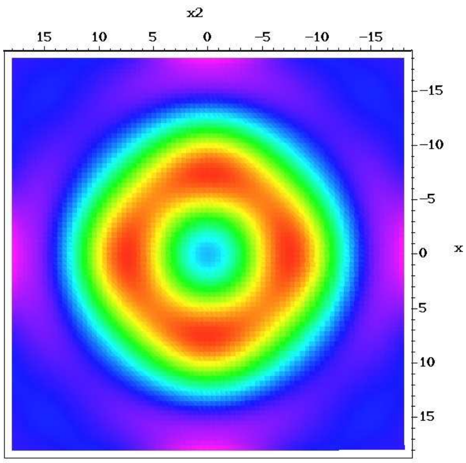
(b) 2-D plot of $E_1(0, x_2, x_3, 20)$



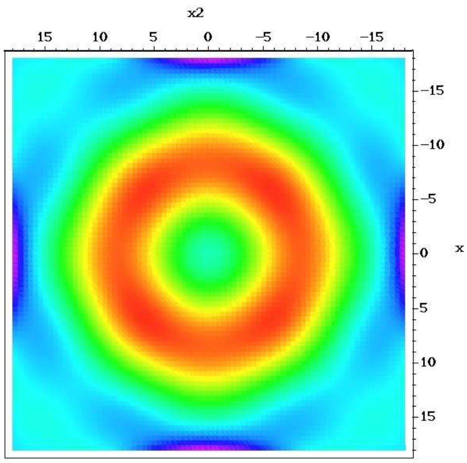
(c) 2-D plot of $E_1(0, x_2, x_3, 24)$



(d) 2-D plot of $E_1(0, x_2, x_3, 28)$

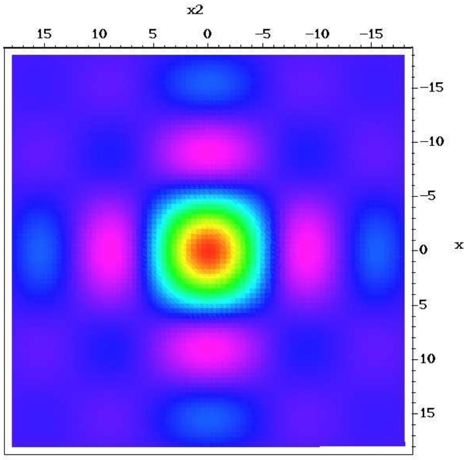


(e) 2-D plot of $E_1(0, x_2, x_3, 32)$

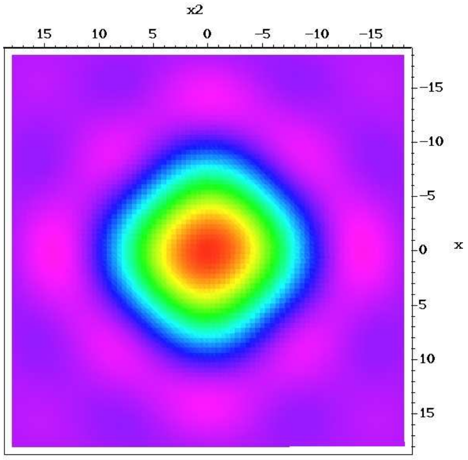


(f) 2-D plot of $E_1(0, x_2, x_3, 36)$

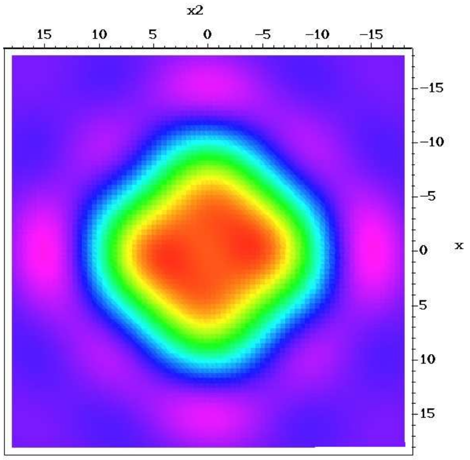
Figure 2.4 2-D plots of the first component of the electric field in the sapphire at $x_1 = 0$, $t = 1,\ 20;\ 24,\ 28;\ 32,\ 36$.



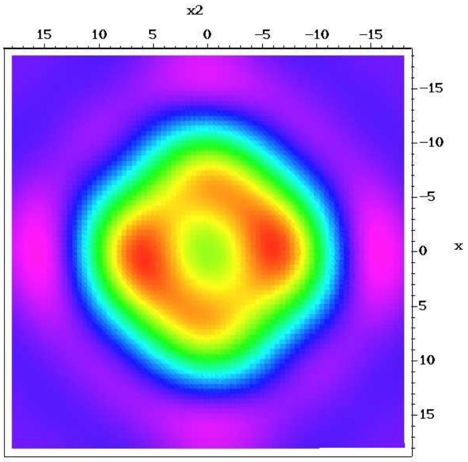
(a) 2-D plot of $E_1(0, x_2, x_3, 1/16)$



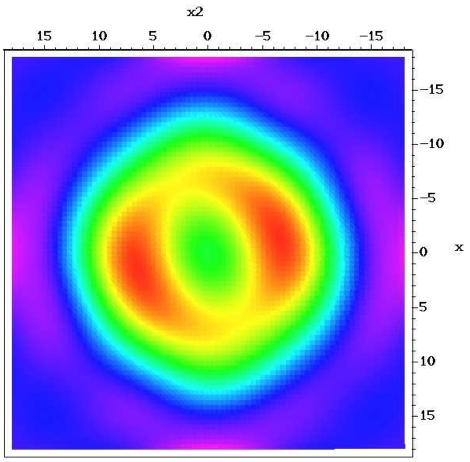
(b) 2-D plot of $E_1(0, x_2, x_3, 3/8)$



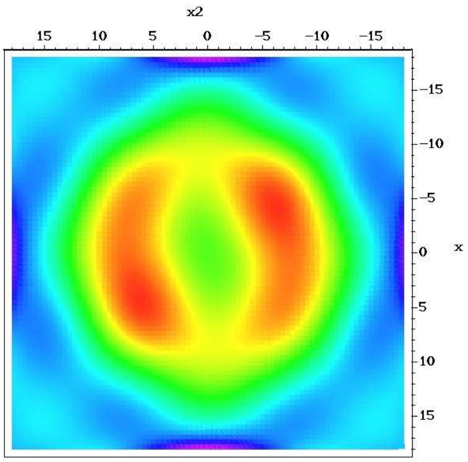
(c) 2-D plot of $E_1(0, x_2, x_3, 7/16)$



(d) 2-D plot of $E_1(0, x_2, x_3, 1/2)$

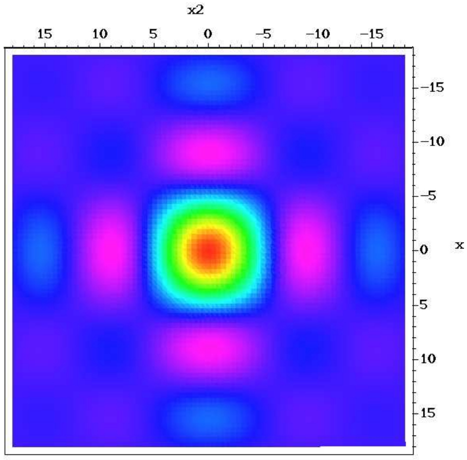


(e) 2-D plot of $E_1(0, x_2, x_3, 9/16)$

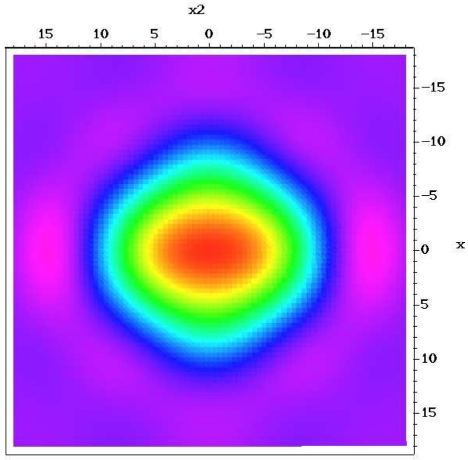


(f) 2-D plot of $E_1(0, x_2, x_3, 5/8)$

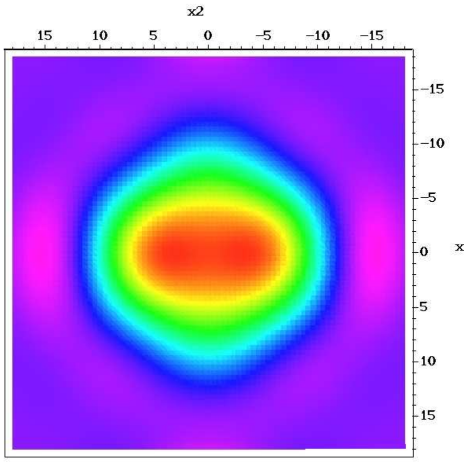
Figure 2.5 2-D plots of the first component of the electric field in the electrically anisotropic medium 1 at $x_1 = 0$, $t = 1/16,\ 3/8;\ 7/16,\ 1/2;\ 9/16,\ 5/8.$



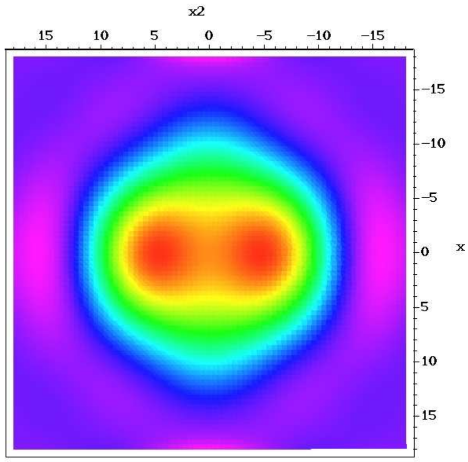
(a) 2-D plot of $E_1(0, x_2, x_3, 1)$



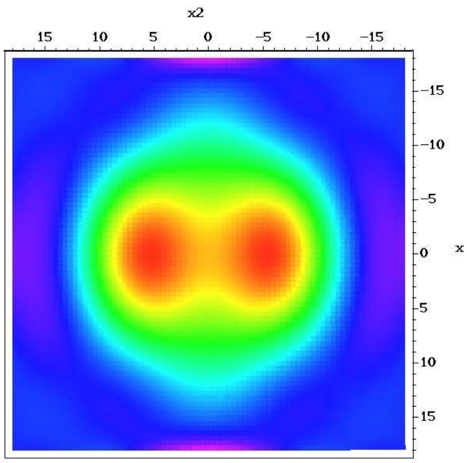
(b) 2-D plot of $E_1(0, x_2, x_3, 24)$



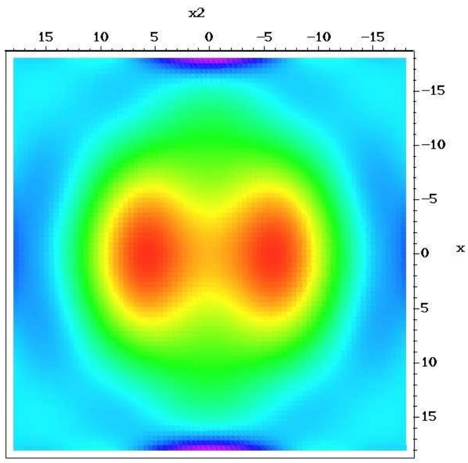
(c) 2-D plot of $E_1(0, x_2, x_3, 26)$



(d) 2-D plot of $E_1(0, x_2, x_3, 28)$

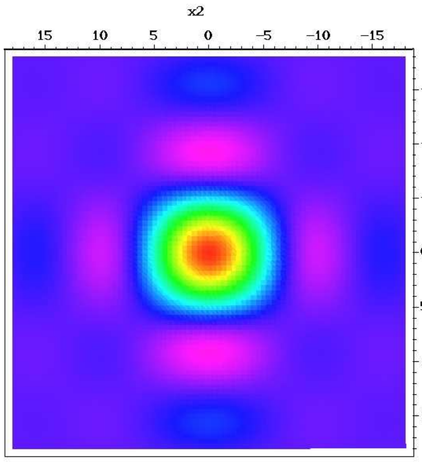


(e) 2-D plot of $E_1(0, x_2, x_3, 30)$

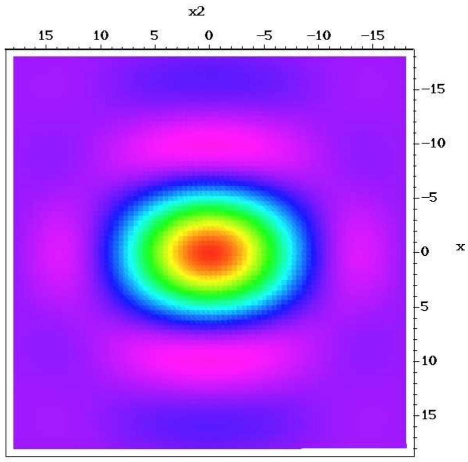


(f) 2-D plot of $E_1(0, x_2, x_3, 32)$

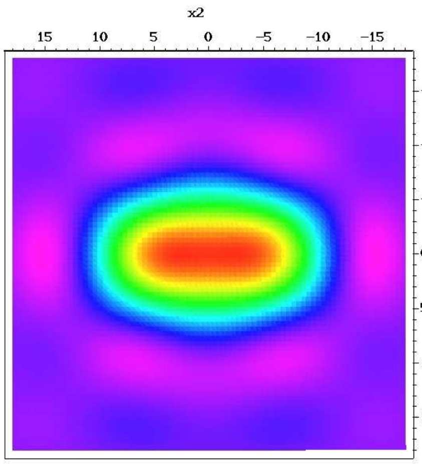
Figure 2.6 2-D plots of the first component of the electric field in the electrically anisotropic medium 2 at $x_1 = 0$, $t = 1;\ 24;\ 26;\ 28;\ 30;\ 32.$



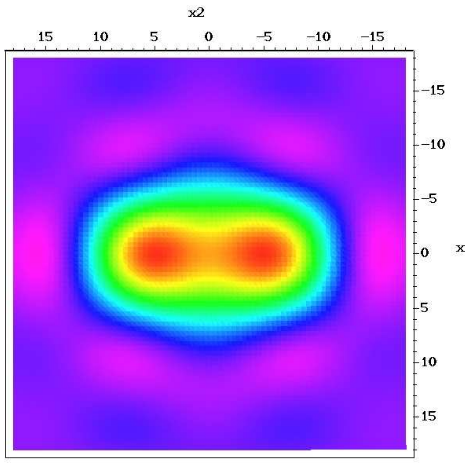
(a) 2-D plot of $E_1(0, x_2, x_3, 40)$



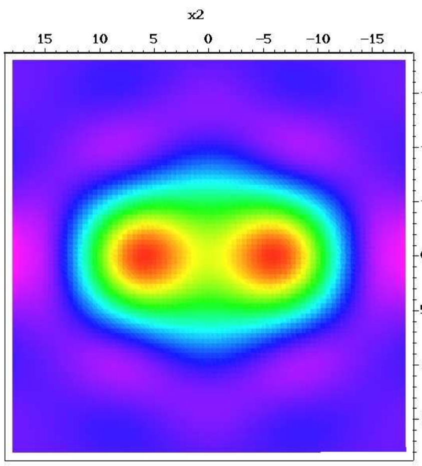
(b) 2-D plot of $E_1(0, x_2, x_3, 64)$



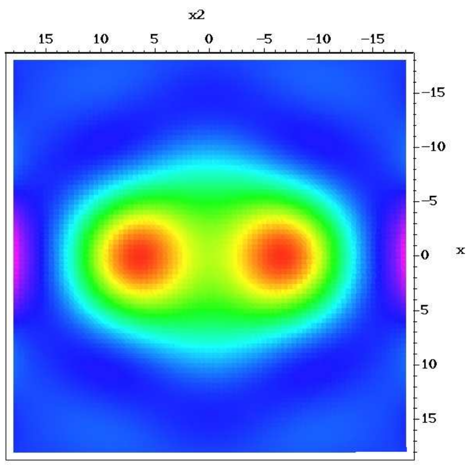
(c) 2-D plot of $E_1(0, x_2, x_3, 80)$



(d) 2-D plot of $E_1(0, x_2, x_3, 88)$



(e) 2-D plot of $E_1(0, x_2, x_3, 96)$



(f) 2-D plot of $E_1(0, x_2, x_3, 104)$

Figure 2.7 2-D plots of the first component of the electric field in the electrically and magnetically anisotropic medium 3 at $x_1 = 0$, $t = 40; 64; 80; 88; 96; 104$.

2.4 Theoretical and Computational Comparison of Polynomial and Non-polynomial Solutions for IVP of Electric Field Equations

The meta-approach of energy estimates for hyperbolic systems is well-known (Courant & Hilbert, 1979). In this Section we adjust this general approach for the description of stability estimates (energy inequalities) of solutions of the initial value problem (2.3.1), (2.3.2). Using these stability estimates we establish that polynomial solutions are approximate solutions of (2.3.1), (2.3.2) with non-polynomial smooth data. The results of a comparison of numerical values of an exact solution of (2.3.1), (2.3.2), corresponding to non-polynomial data, and values of polynomial solutions, which are computed by our method for approximated data, are presented in this Section.

2.4.1 Energy Estimates of IVP of Electric Field Equations

Initial Value Problem (2.3.1), (2.3.2) in the Form of the Cauchy Problem for a Symmetric Hyperbolic System

Letting

$$\mathbf{H}(x,t) = -\mathcal{M}^{-1}\int_0^t curl_x\mathbf{E}(x,\tau)d\tau,$$

$$\mathbf{j}(x,t) = \mathcal{E}\mathbf{g}(x) + \int_0^t \mathbf{f}(x,\tau)d\tau,$$

we find that (2.3.1), (2.3.2) is equivalent to the following Initial Value Problem (IVP) for the first order partial differential equation system

$$\mathcal{E}\frac{\partial \mathbf{E}}{\partial t} - curl_x\mathbf{H} = \mathbf{j}(x,t), \tag{2.4.1}$$

$$\mathcal{M}^{-1}curl_x\mathbf{E} = -\frac{\partial \mathbf{H}}{\partial t}, \tag{2.4.2}$$

$$\mathbf{E}(x,0) = \mathbf{e}(x), \quad \mathbf{H}(x,0) = 0. \tag{2.4.3}$$

The IVP (2.4.1), (2.4.2), (2.4.3) can be written in the form of the Cauchy problem for the symmetric hyperbolic system

$$A_0 \frac{\partial \mathbf{U}}{\partial t} + \sum_{k=1}^{3} A_k \frac{\partial \mathbf{U}}{\partial x_k} = \mathbf{F}, x \in R^3,\ t > 0, \tag{2.4.4}$$

$$\mathbf{U}(x, 0) = \mathbf{\Phi}(x), \tag{2.4.5}$$

where $\mathbf{U} = (U_1, ..., U_6)^T,\ U_k = E_k, U_{3+k} = H_k,\ k = 1, 2, 3$; $\Phi = (e_1, e_2, e_3, 0, 0, 0)^T$; $\mathbf{F} = (j_1, j_2, j_3, 0, 0, 0)^T$,

$$A_0 = \begin{pmatrix} \mathcal{E} & 0_{3\times3} \\ 0_{3\times3} & \mathcal{M} \end{pmatrix}, \quad A_k = \begin{pmatrix} 0_{3\times3} & B_k \\ B_k^T & 0_{3\times3} \end{pmatrix}, \quad k = 1, 2, 3, \tag{2.4.6}$$

$$B_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad B_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Here e_1, e_2, e_3 are components of **e**; j_1, j_2, j_3 are components of **j**; $E_k,\ k = 1, 2, 3$ are components of **E** and $H_k,\ k = 1, 2, 3$ are components of **H**. $I_{3\times3}$ is the identity matrix of the order 3×3, $0_{3\times3}$ is the zero matrix of order 3×3. B_k^T is the transpose matrix of B_k. Since matrices $\mathcal{E}$ and $\mathcal{M}$ are symmetric, positive definite matrices then A_0 defined by (2.4.6) is symmetric, positive definite and therefore there exists a real symmetric, positive definite matrix S (Goldberg, 1992) such that $A_0^{-1} = S^2$ (i.e. $S = A_0^{-\frac{1}{2}}$). Letting

$$\mathbf{U}(x, t) = S\mathbf{V}(x, t) \tag{2.4.7}$$

and substituting (2.4.7) into (2.4.4) and multiplying the obtained relation by S from the left-hand side we have

$$\frac{\partial \mathbf{V}}{\partial t} + \sum_{k=1}^{3} \widetilde{A}_k \frac{\partial \mathbf{V}}{\partial x_k} = \widetilde{\mathbf{F}}, \tag{2.4.8}$$

where

$$\widetilde{A}_k = SA_kS, \quad \widetilde{\mathbf{F}} = S\mathbf{F}.$$

Since S and A_k are real symmetric matrices, we have that $\widetilde{A}_k^T = (SA_kS)^T = S^T(SA_k)^T = S^TA_k^TS^T = SA_kS = \widetilde{A}_k$ and hence $\widetilde{A}_k$ is symmetric. Initial data (2.4.5) may be written as

$$\mathbf{V}(x,0) = \widetilde{\mathbf{\Phi}}(x), \tag{2.4.9}$$

where $\widetilde{\mathbf{\Phi}}(x) = S^{-1}\mathbf{\Phi}(x)$. Therefore (2.3.1), (2.3.2) is written equivalently as the Cauchy problem (2.4.8), (2.4.9).

Stability Estimate for the Cauchy Problem of a Symmetric Hyperbolic System

Let $\mathbf{V}(x,t)$ and $\mathbf{V}^*(x,t)$ be two continuously differentiable solutions of (2.4.8), (2.4.9) corresponding to initial data $\widetilde{\mathbf{\Phi}}(x)$, $\widetilde{\mathbf{\Phi}}^*(x)$ and inhomogeneous terms $\widetilde{\mathbf{F}}(x,t)$, $\widetilde{\mathbf{F}}^*(x,t)$, respectively. Denoting $\hat{\mathbf{V}} = \mathbf{V} - \mathbf{V}^*$, $\hat{\mathbf{\Phi}} = \widetilde{\mathbf{\Phi}} - \widetilde{\mathbf{\Phi}}^*$, $\hat{\mathbf{F}} = \widetilde{\mathbf{F}} - \widetilde{\mathbf{F}}^*$ we find from (2.4.8), (2.4.9)

$$\frac{\partial \hat{\mathbf{V}}}{\partial t} + \sum_{k=1}^{3} \widetilde{A}_k \frac{\partial \hat{\mathbf{V}}}{\partial x_k} = \hat{\mathbf{F}}, \quad x \in R^3, \quad t > 0, \tag{2.4.10}$$

$$\hat{\mathbf{V}}(x,0) = \hat{\mathbf{\Phi}}(x), \ \ x \in R^3. \tag{2.4.11}$$

Let T be a fixed positive number, $\xi = (\xi_1, \xi_2, \xi_3) \in R^3$ be a parameter; $A(\xi)$ be a matrix defined by $A(\xi) = \sum_{k=1}^{3} \widetilde{A}_k \xi_k;\ \lambda_n(\xi),\ n = 1, 2, ..., 6$ be eigenvalues of $A(\xi)$. The positive number M is defined by

$$M = \max_{n=1,2,\ldots,6} \max_{|\xi|=1} |\lambda_n(\xi)|. \tag{2.4.12}$$

Using T and M we introduce the family of spheres in R^3 by

$$S(h) = \{x \in R^3 : |x| \leq M(T-h)\},\ 0 \leq h \leq T. \tag{2.4.13}$$

Applying the reasoning similar to (Courant & Hilbert, 1979) (p. 652-661)(see also Appendix B) we find the following estimate for the solution of (2.4.8), (2.4.9)

$$\int_{S(h)} |\hat{\mathbf{V}}(x,h)|^2 dx \leq e^h \Big[\int_{S(0)} |\hat{\mathbf{\Phi}}(x)|^2 dx + \int_0^h \Big(\int_{S(t)} |\hat{\mathbf{F}}(x,t)|^2 dx \Big) dt \Big]. \tag{2.4.14}$$

Stability Estimates of IVP of Electric Field Equations

Let now $\mathbf{E} = (E_1, E_2, E_3)$ and $\mathbf{E}^* = (E_1^*, E_2^*, E_3^*)$ be two continuous differentiable solutions of (2.3.1), (2.3.2) corresponding to given initial data $\mathbf{e} = (e_1, e_2, e_3)$, $\mathbf{g} = (g_1, g_2, g_3)$, $\mathbf{e}^* = (e_1^*, e_2^*, e_3^*)$, $\mathbf{g}^* = (g_1^*, g_2^*, g_3^*)$ and inhomogeneous terms $\mathbf{f} = (f_1, f_2, f_3)$, $\mathbf{f}^* = (f_1^*, f_2^*, f_3^*)$, respectively.

Let $\hat{E}_k = E_k - E_k^*$, $\hat{e}_k = e_k - e_k^*$, $\hat{g}_k = g_k - g_k^*$, $\hat{f}_k = f_k - f_k^*$, $k = 1, 2, 3$; $\hat{\mathbf{\Phi}} = (\hat{\Phi}_1, ..., \hat{\Phi}_6)$, $\hat{\Phi}_k = \hat{e}_k$, $\hat{\Phi}_{3+k} = 0$, $k = 1, 2, 3$;

$$\hat{\mathbf{F}} = (\hat{F}_1, ..., \hat{F}_6),\ \hat{F}_k = (j_1, j_2, j_3, 0, 0, 0)$$

where j_k are components of $\mathbf{j}$ that is defined as $\mathbf{j}(x,t) = \mathcal{E}\mathbf{g}(x) + \int_0^t \mathbf{f}(x,\tau)d\tau$,

$$\hat{\mathbf{U}} = (\hat{U}_1, ..., \hat{U}_6),\ \hat{U}_k = \hat{E}_k, \hat{U}_{3+k} = H_k$$

where H_k are components of $\mathbf{H}$ that is defined as

$$\mathbf{H}(x,t) = -\mathcal{M}^{-1} \int_0^t curl_x \mathbf{E}(x,\tau)d\tau.$$

Applying results of Section 2.4.1 and using the inequality (2.4.14) we find the stability estimate for solution of (2.3.1), (2.3.2)

$$\int_{S(h)} |A_0^{\frac{1}{2}} \hat{\mathbf{U}}(x,h)|^2 dx \le e^h \Big[\int_{S(0)} |A_0^{\frac{1}{2}} \hat{\mathbf{\Phi}}(x)|^2 dx + \int_0^h \Big(\int_{S(t)} |A_0^{-\frac{1}{2}} \hat{\mathbf{F}}(x,t)|^2 dx \Big) dt \Big]. \qquad (2.4.15)$$

where A_0 is the symmetric positive definite matrix defined by (2.4.6), $A_0^{\frac{1}{2}}$ is the square root of A_0, $A_0^{-\frac{1}{2}}$ is the inverse matrix to $A_0^{\frac{1}{2}}$.

2.4.2 Theoretical Comparison of Polynomial and Non-polynomial Solutions

Let T be a given positive number, M be the number defined by (2.4.12); $S(h)$ be the family of spheres in R^3 defined by (2.4.13); Γ be a cone with the vertex $(0, T)$ defined by

$$\Gamma = \{(x,t) : 0 \leq t \leq T,\ |x| \leq M(T-t)\}. \tag{2.4.16}$$

Let $\mathbf{E}$ be a solution of (2.3.1), (2.3.2) corresponding to polynomial data $\mathbf{e},\ \mathbf{g}$ and polynomial inhomogeneous term $\mathbf{f}$ of the forms (2.3.3), (2.3.4), (2.3.5); $\mathbf{E}^*$ be a continuously differentiable solution of (2.3.1), (2.3.2) corresponding to $\mathbf{e}^*, \mathbf{g}^*, \mathbf{f}^*$ and let $\mathbf{e}^*(x), \mathbf{g}^*(x), \mathbf{f}^*(x,t)$ coincide with $\mathbf{e}(x), \mathbf{g}(x), \mathbf{f}(x,t)$ for $x \in S(0)$ and $(x,t) \in \Gamma$ only, and the behavior of $\mathbf{e}^*(x), \mathbf{g}^*(x), \mathbf{f}^*(x,t)$ be unrestricted outside $S(0)$ and Γ, respectively.

Using polynomial presentations of $\mathbf{e}(x), \mathbf{g}(x), \mathbf{f}(x,t)$ we construct polynomial expansions of $\mathbf{E}(x,t)$ in the form (2.3.6) by the analytic method described in Section 2. Applying the inequality (2.4.15) we find that $\mathbf{E}^*(x,t) = \mathbf{E}(x,t)$ for $(x,t) \in \Gamma$. This means that this analytic method can be used to validate a computational code for finding a solution $\mathbf{E}^*(x,t)$ in the bounded domain Γ if data are polynomial in Γ only. We note also that the stability estimate (2.4.15) shows that a small variation of data $\mathbf{e}(x),\ \mathbf{g}(x)$ for $x \in S(0)$ and inhomogeneous term $\mathbf{f}(x,t)$ for $(x,t) \in \Gamma$ corresponds to a small variation of the solution $\mathbf{E}(x,t)$ for $(x,t) \in \Gamma$. This theoretical result is confirmed by computational examples.

CHAPTER THREE
FUNDAMENTAL SOLUTIONS OF LINEAR ANISOTROPIC ELASTICITY: PROPERTIES, DERIVATION, APPLICATIONS

Fundamental solutions of partial differential equations play an important role in both applied and theoretical studies on physics of solids (see, e.g.(Stokes, 1883); (Poisson, 1829), (Volterra, 1894); (Fredholm, 1908); (Mindlin, 1936); (Mindlin & Cheng, 1950); (Phan-Thien, 1983); (Huang & Wang, 1991)). In this chapter, the system of partial differential equations of anisotropic elasticity is considered. The inhomogeneous term of this system has a finite support. This system is considered for cases when displacement vector depends on one, two or three space and the time variables. A new method is suggested to find fundamental solutions for these cases. This method is based on properties: fundamental solutions of the considered system have finite supports with respect to space variables for any fixed time variable; the Fourier images of solution components are analytic functions with respect to parameters of the Fourier transform and these Fourier images can be expanded in power series. The method consists of following. The system of equations of anisotropic elasticity is written for each cases. These equalities are written in the form of the Fourier images. Using power series presentations with unknown coefficients depending on t we construct the recurrence relations. These unknown coefficients are obtained using a procedure. Using these coefficients Fourier images of solution components can be obtained. Applying inverse Fourier transform to these images, fundamental solutions of the system of anisotropic elasticity can be constructed. Using mathematical tools (Maple 10) simulation of fundamental solutions in different anisotropic materials are presented. Computation examples confirm the robustness of our approach. In the chapter applications of the fundamental solutions for solving the Initial Value Problems (IVP) for the system of anisotropic elasticity is described.

3.1 Definitions and General Equations of Elasticity

In this section we consider basic definitions and equations of linear elasticity. Detailed explanation can be found in the books (Dieulesaint & Royer, 1980), (Fedorov, 1963), (Landau & Lifshitz, 1998).

Consider a solid body which is subject to external forces. This body deforms in shape and points inside the body move. Let x be a point in the undeformed body whose coordinates are (x_1, x_2, x_3) and x' denote the same point after deformation with coordinates (x'_1, x'_2, x'_3). The displacement of the point x at time t is the vector $\mathbf{U} = (U_1, U_2, U_3)$, called displacement vector, with components

$$U_i = x'_i - x_i, \quad i = 1, 2, 3. \tag{3.1.1}$$

Since different points in the body displace differentially during deformation, $\mathbf{U}$ is a vector-function of the coordinates of the point in the body.

Let x and $\tilde{x}$ be points infinitely close to each other, with small displacement $\mathbf{dx} = (dx_1, dx_2, dx_3)$, where

$$dx_i = \tilde{x}_i - x_i.$$

We denote the new locations of these points after deformation with x' and $\tilde{x}'$, and the displacement of them with dx'. Then we have

$$dx'_i = dx_i + dU_i(x, t), \quad i = 1, 2, 3,$$

where $dU_i(x, t) = U_i(\tilde{x}, t) - U_i(x, t)$. Noting the relation

$$dU_i(x, t) \cong \sum_{k=1}^{3} \frac{\partial U_i(x, t)}{\partial x_k} dx_k,$$

the squared difference of the displacements $\mathbf{dx}$ and $\mathbf{dx}'$ can be found as

$$\sum_{i=1}^{3} (dx'_i)^2 - \sum_{i=1}^{3} (dx_i)^2 =$$

$$\sum_{i,k=1}^{3}\left(\frac{\partial U_i(x,t)}{\partial x_k}+\frac{\partial U_k(x,t)}{\partial x_i}+\sum_{l=1}^{3}\frac{\partial U_l(x,t)}{\partial x_k}\frac{\partial U_l(x,t)}{\partial x_i}\right)dx_k dx_i$$

from which we define ϵ_{ik} in the form

$$\epsilon_{ik}=\frac{1}{2}\left(\frac{\partial U_i(x,t)}{\partial x_k}+\frac{\partial U_k(x,t)}{\partial x_i}+\sum_{l=1}^{3}\frac{\partial U_l(x,t)}{\partial x_k}\frac{\partial U_l(x,t)}{\partial x_i}\right), \qquad (3.1.2)$$

$$i,k=1,2,3.$$

The terms ϵ_{ik}, $i,k=1,2,3$ form a second-order tensor with 9 components, which is called strain tensor. It follows from formula (3.1.2) that strain tensor is symmetrical, i.e.

$$\epsilon_{ik}=\epsilon_{ki}.$$

It is usually assumed that the deformation of a solid is small. Therefore we can omit the last term in (3.1.2) to define strain tensor for small deformations with relation

$$\epsilon_{ik}=\frac{1}{2}\left(\frac{\partial U_i(x,t)}{\partial x_k}+\frac{\partial U_k(x,t)}{\partial x_i}\right). \qquad (3.1.3)$$

Let the forces applied to elastic body that cause deformation be removed. In this case it tends to return its original state. This occurs due to internal forces arising in the deformed body. These forces are called internal stresses.

Let $\mathbf{f}=(f_1,f_2,f_3)$ be the force per unit volume. Then the total force $\mathbf{F}=(F_1,F_2,F_3)$ on a volume V can be found by the volume integral

$$\mathbf{F}=\int_V \mathbf{f}dV,$$

where $\mathbf{f}dV$ is the force on the volume element dV. Since these forces also act on the surface bounding that volume, $\mathbf{F}$ can also be written as a surface integral. To do this we define f_i as the divergence of a second-order tensor with the formula

$$f_i=\sum_{k=1}^{3}\frac{\partial \sigma_{ik}}{\partial x_k}, \quad i=1,2,3$$

and apply divergence theorem which gives

$$\int f_i dV = \int \sum_{k=1}^{3} \frac{\partial \sigma_{ik}}{\partial x_k} dV = \oint \sum_{k=1}^{3} \sigma_{ik} n_k dS.$$

The tensor σ_{ik} is called the stress tensor. From the fact that in equilibrium, the sum of the moments of all forces must be zero the symmetry property of the stress tensor follows, i.e.

$$\sigma_{ik} = \sigma_{ki}, \quad i, k = 1, 2, 3.$$

As we stated above when external forces are applied to an elastic material, internal stresses arise to remove the deformation. It is clear that if the deformation is big the stresses will be big. So there is a relation between stress and strain, which can be stated by writing stress as a function of strain

$$\sigma_{jk} = \sigma_{jk}(\epsilon_{lm}).$$

We write the Maclaurin series expansion of each σ_{jk}. Since ϵ_{lm} are small we neglect the terms after first degree and obtain

$$\sigma_{jk} = \sigma_{jk}(0) + \sum_{l,m=1}^{3} \left(\frac{\partial \sigma_{jk}}{\partial \epsilon_{lm}} \right)_{\epsilon_{lm}=0} \epsilon_{lm}. \tag{3.1.4}$$

Noting that if there is no deformation the stress is zero we have $\sigma_{jk}(0) = 0$. So (3.1.4) becomes

$$\sigma_{jk} = \sum_{l,m=1}^{3} c_{jklm} \epsilon_{lm}, \tag{3.1.5}$$

where

$$c_{jklm} = \left(\frac{\partial \sigma_{jk}}{\partial \epsilon_{lm}} \right)_{\epsilon_{lm}=0}. \tag{3.1.6}$$

Formula (3.1.5) is called Hooke's law and c_{jklm} are called elastic moduli, which form a fourth-order tensor. This tensor is called the tensor of elastic moduli and it has $3^4 = 81$ components. Recalling the symmetry properties $\sigma_{jk} = \sigma_{kj}$, $\epsilon_{lm} = \epsilon_{ml}$ of stress and strain tensors we get the following symmetry properties for elastic moduli

$$c_{jklm} = c_{kjlm} = c_{jkml}, \tag{3.1.7}$$

which reduces the number of independent components to 36. Furthermore, due to considerations for the density of potential energy

$$\Phi = \frac{1}{2}\sum_{j,k=1}^{3} \sigma_{jk}\epsilon_{jk} = \frac{1}{2}\sum_{j,k,l,m=1}^{3} c_{jklm}\epsilon_{jk}\epsilon_{lm} \tag{3.1.8}$$

arise an additional symmetry property

$$c_{jklm} = c_{lmjk}, \tag{3.1.9}$$

which decrease the number of independent c_{jklm} to 21. Due to the properties (3.1.7), (3.1.9) it is convenient to represent the fourth-order tensor of elastic moduli in terms of a 6×6 matrix called stiffness matrix, which we denote by $\mathbf{C}$. This representation is realized by replacing the pairs (j, k) of indices $j, k = 1, 2, 3$ with a single index $\alpha = 1, \ldots, 6$ according to the following rules:

$$\begin{aligned} &(1,1) \longleftrightarrow 1, \qquad (2,2) \longleftrightarrow 2, \qquad (3,3) \longleftrightarrow 3, \\ &(2,3),(3,2) \longleftrightarrow 4, \qquad (1,3),(3,1) \longleftrightarrow 5, \qquad (1,2),(2,1) \longleftrightarrow 6. \end{aligned} \tag{3.1.10}$$

Similarly, replacing the pairs (l, m) of indices $l, m = 1, 2, 3$ with index $\beta = 1, \ldots, 6$ in accordance with (3.2.1) gives

$$c_{\alpha\beta} = c_{jklm},$$

where $c_{\alpha\beta}$ are components of the matrix $\mathbf{C}$. From property (3.1.9) we have the symmetry condition $c_{\alpha\beta} = c_{\beta\alpha}$, which implies that $\mathbf{C}$ is a symmetric matrix.

Definition 3.1.1. The tensor of elastic moduli c_{jklm} are positive-definite if the inequality

$$\sum_{j,k,l,m=1}^{3} c_{jklm}\xi_{jk}\xi_{lm} > 0 \tag{3.1.11}$$

is satisfied for arbitrary non-zero second-order tensor $\boldsymbol{\xi} = (\xi_{jk})_{3\times3}$. (Knops & Payne, 1971)

In the thesis we assume that the tensor of elastic moduli is positive definite. Note that (3.1.11) can be written in the form

$$\sum_{\alpha,\beta=1}^{6} c_{\alpha\beta}\xi_\alpha\xi_\beta > 0, \quad \xi \neq 0. \tag{3.1.12}$$

As a result, the tensor of elastic moduli can be written as a 6×6 symmetric, positive-definite matrix

$$\mathbf{C} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} \\ c_{12} & c_{22} & c_{23} & c_{24} & c_{25} & c_{26} \\ c_{13} & c_{23} & c_{33} & c_{34} & c_{35} & c_{36} \\ c_{14} & c_{24} & c_{34} & c_{44} & c_{45} & c_{46} \\ c_{15} & c_{25} & c_{35} & c_{45} & c_{55} & c_{56} \\ c_{16} & c_{26} & c_{36} & c_{46} & c_{56} & c_{66} \end{bmatrix}, \tag{3.1.13}$$

with 21 independent components in general. Anisotropic materials are classified into 7 systems, and (3.1.13) is the stiffness matrix of the most general anisotropic system called triclinic. Each system has special form of stiffness matrix. The other 6 systems and their stiffness matrices are presented in Table 3.1.

Table 3.1 Table of stiffness matrices.

$$\begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & c_{16} \\ c_{12} & c_{22} & c_{23} & 0 & 0 & c_{26} \\ c_{13} & c_{23} & c_{33} & 0 & 0 & c_{36} \\ 0 & 0 & 0 & c_{44} & c_{45} & 0 \\ 0 & 0 & 0 & c_{45} & c_{55} & 0 \\ c_{16} & c_{26} & c_{36} & 0 & 0 & c_{66} \end{bmatrix}$$

Monoclinic

$$\begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{22} & c_{23} & 0 & 0 & 0 \\ c_{13} & c_{23} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66} \end{bmatrix}$$

Orthorhombic

$$\begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} & -c_{25} & 0 \\ c_{12} & c_{11} & c_{13} & -c_{14} & c_{25} & 0 \\ c_{13} & c_{13} & c_{33} & 0 & 0 & 0 \\ c_{14} & -c_{14} & 0 & c_{44} & 0 & c_{25} \\ -c_{25} & c_{25} & 0 & 0 & c_{44} & c_{14} \\ 0 & 0 & 0 & c_{25} & c_{14} & \frac{c_{11}-c_{12}}{2} \end{bmatrix} \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} & 0 & 0 \\ c_{12} & c_{11} & c_{13} & -c_{14} & 0 & 0 \\ c_{13} & c_{13} & c_{33} & 0 & 0 & 0 \\ c_{14} & -c_{14} & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{44} & c_{14} \\ 0 & 0 & 0 & 0 & c_{14} & \frac{c_{11}-c_{12}}{2} \end{bmatrix}$$

Trigonal

$$\begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & c_{16} \\ c_{12} & c_{11} & c_{13} & 0 & 0 & -c_{16} \\ c_{13} & c_{13} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{44} & 0 \\ c_{16} & -c_{16} & 0 & 0 & 0 & c_{66} \end{bmatrix} \qquad \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{11} & c_{13} & 0 & 0 & 0 \\ c_{13} & c_{13} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66} \end{bmatrix}$$

Tetragonal

$$\begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{11} & c_{13} & 0 & 0 & 0 \\ c_{13} & c_{13} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{c_{11}-c_{12}}{2} \end{bmatrix}$$

Hexagonal

$$\begin{bmatrix} c_{11} & c_{12} & c_{12} & 0 & 0 & 0 \\ c_{12} & c_{11} & c_{12} & 0 & 0 & 0 \\ c_{12} & c_{12} & c_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{44} \end{bmatrix}$$

Cubic

The formulation of equations of motion of a deformed elastic body is derived from the fact that the sum of all forces must be equal to zero under equilibrium condition. Denoting the density of external forces with F_j, $j = 1, 2, 3$ and neglecting the body forces the equations of motion can be written in the form

$$\rho \frac{\partial^2 U_j}{\partial t^2} = \sum_{k=1}^{3} \frac{\partial \sigma_{jk}}{\partial x_k} + F_j(x,t), \qquad j = 1, 2, 3, \tag{3.1.14}$$

where $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, $\;t > 0$ and $U_j(x,t);\; (j = 1, 2, 3)$ are the components of the unknown displacement vector $\mathbf{U}(x,t)$. $F_j(x,t);\; (j = 1, 2, 3)$ are the components of nonhomogeneous vector function $\mathbf{F}(x,t)$ depending on x, t.

3.1.1 IVP for the System of Elasticity

The mathematical model of elastic wave propagation in a homogeneous, anisotropic medium [(Royer & Dieulesaint, 2000); (Cohen, 2002)] is described by

$$\rho \frac{\partial^2 U_j}{\partial t^2} = \sum_{k=1}^{3} \frac{\partial \sigma_{jk}}{\partial x_k} + F_j(x,t), \quad j = 1, 2, 3, \tag{3.1.15}$$

$$U_j(x,0) = G_j(x), \quad \left. \frac{\partial U_j(x,t)}{\partial t} \right|_{t=0} = H_j(x), \tag{3.1.16}$$

where $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, $\;t > 0$ and $U_j(x,t);\; (j = 1, 2, 3)$ are the components of the unknown displacement vector $\mathbf{U}(x,t)$. $F_j(x,t);\; (j = 1, 2, 3)$ are the components of nonhomogeneous vector function $\mathbf{F}(x,t)$ depending on x, t and G_j, H_j are components of initial data $\mathbf{G}(x)$; $\mathbf{H}(x)$ depending on x variable only. The constant $\rho > 0$ is the density of the medium. Stress tensor σ_{jk} are defined as

$$\sigma_{jk} = \sum_{l,m=1}^{3} c_{jklm} \epsilon_{lm}, \tag{3.1.17}$$

and the strain tensor is defined as

$$\epsilon_{lm} = \frac{1}{2}\Big(\frac{\partial U_l}{\partial x_m} + \frac{\partial U_m}{\partial x_l} \Big). \tag{3.1.18}$$

$\{c_{jklm}\}_{j,k,l,m=1}^{3}$ are elastic moduli of the medium which is a forth-order positive definite constant tensor that satisfy the symmetry properties $c_{jklm} = c_{lmjk} = c_{kjlm} = c_{jkml}$ [(Fedorov, 1963); (Royer & Dieulesaint, 2000); (Knops & Payne, 1971)].

The system (3.1.15) is hyperbolic. The proof related with hyperbolicity of dynamic elastic system is given in Appendix.

3.2 IVP for the System of Elasticity Depending on x_3 and t Variables

3.2.1 Reduction of System Depending on x_3 and t Variables to a First-Order Symmetric Hyperbolic System

In this section we explain the process of writing (3.1.15), (3.1.16) as a symmetric hyperbolic system when displacement vector depends on x_3 and t variables.

Let us consider IVP of elastic system when inhomogeneous term $\mathbf{F}(x_3, t)$ be given function depending x_3 and t variables and initial data $\mathbf{G}(x_3)$; $\mathbf{H}(x_3)$ be given functions depending x_3 variable. Thus, solution $\mathbf{U}$ of (3.1.15), (3.1.16) depends on x_3 and t variables.

Noting the symmetry properties $\sigma_{jk} = \sigma_{kj}$, $\epsilon_{jk} = \epsilon_{kj}$ of stress and strain tensors and the rule

$$(1,1) \leftrightarrow 1, \quad (2,2) \leftrightarrow 2, \quad (3,3) \leftrightarrow 3,$$

$$(2,3),(3,2) \leftrightarrow 4, \quad (1,3),(3,1) \leftrightarrow 5, \quad (1,2),(2,1) \leftrightarrow 6, \tag{3.2.1}$$

we denote a pair (j,k) of indices $j,k = 1,2,3$ as a single index α, $\alpha = 1,\ldots,6$. Using this renumeration we can write

$$\mathbf{T} = [\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6]^*, \qquad \epsilon = [\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_5, \epsilon_6]^*, \tag{3.2.2}$$

where $*$ is the sign of transposition. Let us define

$$\mathbf{Y} = [\epsilon_1, \epsilon_2, \epsilon_3, 2\epsilon_4, 2\epsilon_5, 2\epsilon_6]^*, \qquad \bar{\mathbf{U}} = [\bar{U}_1, \bar{U}_2, \bar{U}_3]^*, \tag{3.2.3}$$

where

$$\bar{U}_i = \frac{\partial U_i}{\partial t}, \qquad i = 1, 2, 3. \tag{3.2.4}$$

Since

$$\frac{\partial \bar{U}_i}{\partial t} = \frac{\partial^2 U_i}{\partial t^2} \tag{3.2.5}$$

the left-hand side of (3.1.15) can be written in vector form

$$\rho \frac{\partial^2 \mathbf{U}}{\partial t^2} = \rho \frac{\partial \bar{\mathbf{U}}}{\partial t} \tag{3.2.6}$$

Consider the term $\sum_{k=1}^{3} \frac{\partial \sigma_{jk}}{\partial x_k}$ on the right-hand side of (3.1.15). Applying rule (3.2.1) for $j = 1, 2, 3$ gives

$$\sum_{k=1}^{3} \frac{\partial \sigma_{1k}}{\partial x_k} = \frac{\partial \sigma_{13}}{\partial x_3} = \frac{\partial \sigma_5}{\partial x_3}, \tag{3.2.7}$$

$$\sum_{k=1}^{3} \frac{\partial \sigma_{2k}}{\partial x_k} = \frac{\partial \sigma_{23}}{\partial x_3} = \frac{\partial \sigma_4}{\partial x_3}, \tag{3.2.8}$$

$$\sum_{k=1}^{3} \frac{\partial \sigma_{3k}}{\partial x_k} = \frac{\partial \sigma_{33}}{\partial x_3} = \frac{\partial \sigma_3}{\partial x_3}. \tag{3.2.9}$$

Using the vector $\mathbf{T}$, (3.2.7)-(3.2.9) takes the form

$$\begin{aligned} \sum_{k=1}^{3} \frac{\partial \sigma_{1k}}{\partial x_k} &= [0,0,0,0,1,0] \cdot \frac{\partial \mathbf{T}}{\partial x_3}, \\ \sum_{k=1}^{3} \frac{\partial \sigma_{2k}}{\partial x_k} &= [0,0,0,1,0,0] \cdot \frac{\partial \mathbf{T}}{\partial x_3}, \\ \sum_{k=1}^{3} \frac{\partial \sigma_{3k}}{\partial x_k} &= [0,0,1,0,0,0] \cdot \frac{\partial \mathbf{T}}{\partial x_3}. \end{aligned}$$

Noting the coefficient vectors of terms $\dfrac{\partial \mathbf{T}}{\partial x_3}$, we introduce the matrix

$$\mathbf{A}_3^1 = \begin{bmatrix} 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{bmatrix} \tag{3.2.10}$$

and represent $\displaystyle\sum_{k=1}^{3} \frac{\partial \sigma_{jk}}{\partial x_k}$, $j = 1, 2, 3$ in the form

$$\sum_{k=1}^{3} \frac{\partial \sigma_{jk}}{\partial x_k} = -\mathbf{A}_3^1 \frac{\partial \mathbf{T}}{\partial x_3}. \tag{3.2.11}$$

From (3.2.6) and (3.2.11) it follows that we can rewrite (3.1.15) as

$$\rho \frac{\partial \bar{\mathbf{U}}}{\partial t} + \mathbf{A}_3^1 \frac{\partial \mathbf{T}}{\partial x_3} = \mathbf{f}, \tag{3.2.12}$$

where $\mathbf{f} = (f_1, f_2, f_3)$.

Relation (3.1.17) can be written as two summations

$$\sigma_{jk} = \sum_{\substack{l,m=1 \\ l=m}}^{3} c_{jklm} \epsilon_{lm} + \sum_{\substack{l,m=1 \\ l \neq m}}^{3} c_{jklm} \epsilon_{lm}, \quad j, k = 1, 2, 3. \tag{3.2.13}$$

Denoting the pair of indices (j, k) with α, $\alpha = 1, \ldots, 6$, (l, m) with β, $\beta = 1, \ldots, 6$, according to rule (3.2.1), relation (3.2.13) can be written as

$$\sigma_\alpha = \sum_{\beta=1}^{3} c_{\alpha\beta} \epsilon_\beta + 2 \sum_{\beta=4}^{6} c_{\alpha\beta} \epsilon_\beta, \tag{3.2.14}$$

or in terms of vectors $\mathbf{T}$ and $\mathbf{Y}$ as

$$\mathbf{T} = \mathbf{C}\mathbf{Y}, \tag{3.2.15}$$

where $\mathbf{C} = (c_{\alpha\beta})_{6\times 6}$ is stiffness matrix defined with (3.1.13) that is symmetric and positive definite (See, Section 3.1).

Taking derivative of (3.2.15) with respect to t and multiplying both sides by the inverse of $\mathbf{C}$, denoted $\mathbf{C}^{-1}$, we find

$$\mathbf{C}^{-1}\frac{\partial \mathbf{T}}{\partial t} = \frac{\partial \mathbf{Y}}{\partial t}. \tag{3.2.16}$$

Differentiating (3.1.18) with respect to t and using (3.2.4) the following relations can easily be obtained

$$\begin{aligned} &\frac{\partial \epsilon_j}{\partial t} = 0,\ j = 1,2; \qquad \frac{\partial \epsilon_3}{\partial t} = \frac{\partial \bar{U}_3}{\partial x_3}, \\ &2\frac{\partial \epsilon_4}{\partial t} = \frac{\partial \bar{U}_2}{\partial x_3}, \qquad 2\frac{\partial \epsilon_5}{\partial t} = \frac{\partial \bar{U}_1}{\partial x_3}, \qquad 2\frac{\partial \epsilon_6}{\partial t} = 0. \end{aligned} \tag{3.2.17}$$

Using these formulas we get

$$-\frac{\partial \mathbf{Y}}{\partial t} = (\mathbf{A}_3^1)^* \frac{\partial \bar{\mathbf{U}}}{\partial x_3}, \tag{3.2.18}$$

where $\mathbf{A}_3^1$ is given in (3.2.10). Substitution of (3.2.18) into (3.2.16) results the expression

$$\mathbf{C}^{-1}\frac{\partial \mathbf{T}}{\partial t} + (\mathbf{A}_3^1)^* \frac{\partial \bar{\mathbf{U}}}{\partial x_3} = 0. \tag{3.2.19}$$

Let $\mathbf{V}$ and $\mathbf{F}$ be vectors with 9 components in the form

$$\mathbf{V} = \begin{bmatrix} \bar{\mathbf{U}} \\ \mathbf{T} \end{bmatrix}, \qquad \mathbf{F} = \begin{bmatrix} \mathbf{f} \\ \mathbf{0}_{6,1} \end{bmatrix}, \tag{3.2.20}$$

$\mathbf{A}_3$, be the 9×9 matrix

$$\mathbf{A}_0 = \begin{bmatrix} \rho \mathbf{I}_3 & \mathbf{0}_{3,6} \\ \mathbf{0}_{6,3} & \mathbf{C}^{-1} \end{bmatrix}, \quad \mathbf{A}_3 = \begin{bmatrix} \mathbf{0}_{3,3} & \mathbf{A}_3^1 \\ (\mathbf{A}_3^1)^* & \mathbf{0}_{6,6} \end{bmatrix}, \tag{3.2.21}$$

where $\mathbf{I}_m$ is the unit matrix of order $m \times m$ and $\mathbf{0}_{l,m}$ is the zero matrix of order $l \times m$. Since $\mathbf{C}$ is symmetric and positive definite matrix then $\mathbf{C}^{-1}$ is symmetric and positive definite matrix (see, appendix). Notice that the matrix $\mathbf{A}_3$ is also symmetric and positive definite.

Using these notations we can combine (3.2.12) and (3.2.19) to obtain a first-order system

$$\mathbf{A}_0\frac{\partial \mathbf{V}}{\partial t} + \mathbf{A}_3\frac{\partial \mathbf{V}}{\partial x_3} = \mathbf{F} \tag{3.2.22}$$

where $x_3 \in \mathbb{R}, \;\; t > 0$. We finish this section by the following lemma.

Lemma 3.2.1. *(see, (Courant & Hilbert, 1979), p.593-594), (see, (Yakhno & Akmaz, 2005) System (3.2.22) can be transformed into the following form*

$$\mathbf{I}_9\frac{\partial \tilde{\mathbf{V}}}{\partial t} + \tilde{\mathbf{A}}_3\frac{\partial \tilde{\mathbf{V}}}{\partial x_3} = \tilde{\mathbf{F}}, \tag{3.2.23}$$

which is an symmetric hyperbolic system.

Proof. Consider the symmetric positive-definite matrix $\mathbf{C}$. There exists a symmetric positive-definite matrix $\mathbf{M}$ such that $\mathbf{C}^{-1} = \mathbf{M}^2$ (see Theorem A.1.2 of Section A.1), and the matrix $\mathbf{M}^{-1}$, which is inverse of $\mathbf{M}$, is also symmetric (see Theorem A.1.1 of Section A.1). Using these facts we define the matrix

$$\mathbf{S} = \begin{bmatrix} \rho^{-\frac{1}{2}}\mathbf{I}_3 & \mathbf{0}_{3,6} \\ \mathbf{0}_{6,3} & \mathbf{M}^{-1} \end{bmatrix}, \tag{3.2.24}$$

and denote the vector $\mathbf{V}$ as

$$\mathbf{V} = \mathbf{S}\tilde{\mathbf{V}}. \tag{3.2.25}$$

Substituting (3.2.25) into (3.2.22) and multiplying the resulting formula with matrix $\mathbf{S}$ from left-hand side we obtain (3.2.23), where

$$\mathbf{S}\mathbf{A}_0\mathbf{S} = \mathbf{I}_9, \quad \tilde{\mathbf{A}}_3 = \mathbf{S}\mathbf{A}_3\mathbf{S}, \quad \tilde{\mathbf{F}} = \mathbf{S}\mathbf{F}. \tag{3.2.26}$$

Since $\mathbf{S}$ and $\mathbf{A}_3$, are symmetric, the matrices $\tilde{\mathbf{A}}_3$, is also symmetric, which implies that (3.2.23) is symmetric hyperbolic system. □

3.2.2 Existence and Uniqueness of a Classical Solution of IVP for System Depending on x_3, t Variables . Properties of Solutions.

In this section we prove the existence of unique classical solutions depending on x_3, t variables. Also a property for existence of unique solutions with finite support is proved when inhomogeneous term and initial data are infinitely differentiable and have finite support.

Theorem 3.2.2. *Let T be a fixed positive number, $\boldsymbol{G}(x_3)$; $\boldsymbol{H}(x_3)$ and $\boldsymbol{F}(x_3, t)$ be given functions such that $\boldsymbol{G}(x_3) \in H^3(\mathbb{R})$; $\boldsymbol{H}(x_3) \in H^4(\mathbb{R})$ and $\boldsymbol{F}(x_3, t) \in C([0, T]; H^3(\mathbb{R}))$. Then there exists a unique solution of Cauchy problem (3.1.15), (3.1.16)*

$$\mathbf{U}(x_3, t) \in C^1([0, T]; H^3(\mathbb{R})) \cap C^2([0, T]; H^2(\mathbb{R})).$$

Proof. For the case when all functions appearing in (3.1.15), (3.1.16) do not depend on x_1 and x_2 variables, the IVP can be written as an IVP for the following symmetric first order hyperbolic system

$$\frac{\partial \bar{\mathbf{V}}}{\partial t} + \widetilde{A}_3 \frac{\partial \bar{\mathbf{V}}}{\partial x_3} = \widetilde{\mathbf{F}}, \quad x_3 \in \mathbb{R}, \ \ t > 0, \tag{3.2.27}$$

$$\bar{\mathbf{V}}(x, 0) = \mathbf{V}_0(x_3), \tag{3.2.28}$$

where $\bar{\mathbf{V}}$, $\widetilde{\mathbf{F}}$, $\mathbf{V}_0(x_3)$ are defined by formulae

$$\bar{\mathbf{V}} = S \begin{bmatrix} \bar{\mathbf{U}} \\ \mathbf{T} \end{bmatrix}, \qquad \widetilde{\mathbf{F}} = S \begin{bmatrix} \mathbf{F} \\ \mathbf{0}_{6,1} \end{bmatrix}, \qquad \mathbf{V}_0 = S \begin{bmatrix} \bar{\mathbf{U}} \\ \mathbf{T} \end{bmatrix}_{t=0}, \tag{3.2.29}$$

where S is defined by the matrix (3.2.24) and $\mathbf{T}$, $\bar{\mathbf{U}}$, $\widetilde{A}_3$ defined by the equations (3.2.2), (3.2.4), (3.2.26). Using existence theorem (Mizohata, 1973)(p. 335) (also, appendix b) for the symmetric hyperbolic first order system (3.2.27), (3.2.28), it can be shown that there exists a unique solution of (3.1.15), (3.1.16) in the class

$$\mathbf{U}(x_3, t) \in C^1([0, T]; H^3(\mathbb{R})) \cap C^2([0, T]; H^2(\mathbb{R}))$$

for any given initial data $\mathbf{G}(x_3) \in H^3(\mathbb{R})$; $\mathbf{H}(x_3) \in H^4(\mathbb{R})$ and inhomogeneous term

$\mathbf{F}(x_3, t) \in C([0, T]; H^3(\mathbb{R}))$. $\square$

Theorem 3.2.3. *Let T be a fixed positive number, $\boldsymbol{G}(x_3)$; $\boldsymbol{H}(x_3)$ and $\boldsymbol{F}(x_3, t)$ be given functions such that $\boldsymbol{G}(x_3) \in C_0^\infty(\mathbb{R})$; $\boldsymbol{H}(x_3) \in C_0^\infty(\mathbb{R})$ and $\boldsymbol{F}(x_3, t) \in C([0, T]; C_0^\infty(\mathbb{R}))$. Then the solution $\mathbf{U}(x_3, t)$ of Cauchy problem (3.1.15), (3.1.16) belongs to*

$$C^2([0, T]; C_0^\infty(\mathbb{R})).$$

Proof. Using Theorem 3.2.2 it can be found that if $D^\alpha \psi \in H^3(\mathbb{R})$; $D^\alpha \mathbf{H}(x_3) \in H^4(\mathbb{R})$ and $D^\alpha \mathbf{F}(x_3, t) \in C([0, T]; H^3(\mathbb{R}))$, where T is a fixed positive number and $D^\alpha = \dfrac{\partial^{|\alpha|}}{\partial x_3^\alpha}$, $\alpha = 1, 2, ...$. Then $D^\alpha \mathbf{U}$ belongs to the class

$$D^\alpha \mathbf{U}(x_3, t) \in C^1([0, T]; H^3(\mathbb{R})) \cap C^2([0, T]; H^2(\mathbb{R})).$$

Thus, $\forall |\alpha| \leq l, \ l = 1, 2, ...$; $\mathbf{U}(x_3, t) \in C^1([0, T]; H^{3+l}(\mathbb{R})) \cap C^2([0, T]; H^{2+l}(\mathbb{R}))$.

Using this fact and applying Sobolev's lemma (See, appendixA.3), we get $\mathbf{U}(x_3, t) \in C^1([0, T]; C^{1+l}(\mathbb{R})) \cap C^2([0, T]; C^l(\mathbb{R}))$. For arbitrary l we have

$$\mathbf{U}(x_3, t) \in C^2([0, T]; C^\infty(\mathbb{R})).$$

To prove that the function has a compact support, let us consider reduction of the Cauchy problem (3.1.15), (3.1.16) to the first order symmetric hyperbolic system (3.2.27), (3.2.28) that is where 9×9 matrix $\widetilde{A}_3$ is a real, symmetric with constant elements. Let T be a fixed positive number, $\xi \in \mathbb{R}$ be a parameter; $A(\xi)$ be a matrix defined by $A(\xi) = \widetilde{A}_3 \xi$; $\lambda_i(\xi)$, $i = 1, 2, ..., 9$ be eigenvalues of $A(\xi)$. The positive number M is defined by

$$M = \max_{i=1,2,...,9} \max_{|\xi|=1} |\lambda_i(\xi)|. \tag{3.2.30}$$

We claim that M is the upper bound on the speed of waves in any direction.

Using T and M we define the following domains

$$S(x_0, h) = \{x \in \mathbb{R} : |x - x_0| \leq M(T - h)\},\ 0 \leq h \leq T$$
$$\Gamma(x_0, T) = \{(x, t) : 0 \leq t \leq T,\ |x - x_0| \leq M(T - t)\}$$
$$R(x_0, h) = \{(x, t) : 0 \leq t \leq h,\ |x - x_0| = M(T - t)\}$$

Here $\Gamma(x_0, T)$ is the conoid with vertex (x_0, T); $S(x_0, h)$ is the surface constructed by the intersection of the plane $t = h$ and the conoid $\Gamma(x_0, T)$; $R(x_0, h)$ is the lateral surface of the conoid $\Gamma(x_0, T)$ bounded by $S(x_0, 0)$ and $S(x_0, h)$. Let Ω be the region in $\mathbb{R} \times (0, \infty)$ bounded by $S(x_0, 0)$, $S(x_0, h)$ and $R(x_0, h)$ with boundary $\partial\Omega = S(x_0, 0) \cup S(x_0, h) \cup R(x_0, h)$.

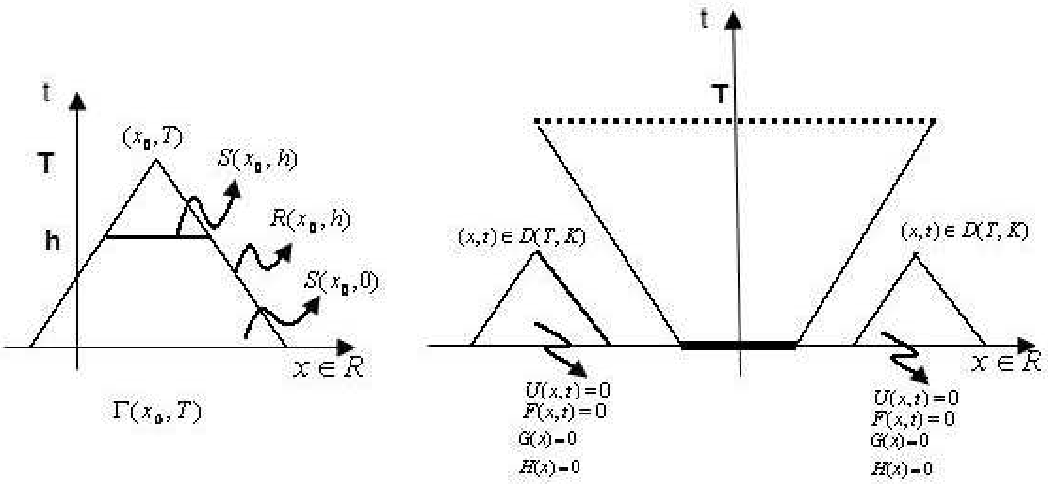


Figure 3.1 Domains of Dependence

Applying the reasoning similar to (Courant & Hilbert, 1979) (p. 652-661)(see also Appendix B) we find the following estimate for the solution of (3.2.27), (3.2.28)

$$\int_{S(h)} |\bar{\mathbf{V}}(x, h)|^2 dx \leq e^h \Big[\int_{S(0)} |\mathbf{V_0}(x)|^2 dx + \int_0^h \Big(\int_{S(t)} |\tilde{\mathbf{F}}(x, t)|^2 dx \Big) dt \Big]. \quad (3.2.31)$$

Let us define $P(K) = \{x_3 \in \mathbb{R} : |x_3| \leq K\}$. Since $\mathbf{G}(x_3) \in C_0^\infty(\mathbb{R})$; $\mathbf{H}(x_3) \in C_0^\infty(\mathbb{R})$; and $\mathbf{F}(x_3, t) \in C([0, T]; C_0^\infty(\mathbb{R}))$ then there exists $K > 0$ such that $supp\ \mathbf{G} \subseteq P(K)$, $supp\ \mathbf{H} \subseteq P(K)$ and $\mathbf{F}(x_3, t)$ as a function of the variable x, has a finite support which is located in $P(K)$ for any fixed t from $[0, T]$. Also let us denote

$$D(T,K) = \{(x_3,t) : 0 \le t \le T,\ \Gamma(x_3,t) \cap P(K) = \emptyset\}.$$

If $(x_3,t) \in D(T,K)$ then $\mathbf{U}(x_3,t) = 0$. This means $\mathbf{U}(x_3,t) = 0$ for any $t \in [0,T]$ and $|x_3| > MT + K$.

Hence, $supp\ \mathbf{U} \subseteq P(MT+K)$. As a result $\mathbf{U}(x_3,t)$ belongs to the class

$$\mathbf{U}(x_3,t) \in C^2([0,T]; C_0^\infty(\mathbb{R})).$$

□

3.2.3 IVP for the System Depending on x_3 and t Variables

In Section 3.2.1 we have shown that IVP of elastic system (3.1.15), (3.1.16) has a unique solution $\mathbf{U}(x_3,t)$ belongs to $C^2([0,T]; C_0^\infty(\mathbb{R}))$ when inhomogeneous term $\mathbf{F}(x_3,t) \in C([0,T]; C_0^\infty(\mathbb{R}))$ be given function depending x_3 and t variables and initial data $\mathbf{G}(x_3) \in C_0^\infty(\mathbb{R})$; $\mathbf{H}(x_3) \in C_0^\infty(\mathbb{R})$ are given functions depending x_3 variable only. In this case we can rewrite IVP of elastic system (3.1.15), (3.1.16)

$$\rho\frac{\partial^2 U_1}{\partial t^2} = c_{55}\frac{\partial^2 U_1}{\partial x_3^2} + c_{54}\frac{\partial^2 U_2}{\partial x_3^2} + c_{53}\frac{\partial^2 U_3}{\partial x_3^2} + F_1(x_3,t), \tag{3.2.32}$$

$$\rho\frac{\partial^2 U_2}{\partial t^2} = c_{45}\frac{\partial^2 U_1}{\partial x_3^2} + c_{44}\frac{\partial^2 U_2}{\partial x_3^2} + c_{43}\frac{\partial^2 U_3}{\partial x_3^2} + F_2(x_3,t), \tag{3.2.33}$$

$$\rho\frac{\partial^2 U_3}{\partial t^2} = c_{35}\frac{\partial^2 U_1}{\partial x_3^2} + c_{34}\frac{\partial^2 U_2}{\partial x_3^2} + c_{33}\frac{\partial^2 U_3}{\partial x_3^2} + F_3(x_3,t), \tag{3.2.34}$$

$$U_j(x_3,0) = G_j(x_3), \quad \left.\frac{\partial U_j(x_3,t)}{\partial t}\right|_{t=0} = H_j(x_3). \tag{3.2.35}$$

Simply, equations (3.2.32)-(3.2.35) can be written as follows

$$\rho\frac{\partial^2 \mathbf{U}}{\partial t^2} = L[\mathbf{U}] + \mathbf{F}(x_3,t), \quad t > 0, \tag{3.2.36}$$

$$\mathbf{U}(x_3,0) = \mathbf{G}(x_3), \quad \left.\frac{\partial \mathbf{U}}{\partial t}\right|_{t=0} = \mathbf{H}(x_3), \tag{3.2.37}$$

where $x_3 \in \mathbb{R},\ \ t > 0$ and $L[\mathbf{U}]$ is the matrix operator defined with components $L_{ij}U_j$ that is given with the formulas

$$\begin{aligned} &L_{11} = c_{55}\partial^2_{x_3}; \quad L_{12} = c_{54}\partial^2_{x_3}; \quad L_{13} = c_{53}\partial^2_{x_3} \\ &L_{21} = c_{45}\partial^2_{x_3}; \quad L_{22} = c_{44}\partial^2_{x_3}; \quad L_{23} = c_{43}\partial^2_{x_3}; \\ &L_{31} = c_{35}\partial^2_{x_3}; \quad L_{32} = c_{34}\partial^2_{x_3}; \quad L_{33} = c_{33}\partial^2_{x_3}. \end{aligned} \tag{3.2.38}$$

3.3 1-D Fundamental Solution of IVP for the System Depending on x_3 and t Variables

A matrix $\mathcal{U}(x_3, t) = [U_{rs}(x_3, t)]_{3\times 3}$ is called 1-D fundamental solution or IVP for the system depending on one space x_3 and the time variable t if $s-$th column

$$\mathbf{U}_s(x_3, t) = \begin{pmatrix} U_{1s}(x_3, t) \\ U_{2s}(x_3, t) \\ U_{3s}(x_3, t) \end{pmatrix}$$

satisfies

$$\rho \frac{\partial^2 \mathbf{U}_s}{\partial t^2} = L[\mathbf{U}_s], \tag{3.3.1}$$

$$\mathbf{U}_s(x_3, 0) = 0, \quad \frac{\partial \mathbf{U}_s}{\partial t} = \frac{1}{\rho}\vec{e}^s \delta(x_3), \tag{3.3.2}$$

where $x_3 \in \mathbb{R},\ \ t \in \mathbb{R}$ and $s = 1, 2, 3$; $\mathbf{e}^1 = (1, 0, 0)$, $\mathbf{e}^2 = (0, 1, 0)$, $\mathbf{e}^3 = (0, 0, 1)$; $U_{js}(x_3, t)$ are the components of the unknown displacement vector $\mathbf{U}_s(x_3, t)$ and $L[\mathbf{U}_s]$ is the matrix operator defined with components $L_{ij}U_{js}$ that is given with the formulas (3.2.38).

3.3.1 Some Properties of 1-D Fundamental Solution

Remark 3.3.1. Let $\mathbf{U}_s(x_3, t)$ be a fundamental solution then $\hat{\mathbf{U}}_s(x_3, t) = \theta(t)\mathbf{U}_s(x_3, t)$ satisfies

$$\rho \frac{\partial^2 \hat{\mathbf{U}}_s}{\partial t^2} = L[\hat{\mathbf{U}}_s] + \mathbf{e}^s \delta(x_3, t), \tag{3.3.3}$$

$$\hat{\mathbf{U}}_s(x_3, t)\Big|_{t<0} = 0, \tag{3.3.4}$$

where $x_3 \in \mathbb{R}$, $t \in \mathbb{R}$ and $s = 1,2,3$; $\mathbf{e}^1 = (1,0,0)$, $\mathbf{e}^2 = (0,1,0)$, $\mathbf{e}^3 = (0,0,1)$; $U_{js}(x_3,t)$ are the components of the unknown displacement vector $\mathbf{U}_s(x_3,t)$ and $L[\mathbf{U}_s]$ is the matrix operator defined with components $L_{ij}U_{js}$ that is given with the formulas (3.2.38).

Proof. Since $\hat{\mathbf{U}}(x_3,t) = \Theta(t)\mathbf{U}(x_3,t)$, derivative of $\hat{\mathbf{U}}(x_3,t)$ with respect to t is

$$\frac{\partial\Big(\hat{\mathbf{U}}(x_3,t)\Big)}{\partial t} = \delta(t)\mathbf{U}(x,0) + \Theta(t)\frac{\partial}{\partial t}\mathbf{U}(x_3,t),$$

$$\frac{\partial^2\Big(\hat{\mathbf{U}}(x_3,t)\Big)}{\partial t^2} = \frac{1}{\rho}\vec{e}^s\delta(t)\delta(x_3) + \Theta(t)\frac{\partial^2}{\partial t^2}\mathbf{U}(x_3,t),$$

and also we have

$$L[\hat{\mathbf{U}}] = \Theta(t)L[\mathbf{U}]$$

then

$$\rho\frac{\partial^2\hat{\mathbf{U}}}{\partial t^2} - L[\hat{\mathbf{U}}] = \rho\frac{1}{\rho}\vec{e}^s\delta(t)\delta(x_3) + \rho\Theta(t)\frac{\partial^2\mathbf{U}}{\partial t^2} - \Theta(t)L[\mathbf{U}],$$

$$\rho\frac{\partial^2\hat{\mathbf{U}}}{\partial t^2} - L[\hat{\mathbf{U}}] = \vec{e}^s\delta(x_3)\delta(t).$$

□

It is well known that (see, Hörmander-Lojasiewicz theorem in appendix) the arbitrary differential equation or system with constant coefficients has a fundamental solution of slow growth. Thus, system with constant coefficients given with equations (3.3.1), (3.3.2) has a fundamental solution

$$\mathbf{U}_s(x_3,t) \in C^2([0,T];\mathcal{S}'(\mathbb{R})).$$

Our aim is to study some of the properties of this fundamental solution and suggest a method to find fundamental solutions.

Let us denote convolution of functions $\mathbf{U}_s(x_3,t)$, with cap-shaped function $w_\varepsilon(x_3)$ is $\mathbf{u}_s^\varepsilon(x_3,t) = (u_{1s}, u_{2s}, u_{3s})$. Taking convolution with cap-shaped function, the problem (3.3.1)-(3.3.2) can be written as

$$\rho \frac{\partial^2 \mathbf{u}_s^\varepsilon}{\partial t^2} = L[\mathbf{u}_s^\varepsilon], \quad x_3 \in \mathbb{R}, \ \ t \in \mathbb{R}, \tag{3.3.5}$$

$$\mathbf{u}_s^\varepsilon(x_3, 0) = 0, \quad \frac{\partial \mathbf{u}_s^\varepsilon}{\partial t} = \frac{1}{\rho}\vec{e}^s w_\varepsilon(x_3). \tag{3.3.6}$$

Using Theorem 3.2.3 of Section 3.2.2, it can be proved that problem (3.3.5), (3.3.6) has a unique solution $\mathbf{u}_s^\varepsilon(x_3,t) \in C^2([0,T]; C_0^\infty(\mathbb{R}))$ where
$supp\ \mathbf{u}_s^\varepsilon(x_3,t) \subseteq P(MT+\varepsilon_0);\ \forall \varepsilon \in (0,\varepsilon_0)$.

Property 1. *As $\varepsilon \to +0$, $\boldsymbol{u}_s^\varepsilon(x_3,t)$ approaches to $\boldsymbol{U}_s(x_3,t)$ in $\mathcal{S}'(\mathbb{R})$;* $\ \forall t \in [0,T]$.

Proof. It can be proved that as $\varepsilon \to +0$, $w_\varepsilon(x)$ approaches to $\delta(x)$ in $\mathcal{S}'(\mathbb{R})$. Using this fact and using the continuity of the convolution $\mathbf{u}(x,t) * w_\varepsilon(x)$ with respect to $w_\varepsilon(x)$ theorem is proved. □

Property 2. *Let T be a fixed positive number. There exists a unique solution of Cauchy problem (3.3.1)-(3.3.2)*

$$\mathbf{U}_s(x_3,t) \in C^2([0,T]; \mathcal{E}'(\mathbb{R})).$$

Proof. We need to show that

$$(\mathbf{U}_s, \varphi) = 0;\ \ \forall \varphi \in \mathcal{S} \quad \text{and } supp\ \varphi \subseteq \mathbb{R} \setminus P(MT+\varepsilon_0).$$

From property 1, we know that

$$(\mathbf{U}_s, \varphi) = \lim_{\varepsilon \to +0} (\mathbf{u}_s^\varepsilon, \varphi);\ \ \forall \varphi \in \mathcal{S}$$

In the beginning of this section we have shown that $\mathbf{u}_s^\varepsilon(x_3,t) \in C^2([0,T]; C_0^\infty(\mathbb{R}))$ where $supp\ \mathbf{u}_s^\varepsilon(x_3,t) \subseteq P(MT+\varepsilon_0);\ \forall \varepsilon \in (0,\varepsilon_0)$. Thus,

$$\lim_{\varepsilon \to +0} (\mathbf{u}_s^\varepsilon, \varphi) = 0;\ \ \forall \varphi \in \mathcal{S}.$$

This means $supp\ \mathbf{U}_s \subseteq P(MT+\varepsilon_0);\ \forall\varepsilon \in (0,\varepsilon_0)$. So we prove that $\mathbf{U}_s(x_3,t)$ is a tempered distribution with compact support that is unique solution of the Cauchy problem i.e.

$$\mathbf{U}_s(x,t) \in C^2([0,T];\mathcal{E}'(\mathbb{R})).$$

□

Property 3. *Let $\boldsymbol{U}_s(x_3,t)$ be solution of the problem (3.3.1), (3.3.2) and $\tilde{\boldsymbol{U}}_s(\upsilon,t) = \left(\tilde{U}_{1s}(\upsilon,t),\tilde{U}_{2s}(\upsilon,t),\tilde{U}_{3s}(\upsilon,t)\right)$ be the Fourier transform image of $\boldsymbol{U}_s(x,t)$ with respect to $x_3 \in \mathbb{R}$. Then the Fourier image $\tilde{\boldsymbol{U}}_s(\upsilon,t)$ is an entire analytic function and satisfies following system of equations:*

$$\rho\frac{\partial^2\tilde{U}_{1s}}{\partial t^2} + c_{55}\upsilon^2\tilde{U}_{1s} + c_{54}\upsilon^2\tilde{U}_{2s} + c_{53}\upsilon^2\tilde{U}_{3s}, \tag{3.3.7}$$

$$\rho\frac{\partial^2\tilde{U}_{2s}}{\partial t^2} + c_{45}\upsilon^2\tilde{U}_{1s} + c_{44}\upsilon^2\tilde{U}_{2s} + c_{43}\upsilon^2\tilde{U}_{3s}, \tag{3.3.8}$$

$$\rho\frac{\partial^2\tilde{U}_{3s}}{\partial t^2} + c_{35}\upsilon^2\tilde{U}_{1s} + c_{34}\upsilon^2\tilde{U}_{2s} + c_{33}\upsilon^2\tilde{U}_{3s}, \tag{3.3.9}$$

$$\tilde{U}_{js}(x_3,0) = 0, \quad \left.\frac{\partial\tilde{U}_{js}(x_3,t)}{\partial t}\right|_{t=0} = \tilde{h}_{js}(x_3), \tag{3.3.10}$$

where $\tilde{h}_{js} = \left(\frac{1}{\rho}\boldsymbol{e}^s\right)_j$.

Proof. Let $\tilde{\mathbf{U}}_s(\upsilon,t)$ be the Fourier transform image of $\mathbf{U}_s(x,t)$ with respect to $x_3 \in \mathbb{R}$, i.e.

$$\tilde{\mathbf{U}}_s(\upsilon,t) = \left(\tilde{U}_{1s}(\upsilon,t),\tilde{U}_{2s}(\upsilon,t),\tilde{U}_{3s}(\upsilon,t)\right)$$

$$\tilde{U}_{ls}(\upsilon,t) = \boldsymbol{\mathcal{F}}_x[U_{ls}];\quad l=1,2,3;\upsilon\in\mathbb{R};$$

where the Fourier operator $\boldsymbol{\mathcal{F}}_x$ is defined by

$$\boldsymbol{\mathcal{F}}_x[U_{ls}] = \int_{-\infty}^{\infty} U_{ls}(x_3,t)e^{i\upsilon x_3}dx_3;\ i^2=-1.$$

Since $\mathbf{U}_s(x,t) \in C^2([0,T];\mathcal{E}'(\mathbb{R}))$, according to Paley-Wiener theorem (Reed & Simon, 1975), Fourier transform of the function $\mathbf{U}_s(x_3,t)$ is an entire analytic function with respect to $\upsilon \in \mathbb{R}$, and can be written as a power series

$$\tilde{\mathbf{U}}_s(\upsilon, t) = \sum_{k=0}^{\infty} \tilde{\mathbf{U}}_s^k(t)\upsilon^k.$$

If we apply Fourier transform with respect to space variable problem (3.3.1), (3.3.2) can be written in terms of Fourier images given with the equations (3.3.7)-(3.3.10). □

3.3.2 Derivation of 1-D Fundamental Solution

Problem in Terms of Coefficients of the Series Expansion

Using property 3, power series expansion of $\tilde{\mathbf{U}}_s(\upsilon, t)$, $\tilde{\mathbf{h}}(\upsilon)$ can be considered i.e.

$$\tilde{\mathbf{U}}_s(\upsilon, t) = \sum_{k=0}^{\infty} \tilde{\mathbf{U}}_s^k(t)\upsilon^k, \tag{3.3.11}$$

$$\tilde{\mathbf{h}}_s(\upsilon) = \sum_{k=0}^{\infty} \tilde{\mathbf{h}}_s^k \upsilon^k, \tag{3.3.12}$$

where $\tilde{\mathbf{h}}_s^k$ are given real numbers; $\tilde{\mathbf{U}}_s^k(t)$ are unknown coefficients we need to find.

Substituting (3.3.11)-(3.3.12) into (3.3.7)-(3.3.10) we obtain

$$\rho \frac{\partial^2 \tilde{U}_{1s}^k}{\partial t^2} + c_{55}\tilde{U}_{1s}^{k-2} + c_{54}\tilde{U}_{2s}^{k-2} + c_{53}\tilde{U}_{3s}^{k-2} = 0, \tag{3.3.13}$$

$$\rho \frac{\partial^2 \tilde{U}_{2s}^k}{\partial t^2} + c_{45}\tilde{U}_{1s}^{k-2} + c_{44}\tilde{U}_{2s}^{k-2} + c_{43}\tilde{U}_{3s}^{k-2} = 0, \tag{3.3.14}$$

$$\rho \frac{\partial^2 \tilde{U}_{3s}^k}{\partial t^2} + c_{35}\tilde{U}_{1s}^{k-2} + c_{34}\tilde{U}_{2s}^{k-2} + c_{33}\tilde{U}_{3s}^{k-2} = 0, \tag{3.3.15}$$

$$\tilde{U}_{js}^k(x_3, 0) = 0, \quad \left. \frac{\partial \tilde{U}_{js}^k(x_3, t)}{\partial t} \right|_{t=0} = \tilde{h}_{js}^k, \tag{3.3.16}$$

where h_{js}^k are given real numbers such that $h_{js}^0 = 1$ and $h_{js}^l = \frac{1}{\rho}\mathbf{e}^s$ for $l = 1, 2, ...$

Equations (3.3.13)-(3.3.16) can be written equivalently as the following recurrence relations:

$$\frac{\partial^2 \tilde{U}_{js}^k}{\partial t^2} = -\frac{1}{\rho}\Upsilon_j^k,\ t > 0,\ j = 1, 2, 3, \tag{3.3.17}$$

$$\tilde{U}_{js}^k(0) = 0, \quad \left.\frac{\partial \tilde{U}_{js}^k(t)}{\partial t}\right|_{t=0} = \tilde{h}_{js}^k,\ j = 1, 2, 3. \tag{3.3.18}$$

where

$$\Upsilon_1^k = c_{55}\tilde{U}_{1s}^{k-2} + c_{54}\tilde{U}_{2s}^{k-2} + c_{53}\tilde{U}_{3s}^{k-2}, \tag{3.3.19}$$

$$\Upsilon_2^k = c_{45}\tilde{U}_{1s}^{k-2} + c_{44}\tilde{U}_{2s}^{k-2} + c_{43}\tilde{U}_{3s}^{k-2}, \tag{3.3.20}$$

$$\Upsilon_3^k = c_{35}\tilde{U}_{1s}^{k-2} + c_{34}\tilde{U}_{2s}^{k-2} + c_{33}\tilde{U}_{3s}^{k-2}. \tag{3.3.21}$$

The solutions of the problems (3.3.17), (3.3.18) for $j = 1, 2, 3$ will be

$$\tilde{U}_{js}^k(t) = \int_0^t (t-\tau)\Upsilon_j(\tau)d\tau + \tilde{h}_{js}^k t, \quad j = 1, 2, 3. \tag{3.3.22}$$

Using (3.3.22) all coefficients $\tilde{\mathbf{U}}_s^k$ of $\tilde{\mathbf{U}}_s$ can be found. Solution of the IVP (3.3.7)-(3.3.10) can be obtained as follows

$$\tilde{U}_{js}(\upsilon, t) = \sum_{k=0}^{\infty} \tilde{U}_{js}^k(t)\upsilon^k \tag{3.3.23}$$

where $\tilde{U}_{js}^k(t), \quad j = 1, 2, 3$ are defined in equations (3.3.22).

Procedure of Finding $\mathbf{U}_s^k$

The procedure of finding $\tilde{\mathbf{U}}_s^k;\ s = 1, 2, 3,$ consists of the sequence of the following iterative steps of constructing some formulae from the others using the relation (3.3.22).

Step 1: $\tilde{\mathbf{U}}_s^{-2} = \tilde{\mathbf{U}}_s^{-1} = 0,$

Step 2: using zero values from step 1 we compute $\tilde{\mathbf{U}}_s^0,$

Step 3: from the relations obtained on previous steps we compute $\tilde{\mathbf{U}}_s^1$,

... ...

Step p**:** from the relations obtained on previous steps we compute $\tilde{\mathbf{U}}_s^p$.

Inverse Fourier Transform and the Solutions of Original IVP

Applying inverse Fourier transform to $\tilde{U}_{js}(\upsilon, t)$ defined by the formula (3.3.23) solution $\mathbf{U}_s(x, t)$; $s = 1, 2, 3$ of problem (3.3.1), (3.3.2) can be obtained for 1-D Case.

3.3.3 Simulation of 1-D Fundamental Solution

In this section we consider problem (3.3.1), (3.3.2) for two different type of anisotropy: hexagonal and monoclinic type of anisotropies (see, Section 3.1). The aim is to create simulations of elastic wave propagations, by the method described in Section 3.3.2.

The hexagonal crystal with density $(\mathrm{gr/cm^3})$, and elastic moduli $(10^{12}\ \mathrm{dyn/cm^2})$ is as follows:
Zinc (Hexagonal): $\rho = 7.134,\ c_{11} = 1.6368,\ c_{12} = 0.3640,\ c_{13} = 0.53,\ c_{33} = 0.6347,\ c_{55} = 0.3879,\ c_{22} = c_{11},\ c_{23} = c_{13},\ c_{44} = c_{55},\ c_{66} = (c_{11} - c_{12})/2.$

The monoclinic crystal with density $(\mathrm{gr/cm^3})$, and elastic moduli $(10^{12}\ \mathrm{dyn/cm^2})$ is as follows:
(Monoclinic): $\rho = 2.649,\ c_{11} = 8.67,\ c_{12} = -0.83,\ c_{13} = 2.71,\ c_{14} = -0.37,\ c_{15} = 0,\ c_{16} = 0,\ c_{21} = -0.83,\ c_{22} = 12.98,\ c_{23} = -0.74,\ c_{24} = 0.57,\ c_{25} = 0,\ c_{26} = 0,\ c_{31} = 2.71,\ c_{32} = -0.74,\ c_{33} = 10.28,\ c_{34} = 0.99,\ c_{35} = 0,\ c_{36} = 0,\ c_{41} = -0.37,\ c_{42} = 0.57,\ c_{43} = 0.99,\ c_{44} = 3.86,\ c_{45} = 0,\ c_{46} = 0,\ c_{51} = 0,\ c_{52} = 0,\ c_{53} = 0,\ c_{54} = 0,\ c_{55} = 6.88,\ c_{56} = 0.25,\ c_{61} = 0,\ c_{62} = 0,\ c_{63} = 0,\ c_{64} = 0,\ c_{65} = 0.25,\ c_{66} = 2.9.$

Using the method of Section 3.3.2, we compute the elements of the fundamental solution matrix $\mathcal{U}(x_3, t)$ whose $s-$th column is $\mathbf{U}_s(x_3, t)$ with the components $\mathbf{U}_s(x_3, t) = (U_{1s}(x_3, t), U_{2s}(x_3, t), U_{3s}(x_3, t))^T$. For simplicity let us consider $s = 1$ for given problem.

Notice that for hexagonal type of anisotropy and $s = 1$; problem given in equations (3.3.1), (3.3.2) becomes

$$\rho \frac{\partial^2 U_{11}}{\partial t^2} = c_{55} \frac{\partial^2 U_{11}}{\partial x_3^2}, \tag{3.3.24}$$

$$\rho \frac{\partial^2 U_{21}}{\partial t^2} = c_{44} \frac{\partial^2 U_{21}}{\partial x_3^2}, \tag{3.3.25}$$

$$\rho \frac{\partial^2 U_{31}}{\partial t^2} = c_{33} \frac{\partial^2 U_{31}}{\partial x_3^2}, \tag{3.3.26}$$

$$U_{j1}(x_3, 0) = 0, \quad \left. \frac{\partial U_{j1}(x_3, t)}{\partial t} \right|_{t=0} = \frac{1}{\rho} \vec{e}^1 \delta(x_3); \; j = 1, 2, 3. \tag{3.3.27}$$

If we apply Fourier transform with respect to space variable to problem (3.3.24)-(3.3.27) then (3.3.24)-(3.3.27) can be written in terms of Fourier images as follows:

$$\rho \frac{\partial^2 \tilde{U}_{11}}{\partial t^2} + c_{55} \upsilon^2 \tilde{U}_{11} = 0, \tag{3.3.28}$$

$$\rho \frac{\partial^2 \tilde{U}_{21}}{\partial t^2} + c_{44} \upsilon^2 \tilde{U}_{21} = 0, \tag{3.3.29}$$

$$\rho \frac{\partial^2 \tilde{U}_{31}}{\partial t^2} + c_{33} \upsilon^2 \tilde{U}_{31} = 0, \tag{3.3.30}$$

$$\tilde{U}_{j1}(\upsilon, 0) = 0, \quad \left. \frac{\partial \tilde{U}_{j1}(\upsilon, t)}{\partial t} \right|_{t=0} = \frac{1}{\rho} \vec{e}^1; \; j = 1, 2, 3. \tag{3.3.31}$$

Solution of Ordinary Differential Equations (ODE) (3.3.28)-(3.3.31) is given by the formula:

$$\tilde{U}_{11}(\upsilon, t) = \frac{\Theta(t)}{\sqrt{\frac{c_{55}}{\rho}} \upsilon} \sin \left(\sqrt{\frac{c_{55}}{\rho}} \upsilon t \right); \;\; \tilde{U}_{21}(\upsilon, t) = 0; \;\; \tilde{U}_{31}(\upsilon, t) = 0 \tag{3.3.32}$$

Table 3.2 shows the comparison of $\tilde{U}_{11}$ and $\tilde{U}_{11}^p$ where $\tilde{U}_{11}$ is the solution of ODE given in equation (3.3.32) and $\tilde{U}_{11}^p$ is the solution of the same problem (3.3.28)-(3.3.31) solved by using method we suggested in Section 3.3.2.

Table 3.2 Values of $\tilde{U}_{11}$ and $\tilde{U}_{11}^p$

p	t	v	$\tilde{U}_{11}$	$\tilde{U}_{11}^p$	Error
50	1	8	0.07190	0.07190	$0.1 * 10^{-10}$
50	5	8	0.00731	0.00731	$0.59 * 10^{-10}$
70	10	8	-0.01456	-0.01456	$0.27 * 10^{-9}$
70	10	9	0.05637	0.05637	$0.11 * 10^{-9}$

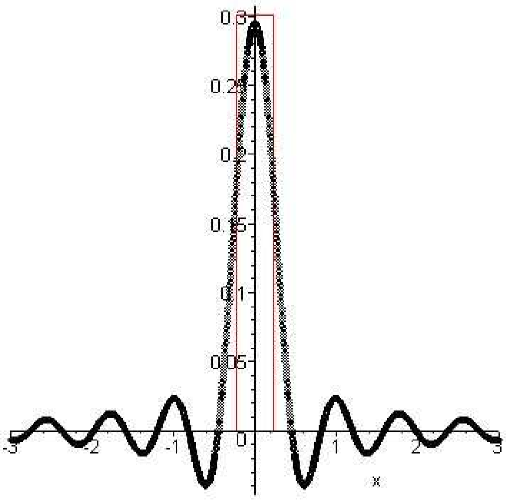
As another example we consider (3.3.1), (3.3.2) when $s = 1$ ($\mathbf{e}^s = (1, 0, 0)$) for monoclinic type of anisotropy. This problem is complicated to find a solution directly. By the method we suggested, we obtain the solutions and these fundamental solutions can be simulated as shown in figures3.3.

Analysis of figures

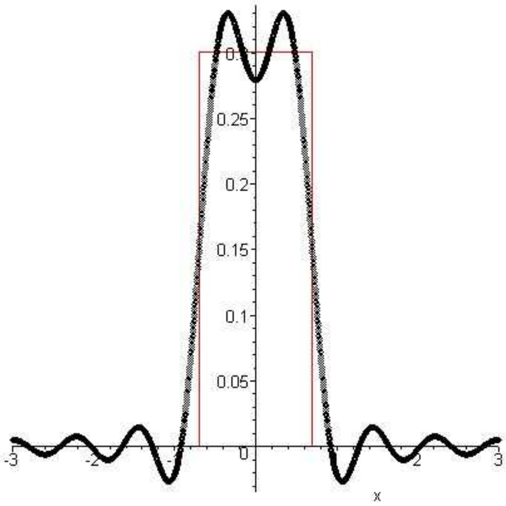
In these figures the horizontal axis is x_3, the vertical axis is density plots of first component of the first column of fundamental solution matrix $\mathcal{U}(x_3, t)$ that is $\mathbf{U}_{11}(x, t)$. This component is presented for varying values of t. In figure3.2, we consider (3.3.1), (3.3.2) for $s = 1$ ($\mathbf{e}^s = (1, 0, 0)$) and we draw the graph of the first component of $\mathbf{U}_s(x_3, t)$ for hexagonal type of anisotropy. In figure3.3 we consider same problem for monoclinic type of anisotropy. Analyzing figures we can see arising of the elastic waves. Wave propagate according to time. Notice that for hexagonal type of anisotropy, problem is given by (3.3.24), (3.3.27). This problem is an IVP for wave equation that has the exact formula:

$$U_{11}(x_3, t) = \frac{\Theta(t)}{2\rho}\sqrt{\frac{c_{55}}{\rho}}\Theta\Big((\sqrt{\frac{c_{55}}{\rho}}t)^2 - x_3^2\Big); \;\; U_{21}(x_3, t) = 0; \;\; U_{31}(x_3, t) = 0. \tag{3.3.33}$$

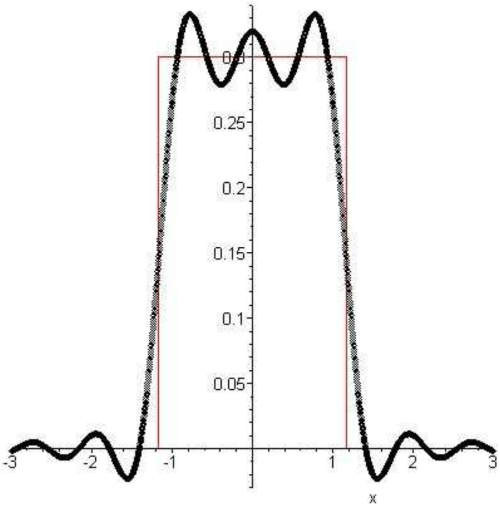
In figure 3.2, the difference between two results can be seen. For monoclinic type of anisotropy, it is complicated to find a solution directly. So we can not do such a comparison for IVP (3.3.1), (3.3.2) in monoclinic case.



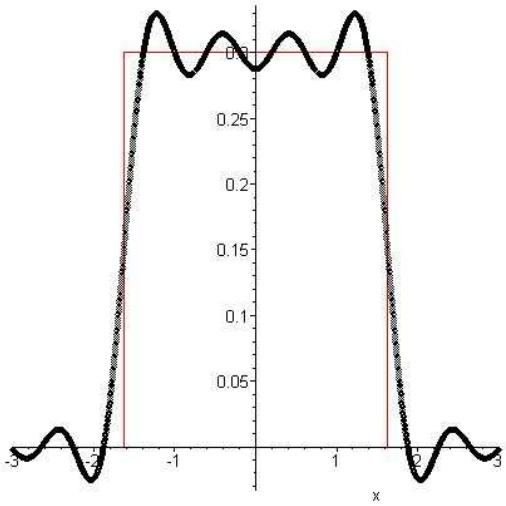
(a) 2-D plot of $U_{11}(x_3, 1)$



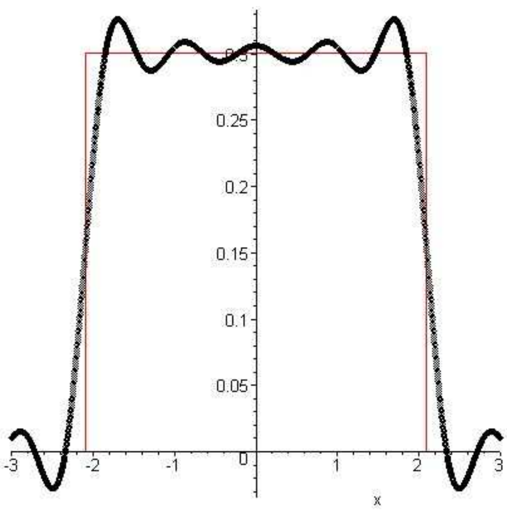
(b) 2-D plot of $U_{11}(x_3, 3)$



(c) 2-D plot of $U_{11}(x_3, 5)$

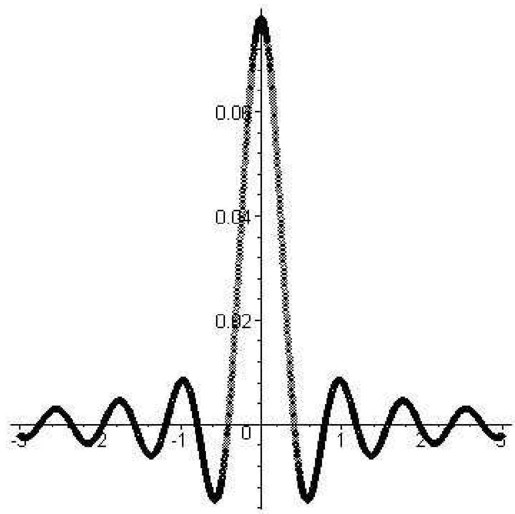


(d) 2-D plot of $U_{11}(x_3, 7)$

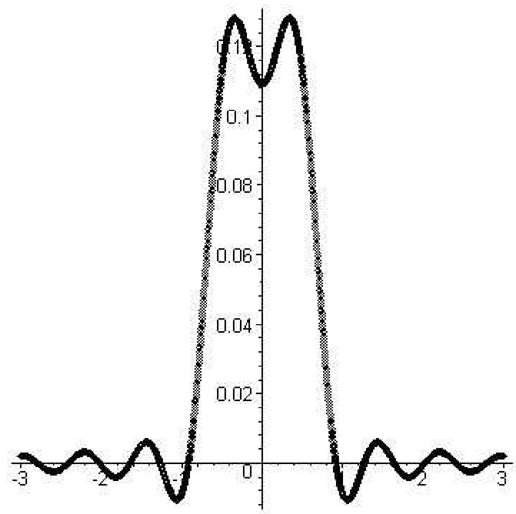


(e) 2-D plot of $U_{11}(x_3, 9)$

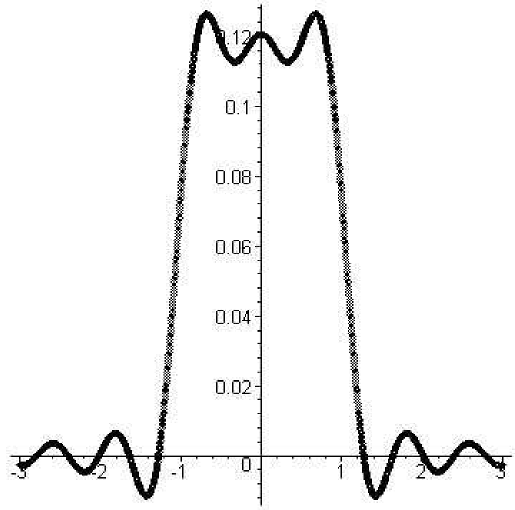
Figure 3.2 2-D plots of U_{11} hexagonal crystal.



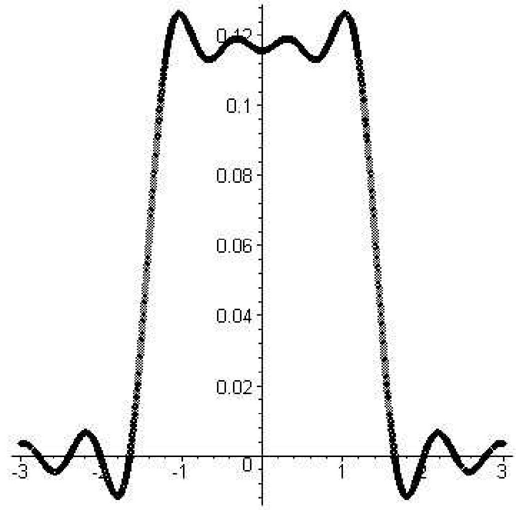
(a) 2-D plot of $U_{11}(x_3, 87/1000)$



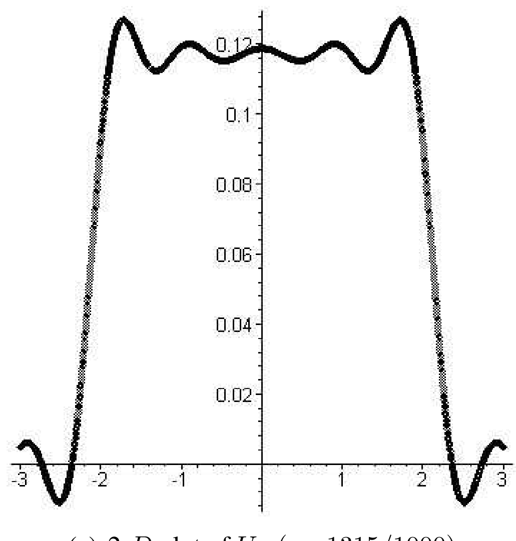
(b) 2-D plot of $U_{11}(x_3, 43/100)$



(c) 2-D plot of $U_{11}(x_3, 65/100)$



(d) 2-D plot of $U_{11}(x_3, 877/1000)$



(e) 2-D plot of $U_{11}(x_3, 1315/1000)$

Figure 3.3 2-D level plots of U_{11} monoclinic media.

3.4 IVP for the System of Elasticity Depending on x_2, x_3 and t Variables

3.4.1 Reduction of System Depending on x_2, x_3 and t Variables to a First-Order Symmetric Hyperbolic System

In this section we explain the process of writing (3.1.15), (3.1.16) as a symmetric hyperbolic system when displacement vector depends on x_2, x_3 and t variables.

Let us consider IVP of elastic system when inhomogeneous term $\mathbf{F}(x_2, x_3, t)$ be given function depending x_2, x_3 and t variables and initial data $\mathbf{G}(x_2, x_3)$; $\mathbf{H}(x_2, x_3)$ be given functions depending x_2, x_3 variable. Thus, solution $\mathbf{U}$ of (3.1.15), (3.1.16) depends on x_2, x_3 and t variables.

Using denotations and renumarations defined in (3.2.1)-(3.2.5), the left-hand side of (3.1.15) can be written in vector form

$$\rho \frac{\partial^2 \mathbf{U}}{\partial t^2} = \rho \frac{\partial \bar{\mathbf{U}}}{\partial t} \tag{3.4.1}$$

Consider the term $\sum_{k=1}^{3} \frac{\partial \sigma_{jk}}{\partial x_k}$ on the right-hand side of (3.1.15). Applying rule (3.2.1) for $j = 1, 2, 3$ gives

$$\sum_{k=1}^{3} \frac{\partial \sigma_{1k}}{\partial x_k} = \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} = \frac{\partial \sigma_1}{\partial x_1} + \frac{\partial \sigma_6}{\partial x_2} + \frac{\partial \sigma_5}{\partial x_3}, \tag{3.4.2}$$

$$\sum_{k=1}^{3} \frac{\partial \sigma_{2k}}{\partial x_k} = \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{23}}{\partial x_3} = \frac{\partial \sigma_6}{\partial x_1} + \frac{\partial \sigma_2}{\partial x_2} + \frac{\partial \sigma_4}{\partial x_3}, \tag{3.4.3}$$

$$\sum_{k=1}^{3} \frac{\partial \sigma_{3k}}{\partial x_k} = \frac{\partial \sigma_{32}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} = \frac{\partial \sigma_5}{\partial x_1} + \frac{\partial \sigma_4}{\partial x_2} + \frac{\partial \sigma_3}{\partial x_3}. \tag{3.4.4}$$

Using the vector $\mathbf{T}$, (3.4.2)-(3.4.4) takes the form

$$\begin{aligned}
\sum_{k=1}^{3}\frac{\partial\sigma_{1k}}{\partial x_k} &= [0,0,0,0,0,1]\cdot\frac{\partial\mathbf{T}}{\partial x_2}+[0,0,0,0,1,0]\cdot\frac{\partial\mathbf{T}}{\partial x_3},\\
\sum_{k=1}^{3}\frac{\partial\sigma_{2k}}{\partial x_k} &= [0,1,0,0,0,0]\cdot\frac{\partial\mathbf{T}}{\partial x_2}+[0,0,0,1,0,0]\cdot\frac{\partial\mathbf{T}}{\partial x_3},\\
\sum_{k=1}^{3}\frac{\partial\sigma_{3k}}{\partial x_k} &= [0,0,0,1,0,0]\cdot\frac{\partial\mathbf{T}}{\partial x_2}+[0,0,1,0,0,0]\cdot\frac{\partial\mathbf{T}}{\partial x_3}.
\end{aligned}$$

Noting the coefficient vectors of terms $\dfrac{\partial\mathbf{T}}{\partial x_k}$, $k=2,3$ we introduce the matrices

$$\mathbf{A}_2^1=\begin{bmatrix}0&0&0&0&0&-1\\0&-1&0&0&0&0\\0&0&0&-1&0&0\end{bmatrix},\qquad \mathbf{A}_3^1=\begin{bmatrix}0&0&0&0&-1&0\\0&0&0&-1&0&0\\0&0&-1&0&0&0\end{bmatrix},\tag{3.4.5}$$

and represent $\displaystyle\sum_{k=1}^{3}\frac{\partial\sigma_{jk}}{\partial x_k}$, $j=1,2,3$ in the form

$$\begin{bmatrix}\sum_{k=1}^{3}\frac{\partial\sigma_{1k}}{\partial x_k}\\ \sum_{k=1}^{3}\frac{\partial\sigma_{2k}}{\partial x_k}\\ \sum_{k=1}^{3}\frac{\partial\sigma_{3k}}{\partial x_k}\end{bmatrix}=-\sum_{k=2}^{3}\mathbf{A}_k^1\frac{\partial\mathbf{T}}{\partial x_k}.\tag{3.4.6}$$

From (3.4.1) and (3.4.6) it follows that we can rewrite (3.1.15) as

$$\rho\frac{\partial\bar{\mathbf{U}}}{\partial t}+\sum_{k=2}^{3}\mathbf{A}_k^1\frac{\partial\mathbf{T}}{\partial x_k}=\mathbf{f},\tag{3.4.7}$$

where $\mathbf{f}=(f_1,f_2,f_3)$.

Relation (3.1.17) can be written as two summations

$$\sigma_{jk}=\sum_{\substack{l,m=1\\ l=m}}^{3}c_{jklm}\epsilon_{lm}+\sum_{\substack{l,m=1\\ l\neq m}}^{3}c_{jklm}\epsilon_{lm},\quad j,k=1,2,3.\tag{3.4.8}$$

Denoting the pair of indices (j,k) with α, $\alpha = 1,\ldots,6$, (l,m) with β, $\beta = 1,\ldots,6$, according to rule (3.2.1), relation (3.4.8) can be written as

$$\sigma_\alpha = \sum_{\beta=1}^{3} c_{\alpha\beta}\epsilon_\beta + 2\sum_{\beta=4}^{6} c_{\alpha\beta}\epsilon_\beta, \tag{3.4.9}$$

or in terms of vectors $\mathbf{T}$ and $\mathbf{Y}$ as

$$\mathbf{T} = \mathbf{C}\mathbf{Y}, \tag{3.4.10}$$

where $\mathbf{C} = (c_{\alpha\beta})_{6\times 6}$ is stiffness matrix defined with (3.1.13) that is symmetric and positive definite (See, Section 3.1).

Taking derivative of (3.4.10) with respect to t and multiplying both sides by the inverse of $\mathbf{C}$, denoted $\mathbf{C}^{-1}$, we find

$$\mathbf{C}^{-1}\frac{\partial \mathbf{T}}{\partial t} = \frac{\partial \mathbf{Y}}{\partial t}. \tag{3.4.11}$$

Differentiating (3.1.18) with respect to t and using (3.2.4) the following relations can easily be obtained

$$\begin{aligned}
&\frac{\partial \epsilon_1}{\partial t} = 0, \qquad \frac{\partial \epsilon_j}{\partial t} = \frac{\partial \bar{U}_j}{\partial x_j}, \qquad j = 2,3; \qquad 2\frac{\partial \epsilon_4}{\partial t} = \frac{\partial \bar{U}_2}{\partial x_3} + \frac{\partial \bar{U}_3}{\partial x_2},\\
&2\frac{\partial \epsilon_5}{\partial t} = \frac{\partial \bar{U}_1}{\partial x_3}, \qquad\qquad 2\frac{\partial \epsilon_6}{\partial t} = \frac{\partial \bar{U}_1}{\partial x_2}.
\end{aligned} \tag{3.4.12}$$

Using these formulas we get

$$-\frac{\partial \mathbf{Y}}{\partial t} = \sum_{j=2}^{3} (\mathbf{A}_j^1)^* \frac{\partial \bar{\mathbf{U}}}{\partial x_j}, \tag{3.4.13}$$

where $\mathbf{A}_j^1$ are given in (3.4.5). Substitution of (3.4.13) into (3.4.11) results the expression

$$\mathbf{C}^{-1}\frac{\partial \mathbf{T}}{\partial t} + \sum_{j=2}^{3} (\mathbf{A}_j^1)^* \frac{\partial \bar{\mathbf{U}}}{\partial x_j} = 0. \tag{3.4.14}$$

Let $\mathbf{V}$ and $\mathbf{F}$ be vectors with 9 components in the form

$$\mathbf{V} = \begin{bmatrix} \bar{\mathbf{U}} \\ \mathbf{T} \end{bmatrix}, \qquad \mathbf{F} = \begin{bmatrix} \mathbf{f} \\ \mathbf{0}_{6,1} \end{bmatrix}, \tag{3.4.15}$$

$\mathbf{A}_j$, $j = 0, 2, 3$ be the 9×9 matrices

$$\mathbf{A}_0 = \begin{bmatrix} \rho\mathbf{I}_3 & \mathbf{0}_{3,6} \\ \mathbf{0}_{6,3} & \mathbf{C}^{-1} \end{bmatrix}, \quad \mathbf{A}_j = \begin{bmatrix} \mathbf{0}_{3,3} & \mathbf{A}_j^1 \\ (\mathbf{A}_j^1)^* & \mathbf{0}_{6,6} \end{bmatrix}, \qquad j = 1, 2, 3, \tag{3.4.16}$$

where $\mathbf{I}_m$ is the unit matrix of order $m \times m$ and $\mathbf{0}_{l,m}$ is the zero matrix of order $l \times m$. Since $\mathbf{C}$ is symmetric and positive definite matrix then $\mathbf{C}^{-1}$ is symmetric and positive definite matrix (see, appendix). Notice that the matrices $\mathbf{A}_j,\ j = 0, 2, 3$ are also symmetric and positive definite.

Using these notations we can combine (3.4.7) and (3.4.14) to obtain a first-order system

$$\mathbf{A}_0 \frac{\partial \mathbf{V}}{\partial t} + \sum_{j=2}^{3} \mathbf{A}_j \frac{\partial \mathbf{V}}{\partial x_j} = \mathbf{F}. \tag{3.4.17}$$

We finish this section by the following lemma.

Lemma 3.4.1. *(see, (Courant & Hilbert, 1979), p.593-594), (see, (Yakhno & Akmaz, 2005) System (3.4.17) can be transformed into the following form*

$$\mathbf{I}_9 \frac{\partial \tilde{\mathbf{V}}}{\partial t} + \sum_{j=2}^{3} \tilde{\mathbf{A}}_j \frac{\partial \tilde{\mathbf{V}}}{\partial x_j} = \tilde{\mathbf{F}}, \tag{3.4.18}$$

which is an symmetric hyperbolic system.

Proof. Consider the symmetric positive-definite matrix $\mathbf{C}$. There exists a symmetric positive-definite matrix $\mathbf{M}$ such that $\mathbf{C}^{-1} = \mathbf{M}^2$ (see Theorem A.1.2 of Section A.1), and the matrix $\mathbf{M}^{-1}$, which is inverse of $\mathbf{M}$, is also symmetric (see Theorem A.1.1 of Section A.1).

Using these facts we define the matrix

$$\mathbf{S} = \begin{bmatrix} \rho^{-\frac{1}{2}}\mathbf{I}_3 & \mathbf{0}_{3,6} \\ \mathbf{0}_{6,3} & \mathbf{M}^{-1} \end{bmatrix}, \tag{3.4.19}$$

and denote the vector $\mathbf{V}$ as

$$\mathbf{V} = \mathbf{S}\tilde{\mathbf{V}}. \tag{3.4.20}$$

Substituting (3.4.20) into (3.4.17) and multiplying the resulting formula with matrix $\mathbf{S}$ from left-hand side we obtain (3.4.18), where

$$\mathbf{S}\mathbf{A}_0\mathbf{S} = \mathbf{I}_9, \quad \tilde{\mathbf{A}}_j = \mathbf{S}\mathbf{A}_j\mathbf{S}, \quad \tilde{\mathbf{F}} = \mathbf{S}\mathbf{F}. \tag{3.4.21}$$

Since $\mathbf{S}$ and $\mathbf{A}_j$, $j = 2, 3$ are symmetric, the matrices $\tilde{\mathbf{A}}_j$, $j = 2, 3$ are also symmetric (see Theorem A.1.3 of Section A.1), which implies that (3.4.18) is symmetric hyperbolic system. □

3.4.2 Existence and Uniqueness of a Classical Solution of IVP for System Depending on x_2, x_3 and t Variables. Properties of Solutions.

In this section we prove the existence of unique classical solutions depending on x_2, x_3 and t variables. Also a property for existence of unique solution with finite support is proved when inhomogeneous term and initial data are infinitely differentiable and have finite support.

Theorem 3.4.2. *Let T be a fixed positive number, $\boldsymbol{G}(x_2, x_3)$; $\boldsymbol{H}(x_2, x_3)$ and $\boldsymbol{F}(x_2, x_3, t)$ be given functions such that $\boldsymbol{G}(x_2, x_3) \in H^4(\mathbb{R}^2)$; $\boldsymbol{H}(x_2, x_3) \in H^5(\mathbb{R}^2)$ and $\boldsymbol{F}(x_2, x_3, t) \in C([0, T]; H^4(\mathbb{R}^2))$. Then there exists a unique solution of Cauchy problem (3.1.15), (3.1.16)*

$$\mathbf{U}(x_2, x_3, t) \in C^1([0, T]; H^4(\mathbb{R}^2)) \cap C^2([0, T]; H^3(\mathbb{R}^2)).$$

Proof. For the case when all functions appearing in (3.1.15), (3.1.16) do not depend on x_1 variable, the IVP can be written as an IVP for the following symmetric first order hyperbolic system

$$\frac{\partial \bar{\mathbf{V}}}{\partial t} + \sum_{k=2}^{3} \widetilde{A}_k \frac{\partial \bar{\mathbf{V}}}{\partial x_k} = \widetilde{\mathbf{F}}, \quad x \in \mathbb{R}^2, \;\; t > 0, \tag{3.4.22}$$

$$\bar{\mathbf{V}}(x,0) = \mathbf{V}_0(x_2, x_3), \tag{3.4.23}$$

where $\bar{\mathbf{V}},\ \widetilde{\mathbf{F}},\ \mathbf{V}_0(x_2, x_3)$ are defined by formulae

$$\bar{\mathbf{V}} = S \begin{bmatrix} \bar{\mathbf{U}} \\ \mathbf{T} \end{bmatrix}, \qquad \widetilde{\mathbf{F}} = S \begin{bmatrix} \mathbf{F} \\ \mathbf{0}_{6,1} \end{bmatrix}, \qquad \mathbf{V}_0 = S \begin{bmatrix} \bar{\mathbf{U}} \\ \mathbf{T} \end{bmatrix}_{t=0}, \tag{3.4.24}$$

where S is defined by the matrix (3.4.19) and $\mathbf{T},\ \bar{\mathbf{U}},\ \widetilde{A}_k;\ k = 2, 3$ are defined by the equations (3.2.2), (3.2.4), (3.4.21).

Using existence theorem (Mizohata, 1973)(p. 335) (also, appendix b) for the symmetric hyperbolic first order system (3.4.22), (3.4.23), it can be shown that there exists a unique solution of (3.1.15), (3.1.16) in the class

$$\mathbf{U}(x_2, x_3, t) \in C^1([0, T]; H^4(\mathbb{R}^2)) \cap C^2([0, T]; H^3(\mathbb{R}^2))$$

for any given initial data $\mathbf{G}(x_2, x_3) \in H^4(\mathbb{R}^2)$; $\mathbf{H}(x_2, x_3) \in H^5(\mathbb{R}^2)$ and inhomogeneous term $\mathbf{F}(x_2, x_3, t) \in C([0, T]; H^4(\mathbb{R}^2))$. □

Theorem 3.4.3. *Let T be a fixed positive number, $\boldsymbol{G}(x_2, x_3)$; $\boldsymbol{H}(x_2, x_3)$ and $\boldsymbol{F}(x_2, x_3, t)$ be given functions such that $\boldsymbol{G}(x_2, x_3) \in C_0^\infty(\mathbb{R}^2)$; $\boldsymbol{H}(x_2, x_3) \in C_0^\infty(\mathbb{R}^2)$ and $\boldsymbol{F}(x_2, x_3, t) \in C([0, T]; C_0^\infty(\mathbb{R}^2))$. Then the solution $\mathbf{U}(x_2, x_3, t)$ of Cauchy problem (3.1.15), (3.1.16) belongs to*

$$C^2([0, T]; C_0^\infty(\mathbb{R}^2)).$$

Proof. Using Theorem 3.4.2 it can be found that if $D^\alpha \psi \in H^4(\mathbb{R}^2)$; $D^\alpha \mathbf{H}(x_2, x_3) \in H^5(\mathbb{R}^2)$ and $D^\alpha \mathbf{F}(x_2, x_3, t) \in C([0, T]; H^4(\mathbb{R}^2))$, where T is a fixed positive number and for an arbitrary multi-index $\alpha = (\alpha_1, \alpha_2)$ where $|\alpha| = \alpha_1 + \alpha_2$, $\alpha_i;\ i = 1, 2$ are nonnegative integers, $D^\alpha = \dfrac{\partial^{|\alpha|}}{\partial x_2^{\alpha_1} \partial x_3^{\alpha_2}}$. Then $D^\alpha \mathbf{U}$ belongs to the class

$$D^\alpha \mathbf{U}(x_2, x_3, t) \in C^1([0,T]; H^4(\mathbb{R}^2)) \cap C^2([0,T]; H^3(\mathbb{R}^2)).$$

Using this fact and applying Sobolev's lemma (See, appendixA.3) it can be proved that

$$\mathbf{U}(x_2, x_3, t) \in C^2([0,T]; C^\infty(\mathbb{R}^2)).$$

To prove that the function has a compact support, let us consider reduction of the Cauchy problem (3.1.15), (3.1.16) to the first order symmetric hyperbolic system (3.4.22), (3.4.23) that is where all matrices $\widetilde{A}_k$ are real symmetric matrices with constant elements. Let T be a fixed positive number, $\xi = (\xi_2, \xi_3) \in \mathbb{R}^2$ be a parameter; $A(\xi)$ be a matrix defined by $A(\xi) = \sum_{k=2}^{3} \widetilde{A}_k \xi_k;\ \lambda_i(\xi),\ i = 1, 2, ..., 9$ be eigenvalues of $A(\xi)$. The positive number M is defined by

$$M = \max_{i=1,2,\ldots,9} \max_{|\xi|=1} |\lambda_i(\xi)|. \tag{3.4.25}$$

We claim that M is the upper bound on the speed of waves in any direction.

Using T and M we define the following domains

$$\begin{aligned} &S(x_0, h) = \{x \in \mathbb{R}^2 : |x - x_0| \le M(T - h)\},\ 0 \le h \le T \\ &\Gamma(x_0, T) = \{(x, t) : 0 \le t \le T,\ |x - x_0| \le M(T - t)\} \\ &R(x_0, h) = \{(x, t) : 0 \le t \le h,\ |x - x_0| = M(T - t)\} \end{aligned}$$

Here $\Gamma(x_0, T)$ is the conoid with vertex (x_0, T); $S(x_0, h)$ is the surface constructed by the intersection of the plane $t = h$ and the conoid $\Gamma(x_0, T)$; $R(x_0, h)$ is the lateral surface of the conoid $\Gamma(x_0, T)$ bounded by $S(x_0, 0)$ and $S(x_0, h)$. Let Ω be the region in $\mathbb{R}^2 \times (0, \infty)$ bounded by $S(x_0, 0)$, $S(x_0, h)$ and $R(x_0, h)$ with boundary $\partial\Omega = S(x_0, 0) \cup S(x_0, h) \cup R(x_0, h)$.

Applying the reasoning similar to (Courant & Hilbert, 1979) (p. 652-661)(see,

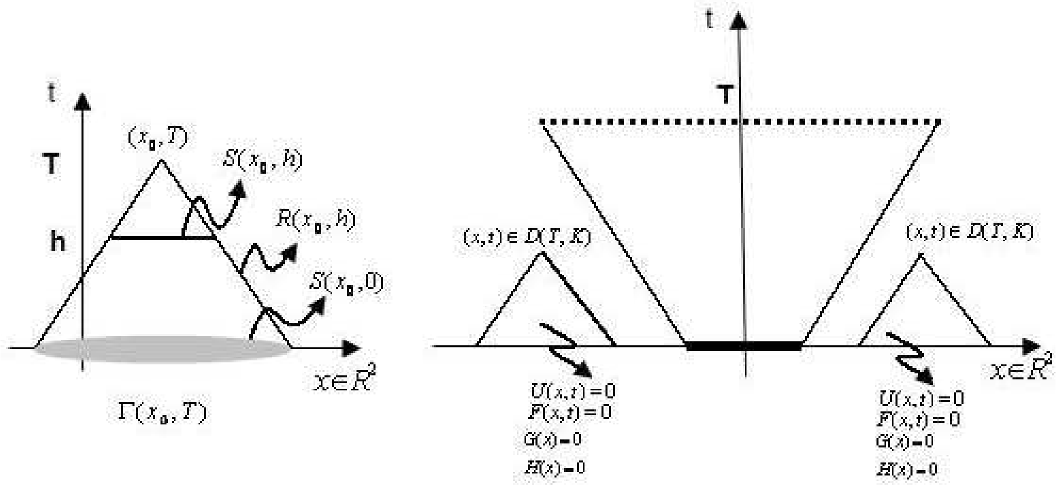


Figure 3.4 Domains of Dependence

Appendix B) we find the following estimate for the solution of (3.2.27), (3.2.28)

$$\int_{S(h)} |\bar{\mathbf{V}}(x,h)|^2 dx \le e^h \Big[\int_{S(0)} |\mathbf{V_0}(x)|^2 dx + \int_0^h \Big(\int_{S(t)} |\tilde{\mathbf{F}}(x,t)|^2 dx \Big) dt \Big]. \quad (3.4.26)$$

Let us define $P(K) = \{x = (x_2, x_3) \in \mathbb{R}^2 : |x| \le K\}$. Since $\mathbf{G}(x_2, x_3) \in C_0^\infty(\mathbb{R}^2)$; $\mathbf{H}(x_2, x_3) \in C_0^\infty(\mathbb{R}^2)$; and $\mathbf{F}(x_2, x_3, t) \in C([0,T]; C_0^\infty(\mathbb{R}^2))$ then there exists $K > 0$ such that *supp* $\mathbf{G} \subseteq P(K)$, *supp* $\mathbf{H} \subseteq P(K)$ and $\mathbf{F}(x_2, x_3, t)$ as a function of the variable (x_2, x_3), has a finite support which is located in $P(K)$ for any fixed t from $[0, T]$.

Also let us denote

$$D(T, K) = \{(x_2, x_3, t) : 0 \le t \le T,\ \Gamma(x_2, x_3, t) \cap P(K) = \emptyset\}.$$

If $(x_2, x_3, t) \in D(T, K)$ then $\mathbf{U}(x_2, x_3, t) = 0$. This means $\mathbf{U}(x_2, x_3, t) = 0$ for any $t \in [0, T]$ and $|x| > MT + K$.

Hence, *supp* $\mathbf{U} \subseteq P(MT + K)$. As a result $\mathbf{U}(x_2, x_3, t)$ belongs to the class

$$\mathbf{U}(x_2, x_3, t) \in C^2([0, T]; C_0^\infty(\mathbb{R}^2)).$$

□

3.4.3 IVP for the System Depending on x_2, x_3 and t Variables

In Section 3.4.2 we have shown that IVP of elastic system (3.1.15), (3.1.16) has a unique solution $\mathbf{U}(x_2, x_3, t)$ belongs to $C^2([0,T]; C_0^\infty(\mathbb{R}^2))$ when inhomogeneous term $\mathbf{F}(x_2, x_3, t) \in C([0,T]; C_0^\infty(\mathbb{R}^2))$ be given function depending x_2, x_3 and t variables and initial data $\mathbf{G}(x_2, x_3) \in C_0^\infty(\mathbb{R}^2)$; $\mathbf{H}(x_2, x_3) \in (x_2, x_3) \in C_0^\infty(\mathbb{R}^2)$ are given functions depending x_2, x_3 variable only. In this case we can rewrite IVP of elastic system (3.1.15), (3.1.16)

$$\rho\frac{\partial^2 U_1}{\partial t^2} = c_{62}\frac{\partial^2 U_2}{\partial x_2^2} + c_{63}\frac{\partial^2 U_3}{\partial x_2 \partial x_3} + c_{66}\frac{\partial^2 U_1}{\partial x_2^2}$$

$$+c_{65}\frac{\partial^2 U_1}{\partial x_2 \partial x_3} + c_{64}\Big(\frac{\partial^2 U_2}{\partial x_2 \partial x_3} + \frac{\partial^2 U_3}{\partial x_2^2}\Big) + c_{52}\frac{\partial^2 U_2}{\partial x_2 \partial x_3} + c_{53}\frac{\partial^2 U_3}{\partial x_3^2}$$

$$+c_{56}\frac{\partial^2 U_1}{\partial x_2 \partial x_3} + c_{55}\frac{\partial^2 U_1}{\partial x_3^2} + c_{54}\Big(\frac{\partial^2 U_2}{\partial x_3^2} + \frac{\partial^2 U_3}{\partial x_2 \partial x_3}\Big) + F_1, \tag{3.4.27}$$

$$\rho\frac{\partial^2 U_2}{\partial t^2} = c_{22}\frac{\partial^2 U_2}{\partial x_2^2} + c_{23}\frac{\partial^2 U_3}{\partial x_2 \partial x_3} + c_{26}\frac{\partial^2 U_1}{\partial x_2^2}$$

$$+c_{25}\frac{\partial^2 U_1}{\partial x_2 \partial x_3} + c_{24}\Big(\frac{\partial^2 U_2}{\partial x_2 \partial x_3} + \frac{\partial^2 U_3}{\partial x_2^2}\Big) + c_{42}\frac{\partial^2 U_2}{\partial x_2 \partial x_3} + c_{43}\frac{\partial^2 U_3}{\partial x_3^2}$$

$$+c_{46}\frac{\partial^2 U_1}{\partial x_2 \partial x_3} + c_{45}\frac{\partial^2 U_1}{\partial x_3^2} + c_{44}\Big(\frac{\partial^2 U_2}{\partial x_3^2} + \frac{\partial^2 U_3}{\partial x_2 \partial x_3}\Big) + F_2, \tag{3.4.28}$$

$$\rho\frac{\partial^2 U_3}{\partial t^2} = c_{42}\frac{\partial^2 U_2}{\partial x_2^2} + c_{43}\frac{\partial^2 U_3}{\partial x_2 \partial x_3} + c_{46}\frac{\partial^2 U_1}{\partial x_2^2}$$

$$+c_{45}\frac{\partial^2 U_1}{\partial x_2 \partial x_3} + c_{44}\Big(\frac{\partial^2 U_2}{\partial x_2 \partial x_3} + \frac{\partial^2 U_3}{\partial x_2^2}\Big) + c_{32}\frac{\partial^2 U_2}{\partial x_2 \partial x_3} + c_{33}\frac{\partial^2 U_3}{\partial x_3^2}$$

$$+c_{36}\frac{\partial^2 U_1}{\partial x_2 \partial x_3} + c_{35}\frac{\partial^2 U_1}{\partial x_3^2} + c_{34}\Big(\frac{\partial^2 U_2}{\partial x_3^2} + \frac{\partial^2 U_3}{\partial x_2 \partial x_3}\Big) + F_3, \tag{3.4.29}$$

$$U_j(x_2, x_3, 0) = G_j(x_2, x_3), \quad \left.\frac{\partial U_j(x_2, x_3, t)}{\partial t}\right|_{t=0} = H_j(x_2, x_3), \tag{3.4.30}$$

where $x = (x_2, x_3) \in \mathbb{R}^2, \;\; t > 0$. Simply, equations (3.4.27)-(3.4.30) can be written as follows

$$\rho \frac{\partial^2 \mathbf{U}}{\partial t^2} = L[\mathbf{U}] + \mathbf{F}(x_2, x_3, t), \quad t > 0, \tag{3.4.31}$$

$$\mathbf{U}(x_2, x_3, 0) = \mathbf{G}(x_2, x_3), \quad \left.\frac{\partial \mathbf{U}}{\partial t}\right|_{t=0} = \mathbf{H}(x_2, x_3), \tag{3.4.32}$$

where $x = (x_2, x_3) \in \mathbb{R}^2, \quad t > 0$ and $L[\mathbf{U}]$ is the matrix operator defined with components $L_{ij}U_j$ that is given with the formulas

$$\begin{aligned}
L_{11} &= c_{66}\partial^2_{x_2} + c_{65}\partial^2_{x_2x_3} + c_{56}\partial^2_{x_2x_3} + c_{55}\partial^2_{x_3},\\
L_{12} &= c_{62}\partial^2_{x_2} + c_{64}\partial^2_{x_2x_3} + c_{52}\partial^2_{x_2x_3} + c_{54}\partial^2_{x_3},\\
L_{13} &= c_{63}\partial^2_{x_2x_3} + c_{64}\partial^2_{x_2} + c_{53}\partial^2_{x_3} + c_{54}\partial^2_{x_2x_3},\\
L_{21} &= c_{26}\partial^2_{x_2} + c_{25}\partial^2_{x_2x_3} + c_{46}\partial^2_{x_2x_3} + c_{45}\partial^2_{x_3},\\
L_{22} &= c_{22}\partial^2_{x_2} + c_{24}\partial^2_{x_2x_3} + c_{42}\partial^2_{x_2x_3} + c_{44}\partial^2_{x_3},\\
L_{23} &= c_{23}\partial^2_{x_2x_3} + c_{24}\partial^2_{x_2} + c_{43}\partial^2_{x_3} + c_{44}\partial^2_{x_2x_3},\\
L_{31} &= c_{46}\partial^2_{x_2} + c_{45}\partial^2_{x_2x_3} + c_{36}\partial^2_{x_2x_3} + c_{35}\partial^2_{x_3},\\
L_{32} &= c_{42}\partial^2_{x_2} + c_{44}\partial^2_{x_2x_3} + c_{32}\partial^2_{x_2x_3} + c_{34}\partial^2_{x_3},\\
L_{33} &= c_{43}\partial^2_{x_2x_3} + c_{44}\partial^2_{x_2} + c_{33}\partial^2_{x_3} + c_{34}\partial^2_{x_2x_3}.
\end{aligned} \tag{3.4.33}$$

3.5 2-D Fundamental Solutions of IVP for the System Depending on x_2, x_3 and t Variables

A matrix $\mathcal{U}(x_2, x_3, t) = [U_{rs}(x_2, x_3, t)]_{3\times 3}$ is called 2-D fundamental solution or IVP for the system depending on two space x_2, x_3 and the time variable t if $s-$th column

$$\mathbf{U}_s(x_2, x_3, t) = \begin{pmatrix} U_{1s}(x_2, x_3, t) \\ U_{2s}(x_2, x_3, t) \\ U_{3s}(x_2, x_3, t) \end{pmatrix}$$

satisfies

$$\rho \frac{\partial^2 \mathbf{U}_s}{\partial t^2} = L[\mathbf{U}_s], \tag{3.5.1}$$

$$\mathbf{U}_s(x_2, x_3, 0) = 0, \quad \frac{\partial \mathbf{U}_s}{\partial t} = \frac{1}{\rho}\vec{e}^s \delta(x_2, x_3), \tag{3.5.2}$$

where $x = (x_2, x_3) \in \mathbb{R}^2, \quad t \in \mathbb{R}$; $\mathbf{e}^1 = (1,0,0)$, $\mathbf{e}^2 = (0,1,0)$, $\mathbf{e}^3 = (0,0,1)$; $U_{js}(x_2, x_3, t)$ are the components of the unknown displacement vector $\mathbf{U}_s(x_2, x_3, t)$ and $L[\mathbf{U}_s]$ is the matrix operator defined with components $L_{ij}U_{js}$ that is given with the formulas (3.4.33).

3.5.1 Some properties of 2-D Fundamental Solution

Remark 3.5.1. Let $\mathbf{U}_s(x_2, x_3, t)$ be a fundamental solution then $\hat{\mathbf{U}}_s(x_2, x_3, t) = \theta(t)\mathbf{U}_s(x_2, x_3, t)$ satisfies

$$\rho \frac{\partial^2 \hat{\mathbf{U}}_s}{\partial t^2} = L[\hat{\mathbf{U}}_s] + \mathbf{e}^s \delta(x_2, x_3, t), \tag{3.5.3}$$

$$\hat{\mathbf{U}}_s(x_2, x_3, t)\Big|_{t<0} = 0, \tag{3.5.4}$$

$s = 1, 2, 3$; where $x = (x_2, x_3) \in \mathbb{R}^2, \quad t \in \mathbb{R}$; $\mathbf{e}^1 = (1,0,0)$, $\mathbf{e}^2 = (0,1,0)$, $\mathbf{e}^3 = (0,0,1)$; $U_{js}(x_2, x_3, t)$ are the components of the unknown displacement vector $\mathbf{U}_s(x_2, x_3, t)$ and $L[\mathbf{U}_s]$ is the matrix operator defined with components $L_{ij}U_{js}$ that is given with the formulas (3.4.33).

Proof. Since $\hat{\mathbf{U}}(x_2, x_3, t) = \Theta(t)\mathbf{u}(x_2, x_3, t)$, derivative of $\hat{\mathbf{U}}(x_2, x_3, t)$ with respect to t is

$$\frac{\partial\Big(\hat{\mathbf{U}}(x_2, x_3, t)\Big)}{\partial t} = \delta(t)\mathbf{U}(x, 0) + \Theta(t)\frac{\partial}{\partial t}\mathbf{U}(x_2, x_3, t),$$

$$\frac{\partial^2\Big(\hat{\mathbf{U}}(x_2, x_3, t)\Big)}{\partial t^2} = \frac{1}{\rho}\vec{e}^s \delta(t)\delta(x_2, x_3) + \Theta(t)\frac{\partial^2}{\partial t^2}\mathbf{U}(x_2, x_3, t),$$

and also we have

$$L[\hat{\mathbf{U}}] = \Theta(t)L[\mathbf{U}]$$

then

$$\rho\frac{\partial^2 \hat{\mathbf{U}}}{\partial t^2} - L[\hat{\mathbf{U}}] = \rho\frac{1}{\rho}\vec{e}^s \delta(t)\delta(x_2, x_3) + \rho\Theta(t)\frac{\partial^2 \mathbf{U}}{\partial t^2} - \Theta(t)L[\mathbf{U}],$$

$$\rho \frac{\partial^2 \hat{\mathbf{U}}}{\partial t^2} - L[\hat{\mathbf{U}}] = \vec{e}^s \delta(x_2, x_3)\delta(t).$$

□

It is well known that (see, Hörmander-Lojasiewicz theorem in appendix) the arbitrary differential equation or system with constant coefficients has a fundamental solution of slow growth. Thus, system with constant coefficients given with equations (3.5.1), (3.5.2) has a fundamental solution

$$\mathbf{U}_s(x_2, x_3, t) \in C^2([0, T]; \mathcal{S}'(\mathbb{R}^2)).$$

Our aim is to study some of the properties of this fundamental solution and suggest a method to find fundamental solutions.

Let us denote convolution of functions $\mathbf{U}_s(x, t)$, with cap-shaped function $w_\varepsilon(x)$ is $\mathbf{u}_s^\varepsilon(x, t) = (u_{1s}, u_{2s}, u_{3s})$. Taking convolution with cap-shaped function, the generalized Cauchy problem (3.5.1)-(3.5.2) can be written as

$$\rho \frac{\partial^2 \mathbf{u}_s^\varepsilon}{\partial t^2} = L[\mathbf{u}_s^\varepsilon], \quad (x_2, x_3) \in \mathbb{R}^2, \ \ t \in \mathbb{R}, \tag{3.5.5}$$

$$\mathbf{u}_s^\varepsilon(x_2, x_3, 0) = 0, \quad \frac{\partial \mathbf{u}_s^\varepsilon}{\partial t} = \frac{1}{\rho} \vec{e}^s w_\varepsilon(x_2, x_3). \tag{3.5.6}$$

Using Theorem 3.2.3 of Section 3.4.2, it can be proved that problem (3.5.1), (3.5.2) has a unique solution $\mathbf{u}_s^\varepsilon(x_2, x_3, t) \in C^2([0, T]; C_0^\infty(\mathbb{R}^2))$ and
$supp\ \mathbf{u}_s^\varepsilon(x_2, x_3, t) \subseteq P(MT + \varepsilon_0)\ \forall \varepsilon \in (0, \varepsilon_0)$.

Property 4. *As $\varepsilon \to +0$, $\boldsymbol{u}_s^\varepsilon(x_2, x_3, t)$ approaches to $\boldsymbol{U}_s(x_2, x_3, t)$ in $\mathcal{S}'(\mathbb{R}^2)$; $\forall t \in [0, T]$.*

Proof. It can be proved that as $\varepsilon \to +0$, $w_\varepsilon(x)$ approaches to $\delta(x)$ in $\mathcal{S}'(\mathbb{R}^2)$. Using this fact and using the continuity of the convolution $\mathbf{u}(x_2, x_3, t) * w_\varepsilon(x)$ with respect to $w_\varepsilon(x)$ theorem is proved. □

Property 5. *Let T be a fixed positive number. There exists a unique solution of Cauchy problem (3.5.1), (3.5.2)*

$$\mathbf{U}_s(x_2, x_3, t) \in C^2([0, T]; \mathcal{E}'(\mathbb{R}^2)).$$

Proof. We need to show that

$$(\mathbf{U}_s, \varphi) = 0; \quad \forall \varphi \in \mathcal{S} \quad \text{and } supp\ \varphi \subseteq \mathbb{R} \setminus P(MT + \varepsilon_0)$$

$$\begin{aligned}(\mathbf{U}_s, \varphi) &= \lim_{\varepsilon \to +0} (\mathbf{u}_s^\varepsilon, \varphi); \quad \forall \varphi \in \mathcal{S} \\ &= 0.\end{aligned}$$

This means $supp\ \mathbf{U}_s \subseteq P(MT + \varepsilon_0)\ \forall \varepsilon \in (0, \varepsilon_0)$. Also using property(4) we conclude that there exist a unique solution of the Cauchy problem (3.5.1), (3.5.2)

$$\mathbf{U}_s(x_2, x_3, t) \in C^2([0, T]; \mathcal{E}'(\mathbb{R}^2)).$$

□

Property 6. *Let $\boldsymbol{U}_s(x_2, x_3, t)$ be solution of the problem (3.5.1), (3.5.2) and $\tilde{\boldsymbol{U}}_s(\upsilon_2, \upsilon_3, t) = \left(\tilde{U}_{1s}, \tilde{U}_{2s}, \tilde{U}_{3s}\right)$ be the Fourier transform image of $\boldsymbol{U}_s(x_2, x_3, t)$ with respect to $(x_2, x_3) \in \mathbb{R}^2$. Then the Fourier image $\tilde{\boldsymbol{U}}_s(\upsilon_2, \upsilon_3, t)$ is an entire analytic function and satisfies following system of equations:*

$$\rho \frac{\tilde{U}_{1s}}{\partial t^2} = c_{62}\upsilon_2^2 \tilde{U}_{2s} + c_{63}\upsilon_2\upsilon_3 \tilde{U}_{3s} + c_{66}\upsilon_2^2 \tilde{U}_{1s}$$

$$+ c_{65}\upsilon_2\upsilon_3 \tilde{U}_{1s} + c_{64}\left(\upsilon_2\upsilon_3 \tilde{U}_{2s} + \upsilon_2^2 \tilde{U}_{3s}\right) + c_{52}\upsilon_2\upsilon_3 \tilde{U}_{2s} + c_{53}\upsilon_3^2 \tilde{U}_{3s}$$

$$+ c_{56}\upsilon_2\upsilon_3 \tilde{U}_{1s} + c_{55}\upsilon_3^2 \tilde{U}_{1s} + c_{54}\left(\upsilon_3^2 \tilde{U}_{2s} + \upsilon_2\upsilon_3 \tilde{U}_{3s}\right), \qquad (3.5.7)$$

$$\rho \frac{\tilde{U}_{2s}}{\partial t^2} = c_{22}\upsilon_2^2 \tilde{U}_{2s} + c_{23}\upsilon_2\upsilon_3 \tilde{U}_{3s} + c_{26}\upsilon_2^2 \tilde{U}_{1s}$$

$$+c_{25}\upsilon_2\upsilon_3\tilde{U}_{1s}+c_{24}\Big(\upsilon_2\upsilon_3\tilde{U}_{2s}+\upsilon_2^2\tilde{U}_{3s}\Big)+c_{42}\upsilon_2\upsilon_3\tilde{U}_{2s}+c_{43}\upsilon_3^2\tilde{U}_{3s}$$

$$+c_{46}\upsilon_2\upsilon_3\tilde{U}_{1s}+c_{45}\upsilon_3^2\tilde{U}_{1s}+c_{44}\Big(\upsilon_3^2\tilde{U}_{2s}+\upsilon_2\upsilon_3\tilde{U}_{3s}\Big), \tag{3.5.8}$$

$$\rho\frac{\tilde{U}_{3s}}{\partial t^2}=c_{42}\upsilon_2^2\tilde{U}_{2s}+c_{43}\upsilon_2\upsilon_3\tilde{U}_{3s}+c_{46}\upsilon_2^2\tilde{U}_{1s}$$

$$+c_{45}\upsilon_2\upsilon_3\tilde{U}_{1s}+c_{44}\Big(\upsilon_2\upsilon_3\tilde{U}_{2s}+\upsilon_2^2\tilde{U}_{3s}\Big)+c_{32}\upsilon_2\upsilon_3\tilde{U}_{2s}+c_{33}\upsilon_3^2\tilde{U}_{3s}$$

$$+c_{36}\upsilon_2\upsilon_3\tilde{U}_{1s}+c_{35}\upsilon_3^2\tilde{U}_{1s}+c_{34}\Big(\upsilon_3^2\tilde{U}_{2s}+\upsilon_2\upsilon_3\tilde{U}_{3s}\Big), \tag{3.5.9}$$

$$\tilde{U}_{js}(x_2,x_3,0)=0,\quad \frac{\partial\tilde{U}_{js}(x_2,x_3,t)}{\partial t}\bigg|_{t=0}=\tilde{h}_{js}(x_2,x_3), \tag{3.5.10}$$

where $\tilde{h}_{js}=\left(\frac{1}{\rho}\boldsymbol{e}^s\right)_j$.

Proof. Let $\tilde{\mathbf{U}}_s(\upsilon,t)$ be the Fourier transform image of $\mathbf{U}_s(x,t)$ with respect to $x=(x_2,x_3)\in\mathbb{R}^2$, i.e.

$$\tilde{\mathbf{U}}_s(\upsilon,t)=\Big(\tilde{U}_{1s}(\upsilon,t),\tilde{U}_{2s}(\upsilon,t),\tilde{U}_{3s}(\upsilon,t)\Big)$$

$$\tilde{U}_{ls}(\upsilon,t)=\boldsymbol{\mathcal{F}}_x[U_{ls}];\quad l=1,2,3;\upsilon=(\upsilon_2,\upsilon_3)\in\mathbb{R}^2;$$

where the Fourier operator $\boldsymbol{\mathcal{F}}_x$ is defined by

$$\boldsymbol{\mathcal{F}}_x[U_{ls}]=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}U_{ls}(x,t)e^{i\upsilon x}dx_1dx_2;$$

$$x\upsilon=x_1\upsilon_1+x_2\upsilon_2;\quad i^2=-1.$$

Since $\mathbf{U}_s(x_2,x_3,t)\in C^2([0,T];\mathcal{E}'(\mathbb{R}^2))$, according to Paley-Wiener theorem (Reed & Simon, 1975), Fourier transform of the function $\mathbf{U}_s(x_2,x_3,t)$ is an entire analytic function with respect to $\upsilon=(\upsilon_2,\upsilon_3)\in\mathbb{R}^2$, and can be written as a power series

$$\tilde{\mathbf{U}}_s(\upsilon,t)=\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\tilde{\mathbf{U}}_s^{k,m}(t)\upsilon_2^k\upsilon_3^m.$$

If we apply Fourier transform with respect to $x = (x_2, x_3) \in \mathbb{R}^2$, problem (3.5.1), (3.5.2) can be written in terms of Fourier images given with the equations (3.5.7)-(3.5.10). □

3.5.2 Derivation of 2-D Fundamental Solution

Problem in Terms of Coefficients of the Series Expansion

According to Paley-Wiener Theorem, power series expansion of $\tilde{\mathbf{U}}_s(\upsilon, t)$, $\tilde{\mathbf{h}}_s(\upsilon)$ can be considered i.e.

$$\tilde{\mathbf{U}}_s(\upsilon, t) = \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \tilde{\mathbf{U}}_s^{k,m}(t) \upsilon_2^k \upsilon_3^m, \tag{3.5.11}$$

$$\tilde{\mathbf{h}}_s(\upsilon) = \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \tilde{\mathbf{h}}_s^{k,m} \upsilon_2^k \upsilon_3^m, \tag{3.5.12}$$

where $\tilde{\mathbf{h}}_s^{k,m}$ are given real numbers; $\tilde{\mathbf{U}}_s^{k,m}(t)$ are unknown coefficients we need to find. Substituting (3.5.11)-(3.5.12) into (3.5.7)-(3.5.10) we obtain

$$\begin{aligned}
&\rho \frac{\tilde{U}_{1s}^{k,m}}{\partial t^2} + c_{62} \tilde{U}_{2s}^{k-2,m} + c_{63} \tilde{U}_{3s}^{k-1,m-1} + c_{66} \tilde{U}_{1s}^{k-2,m} \\
&+ c_{65} \tilde{U}^{k-1,m-1} + c_{64} \Big(\tilde{U}_{2s}^{k-1,m-1} + \tilde{U}_{3s}^{k-2,m} \Big) + c_{52} \tilde{U}_{2s}^{k-1,m-1} + c_{53} \tilde{U}_{3s}^{k,m-2} \\
&\qquad + c_{56} \tilde{U}_{1s}^{k-1,m-1} + c_{55} \tilde{U}_{1s}^{k,m-2} + c_{54} \Big(\tilde{U}_{2s}^{k,m-2} + \tilde{U}_{3s}^{k-1,m-1} \Big) = 0,
\end{aligned} \tag{3.5.13}$$

$$\begin{aligned}
&\rho \frac{\tilde{U}_{2s}^{k,m}}{\partial t^2} + c_{22} \tilde{U}_{2s}^{k-2,m} + c_{23} \tilde{U}_{3s}^{k-1,m-1} + c_{26} \tilde{U}_{1s}^{k-2,m} \\
&+ c_{25} \tilde{U}_{1s}^{k-1,m-1} + c_{24} \Big(\tilde{U}_{2s}^{k-1,m-1} + \tilde{U}_{3s}^{k-2,m} \Big) + c_{42} \tilde{U}_{2s}^{k-1,m-1} + c_{43} \tilde{U}_{3s}^{k,m-2} \\
&\qquad + c_{46} \tilde{U}_{1s}^{k-1,m-1} + c_{45} \tilde{U}_{1s}^{k,m-2} + c_{44} \Big(\tilde{U}_{2s}^{k,m-2} + \tilde{U}_{3s}^{k-1,m-1} \Big) = 0,
\end{aligned} \tag{3.5.14}$$

$$\rho\frac{\tilde U^{k,m}_{3s}}{\partial t^2}+c_{42}\tilde U^{k-2,m}_{2s}+c_{43}\tilde U^{k-1,m-1}_{3s}+c_{46}\tilde U^{k-2,m}_{1s}$$

$$+c_{45}\tilde U^{k-1,m-1}_{1s}+c_{44}\Big(\tilde U^{k-1,m-1}_{2s}+\tilde U^{k-2,m}_{3s}\Big)+c_{32}\tilde U^{k-1,m-1}_{2s}+c_{33}\tilde U^{k,m-2}_{3s}$$

$$+c_{36}\tilde U^{k-1,m-1}_{1s}+c_{35}\tilde U^{k,m-2}_{1s}+c_{34}\Big(\tilde U^{k,m-2}_{2s}+\tilde U^{k-1,m-1}_{3s}\Big)=0, \tag{3.5.15}$$

$$\tilde U^{k,m}_{js}(x_2,x_3,0)=0,\quad \frac{\partial \tilde U^{k,m}_{js}(x_2,x_3,t)}{\partial t}\bigg|_{t=0}=\tilde h^{k,m}_{js}, \tag{3.5.16}$$

where h^k_{js} are given real numbers such that $h^0_{js}=1$ and $h^l_{js}=\frac{1}{\rho}\mathbf{e}^s$ for $l=1,2,....$ Equations (3.5.13)-(3.5.16) can be written equivalently as the following recurrence relations:

$$\frac{\partial^2\tilde U^{k,m}_{js}}{\partial t^2}=-\frac{1}{\rho}\Upsilon^{k,m}_j,\ t>0,\ j=1,2,3, \tag{3.5.17}$$

$$\tilde U^{k,m}_{js}(0)=0,\quad \frac{\partial \tilde U^{k,m}_{js}(t)}{\partial t}\bigg|_{t=0}=\tilde h^{k,m}_{js},\ j=1,2,3. \tag{3.5.18}$$

where

$$\Upsilon^{k,m}_1=c_{62}\tilde U^{k-2,m}_{2s}+c_{63}\tilde U^{k-1,m-1}_{3s}+c_{66}\tilde U^{k-2,m}_{1s}+c_{65}\tilde U^{k-1,m-1}$$

$$+c_{64}\Big(\tilde U^{k-1,m-1}_{2s}+\tilde U^{k-2,m}_{3s}\Big)+c_{52}\tilde U^{k-1,m-1}_{2s}+c_{53}\tilde U^{k,m-2}_{3s}+c_{56}\tilde U^{k-1,m-1}_{1s}$$

$$+c_{55}\tilde U^{k,m-2}_{1s}+c_{54}\Big(\tilde U^{k,m-2}_{2s}+\tilde U^{k-1,m-1}_{3s}\Big), \tag{3.5.19}$$

$$\Upsilon^{k,m}_2=c_{22}\tilde U^{k-2,m}_{2s}+c_{23}\tilde U^{k-1,m-1}_{3s}+c_{26}\tilde U^{k-2,m}_{1s}+c_{25}\tilde U^{k-1,m-1}_{1s}$$

$$+c_{24}\Big(\tilde U^{k-1,m-1}_{2s}+\tilde U^{k-2,m}_{3s}\Big)+c_{42}\tilde U^{k-1,m-1}_{2s}+c_{43}\tilde U^{k,m-2}_{3s}+c_{46}\tilde U^{k-1,m-1}_{1s}$$

$$+c_{45}\tilde U^{k,m-2}_{1s}+c_{44}\Big(\tilde U^{k,m-2}_{2s}+\tilde U^{k-1,m-1}_{3s}\Big), \tag{3.5.20}$$

$$\Upsilon^{k,m}_3=c_{42}\tilde U^{k-2,m}_{2s}+c_{43}\tilde U^{k-1,m-1}_{3s}+c_{46}\tilde U^{k-2,m}_{1s}+c_{45}\tilde U^{k-1,m-1}_{1s}+c_{32}\tilde U^{k-1,m-1}_{2s}$$

$$+c_{44}\Big(\tilde U^{k-1,m-1}_{2s}+\tilde U^{k-2,m}_{3s}\Big)+c_{33}\tilde U^{k,m-2}_{3s}+c_{36}\tilde U^{k-1,m-1}_{1s}$$

$$+c_{35}\tilde U^{k,m-2}_{1s}+c_{34}\Big(\tilde U^{k,m-2}_{2s}+\tilde U^{k-1,m-1}_{3s}\Big). \tag{3.5.21}$$

The solutions of the problems (3.5.17)- (3.5.18) for $j = 1, 2, 3$ will be

$$\tilde{U}_{js}^{k,m}(t) = \int_0^t (t-\tau)\Upsilon_j(\tau)d\tau + \tilde{h}_{js}^{k,m}t, \quad j = 1, 2, 3. \tag{3.5.22}$$

Using (3.5.22) all coefficients $\tilde{\mathbf{U}}_s^{k,m}$ of $\tilde{\mathbf{U}}_s$ can be found. Solution of the IVP (3.5.7)-(3.5.10) can be obtained as follows

$$\tilde{U}_{js}(\upsilon_2, \upsilon_3, t) = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \tilde{U}_{js}^{k,m}(t)\upsilon_2^k\upsilon_3^m \tag{3.5.23}$$

where $\tilde{U}_{js}^{k,m}(t), \quad j = 1, 2, 3$ are defined in equations (3.5.22).

Procedure of Finding $\mathbf{U}^{k,m}$

The procedure of finding $\tilde{\mathbf{U}}_s^{k,m}$; $s = 1, 2, 3$, consists of the sequence of the following iterative steps of constructing some formulae from the others using the relation (3.5.22).

Step 1:

$$\tilde{\mathbf{U}}_s^{-2,m} = \tilde{\mathbf{U}}_s^{k,-2} = \tilde{\mathbf{U}}_s^{-1,m} = \tilde{\mathbf{U}}_s^{k,-1} = 0$$

when $k = -2,\ -1,\ 0, ...; m = -2,\ -1,\ 0, ...$

Step 2: using zero values from step 1 we compute

$$\tilde{\mathbf{U}}_s^{0,m},\ \tilde{\mathbf{U}}_s^{k,0},\ k = 0,\ 1,\ 2,\ ...;\ m = 0,\ 1,\ 2,\ ....$$

Step 3: from the relations obtained on previous steps we compute

$$\tilde{\mathbf{U}}_s^{1,m},\ \tilde{\mathbf{U}}_s^{k,1}, \quad k = 1,\ 2,\ ...;\ m = 1,\ 2,\ ....$$

... ...

Step p**:** from the relations obtained on previous steps we compute

$$\tilde{\mathbf{U}}_s^{p,m},\ \tilde{\mathbf{U}}_s^{k,p}, \quad \text{for} \quad k = p,\ p+1,\ p+2,\ ...;\ m = p,\ p+1,\ p+2,\ ....$$

Inverse Fourier Transform and the Solutions of Original IVP

Applying inverse Fourier transform to $\tilde{U}_{js}(\upsilon_2, \upsilon_3, t)$ defined by the formula (3.5.23), solution $\mathbf{U}_s(x,t)$; $s = 1,2,3$ of problem (3.5.1), (3.5.2) can be obtained for 2-D Case with with $\mathbf{h}_s(x,t) = \frac{1}{\rho}\mathbf{e}^s\delta(x_2, x_3)$.

3.5.3 Simulation of 2-D Fundamental Solution

In this part of the section we consider problem (3.5.1), (3.5.2) for different types of anisotropy. The aim is to create simulations of elastic wave propagations, obtained according to the method we have explained, in different crystals under same conditions.

We study problem (3.5.1), (3.5.2) for $\mathbf{h}_1(x,t) = \frac{1}{\rho}\mathbf{e}^1\delta(x_2, x_3)$ and in some crystals from different types of anisotropy. The name of these crystals, their densities $(\mathrm{gr/cm^3})$, elastic moduli $(10^{12}\ \mathrm{dyn/cm^2})$ and type of anisotropy are as follows:

1. Zinc (Hexagonal): $\rho = 7.134$, $c_{11} = 1.6368$, $c_{12} = 0.3640$, $c_{13} = 0.53$, $c_{33} = 0.6347$, $c_{55} = 0.3879$, $c_{22} = c_{11}$, $c_{23} = c_{13}$, $c_{44} = c_{55}$, $c_{66} = (c_{11} - c_{12})/2$.

2. (Monoclinic): $\rho = 2.649$, $c_{11} = 8.67$, $c_{12} = -0.83$, $c_{13} = 2.71$, $c_{14} = -0.37$, $c_{15} = 0$, $c_{16} = 0$, $c_{21} = -0.83$, $c_{22} = 12.98$, $c_{23} = -0.74$, $c_{24} = 0.57$, $c_{25} = 0$, $c_{26} = 0$, $c_{31} = 2.71$, $c_{32} = -0.74$, $c_{33} = 10.28$, $c_{34} = 0.99$, $c_{35} = 0$, $c_{36} = 0$, $c_{41} = -0.37$, $c_{42} = 0.57$, $c_{43} = 0.99$, $c_{44} = 3.86$, $c_{45} = 0$, $c_{46} = 0$, $c_{51} = 0$, $c_{52} = 0$, $c_{53} = 0$, $c_{54} = 0$, $c_{55} = 6.88$, $c_{56} = 0.25$, $c_{61} = 0$, $c_{62} = 0$, $c_{63} = 0$, $c_{64} = 0$, $c_{65} = 0.25$, $c_{66} = 2.9$.

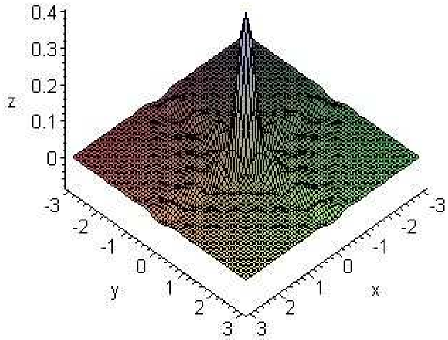
3. Copper Sulphate Pentahydrate (Triclinic): $\rho = 2.649$, $c_{11} = 5.65$, $c_{12} = 2.65$, $c_{13} = 3.21$, $c_{14} = -0.33$, $c_{15} = -0.08$, $c_{16} = -0.39$, $c_{21} = 2.65$, $c_{22} = 4.33$, $c_{23} = 3.47$, $c_{24} = -0.07$, $c_{25} = -0.21$, $c_{26} = 0.02$, $c_{31} = 3.21$, $c_{32} = 3.47$, $c_{33} = 5.69$, $c_{34} = -0.44$, $c_{35} = -0.21$, $c_{36} = -0.16$, $c_{41} = -0.33$, $c_{42} = -0.07$, $c_{43} = -0.44$, $c_{44} = 1.73$, $c_{45} = 0.09$, $c_{46} = 0.03$, $c_{51} = -0.08$, $c_{52} = -0.21$, $c_{53} = -0.21$, $c_{54} = 0.09$, $c_{55} = 1.22$, $c_{56} = -0.26$, $c_{61} = -0.39$, $c_{62} = 0.02$, $c_{63} = -0.16$, $c_{64} = 0.03$, $c_{65} = -0.26$, $c_{66} = 1$.

Using the method of the Section (3.5), we compute the elements of the fundamental solution matrix $\mathcal{U}(x_2, x_3, t)$ whose $s-$th column is $\mathbf{U}_s(x_2, x_3, t)$ with the components $\mathbf{U}_s(x_2, x_3, t) = (U_{1s}(x_2, x_3, t), U_{2s}(x_2, x_3, t), U_{3s}(x_2, x_3, t))^T$. In figure3.5, we consider (3.3.3), (3.3.4) for $s = 1$ ($\mathbf{e}^s = (1, 0, 0)$) and we draw the graph of the first component of $\mathbf{U}_s(x_2, x_3, t)$.

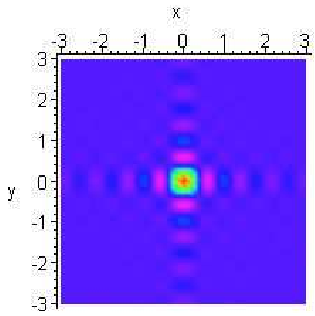
Example: We study problem (3.5.1), (3.5.2) inside the crystal Zinc for $\mathbf{h}_1(x, t) = \frac{1}{\rho}\mathbf{e}^1\delta(x_2, x_3)$.

Example: We study problem (3.5.1), (3.5.2) inside the monoclinic crystal for $\mathbf{h}_1(x, t) = \frac{1}{\rho}\mathbf{e}^1\delta(x_2, x_3)$.

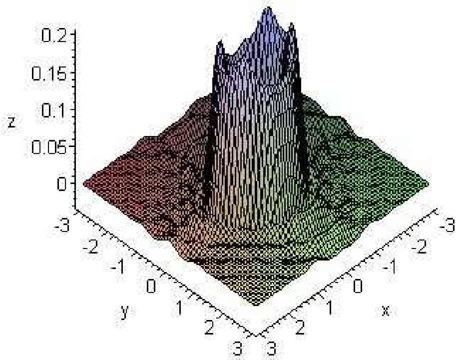
Example: We study problem (3.5.1), (3.5.2) inside the triclinic crystal for $\mathbf{h}_1(x, t) = \frac{1}{\rho}\mathbf{e}^1\delta(x_2, x_3)$.



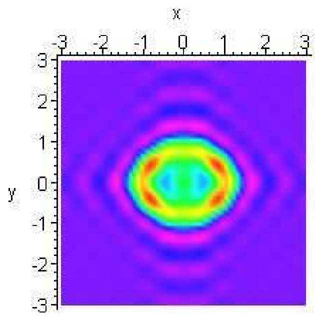
(a) 3-D plot of $U_{11}(x_2, x_3, 1/2)$



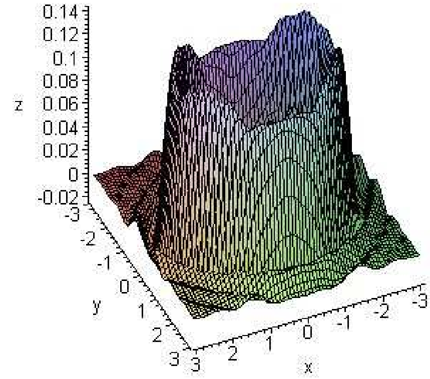
(b) 2-D plot of $U_{11}(x_2, x_3, 1/2)$



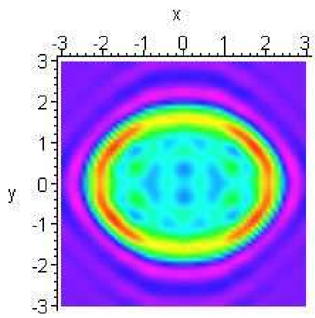
(c) 3-D plot of $U_{11}(x_2, x_3, 394/100)$



(d) 2-D plot of $U_{11}(x_2, x_3, 394/100)$

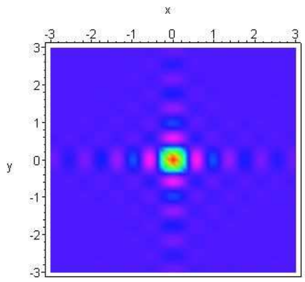


(e) 3-D plot of $U_{11}(x_2, x_3, 75/10)$

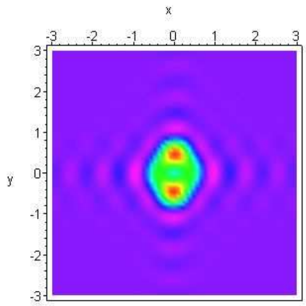


(f) 2-D plot of $U_{11}(x_2, x_3, 75/10)$

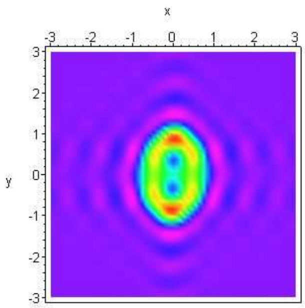
Figure 3.5 3-D and 2-D level plots of U_{11} hexagonal media.



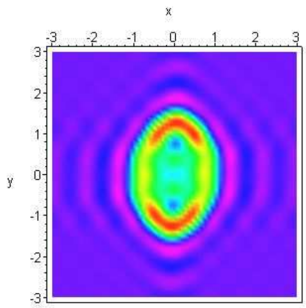
(a) 2-D plot of $U_{11}(x_2, x_3, 87/1000)$



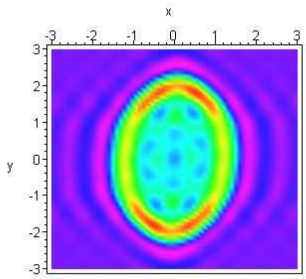
(b) 2-D plot of $U_{11}(x_2, x_3, 43/100)$



(c) 2-D plot of $U_{11}(x_2, x_3, 65/100)$

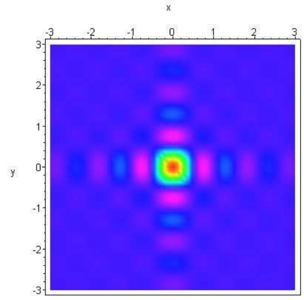


(d) 2-D plot of $U_{11}(x_2, x_3, 877/1000)$

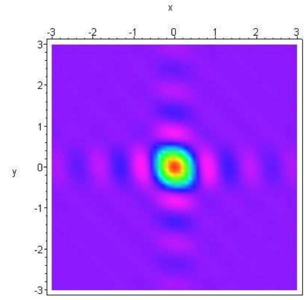


(e) 2-D plot of $U_{11}(x_2, x_3, 1315/1000)$

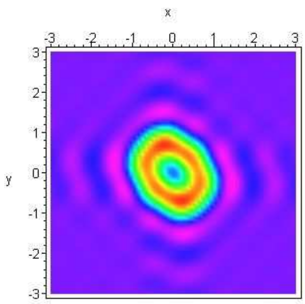
Figure 3.6 2-D level plots of U_{11} monoclinic media.



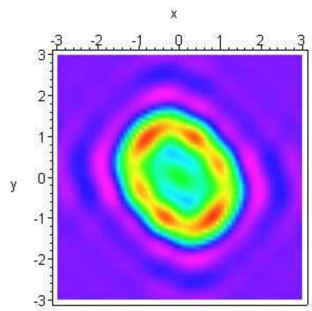
(a) 2-D plot of $U_{11}(x_2, x_3, 1/10)$



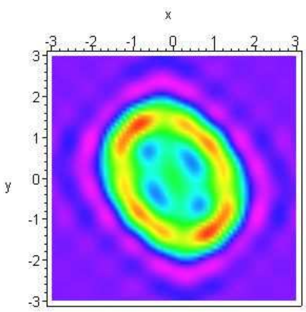
(b) 2-D plot of $U_{11}(x_2, x_3, 1/2)$



(c) 2-D plot of $U_{11}(x_2, x_3, 138/100)$



(d) 2-D plot of $U_{11}(x_2, x_3, 21/10)$



(e) 2-D plot of $U_{11}(x_2, x_3, 25/10)$

Figure 3.7   2-D level plots of U_{11} triclinic media.

Analysis of figures

In these figures 3.5-3.7 the first component of the first column of fundamental solution matrix $\mathcal{U}(x_2, x_3, t)$ that is $\mathbf{U}_{11}(x_2, x_3, t)$ in different media is presented. In figure 3.5, simulations of $\mathbf{U}_{11}(x_2, x_3, t)$ inside Hexagonal crystal are presented for varying values of t. 3-D plots of density of $\mathbf{U}_{11}(x_2, x_3, t)$ are presented in figure 3.5; (a),(c),(e) with horizontal axes x_2 and x_3, respectively. The vertical axis is the magnitude of $\mathbf{U}_{11}(x_2, x_3, t)$ for varying values of t. Figures 3.5; (b),(d),(f) are screen plot of 2-D level plots of the same surface $\mathbf{U}_{11}(x_2, x_3, t)$ i.e. a view on the surface $z = \mathbf{U}_{11}(x_2, x_3, 1/2)$ is presented in figures 3.5; (a) from the top of z-axis. In figures 3.6-3.7, fundamental solution of elastic system is considered in monoclinic and triclinic media respectively. These figures are screen shot of 2-D level plot of the surface $\mathbf{U}_{11}(x_2, x_3, t)$. In each figures wave front can be seen as a boundary and we can observe wave propagates according to time.

3.6 IVP for the System of Elasticity Depending on x_1, x_2, x_3 and t Variables

3.6.1 Reduction of System Depending on x_1, x_2, x_3 and t Variables to a First-Order Symmetric Hyperbolic System

In this section we explain the process of writing (3.1.15), (3.1.16) as a symmetric hyperbolic system when displacement vector depends on x_1, x_2, x_3 and t variables.

Let us consider IVP of elastic system when inhomogeneous term $\mathbf{F}(x_1, x_2, x_3, t)$ be given function depending x_2, x_3 and t variables and initial data $\mathbf{G}(x_1, x_2, x_3)$; $\mathbf{H}(x_1, x_2, x_3)$ be given functions depending x_2, x_3 variable.

Using denotations and renumarations defined in (3.2.1)-(3.2.5), the left-hand side of (3.1.15) can be written in vector form

$$\rho \frac{\partial^2 \mathbf{U}}{\partial t^2} = \rho \frac{\partial \bar{\mathbf{U}}}{\partial t} \tag{3.6.1}$$

Consider the term $\sum_{k=1}^{3} \frac{\partial \sigma_{jk}}{\partial x_k}$ on the right-hand side of (3.1.15). Applying rule (3.2.1) for $j = 1, 2, 3$ gives

$$\sum_{k=1}^{3} \frac{\partial \sigma_{1k}}{\partial x_k} = \frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} = \frac{\partial \sigma_1}{\partial x_1} + \frac{\partial \sigma_6}{\partial x_2} + \frac{\partial \sigma_5}{\partial x_3}, \tag{3.6.2}$$

$$\sum_{k=1}^{3} \frac{\partial \sigma_{2k}}{\partial x_k} = \frac{\partial \sigma_{21}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{23}}{\partial x_3} = \frac{\partial \sigma_6}{\partial x_1} + \frac{\partial \sigma_2}{\partial x_2} + \frac{\partial \sigma_4}{\partial x_3}, \tag{3.6.3}$$

$$\sum_{k=1}^{3} \frac{\partial \sigma_{3k}}{\partial x_k} = \frac{\partial \sigma_{31}}{\partial x_1} + \frac{\partial \sigma_{32}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} = \frac{\partial \sigma_5}{\partial x_1} + \frac{\partial \sigma_4}{\partial x_2} + \frac{\partial \sigma_3}{\partial x_3}. \tag{3.6.4}$$

Using the vector $\mathbf{T}$, (3.6.2)-(3.6.4) takes the form

$$\begin{aligned}
\sum_{k=1}^{3} \frac{\partial \sigma_{1k}}{\partial x_k} &= [1,0,0,0,0,0] \cdot \frac{\partial \mathbf{T}}{\partial x_1} + [0,0,0,0,0,1] \cdot \frac{\partial \mathbf{T}}{\partial x_2} + [0,0,0,0,1,0] \cdot \frac{\partial \mathbf{T}}{\partial x_3}, \\
\sum_{k=1}^{3} \frac{\partial \sigma_{2k}}{\partial x_k} &= [0,0,0,0,0,1] \cdot \frac{\partial \mathbf{T}}{\partial x_1} + [0,1,0,0,0,0] \cdot \frac{\partial \mathbf{T}}{\partial x_2} + [0,0,0,1,0,0] \cdot \frac{\partial \mathbf{T}}{\partial x_3}, \\
\sum_{k=1}^{3} \frac{\partial \sigma_{3k}}{\partial x_k} &= [0,0,0,0,1,0] \cdot \frac{\partial \mathbf{T}}{\partial x_1} + [0,0,0,1,0,0] \cdot \frac{\partial \mathbf{T}}{\partial x_2} + [0,0,1,0,0,0] \cdot \frac{\partial \mathbf{T}}{\partial x_3}.
\end{aligned}$$

Noting the coefficient vectors of terms $\dfrac{\partial \mathbf{T}}{\partial x_k}$, $k = 1, 2, 3$ we introduce the matrices

$$\mathbf{A}_1^1 = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix}, \qquad \mathbf{A}_2^1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \end{bmatrix},$$

$$\mathbf{A}_3^1 = \begin{bmatrix} 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{bmatrix} \tag{3.6.5}$$

and represent $\sum_{k=1}^{3} \dfrac{\partial \sigma_{jk}}{\partial x_k}$, $j = 1, 2, 3$ in the form

$$\begin{bmatrix} \sum_{k=1}^{3} \dfrac{\partial \sigma_{1k}}{\partial x_k} \\ \sum_{k=1}^{3} \dfrac{\partial \sigma_{2k}}{\partial x_k} \\ \sum_{k=1}^{3} \dfrac{\partial \sigma_{3k}}{\partial x_k} \end{bmatrix} = -\sum_{k=1}^{3} \mathbf{A}_k^1 \frac{\partial \mathbf{T}}{\partial x_k}. \tag{3.6.6}$$

From (3.6.1) and (3.6.6) it follows that we can rewrite (3.1.15) as

$$\rho \frac{\partial \bar{\mathbf{U}}}{\partial t} + \sum_{k=1}^{3} \mathbf{A}_k^1 \frac{\partial \mathbf{T}}{\partial x_k} = \mathbf{f}, \tag{3.6.7}$$

where $\mathbf{f} = (f_1, f_2, f_3)$.

Relation (3.1.17) can be written as two summations

$$\sigma_{jk} = \sum_{\substack{l,m=1 \\ l=m}}^{3} c_{jklm}\epsilon_{lm} + \sum_{\substack{l,m=1 \\ l\neq m}}^{3} c_{jklm}\epsilon_{lm}, \quad j, k = 1, 2, 3. \tag{3.6.8}$$

Denoting the pair of indices (j, k) with α, $\alpha = 1, \ldots, 6$, (l, m) with β, $\beta = 1, \ldots, 6$, according to rule (3.2.1), relation (3.6.8) can be written as

$$\sigma_\alpha = \sum_{\beta=1}^{3} c_{\alpha\beta}\epsilon_\beta + 2\sum_{\beta=4}^{6} c_{\alpha\beta}\epsilon_\beta, \tag{3.6.9}$$

or in terms of vectors $\mathbf{T}$ and $\mathbf{Y}$ as

$$\mathbf{T} = \mathbf{C}\mathbf{Y}, \tag{3.6.10}$$

where $\mathbf{C} = (c_{\alpha\beta})_{6\times 6}$ is stiffness matrix defined with (3.1.13) that is symmetric and positive definite (See, Section 3.1).

Taking derivative of (3.6.10) with respect to t and multiplying both sides by the inverse of $\mathbf{C}$, denoted $\mathbf{C}^{-1}$, we find

$$\mathbf{C}^{-1}\frac{\partial \mathbf{T}}{\partial t} = \frac{\partial \mathbf{Y}}{\partial t}. \tag{3.6.11}$$

Differentiating (3.1.18) with respect to t and using (3.2.4) the following relations can easily be obtained

$$\begin{aligned} &\frac{\partial \epsilon_j}{\partial t} = \frac{\partial \bar{U}_j}{\partial x_j}, \qquad j = 1,2,3; \qquad && 2\frac{\partial \epsilon_4}{\partial t} = \frac{\partial \bar{U}_2}{\partial x_3} + \frac{\partial \bar{U}_3}{\partial x_2}, \\ &2\frac{\partial \epsilon_5}{\partial t} = \frac{\partial \bar{U}_1}{\partial x_3} + \frac{\partial \bar{U}_3}{\partial x_1}, && 2\frac{\partial \epsilon_6}{\partial t} = \frac{\partial \bar{U}_1}{\partial x_2} + \frac{\partial \bar{U}_2}{\partial x_1}. \end{aligned} \tag{3.6.12}$$

Using these formulas we get

$$-\frac{\partial \mathbf{Y}}{\partial t} = \sum_{j=1}^{3} (\mathbf{A}_j^1)^* \frac{\partial \bar{\mathbf{U}}}{\partial x_j}, \tag{3.6.13}$$

where $\mathbf{A}_j^1$ are given in (3.6.5). Substitution of (3.6.13) into (3.6.11) results the expression

$$\mathbf{C}^{-1}\frac{\partial \mathbf{T}}{\partial t} + \sum_{j=1}^{3} (\mathbf{A}_j^1)^* \frac{\partial \bar{\mathbf{U}}}{\partial x_j} = 0. \tag{3.6.14}$$

Let $\mathbf{V}$ and $\mathbf{F}$ be vectors with 9 components in the form

$$\mathbf{V} = \begin{bmatrix} \bar{\mathbf{U}} \\ \mathbf{T} \end{bmatrix}, \qquad \mathbf{F} = \begin{bmatrix} \mathbf{f} \\ \mathbf{0}_{6,1} \end{bmatrix}, \tag{3.6.15}$$

$\mathbf{A}_j$, $j = 0, 1, 2, 3$ be the 9×9 matrices

$$\mathbf{A}_0 = \begin{bmatrix} \rho \mathbf{I}_3 & \mathbf{0}_{3,6} \\ \mathbf{0}_{6,3} & \mathbf{C}^{-1} \end{bmatrix}, \quad \mathbf{A}_j = \begin{bmatrix} \mathbf{0}_{3,3} & \mathbf{A}_j^1 \\ (\mathbf{A}_j^1)^* & \mathbf{0}_{6,6} \end{bmatrix}, \qquad j = 1, 2, 3, \tag{3.6.16}$$

where $\mathbf{I}_m$ is the unit matrix of order $m \times m$ and $\mathbf{0}_{l,m}$ is the zero matrix of order $l \times m$. Since $\mathbf{C}$ is symmetric and positive definite matrix then $\mathbf{C}^{-1}$ is symmetric and positive definite matrix (see, appendix). Notice that the matrices $\mathbf{A}_j$, $j = 0, 1, 2, 3$ are also symmetric and positive definite.

Using these notations we can combine (3.6.7) and (3.6.14) to obtain a first-order system

$$\mathbf{A}_0 \frac{\partial \mathbf{V}}{\partial t} + \sum_{j=1}^{3} \mathbf{A}_j \frac{\partial \mathbf{V}}{\partial x_j} = \mathbf{F}. \tag{3.6.17}$$

We finish this section by the following lemma.

Lemma 3.6.1. *(see, (Courant & Hilbert, 1979), p.593-594), (see, (Yakhno & Akmaz, 2005) System (3.6.17) can be transformed into the following form*

$$\mathbf{I}_9 \frac{\partial \tilde{\mathbf{V}}}{\partial t} + \sum_{j=1}^{3} \tilde{\mathbf{A}}_j \frac{\partial \tilde{\mathbf{V}}}{\partial x_j} = \tilde{\mathbf{F}}, \tag{3.6.18}$$

which is an symmetric hyperbolic system.

Proof. Consider the symmetric positive-definite matrix $\mathbf{C}$. There exists a symmetric positive-definite matrix $\mathbf{M}$ such that $\mathbf{C}^{-1} = \mathbf{M}^2$ (see Theorem A.1.2 of Section A.1), and the matrix $\mathbf{M}^{-1}$, which is inverse of $\mathbf{M}$, is also symmetric (see Theorem A.1.1 of Section A.1).

Using these facts we define the matrix

$$\mathbf{S} = \begin{bmatrix} \rho^{-\frac{1}{2}} \mathbf{I}_3 & \mathbf{0}_{3,6} \\ \mathbf{0}_{6,3} & \mathbf{M}^{-1} \end{bmatrix}, \tag{3.6.19}$$

and denote the vector $\mathbf{V}$ as

$$\mathbf{V} = \mathbf{S}\tilde{\mathbf{V}}. \tag{3.6.20}$$

Substituting (3.6.20) into (3.6.17) and multiplying the resulting formula with matrix $\mathbf{S}$ from left-hand side we obtain (3.6.18), where

$$\mathbf{S}\mathbf{A}_0\mathbf{S} = \mathbf{I}_9, \quad \tilde{\mathbf{A}}_j = \mathbf{S}\mathbf{A}_j\mathbf{S}, \quad \tilde{\mathbf{F}} = \mathbf{S}\mathbf{F}. \tag{3.6.21}$$

Since $\mathbf{S}$ and $\mathbf{A}_j,\ j = 1,2,3$ are symmetric, the matrices $\tilde{\mathbf{A}}_j,\ j = 1,2,3$ are also symmetric (see Theorem A.1.3 of Section A.1), which implies that (3.6.18) is symmetric hyperbolic system. □

3.6.2 Existence and Uniqueness of a Classical Solution of IVP for System Depending on x_1, x_2, x_3 and t Variables. Properties of Solutions.

In this section we prove the existence of unique classical solutions depending on x_2, x_3 and t variables. Also a property for existence of unique solution with finite support is proved when inhomogeneous term and initial data are infinitely differentiable and have finite support.

Theorem 3.6.2. *Let T be a fixed positive number, $\boldsymbol{G}(x_1, x_2, x_3)$; $\boldsymbol{H}(x_1, x_2, x_3)$ and $\boldsymbol{F}(x_1, x_2, x_3, t)$ be given functions such that $\boldsymbol{G}(x_1, x_2, x_3) \in H^4(\mathbb{R}^3)$; $\boldsymbol{H}(x_1, x_2, x_3) \in H^5(\mathbb{R}^3)$ and $\boldsymbol{F}(x_1, x_2, x_3, t) \in C([0,T]; H^4(\mathbb{R}^3))$. Then there exists a unique solution of Cauchy problem (3.1.15), (3.1.16)*

$$\mathbf{U}(x_1, x_2, x_3, t) \in C^1([0,T]; H^4(\mathbb{R}^3)) \cap C^2([0,T]; H^3(\mathbb{R}^3)).$$

Proof. Cauchy problem defined with the equations (3.1.15), (3.1.16) can be written as an IVP for symmetric first order hyperbolic system

$$\frac{\partial \bar{\mathbf{V}}}{\partial t} + \sum_{k=1}^{3} \widetilde{A}_k \frac{\partial \bar{\mathbf{V}}}{\partial x_k} = \widetilde{\mathbf{F}}, \quad x \in \mathbb{R}^3, \ \ t > 0, \tag{3.6.22}$$

$$\bar{\mathbf{V}}(x, 0) = \mathbf{V}_0(x_1, x_2, x_3), \tag{3.6.23}$$

where $\bar{\mathbf{V}},\ \widetilde{\mathbf{F}},\ \mathbf{V}_0(x_1, x_2, x_3)$ are defined by formulae

$$\bar{\mathbf{V}} = S\begin{bmatrix} \bar{\mathbf{U}} \\ \mathbf{T} \end{bmatrix}, \qquad \widetilde{\mathbf{F}} = S\begin{bmatrix} \mathbf{F} \\ \mathbf{0}_{6,1} \end{bmatrix}, \qquad \mathbf{V}_0 = S\begin{bmatrix} \bar{\mathbf{U}} \\ \mathbf{T} \end{bmatrix}_{t=0}, \tag{3.6.24}$$

where S is defined by the matrix (3.6.19) and $\mathbf{T},\ \bar{\mathbf{U}},\ \widetilde{A}_k;\ k = 1, 2, 3$ defined by the equations (3.2.2), (3.2.4), (3.6.21).

Using existence theorem (Mizohata, 1973)(p. 335) (also, appendix b) for the symmetric hyperbolic first order system (3.6.22), (3.6.23), it can be shown that there exists a unique solution of (3.1.15), (3.1.16) in the class

$$\mathbf{U}(x,t) \in C^1([0,T]; H^4(\mathbb{R}^n)) \cap C^2([0,T]; H^3(\mathbb{R}^n))$$

for any given initial data $\mathbf{G}(x) \in H^4(\mathbb{R}^n);\ \mathbf{H}(x) \in H^5(\mathbb{R}^n)$ and inhomogeneous term $\mathbf{F}(x,t) \in C([0,T]; H^4(\mathbb{R}^n))$. □

Theorem 3.6.3. *Let T be a fixed positive number, $\boldsymbol{G}(x);\ \boldsymbol{H}(x)$ and $\boldsymbol{F}(x,t)$ be given functions such that $\boldsymbol{G}(x) \in C_0^\infty(\mathbb{R}^3);\ \boldsymbol{H}(x) \in C_0^\infty(\mathbb{R}^3)$ and $\boldsymbol{F}(x,t) \in C([0,T]; C_0^\infty(\mathbb{R}^3))$. Then the solution $\mathbf{U}(x,t)$ of Cauchy problem (3.1.15), (3.1.16) belongs to*

$$C^2([0,T]; C_0^\infty(\mathbb{R}^3)).$$

Proof. Using Theorem 3.6.2 it can be found that if $D^\alpha\psi \in H^4(\mathbb{R}^3);\ D^\alpha\mathbf{H}(x) \in H^5(\mathbb{R}^3)$ and $D^\alpha\mathbf{F}(x,t) \in C([0,T]; H^4(\mathbb{R}^3))$ where T is a fixed positive number and for an arbitrary multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ where $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$, $\alpha_i;\ i = 1, 2, 3$ are nonnegative integers, $D^\alpha = \dfrac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\partial x_3^{\alpha_3}}$. Then $D^\alpha\mathbf{U}$ belongs to the class

$$D^\alpha\mathbf{U}(x,t) \in C^1([0,T]; H^4(\mathbb{R}^3)) \cap C^2([0,T]; H^3(\mathbb{R}^3)).$$

Using this fact and applying Sobolev's lemma (See, appendixA.3) it can be proved that

$$\mathbf{U}(x,t) \in C^2([0,T]; C^\infty(\mathbb{R}^3)).$$

To prove that the function has a compact support, let us consider reduction of the Cauchy problem (3.1.15), (3.1.16) to the first order symmetric hyperbolic system (3.6.22), (3.6.23) that is where all matrices $\widetilde{A}_k$ are real symmetric matrices with constant elements. Let T be a fixed positive number, $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$ be a parameter; $A(\xi)$ be a matrix defined by $A(\xi) = \sum_{k=1}^{3} \widetilde{A}_k \xi_k;\ \lambda_i(\xi),\ i = 1, 2, ..., 9$ be eigenvalues of $A(\xi)$. The positive number M is defined by

$$M = \max_{i=1,2,...,9} \max_{|\xi|=1} |\lambda_i(\xi)|. \tag{3.6.25}$$

We claim that M is the upper bound on the speed of waves in any direction.

Using T and M we define the following domains

$$S(x_0, h) = \{x \in \mathbb{R}^3 : |x - x_0| \leq M(T - h)\},\ 0 \leq h \leq T$$
$$\Gamma(x_0, T) = \{(x, t) : 0 \leq t \leq T,\ |x - x_0| \leq M(T - t)\}$$
$$R(x_0, h) = \{(x, t) : 0 \leq t \leq h,\ |x - x_0| = M(T - t)\}$$

Here $\Gamma(x_0, T)$ is the conoid with vertex (x_0, T); $S(x_0, h)$ is the surface constructed by the intersection of the plane $t = h$ and the conoid $\Gamma(x_0, T)$; $R(x_0, h)$ is the lateral surface of the conoid $\Gamma(x_0, T)$ bounded by $S(x_0, 0)$ and $S(x_0, h)$. Let Ω be the region in $\mathbb{R}^3 \times (0, \infty)$ bounded by $S(x_0, 0)$, $S(x_0, h)$ and $R(x_0, h)$ with boundary $\partial\Omega = S(x_0, 0) \cup S(x_0, h) \cup R(x_0, h)$.

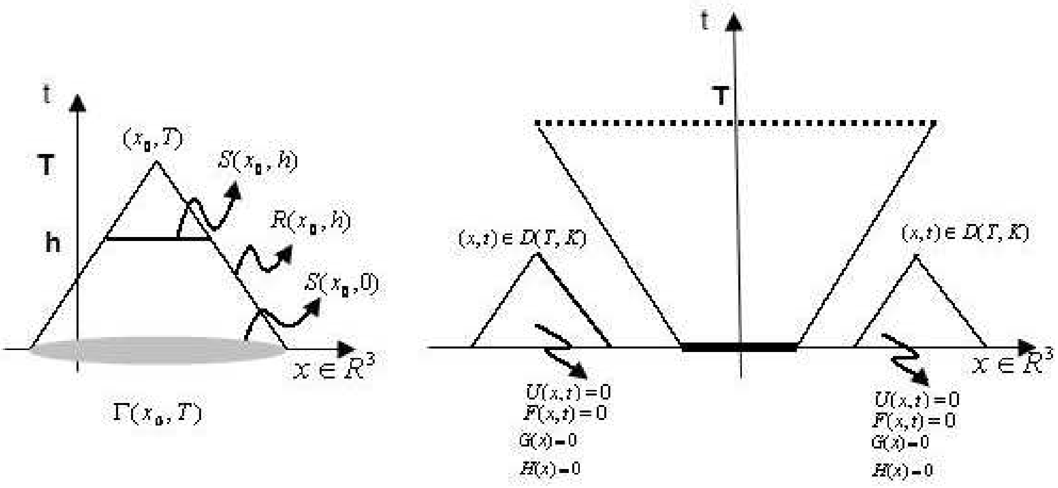


Figure 3.8 Domains of Dependence

Applying the reasoning similar to (Courant & Hilbert, 1979) (p. 652-661)(see also

Appendix B) we find the following estimate for the solution of (3.2.27), (3.2.28)

$$\int_{S(h)} |\bar{\mathbf{V}}(x,h)|^2 dx \le e^h \Big[\int_{S(0)} |\mathbf{V_0}(x)|^2 dx + \int_0^h \Big(\int_{S(t)} |\tilde{\mathbf{F}}(x,t)|^2 dx \Big) dt \Big]. \quad (3.6.26)$$

Let us define $P(K) = \{x \in \mathbb{R}^3 : |x| \le K\}$. Since $\mathbf{G}(x) \in C_0^\infty(\mathbb{R}^3)$; $\mathbf{H}(x) \in C_0^\infty(\mathbb{R}^3)$; and $\mathbf{F}(x,t) \in C([0,T]; C_0^\infty(\mathbb{R}^3))$ then there exists $K > 0$ such that $supp\ \mathbf{G} \subseteq P(K)$, $supp\ \mathbf{H} \subseteq P(K)$ and $\mathbf{F}(x,t)$ as a function of the variable x, has a finite support which is located in $P(K)$ for any fixed t from $[0,T]$.

Also let us denote

$$D(T,K) = \{(x,t) : 0 \le t \le T,\ \Gamma(x,t) \cap P(K) = \emptyset\}.$$

If $(x,t) \in D(T,K)$ then $\mathbf{U}(x,t) = 0$. This means $\mathbf{U}(x,t) = 0$ for any $t \in [0,T]$ and $|x| > MT + K$.

Hence, $supp\ \mathbf{U} \subseteq P(MT+K)$. As a result $\mathbf{U}(x,t)$ belongs to the class

$$\mathbf{U}(x,t) \in C^2([0,T]; C_0^\infty(\mathbb{R}^3)).$$

□

3.6.3 IVP for the System Depending on x_1, x_2, x_3 and t Variables

In Section 3.6.2 we have shown that IVP of elastic system (3.1.15), (3.1.16) has a unique solution $\mathbf{U}(x_1,x_2,x_3,t)$ belongs to $C^2([0,T]; C_0^\infty(\mathbb{R}^3))$ when inhomogeneous term $\mathbf{F}(x_1,x_2,x_3,t) \in C([0,T]; C_0^\infty(\mathbb{R}^3))$ be given function depending x_2, x_3 and t variables and initial data $\mathbf{G}(x_1,x_2,x_3) \in C_0^\infty(\mathbb{R}^3)$; $\mathbf{H}(x_1,x_2,x_3) \in C_0^\infty(\mathbb{R}^3)$ are given functions depending x_1, x_2, x_3 variables. In this case we can rewrite IVP of elastic system (3.1.15), (3.1.16)

$$\rho\frac{\partial^2 U_1}{\partial t^2} = c_{11}\frac{\partial^2 U_1}{\partial x_1^2} + c_{12}\frac{\partial^2 U_2}{\partial x_1 \partial x_2} + c_{13}\frac{\partial^2 U_3}{\partial x_1 \partial x_3}$$
$$+c_{16}\Big(\frac{\partial^2 U_1}{\partial x_2 \partial x_1} + \frac{\partial^2 U_2}{\partial x_1^2}\Big) + c_{15}\Big(\frac{\partial^2 U_1}{\partial x_1 \partial x_3} + \frac{\partial^2 U_3}{\partial x_1^2}\Big)$$
$$+c_{14}\Big(\frac{\partial^2 U_2}{\partial x_1 \partial x_3} + \frac{\partial^2 U_3}{\partial x_1 \partial x_2}\Big)$$

$$+c_{61}\frac{\partial^2 U_1}{\partial x_1 \partial x_2} + c_{62}\frac{\partial^2 U_2}{\partial x_2^2} + c_{63}\frac{\partial^2 U_3}{\partial x_2 \partial x_3}$$
$$+c_{66}\Big(\frac{\partial^2 U_2}{\partial x_1 \partial x_2} + \frac{\partial^2 U_1}{\partial x_2^2}\Big) + c_{65}\Big(\frac{\partial^2 U_1}{\partial x_2 \partial x_3} + \frac{\partial^2 U_3}{\partial x_1 \partial x_2}\Big)$$
$$+c_{64}\Big(\frac{\partial^2 U_2}{\partial x_2 \partial x_3} + \frac{\partial^2 U_3}{\partial x_2^2}\Big)$$

$$+c_{51}\frac{\partial^2 U_1}{\partial x_1 \partial x_3} + c_{52}\frac{\partial^2 U_2}{\partial x_2 \partial x_3} + c_{53}\frac{\partial^2 U_3}{\partial x_3^2}$$
$$+c_{56}\Big(\frac{\partial^2 U_1}{\partial x_2 \partial x_3} + \frac{\partial^2 U_2}{\partial x_1 \partial x_3}\Big) + c_{55}\Big(\frac{\partial^2 U_1}{\partial x_3^2} + \frac{\partial^2 U_3}{\partial x_1 \partial x_3}\Big) \tag{3.6.27}$$
$$+c_{54}\Big(\frac{\partial^2 U_2}{\partial x_3^2} + \frac{\partial^2 U_3}{\partial x_2 \partial x_3}\Big) + F_1,$$

$$\rho\frac{\partial^2 U_2}{\partial t^2} = c_{61}\frac{\partial^2 U_1}{\partial x_1^2} + c_{62}\frac{\partial^2 U_2}{\partial x_1 \partial x_2} + c_{63}\frac{\partial^2 U_3}{\partial x_1 \partial x_3}$$
$$+c_{66}\Big(\frac{\partial^2 U_1}{\partial x_2 \partial x_1} + \frac{\partial^2 U_2}{\partial x_1^2}\Big) + c_{65}\Big(\frac{\partial^2 U_1}{\partial x_1 \partial x_3} + \frac{\partial^2 U_3}{\partial x_1^2}\Big)$$
$$+c_{64}\Big(\frac{\partial^2 U_2}{\partial x_1 \partial x_3} + \frac{\partial^2 U_3}{\partial x_1 \partial x_2}\Big)$$

$$+c_{21}\frac{\partial^2 U_1}{\partial x_1 \partial x_2}+c_{22}\frac{\partial^2 U_2}{\partial x_2^2}+c_{23}\frac{\partial^2 U_3}{\partial x_2 \partial x_3}$$

$$+c_{26}\Big(\frac{\partial^2 U_2}{\partial x_1 \partial x_2}+\frac{\partial^2 U_1}{\partial x_2^2}\Big)+c_{25}\Big(\frac{\partial^2 U_1}{\partial x_2 \partial x_3}+\frac{\partial^2 U_3}{\partial x_1 \partial x_2}\Big)$$

$$+c_{24}\Big(\frac{\partial^2 U_2}{\partial x_2 \partial x_3}+\frac{\partial^2 U_3}{\partial x_2^2}\Big)$$

$$+c_{41}\frac{\partial^2 U_1}{\partial x_1 \partial x_3}+c_{42}\frac{\partial^2 U_2}{\partial x_2 \partial x_3}+c_{43}\frac{\partial^2 U_3}{\partial x_3^2}$$

$$+c_{46}\Big(\frac{\partial^2 U_1}{\partial x_2 \partial x_3}+\frac{\partial^2 U_2}{\partial x_1 \partial x_3}\Big)+c_{45}\Big(\frac{\partial^2 U_1}{\partial x_3^2}+\frac{\partial^2 U_3}{\partial x_1 \partial x_3}\Big) \qquad (3.6.28)$$

$$+c_{44}\Big(\frac{\partial^2 U_2}{\partial x_3^2}+\frac{\partial^2 U_3}{\partial x_2 \partial x_3}\Big)+F_2,$$

$$\rho\frac{\partial^2 U_3}{\partial t^2}=c_{51}\frac{\partial^2 U_1}{\partial x_1^2}+c_{52}\frac{\partial^2 U_2}{\partial x_1 \partial x_2}+c_{53}\frac{\partial^2 U_3}{\partial x_1 \partial x_3}$$

$$+c_{56}\Big(\frac{\partial^2 U_1}{\partial x_2 \partial x_1}+\frac{\partial^2 U_2}{\partial x_1^2}\Big)+c_{55}\Big(\frac{\partial^2 U_1}{\partial x_1 \partial x_3}+\frac{\partial^2 U_3}{\partial x_1^2}\Big)$$

$$+c_{54}\Big(\frac{\partial^2 U_2}{\partial x_1 \partial x_3}+\frac{\partial^2 U_3}{\partial x_1 \partial x_2}\Big)$$

$$+c_{41}\frac{\partial^2 U_1}{\partial x_1 \partial x_2}+c_{42}\frac{\partial^2 U_2}{\partial x_2^2}+c_{43}\frac{\partial^2 U_3}{\partial x_2 \partial x_3}$$

$$+c_{46}\Big(\frac{\partial^2 U_2}{\partial x_1 \partial x_2}+\frac{\partial^2 U_1}{\partial x_2^2}\Big)+c_{45}\Big(\frac{\partial^2 U_1}{\partial x_2 \partial x_3}+\frac{\partial^2 U_3}{\partial x_1 \partial x_2}\Big)$$

$$+c_{44}\Big(\frac{\partial^2 U_2}{\partial x_2 \partial x_3}+\frac{\partial^2 U_3}{\partial x_2^2}\Big)$$

$$+c_{31}\frac{\partial^2 U_1}{\partial x_1 \partial x_3} + c_{32}\frac{\partial^2 U_2}{\partial x_2 \partial x_3} + c_{33}\frac{\partial^2 U_3}{\partial x_3^2}$$

$$+c_{36}\Big(\frac{\partial^2 U_1}{\partial x_2 \partial x_3} + \frac{\partial^2 U_2}{\partial x_1 \partial x_3}\Big) + c_{35}\Big(\frac{\partial^2 U_1}{\partial x_3^2} + \frac{\partial^2 U_3}{\partial x_1 \partial x_3}\Big) \tag{3.6.29}$$

$$+c_{34}\Big(\frac{\partial^2 U_2}{\partial x_3^2} + \frac{\partial^2 U_3}{\partial x_2 \partial x_3}\Big) + F_3.$$

$$U_j(x_1, x_2, x_3, 0) = G_j(x_1, x_2, x_3), \quad \frac{\partial U_j(x_1, x_2, x_3, t)}{\partial t}\Big|_{t=0} = H_j(x_1, x_2, x_3), \tag{3.6.30}$$

where $x = (x_1, x_2, x_3) \in \mathbb{R}^3, \quad t > 0$. Simply, equations (3.6.27)-(3.6.30) can be written as follows

$$\rho\frac{\partial^2 \mathbf{U}}{\partial t^2} = L[\mathbf{U}] + \mathbf{F}(x_1, x_2, x_3, t), \quad t > 0, \tag{3.6.31}$$

$$\mathbf{U}(x_1, x_2, x_3, 0) = \mathbf{G}(x_1, x_2, x_3), \quad \frac{\partial \mathbf{U}}{\partial t}\Big|_{t=0} = \mathbf{H}(x_1, x_2, x_3), \tag{3.6.32}$$

where $x = (x_1, x_2, x_3) \in \mathbb{R}^3, \quad t > 0$ and $L[\mathbf{U}]$ is the matrix operator defined with components $L_{ij}U_j$ that is given with the formulas

$$\begin{aligned}
L_{11} &= c_{11}\partial^2_{x_1} + c_{16}\partial^2_{x_1x_2} + c_{15}\partial^2_{x_1x_3} + c_{14}\partial^2_{x_1x_3} + c_{61}\partial^2_{x_1x_2} + c_{66}\partial^2_{x_2} + c_{65}\partial^2_{x_2x_3} \\
&\quad + c_{51}\partial^2_{x_1x_3} + c_{56}\partial^2_{x_2x_3} + c_{55}\partial^2_{x_3}, \\
L_{12} &= c_{12}\partial^2_{x_1x_2} + c_{16}\partial^2_{x_1} + c_{62}\partial^2_{x_2} + c_{66}\partial^2_{x_1x_2} + c_{64}\partial^2_{x_2x_3} + c_{52}\partial^2_{x_2x_3} + c_{56}\partial^2_{x_1x_3} + c_{54}\partial^2_{x_3}, \\
L_{13} &= c_{13}\partial^2_{x_1x_3} + c_{15}\partial^2_{x_1} + c_{14}\partial^2_{x_1x_2} + c_{63}\partial^2_{x_2x_3} + c_{65}\partial^2_{x_1x_2} + c_{64}\partial^2_{x_2} + c_{53}\partial^2_{x_3} + c_{55}\partial^2_{x_1x_3} \\
&\quad + c_{54}\partial^2_{x_2x_3}, \\
L_{21} &= c_{61}\partial^2_{x_1} + c_{66}\partial^2_{x_1x_2} + c_{65}\partial^2_{x_1x_3} + c_{64}\partial^2_{x_1x_3} + c_{21}\partial^2_{x_1x_2} + c_{26}\partial^2_{x_2} + c_{25}\partial^2_{x_2x_3} \\
&\quad + c_{41}\partial^2_{x_1x_3} + c_{46}\partial^2_{x_2x_3} + c_{45}\partial^2_{x_3}, \\
L_{22} &= c_{62}\partial^2_{x_1x_2} + c_{66}\partial^2_{x_1} + c_{22}\partial^2_{x_2} + c_{26}\partial^2_{x_1x_2} + c_{24}\partial^2_{x_2x_3} + c_{42}\partial^2_{x_2x_3} + c_{46}\partial^2_{x_1x_3} + c_{44}\partial^2_{x_3}, \\
L_{23} &= c_{63}\partial^2_{x_1x_3} + c_{65}\partial^2_{x_1} + c_{64}\partial^2_{x_1x_2} + c_{23}\partial^2_{x_2x_3} + c_{25}\partial^2_{x_1x_2} + c_{24}\partial^2_{x_2} + c_{43}\partial^2_{x_3} + c_{45}\partial^2_{x_1x_3} \\
&\quad + c_{44}\partial^2_{x_2x_3}, \\
L_{31} &= c_{51}\partial^2_{x_1} + c_{56}\partial^2_{x_1x_2} + c_{55}\partial^2_{x_1x_3} + c_{54}\partial^2_{x_1x_3} + c_{41}\partial^2_{x_1x_2} + c_{46}\partial^2_{x_2} + c_{45}\partial^2_{x_2x_3} + c_{31}\partial^2_{x_1x_3} \\
&\quad + c_{36}\partial^2_{x_2x_3} + c_{35}\partial^2_{x_3}, \\
L_{32} &= c_{52}\partial^2_{x_1x_2} + c_{56}\partial^2_{x_1} + c_{42}\partial^2_{x_2} + c_{46}\partial^2_{x_1x_2} + c_{44}\partial^2_{x_2x_3} + c_{32}\partial^2_{x_2x_3} + c_{36}\partial^2_{x_1x_3} + c_{34}\partial^2_{x_3}, \\
L_{33} &= c_{53}\partial^2_{x_1x_3} + c_{55}\partial^2_{x_1} + c_{54}\partial^2_{x_1x_2} + c_{43}\partial^2_{x_2x_3} + c_{45}\partial^2_{x_1x_2} + c_{44}\partial^2_{x_2} + c_{33}\partial^2_{x_3} + c_{35}\partial^2_{x_1x_3} \\
&\quad + c_{34}\partial^2_{x_2x_3}.
\end{aligned} \tag{3.6.33}$$

3.7 3-D Fundamental Solution of IVP for the System Depending on x_1, x_2, x_3 and t Variables

A matrix $\mathcal{U}(x,t) = [U_{rs}(x,t)]_{3\times 3}$ is called 3-D fundamental solution or IVP for the system depending on space $x \in \mathbb{R}^3$ and the time variable t if $s-$th column

$$\mathbf{U}_s(x,t) = \begin{pmatrix} U_{1s}(x,t) \\ U_{2s}(x,t) \\ U_{3s}(x,t) \end{pmatrix}$$

satisfies

$$\rho \frac{\partial^2 \mathbf{U}_s}{\partial t^2} = L[\mathbf{U}_s], \quad x \in \mathbb{R}^3,\ t > 0, \tag{3.7.1}$$

$$\mathbf{U}_s(x_1,x_2,x_3,0) = 0, \quad \frac{\partial \mathbf{U}_s}{\partial t} = \frac{1}{\rho}\vec{e}^s \delta(x_1,x_2,x_3), \tag{3.7.2}$$

where $x = (x_1, x_2, x_3) \in \mathbb{R}^3, \quad t \in \mathbb{R}$; $\mathbf{e}^1 = (1,0,0)$, $\mathbf{e}^2 = (0,1,0)$, $\mathbf{e}^3 = (0,0,1)$;and $U_{js}(x_1, x_2, x_3, t)$ are the components of the unknown displacement vector $\mathbf{U}_s(x_1, x_2, x_3, t)$ and $L[\mathbf{U}_s]$ is the matrix operator defined with components $L_{ij}U_{js}$ that is given with the formulas (3.6.33).

3.7.1 Some Properties of 3-D Fundamental Solution

Remark 3.7.1. Let $\mathbf{U}_s(x,t)$ be a fundamental solution then $\hat{\mathbf{U}}_s(x,t) = \theta(t)\mathbf{U}_s(x,t)$ satisfies

$$\rho \frac{\partial^2 \hat{\mathbf{U}}_s}{\partial t^2} = L[\hat{\mathbf{U}}_s] + \mathbf{e}^s \delta(x,t), \tag{3.7.3}$$

$$\hat{\mathbf{U}}_s(x,t)\Big|_{t<0} = 0, \tag{3.7.4}$$

$s = 1,2,3$; where $x = (x_1, x_2, x_3) \in \mathbb{R}^3, \quad t \in \mathbb{R}$; $\mathbf{e}^1 = (1,0,0)$, $\mathbf{e}^2 = (0,1,0)$, $\mathbf{e}^3 = (0,0,1)$;and $U_{js}(x,t)$ are the components of the unknown displacement vector $\mathbf{U}_s(x,t)$ and $L[\mathbf{U}_s]$ is the matrix operator defined with components $L_{ij}U_{js}$ that is given with the formulas (3.6.33).

Proof. Since $\hat{\mathbf{U}}(x,t) = \Theta(t)\mathbf{u}(x,t)$, derivative of $\hat{\mathbf{U}}(x,t)$ with respect to t is

$$\frac{\partial\Big(\hat{\mathbf{U}}(x,t)\Big)}{\partial t} = \delta(t)\mathbf{U}(x,0) + \Theta(t)\frac{\partial}{\partial t}\mathbf{U}(x,t),$$

$$\frac{\partial^2\Big(\hat{\mathbf{U}}(x,t)\Big)}{\partial t^2} = \frac{1}{\rho}\vec{e^s}\delta(t)\delta(x) + \Theta(t)\frac{\partial^2}{\partial t^2}\mathbf{U}(x,t),$$

and also we have

$$L[\hat{\mathbf{U}}] = \Theta(t)L[\mathbf{U}]$$

then

$$\rho\frac{\partial^2 \hat{\mathbf{U}}}{\partial t^2} - L[\hat{\mathbf{U}}] = \rho\frac{1}{\rho}\vec{e^s}\delta(t)\delta(x) + \rho\Theta(t)\frac{\partial^2 \mathbf{U}}{\partial t^2} - \Theta(t)L[\mathbf{U}],$$

$$\rho\frac{\partial^2 \hat{\mathbf{U}}}{\partial t^2} - L[\hat{\mathbf{U}}] = \vec{e^s}\delta(x)\delta(t).$$

□

It is well known that (see, Hörmander-Lojasiewicz theorem in appendix) the arbitrary differential equation or system with constant coefficients has a fundamental solution of slow growth. Thus, system with constant coefficients given with equations (3.7.1), (3.7.2) has a fundamental solution

$$\mathbf{U}_s(x,t) \in C^2([0,T];\mathcal{S}'(\mathbb{R}^3)).$$

Our aim is to study some of the properties of this fundamental solution and suggest a method to find fundamental solutions.

Let us denote convolution of functions $\mathbf{U}_s(x,t)$, with cap-shaped function $w_\varepsilon(x)$ is $\mathbf{u}_s^\varepsilon(x,t) = (u_{1s}, u_{2s}, u_{3s})$. Taking convolution with cap-shaped function, the generalized Cauchy problem (3.7.1)-(3.7.2) can be written as

$$\rho\frac{\partial^2 \mathbf{u}_s^\varepsilon}{\partial t^2} = L[\mathbf{u}_s^\varepsilon], \quad (x_1,x_2,x_3) \in \mathbb{R}^3, \quad t \in \mathbb{R}, \tag{3.7.5}$$

$$\mathbf{u}_s^\varepsilon(x,0) = 0, \quad \frac{\partial \mathbf{u}_s^\varepsilon}{\partial t} = \frac{1}{\rho}\vec{e}^s w_\varepsilon(x_1)w_\varepsilon(x_2)w_\varepsilon(x_3). \tag{3.7.6}$$

Using Theorem 3.2.3 of Section 3.6.2, it can be proved that problem (3.7.1), (3.7.2) has a unique solution $\mathbf{u}_s^\varepsilon(x,t) \in C^2([0,T];C_0^\infty(\mathbb{R}^3))$ and
$supp\ \mathbf{u}_s^\varepsilon(x,t) \subseteq P(MT+\varepsilon_0)\ \forall\varepsilon \in (0,\varepsilon_0)$.

Property 7. *As $\varepsilon \to +0$, $\boldsymbol{u}_s^\varepsilon(x,t)$ approaches to $\boldsymbol{U}_s(x,t)$ in $\mathcal{S}'(\mathbb{R}^3)$; $\forall t \in [0,T]$.*

Proof. It can be proved that as $\varepsilon \to +0$, $w_\varepsilon(x)$ approaches to $\delta(x)$ in $\mathcal{S}'(\mathbb{R}^3)$. Using this fact and using the continuity of the convolution $\mathbf{u}(x,t) * w_\varepsilon(x)$ with respect to $w_\varepsilon(x)$ theorem is proved. □

Property 8. *Let T be a fixed positive number. There exists a unique solution of Cauchy problem (3.7.1), (3.7.2)*

$$\mathbf{U}_s(x,t) \in C^2([0,T];\mathcal{E}'(\mathbb{R}^3)).$$

Proof. We need to show that

$$(\mathbf{U}_s, \varphi) = 0; \ \ \forall \varphi \in \mathcal{S} \quad \text{and } supp\ \varphi \subseteq \mathbb{R}^3 \setminus P(MT + \varepsilon_0)$$

$$\begin{aligned}(\mathbf{U}_s, \varphi) &= \lim_{\varepsilon \to +0} (\mathbf{u}_s^\varepsilon, \varphi); \ \ \forall \varphi \in \mathcal{S} \\ &= 0.\end{aligned}$$

This means $supp\ \mathbf{U}_s \subseteq P(MT + \varepsilon_0)\ \forall \varepsilon \in (0, \varepsilon_0)$. Also using property(7) we conclude that there exist a unique solution of the Cauchy problem (3.7.1), (3.7.2)

$$\mathbf{U}_s(x,t) \in C^2([0,T]; \mathcal{E}'(\mathbb{R}^3)).$$

□

Property 9. *Let $\boldsymbol{U}_s(x,t)$ be solution of the problem (3.7.1), (3.7.2) and $\tilde{\boldsymbol{U}}_s(\upsilon,t) = \Big(\tilde{U}_{1s}(\upsilon,t), \tilde{U}_{2s}(\upsilon,t), \tilde{U}_{3s}(\upsilon,t)\Big)$ where $\upsilon = (\upsilon_1, \upsilon_2, \upsilon_3) \in \mathbb{R}^3$ be the Fourier transform image of $\boldsymbol{U}_s(x,t)$ with respect to $(x_1, x_2, x_3) \in \mathbb{R}^3$. Then the Fourier image $\tilde{\boldsymbol{U}}_s(\upsilon,t)$ is an entire analytic function and satisfies following system of equations:*

$$\begin{aligned}&\rho\frac{\partial^2 \tilde{U}_{1s}}{\partial t^2} + c_{11}\upsilon_1^2\tilde{U}_{1s} + c_{12}\upsilon_1\upsilon_2\tilde{U}_{2s} + c_{13}\upsilon_1\upsilon_3\tilde{U}_{3s} + c_{16}\Big(\upsilon_2\upsilon_1\tilde{U}_{1s} + \upsilon_1^2\tilde{U}_{2s}\Big) + c_{15}\Big(\upsilon_1\upsilon_3\tilde{U}_{1s} + \upsilon_1^2\tilde{U}_{3s}\Big) \\ &+ c_{14}\Big(\upsilon_1\upsilon_3\tilde{U}_{2s} + \upsilon_1\upsilon_2\tilde{U}_{3s}\Big) + c_{61}\upsilon_1\upsilon_2\tilde{U}_{1s} + c_{62}\upsilon_2^2\tilde{U}_{2s} + c_{63}\upsilon_2\upsilon_3\tilde{U}_{3s} + c_{66}\Big(\upsilon_1\upsilon_2\tilde{U}_{2s} + \upsilon_2^2\tilde{U}_{1s}\Big) \\ &+ c_{65}\Big(\upsilon_2\upsilon_3\tilde{U}_{1s} + \upsilon_1\upsilon_2\tilde{U}_{3s}\Big) + c_{64}\Big(\upsilon_2\upsilon_3\tilde{U}_{2s} + \upsilon_2^2\tilde{U}_{3s}\Big) + c_{51}\upsilon_1\upsilon_3\tilde{U}_{1s} + c_{52}\upsilon_2\upsilon_3\tilde{U}_{2s} + c_{53}\upsilon_3^2\tilde{U}_{3s} \\ &+ c_{56}\Big(\upsilon_2\upsilon_3\tilde{U}_{1s} + \upsilon_1\upsilon_3\tilde{U}_{2s}\Big) + c_{55}\Big(\upsilon_3^2\tilde{U}_{1s} + \upsilon_1\upsilon_3\tilde{U}_{3s}\Big) + c_{54}\Big(\upsilon_3^2\tilde{U}_{2s} + \upsilon_2\upsilon_3\tilde{U}_{3s}\Big), \quad (3.7.7)\end{aligned}$$

$$\begin{aligned}&\rho\frac{\partial^2 \tilde{U}_{2s}}{\partial t^2} + c_{61}\upsilon_1^2\tilde{U}_{1s} + c_{62}\upsilon_1\upsilon_2\tilde{U}_{2s} + c_{63}\upsilon_1\upsilon_3\tilde{U}_{3s} + c_{66}\Big(\upsilon_2\upsilon_1\tilde{U}_{1s} + \upsilon_1^2\tilde{U}_{2s}\Big) + c_{65}\Big(\upsilon_1\upsilon_3\tilde{U}_{1s} + \upsilon_1^2\tilde{U}_{3s}\Big) \\ &+ c_{64}\Big(\upsilon_1\upsilon_3\tilde{U}_{2s} + \upsilon_1\upsilon_2\tilde{U}_{3s}\Big) + c_{21}\upsilon_1\upsilon_2\tilde{U}_{1s} + c_{22}\upsilon_2^2\tilde{U}_{2s} + c_{23}\upsilon_2\upsilon_3\tilde{U}_{3s} + c_{26}\Big(\upsilon_1\upsilon_2\tilde{U}_{2s} + \upsilon_2^2\tilde{U}_{1s}\Big) \\ &+ c_{25}\Big(\upsilon_2\upsilon_3\tilde{U}_{1s} + \upsilon_1\upsilon_2\tilde{U}_{3s}\Big) + c_{24}\Big(\upsilon_2\upsilon_3\tilde{U}_{2s} + \upsilon_2^2\tilde{U}_{3s}\Big) + c_{41}\upsilon_1\upsilon_3\tilde{U}_{1s} + c_{42}\upsilon_2\upsilon_3\tilde{U}_{2s} + c_{43}\upsilon_3^2\tilde{U}_{3s}\end{aligned}$$

$$+c_{46}\Big(v_2v_3\tilde{U}_{1s}+v_1v_3\tilde{U}_{2s}\Big)+c_{45}\Big(v_3^2\tilde{U}_{1s}+v_1v_3\tilde{U}_{3s}\Big)+c_{44}\Big(v_3^2\tilde{U}_{2s}+v_2v_3\tilde{U}_{3s}\Big), \quad (3.7.8)$$

$$\rho\frac{\partial^2\tilde{U}_{3s}}{\partial t^2}+c_{51}v_1^2\tilde{U}_{1s}+c_{52}v_1v_2\tilde{U}_{2s}+c_{53}v_1v_3\tilde{U}_{3s}+c_{56}\Big(v_2v_1\tilde{U}_{1s}+v_1^2\tilde{U}_{2s}\Big)+c_{55}\Big(v_1v_3\tilde{U}_{1s}+v_1^2\tilde{U}_{3s}\Big)$$

$$+c_{54}\Big(v_1v_3\tilde{U}_{2s}+v_1v_2\tilde{U}_{3s}\Big)+c_{41}v_1v_2\tilde{U}_{1s}+c_{42}v_2^2\tilde{U}_{2s}+c_{43}v_2v_3\tilde{U}_{3s}+c_{46}\Big(v_1v_2\tilde{U}_{2s}+v_2^2\tilde{U}_{1s}\Big)$$

$$+c_{45}\Big(v_2v_3\tilde{U}_{1s}+v_1v_2\tilde{U}_{3s}\Big)+c_{44}\Big(v_2v_3\tilde{U}_{2s}+v_2^2\tilde{U}_{3s}\Big)+c_{31}v_1v_3\tilde{U}_{1s}+c_{32}v_2v_3\tilde{U}_{2s}+c_{33}v_3^2\tilde{U}_{3s}$$

$$+c_{36}\Big(v_2v_3\tilde{U}_{1s}+v_1v_3\tilde{U}_{2s}\Big)+c_{35}\Big(v_3^2\tilde{U}_{1s}+v_1v_3\tilde{U}_{3s}\Big)+c_{34}\Big(v_3^2\tilde{U}_{2s}+v_2v_3\tilde{U}_{3s}\Big), \quad (3.7.9)$$

$$\tilde{U}_{js}(x,0)=0, \quad \left.\frac{\partial\tilde{U}_{js}(x,t)}{\partial t}\right|_{t=0}=\Big(\frac{1}{\rho}\boldsymbol{e}^s\Big)_j. \quad (3.7.10)$$

Proof. Let $\tilde{\mathbf{U}}_s(\upsilon,t)$ be the Fourier transform images of $\mathbf{U}_s(x,t)$ with respect to $x=(x_1,x_2,x_3)\in\mathbb{R}^3$ i.e.

$$\tilde{\mathbf{U}}_s(\upsilon,t)=\Big(\tilde{U}_{1s}(\upsilon,t),\tilde{U}_{2s}(\upsilon,t),\tilde{U}_{3s}(\upsilon,t)\Big)$$

$$\tilde{U}_{ls}(\upsilon,t)=\boldsymbol{\mathcal{F}}_x[U_{ls}]; \quad l=1,2,3; \upsilon=(\upsilon_1,\upsilon_2,\upsilon_3)\in\mathbb{R}^3;$$

where the Fourier operator $\boldsymbol{\mathcal{F}}_x$ is defined by

$$\boldsymbol{\mathcal{F}}_x[u_{ls}]=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}u_{ls}(x,t)e^{i\upsilon x}dx_1dx_2dx_3;$$

$$x\upsilon=x_1\upsilon_1+x_2\upsilon_2+x_3\upsilon_3; \quad i^2=-1.$$

Since $\mathbf{U}_s(x,t)\in C^2([0,T];\mathcal{E}'(\mathbb{R}^3))$, according to Paley-Wiener theorem (Reed & Simon, 1975), Fourier transform of the function $\mathbf{U}_s(x,t)$ is an entire analytic function with respect to $\upsilon=(\upsilon_1,\upsilon_2,\upsilon_3)\in\mathbb{R}^3$, and can be written as a power series

$$\tilde{\mathbf{U}}_s(\upsilon,t)=\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\tilde{\mathbf{U}}_s^{k,m,n}(t)\upsilon_1^k\upsilon_2^m\upsilon_3^n.$$

If we apply Fourier transform with respect to space variable problem (3.7.1)-(3.7.2) can be written in terms of Fourier images given with the equations (3.7.7)-(3.7.10).

□

3.7.2 Derivation of 3-D Fundamental Solution

Problem in Terms of Coefficients of the Series Expansion

According to Paley-Wiener Theorem, power series expansion of $\tilde{\mathbf{U}}_s(\upsilon,t)$, $\tilde{\mathbf{h}}_s(\upsilon)$ can be considered i.e.

$$\tilde{\mathbf{U}}_s(\upsilon,t)=\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\tilde{\mathbf{U}}_s^{k,m,n}(t)\upsilon_1^k\upsilon_2^m\upsilon_3^n, \tag{3.7.11}$$

$$\tilde{\mathbf{h}}_s(\upsilon)=\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\tilde{\mathbf{h}}_s^{k,m,n}\upsilon_1^k\upsilon_2^m\upsilon_3^n, \tag{3.7.12}$$

where $\tilde{\mathbf{h}}_s^{k,m,n}$ are given real numbers; $\tilde{\mathbf{U}}_s^{k,m,n}(t)$ are unknown coefficients we need to find.
Substituting (3.7.11-(3.7.12) into (3.7.7)-(3.7.10) we obtain

$$\rho\frac{\partial^2\tilde{U}_{1s}^{k,m,n}}{\partial t^2}+c_{11}\tilde{U}_{1s}^{k-2,m,n}+c_{12}\tilde{U}_{2s}^{k-1,m-1,n}+c_{13}\tilde{U}_{3s}^{k-1,m,n-1}+c_{16}\left(\tilde{U}_{1s}^{k-1,m-1,n}+\tilde{U}_{2s}^{k-2,m,n}\right)$$
$$+c_{15}\left(\tilde{U}_{1s}^{k-1,m,n-1}+\tilde{U}_{3s}^{k-2,m,n}\right)+c_{14}\left(\tilde{U}_{2s}^{k-1,m,n-1}+\tilde{U}_{3s}^{k-1,m-1,n}\right)+c_{61}\tilde{U}_{1s}^{k-1,m-1,n}$$
$$+c_{62}\tilde{U}_{2s}^{k,m-2,n}+c_{63}\tilde{U}_{3s}^{k,m-1,n-1}+c_{66}\left(\tilde{U}_{2s}^{k-1,m-1,n}+\tilde{U}_{1s}^{k,m-2,n}\right)$$
$$+c_{65}\left(\tilde{U}_{1s}^{k,m-1,n-1}+\tilde{U}_{3s}^{k-1,m-1,n}\right)+c_{64}\left(\tilde{U}_{2s}^{k,m-1,n-1}+\tilde{U}_{3s}^{k,m-2,n}\right)+c_{51}\tilde{U}_{1s}^{k-1,m,n-1}$$
$$+c_{52}\tilde{U}_{2s}^{k,m-1,n-1}+c_{53}\tilde{U}_{3s}^{k,m,n-2}+c_{56}\left(\tilde{U}_{1s}^{k,m-1,n-1}+\tilde{U}_{2s}^{k-1,m,n-1}\right)$$
$$+c_{55}\left(\tilde{U}_{1s}^{k,m,n-2}+\tilde{U}_{3s}^{k-1,m,n-1}\right)+c_{54}\left(\tilde{U}_{2s}^{k,m,n-2}+\tilde{U}_{3s}^{k,m-1,n-1}\right)=0, \tag{3.7.13}$$

$$\rho\frac{\partial^2\tilde{U}_{2s}^{k,m,n}}{\partial t^2}+c_{61}\tilde{U}_{1s}^{k-2,m,n}+c_{62}\tilde{U}_{2s}^{k-1,m-1,n}+c_{63}\tilde{U}_{3s}^{k-1,m,n-1}+c_{66}\left(\tilde{U}_{1s}^{k-1,m-1,n}+\tilde{U}_{2s}^{k-2,m,n}\right)$$
$$+c_{65}\left(\tilde{U}_{1s}^{k-1,m,n-1}+\tilde{U}_{3s}^{k-2,m,n}\right)+c_{64}\left(\tilde{U}_{2s}^{k-1,m,n-1}+\tilde{U}_{3s}^{k-1,m-1,n}\right)+c_{21}\tilde{U}_{1s}^{k-1,m-1,n}$$
$$+c_{22}\tilde{U}_{2s}^{k,m-2,n}+c_{23}\tilde{U}_{3s}^{k,m-1,n-1}+c_{26}\left(\tilde{U}_{2s}^{k-1,m-1,n}+\tilde{U}_{1s}^{k,m-2,n}\right)$$
$$+c_{25}\left(\tilde{U}_{1s}^{k,m-1,n-1}+\tilde{U}_{3s}^{k-1,m-1,n}\right)+c_{24}\left(\tilde{U}_{2s}^{k,m-1,n-1}+\tilde{U}_{3s}^{k,m-2,n}\right)+c_{41}\tilde{U}_{1s}^{k-1,m,n-1}$$
$$+c_{42}\tilde{U}_{2s}^{k,m-1,n-1}+c_{43}\tilde{U}_{3s}^{k,m,n-2}+c_{46}\left(\tilde{U}_{1s}^{k,m-1,n-1}+\tilde{U}_{2s}^{k-1,m,n-1}\right)$$
$$+c_{45}\left(\tilde{U}_{1s}^{k,m,n-2}+\tilde{U}_{3s}^{k-1,m,n-1}\right)+c_{44}\left(\tilde{U}_{2s}^{k,m,n-2}+\tilde{U}_{3s}^{k,m-1,n-1}\right)=0, \quad (3.7.14)$$

$$\rho\frac{\partial^2\tilde{U}_{3s}^{k,m,n}}{\partial t^2}+c_{51}\tilde{U}_{1s}^{k-2,m,n}+c_{52}\tilde{U}_{2s}^{k-1,m-1,n}+c_{53}\tilde{U}_{3s}^{k-1,m,n-1}+c_{56}\left(\tilde{U}_{1s}^{k-1,m-1,n}+\tilde{U}_{2s}^{k-2,m,n}\right)$$
$$+c_{55}\left(\tilde{U}_{1s}^{k-1,m,n-1}+\tilde{U}_{3s}^{k-2,m,n}\right)+c_{54}\left(\tilde{U}_{2s}^{k-1,m,n-1}+\tilde{U}_{3s}^{k-1,m-1,n}\right)+c_{41}\tilde{U}_{1s}^{k-1,m-1,n}$$
$$+c_{42}\tilde{U}_{2s}^{k,m-2,n}+c_{43}\tilde{U}_{3s}^{k,m-1,n-1}+c_{46}\left(\tilde{U}_{2s}^{k-1,m-1,n}+\tilde{U}_{1s}^{k,m-2,n}\right)$$
$$+c_{45}\left(\tilde{U}_{1s}^{k,m-1,n-1}+\tilde{U}_{3s}^{k-1,m-1,n}\right)+c_{44}\left(\tilde{U}_{2s}^{k,m-1,n-1}+\tilde{U}_{3s}^{k,m-2,n}\right)+c_{31}\tilde{U}_{1s}^{k-1,m,n-1}$$
$$+c_{32}\tilde{U}_{2s}^{k,m-1,n-1}+c_{33}\tilde{U}_{3s}^{k,m,n-2}+c_{36}\left(\tilde{U}_{1s}^{k,m-1,n-1}+\tilde{U}_{2s}^{k-1,m,n-1}\right)$$
$$+c_{35}\left(\tilde{U}_{1s}^{k,m,n-2}+\tilde{U}_{3s}^{k-1,m,n-1}\right)+c_{34}\left(\tilde{U}_{2s}^{k,m,n-2}+\tilde{U}_{3s}^{k,m-1,n-1}\right)=0. \quad (3.7.15)$$

$$\tilde{U}_{js}^{k,m,n}(x,0)=0,\quad \left.\frac{\partial\tilde{U}_{js}^{k,m,n}(x,t)}{\partial t}\right|_{t=0}=\tilde{h}_{js}^{k,m,n}, \quad (3.7.16)$$

where h_{js}^{k} are given real numbers such that $h_{js}^{0}=1$ and $h_{js}^{l}=\frac{1}{\rho}\mathbf{e}^{s}$ for $l=1,2,....$

Equations (3.7.13)-(3.7.16) can be written equivalently as the following recurrence relations:

$$\frac{\partial^2\tilde{U}_{js}^{k,m,n}}{\partial t^2}=-\frac{1}{\rho}\Upsilon_j^{k,m,n},\ t>0,\ j=1,2,3, \quad (3.7.17)$$

$$\tilde{U}_{js}^{k,m,n}(0) = 0, \quad \left.\frac{\partial \tilde{U}_j^{k,m,n}(t)}{\partial t}\right|_{t=0} = \tilde{h}_{js}^{k,m,n}, \ j = 1, 2, 3. \tag{3.7.18}$$

where

$$\begin{aligned}
\Upsilon_1^{k,m,n} &= c_{11}\tilde{U}_{1s}^{k-2,m,n} + c_{12}\tilde{U}_{2s}^{k-1,m-1,n} + c_{13}\tilde{U}_{3s}^{k-1,m,n-1} + c_{16}\Big(\tilde{U}_{1s}^{k-1,m-1,n} + \tilde{U}_{2s}^{k-2,m,n}\Big) \\
&+ c_{15}\Big(\tilde{U}_{1s}^{k-1,m,n-1} + \tilde{U}_{3s}^{k-2,m,n}\Big) + c_{14}\Big(\tilde{U}_{2s}^{k-1,m,n-1} + \tilde{U}_{3s}^{k-1,m-1,n}\Big) + c_{61}\tilde{U}_{1s}^{k-1,m-1,n} \\
&+ c_{62}\tilde{U}_{2s}^{k,m-2,n} + c_{63}\tilde{U}_{3s}^{k,m-1,n-1} + c_{66}\Big(\tilde{U}_{2s}^{k-1,m-1,n} + \tilde{U}_{1s}^{k,m-2,n}\Big) \\
&+ c_{65}\Big(\tilde{U}_{1s}^{k,m-1,n-1} + \tilde{U}_{3s}^{k-1,m-1,n}\Big) + c_{64}\Big(\tilde{U}_{2s}^{k,m-1,n-1} + \tilde{U}_{3s}^{k,m-2,n}\Big) + c_{51}\tilde{U}_{1s}^{k-1,m,n-1} \\
&+ c_{52}\tilde{U}_{2s}^{k,m-1,n-1} + c_{53}\tilde{U}_{3s}^{k,m,n-2} + c_{56}\Big(\tilde{U}_{1s}^{k,m-1,n-1} + \tilde{U}_{2s}^{k-1,m,n-1}\Big) \\
&+ c_{55}\Big(\tilde{U}_{1s}^{k,m,n-2} + \tilde{U}_{3s}^{k-1,m,n-1}\Big) + c_{54}\Big(\tilde{U}_{2s}^{k,m,n-2} + \tilde{U}_{3s}^{k,m-1,n-1}\Big),
\end{aligned} \tag{3.7.19}$$

$$\begin{aligned}
\Upsilon_2^{k,m,n} &= c_{61}\tilde{U}_{1s}^{k-2,m,n} + c_{62}\tilde{U}_{2s}^{k-1,m-1,n} + c_{63}\tilde{U}_{3s}^{k-1,m,n-1} + c_{66}\Big(\tilde{U}_{1s}^{k-1,m-1,n} + \tilde{U}_{2s}^{k-2,m,n}\Big) \\
&+ c_{65}\Big(\tilde{U}_{1s}^{k-1,m,n-1} + \tilde{U}_{3s}^{k-2,m,n}\Big) + c_{64}\Big(\tilde{U}_{2s}^{k-1,m,n-1} + \tilde{U}_{3s}^{k-1,m-1,n}\Big) + c_{21}\tilde{U}_{1s}^{k-1,m-1,n} \\
&+ c_{22}\tilde{U}_{2s}^{k,m-2,n} + c_{23}\tilde{U}_{3s}^{k,m-1,n-1} + c_{26}\Big(\tilde{U}_{2s}^{k-1,m-1,n} + \tilde{U}_{1s}^{k,m-2,n}\Big) \\
&+ c_{25}\Big(\tilde{U}_{1s}^{k,m-1,n-1} + \tilde{U}_{3s}^{k-1,m-1,n}\Big) + c_{24}\Big(\tilde{U}_{2s}^{k,m-1,n-1} + \tilde{U}_{3s}^{k,m-2,n}\Big) + c_{41}\tilde{U}_{1s}^{k-1,m,n-1} \\
&+ c_{42}\tilde{U}_{2s}^{k,m-1,n-1} + c_{43}\tilde{U}_{3s}^{k,m,n-2} + c_{46}\Big(\tilde{U}_{1s}^{k,m-1,n-1} + \tilde{U}_{2s}^{k-1,m,n-1}\Big) \\
&+ c_{45}\Big(\tilde{U}_{1s}^{k,m,n-2} + \tilde{U}_{3s}^{k-1,m,n-1}\Big) + c_{44}\Big(\tilde{U}_{2s}^{k,m,n-2} + \tilde{U}_{3s}^{k,m-1,n-1}\Big),
\end{aligned} \tag{3.7.20}$$

$$\begin{aligned}
\Upsilon_3^{k,m,n} &= c_{51}\tilde{U}_{1s}^{k-2,m,n} + c_{52}\tilde{U}_{2s}^{k-1,m-1,n} + c_{53}\tilde{U}_{3s}^{k-1,m,n-1} + c_{56}\Big(\tilde{U}_{1s}^{k-1,m-1,n} + \tilde{U}_{2s}^{k-2,m,n}\Big) \\
&+ c_{55}\Big(\tilde{U}_{1s}^{k-1,m,n-1} + \tilde{U}_{3s}^{k-2,m,n}\Big) + c_{54}\Big(\tilde{U}_{2s}^{k-1,m,n-1} + \tilde{U}_{3s}^{k-1,m-1,n}\Big) + c_{41}\tilde{U}_{1s}^{k-1,m-1,n} \\
&+ c_{42}\tilde{U}_{2s}^{k,m-2,n} + c_{43}\tilde{U}_{3s}^{k,m-1,n-1} + c_{46}\Big(\tilde{U}_{2s}^{k-1,m-1,n} + \tilde{U}_{1s}^{k,m-2,n}\Big)
\end{aligned}$$

$$+c_{45}\left(\tilde{U}_{1s}^{k,m-1,n-1}+\tilde{U}_{3s}^{k-1,m-1,n}\right)+c_{44}\left(\tilde{U}_{2s}^{k,m-1,n-1}+\tilde{U}_{3s}^{k,m-2,n}\right)+c_{31}\tilde{U}_{1s}^{k-1,m,n-1}$$
$$+c_{32}\tilde{U}_{2s}^{k,m-1,n-1}+c_{33}\tilde{U}_{3s}^{k,m,n-2}+c_{36}\left(\tilde{U}_{1s}^{k,m-1,n-1}+\tilde{U}_{2s}^{k-1,m,n-1}\right)$$
$$+c_{35}\left(\tilde{U}_{1s}^{k,m,n-2}+\tilde{U}_{3s}^{k-1,m,n-1}\right)+c_{34}\left(\tilde{U}_{2s}^{k,m,n-2}+\tilde{U}_{3s}^{k,m-1,n-1}\right). \quad (3.7.21)$$

The solutions of the problems (3.7.17), (3.7.18) for $j=1,2,3$ will be

$$\tilde{U}_{js}^{k,m,n}(t)=\int_0^t (t-\tau)\Upsilon_j(\tau)d\tau+\tilde{h}_{js}^{k,m,n}t,\ \ j=1,2,3. \quad (3.7.22)$$

Using (3.7.22) all coefficients $\tilde{\mathbf{U}}_s^{k,m,n}$ of $\tilde{\mathbf{U}}_s$ can be found. Solution of the IVP (3.7.7)-(3.7.10) can be obtained as follows

$$\tilde{U}_{js}(\upsilon_1,\upsilon_2,\upsilon_3,t)=\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\tilde{U}_{js}^{k,m,n}(t)\upsilon_1^k\upsilon_2^m\upsilon_3^n \quad (3.7.23)$$

where $\tilde{U}_{js}^{k,m,n}(t),\ \ j=1,2,3$ are defined in equations (3.7.22).

Procedure of Finding $\mathbf{U}_s^{k,m,n}$

The procedure of finding $\tilde{\mathbf{U}}_s^{k,m,n}$; $s=1,2,3$, consists of the sequence of the following iterative steps of constructing some formulae from the others using the relation (3.7.22).

Step 1:

$$\tilde{\mathbf{U}}_s^{-2,m,n}=\tilde{\mathbf{U}}_s^{k,-2,n}=\tilde{\mathbf{U}}_s^{k,m,-2}=\tilde{\mathbf{U}}_s^{-1,m,n}=\tilde{\mathbf{U}}_s^{k,-1,n}=\tilde{\mathbf{U}}_s^{k,m,-1}=0$$

when $k=-2,\,-1,\,0,...$; $m=-2,\,-1,\,0,...$; $n=-2,\,-1,\,0,....$

Step 2: using zero values from step 1 we compute compute

$$\tilde{\mathbf{U}}_s^{0,m,n},\ \tilde{\mathbf{U}}_s^{k,0,n},\ \tilde{\mathbf{U}}_s^{k,m,0},\ k=0,\,1,\,2,\,...;\ \ m=0,\,1,\,2,\,...;\ \ n=0,\,1,\,2,\,....$$

Step 3: from the relations obtained on previous steps we compute

$$\tilde{\mathbf{U}}_s^{1,m,n},\ \tilde{\mathbf{U}}_s^{k,1,n},\ \tilde{\mathbf{U}}_s^{k,m,1}, \quad k = 1,\ 2,\ ...;\ m = 1,\ 2,\ ...;\ n = 1,\ 2,\ ....$$

... ...

Step p: from the relations obtained on previous steps we compute

$$\tilde{\mathbf{U}}_s^{p,m,n},\ \tilde{\mathbf{U}}_s^{k,p,n},\ \tilde{\mathbf{U}}_s^{k,m,p}, \quad \text{for} \quad k = p,\ p+1,\ p+2,\ ...;\ m = p,\ p+1,\ p+2,\ ...;$$

$n = p,\ p+1,\ p+2,\ ....$

Inverse Fourier Transform and the Solutions of Original IVP

Applying inverse Fourier transform to $\tilde{U}_{js}(v_1, v_2, v_3, t)$ defined by the formula (3.7.23), solution of problem (3.7.1), (3.7.2) can be obtained for 3-D Case with with $\mathbf{h}_s(x,t) = \frac{1}{\rho}\mathbf{e}^s\delta(x_1, x_2, x_3)$; $s = 1$.

3.7.3 Simulation of 3-D Fundamental Solution

In this part of the section we consider problem (3.7.1), (3.7.2) for monoclinic type of anisotropy (see, Section (3.1)). The aim is to create simulations of elastic wave propagations, by the method described in Section (3.7.2).

The monoclinic crystal with density $(\mathrm{gr/cm}^3)$ and elastic moduli $(10^{12}\ \mathrm{dyn/cm}^2)$ is as follows:

Monoclinic: $\rho = 2.649$, $c_{11} = 8.67$, $c_{12} = -0.83$, $c_{13} = 2.71$, $c_{14} = -0.37$, $c_{15} = 0$, $c_{16} = 0$, $c_{21} = -0.83$, $c_{22} = 12.98$, $c_{23} = -0.74$, $c_{24} = 0.57$, $c_{25} = 0$, $c_{26} = 0$, $c_{31} = 2.71$, $c_{32} = -0.74$, $c_{33} = 10.28$, $c_{34} = 0.99$, $c_{35} = 0$, $c_{36} = 0$, $c_{41} = -0.37$, $c_{42} = 0.57$,

$c_{43} = 0.99,\ c_{44} = 3.86,\ c_{45} = 0,\ c_{46} = 0,\ c_{51} = 0,\ c_{52} = 0,\ c_{53} = 0,$ $c_{54} = 0,\ c_{55} = 6.88,\ c_{56} = 0.25,\ c_{61} = 0,\ c_{62} = 0,\ c_{63} = 0,\ c_{64} = 0,$ $c_{65} = 0.25,\ c_{66} = 2.9.$

Using the method of the Section (3.7), we compute the elements of the fundamental solution matrix $\mathcal{U}(x,t)$ whose $s-$th column is $\mathbf{U}_s(x,t)$ with the components $\mathbf{U}_s(x,t) = (U_{1s}(x,t), U_{2s}(x,t), U_{3s}(x,t))^T$. In figure3.2, we consider (3.7.1), (3.7.2) for $s = 1$ ($\mathbf{e}^s = (1,0,0)$) and we draw the graph of the first component of $\mathbf{U}_s(x,t)$.

Example: We study problem (3.7.1), (3.7.2) inside the Monoclinic crystal for $\mathbf{h}_1(x,t) = \frac{1}{\rho}\mathbf{e}^1\delta(x_1,x_2,x_3)$. In Figures the density plots of first, second and third components of the first column $\mathbf{U}_1(x,t)$ of fundamental solution matrix is presented at $t = 11/10$.

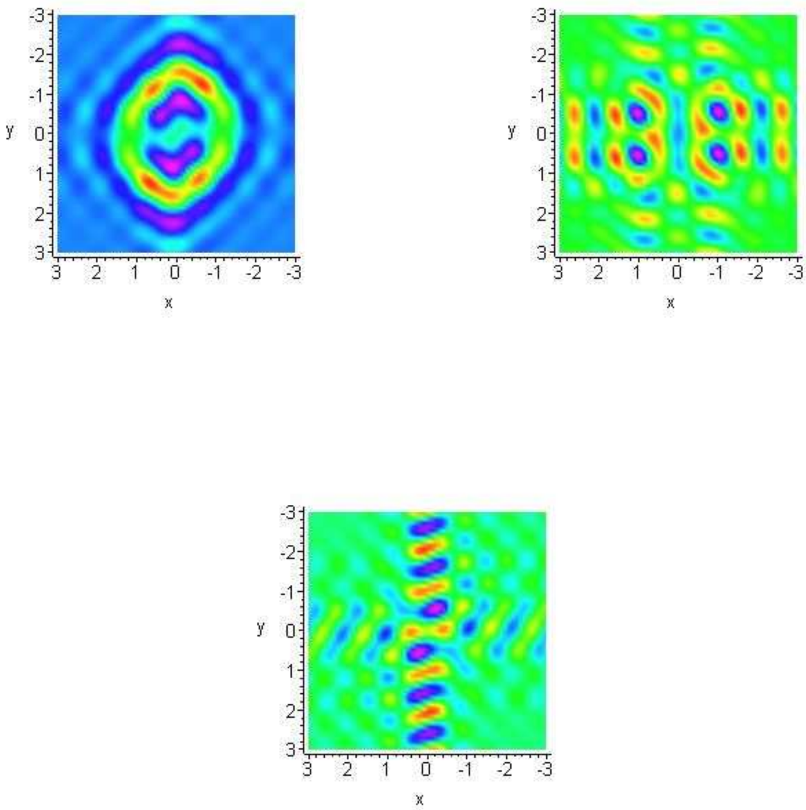


Figure 3.9 2-D level plots of $U_{11}(0, x_2, x_3, 11/10)$, $U_{21}(0, x_2, x_3, 11/10)$ and $U_{31}(0, x_2, x_3, 11/10)$ when $x_1 = 0$ of Monoclinic Crystal

Analysis of figures

In figure 3.9, 2-D level plots of first, second and third components of first column of fundamental solution matrix of elastic system depending on four variables x_1, x_2, x_3, t in monoclinic crystal is presented for time $t = 11/10$ and $x_1 = 0$. At time $t = 11/10$, the difference of wave propagations of different components of $\mathbf{U}_1(x,t)$ can be observed.

3.8 Application

In previous sections, IVP for system of anisotropic elasticity is described. We consider the system for cases when displacement vector depends on one, two or three space and time variables. For each case, generalized Cauchy problem is stated and derivation of fundamental solution is explained.

Fundamental solution plays an important role in constructing solutions to various kinds of boundary-value and IVPs. In this section we consider IVP for the system of anisotropic elasticity as an application and by the help of fundamental solution we will show that solution of IVP can be obtained.

Let us consider the Cauchy problem for system of anisotropic elasticity that is given with equations

$$\rho \frac{\partial^2 u_j}{\partial t^2} = \sum_{k=1}^{3} \frac{\partial \sigma_{jk}[\mathbf{u}]}{\partial x_k} + f_j(x,t), \quad j = 1,2,3, \tag{3.8.1}$$

$$u_j(x,0) = g_j(x), \quad \left.\frac{\partial u_j(x,t)}{\partial t}\right|_{t=0} = h_j(x), \tag{3.8.2}$$

where $x = (x_1, x_2, x_3) \in \mathbb{R}^3, \ \ t > 0$ and $u_j(x,t); \ (j = 1,2,3)$ are the components of the unknown displacement vector $\mathbf{u}(x,t)$. $f_j(x,t); \ (j = 1,2,3)$ are the components of nonhomogeneous vector function depending on x, t and g_j, h_j are components of initial data depending on x variable only.

The mathematical model given with the equations (3.8.1), (3.8.2) can be written as follows

$$\rho\frac{\partial^2\mathbf{u}}{\partial t^2} = L[\mathbf{u}] + \mathbf{f}(x,t), \quad x \in \mathbb{R}^3, t > 0, \tag{3.8.3}$$

$$\mathbf{u}(x,0) = \mathbf{g}(x), \quad \left.\frac{\partial\mathbf{u}}{\partial t}\right|_{t=0} = \mathbf{h}(x), \tag{3.8.4}$$

where $x = (x_1, x_2, x_3) \in \mathbb{R}^3, \quad t > 0$ and $L[\mathbf{U}]$ is the matrix operator defined with components $L_{ij}U_j$ that is given with the formulas (3.6.33).

Lemma 3.8.1. *If $\mathbf{u}(x,t)$ is a solution of (3.8.3), (3.8.4); then $\mathbf{v}(x,t) = \Theta(t)\mathbf{u}(x,t)$ is a solution of the following problem*

$$\rho\frac{\partial^2\mathbf{v}}{\partial t^2} = L[\mathbf{v}] + \mathbf{F}(x,t), \tag{3.8.5}$$

$$\left.\mathbf{v}(x,t)\right|_{t<0} = 0, \tag{3.8.6}$$

where $x \in \mathbb{R}^3, t \in \mathbb{R}$, $\mathbf{v}(x,t)$ is unknown displacement vector with the components $v_j(x,t)$ and $L[\mathbf{u}]$ is the matrix operator defined with components $L_{ij}u_j$ defined in (3.6.33) and nonhomogeneous term is defined as $\mathbf{F}(x,t) = \Theta(t)\mathbf{f}(x,t) + \rho\delta'(t)\mathbf{g}(x) + \rho\delta(t)\mathbf{h}(x)$.

Proof. Since $\mathbf{v}(x,t) = \Theta(t)\mathbf{u}(x,t)$, derivative of $\mathbf{v}(x,t)$ with respect to t is

$$\Big(\mathbf{v}(x,t)\Big)_t = \delta(t)\mathbf{u}(x,0) + \Theta(t)\frac{\partial}{\partial t}\mathbf{u}(x,t),$$

$$\Big(\mathbf{v}(x,t)\Big)_t t = \delta'(t)\mathbf{g}(x) + \delta(t)\mathbf{h}(x) + \Theta(t)\frac{\partial^2}{\partial t^2}\mathbf{u}(x,t),$$

and also we have

$$L[\mathbf{v}] = \Theta(t)L[\mathbf{u}]$$

then

$$\rho\frac{\partial^2\mathbf{v}}{\partial t^2} - L[\mathbf{v}] = \rho\delta'(t)\mathbf{g}(x) + \rho\delta(t)\mathbf{h}(x) + \rho\Theta(t)\frac{\partial^2\mathbf{u}}{\partial t^2} - \Theta(t)L[\mathbf{u}],$$

$$\rho\frac{\partial^2\mathbf{v}}{\partial t^2} - L[\mathbf{v}] = \rho\delta'(t)\mathbf{g}(x) + \rho\delta(t)\mathbf{h}(x) + \Theta(t)\mathbf{f}(x,t).$$

□

Lemma 3.8.2. *Let us denote fundamental solution as* $\mathbf{G}(x,t)$ *that is solution of the following problem*

$$\rho\frac{\partial^2\mathbf{G}}{\partial t^2} = L[\mathbf{G}] + \delta(x,t), \tag{3.8.7}$$

$$\mathbf{G}(x,t)\Big|_{t<0} = 0, \tag{3.8.8}$$

then $\mathbf{v}(x,t) = (\mathbf{G} * \mathbf{F})(x,t)$ *is a solution of the Cauchy problem given with the equations (3.8.5), (3.8.6).*

Proof. Consider

$$\frac{\partial^2\mathbf{v}}{\partial t^2} = \frac{\partial^2}{\partial t^2}(\mathbf{G} * \mathbf{F}) = \frac{\partial^2\mathbf{G}}{\partial t^2} * \mathbf{F},$$

$$\frac{\partial^2\mathbf{v}}{\partial x_j^2} = \frac{\partial^2}{\partial x_j^2}(\mathbf{G} * \mathbf{F}) = \frac{\partial^2\mathbf{G}}{\partial x_j^2} * \mathbf{F}$$

so

$$\rho\frac{\partial^2\mathbf{v}}{\partial t^2} - L[\mathbf{v}] = \rho\frac{\partial^2(\mathbf{G} * \mathbf{F})}{\partial t^2} - L[(\mathbf{G} * \mathbf{F})] = (\rho\frac{\partial^2\mathbf{G}}{\partial t^2} - L[\mathbf{G}]) * \mathbf{F} = \delta * \mathbf{F} = \mathbf{F},$$

$$\mathbf{v}(x,t)\Big|_{t<0} = 0.$$

□

Using Lemma 3.8.1 and Lemma 3.8.2 we conclude that solution of the IVP (3.8.1), (3.8.2) can be obtained using fundamental solution that we have mention in Chapter 2.

In this section, we have an application that shows importance of fundamental solution constructing solution to the given Cauchy problem. There are various kinds of problems and methods (for example, boundary element method) where fundamental solution takes an important place for constructing solutions.

3.9 Concluding Remarks

In this chapter of the thesis, a new method for finding fundamental solutions of elastic system depending on two (x_3,t), three (x_2,x_3,t) and four (x_1,x_2,x_3,t) variables with different inhomogeneous term and initial data that have finite support is described. This method is based on Fourier transformation, Paley-Wiener theorem and some properties of fundamental solutions. Computational examples confirm robustness of the method. By theoretical study and computational examples we conclude that this method can be applied modeling fundamental solutions of elastic system in different types of homogeneous anisotropic media.

CHAPTER FOUR
CONCLUSION

In the thesis two new analytic methods are applied for solving initial value problems for equations of anisotropic electrodynamics and elasticity.

The first method is PS method. This method is based on polynomial presentation of initial data and inhomogeneous terms of the systems. The solutions are found also in the form of polynomials with respect to space variables. This method uses essentially symbolic calculations. In the thesis PS method has been applied for solving an initial value problem of the system of electrodynamics related with recovering the electric field in electrically and magnetically anisotropic materials. In the thesis the robustness of this method has been checked by computational examples. Using PS method the simulation of electric fields has been obtained in different electrically and magnetically anisotropic materials and presented in the form of pictures. In the thesis we show that PS method can be used for construction of approximate solutions of electrodynamic system if initial data and inhomogeneous term have non polynomial presentation (for example, smooth or continuous functions). Theorem about the estimate of approximate and exact solutions has been proved in the thesis.

The second analytic method is based on properties of hyperbolic systems: solutions of hyperbolic system have finite supports with respect to space variables for any fixed time variable if initial data and inhomogeneous terms have finite supports. This property is proved in the thesis for the system of anisotropic elasticity. Using Paley-Wiener theorem we find that the Fourier image of the solutions with finite supports with respect to space variables are analytic functions with respect to Fourier parameters. Using the presentation of unknown solution in the form of a power series with respect to Fourier parameters we can find the recurrence relations for unknown coefficients of the power series from the considered system of anisotropic elasticity and initial data. Using these relations we recover all unknown coefficients. Finally, the inverse Fourier transform is applied for the constructed power series for Fourier image of the solution. This method has been applied in the thesis for the construction of the fundamental solutions for equations of anisotropic elasticity. The method has been tested and its robustness has been checked. The simulation of elastic wave propagations in different anisotropic materials has been obtained in the thesis.

REFERENCES

At′ya, M. F. and Bott, R. and Gording, L. (1984). Lacunas for hyperbolic differential operators with constant coefficients.II. *Akademiya Nauk SSSR i Moskovskoe Matematicheskoe Obshchestvo*. Uspekhi Matematicheskikh Nauk, 39, N.3(237), 171-224.

Atiyah, M. F. (1970). Resolution of singularities and division of distributions. *Communications on Pure and Applied Mathematics*, 23, 145-150.

Barnett, D. M.& Lothe, J. (1975). Line force loadings on anisotropic half-spaces and wedges. *Phys. Norv.* 8, 13-22.

Barnett, D. M. (1972). *The precise evaluation of derivatives of the anisotropic elastic Greens function.* Phys. Status Solidi B. 49, 741-748.

Beltzer, A.I. (1990). Engineering analysis via symbolic computation - a breakthrough. *Appl Mech Rev.* 43, 119-127.

Berini, P. & Wu, K. (1996). Modelling lossy anisotropic dielectric waveguides with the method of lines. *IEEE transactions on microwave and techniques*; 44 (5), 749-759.

Bernšteĭn, I. N. (1971). Modules over a ring of differential operators. An investigation of the fundamental solutions of equations with constant coefficients. *Akademija Nauk SSSR.* Funkcional′ nyi Analiz i ego Priloženija, 5(2), 1-16.

Bunciu, C., Leger, C., Thiel, J. A. (1998). *Fuorier-Shannon approach to closed contours modelling*. Bioimaging, 6, 111-125.

Borovikov, A. (1959). Fundamental solutions of linear partial differential equations with constant coefficients. *Trudy Moskovskogo Matematičeskogo Obščestva*, 8, 199-257.

Burridge, R. & Qian, J. (2006). The fundamental solution of the time-dependent system of crystal optics. *European J. Appl. Math.* 17(1), 63-94.

Carcione, J. M. & Kosloff, D. &Kosloff, R. (1988). *Wave-propagation simulation in an elastic anisotropic (transversely isotropic) solid*, Vol. 41. The Quarterly Journal of Mechanics and Applied Mathematics.

Cohen, Gary and Fauqueux, Sandrine. (2005). Mixed spectral finite elements for the linear elasticity system in unbounded domains. *SIAM Journal on Scientific Computing*,26(3): 864-884 (electronic).

Cohen, GC, Heikkola, E, Joly, P, Neittaan MP,(2003). *Mathematical and numerical aspects of waves propagation*. Springer Verlag, Berlin.

Cohen, Gary C. (2002). *Higher-order numerical methods for transient wave equations*. Scientific Computation. With a foreword by R. Glowinski. Springer-Verlag. Berlin.

Courant, R. and Hilbert, D. (1962). *Methods of Mathematical Physics*. International Science. New York.

Dieulesaint, E., & Royer, D. (1980). *Elastic waves in solids*. (A. Bastin, & M. Motz, Trans.). Chichester: John Wiley & Sons. (Original work published 1974).

Ehrenpreis, L. (1960). Solution of some problems of division. IV. Invertible and elliptic operators. *American Journal of Mathematics*, 82: 522-588.

Eom, H.J. (2004). *Electromagnetic wave theory for boundary-value problems*. Springer, Berlin.

Evans, Lawrence C. (1998). *Partial differential equations*. Graduate Studies in Mathematics. V.19. American Mathematical Society. Providence, RI.

Fedorov F. I. (1963). *The theory of elastic waves in crystals. Comparison with isotropic medium*. Soviet Physics Crystallography,8: 159-163.

Folland, Gerald B. (1995). *Introduction to partial differential equations*. Princeton University Press.

Fourier, J. (1818). Note relative aux vibrations des surfaces lastiques et au mouvement des ondes. *Bull. Sci. Soc.* Philomathique Paris, 129-136.

Fredholm, Ivar. (1900). Sur les équations de l'équilibre d'un corps solide élastique. *Acta Mathematica*, 23(1): 1-42.

Fredholm, I. (1908). Sur l'intgrale fondamentale d'une quation diffrentielle elliptique coefficients constants. *Rend. Circ. Mat. Palermo* 25: 346-351.

Gel'fand, I. M. and Shilov, G. E. (1964). *Generalized functions. Vol. I: Properties and operations. Translated by Eugene Saletan.* Academic Press. New York.

Goldberg, J.L. (1992). *Matrix theory with applications*. McGraw-Hill International Editions.

Gottis, P.G. & Kondylis, G.D. (1995). Properties of the dyadic Green's function for unbounded anisotropic medium. *IEEE transactions on antennas and propagation*, 45: 154-161.

Hörmander, Lars. (1955). On the theory of general partial differential operators. *Acta Mathematica*, 94: 161-248.

Hörmander, Lars. (1957). Local and global properties of fundamental solutions. *Mathematica Scandinavica*, 5: 27-39.

Hörmander, Lars. (1958). On the division of distributions by polynomials. *Arkiv för Matematik*, 3: 555-568.

Hörmander, Lars. (1963). *Linear partial differential operators.* Academic Press Inc. Publishers, New York.

Hörmander, Lars. *The analysis of linear partial differential operators.* I-IV. Springer-Verlag. Berlin. (1983,1985).

Haba, Z. (2004). Green functions and propagation of waves in strongly inhomogeneous media. *Journal of Physics A: Mathematical and General*, 37: 9295-9302.

Herglotz, G. (1926). Über die Integration linearer partieller Differentialgleichungen mit konstanten Koeffizienten IûIII. *Ber. Schs. Akademie Wiss*. 78: 93-126.

Horváth, John. (1966). *Topological vector spaces and distributions.* Vol. I. Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont.

Horváth, John. (1970). An introuction to distributions. *The American Mathematical Monthly*, 77: 227-240.

Horváth, J. (1972). Finite parts of distributions. Linear operators and approximation (Proc. Conf., Oberwolfach, 1971)). *Internat. Ser. Numer. Math. 20,Birkhäuser. Basel,* 142-158.

Horváth, Juan. (1974). Distributions defined by analytic continuation. *Revista Colombiana de Matemáticas*, 8: 47-95.

Horváth, Jean. (1974). Sur la convolution des distributions. Bull. Sci. Math. (2). *Bulletin des Sciences Mathématiques*. 2e Série, 98(3): 183-192.

Horváth, John. (1977). Composition of hypersingular integral operators. Applicable Analysis. *An International Journal*, 7(3): 171-190.

Huang, K. & Wang, M. (1991). Fundamental solution of bi-material elastic space. *Sci. China,* 34: 309-315.

Ikawa, Mitsuru. (2000). *Hyperbolic partial differential equations and wave phenomena. Translations of Mathematical Monographs*. V.189. Translated from the 1997 Japanese original by Bohdan I. Kurpita, Iwanami Series in Modern Mathematics. American Mathematical Society. Providence, RI.

Karchevsky, A. L. & Yakhno, V. G. (1999). One-dimensional inverse problems for systems of elasticity with a source of explosion type. *Journal of Inverse and Ill-Posed Problems*, 7(4): 329-346.

Knops, R. J. & Payne, L. E. (1971). *Uniqueness theorems in linear elasticity*. Springer Tracts in Natural Philosophy, Vol. 19. Springer-Verlag. New York.

Kong, J.A. (1986). *Electromagnetic wave theory*. John Wiley and Sons, New York.

Landau, L.D., & Lifshitz, E.M. (1998). *Theory of elasticity* (3rd ed.). (J.B. Sykes, & W.H. Reid, Trans.). Oxford: Butterworth-Heinemann. (Original work published 1986).

Laplace, S. (1787). Mmoire sur la thorie de l'anneau de Saturne. *Mm. Acad. Roy. Sci.* Paris, 201-234.

Laplace, S. and cole Polytch, J. (1809). *quoted in Enc. Math. Wiss. Band II, 1. Teil, 2. Halfte*, 240.

Leray, Jean. (1955). *Hyperbolic differential equations.* The Institute for Advanced Study, Princeton, N. J. 238, Reprinted November.

Li, L.W. at al. (2001). Circular cylindrical waveguide filled with uniaxial anisotropic media-electromagnetic fields and dyadic Green's functions. *IEEE transactions on microwave and techniques*, 49(7): 1361-1364.

Lindell, I.V. (1990). Time-domain TE/TM decomposition of electromagnetic sources. *IEEE transactions on antennas and propagation*, 38(3): 353-358.

Love, A. E. H. (1944). *A treatise on the Mathematical Theory of Elasticity.* Dover Publications, New York.Fourth Ed.

Łojasiewicz, S. (1959). Sur le problème de la division. *Polska Akademia Nauk. Instytut Matematyczny. Studia Mathematica,* 18: 87-136.

Lützen, Jesper. (1982). *The prehistory of the theory of distributions. Studies in the History of Mathematics and Physical Sciences, 7.* Springer-Verlag. New York.

Ma, C. C. & Lin , R. L. (2001). Image singularities of Greens functions for an isotropic elastic half-plane subjected to forces and dislocations. *Math. Mech. Solids*, 6: 503-524.

Malgrange, Bernard. (1955-1956). Existence et approximation des solutions des équations aux dérivées partielles et des équations de convolution. *Ann. Inst. Fourier, Grenoble. Université de Grenoble. Annales de l'Institut Fourier*, 6, 271-355.

Meise, R. and Taylor, B. A. and Vogt, D. (1990). Characterization of the linear partial differential operators with constant coefficients that admit a continuous linear right inverse. *Université de Grenoble. Annales de l'Institut Fourier*, 40(3): 619-655.

Menahem, A.B., & Singh, S. J. (2000). *Seismic Waves and Sources (2nd ed.).* New York: Dover.

Meyer, Carl. (2000). *Matrix analysis and applied linear algebra*. With 1 CD-ROM (Windows, Macintosh and UNIX) and a solutions manual . Society for Industrial and Applied Mathematics (SIAM). Philadelphia.

Mindlin, R. D. &Cheng, D. H. (1950). Nuclei of strain in the semi-infinite solid. *Jour. Appl. Phys.*, 21: 926-930.

Mindlin, R. D. (1936). Force at point in the interior of a semi-infinite solid. *Physics*. 7, 195-202.

Mizohata, Sigeru. (1973). *The theory of partial differential equations*. Translated from the Japanese by Katsumi Miyahara. New York: Cambridge University Press.

Monk, Peter. (2003). *Finite element methods for Maxwell's equations*. Numerical Mathematics and Scientific Computation. Oxford University Press. New York.

Mura, T. (1987). *Micromechanics of Defects in Solids, 2nd edn.* Phys. The Netherlands: Martinus Nijhoff.

Nye, J.F. (1967). *Physical Properties of Crystals: Their Presentation by Tensors and Matrices*. Clarendon Press, Oxford.

Ortner, N. & Wagner, P. (2004). Fundamental matrices of homogeneous hyperbolic system. Applications to crystal optics, elastodynamics, and piezoelectromagnetism. *Z. Angew. Math. Mech.*, 84(5): 314-346.

Ortner, Norbert. Regularisierte Faltung von Distributionen. II. (1980). Eine Tabelle von Fundamentallösungen. *Zeitschrift für Angewandte Mathematik und Physik. ZAM Journal of Applied Mathematics and Physics. Journal de Mathématiques et de Physiques Appliquées,* 31(1): 155-173.

Ortner, Norbert and Wagner, Peter. (1997). A survey on explicit representation formulae for fundamental solutions of linear partial differential operators. *Acta Applicandae Mathematicae. An International Survey Journal on Applying Mathematics and Mathematical Applications,* 47(1): 101-124.

Ortner, Norbert and Wagner, Peter. (2004). Fundamental matrices of homogeneous hyperbolic systems. Applications to crystal optics, elastodynamics, and piezoelectromagnetism. *ZAMM. Zeitschrift für Angewandte Mathematik und Mechanik. Journal of Applied Mathematics and Mechanics,* 84(5), 314-346.

Pan, E. & Yuan, F. G. (2000). Three-dimensional Greens function in anisotropic bimaterials. *Int. J. Solids Struct.*, 37: 5329-5351.

Pavlovic, M.N. & Sapountzakis, E.J. (1986). Computers and structures: non-numerical applications. *Computers and Structures,* 24: 119-127.

Pavlovic, M.N. (2003). Symbolic computation in structural engineering. *Computers and Structures,* 81: 2121-2136.

Petrowsky, I. (1945). On the diffusion of waves and the lacunas for hyperbolic equations. *Rec. Math. [Mat. Sbornik] N. S.*, 17(59): 289-370.

Phan-Thien, N. (1983). On the image system for Kelvin-state. *Jour. Elasticity*, 13: 231-235.

Poisson, S.D. (1813). Remarques sur une quation qui se prsente dans la thorie de l'attraction des sphrodes. *Bulletin de la Socit, Philomathique de Paris*, 3: 388-392.

Poisson, S.D. (1818). Sur l'intgrale de l'quation relative aux vibrations des surfaces lastiques et au mouvement des ondes. *Bulletin de la Socit*, Philomathique de Paris, 125-128.

Poisson, S.D.(1818). *Mm. Acad. Sci*. Paris 3, 131.

Poisson, S.D. (1829). Mmoire sur l'quilibre et le mouvement des corps lastiques. *Mm. Acad. Sci.* Paris 8, 357-570.

Qu, J. M. & Li, Q. Q. (1991). Interfacial dislocation and its application to interface cracks in anisotropic bimaterials. *Jour. Elast.*, 26: 169-195.

Ramo, S., Whinnery, J.R., Duzer T. (1994). *Fields and waves in communication electronics*. New York: John Wiley and Sons, New York.

Cohen, G.C., Heikkola, E., Joly, P., Neittaanmäki, P. (Eds.). (2003). *Mathematical and Numerical Aspects of Wave Propagation*. Springer, Berlin.

Reed, Michael and Simon, Barry. (1975). *Methods of modern mathematical physics. II. Fourier analysis, self-adjointness*. Academic Press [Harcourt Brace Jovanovich Publishers]. New York.

Royer, Daniel and Dieulesaint, Eugène. (2000). *Elastic waves in solids. II*. Advanced Texts in Physics. Generation, acousto-optic interaction, applications, Translated from the 1999 French original by Stephen N. Lyle. Springer-Verlag. Berlin.

Nagle, R.K., Saff, E.B., Snider, A.D. (2004). *Fundamentals of Differential Equations and Boundary Value Problems*. Addison Wesley, Boston.

Schwartz, L. (1966). *Thorie des distributions.* (Nou d. ed.), Hermann. Paris, 357-570.

Stokes, G.G. (1849). On the dynamical theory of diffraction. *Trans. Cambridge Phil. Soc.*, 9: 1-62.

Stokes, G. G. (1883). *Mathematical and physical papers.* Cambridge University Press.

Suo, Z. G. (1990). Singularities, interfaces and cracks in dissimilar anisotropic media. *Proc. R. Soc.* Lond. Ser. A. 427: 331-358.

Synge, J. L. (1957). *The Hypercircle in Mathematical Physics.* Cambridge: Cambridge University Press.

Ting, T.C.T., & Barnett D.M., & Wu, J.J. (1990). *Modern Theory of Anisotropic Elasticity and Applications*. Philadelphia: SIAM.

Ting, T. C. T. & Lee, V. G. (1997). The three-dimensional elastostatic Greens function for general anisotropic linear elastic solids. *Q. J. Mech. Appl. Math.*, 50: 407-426.

Ting, T. C. T. (1991). *The Stroh formalism and certain invariances in two-dimensional anisotropic elasticity. Modern theory of anisotropic elasticity and applications (Research Triangle Park, NC, 1990)*, 3-32. SIAM. Philadelphia, PA.

Ting, T. C. T. (1996). *Anisotropic elasticity*. Oxford Engineering Science Series. V.45. Theory and applications. Oxford University Press. New York.

Ting, T. C. T. (1999). Recent developments in anisotropic elasticity. Research trends in solid mechanics. *Internat. J. Solids Structures. International Journal of Solids and Structures,*37:401-409.

Trèves, François. (1962). Fundamental solutions of linear partial differential equations with constant coefficients depending on parameters. *American Journal of Mathematics*, 84: 561-577.

Trèves, François. (1962). Un théorème sur les équations aux dérivées partielles à coefficients constants dépendant de paramètres. *Bulletin de la Société Mathématique de France*, 90: 473-486.

Trèves, François. (1966). *Linear partial differential equations with constant coefficients: Existence, approximation and regularity of solutions.* Mathematics and its Applications, Vol. 6, *Gordon and Breach Science Publishers*. New York.

Vladimirov, V. S. (1979). *Generalized functions in mathematical physics*. Translated from the second Russian edition by G. Yankovskiĭ. Moscow: "Mir".

Volterra, Vito. (1894).Sur les vibrations des corps élastiques isotropes. *Acta Mathematica*, 18(1):161-232.

Wagner, Peter. (1999). Fundamental solutions of real homogeneous cubic operators of principal type in three dimensions. *Acta Mathematica*, 182(2): 283-300.

Wagner, Peter. (2000). A fundamental solution of N. Zeilon's operator. *Mathematica Scandinavica,* 86(2): 273-287.

Wagner. (2001). *Zeilon's operator and lacunae. H. Florian, N. Ortner, F. Schnitzer and W. Tutschke, Editors, FunctionalûAnalytic and Complex Methods, their Interactions and Applications to Partial Differential EquationsProc.* PDE-Workshop 2001 in Graz, , World Scientific, Singapore.

Walker, K. P. (1993). Fourier integral representation of the Green function for an anisotropic elastic half-space. *Proc. R. Soc.* Lond. Ser. A. 443: 367-389.

Wei, G.W. (2001). Discrete singular convolution for beam analysis. *Engineering Structures*, 23, 1045-1053.

Werner, G.R. & Cary, J.R. (2007). *A stable FDTD algorithm for non-diagonal, anisotropic dielectrics*. Journal of Computational Physics.

Wijnands, F. at al. (1997). Green's functions for Maxwell's equations: application to spontaneous emission. *Optical and Quantum Electronics*, 29: 199-216.

Willis, J. R. (1966). Hertian contact of anisotropic bodies. *Int. J. Mech. Phys. Solids*, 50(14): 163-176.

Wu, K. C. (1998). Generalization of the Stroh formalism to 3-dimensional anisotropic elasticity. *J. Elast.* 51: 213-225.

Yakhno, V. G. & Akmaz, H. K. (2005). Initial value problem for the dynamic system of anisotropic elasticity. *International Journal of Solids and Structures*,42:855-876.

Yakhno, V.G., Yakhno, T.M., Kasap, M. (2006). A novel approach for modelling and simulation of electromagnetic waves in anisotropic dielectrics. *International Journal of Solids and Structures*, 43: 6261-6276.

Yakhno, V. G. & Merazhov, I. Z.. Direct problems and a one-dimensional inverse problem of electroelasticity for “slow” waves [translation of Mat. Tr. **2** (1999), no. 2, 148-213; MR1767828 (2001k:74052a)]. *Siberian Advances in Mathematics*, 10(1): 87-150.

Yakhno, V. G. (1992). Inverse problems for a system of differential equations of elasticity. Conditionally well-posed problems in mathematical physics and analysis (Russian) *Ross. Akad. Nauk Sib. Otd., Inst. Mat.*, Novosibirsk, , 255-266.

Yakhno, Valery G. (1998). Inverse problem for differential equations system of electromagnetoelasticity in linear approximation. *Inverse problems, tomography, and image processing (Newark, DE, 1997)*, 211-240. Plenum. New York.

Yakhno, V. G. (2002). *Multidimensional inverse problems for hyperbolic equations with point sources. Ill-posed and inverse problems*.443-468. VSP. Zeist.

Yakhno, Valery G. (2003). *1-D inverse problem of thermoelasticity. Mathematical and numerical aspects of wave propagation–WAVES*. Springer. Berlin.

Yakhno, V. G. & Akmaz, H. K. (2005). Anisotropic elastodynamics in a half space: An analytic method for polynomial data. *Journal Computational and Applied Mathematics.*

Yakhno, V. G. & Akmaz, H. K. (2005). Modeling and simulating waves in anisotropic elastic solids. *Lecture Notes in Computer Science*, 3401: 574 - 582.

Yakhno, V.G. (2005). Constructing Green's function for the time-dependent Maxwell system in anisotropic dielectrics. *Journal of Physics A: Mathematical and General*, 38(10): 2277-2287.

Yakhno, V.G., Yakhno, T.M. (2007). Modeling and simulation of electric and magnetic fields in homogeneouis non-dispersive anisotropic materials. *Computers and Structures*, 85: 1623-1634.

Zachariasen, W. H. (1952). A new analytical method for solving complex crystal structures. *Acta Cryst.*, 5, 68-73.

Zeilon, N. (1911). Das Fundamental integral der Allgemeinen Partiellen Linearen Differentialgleichung mit konstanten Koeffizienten. *Arkiv f. Mat. Astr. o. Fys.*, 6: 1-32.

Zeilon, N. (1913). Sur les intgrales fondamentales des quations charactristique relle de la Physique Mathmatique. *Arkiv f. Mat. Astr. o. Fys.*, 9 (18): 1-70.

Zienkiewicz, O. C. & Taylor, R. L. (2000). *The finite element method*. Vol. 3. Ed.5th. Fluid dynamics. Butterworth-Heinemann. Oxford.

APPENDICES

A.1 Some Facts From Matrix Theory

This section contains some basic facts from matrix theory related with symmetric and positive-definite matrices, which are used inside the thesis (Goldberg, 1992).

Theorem A.1.1. *Let* $\mathbf{C}$ *be a real, symmetric, positive-definite matrix of dimension* $m \times m$*. Then* $\mathbf{C}^{-1}$ *is a real, symmetric, positive-definite matrix.*

Proof. Since $\mathbf{C}^{-1}\mathbf{C} = \mathbf{C}\mathbf{C}^{-1} = \mathbf{I}$, using the symmetry property of $\mathbf{C}$ and the rule $(\mathbf{AB})^* = \mathbf{B}^*\mathbf{A}^*$ we get $\mathbf{I} = \mathbf{C}(\mathbf{C}^{-1})^*$. Multiplying both sides of the last equality by $\mathbf{C}^{-1}$ from left-hand side we get $\mathbf{C}^{-1} = (\mathbf{C}^{-1})^*$, which implies symmetry property of $\mathbf{C}^{-1}$.

If λ is an eigenvalue of $\mathbf{C}$, then $\frac{1}{\lambda}$ is an eigenvalue of $\mathbf{C}^{-1}$. Since the eigenvalues of a positive-definite matrix are positive, all eigenvalues of $\mathbf{C}$ are positive, which implies that $\mathbf{C}^{-1}$ has all positive eigenvalues. Hence $\mathbf{C}^{-1}$ is positive-definite. □

Theorem A.1.2. *Let* $\mathbf{C}$ *be a real, symmetric, positive-definite matrix of dimension* $m \times m$*. Then there exists a real, symmetric, positive-definite matrix* $\mathbf{M}$ *such that* $\mathbf{C}^{-1} = \mathbf{M}^2$*.*

Proof. According to Theorem A.1.1, $\mathbf{C}^{-1}$ is real, symmetric, positive-definite and is congruent to a diagonal matrix of its eigenvalues. That is, there exists an orthogonal matrix $\mathbf{Q}$ such that

$$\mathbf{Q}^*\mathbf{C}^{-1}\mathbf{Q} = \Lambda, \qquad \mathbf{Q}^* = \mathbf{Q}^{-1}. \tag{A.1.1}$$

Since $\mathbf{C}^{-1}$ is symmetric and positive-definite, its eigenvalues λ_i, $i = 1, 2, \ldots, m$ are real and positive. Let $\Lambda^{\frac{1}{2}}$ and $\mathbf{M}$ be defined as follows

$$\Lambda^{\frac{1}{2}} = \text{diag}(\lambda_i^{\frac{1}{2}}, i = 1, 2, \ldots, m), \qquad \mathbf{M} = \mathbf{Q}\Lambda^{\frac{1}{2}}\mathbf{Q}^*. \tag{A.1.2}$$

Noting that $\mathbf{Q}$ is orthogonal we have $\mathbf{Q}^*\mathbf{Q} = \mathbf{I}$, and therefore

$$\mathbf{M}^2 = (\mathbf{Q}\Lambda^{\frac{1}{2}}\mathbf{Q}^*)(\mathbf{Q}\Lambda^{\frac{1}{2}}\mathbf{Q}^*) = \mathbf{Q}\Lambda\mathbf{Q}^* = \mathbf{C}^{-1}. \tag{A.1.3}$$

Clearly, $\mathbf{M} = \mathbf{Q}\Lambda^{\frac{1}{2}}\mathbf{Q}^*$ is positive-definite. □

Theorem A.1.3. *Let* $\mathbf{A}_j$, $\mathbf{S}$ *be real and symmetric matrices of dimension* $m \times m$. *Then the matrix* $\tilde{\mathbf{A}}_j = \mathbf{S}\mathbf{A}_j\mathbf{S}$ *is real and symmetric.*

Proof. The proof follows from equalities

$$\tilde{\mathbf{A}}_j^* = (\mathbf{S}\mathbf{A}_j\mathbf{S})^* = \mathbf{S}^*(\mathbf{S}\mathbf{A}_j)^* = \mathbf{S}^*\mathbf{A}_j^*\mathbf{S}^* = \mathbf{S}\mathbf{A}_j\mathbf{S} = \tilde{\mathbf{A}}_j. \tag{A.1.4}$$

□

A.2 Hyperbolicity of Elastic and Electromagnetic Systems

A.2.1 Hyperbolicity of Elastic system for Isotropic Media

In isotropic medium elastic moduli $\{c_{jklm}\}_{j,k,l,m=1}^3$ which is a forth-order positive definite constant tensor defined as (see, (Dieulesaint & Royer, 1980), p.141).

$$c_{jklm} = \lambda\delta_{jk}\delta_{lm} + \mu(\delta_{jl}\delta_{km} + \delta_{jm}\delta_{kl}), \tag{A.2.1}$$

and satisfy the symmetry conditions $c_{jklm} = c_{kjlm} = c_{jkml}$.

Lemma A.2.1. *If* $\{c_{jklm}\}_{j,k,l,m=1}^3$ *is positive definite, then* $\sum\limits_{j,k,l,m=1}^{3} c_{jklm}\xi_k\xi_m\eta_j\eta_l > 0$.

Proof. Let $\epsilon_{jk} = \frac{1}{2}(\xi_j\eta_k + \xi_k\eta_j)$; $j,k = 1,2,3$ are non-zero vectors. Since $\{c_{jklm}\}_{j,k,l,m=1}^3$ is positive-definite then we have

$$\sum_{j,k,l,m=1}^{3} c_{jklm}\epsilon_{jk}\epsilon_{lm} > 0 \ ,$$

$$\frac{1}{2}\sum_{j,k,l,m=1}^{3} c_{jklm}(\xi_j\eta_k+\xi_k\eta_j)\epsilon_{lm} > 0\ ,$$

$$\frac{1}{2}\sum_{j,k,l,m=1}^{3} c_{jklm}\xi_j\eta_k\epsilon_{lm} + \frac{1}{2}\sum_{j,k,l,m=1}^{3} c_{jklm}\xi_k\eta_j\epsilon_{lm} > 0\ ,$$

changing $j \leftrightarrow k$ to the first part of the summation and since $c_{jklm} = c_{kjlm}$ we have

$$\frac{1}{2}\sum_{j,k,l,m=1}^{3} c_{kjlm}\xi_k\eta_j\epsilon_{lm} + \frac{1}{2}\sum_{j,k,l,m=1}^{3} c_{jklm}\xi_k\eta_j\epsilon_{lm} > 0\ ,$$

$$\frac{1}{2}\sum_{j,k,l,m=1}^{3} c_{jklm}\xi_k\eta_j\epsilon_{lm} + \frac{1}{2}\sum_{j,k,l,m=1}^{3} c_{jklm}\xi_k\eta_j\epsilon_{lm} > 0\ ,$$

$$\sum_{j,k,l,m=1}^{3} c_{jklm}\xi_k\eta_j\epsilon_{lm} > 0\ ,$$

and setting $\epsilon_{lm} = \frac{1}{2}(\xi_l\eta_m + \xi_m\eta_l)$; $l,m = 1,2,3$ are non-zero vectors. We have

$$\sum_{j,k,l,m=1}^{3} c_{jklm}\xi_k\eta_j\epsilon_{lm} > 0\ ,$$

$$\sum_{j,k,l,m=1}^{3} c_{jklm}\xi_k\eta_j\frac{1}{2}(\xi_l\eta_m + \xi_m\eta_l) > 0\ ,$$

$$\frac{1}{2}\sum_{j,k,l,m=1}^{3} c_{jklm}\xi_k\eta_j\xi_l\eta_m + \frac{1}{2}\sum_{j,k,l,m=1}^{3} c_{jklm}\xi_k\eta_j\xi_m\eta_l > 0\ ,$$

changing $l \leftrightarrow m$ to the first part of the summation and since $c_{jklm} = c_{jkml}$ we have

$$\frac{1}{2}\sum_{j,k,l,m=1}^{3} c_{jkml}\xi_k\eta_j\xi_m\eta_l + \frac{1}{2}\sum_{j,k,l,m=1}^{3} c_{jklm}\xi_k\eta_j\xi_m\eta_l > 0\ ,$$

$$\frac{1}{2}\sum_{j,k,l,m=1}^{3} c_{jklm}\xi_k\eta_j\xi_m\eta_l + \frac{1}{2}\sum_{j,k,l,m=1}^{3} c_{jklm}\xi_k\eta_j\xi_m\eta_l > 0\ ,$$

$$\sum_{j,k,l,m=1}^{3} c_{jklm}\xi_k\eta_j\xi_m\eta_l > 0\ ,$$

$$\sum_{j,k,l,m=1}^{3} c_{jklm}\xi_k\eta_j\xi_m\eta_l \neq 0.$$

□

Using (A.2.1) system of elasticity in isotropic medium can be rewritten as follows

$$\rho\frac{\partial^2\mathbf{u}}{\partial t^2} = \sum_{i=1}^{3}\sum_{j=1}^{3} A_{ij}\frac{\partial^2\mathbf{u}}{\partial x_i\partial x_j} + \mathbf{f}, \tag{A.2.2}$$

where $\mathbf{f} = (f_1, f_2, f_3)$ and $A_{ij}(i,j = 1,2,3)$ are symmetric matrices whose components are elastic moduli. The formulas of these matrices are as follows

$$A_{11} = \begin{pmatrix} \lambda+2\mu & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \mu \end{pmatrix}, \quad A_{22} = \begin{pmatrix} \mu & 0 & 0 \\ 0 & \lambda+2\mu & 0 \\ 0 & 0 & \mu \end{pmatrix},$$

$$A_{33} = \begin{pmatrix} \mu & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \lambda+2\mu \end{pmatrix}, \quad A_{12} = \frac{1}{2}\begin{pmatrix} 0 & \lambda+\mu & 0 \\ \lambda+\mu & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$A_{13} = \frac{1}{2}\begin{pmatrix} 0 & 0 & \lambda+\mu \\ 0 & 0 & 0 \\ \lambda+\mu & 0 & 0 \end{pmatrix}, \quad A_{23} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \lambda+\mu \\ 0 & \lambda+\mu & 0 \end{pmatrix}.$$

Characteristic polynomial of equation (A.2.2) has the form

$$det\Big(\rho p^2 I_3 - A_{11}\xi_1^2 - A_{22}\xi_2^2 - A_{33}\xi_3^2 - A_{12}\xi_1\xi_2 - A_{13}\xi_1\xi_3 - A_{23}\xi_2\xi_3\Big), \tag{A.2.3}$$

where I_3 is 3×3 identity matrix.We will find characteristic roots of (A.2.3). Let us denote $\lambda+\mu=\beta,\ \ \mu=\alpha$ then (A.2.3) can be written in the form

$$\begin{vmatrix} \rho p^2 - \alpha|\xi|^2 - \beta\xi_1^2 & -\beta\xi_1\xi_2 & -\beta\xi_1\xi_3 \\ -\beta\xi_1\xi_2 & \rho p^2 - \alpha|\xi|^2 - \beta\xi_2^2 & -\beta\xi_2\xi_3 \\ -\beta\xi_1\xi_3 & -\beta\xi_2\xi_3 & \rho p^2 - \alpha|\xi|^2 - \beta\xi_3^2 \end{vmatrix} = 0.$$

Doing analysis we obtain

$$p_{1,2} = \sqrt{\frac{\mu}{\rho}}|\xi|, \quad p_{3,4} = -\sqrt{\frac{\mu}{\rho}}|\xi|,$$

$$p_5 = \sqrt{\frac{\lambda + 2\mu}{\rho}}|\xi|, \quad p_6 = -\sqrt{\frac{\lambda + 2\mu}{\rho}}|\xi|.$$

If $\mu > 0, \quad \rho > 0, \quad \lambda + 2\mu > 0$ then all eigenvalues will be real so given system is hyperbolic.

A.2.2 Hyperbolicity of Elastic system for General Anisotropic Media

In anisotropic medium $\{c_{jklm}\}_{j,k,l,m=1}^{3}$ are elastic moduli which is a forth-order positive definite constant tensor that satisfy the symmetry conditions $c_{jklm} = c_{kjlm} = c_{jkml}$. System of elasticity in anisotropic medium can be written as follows

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} = \sum_{k,m=1}^{3} A_{km} \frac{\partial^2 \mathbf{u}}{\partial x_k \partial x_m} + \mathbf{f}(x,t) \ , \qquad \text{(A.2.4)}$$

where $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{f} = (f_1, f_2, f_3)$ and A_{km} are 3×3 matrices with (j,l) component

$$A_{km}^{jl} = \frac{1}{2}(c_{jklm} + c_{lkjm}) \ .$$

Principal part of (A.2.4) is given by the formula

$$P_0 = \rho \frac{\partial^2}{\partial t^2} - \sum_{k,m=1}^{3} A_{km} \frac{\partial^2}{\partial x_k \partial x_m}$$

and for hyperbolicity we look for characteristic roots of characteristic polynomial that is

$$\det(\rho\lambda^2 I_3 - \sum_{k,m=1}^{3} A_{km}\xi_k\xi_m) \ . \qquad \text{(A.2.5)}$$

Let us denote

$$A = \sum_{k,m=1}^{3} A_{km}\xi_k\xi_m \ . \qquad \text{(A.2.6)}$$

If A is symmetric positive-definite, characteristic roots of (A.2.5) are real and positive. This shows that (A.2.4) is a hyperbolic system.

Remark A.2.2. The matrix A defined in (A.2.6) is symmetric.

Proof.

$$A^{jl} = \sum_{k,m=1}^{3} A^{jl}_{km}\xi_k\xi_m = \sum_{k,m=1}^{3} \frac{1}{2}(c_{jklm} + c_{lkjm})\xi_k\xi_m \ ,$$

changing $j \longleftrightarrow l$ we have

$$A^{jl} = \sum_{k,m=1}^{3} \frac{1}{2}(c_{lkjm} + c_{jklm})\xi_k\xi_m = \sum_{k,m=1}^{3} A^{lj}_{km}\xi_k\xi_m = A^{lj} \ .$$

□

Remark A.2.3. The matrix A defined defined as

$$A = \sum_{k,m=1}^{3} A_{km}\xi_k\xi_m$$

is positive-definite .

Proof. Let $\eta = (\eta_1, \eta_2, \eta_3)$ be an arbitrary vector. We are trying to show

$$\eta^T A\eta = \sum_{j,l=1}^{3} A^{jl}\eta_j\eta_l > 0 \ ,$$

substituting A^{jl} into this we get

$$\eta^T A\eta = \sum_{j,l=1}^{3} \eta_j\eta_l(\sum_{k,m=1}^{3} A^{jl}_{km}\xi_k\xi_m) = \frac{1}{2}\sum_{j,k,l,m=1}^{3} (c_{lkjm} + c_{jklm})\xi_k\xi_m\eta_j\eta_l \ ,$$

$$\eta^T A\eta = \frac{1}{2}\sum_{j,k,l,m=1}^{3} c_{lkjm}\xi_k\xi_m\eta_j\eta_l + \frac{1}{2}\sum_{j,k,l,m=1}^{3} c_{jklm}\xi_k\xi_m\eta_j\eta_l \ ,$$

changing $l \longleftrightarrow j$ to the first part and since $c_{jklm} = c_{kjlm}$ we have

$$\eta^T A\eta = \frac{1}{2} \sum_{j,k,l,m=1}^{3} c_{jklm}\xi_k\xi_m\eta_j\eta_l + \frac{1}{2} \sum_{j,k,l,m=1}^{3} c_{jklm}\xi_k\xi_m\eta_l\eta_j =$$

$$\sum_{j,k,l,m=1}^{3} c_{jklm}\xi_k\xi_m\eta_j\eta_l.$$

Since $\{c_{jklm}\}_{j,k,l,m=1}^{3}$ is positive definite, then by lemma (A.2.1) we have

$$\sum_{j,k,l,m=1}^{3} c_{jklm}\xi_k\xi_m\eta_j\eta_l > 0.$$

Thus

$$\eta^T A\eta = \sum_{j,l=1}^{3} A^{jl}\eta_j\eta_l > 0,$$

so A is positive definite. □

According to given definitions and theorems the matrix A defined in (A.2.6) is symmetric and positive definite which implies that its real eigenvalues are positive and there exists orthogonal T such that $T^T AT = D$ where $D = diag(d_i;\ i = 1, 2, 3)$ are eigenvalues of matrix A. So all eigenvalues of D are positive and real. Using theorems, the characteristic polynomial (A.2.5) can be rewritten in the form

$$\det(\rho\lambda^2 I_3 - A) = \det(T^T(\rho\lambda^2 I_3 - A)T) = \det(\rho\lambda^2 T^T T - T^T AT),$$

since $T^T T = I$ then

$$\det(\rho\lambda^2 I_3 - A) = \det(\rho\lambda^2 I_3 - D) = (\rho\lambda^2 - d_1)(\rho\lambda^2 - d_2)(\rho\lambda^2 - d_3),$$

where $d_j,\ j = 1, 2, 3$ are eigenvalues of D that are positive and real. Thus all roots of characteristic equation (A.2.5) are positive and real. According to the definition given elastic system is hyperbolic in anisotropic medium.

A.2.3 Hyperbolicity of the Electromagnetic System for the Isotropic Media

Let $x \in \mathbb{R}^3, t > 0$. Let $\mathbf{E} = (E_1, E_2, E_3)$, $\mathbf{f} = (f_1, f_2, f_3)$ and $\varepsilon = \varepsilon_0 I$, $\mu = I$ is a symmetric, positive-definite matrix where I is identity matrix and $\varepsilon_0 > 0$.

Consider the system (2.2.1). Principal parts of (2.2.1) for isotropic case will be

$$P_0 = \varepsilon_0 \frac{\partial^2}{\partial t^2} + curl_x curl_x. \tag{A.2.7}$$

Setting

$$\frac{\partial}{\partial t} \longleftrightarrow \lambda, \qquad \frac{\partial}{\partial x_j} \longleftrightarrow \xi_j, \qquad \frac{\partial^2}{\partial x_j \partial x_k} \longleftrightarrow \xi_j \xi_k \ ,$$

the characteristic polynomial of (A.2.7) is

$$\det(\lambda^2 \varepsilon_0 I - A(\xi)), \tag{A.2.8}$$

where

$$A(\xi) = \begin{pmatrix} -\xi_2^2 - \xi_3^2 & \xi_1\xi_2 & \xi_1\xi_3 \\ \xi_1\xi_2 & -\xi_1^2 - \xi_3^2 & \xi_2\xi_3 \\ \xi_1\xi_3 & \xi_2\xi_3 & -\xi_1^2 - \xi_2^2 \end{pmatrix} .$$

Characteristic roots of (A.2.8) can be found by solving

$$\begin{vmatrix} \varepsilon_0\lambda^2 - \xi_2^2 - \xi_3^2 & \xi_1\xi_2 & \xi_1\xi_3 \\ \xi_1\xi_2 & \varepsilon_0\lambda^2 - \xi_1^2 - \xi_3^2 & \xi_2\xi_3 \\ \xi_1\xi_3 & \xi_2\xi_3 & \varepsilon_0\lambda^2 - \xi_1^2 - \xi_2^2 \end{vmatrix} = 0 \ .$$

Taking $|\xi|^2 = \xi_1^2 + \xi_2^2 + \xi_3^2$ and using it we can rearrange the last determinant that is

$$\begin{vmatrix} \varepsilon_0\lambda^2 - |\xi|^2 - \xi_1^2 & \xi_1\xi_2 & \xi_1\xi_3 \\ \xi_1\xi_2 & \varepsilon_0\lambda^2 - |\xi|^2 - \xi_2^2 & \xi_2\xi_3 \\ \xi_1\xi_3 & \xi_2\xi_3 & \varepsilon_0\lambda^2 - |\xi|^2 - \xi_3^2 \end{vmatrix} = 0 \ ,$$

and setting $\rho = \varepsilon_0\lambda^2 + |\xi|^2$ we get

$$\begin{vmatrix} \rho + \xi_1^2 & +\xi_1\xi_2 & \xi_1\xi_3 \\ \xi_1\xi_2 & \rho + \xi_2^2 & \xi_2\xi_3 \\ \xi_1\xi_3 & +\xi_2\xi_3 & \rho + \xi_3^2 \end{vmatrix} = 0 \; .$$

Doing analysis we obtain

$$\rho^3 + \rho^2|\xi|^2 = 0 \; .$$

Solving the last equation we get

$$\rho_1 = -|\xi|^2, \quad \rho_2 = 0, \quad \rho_3 = 0 \; ,$$

and also we know that $\rho = \varepsilon_0\lambda^2 - |\xi|^2$ using this we have

$$\varepsilon_0\lambda^2 - |\xi|^2 = -|\xi|^2, \quad \varepsilon_0\lambda^2 - |\xi|^2 = 0, \quad \varepsilon_0\lambda^2 - |\xi|^2 = 0 \; .$$

Thus,

$$\lambda_{1,2} = 0,$$
$$\lambda_{3,4} = \frac{|\xi|}{\sqrt{\varepsilon_0}},$$
$$\lambda_{5,6} = -\frac{|\xi|}{\sqrt{\varepsilon_0}},$$

since $\varepsilon_0 > 0$ characteristic roots are all real and positive then according to the given definition magnetic system for isotropic case is hyperbolic.

A.2.4 Hyperbolicity of the Electromagnetic System for General Anisotropic Media

Let $x \in \mathbb{R}^3, t > 0$. Let $\mathbf{E} = (E_1, E_2, E_3)$, $\mathbf{f} = (f_1, f_2, f_3)$ and $\mu = diag(\mu_{11}, \mu_{22}, \mu_{33})$ and ε is a symmetric, positive definite matrix with elements depending on x. Assume that $\frac{\varepsilon(x)}{\mu(x)}$ is also positive definite.

Consider the system (2.3.1). Principal parts of (2.3.1) for anisotropic case will be

$$P_0 = \frac{\varepsilon(x)}{\mu(x)}\frac{\partial^2}{\partial t^2} + curl_x curl_x. \tag{A.2.9}$$

Setting

$$\frac{\partial}{\partial t} \longleftrightarrow \lambda, \qquad \frac{\partial}{\partial x_j} \longleftrightarrow \xi_j, \qquad \frac{\partial^2}{\partial x_j \partial x_k} \longleftrightarrow \xi_j\xi_k\ ,$$

the characteristic polynomial of equation (A.2.9) is

$$P_0(\xi, x, t) = det(\lambda^2 \frac{\varepsilon(x)}{\mu(x)} + A(\xi)), \tag{A.2.10}$$

where

$$A(\xi) = \begin{pmatrix} -\xi_2^2 - \xi_3^2 & \xi_1\xi_2 & \xi_1\xi_3 \\ \xi_1\xi_2 & -\xi_1^2 - \xi_3^2 & \xi_2\xi_3 \\ \xi_1\xi_3 & \xi_2\xi_3 & -\xi_1^2 - \xi_2^2 \end{pmatrix}.$$

Let us denote $\frac{\varepsilon(x)}{\mu(x)} = S(x)$.

Theorem A.2.4. *(see, (Goldberg, 1992), p.366-383) If S is positive definite and A is symmetric, positive semi-definite matrix then there exists a nonsingular matrix M such that*

$$M^T S M = I,$$

$$M^T A(\xi) M = D,$$

where $D = (d_1, d_2, d_3)$, $d_j \geqslant 0$ $(j = 1, 2, 3)$ is the diagonal matrix with eigenvalues of $S^{-\frac{1}{2}} A S^{-\frac{1}{2}}$ on diagonal.

$S^{-\frac{1}{2}} A S^{-\frac{1}{2}}$ is symmetric and positive definite which implies that its real eigenvalues are positive. So $d_j;\ \ (j = 1, 2, 3)$ are positive and real. Using this theorem (A.2.4) and below relation

$$\Big|M^T(\lambda^2 S - A)M\Big| = \Big|M^T\Big| \cdot \Big|\lambda^2 S - A\Big| \cdot \Big|M\Big|,$$

the characteristic polynomial (A.2.10) becomes

$$|\lambda^2 S - A| = \frac{1}{|M^T| \cdot |M|}\Big|M^T(\lambda^2 S - A)M\Big|$$

$$= \frac{1}{|M^T| \cdot |M|} \Big| \lambda^2 (M^T S M) - (M^T A M) \Big|$$

$$= \frac{1}{|M^T| \cdot |M|} \Big| \lambda^2 - D \Big|,$$

since $|M^T| \cdot |M| \neq 0$, the characteristic polynomial can be obtained by solving

$$|\lambda^2 - D| = 0.$$

Thus all roots of characteristic equation are positive and real. According to the definition of hyperbolicity, given magnetic system is hyperbolic in anisotropic medium.

A.3 Definitions of L^2 and Sobolev Spaces

L^2 Spaces

Let $f(x)$ denote the function defined on $\mathbb{R}^n$ and that satisfies

$$\int_{\mathbb{R}^n} |f(x)|^2 dx < \infty$$

The collection of all such functions will be denoted by $L^2(\mathbb{R}^n)$ and called as square integrable functions.

Norm of a function $f(x)$ in $L^2(\mathbb{R}^n)$ is defined as

$$||f||_{L^2} = \Big(\int_{\mathbb{R}^n} |f(x)|^2 dx \Big)^{\frac{1}{2}}.$$

The functions f and g from $L^2(\mathbb{R}^n)$ are said to be equivalent if

$$||f - g||_{L^2} = 0 \;\; i.e. \int_{\mathbb{R}^n} |f(x) - g(x)|^2 dx = 0.$$

Roughly speaking, this means that they are equal almost everywhere. It is easy to verify that this is an equivalence relation, and it follows that functions can be divided into equivalence classes, each class consisting of all functions equivalent to a given function.

Sobolev Space on $\mathbb{R}^n$ Let α_i be nonnegative integers, $\alpha = (\alpha_1, ..., \alpha_n)$, be an arbitrary multi-index where $|\alpha| = \alpha_1 + ... + \alpha_n$ and let us use notation D^α for partial derivative defined as follows

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} ... \partial x_n^{\alpha_n}}.$$

If s is a nonnegative integer, we define the Sobolev space $H^s = H^s(\mathbb{R}^n)$ of order s to be the set of all $u \in L^2(\mathbb{R}^n)$ whose derivatives $D^\alpha u$ belongs to $L^2(\mathbb{R}^n)$ for $|\alpha| \leq s$: (SeeFolland, 1995)

$$H^s = \{u \in L^2 : D^\alpha u \in L^2 \text{ for } |\alpha| \leq s\}.$$

There is an easier equivalent characterization of H^s in terms of the Fourier transform that is easier to work with:(SeeFolland, 1995)

$$u \in H^s(\mathbb{R}^n) \text{ iff } (1 + |\xi|^2)^{s/2} \hat{u} \in L^2(\mathbb{R}^n)$$

Sobolev Lemma: Let k and n be naturel numbers and s be a real number such that $s > k + \left[\frac{n}{2}\right]$. Then any function $f \in H^s(\mathbb{R}^n)$ belongs to C^k.

Definition A.3.1. The spaces $\mathcal{L}^2(\mathbb{R}^n; \mathbb{R}^m)$, $\mathcal{C}^k(\mathbb{R}^n; \mathbb{R}^m)$, $\mathcal{H}^k(\mathbb{R}^n; \mathbb{R}^m)$, $(k = 0, 1, \ldots)$ consist of all vector functions $\mathbf{w} = (w_1, w_2, \ldots, w_m)$ such that w_j belongs to $\mathcal{L}^2(\mathbb{R}^n)$, $\mathcal{C}^k(\mathbb{R}^n)$, $\mathcal{H}^k(\mathbb{R}^n)$, $j = 1, \ldots, m$, respectively.

Following theorem and corollary is from (Vladimirov, 1979)

Theorem A.3.2. *Hörmander-Lojasiewicz Theorem The equation*

$$P(D)u = f,$$

where $P(D) \neq 0$, *is solvable in* $\mathbb{S}'$ *for all* $f \in \mathbb{S}'$.

Corollary: Every nonzero linear differential operator with constant coefficients has a fundamental solution of slow growth.

Following theorem is from (Reed & Simon, 1975)

Theorem A.3.3. *Paley-Wiener Theorem A distribution $T \in \mathcal{S}'(\mathbb{R}^n)$ has compact support if and only if Fourier transform of T, $\hat{T}$, has an analytic continuation to an entire analytic function of n variables $\hat{T}(\eta)$ that satisfies*

$$|\hat{T}(\eta)| \leq C(1 + |\eta|)^N e^{R|Im\eta|} \tag{A.3.1}$$

for all $\eta \in \mathbb{C}^n$ and some constants C, N, R. Moreover, if (A.3.1) holds, the support of T is contained in the ball of radius R.

A.4 Some Existence and Uniqueness Theorems for Symmetric Hyperbolic Systems

In this section we present the existence and uniqueness theorems for first order symmetric hyperbolic systems. Consider the initial value problem for the first order symmetric hyperbolic system Let the symmetric hyperbolic system be written in the form

$$\frac{\partial \mathbf{V}}{\partial t} + \sum_{j=1}^{n} \mathbf{A}_j \frac{\partial \mathbf{V}}{\partial x_j} = \tilde{\mathbf{F}}, \quad x \in \mathbb{R}^n, \quad t \in (0, T), \tag{A.4.1}$$

$$\mathbf{V}(x, 0) = \tilde{\mathbf{\Phi}}(x) \qquad x \in \mathbb{R}^n, \tag{A.4.2}$$

where T is a fixed positive number, $\mathbf{V} = (V_1, V_2, \ldots, V_n)$ is the vector-function with components $V_j = V_j(x, t)$, $j = 1, 2, \ldots, n$, $\tilde{\mathbf{A}}_j$, $j = 1, \ldots, n$ are real, symmetric, $m \times m$ matrices with constant elements.

The statement and the proof of this theorem can be found in the book (Mizohata, 1973)

Theorem A.4.1. *Let $\mathbf{A}_j(x, t)$, $j = 1, \ldots, n$, $\mathbf{A}_j(x, t)$ be $N \times N$ symmetric matrices, $\tilde{\mathbf{\Phi}}(x) \in \mathcal{H}^m(\mathbb{R}^n; \mathbb{R}^m)$, $\tilde{\mathbf{F}}(x, t) \in \mathcal{C}([0, T]; \mathcal{H}^m(\mathbb{R}^n; \mathbb{R}^m))$, where $m = 2, 3, \ldots$. Then* *there exists a unique solution* $\mathbf{V}(x,t)$ *of the problem (A.4.1), (A.4.2) such that*

$$\mathbf{V}(x,t) \in \mathcal{C}([0,T];\mathcal{H}^m(\mathbb{R}^n;\mathbb{R}^m)) \bigcap \mathcal{C}^1([0,T];\mathcal{H}^{m-1}(\mathbb{R}^n;\mathbb{R}^m)). \tag{A.4.3}$$

Theorem A.4.2. *Let* $m \geq \left[\left|\frac{n}{2}\right|\right] + 2$ *in Theorem A.4.1. Then* $\mathbf{V}(x,t)$ *is a classical solution of (A.4.1), (A.4.2) such that*

$$\mathbf{V}(x,t) \in \mathcal{C}([0,T];\mathcal{C}^{m-\left[\left|\frac{n}{2}\right|\right]-1}(\mathbb{R}^n;\mathbb{R}^m)) \bigcap \mathcal{C}^1([0,T];\mathcal{C}^{m-\left[\left|\frac{n}{2}\right|\right]-2}(\mathbb{R}^n;\mathbb{R}^m)). \tag{A.4.4}$$

In this part of the section, we adjust the general approach Courant & Hilbert (1979) (see, pages 652-661) for finding stability estimates of solutions for the symmetric hyperbolic system. Let the symmetric hyperbolic system be written in the form given with the equations (A.4.1), (A.4.2)

Let $\xi = (\xi_1, ..., \xi_n) \in \mathbb{R}^n$ and $A(\xi) = \sum_{j=1}^{n} \mathbf{A}_j \xi_j$; $\lambda_i(\xi),\ i = 1, 2, ..., 9$ be the eigenvalues of $A(\xi)$. We defined the constant M as

$$M = \max_{i=1,2,...,9} \max_{|\xi|=1} |\,\lambda_i(\xi)|. \tag{A.4.5}$$

Let T be a given positive number. Using M and T we define the following domains

$$\Gamma = \{(x,t) : 0 \leq t \leq T, |x| \leq M(T-t)\}\,, \tag{A.4.6}$$

$$S(h) = \{x \in \mathbb{R}^n : |x| \leq M(T-h)\}\,, \quad 0 \leq h \leq T, \tag{A.4.7}$$

$$R(h) = \{(x,t) : 0 \leq t \leq h, |x| = M(T-t)\}\,. \tag{A.4.8}$$

Here Γ is the conoid with vertex $(0,T)$; $S(h)$ is the surface constructed by the intersection of the plane $t = h$ and the conoid Γ; $R(h)$ is the lateral surface of the conoid Γ bounded by $S(0)$ and $S(h)$. Let Ω be the region in $\mathbb{R}^n \times (0,\infty)$ bounded by $S(0)$, $S(h)$, $R(h)$ with the boundary $\partial\Omega = S(0) \cup S(h) \cup R(h)$. Further we assume that $\tilde{\mathbf{\Phi}}(x)$, $\tilde{\mathbf{F}}(x,t)$, $\mathbf{V}(x,t)$ are vector functions with continuously differentiable components in $S(0)$ and Γ, respectively. Multiplying (A.4.1) with $\mathbf{V}$ and integrating over Ω we have

$$\int_{\Omega}\left\langle \mathbf{V}, \frac{\partial \mathbf{V}}{\partial t}\right\rangle + \left\langle \mathbf{V}, \left(\sum_{j=1}^{n} \mathbf{A}_j \frac{\partial \mathbf{V}}{\partial x_j}\right)\right\rangle dxdt = \int_{\Omega}\left\langle \tilde{\mathbf{F}}, \mathbf{V}\right\rangle dxdt. \tag{A.4.9}$$

Noting the relations

$$\left\langle \mathbf{V}, \frac{\partial \mathbf{V}}{\partial t}\right\rangle = \frac{1}{2}\frac{\partial |\mathbf{V}|^2}{\partial t}, \tag{A.4.10}$$

$$\left\langle \mathbf{V}, \left(\sum_{j=1}^{n} \mathbf{A}_j \frac{\partial \mathbf{V}}{\partial x_j}\right)\right\rangle = \frac{1}{2}\sum_{j=1}^{n} \frac{\partial}{\partial x_j}\left\langle \mathbf{V}, \mathbf{A}_j \mathbf{V}\right\rangle \tag{A.4.11}$$

we rewrite (A.4.9) as

$$\frac{1}{2}\int_{\Omega}\left(\frac{\partial |\mathbf{V}|^2}{\partial t} + \sum_{j=1}^{n} \frac{\partial}{\partial x_j}\left\langle \mathbf{V}, \mathbf{A}_j \mathbf{V}\right\rangle\right) dxdt = \int_{\Omega}\left\langle \tilde{\mathbf{F}}, \mathbf{V}\right\rangle dxdt \tag{A.4.12}$$

Applying divergence theorem to left hand side of the last equality we find

$$\frac{1}{2}\int_{\partial\Omega}\left(|\mathbf{V}|^2 \nu_t + \sum_{j=1}^{n} \left\langle \mathbf{V}, \mathbf{A}_j \mathbf{V}\right\rangle \nu_j\right) dS = \int_{\Omega}\left\langle \tilde{\mathbf{F}}, \mathbf{V}\right\rangle dxdt, \tag{A.4.13}$$

where $\nu = (\nu_1, ..., \nu_n, \nu_t)$ is the outward unit normal on $\partial\Omega$. Since $\partial\Omega = S(0)\cup S(h)\cup R(h)$ and

$$\nu = (0, ..., 0, 1) \quad \text{on} \quad S(h), \tag{A.4.14}$$

$$\nu = (0, ..., 0, -1) \quad \text{on} \quad S(0), \tag{A.4.15}$$

$$\nu = \frac{(x_1, ..., x_n, M^2(T-t))}{(T-t)M\sqrt{1+M^2}} \quad \text{on} \quad R(h), \tag{A.4.16}$$

formula (A.4.13) takes the form

$$\frac{1}{2}\int_{S(h)} |\mathbf{V}(x,h)|^2 dx - \frac{1}{2}\int_{S(0)} |\mathbf{V}(x,0)|^2 dx + \frac{1}{2}\int_{R(h)} |\mathbf{V}(x,t)|^2 \frac{M}{\sqrt{1+M^2}} dS$$

$$+\frac{1}{2}\int_{R(h)} \sum_{j=1}^{n} \left\langle \mathbf{V}, \mathbf{A}_j \mathbf{V}\right\rangle \frac{x_j}{(T-t)M\sqrt{1+M^2}} dS$$

$$= \int_0^h \int_{S(t)} \left\langle \tilde{\mathbf{F}}(x,t), \mathbf{V}(x,t) \right\rangle dxdt. \tag{A.4.17}$$

Let us denote $\xi = (\xi_1, ..., \xi_n)$, where ξ_j are defined by

$$\xi_j = \frac{(x_j)}{(T-t)M}\ , \quad j = 1, ..., n.$$

We have $|\xi| = 1$ and the following equality is satisfied

$$\frac{1}{2} \int_{R(h)} \sum_{j=1}^{n} \langle \mathbf{V}, \mathbf{A}_j \mathbf{V} \rangle \frac{x_j}{(T-t)M\sqrt{1+M^2}} dS$$
$$= \frac{1}{2\sqrt{1+M^2}} \int_{R(h)} \langle \mathbf{V}, \mathbf{A}(\xi)\mathbf{V} \rangle \, dS \tag{A.4.18}$$

Substituting (A.4.18) into (A.4.17) we find

$$\frac{1}{2} \int_{S(h)} |\mathbf{V}(x,h)|^2 dx - \frac{1}{2} \int_{S(0)} |\mathbf{V}(x,0)|^2 dx$$

$$+ \frac{1}{2\sqrt{1+M^2}} \int_{R(h)} \left\langle \mathbf{V}, [M\mathbf{I} + \mathbf{A}(\xi)]\, \mathbf{V} \right\rangle dS$$

$$= \int_0^h \int_{S(t)} \left\langle \mathbf{V}, \tilde{\mathbf{F}}(x,t) \right\rangle dxdt. \tag{A.4.19}$$

Let us consider the matrix $M\mathbf{I} + \mathbf{A}(\xi)$. Since $\mathbf{A}(\xi)$ is diagonalizable then we can find a matrix $\mathbf{Z}$ which reduces $\mathbf{A}(\xi)$ to a diagonal matrix of its eigenvalues, denoted $\Lambda = \mathrm{diag}(\lambda_1, ..., \lambda_9)$, i.e.

$$\mathbf{Z}^{-1} \left(M\mathbf{I} + \mathbf{A}(\xi) \right) \mathbf{Z} \;\; = \;\; M\mathbf{I} + \Lambda, \tag{A.4.20}$$

Noting formula (A.4.5), we conclude that the matrix $M\mathbf{I}+\Lambda$ has non-negative elements on the diagonal. It means that the matrix $M\mathbf{I}+\mathbf{A}(\xi)$ has non-negative eigenvalues. This fact Goldberg (1992) (see, pages 365-367) implies the positive semi-definiteness of the matrix $M\mathbf{I} + \mathbf{A}(\xi)$, i.e. $\left\langle \mathbf{V}, [M\mathbf{I} + \mathbf{A}(\xi)]\, \mathbf{V} \right\rangle \geq 0$ for any $\xi \in \mathbb{R}^n,\ |\xi| = 1$ and any vector-function $\mathbf{V}(x,t)$. Using this remark and denoting $\frac{1}{2}\int_{S(\tau)}|\mathbf{V}(x,\tau)|^2dx = w(\tau)$, we find from (A.4.19)

$$w(h) \leq w(0) + \int_0^h w(\tau)d\tau + \frac{1}{2}\int_0^h\int_{S(t)}|\tilde{\mathbf{F}}|^2dxdt,$$

or

$$w(h) \leq g(h) + \int_0^h w(\tau)d\tau, \tag{A.4.21}$$

where $g(h) = \frac{1}{2}\int_0^h\int_{S(t)}|\tilde{\mathbf{F}}|^2dxdt + \frac{1}{2}\int_{S(0)}|\tilde{\mathbf{\Phi}}(x)|^2dx,$

Using Gronwall's Lemma Saff & Nagle& Snider (2004) we find from (A.4.21)

$$w(h) \leq g(h)e^h, \tag{A.4.22}$$

or

$$\int_{S(h)}|\mathbf{V}(x,h)|^2dx \leq e^h\Big[\int_{S(0)}|\tilde{\mathbf{\Phi}}(x)|^2dx + \int_0^h\int_{S(t)}|\tilde{\mathbf{F}}(x,t)|^2dxdt\Big].$$