DOKUZ EYLÜL UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

RAD-SUPPLEMENTED MODULES AND FLAT COVERS OF QUIVERS

by
Salahattin ÖZDEMİR

July, 2011
İZMİR

RAD-SUPPLEMENTED MODULES
AND
FLAT COVERS OF QUIVERS

A Thesis Submitted to the

Graduate School of Natural and Applied Sciences of

Dokuz Eylül University

In Partial Fulfilment of the Requirements for

the Degree of Doctor of Philosophy in Mathematics

by

Salahattin ÖZDEMİR

July, 2011

İZMİR

Ph.D. THESIS EXAMINATION RESULT FORM

We have read the thesis entitled **"RAD-SUPPLEMENTED MODULES AND FLAT COVERS OF QUIVERS"** completed by **SALAHATTİN ÖZDEMİR** under supervision of **ASSIST. PROF. DR. ENGİN MERMUT** with the contribution of **ASSOC. PROF. DR. SERGIO ESTRADA DOMÍNGUEZ** as the co-supervisor and we certify that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Doctor of Philosophy.

Assist. Prof. Dr. Engin MERMUT

Supervisor

Assoc. Prof. Dr. Gökhan BİLHAN

Thesis Committee Member

Assist. Prof. Dr. Özlem EGE ORUÇ

Thesis Committee Member

Prof. Dr. Ayşe Çiğdem ÖZÇAN

Examining Committee Member

Prof. Dr. Rafail ALİZADE

Examining Committee Member

Assoc. Prof. Dr. Engin BÜYÜKAŞIK

Examining Committee Member

Assist. Prof. Dr. İlhan KARAKILIÇ

Examining Committee Member

Prof.Dr. Mustafa SABUNCU
Director
Graduate School of Natural and Applied Sciences

ACKNOWLEDGEMENTS

I would like to express my deepest gratitude to my supervisor Engin Mermut (Dokuz Eylül University, İzmir) for his guidance, encouragement, advice and help during my study with him. I have always found myself lucky to study with such an ambitious and diligent advisor.

I would like to thank Pedro A. Guil Asensio, Sergio Estrada Domínguez (University of Murcia, Spain) and José Goméz-Torrecillas (University of Granada, Spain) who gave a two-week beneficial seminar entitled "Lectures on Categorical and Homological Methods in Non-Commutative Algebra" on April, 2009 in İzmir Institute of Technology (İYTE, İzmir Yüksek Teknoloji Enstitütsü). Also, I would like to thank Dilek Pusat Yılmaz (İYTE) for her organizing this seminar.

I have been to University of Murcia for six months for a research related to my PhD thesis in cooperation with my co-advisor Sergio Estrada who really got me to have a different viewpoint of academic research in that period. Chapter 5 of the thesis has been completed during this visit. Not only for his valuable guidance throughout this study, but also for his support and endless patience in my getting used to living in Murcia, I would like to express my deepest gratitude to Sergio Estrada.

I would like to thank Engin Büyükaşık (İYTE) for many reasons. Firstly, for his help and valuable suggestions in this PhD thesis; especially in Chapter 3 of the thesis, which is a joint work with him, he made a great effort. He also had valuable advice in Chapter 4. Besides, when I had any question about this work, he did his best to help me.

I would also like to thank Rafail Alizade (Yaşar University, İzmir) who suggested the relative homological algebra approach to some problems related to complements and supplements and who has introduced the coneat concept.

I would like to thank all people who have worked in translations (from German into English) of Helmut Zöschinger's articles which have been used for some of the results of this thesis: Rafail Alizade, Dilek Pusat Yılmaz, O.Can Hatipoğlu (University of Porto, Portugal), S.Eylem Erdoğan Toksoy (İYTE), Zafer Çeliköz. I would also like to thank Helmut Zöschinger (University of Munich, Germany) for his careful checking of these translations and answering our questions about his articles.

I would like to express my gratitude to TÜBİTAK (The Scientific and Technological Research Council of Turkey) for its support during my PhD research, and to YÖK (The Council of Higher Education) for its support during my stay in Spain for a research.

The last but not the least, special thanks to my dearest wife Hatice for her support in any aspect.

Salahattin ÖZDEMİR

RAD-SUPPLEMENTED MODULES AND FLAT COVERS OF QUIVERS

ABSTRACT

Let R be an arbitrary ring with unity, M be a left R-module and τ be a radical for the category of left R-modules. If V is a τ-supplement in M, then the intersection of V and $\tau(M)$ is $\tau(V)$; in particular, if V is a Rad-supplement in M, then the intersection of V and $\operatorname{Rad} M$ is $\operatorname{Rad} V$. M is τ-supplemented if and only if the factor module of M by $P_\tau(M)$ is τ-supplemented, where $P_\tau(M)$ is the sum of all τ-torsion submodules of M. If V is both a τ-supplement in M and τ-coatomic, then it is a supplement in M. Every left R-module is Rad-supplemented if and only if $R/P(R)$ is left perfect, where $P(R)$ is the sum of all left ideals I of R such that $\operatorname{Rad} I = I$. For a left duo ring R, R is Rad-supplemented as a left R-module if and only if $R/P(R)$ is semiperfect. For a Dedekind domain R, M is Rad-supplemented if and only if M/D is supplemented, where D is the divisible part of M. Max-injective R-modules and $\mathcal{N}eat$-coinjective R-modules coincide, where $\mathcal{N}eat$ is the proper class projectively generated by all simple R-modules. A ring R is a left C-ring if and only if all left max-injective R-modules are injective. Over a Dedekind domain, a homomorphism f from A to B of modules is neat in the sense of Enochs if and only if the kernel of f is in $\operatorname{Rad} A$ and the image of f is closed in B. The class of all short exact sequences determined by coclosed submodules forms a proper class. Those determined by neat epimorphisms of Enochs does not form a proper class. Torsion free covers, relative to a torsion theory, exist in the category of representations by modules of a quiver for a wide class of quivers included in the class of the source injective representation quivers provided that any direct sum of torsion free injective modules is injective. For any quiver Q, $\mathscr{F}_{cw}$-covers, that is "componentwise" flat covers, and $\mathscr{F}_{cw}^{\perp}$-envelopes exist, where $\mathscr{F}_{cw}$ is the class of all componentwise flat representations of Q. Finally, "categorical" flat covers and "componentwise" flat covers do not coincide in general, where by "categorical" flat object we mean Stenström's concept of flat object defined in terms of purity.

Keywords: supplement, complement, neat submodule, coneat submodule, Rad-supplement, coatomic, reduced, radical on modules, neat-coinjective, coclosed submodule, injectively generated proper class, neat homomorphism, coneat homomorphism, max-injective, cover, envelope, torsion free cover, flat cover, quiver, representations of a quiver, flat representation.

RAD-TÜMLENMİŞ MODÜLLER VE KUİVERLERİN DÜZ ÖRTÜLERİ

ÖZ

R birim elemanı olan herhangi bir halka, M bir sol R-modül ve τ, sol R-modüller kategorisi için bir radikal olsun. Eğer V, M'de bir τ-tümleyen ise, o zaman V ile $\tau(M)$'nin kesişimi $\tau(V)$ olur; özellikle eğer V, M'de bir Rad-tümleyen ise, o zaman V ile Rad$M$'nin kesişimi Rad$V$ olur. M τ-tümlenmişdir ancak ve ancak M'nin $P_\tau(M)$'e göre çarpan modülü τ-tümlenmiş ise, burada $P_\tau(M)$, M'nin bütün τ-burulma alt modüllerinin toplamıdır. Eğer V, M'de τ-tümleyen ve τ-koatomik ise, o zaman V, M'de tümleyendir. Her R-modül Rad-tümlenmişdir ancak ve ancak $R/P(R)$ sol mükemmel ise, burada $P(R)$, Rad$I = I$ şeklindeki R'nin sol ideallerinin toplamıdır. R sol duo halkası ise, R, bir sol R-modül olarak, Rad-tümlenmişdir ancak ve ancak $R/P(R)$ yarı-mükemmel halka ise. R Dedekind tamlık bölgesi ise, M Rad-tümlenmişdir ancak ve ancak M/D tümlenmiş ise, burada D, M'nin bölünebilir kısmıdır. Maks-injektif ile $\mathcal{N}eat$-koinjektif modüller çakışmaktadır, burada $\mathcal{N}eat$, bütün basit modüller tarafından projektif olarak üretilen bir öz sınıftır. R sol C-halka'dır ancak ve ancak bütün maks-injektif R-modüller injektif ise. Bir Dedekind tamlık bölgesi üzerinde, A'dan B'ye bir f modül homomorfizması Enochs'un tanımına göre düzenlidir ancak ve ancak f'nin çekirdeği Rad$A$'nın içinde ise ve görüntüsü B'de kapalı ise. Eşkapalı altmodüller ile tanımlanan bütün kısa tam dizilerin sınıfı bir öz sınıf biçimindedir, ama Enochs'un düzenli epimorfizmaları ile tanımlanan sınıf bir öz sınıf biçiminde değildir. Burulmasız ve injektif R-modüllerin direkt toplamının yine injektif olması durumunda, kaynak injektif temsil kuiverler sınıfında yer alan geniş bir kuiverler sınıfı icin, kuiverlerin temsilleri kategorisinde, bir burulma teorisine göre burulmasız örtüler vardır. Herhangi bir Q kuiveri için, $\mathscr{F}_{cw}$-örtüler ve $\mathscr{F}_{cw}^{\perp}$-bürümler vardır, burada $\mathscr{F}_{cw}$, Q'nun bileşenlere göre düz temsillerinin sınıfıdır. Kategorik düz örtüler ve bileşenlere göre düz örtüler genelde çakışmaz, burada "kategorik" düz nesne, Stenström'ün pür altnesneler cinsinden tanımladığı düz nesnedir.

Anahtar sözcükler: tümleyen, tamamlayan, düzenli altmodül, kodüzenli altmodül, Rad-tümleyen, koatomik, indirgenmiş, modüllerde radikal, düzenli-koinjektif, eşkapalı altmodül, injektif olarak üretilmiş öz sınıf, düzenli homomorfizma, kodüzenli homomorfizma, maks-injektif, örtü, bürüm, burulmasız örtü, düz örtü, kuiver, bir kuiverin temsilleri, düz temsil.

CONTENTS

Page

CHAPTER ONE

INTRODUCTION

In this introductory chapter, we will give the motivating ideas for our thesis problems and the main results of this thesis. In Section 1.1, the motivation for considering Rad-supplements (=coneat submodules) and in general τ-supplements for a radical τ on the category of left R-modules will be explained. See Section 1.2, for the reason for considering torsion free covers and neat homomorphisms of Enochs, C-rings of Renault and max-injective modules. In Section 1.3, we explain the motivation for the study of covers and envelopes in categories of representations by modules over quivers. To explain these problems and results, we need some basic definitions, results, preliminary notions and notation; see Chapter 2. In particular, see Sections 2.2, 2.3, 2.4, 2.5 and 2.6 for preliminary notions needed for Chapter 3 and Chapter 4; see Sections 2.7, 2.8, 2.9 and 2.10 for Chapter 5. In Chapter 3, we study in the category of left R-modules; we deal with Rad-supplemented modules and in general τ-supplemented modules, where τ is a radical for the category of left R-modules. In Chapter 4, we review some results of torsion free covers and neat homomorphisms of Enochs, and study left C-rings of Renault which turn out to be the rings where all max-injective modules are injective. In Chapter 5, we study in the category of representations by modules of a quiver; we deal with the existence of torsion free covers, relative to a torsion theory, and componentwise flat covers in this category.

Throughout this thesis, R denotes an associative ring with unity. An R-module or just a module will be a unital *left* R-module, and R-$\mathcal{M}od$ will denote the category of left R-modules. For modules A and C, $\mathrm{Ext}^1_R(C,A)$ will mean the equivalence classes of all short exact sequences starting with A and ending with C; for abelian groups, we use the notation $\mathrm{Ext}(C,A)$.

1.1 Rad-supplemented Modules

Neat subgroups of abelian groups have been introduced in Honda (1956, pp. 43-44): A subgroup A of an abelian group B is said to be *neat* in B if $A \cap pB = pA$ for every prime number p (see also Fuchs (1970, §31, p. 131)). After that, they have been generalized to modules by Stenström (1967a, 9.6) and Stenström (1967b, §3): A monomorphism $f : K \longrightarrow L$ of modules is called *neat* if every simple module S is *projective* relative to the natural epimorphism $L \longrightarrow L/\operatorname{Im} f$, that is, the Hom sequence

$$\operatorname{Hom}_R(S, L) \to \operatorname{Hom}_R(S, L/\operatorname{Im} f) \to 0$$

obtained by applying the functor $\operatorname{Hom}_R(S, -)$ to the exact sequence $L \longrightarrow L/\operatorname{Im} f \longrightarrow 0$ is exact. See Özdemir (2007) for a survey of related results on neat subgroups and neat submodules. Dually, the class of coneat submodules has been introduced in Mermut (2004) and Alizade & Mermut (2004): A monomorphism $f : K \to L$ of modules is called *coneat* if every module M with $\operatorname{Rad} M = 0$ is *injective* with respect to it, that is, the Hom sequence

$$\operatorname{Hom}_R(L, M) \to \operatorname{Hom}_R(K, M) \to 0$$

obtained by applying the functor $\operatorname{Hom}_R(-, M)$ to the exact sequence $0 \longrightarrow K \longrightarrow L$ is exact. A submodule A of a module B is said to be a *neat submodule* (respectively *coneat submodule*) if the inclusion monomorphism $A \hookrightarrow B$ is neat (resp. coneat). See Mermut (2004, Proposition 3.4.2) or Clark et al. (2006, 10.14) or Al-Takhman et al. (2006, 1.14) for a characterization of coneat submodules. This characterization is the particular case $\tau = \operatorname{Rad}$ in Proposition 1.1.1 given below and this is the reason for considering Rad-supplements and in general τ-supplements for a radical τ for $R\text{-}\mathcal{M}od$. For more results on coneat submodules see Mermut (2004), Alizade & Mermut (2004), Clark et al. (2006, §10 and 20.7–8), Al-Takhman et al. (2006) and Özdemir (2007).

A *preradical* τ for R-$\mathcal{M}od$ is defined to be a subfunctor of the identity functor on R-$\mathcal{M}od$, that is, for every module N, $\tau(N) \subseteq N$ and every homomorphism $f : N \longrightarrow M$ induces a homomorphism $\tau(N) \longrightarrow \tau(M)$ by restriction. τ is said to be *idempotent* if $\tau(\tau(N)) = \tau(N)$, and a *radical* if $\tau(N/\tau(N)) = 0$ for every module N. τ is a left exact functor if and only if $\tau(K) = K \cap \tau(N)$ for every submodule $K \subseteq N$, and in this case τ is said to be *hereditary*. For the main elementary properties that we shall use frequently for a (pre)radical, see Section 2.4. The following module classes are defined for a preradical τ for R-$\mathcal{M}od$: the *(pre)torsion class* and the *(pre)torsion free class* of τ are respectively

$$\mathbb{T}_\tau = \{N \in R\text{-}\mathcal{M}od \mid \tau(N) = N\} \quad \text{and} \quad \mathbb{F}_\tau = \{N \in R\text{-}\mathcal{M}od \mid \tau(N) = 0\}.$$

The modules in $\mathbb{T}_\tau$ are said to be τ-*torsion* and the modules in $\mathbb{F}_\tau$ are said to be τ-*torsion free*. $\mathbb{T}_\tau$ is closed under quotient modules and direct sums, while $\mathbb{F}_\tau$ is closed under submodules and direct products.

Proper classes of short exact sequences of modules were introduced in Buschbaum (1959) to do *relative* homological algebra (see Section 2.3 for the definition). We use the language of proper classes of short exact sequences of modules to investigate the relations among the concepts like complement, supplement, neat and coneat, by considering the corresponding class of short exact sequences. $\mathcal{N}eat$ is the proper class which consists of all short exact sequences of modules such that every simple module is projective with respect to it, and the proper class $\mathcal{C}ompl$ consists of all short exact sequences of modules where the monomorphism has closed image. In Stenström (1967b, Remark after Proposition 6), it is pointed out that supplement submodules induce a proper class of short exact sequences (the term 'low' is used for supplements dualizing the term 'high' used in abelian groups for complements). Generalov uses the terminology 'cohigh' for supplements and gives more general definitions for proper classes of supplements related to another given proper class motivated by the considerations as pure-high extensions and neat-high extensions in Harrison et al. (1963); see Generalov (1983).

A submodule A of a module B is *small (or superfluous)* in B, denoted by $A \ll B$, if $A+K \neq B$ for every proper submodule $K \subseteq B$. An epimorphism $f : M \longrightarrow N$ of modules is said to be *small* (or *superfluous*) if $\operatorname{Ker} f \ll M$.

Denote by $\mathcal{S}uppl$ the class of all short exact sequences induced by supplement submodules, that is, $\mathcal{S}uppl$ is the class of all short exact sequences

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0 \tag{1.1.1}$$

of modules such that $\operatorname{Im} f$ is a supplement in B, where a submodule $A \subseteq B$ is called a *supplement* in B if there is a submodule $K \subseteq B$ such that $A+K=B$ and $A \cap K \ll A$. Then as mentioned above, the class $\mathcal{S}uppl$ forms a *proper* class, see Clark et al. (2006, 20.7) and Erdoğan (2004). Every module M with $\operatorname{Rad} M = 0$ is $\mathcal{S}uppl$-injective, that is, M is injective with respect to every short exact sequence in $\mathcal{S}uppl$. Thus supplement submodules are coneat submodules by the definition of coneat submodules. In the definition of coneat submodules, using any radical τ for R-$\mathcal{M}od$ instead of Rad, the following proposition is obtained (see Proposition 2.3.4 for the characterization of coneat submodules). It gives us the definition of a τ-*supplement* in a module because the last condition is like the usual supplement condition except that, instead of $U \cap V \ll V$, the condition $U \cap V \subseteq \tau(V)$ is required.

Proposition 1.1.1. *(see Clark et al. (2006, 10.11) or Al-Takhman et al. (2006, 1.11)) Let τ be a radical for R-$\mathcal{M}od$. For a submodule V of a module M, the following statements are equivalent:*

(i) Every module N with $\tau(N) = 0$ is injective with respect to the inclusion $V \hookrightarrow M$;

(ii) there exists a submodule $U \subseteq M$ such that

$$U + V = M \text{ and } U \cap V = \tau(V);$$

(iii) there exists a submodule $U \subseteq M$ such that

$$U + V = M \text{ and } U \cap V \subseteq \tau(V).$$

If these conditions are satisfied, then V is called a τ-supplement *in M.*

Denote by τ-$\mathcal{S}uppl$ the class induced by τ-supplement submodules, that is, it consists of all short exact sequences (1.1.1) of modules such that $\operatorname{Im} f$ is a τ-supplement in B. By the above characterization of τ-supplements, the class τ-$\mathcal{S}uppl$ is the proper class injectively generated by all modules M such that $\tau(M) = 0$.

The usual definitions are then given as follows for a radical τ for R-$\mathcal{M}od$: For submodules U and V of a module M, the submodule V is said to be a τ-*supplement* of U in M or U is said to *have a* τ-*supplement* V in M if $U + V = M$ and $U \cap V \subseteq \tau(V)$. M is called a τ-*supplemented module* if every submodule of M has a τ-supplement in M. We call M *totally* τ-*supplemented* if every submodule of M is τ-supplemented. A submodule N of M is said to have *ample* τ-*supplements in* M if for every $L \subseteq M$ with $N + L = M$, there is a τ-supplement L' of N in M with $L' \subseteq L$. A module M is said to be *amply* τ-*supplemented* if every submodule of M has ample τ-supplements in M. For $\tau = \operatorname{Rad}$, the above definitions give *Rad-supplement submodules* of a module, *Rad-supplemented modules*, etc. By these definitions, we have: A submodule V of a module M is a *coneat* submodule of M if and only if V is a *Rad-supplement* of a submodule U of M in M.

The main results of Chapter 3 are given as follows. We shall investigate some properties of Rad-supplemented modules and in general τ-supplemented modules where τ is a radical for R-$\mathcal{M}od$. Rad-supplemented modules are also called *generalized supplemented* modules in Wang & Ding (2006). For a survey of related results on Rad-supplemented modules, see Özdemir (2007, Chap. 6). Remember that all R-modules are (amply) supplemented if and only if R is a left perfect ring by characterization of left perfect rings in Wisbauer (1991, 43.9); see Section 2.5 for perfect rings. One of our main questions is to characterize the rings R for which every left R-module is Rad-supplemented. In the investigation of this problem, the notions of radical modules, reduced modules and coatomic modules turn out to be useful (see Zöschinger (1974).

A module M is said to be a *radical* module if $\mathrm{Rad} M = M$. M is called *reduced* if it has no *nonzero* radical submodule, and M is called *coatomic* if it has no nonzero radical factor module.

We prove that the following are equivalent (Theorem 3.5.1):

(i) every left R-module is Rad-supplemented;

(ii) the direct sum of countably many copies of R is a Rad-supplemented left R-module;

(iii) $R/P(R)$ is left perfect, where $P(R)$ is the sum of all left ideals I of R such that $\mathrm{Rad} I = I$.

We also show that a reduced module M is totally Rad-supplemented if and only if M is totally supplemented (Corollary 3.3.20).

In Büyükaşık & Lomp (2008), it was proved that the class of Rad-supplemented rings lies properly between those of the semiperfect and the semilocal rings. We show that a *left duo ring* R (that is, a ring whose all left ideals is a two-sided ideal) is Rad-supplemented as a left R-module if and only if $R/P(R)$ is semiperfect (Theorem 3.5.6).

Whenever possible the related results are given in general for a radical τ for $R\text{-}\mathcal{M}od$. See Al-Takhman et al. (2006) and Clark et al. (2006, §10) for some results on τ-supplements and τ-supplemented modules. In Koşan & Harmanci (2004) and Koşan (2007), supplemented modules relative to a hereditary torsion theory have been studied. There is a bijective correspondence between hereditary torsion theories and left exact radicals (i.e. hereditary radicals) in $R\text{-}\mathcal{M}od$. For a hereditary torsion theory $\tau = (\mathcal{T}, \mathcal{F})$ in $R\text{-}\mathcal{M}od$, our definition of τ-supplemented modules coincide with the definition of τ-*weakly supplemented* modules introduced in Koşan (2007), but in our case, τ need not be hereditary; in particular, Rad is not hereditary. In the definitions and properties of reduced and coatomic modules, instead of Rad, we can use any (pre)radical τ on $R\text{-}\mathcal{M}od$ (see Section 3.1), and these will be useful in the investigation of the properties of τ-supplemented modules. We show that if a module M is τ-*coatomic*, that is, if M has no nonzero τ-torsion factor module, then $\tau(M)$ is small in M (Proposition 3.1.3). We also show that if a submodule V of a module M is both a τ-supplement in M and τ-coatomic, then V is a supplement in M (Proposition 3.3.18). We prove that a module M is τ-supplemented if and only if $M/P_\tau(M)$ is τ-supplemented, where $P_\tau(M)$ is the sum of all τ-torsion submodules of M (Proposition 3.3.16). For some rings R, we also determine when all left R-modules are τ-supplemented. For a ring R with $P_\tau(R) \subseteq J(R)$, every left R-module is τ-supplemented if and only if the quotient ring $R/P_\tau(R)$ is left perfect and $\tau(R) = J(R)$, where $J(R)$ is the Jacobson radical of R (Theorem 3.4.6). We also investigate the property $\operatorname{Rad} V = V \cap \operatorname{Rad} M$ for a submodule V of a module M. It is known that this property holds if V is a supplement in M (Wisbauer, 1991, 41.1) and moreover if V is coclosed in M (Clark et al., 2006, 3.7). We show that this property holds when V is a Rad-supplement in M; in general for a radical τ for $R\text{-}\mathcal{M}od$, we show that if V is a τ-supplement in M, then $\tau(V) = V \cap \tau(M)$ (Theorem 3.3.2).

Every abelian group A can be expressed as the direct sum of a divisible subgroup D and a reduced subgroup C: $A = D \oplus C$. Here D is a uniquely determined subgroup of A, it is the sum of all divisible subgroups of A and indeed it is the largest divisible subgroup of A. The subgroup C is unique up to isomorphism, and C is *reduced* means that C has no divisible subgroup other than 0. See, for example, Fuchs (1970, Theorem 21.3). This notion is also generalized to modules over Dedekind domains. Over Dedekind domains, divisible modules coincide with injective modules as in abelian groups. Note that for a module M over a Dedekind domain R, M is divisible if and only if M is a radical module (that is, $\operatorname{Rad} M = M$), and this holds if and only if M is injective; see, for example, Alizade et al. (2001, Lemma 4.4). This is the motivation for the definition of reduced modules in general. A module over a Dedekind domain is *reduced* if it has no nonzero divisible submodules (that is, if it has no nonzero radical submodules). As in abelian groups every module M over a Dedekind domain possesses a unique largest divisible submodule D and $M = D \oplus C$ for a reduced submodule C of M (see Kaplansky (1952, Theorem 8)); this D is called the *divisible part* of M, and $D = P(M)$.

We show that for a commutative noetherian ring R, a reduced R-module M is Rad-supplemented if and only if it is supplemented (Proposition 3.6.3). It is clear that every supplemented module is Rad-supplemented, but the converse is not true always. For example, the $\mathbb{Z}$-module $\mathbb{Q}$ is Rad-supplemented but not supplemented. Since $\operatorname{Rad} \mathbb{Q} = \mathbb{Q}$ (see, for example, Kasch (1982, 2.3.7)), $\mathbb{Q}$ is Rad-supplemented (by Proposition 3.3.13-(i)). But $\mathbb{Q}$ is not supplemented by Clark et al. (2006, 20.12). Moreover, we understand this example clearly and give the structure of Rad-supplemented modules over Dedekind domains in terms of supplemented modules which have been characterized in Zöschinger (1974). Over a Dedekind domain R, an R-module M is Rad-supplemented if and only if $M/P(M)$ is (Rad-)supplemented, where $P(M)$ is the divisible part of M. In fact, $P(M)$ is the sum of all submodules U of M such that $\operatorname{Rad} U = U$, that is, $P(M)$ is the largest radical submodule of M and this equals $P(M)$ to be the divisible part of M over a Dedekind domain.

1.2 Enochs' Neat Homomorphisms and Max-injective Modules

In the first part of Chapter 4, motivated by Theorem 1.2.1 given in Enochs & Jenda (2000, Chap. 4) related to torsion free covers over commutative domains, we deal with max-injective modules. Torsion free covers were first defined in Enochs (1963) and shown to exist for the usual torsion theory over a commutative domain: Over a commutative domain R, a homomorphism $\varphi : T \longrightarrow M$, where T is a torsion free R-module, is called a *torsion free cover* of M if

(i) for every torsion free R-module G and a homomorphism $f : G \longrightarrow M$ there is a homomorphism $g : G \longrightarrow T$ such that $\varphi g = f$ and,

(ii) $\operatorname{Ker}\varphi$ contains *no* non-trivial submodule S of T such that $rS = rT \cap S$ for all $r \in R$, that is, S is a *relatively divisible* submodule or shortly an *RD-submodule* of T; see Section 2.3.

If φ satisfies (i) and maybe not (ii) above, then it is called a *torsion free precover*.

It is known that given a family $\varphi_i : T_i \longrightarrow M_i$ of torsion free covers, for $i = 1, 2, \dots n$, $\bigoplus_{i=1}^{n} T_i \longrightarrow \bigoplus_{i=1}^{n} M_i$ is also a torsion free cover (see, for example, Enochs & Jenda (2000, Proposition 5.5.4)). So, the corresponding question for infinite direct products has been considered in Enochs & Jenda (2000).

Theorem 1.2.1. *(Enochs & Jenda, 2000, Theorem 4.4.1) The following are equivalent for a commutative domain R:*

(i) Every torsion R-module $G \neq 0$ has a simple submodule;

(ii) $\prod_{i \in A} \varphi_i : \prod_{i \in A} T_i \longrightarrow \prod_{i \in A} M_i$ is a torsion free cover for every family $\{\varphi_i : T_i \longrightarrow M_i\}_{i \in A}$ of torsion free covers of R-modules;

(iii) An R-module E is injective if and only if $\operatorname{Ext}^1_R(S, E) = 0$ for every simple R-module S.

The notion of C-ring has been introduced in Renault (1964): A ring R is said to be a *left* C-ring if for every (left) R-module B and for every essential proper submodule A of B, $\mathrm{Soc}(B/A) \neq 0$, that is B/A has a simple submodule. Similarly *right* C-rings are defined.

The notion of max-injectivity (a weakened injectivity in view of Baer's criterion) has been studied recently by several authors; see, for example, Crivei (1998), Crivei (2000) and Wang & Zhao (2005): A module M is said to be *maximally injective* (or *max-injective* for short) if for every maximal left ideal P of R, every homomorphism $f : P \longrightarrow M$ can be extended to a homomorphism $g : R \longrightarrow M$. A module M is max-injective if and only if $\mathrm{Ext}^1_R(S,M) = 0$ for every simple module S (see Crivei (1998, Theorem 2)). Max-injective modules are called *m-injective modules* in Crivei (1998).

In, for example, Mermut (2004, Proposition 3.3.9), it has been proved that a commutative domain R is a C-ring if and only if every nonzero torsion R-module has a simple submodule. So we observe, by Theorem 1.2.1, that for a commutative domain R, the following are equivalent:

(i) R is a C-ring;

(ii) Every direct product of torsion free covers is again a torsion free cover;

(iii) Every max-injective module is injective.

It has been proved by Patrick F. Smith that for a ring R, $\mathrm{Soc}(R/I) \neq 0$ for every essential proper left ideal I of R (that is, R is a left C-ring by Proposition 4.1.2) if and only if every max-injective module is injective (Smith, 1981, Lemma 4). This result also stated in Ding & Chen (1993) and for its proof the reference to Smith (1981) has been given. In Section 4.2, we shall give a proof of this result with our interest in the proper classes $\mathcal{N}eat$ and $\mathcal{C}ompl$, and with further observations (Theorem 4.2.14). In the articles Crivei (2000) and Wang & Zhao (2005), this result is not known; all the given examples in these articles for rings over which every max-injective module is injective are indeed left C-rings. For instance, in Crivei (1998) and Wang & Zhao (2005), it was shown that if R is a left *semi-artinian* ring (that is, $\mathrm{Soc}(R/I) \neq 0$ for every proper left ideal I of R), then every max-injective R-module is injective. But, of course, a left semi-artinian ring is a left C-ring.

For a proper class $\mathcal{P}$, a module A is called $\mathcal{P}$-coinjective if every short exact sequences of modules starting with A is in $\mathcal{P}$. See Section 2.3 for proper classes of modules. A module M is $\mathcal{P}$-coinjective if and only if it is a $\mathcal{P}$-submodule of $\mathrm{E}(M)$, the injective envelope of M. So, a module M is $\mathcal{C}ompl$-coinjective if and only if M is a complement submodule (=closed submodule) of $\mathrm{E}(M)$. Since M is essential in $\mathrm{E}(M)$, we obtain that $\mathcal{C}ompl$-coinjective modules are just injective modules. $\mathcal{N}eat$-coinjective modules and max-injective modules coincide. In Generalov (1978, Theorem 5), it was proved that a ring R is a left C-ring if and only if $\mathcal{C}ompl = \mathcal{N}eat$. So, it can be easily seen that if R is a left C-ring, then

$$\textit{max-injectives} = \mathcal{N}\textit{eat-coinjectives} = \mathcal{C}\textit{ompl-coinjectives} = \textit{injectives}$$

Conversely, we prove that if all $\mathcal{N}eat$-coinjective modules are injective, then R is a left C-ring. As a result, we have that the following are equivalent for a ring R (Theorem 4.2.18):

(i) R is a left C-ring;

(ii) All $\mathcal{N}eat$-coinjective (=max-injective) R-modules are injective;

(iii) The direct sum of all simple R-modules is a left Whitehead test module for injectivity, where a module N is called *Whitehead test module for injectivity* if for every module M, $\mathrm{Ext}^1_R(N,M) = 0$ implies M is injective.

We devote the second part of Chapter 4 to neat homomorphisms of Enochs. The study of neat homomorphisms, due to Enochs (1971) and Bowe (1972), originated with a generalization of neat subgroups and torsion free covers of modules; in Enochs

& Jenda (2000, Proposition 4.3.9) it was shown that torsion free covers are neat homomorphisms, over a commutative domain.

The concept of neat homomorphism is indeed a natural concept to consider by the following characterization: A homomorphism $f : M \longrightarrow N$ of modules is a *neat* homomorphism in the sense of Enochs if and only there are no proper extensions of f in the injective envelope $\mathrm{E}(M)$ of M, that is, there exists *no* homomorphism $g : M' \longrightarrow N$ such that $M \subsetneqq M' \subseteq \mathrm{E}(M)$ and $g\,|_M = f$. This is not the original definition, but one of the equivalent conditions of being a neat homomorphism given in Bowe (1972) (see Theorem 4.3.3). We call such homomorphisms *E-neat homomorphisms*. These E-neat homomorphisms need not be one-to-one or onto. A monomorphism $f : A \longrightarrow B$ is E-neat if and only if $\operatorname{Im} f$ is a closed submodule (=complement submodule) of B (see Lemma 4.3.4). Thus, the class of all short exact sequences

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0 \tag{1.2.1}$$

of modules such that the monomorphism α is E-neat forms the proper class $\mathcal{C}ompl$ that we have already mentioned. So, we investigate the class of all short exact sequences (1.2.1) such that the epimorphism $\beta : B \longrightarrow C$ is E-neat. We denote this class by $\mathcal{EN}eat$. We show that $\mathcal{EN}eat$ forms a proper class if and only if R is a semisimple ring (Theorem 4.5.2).

Zöschinger gave the definition of E-neat homomorphisms for abelian groups by considering the equivalent condition (iv) for being E-neat homomorphism given in Theorem 4.3.3: A homomorphism $f : M \to N$ of modules is *E-neat* if for every decomposition $f = \beta\alpha$ where α is an essential monomorphism, α is an isomorphism. See Proposition 4.4.1 for this equivalence for modules over arbitrary rings.

See Theorem 4.4.2 for the proof of the theorem given below which has been explained by Zöschinger.

Theorem 1.2.2. *(Zöschinger, 1978, Satz 2.3*) Let A and A' be abelian groups. For a homomorphism $f : A \longrightarrow A'$, the following are equivalent:*

(i) f *is* E*-neat;*

(ii) $\operatorname{Im} f$ *is closed in* A' *and* $\operatorname{Ker} f \subseteq \operatorname{Rad} A$*;*

(iii) $f^{-1}(pA') = pA$ *for all prime numbers* p*;*

(iv) If the following diagram is a pushout diagram of abelian groups and α *is an essential monomorphism, then* α' *is also an essential monomorphism:*

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & B \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ A' & \xrightarrow[\alpha']{} & B'. \end{array}$$

By considering the first two equivalent conditions for abelian groups in the previous theorem, we define Z-neat homomorphisms in general for modules over arbitrary rings: We call a homomorphism $f : A \longrightarrow A'$ of modules *Z-neat* if $\operatorname{Im} f$ is closed in A' and $\operatorname{Ker} f \subseteq \operatorname{Rad} A$. So we wonder if E-neat and Z-neat homomorphisms coincide in general for arbitrary rings. In the investigation of this problem the following result plays an important role: The natural epimorphism $f : A \longrightarrow A/K$ of modules with $K \subseteq A$ is E-neat if and only if $(A/K) \trianglelefteq (E(A)/K)$ (Corollary 4.4.5). Over a Dedekind domain, we prove that the natural epimorphism $f : A \longrightarrow A/K$ is E-neat if and only if $K \subseteq \operatorname{Rad} A$ (Proposition 4.4.14). Using these results, finally, we prove that E-neat homomorphisms and Z-neat homomorphisms coincide over Dedekind domains (Theorem 4.4.17).

As a dual to E-neat homomorphisms, Zöschinger has introduced and studied *coneat homomorphisms* when he was studying the submodules that have supplements for abelian groups in Zöschinger (1978, §2): A homomorphism $g : C' \to C$ of modules is called *Z-coneat* if for every decomposition $g = \beta\alpha$ where β is small epimorphism,

β is an isomorphism. The reason for his studying such homomorphisms, which we call Z-coneat homomorphisms not to mix them with our concept *coneat*, is that $g^* : \mathrm{Ext}(C, A) \to \mathrm{Ext}(C', A)$ preserves κ-elements for every *Z-coneat homomorphism* $g : C' \to C$. κ-*elements* of $\mathrm{Ext}(C,A)$ are the equivalence classes of κ-exact short exact sequences starting with A and ending with C; a short exact sequence (1.2.1) is called κ-*exact* if $\mathrm{Im}\,\alpha$ has a *supplement* in B. In Zöschinger (1978, Hilfssatz 2.2 (a)), it was proved that an epimorphism $f : A \longrightarrow B$ of abelian groups is Z-coneat if and only if $\mathrm{Ker} f$ is coclosed in A. We devote the last part of Chapter 4 to investigate coclosed monomorphisms of modules.

Given submodules $K \subseteq L \subseteq M$, the inclusion $K \subseteq L$ is called *cosmall in M*, denoted by $K \overset{cs}{\underset{M}{\hookrightarrow}} L$, if $L/K \ll M/K$. A submodule L of a module M is called *coclosed in M*, denoted by $L \overset{cc}{\hookrightarrow} M$, if L has no proper submodule K for which $K \overset{cs}{\underset{M}{\hookrightarrow}} L$. See Clark et al. (2006, §3.1 and §3.6) for cosmall inclusions and coclosed submodules.

We show that the class of all short exact sequences (1.2.1) such that $\mathrm{Im}\,\alpha$ is coclosed in B forms a proper class, denoted by $\mathcal{C}oclosed$ (Theorem 4.6.4). Note that Zöschinger calls a module M *weakly injective* if for every extension $M \subseteq X$, M is coclosed in X, that is, M is $\mathcal{C}oclosed$-coinjective (see Zöschinger (2006)). In Zöschinger (2006), it was shown for every noetherian, local, one-dimensional commutative domain R with field of fractions K and completion $\widehat{R}$ that $\widehat{R} \otimes_R K$ as $\widehat{R}$-module and K/R as R-module are weakly injective.

1.3 Torsion Free and Componentwise Flat Covers in Categories of Quivers

Given a class $\mathscr{F}$ of objects in an abelian category $\mathcal{A}$, recall from Enochs (1981) that, an $\mathscr{F}$-precover of an object C is a morphism $\varphi : F \longrightarrow C$ with $F \in \mathscr{F}$ such that $\mathrm{Hom}_{\mathcal{A}}(F', F) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(F', C) \longrightarrow 0$ is exact for every $F' \in \mathscr{F}$, that is, the following diagram commutes:

$$\begin{tikzcd} & F' \arrow[dl, dashed] \arrow[d] \\ F \arrow[r, "\varphi"] & C \end{tikzcd}.$$

If, moreover, every morphism $f : F \longrightarrow F$ such that $\varphi f = \varphi$ is an automorphism, then φ is said to be an $\mathscr{F}$-cover.

Dually, an $\mathscr{F}$*-preenvelope* of M is a morphism $\varphi : M \longrightarrow F$ with $F \in \mathscr{F}$ such that $\mathrm{Hom}_{\mathscr{A}}(F,F') \longrightarrow \mathrm{Hom}_{\mathscr{A}}(M,F')$ is surjective for every $F' \in \mathscr{F}$, that is, the following diagram commutes:

$$\begin{tikzcd} M \arrow[r, "\varphi"] \arrow[d] & F \arrow[dl, dashed] \\ F' & \end{tikzcd}.$$

An $\mathscr{F}$-preenvelope φ is said to be an $\mathscr{F}$*-envelope* if every endomorphism $f : F \longrightarrow F$ such that $f\varphi = \varphi$ is an automorphism.

So, for instance, if we take $\mathscr{F}$ to be the class of all flat modules, then a flat cover of a module will be an $\mathscr{F}$-cover. See Section 2.7 for details for abelian categories, and Section 2.10 for covers and envelopes.

The study of covers and envelopes started in 1953, when Eckman and Schopf proved that each module over an associative ring has an injective envelope (Eckmann & Schopf, 1953). On the other hand, Bass characterized rings over which every module has a projective cover: perfect rings (Bass, 1960). Other authors studied different types of covers and envelopes, for example, Kiełpiǹski proved the existence of pure-injective envelopes in the category R-$\mathcal{M}od$ (Kiełpiński, 1967), and Warfield gave another proof of the existence of pure-injective envelopes of modules (Warfield, 1969). Enochs studied torsion free covers and proved the existence of torsion free covers of modules over a commutative domain (Enochs, 1963). In the arguments after Enochs & Jenda (2000, Definition 5.1.1), it has been pointed out that torsion free covers and $\mathscr{F}$-covers coincide over a commutative domain R, where $\mathscr{F}$ is the class of torsion free R-modules. Moreover, in 1981, Enochs conjectured that every module over an associative ring admits a flat cover (Enochs, 1981). This is known as the "flat cover conjecture". In the same paper, he noticed the categorical version of injective cover, and then gave a general definition of covers and envelopes in terms of commutative diagrams, for a given class of modules. Independently, this definition of covers and envelopes was given by Auslander and Smalø in terms of *minimal left and right approximations* (Auslander & Smalø, 1980). Enochs gave the general definition for a class of modules over arbitrary rings, while Auslander and Smalø considered finitely generated modules over finite dimensional algebras. The main idea for studying covers and envelopes is to use certain aspects of a special class of modules, or more generally objects to study entire category. Because, once we understand the structure of a class of objects, we may approximate arbitrary objects by the objects from this class. In 2001, once the "flat cover conjecture" has been proved in Bican et al. (2001), in a natural way, flat covers and covers by more general classes of objects have been studied in more general settings than that of modules. For example, the existence of flat covers was shown for categories of complexes of modules over a ring R (Aldrich et al., 2001) and of quasi-coherent sheaves over a scheme (Enochs & Estrada, 2005b). Also, the existence of flat covers has been studied for the category of representations by modules of some class of quivers.

A *quiver* is a directed graph whose edges are called arrows. As usual we denote a quiver by Q understanding that $Q = (V, E)$ where V is the set of vertices (points) and E is the set of arrows (directed edges). An *arrow* of a quiver from a vertex v_1 to a vertex v_2 is denoted by

$$a : v_1 \longrightarrow v_2 \quad \text{or} \quad v_1 \xrightarrow{a} v_2 .$$

In this case, we write $s(a) = v_1$ and call the *starting (initial)* vertex of the arrow a, and $t(a) = v_2$ and call the *terminal (ending)* vertex of a.

A *path p of length $n \geq 1$* from the vertex v_0 to the vertex v_n of a quiver Q is a sequence of arrows

$$(v_0 \mid a_1, a_2, \ldots, a_n \mid v_n)$$

where $a_i \in E$ for all $1 \leq i \leq n$, and we have $s(a_1) = v_0$, $t(a_i) = s(a_{i+1})$ for each $1 \leq i < n$, and finally $t(a_n) = v_n$. Such a path is denoted briefly by $a_n a_{n-1} \ldots a_1$ and may be visualised as follows:

$$p: \; v_0 \xrightarrow{a_1} v_1 \xrightarrow{a_2} v_2 \longrightarrow \cdots \xrightarrow{a_n} v_n$$

For this path p, define the starting vertex $s(p) = s(a_1) = v_0$ and the ending vertex $t(p) = t(a_n) = v_n$. In this case, we will write, shortly, $p : v_0 \longrightarrow v_n$. An arrow $a : v \longrightarrow w$ of Q is also considered as a path of length 1. We also agree to associate with each vertex $v \in V$ a path of length $n = 0$, called the *trivial path* at v, and denoted by p_v. It has no arrows and we take$s(p_v) = t(p_v) = v$. Thus, a vertex $v \in V$ can be considered as a trivial path p_v. Instead of p_v, we usually write just v. If $p = a_n a_{n-1} \ldots a_1$ and $q = b_m b_{m-1} \ldots b_1$ are two paths of Q such that $s(a_n) = t(b_1)$, where $a_i, b_j \in E$ for all $1 \leq i \leq n$ and $1 \leq j \leq m$, then the *composition* of p and q is defined as $qp = b_m \ldots b_1 a_n \ldots a_1$. Thus, two paths p and q can be composed, getting another path qp whenever $t(p) = s(q)$. So, given a path $p : v_1 \longrightarrow v_2$, we have that $pp_{v_1} = p_{v_2} p = p$.

Therefore, any quiver Q is thought as a category in which the objects are the vertices of Q, and the morphisms are the paths of Q. Clearly, every object (i.e. vertex) v of Q has an identity morphism p_v (trivial path).

A *representation* by modules of a given quiver $Q = (V, E)$ is a functor $X : Q \longrightarrow R\text{-}\mathcal{M}od$. So such a representation is determined by giving a module $X(v)$ to each vertex v of Q and a homomorphism $X(a) : X(v_1) \longrightarrow X(v_2)$ to each arrow $a : v_1 \longrightarrow v_2$ of Q. A *morphism* η between two representations X and Y is a *natural transformation*, so it will be a family $\{\eta_v\}_{v \in V}$ of module homomorphisms such that $Y(a)\eta_{v_1} = \eta_{v_2} X(a)$

for every arrow $a : v_1 \longrightarrow v_2$ of Q, that is, the following diagram commutes for every arrow $a : v_1 \longrightarrow v_2$ of Q:

$$\begin{tikzcd} X(v_1) \arrow[r, "X(a)"] \arrow[d, "\eta_{v_1}"'] & X(v_2) \arrow[d, "\eta_{v_2}"] \\ Y(v_1) \arrow[r, "Y(a)"] & Y(v_2) \end{tikzcd} .$$

Thus, the representations by modules of a quiver Q over a ring R form a category, denoted by $(Q, R\text{-}\mathcal{M}od)$. This is a locally finitely presented Grothendieck category with enough projectives and injectives (see Section 5.1 for details). By a representation of a quiver we will mean a representation by modules of a quiver over a ring R.

In Chapter 5, we continue with the program initiated in Enochs & Herzog (1999) and continued in Enochs et al. (2002), Enochs et al. (2003a), Enochs et al. (2004b), Enochs & Estrada (2005a), Enochs et al. (2007) and Enochs et al. (2009) to develop new techniques on the study of representations by modules over (possibly infinite) quivers. In contrast to the classical representation theory of quivers motivated by Gabriel (1972b), we do not assume that the base ring is an algebraically closed field and that all vector spaces involved are finite dimensional. Techniques on representation theory of infinite quivers have recently proved to be very useful in leading to simplifications of proofs as well as the descriptions of objects in related categories. For instance, in Enochs & Estrada (2005b) it was shown that the category of quasi-coherent sheaves on an arbitrary scheme is equivalent to a category of representations of a quiver (with certain modifications on the representations). Note that in this thesis we do not deal with the category of quasi-coherent sheaves on an arbitrary scheme; see, for example, Hartshorne (1977, Chap. II) for the definitions of the related concepts. This point of view allows to introduce new versions of homological algebra in such categories (see Enochs & Estrada (2005b, §5) and Enochs et al. (2003b)). Infinite quivers also appear when the category of $\mathbb{Z}$-graded modules is considered over the graded ring $R[x]$ as explained below.

Recall that a commutative ring R is called a *graded ring* (or more precisely, a $\mathbb{Z}$-*graded* ring) if R can be expressed as a direct sum $R = \bigoplus_{n\in\mathbb{Z}} R_n$ of its additive subgroups such that the ring multiplication satisfies $R_n \cdot R_m \subseteq R_{n+m}$ for all $m,n \in \mathbb{Z}$. In particular, R_0 is a subring of R and each component R_n is an R_0-module. For example, the polynomial ring $R[x]$ is a graded ring with

$$R[x] = \bigoplus_{n\in\mathbb{Z}} R_n, \qquad \text{where} \qquad R_n = Rx^n \text{ if } n \geq 0 \text{ and } R_n = 0 \text{ otherwise} .$$

Let $R = \bigoplus\limits_{n\in\mathbb{Z}} R_n$ be a graded ring. An R-module M is called a *graded module* (or is said to have an R-*grading*) if M can be expressed as a direct sum $\bigoplus_{n\in\mathbb{Z}} M_n$ of its additive subgroups such that $R_n \cdot M_m \subseteq M_{n+m}$ for all $m,n \in \mathbb{Z}$. In particular, M_n is an R_0-module for every $n \in \mathbb{Z}$. See, for example, Lang (2002, Chap. X, §5) for graded modules.

The category of graded modules over the graded ring $R[x]$ is equivalent to the category of representations over R of the quiver

$$A_\infty^\infty \equiv \cdots \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \cdots .$$

Indeed, a representation $\cdots \longrightarrow A_{-1} \xrightarrow{f_{-1}} A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} \cdots$ of A_∞^∞ can be thought of as a graded module $\bigoplus_{n\in\mathbb{Z}} A_n$ over the polynomial ring $R[x]$, the action of x being given by module homomorphisms $A_n \xrightarrow{f_n} A_{n+1}$:

$$R_n \cdot A_m = Rx^n \cdot A_m = Rx^{n-1} \cdot f_m(A_m) = Rf_{n+m-1} \cdots f_m(A_m) \subseteq RA_{n+m} \subseteq A_{n+m}$$

for all $n,m \in \mathbb{Z}$. Conversely, as a graded ring

$$R[x] \equiv \cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow R \xrightarrow{\cdot x} Rx \xrightarrow{\cdot x} Rx^2 \longrightarrow \cdots$$

and as a graded module

$$_{R[x]}M \equiv \cdots \longrightarrow A_{-2} \xrightarrow{\cdot x} A_{-1} \xrightarrow{\cdot x} A_0 \xrightarrow{\cdot x} A_1 \xrightarrow{\cdot x} A_2 \longrightarrow \cdots$$

In Chapter 5, we introduce new classes in the category of representations of a (possibly infinite) quiver to compute (unique up to homotopy) resolutions which give rise to new versions of homological algebra on it. The first of such versions turns to Enochs' proof on the existence of torsion free covers of modules over a commutative domain (see Enochs (1963)) and its subsequent generalization in Teply (1969) and Golan & Teply (1973) to more general torsion theories in $R\text{-}\mathcal{M}od$.

Given a hereditary torsion theory $(\mathcal{T},\mathcal{F})$ for $R\text{-}\mathcal{M}od$, we define a torsion theory $(\mathcal{T}_{cw},\mathcal{F}_{cw})$ for $(Q,R\text{-}\mathcal{M}od)$, by defining the torsion class $\mathcal{T}_{cw}$ as follows:

$$\mathcal{T}_{cw} = \{X \in (Q,R\text{-}\mathcal{M}od) \mid X(v) \in \mathcal{T} \text{ for every vertex } v \text{ of } Q\}.$$

Then the torsion free class $\mathcal{F}_{cw}$ will be as follows:

$$\mathcal{F}_{cw} = \{X \in (Q,R\text{-}\mathcal{M}od) \mid X(v) \in \mathcal{F} \text{ for every vertex } v \text{ of } Q\};$$

see Proposition 5.2.4. Note that the torsion theory $(\mathcal{T}_{cw},\mathcal{F}_{cw})$ is hereditary, that is, it closed under subrepresentations since the torsion class $\mathcal{T}$ is closed under submodules.

In the first part of Chapter 5, we prove that torsion free covers exist in $(Q,R\text{-}\mathcal{M}od)$ relative to the torsion theory $(\mathcal{T}_{cw},\mathcal{F}_{cw})$, for a wide class of quivers included in the class of the so-called source injective representation quivers as introduced in Enochs et al. (2009) (Theorem 5.2.16). This important class of quivers includes all finite quivers with no oriented cycles, but also includes infinite line quivers:

$$A_\infty \equiv \qquad \cdots \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet\ ,$$

$$A^\infty \equiv \qquad \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \cdots\ ,$$

$$A_\infty^\infty \equiv \qquad \cdots \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \cdots$$

On the second part, we will focus on the existence of a version of relative homological algebra by using the class of componentwise flat representations in $(Q, R\text{-}\mathcal{M}od)$. Recently, it has been proved in Rump (2010) that flat covers do exist on each abelian locally finitely presented category. Here by "flat" the author means Stenström's concept of flat object given in Stenström (1968) in terms of *the theory of purity* that one can always define in locally finitely presented additive categories (see Crawley-Boevey (1994)). It is well-known that a short exact sequence of modules is pure if and only every finitely presented module is projective relative to it (see Example 2.3.3). Using this characterization of pure-exact sequences, Stenström (1968) defined purity in locally finitely generated Grothendieck categories.

Let $\mathcal{C}$ be a Grothendieck category and C be an object in $\mathcal{C}$. The object C is called *finitely generated* if whenever $C = \sum_{i\in I} C_i$ for a direct family $(C_i)_{i\in I}$ of subobjects of C (where I is some index set), there is an index $i_0 \in I$ such that $C = C_{i_0}$. The object C is called *finitely presented* if it is finitely generated and every epimorphism $B \longrightarrow C$, where B is a finitely generated object in $\mathcal{C}$, has a finitely generated kernel. The category $\mathcal{C}$ is called *locally finitely generated* (respectively *locally finitely presented*) if it has a family of finitely generated (resp. finitely presented) generators.

Let $\mathcal{C}$ be a locally finitely generated Grothendieck category. A short exact sequence in $\mathcal{C}$ is said to be *pure* if every finitely presented object P of $\mathcal{C}$ is projective relative to it. An object F of $\mathcal{C}$ is said to be a *flat object* in the sense of Stenström if every short exact sequence ending with F is pure. We call such flat objects "categorical flat". For abelian locally finitely presented categories with enough projectives, this notion of "flatness" is equivalent to being the direct limit of certain projective objects.

As $(Q, R\text{-}\mathcal{M}od)$ is a locally finitely presented Grothendieck category with enough projectives, we infer by using Rump's result that $(Q, R\text{-}\mathcal{M}od)$ admits "categorical flat" covers for every quiver Q and any associative ring R with unity. But there are categories in which there is a classical notion of flatness having nothing to do with respect to the theory of purity. This is the case of the notion of "flatness" in categories of presheaves or quasi-coherent sheaves, where "flatness" is more related with a "componentwise" notion. Those categories may be viewed as certain categories of representations of quivers.

We proved the existence of "componentwise" flat covers for every quiver and any ring R with unity (Theorem 5.3.6), where we call a representation X of $(Q, R\text{-}\mathcal{M}od)$ *componentwise flat* if $X(v)$ is a flat R-module for each vertex v of Q. In particular if X is a topological space, an easy modification of our techniques can prove the existence of a flat cover (in the algebraic geometrical sense) for every presheaf on X over $R\text{-}\mathcal{M}od$. Finally, the last part of Chapter 5 contains some examples for comparing "categorical" flat covers with "componentwise" flat covers which show that these two kinds of covers do not coincide in general (see Section 5.4).

CHAPTER TWO

PRELIMINARIES

In this chapter, we give the basic definitions, results, tools and notation which will be used throughout this thesis. We will give further notions and notation when they are needed. The terminology, notation and our main references are sketched in Section 2.1; we give the definition of proper classes and some related properties in Section 2.3. Some elementary properties of preradicals and torsion theories for R-$\mathcal{M}od$ are given in Section 2.4. Section 2.5 contains some properties of projective covers and perfect rings. In Section 2.6, we give the definition of torsion free covers of R-modules over a commutative domain R. See Section 2.2 for the definition of complements and supplements, and Section 2.10 for the definition of covers and envelopes. For details for abelian categories, see Section 2.7 and see Section 2.8 for torsion theories in abelian categories. In Section 2.9, we will give some basic definitions and results of cotorsion theories, and explain the method of the proof of flat cover conjecture given by Enochs that uses cotorsion theories (see Bican et al. (2001)).

2.1 Notation and Terminology

Unless otherwise stated, all rings considered will be associative with identity and not necessarily commutative. R will denote an arbitrary ring. So, *if nothing is said about R in the statement of a theorem, proposition, etc., then that means R is just an arbitrary ring.* An R-module or just a module will be a unital *left* R-module. R-$\mathcal{M}od$ (respectively $\mathcal{M}od$-R) denotes the category of all *left* (resp. *right*) R-modules. A *commutative domain* will mean a nonzero commutative ring in which there is *no* zero divisor other than zero. $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$ denotes the set of positive natural numbers, the ring of integers and the field of rational numbers, respectively. $\mathcal{A}b$, or $\mathbb{Z}$-$\mathcal{M}od$, denotes the category of abelian groups (i.e. $\mathbb{Z}$-modules). Group will mean abelian group. As usual, $J(R)$ denotes the Jacobson radical of R, and $\mathrm{Rad}M$ (respectively $\mathrm{Soc}M$) denotes the radical (resp. the socle) of a module M. $\mathrm{E}(M)$ will denote the injective envelope of a module M. We denote by $X \subseteq M$ that X is a submodule of M. For any modules A and B, $\mathrm{Hom}_R(A,B)$ denotes the set of all homomorphisms from A to B. We denote by $1_M : M \longrightarrow M$ the identity map. By a homomorphism $f : A \longrightarrow B$ we will mean a homomorphism of modules from A to B, unless otherwise stated. $\mathrm{Ext}^1_R(C,A)$ denotes the equivalence classes of extensions of an R-module A by an R-module C. For abelian groups we will use the notation $\mathrm{Ext}(C,A)$. For the definition of $\mathrm{Ext}^1_R(C,A)$, see Maclane (1963, Chap. III).

We do not delve into the details of definitions of every term used in this thesis. We refer to Enochs & Jenda (2000), Stenström (1975), Freyd (1964) and Assem et al. (2006) for details on covers and envelopes, abelian categories or quivers. For fundamentals of module theory see, for example, Anderson & Fuller (1992), Lam (1999), Facchini (1998), Kasch (1982), Wisbauer (1991) and Clark et al. (2006); for details in homological algebra see the books Cartan & Eilenberg (1956), Maclane (1963) and Rotman (2009); for relative homological algebra, our main references are the books Maclane (1963), Enochs & Jenda (2000) and the article Sklyarenko (1978); for abelian groups, see Fuchs (1970).

*The **notation** we use have been given on pages (173-177) and an **index** will be given at the end of this thesis.*

2.2 Complements and Supplements

Let M be an R-module and A be a submodule of M. It would be best if A is a *direct summand* of M, that is, if there exists a submodule B of M such that $M = A \oplus B$; that means,

$$M = A + B \quad \text{and} \quad A \cap B = 0.$$

If A is not a direct summand, then we wish at least one of these conditions to hold. These give rise two concepts: *complement* and *supplement*.

Let M be a module and A, B be submodules of M such that $M = A + B$ (that is, the above first condition for direct sum holds). If A is *minimal* with respect to this property, that is, there is *no* submodule $\tilde{A}$ of M such that $\tilde{A} \subsetneqq A$ but still $M = \tilde{A} + B$, then A is called a *supplement of B in M* and B is said to *have a supplement A in M*.

A submodule B of a module M need *not* have a supplement in M. If a module M is such that every submodule of it has a supplement in M, then it is called a *supplemented module*. For the definitions and related properties see Wisbauer (1991, §41) and Clark et al. (2006, Chap. 4).

Let M be a module and A, B be submodules of M such that $A \cap B = 0$ (that is, the above second condition for direct sum holds). If A is *maximal* with respect to this property, that is, there is *no* submodule $\tilde{A}$ of M such that $\tilde{A} \supsetneqq A$ but still $\tilde{A} \cap B = 0$, then A is called a *complement of B in M* and B is said to *have a complement A in M*.

Remark 2.2.1. By Zorn's Lemma, it can be seen that a submodule B of a module M always has a complement A in M (unlike the case for supplements). In fact, by Zorn's Lemma, we know that if we have a submodule $\widetilde{A}$ of M such that $B \cap \widetilde{A} = 0$, then there exists a complement A of B in M such that $A \supseteq \widetilde{A}$. See the monograph Dung et al. (1994) for a survey of results in the related concepts.

We are interested in the collection of submodules each of which is a complement of some submodule or supplement of some submodule.

A submodule A of a module M is said to be a *complement in M* if A is a complement of some submodule of M; shortly, we also say that A is a *complement submodule of* M in this case. Dually, A is said to be a *supplement in* M if A is a supplement of some submodule of M; shortly, we also say that A is a *supplement submodule of* M in this case.

A submodule A of a module B is *essential* (or *large*) in A, denoted by $A \trianglelefteq B$, if for every *nonzero* submodule K of B, we have $A \cap K \neq 0$. A monomorphism $f : M \longrightarrow N$ of modules is called *essential* if $\operatorname{Im} f \trianglelefteq M$.

A submodule A of a module M is said to be *closed in* M if A has *no* proper essential extension in M, that is, there exists *no* submodule $\tilde{A}$ of M such that $A \subsetneqq \tilde{A}$ and A is *essential* in $\tilde{A}$. We also say in this case that A is a *closed submodule*.

Note that closed submodules and complement submodules in a module coincide (see Dung et al. (1994, §1)).

Proposition 2.2.2. *(Anderson & Fuller, 1992, Proposition 5.21) Let M be a module and B a submodule of M. Then B has a complement A in M, and*

(i) $B \oplus A \trianglelefteq M$*;*

(ii) $(B \oplus A)/A \trianglelefteq M/A$.

A module M is said to be *semi-artinian* if for every proper submodule U of M, $\operatorname{Soc}(M/U) \neq 0$, that is, M/U contains a simple submodule.

See Dung et al. (1994, 3.12, 3.13) for some properties of semi-artinian modules and rings. The following characterization is also given as the definition of semi-artinian modules there; we give its elementary proof for completeness:

Proposition 2.2.3. *A module M semi-artinian if and only if* $\operatorname{Soc}(M/U)$ *is essential in M/U for every proper submodule U of M.*

Proof. Let M be a semi-artinian module and let $U \subseteq M$ be a proper submodule. Since the factor module M/U is also semi-artinian, it suffices to show that $\operatorname{Soc} M \trianglelefteq M$. Let $0 \neq K$ be a submodule of M and let K' be a complement of K in M. Then $K \cap K' = 0$ and $(K \oplus K')/K' \trianglelefteq M/K'$ by the previous proposition. So

$$\operatorname{Soc}(M/K') = \operatorname{Soc}((K \oplus K')/K') \cong \operatorname{Soc} K.$$

Since $K' \neq M$ as $K \neq 0$ and M is semi-artinian, we obtain $\operatorname{Soc} K \neq 0$. Thus $K \cap \operatorname{Soc} M = \operatorname{Soc} K \neq 0$, that is, $\operatorname{Soc} M \trianglelefteq M$. Conversely, if $\operatorname{Soc}(M/U) \trianglelefteq M/U$ for every proper submodule U of M, then obviously $\operatorname{Soc}(M/U) \neq 0$. □

For a module M, a well-ordered sequence of fully invariant submodules $\operatorname{Soc}_\alpha(M)$ of M is defined inductively for each ordinal α as follows:

$$\operatorname{Soc}_0(M) = 0,$$

$$\operatorname{Soc}_{\alpha+1}(M)/\operatorname{Soc}_\alpha(M) = \operatorname{Soc}(M/\operatorname{Soc}_\alpha(M)),$$

for every ordinal α, and

$$\operatorname{Soc}_\beta(M) = \bigcup_{\alpha<\beta} \operatorname{Soc}_\alpha(M)$$

for every limit ordinal β. The chain

$$\operatorname{Soc}_0(M) \subseteq \operatorname{Soc}_1(M) \subseteq \operatorname{Soc}_2(M) \subseteq \cdots \subseteq \operatorname{Soc}_\alpha(M) \subseteq \dots$$

is called the *(ascending) Loewy series* of M. The module M is said to be a *Loewy module* if there is an ordinal α such that $M = \operatorname{Soc}_\alpha(M)$, and in this case the least ordinal α such that $M = \operatorname{Soc}_\alpha(M)$ is called the *Loewy length* of M (see, for example, Facchini (1998, §2.11) for Loewy modules).

Proposition 2.2.4. *(see, for example, Facchini (1998, Lemma 2.58)) A module M is a Loewy module if and only if M is semi-artinian.*

2.3 Proper Classes of R-modules

In this section, we give the definition of proper classes in R-$\mathcal{M}od$ (since our investigations are in the proper classes of modules) and some important examples of proper classes that we are interested in. We also give the definitions for projectives, injectives, flats, coprojectives, coinjectives with respect to a proper class, and projectively generated, injectively generated, flatly generated proper classes. See, for example, Maclane (1963, Chap. XII) for the general definition of proper classes in an abelian category. Proper classes of monomorphisms and short exact sequences were introduced in Buschbaum (1959). For further details we refer to Maclane (1963, Chap. XII), Stenström (1967a), Mishina & Skornyakov (1976) and Sklyarenko (1978).

Let $\mathcal{P}$ be a class of short exact sequences of R-modules and R-module homomorphisms. If a short exact sequence

$$\mathbb{E} : 0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0 \tag{2.3.1}$$

belongs to $\mathcal{P}$, then f is said to be a *$\mathcal{P}$-monomorphism* and g is said to be a *$\mathcal{P}$-epimorphism* (both are said to be *$\mathcal{P}$-proper* and the short exact sequence is said to be a *$\mathcal{P}$-proper* short exact sequence). The class $\mathcal{P}$ is said to be *proper* (in the sense of Buchsbaum) if it satisfies the following conditions:

P1. If a short exact sequence $\mathbb{E}$ is in $\mathcal{P}$, then $\mathcal{P}$ contains every short exact sequence isomorphic to $\mathbb{E}$.

P2. $\mathcal{P}$ contains all splitting short exact sequences.

P3. (i) The composite of two $\mathcal{P}$-monomorphisms is a $\mathcal{P}$-monomorphism if this composite is defined.

(ii) The composite of two $\mathcal{P}$-epimorphisms is a $\mathcal{P}$-epimorphism if this composite is defined.

P4. (i) If g and f are monomorphisms, and gf is a $\mathcal{P}$-monomorphism, then f is a $\mathcal{P}$-monomorphism.

(ii) If g and f are epimorphisms, and gf is a $\mathcal{P}$-epimorphism, then g is a $\mathcal{P}$-epimorphism.

For a proper class $\mathcal{P}$ of R-modules, a submodule A of a module B is called a $\mathcal{P}$-submodule of B, if the *inclusion* monomorphism $i_A : A \longrightarrow B$, $i_A(a) = a$, $a \in A$, is a $\mathcal{P}$-monomorphism.

A module F is said to be *flat* if for every exact sequence $0 \longrightarrow A \longrightarrow B$ of *right* modules, the tensored sequence $0 \longrightarrow A \otimes_R F \longrightarrow B \otimes_R F$ is exact.

Definition 2.3.1. Let $\mathcal{P}$ be a proper class of modules.

(i) A module M is said to be *$\mathcal{P}$-projective* (respectively *$\mathcal{P}$-injective*) if it is projective (resp. injective) with respect to all short exact sequences in $\mathcal{P}$.

(ii) A *right* module M is said to be *$\mathcal{P}$-flat* if M is flat with respect to every short exact sequence $\mathbb{E} \in \mathcal{P}$, that is, $M \otimes \mathbb{E}$ is exact for every $\mathbb{E}$ in $\mathcal{P}$.

(iii) A module C is said to be *$\mathcal{P}$-coprojective* if *every* short exact sequence (2.3.1) of modules *ending* with C is in the proper class $\mathcal{P}$. Dually, a module A is said to be *$\mathcal{P}$-coinjective* if *every* short exact sequence (2.3.1) of modules *starting* with A is in the proper class $\mathcal{P}$.

Definition 2.3.2. For a given class $\mathcal{M}$ of modules,

(i) the class of all short exact sequences $\mathbb{E}$ of modules such that $\mathrm{Hom}_R(M,\mathbb{E})$ is exact for all $M \in \mathcal{M}$ is the largest proper class $\mathcal{P}$ for which each $M \in \mathcal{M}$ is

$\mathcal{P}$-projective, and it is called the proper class *projectively generated* by $\mathcal{M}$ and denoted by $\pi^{-1}(\mathcal{M})$.

(ii) the class of all short exact sequences $\mathbb{E}$ of modules such that $\mathrm{Hom}_R(\mathbb{E}, M)$ is exact for all $M \in \mathcal{M}$ is the largest proper class $\mathcal{P}$ for which each $M \in \mathcal{M}$ is $\mathcal{P}$-injective, and it is called the proper class *injectively generated* by $\mathcal{M}$ and denoted by $\iota^{-1}(\mathcal{M})$.

(iii) Let $\mathcal{M}$ be a class of *right* modules. The class of all short exact sequences $\mathbb{E}$ of modules such that $M \otimes \mathbb{E}$ is exact for all $M \in \mathcal{M}$ is the largest proper class $\mathcal{P}$ of (left) R-modules for which each $M \in \mathcal{M}$ is $\mathcal{P}$-flat. It is called the proper class *flatly generated* by the class $\mathcal{M}$ of *right* modules and denoted by $\tau^{-1}(\mathcal{M})$.

A module M is said to be *finitely presented* if there is a an exact sequence

$$R^m \longrightarrow R^n \longrightarrow M \longrightarrow 0$$

for some positive integers m and n.

The *character module functor* is the functor

$$(-)^\flat = \mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z}) : R\text{-}\mathcal{M}od \longrightarrow \mathcal{M}od\text{-}R.$$

So, for a (left) R-module M, $M^\flat = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ is a *right* R-module.

For a functor T from a category $\mathcal{A}$ of left or right R-modules to a category $\mathcal{B}$ of left or right S-modules (where R, S are rings), and for a given class $\mathcal{F}$ of short exact sequences in $\mathcal{B}$, let $T^{-1}(\mathcal{F})$ be the class of those short exact sequences of $\mathcal{A}$ which are carried into $\mathcal{F}$ by the functor T. If the functor T is left or right exact, then $T^{-1}(\mathcal{F})$ is a proper class; see Stenström (1967a, Proposition 2.1).

We give some examples of proper classes, which are interesting for the purpose of this thesis:

Example 2.3.3. The proper classes $\mathcal{P}ure_{\mathbb{Z}}$ and its generalization $\mathcal{P}ure_R$ form the origins of *relative* homological algebra; this is the reason why proper classes are also called purities (for example, in Mishina & Skornyakov (1976), Generalov (1972, 1978, 1983)).

(i) $\mathcal{S}plit_R$: The smallest proper class of modules consists of only *splitting* short exact sequences of modules.

(ii) $\mathcal{A}bs_R$ (*absolute purity*): The largest proper class of modules consists of *all* short exact sequences of modules.

(iii) $\mathcal{P}ure_{\mathbb{Z}}$: The proper class of all short exact sequences (2.3.1) of abelian groups and abelian group homomorphisms such that $\operatorname{Im} f$ is a pure subgroup of B, where a subgroup A of a group B is *pure* in B if $A \cap nB = nA$ for all integers n. The short exact sequences in $\mathcal{P}ure_{\mathbb{Z}}$ are called *pure-exact sequences* of abelian groups (see Fuchs (1970, §29)).

(iv) $\mathcal{P}ure_R$ is the classical Cohn's purity; it was introduced by Cohn (1959) for arbitrary rings as a generalization of purity in abelian groups:

$$\begin{aligned}
\mathcal{P}ure_R &= \pi^{-1}(\text{ all finitely presented } R\text{-modules }) \\
&= \tau^{-1}(\text{ all finitely presented } \textit{right } R\text{-modules }) \\
&= \tau^{-1}(\text{ all } \textit{right } R\text{-modules }) \\
&= [(-)^{\flat}]^{-1}(\mathcal{S}plit_R). \\
&= \iota^{-1}(\{M^{\flat} \mid M \text{ is a finitely presented } \textit{right } R\text{-module}\})
\end{aligned}$$

See, for example, Facchini (1998, §1.4) for the proof of first four of these equalities. See Sklyarenko (1978, Proposition 6.2) for the last equality.

(v) $\mathcal{C}ompl$ and $\mathcal{S}uppl$: The class of all short exact sequences (2.3.1) of modules such that $\operatorname{Im} f$ is a complement (respectively supplement) in B forms a proper class as has been shown more generally by Stenström (1967b), Generalov (1978), Generalov (1983). See also Erdoğan (2004) and Clark et al. (2006, 10.5 and 20.7) for the proofs of $\mathcal{C}ompl$ and $\mathcal{S}uppl$ being proper classes.

(vi) $\mathcal{N}eat$: The class of all short exact sequences (2.3.1) of modules such that $\operatorname{Im} f$ is a neat submodule of B (that is, f is a neat monomorphism) forms a proper class following Stenström (1967a) and Stenström (1967b):

$$\begin{aligned}\mathcal{N}eat &= \pi^{-1}(\text{all simple } R\text{-modules})\\ &= \pi^{-1}(\{R/P \mid P \text{ maximal left ideal of } R\})\\ &= \pi^{-1}(\{M \mid \operatorname{Soc} M = M,\ M \text{ an } R\text{-module}\}).\end{aligned}$$

Dually, the class of coneat submodules has been introduced in Mermut (2004) and Alizade & Mermut (2004):

(vii) $\mathcal{C}o\text{-}\mathcal{N}eat$: The class of all short exact sequences (2.3.1) of modules such that $\operatorname{Im} f$ is a coneat submodule of B (that is, f is a coneat monomorphism) forms a proper class:

$$\begin{aligned}\mathcal{C}o\text{-}\mathcal{N}eat &= \iota^{-1}(\text{all } R\text{-modules with zero radical})\\ &= \iota^{-1}(\{M \in R\text{-}\mathcal{M}od \mid \operatorname{Rad} M = 0\}).\end{aligned}$$

Fuchs calls a ring R to be an *N-domain* if R is a commutative domain and $\mathcal{N}eat = \tau^{-1}($ all simple R-modules $)$. He proved that a ring R is an N-domain if and only if R is a commutative domain whose all maximal ideals are projective (and so all maximal ideals invertible and finitely generated); see Fuchs (2010).

The criterion for being a coneat submodule is like being a supplement in the following weaker sense:

Proposition 2.3.4. *(Mermut, 2004, Proposition 3.4.2) For a submodule A of a module B, the following are equivalent:*

(i) A is coneat in B,

(ii) There exists a submodule $K \subseteq B$ such that ($K \geq \operatorname{Rad} A$ and,)

$$A + K = B \quad \text{and} \quad A \cap K = \operatorname{Rad} A.$$

(iii) There exists a submodule $K \subseteq B$ such that

$$A + K = B \quad \text{and} \quad A \cap K \subseteq \operatorname{Rad} A.$$

One of the generalizations of pure subgroups of abelian groups to modules over arbitrary rings is relative divisibility: A submodule A of a module B is called *relatively divisible* or briefly *RD-submodule* if $rA = A \cap rB$ for every $r \in R$. This terminology is due to Warfield (1969) See, for example, Fuchs & Salce (2001, Chap I, §7) for properties of *RD*-submodules.

Proposition 2.3.5. *(Warfield, 1969, Proposition 2) Let R be a ring and let $r \in R$. The following are equivalent for a short exact sequence*

$$\mathbb{E}: \qquad 0 \longrightarrow A \xrightarrow{i_A} B \xrightarrow{g} C \longrightarrow 0$$

of R-modules where A is a submodule of B and i_A is the inclusion map:

(i) $\operatorname{Hom}_R(R/Rr, B) \xrightarrow{g_*} \operatorname{Hom}_R(R/Rr, C)$ *is epic (that is, R/Rr is projective relative to $\mathbb{E}$);*

(ii) $R/rR \otimes A \xrightarrow{1_{R/rR} \otimes i_A} R/rR \otimes B$ *is monic (that is, R/rR is flat relative to $\mathbb{E}$);*

(iii) $rA = A \cap rB$ *(that is, A is an RD-submodule of B).*

Note that the notion of Cohn's purity is a strengthened version of the concept of *RD*-submodule. Note also that Enochs calls *RD*-submodules pure submodules in his definition of torsion free covers (see Section 1.2). By pure submodules in this thesis, we will mean pure submodules in the sense of Cohn.

A ring R is said to be *left semihereditary* if every finitely generated ideal of R is projective as a left R-module. A semihereditary commutative domain is called a *Prüfer domain*.

Over Prüfer domains, pure submodules and *RD*-submodules of a module coincide (see Warfield (1969, Corollary 5) or, for example, Fuchs & Salce (2001, Theorem 8.11)).

The following proposition that gives a basic relationship between flat modules and pure-exact sequences will be useful:

Proposition 2.3.6. *(by Lam (1999, Corollary 4.86))*
Let $\mathbb{E}: 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ *be a short exact sequence of modules.*

(i) Assume B is flat. Then $\mathbb{E}$ is pure if and only if C is flat.

(ii) Assume C is flat. Then B is flat if and only if A is flat.

(iii) C is flat if and only if every short exact sequences ending with C is pure, that is, C is $\mathcal{P}ure_R$-coprojective.

Definition 2.3.7. A proper class $\mathcal{P}$ is called $\prod$-*closed* (respectively $\oplus$-*closed*) if for *every* collection $\{\mathbb{E}_\lambda : 0 \longrightarrow A_\lambda \longrightarrow B_\lambda \longrightarrow C_\lambda \longrightarrow 0\}_{\lambda\in\Lambda}$ in $\mathcal{P}$, the direct product

$$\prod_{\lambda\in\Lambda} \mathbb{E}_\lambda : 0 \longrightarrow \textstyle\prod_\lambda A_\lambda \longrightarrow \prod_\lambda B_\lambda \longrightarrow \prod_\lambda C_\lambda \longrightarrow 0$$

$\left(\text{resp. the direct sum } \bigoplus_{\lambda\in\Lambda} \mathbb{E}_\lambda : 0 \longrightarrow \bigoplus_\lambda A_\lambda \longrightarrow \bigoplus_\lambda B_\lambda \longrightarrow \bigoplus_\lambda C_\lambda \longrightarrow 0\right)$ is in $\mathcal{P}$.

Proposition 2.3.8. *(Sklyarenko, 1978, Propositions 1.2 and 3.2) Every projectively (respectively injectively) generated proper class is $\prod$-closed (resp. $\oplus$-closed).*

Proposition 2.3.9. *(by Sklyarenko (1978, Proposition 9.3)) Let $\mathcal{P}$ be a proper class.*

(i) If $\mathcal{P}$ is $\prod$-closed, then every product of $\mathcal{P}$-coinjective modules is $\mathcal{P}$-coinjective.

(ii) If $\mathcal{P}$ is $\oplus$-closed, then every direct sum of $\mathcal{P}$-coprojective modules is $\mathcal{P}$-coprojective.

2.4 Preradicals and Torsion Theories for R-$\mathcal{M}od$

In this section, we give the definition a torsion theory and some properties of (pre)radicals for R-$\mathcal{M}od$ (since we are interested in radicals on R-$\mathcal{M}od$ in Chapter 3). Preradicals were first introduced in Maranda (1964). We refer to Clark et al. (2006, §6) for elementary properties of preradicals and torsion theories for R-$\mathcal{M}od$. See also Crivei (2004); injective modules relative to a torsion theory have been studied.

See Section 1.1 for the definition of (pre)radicals on R-$\mathcal{M}od$.

We collect the main elementary properties that we shall use frequently for a (pre)radical on R-$\mathcal{M}od$ in the following proposition:

Proposition 2.4.1. *(Clark et al., 2006, p. 55) Let τ be a preradical on R-$\mathcal{M}od$, M be a submodule of a module N and $(M_\lambda)_{\lambda\in\Lambda}$ be a family of modules. Then*

(i) if $\tau(M) = M$, then $M \subseteq \tau(N)$,

(ii) if $\tau(N/M) = 0$, then $\tau(N) \subseteq M$,

(iii) $\tau\left(\bigoplus_{\lambda\in\Lambda} M_\lambda\right) = \bigoplus_{\lambda\in\Lambda} \tau(M_\lambda)$,

(iv) $\tau\left(\prod_{\lambda\in\Lambda} M_\lambda\right) \subseteq \prod_{\lambda\in\Lambda} \tau(M_\lambda)$.

For completeness, note also the following properties of (pre)radicals with their proofs:

Proposition 2.4.2. *For a preradical τ on R-$\mathcal{M}od$ and a homomorphism $f : M \longrightarrow N$ of modules, we have:*

(i) $f(\tau(U)) \subseteq \tau(f(U))$ *for every submodule U of M. In particular,* $f(\tau(M)) \subseteq \tau(f(M))$.

(ii) If U is a submodule of M such that $U = \tau(U)$, then $f(U) = \tau(f(U))$.

(iii) For every submodule K of M, if U is a submodule of M such that $U = \tau(U)$, then $(U+K)/K = \tau((U+K)/K)$. In particular, if $K \subseteq U \subseteq M$ and $U = \tau(U)$, then $U/K = \tau(U/K)$.

Proof. (i) It follows by considering the restriction $f' : U \longrightarrow f(U)$. Since τ is a preradical $f(\tau(U)) = f'(\tau(U)) \subseteq \tau(f(U))$.

(ii) $U = \tau(U)$ (by hypothesis) implies that $f(U) = f(\tau(U)) \subseteq \tau(f(U)) \subseteq f(U)$ by part (i), and so $f(U) = \tau(f(U))$.

(iii) It follows by taking the canonical epimorphism $f : M \longrightarrow M/K$ in (ii); because then $(U+K)/K = f(U) = \tau(f(U)) = \tau((U+K)/K)$. □

Proposition 2.4.3. *For a preradical τ on R-$\mathcal{M}od$ and modules $K \subseteq M$, we have:*

(i) $(\tau(M)+K)/K \subseteq \tau(M/K)$.

(ii) If τ is a radical *and $K \subseteq \tau(M)$, then $\tau(M/K) = \tau(M)/K$ (see, for example, Stenström (1975, Chap. VI, Lemma 1.1)).*

Proof. (i) Let $f : M \longrightarrow M/K$ be the natural epimorphism. Since τ is a preradical,

$$(\tau(M)+K)/K = f(\tau(M)) \subseteq \tau(f(M)) = \tau(M/K).$$

(ii) Since $K \subseteq \tau(M)$ it follows, by (i), that

$$\tau(M)/K = (\tau(M)+K)/K \subseteq \tau(M/K).$$

Conversely, since τ is a radical we have

$$\tau\,[(M/K)/(\tau(M)/K)] \cong \tau(M/\tau(M)) = 0\,.$$

Thus $\tau(M/K) \subseteq \tau(M)/K$ by Proposition 2.4.1-(ii). □

When we consider a ring R as a left R-module, we already have that $A = \tau({}_RR)$ is a left ideal of R; in fact, it is a two-sided ideal of R (as the following theorem shows), so that we can consider the quotient ring R/A which we shall use in the results for τ-supplemented modules.

Theorem 2.4.4. *(Stenström, 1975, Chap. VI, §1, Examples (3)) For each ring R and any preradical τ for R-$\mathcal{M}od$, when R is considered as a left R-module, the left ideal $\tau({}_RR)$ is a two-sided ideal of R.*

Proof. Let $A = \tau({}_RR)$. Since τ is a preradical, $A = \tau({}_RR)$ is a *fully invariant* submodule of the left R-module R (that is, $f(A) \subseteq A$ for every endomorphism $f : {}_RR \longrightarrow {}_RR$). Let $r \in R$. For the R-module endomorphism $f : {}_RR \longrightarrow {}_RR$, defined by $f(x) = xr$ for every $x \in R$, we must have $Ar = f(A) \subseteq A$ as required. □

Let τ be a preradical for R-$\mathcal{M}od$. For a free R-module F, the property $\tau(F) = \tau(R)F$ is easily obtained. Indeed, if $F = \bigoplus_{i \in I} R$ for some index set I, then

$$\tau(F) = \bigoplus_{i\in I} \tau(R) = \bigoplus_{i\in I} \tau(R)R = \tau(R)F.$$

This also holds for projective modules since a projective module is a direct summand of a free module: $\tau(P) = \tau(R)P$ for a projective module P.

Definition 2.4.5. A *torsion theory* for R-$\mathcal{M}od$ is a pair $(\mathcal{T}, \mathcal{F})$ of classes of modules such that

(i) $\mathrm{Hom}_R(T, F) = 0$ for all $T \in \mathcal{T}$, $F \in \mathcal{F}$.

(ii) If $\mathrm{Hom}_R(C, F) = 0$ for all $F \in \mathcal{F}$, then $C \in \mathcal{T}$.

(iii) If $\mathrm{Hom}_R(T, C) = 0$ for all $T \in \mathcal{T}$, then $C \in \mathcal{F}$.

Here, $\mathcal{T}$ is called a *torsion class* and its modules are called *torsion modules*, while $\mathcal{F}$ is called a *torsion free class* and its modules are called *torsion free modules*.

Every class of modules $\mathcal{A}$ *generates* and *cogenerates* a torsion theory in the following sense.

Definition 2.4.6. Let $\mathcal{A}$ be a class of modules in R-$\mathcal{M}od$.

(i) Consider the classes of modules

$$\mathcal{F}_1 = \{Y \in R\text{-}\mathcal{M}od \mid \mathrm{Hom}_R(A, Y) = 0, \text{ for all } A \in \mathcal{A}\}$$

and

$$\mathcal{T}_1 = \{X \in R\text{-}\mathcal{M}od \mid \mathrm{Hom}_R(X, F) = 0, \text{ for all } F \in \mathcal{F}_1\}.$$

Then $(\mathcal{T}_1, \mathcal{F}_1)$ is a torsion theory called the *torsion theory generated by* $\mathcal{A}$.

(ii) Consider the classes of modules

$$\mathcal{T}_2 = \{X \in R\text{-}\mathcal{M}od \mid \mathrm{Hom}_R(X, A) = 0, \text{ for all } A \in \mathcal{A}\}$$

and

$$\mathcal{F}_2 = \{Y \in R\text{-}\mathcal{M}od \mid \mathrm{Hom}_R(T, Y) = 0, \text{ for all } T \in \mathcal{T}_2\}.$$

Then $(\mathcal{T}_2, \mathcal{F}_2)$ is a torsion theory called the *torsion theory cogenerated by* $\mathcal{A}$.

Note that $\mathcal{T}_1$ is the least torsion class containing $\mathcal{A}$, whereas $\mathcal{F}_2$ is the least torsionfree class containing $\mathcal{A}$.

Recall that a torsion theory $(\mathcal{T},\mathcal{F})$ for R-$\mathcal{M}od$ is *hereditary* if the torsion class $\mathcal{T}$ is closed under submodules. Equivalently, the torsion free class $\mathcal{F}$ is closed under injective envelopes (by Proposition 2.8.7 since R-$\mathcal{M}od$ has enough injectives).

Let M be a module and let $a \in M$. Then the set $(M : a) = \{r \in R \mid ra \in M\}$ is a *left* ideal of R. For a submodule N of M, $(0 : N) = \{r \in R \mid rN = 0\}$ is the *annihilator* of N, denoted by $\mathrm{Ann}_R(N)$. For an element $x \in M$, $(0 : x) = \{r \in R \mid rx = 0\}$ is the *annihilator* of x, denoted by $\mathrm{Ann}_R(x)$.

Definition 2.4.7. A non-empty set $F(R)$ of left ideals of R is called a *Gabriel filter* if

(i) for every $I \in F(R)$ and every $a \in R$, we have $(I : a) \in F(R)$ and,

(ii) for every $J \in F(R)$ and every left ideal I of R with $(I : a) \in F(R)$ for all $a \in J$, we have $I \in F(R)$.

The following result can be found, for example, in Crivei (2004):

Theorem 2.4.8. *(see the proof of Crivei (2004, Theorem 1.3.3)) Let $(\mathcal{T},\mathcal{F})$ be a hereditary torsion theory for R-$\mathcal{M}od$. Then for the Gabriel filter $F(R)$ for $(\mathcal{T},\mathcal{F})$ we have*

$$F(R) = \{I \mid I \text{ is a left ideal of } R \text{ and } R/I \in \mathcal{T}\}.$$

Note that $Rx \in \mathcal{T}$ if and only if $Ix = 0$ for some $I \in F(R)$. Equivalently, $Rx \in \mathcal{F}$ if and only if $Ix \neq 0$ for every $I \in F(R)$.

Theorem 2.4.9. *(by Crivei (2004, Theorem 2.1.1)) Let $\tau = (\mathcal{T},\mathcal{F})$ be a hereditary torsion theory for R-$\mathcal{M}od$. The following are equivalent for a module M:*

(i) M is injective with respect to every short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

such that $C \in \mathcal{T}$;

(ii) Any homomorphism from a left ideal I of R such that $R/I \in \mathcal{T}$ to M can be extended to a homomorphism from R to M;

(iii) $\mathrm{Ext}^1_R(C,M) = 0$ *for every module* $C \in \mathcal{T}$.

A module M satisfying these equivalent conditions is called τ-*injective*.

Let M be a module. An element $m \in M$ is said to be a *singular element* of M if $\mathrm{Ann}_R(m) \trianglelefteq R$. The set of all singular elements of M is denoted by $Z(M)$, that is,

$$Z(M) = \{m \in M \mid \mathrm{Ann}_R(m) \trianglelefteq R\}.$$

The submodule $Z(M)$ is called the *singular submodule* of M. The module M is said to be a *singular module* if $Z(M) = M$, and is said to be a *nonsingular module* if $Z(M) = 0$. See, for example, Lam (1999, Chap. 3, §7) for some properties related to singular modules.

Proposition 2.4.10. *(Lam, 1999, Chap. 3, 7.6-(3)) A module M is singular if and only if $M \cong B/A$ for some modules $A \subseteq B$ such that $A \trianglelefteq B$.*

We give some examples of torsion theories which will be used in Chapter 4 (see, for example, Crivei (2004, Example 1.2.16)).

Example 2.4.11. Let τ_D be the torsion theory generated by the class of semisimple (or even simple) modules. Then τ_D is a hereditary torsion theory, called the *Dickson torsion theory*. Its torsion and torsion free classes are respectively

$$\mathcal{T}_D = \{A \in R\text{-}\mathcal{M}od \mid A \text{ is semi-artinian}\} \quad \text{and} \quad \mathcal{F}_D = \{A \in R\text{-}\mathcal{M}od \mid \mathrm{Soc}A = 0\}.$$

In a torsion theory $\tau = (\mathcal{T}, \mathcal{F})$, the torsion class $\mathcal{T}$ need not be closed under taking injective envelopes. τ is called *stable* if the torsion class $\mathcal{T}$ is closed under taking injective envelopes. τ is said to be *faithful* if $R \in \mathcal{F}$.

Let $Z_2(M) = \{x \in M \mid x + Z(M) \in Z(Z/Z(M))\}$ for a module M.

Example 2.4.12. Let τ_G be the torsion theory generated by all singular modules. Then τ_G is a stable hereditary torsion theory, called the *Goldie torsion theory*. Its torsion and torsion free classes are respectively

$$\mathcal{T}_G = \{A \mid Z_2(A) = A\} \quad \text{and} \quad \mathcal{F}_G = \{A \mid A \text{ is nonsingular }\}.$$

Note that if $R \in \mathcal{F}_G$, that is, if R is nonsingular, then $\mathcal{T}_G$ consists of all singular modules.

Remark 2.4.13. In Thorem 4.3.7, Bowe has given his result for Goldie torsion theory with R nonsingular; he has called this torsion theory *Singular Theory*.

2.5 Projective Covers and Perfect Rings

In this section, we give some elementary definitions and properties for projective covers of modules and perfect rings which are needed in Chapter 3.

For modules P and M, an epimorphism $f : P \to M$ is said to be a *projective cover* if P is projective and $\operatorname{Ker} f \ll P$.

Definition 2.5.1. Let R be a ring.

(i) R is called *left perfect* (*semiperfect*) if every (finitely generated) left R-module has a projective cover.

(ii) R is said to be a *left max ring* if every left R-module has a maximal submodule or equivalently, $\operatorname{Rad} M \ll M$ for every left R-module M.

(iii) R is said to be a *semilocal* ring if $R/J(R)$ is a semisimple ring (that is a left (and right) semisimple R-module) (see Lam (2001, §20)). Semilocal rings are also referred to as rings semisimple modulo their radical (see Anderson & Fuller (1992, §15, pp. 170-172)).

Proposition 2.5.2. *Let R an arbitrary ring.*

(i) If P is a projective module and U is a submodule of P such that P/U has a projective cover, then U has a supplement V in P such that V is a direct summand of P (and hence projective) (see Wisbauer (1991, 42.1)).

(ii) A ring R is (left or right) semiperfect if and only if the left (or right) R-module R is supplemented (see Wisbauer (1991, 42.6)).

(iii) For a semilocal ring R, $\operatorname{Rad} M = JM$ for every left R-module M where $J = J(R)$ (see, for example, Anderson & Fuller (1992, Corollary 15.18)).

Recall that a subset I of a ring R is said to be *left T-nilpotent* in case for every sequence $\{a_k\}_{k=1}^{\infty}$ in I there is a positive integer n such that $a_1 \cdots a_n = 0$.

Some of the principal characterizations of left perfect rings given by Bass are contained in the following theorem:

Theorem 2.5.3. *(see, for example, Anderson & Fuller (1992, Theorem 28.4)) The following are equivalent for a ring R:*

(i) R is left perfect;

(ii) R is a semilocal ring and $J(R)$ is left T-nilpotent;

(iii) R is a semilocal left max ring.

2.6 Torsion Free Covering Modules over Commutative Domains

In this section, we review some basic properties of torsion free covers for the usual torsion theory over a commutative domain, and in general for hereditary torsion theories.

See Section 1.2 for the definition of torsion free covers of modules over commutative domains.

If $\mathcal{F}$ is the class of all torsion free modules, then $\mathcal{F}$-covers coincide with torsion free covers defined in Section 1.2. The following is not the usual definition of torsion free covers, but agree with it.

Proposition 2.6.1. *(see arguments after Enochs & Jenda (2000, Definition 5.1.1)) Let $\mathcal{F}$ be the class of all torsion free R-modules over a commutative domain R. A homomorphism $\varphi : T \longrightarrow M$ of modules is torsion free cover of M if and only if φ is an $\mathcal{F}$-cover of M.*

In Enochs (1963, Theorem 1) it was proved that every module over a commutative domain has a torsion free cover (see also Enochs & Jenda (2000, Theorem 4.2.1)). That is, he proved the existence of torsion free covers for the usual torsion theory over a commutative domain. In Teply (1976) and Golan & Teply (1973), this result has been generalized to faithful hereditary torsion theories for R-$\mathcal{M}od$.

Let $(\mathcal{T},\mathcal{F})$ be a faithful hereditary torsion theory for R-$\mathcal{M}od$. A torsion free precover $\varphi : F \longrightarrow M$ (i.e., φ satisfies (i) in the definition of torsion free covers given in Section 1.2) is called a *torsion free cover* of M if $\operatorname{Ker}\varphi$ contains no non-trivial submodule N of F such that $F/N \in \mathcal{F}$.

For a partially ordered set L, a subset $K \subseteq L$ is said to be *cofinal* if, for every $n \in L$, there exists $k \in K$ such that $n \leq k$.

A Gabriel filter $F(R)$ for a hereditary torsion theory $\tau = (\mathcal{T},\mathcal{F})$ for R-$\mathcal{M}od$ is said to *have a cofinal subset of finitely generated left ideals* if, for every $I \in F(R)$, there exists a finitely generated left ideal $J \subseteq I$ such that $J \in F(R)$. Note that if a hereditary torsion theory τ has a cofinal subset of finitely generated left ideals, then τ is said to be of *finite type* (see Golan (1986, Chap. 42)).

Theorem 2.6.2. *(Teply, 1969, Theorem 1.5) Let $(\mathcal{T},\mathcal{F})$ be a hereditary torsion theory. If any direct sum of torsion free injective modules is injective, then $F(R)$ has a cofinal subset of finitely generated left ideals.*

Theorem 2.6.3. *(Teply, 1976, Theorem) Let $(\mathcal{T},\mathcal{F})$ be a faithful hereditary torsion theory. If the Gabriel filter $F(R)$ has a cofinal subset of finitely generated left ideals, then every R-module has a unique torsion free cover.*

Corollary 2.6.4. *Let $(\mathcal{T},\mathcal{F})$ be a faithful hereditary torsion theory. If any direct sum of torsion free injective modules is injective, then every R-module has a unique torsion free cover.*

2.7 Abelian Categories

In this section we recall some definitions and elementary properties of abelian categories. For more details we refer to Stenström (1975, Chaps. IV-V) or Freyd (1964).

A *category* $\mathcal{C}$ is defined to consist of three ingredients: a class Obj($\mathcal{C}$) of *objects* of $\mathcal{C}$, a set $\mathrm{Hom}_{\mathcal{C}}(A,B)$, whose elements are called *morphisms* from A to B for every ordered pair (A,B) of objects, and *composition* $\mathrm{Hom}_{\mathcal{C}}(C',C'') \times \mathrm{Hom}_{\mathcal{C}}(C,C') \longrightarrow \mathrm{Hom}_{\mathcal{C}}(C,C'')$ for every ordered triple (C,C',C'') of objects. These ingredients subject to the following axioms (note that we often write $\alpha : C \longrightarrow C'$ or $C \xrightarrow{\alpha} C'$ instead of $\alpha \in \mathrm{Hom}_{\mathcal{C}}(C,C'')$, and the composition of $\alpha \in \mathrm{Hom}(C,C')$ and $\beta \in \mathrm{Hom}(C',C'')$ is denoted by $\beta\alpha$):

(i) $\mathrm{Hom}_{\mathcal{C}}(C,C')$ and $\mathrm{Hom}_{\mathcal{C}}(D,D')$ are disjoint sets if $(C,C') \neq (D,D')$,

(ii) composition is associative: if $\alpha : C \longrightarrow C'$, $\beta : C' \longrightarrow C''$ and $\gamma : C'' \longrightarrow C'''$ are morphisms, then $\gamma(\beta\alpha) = (\gamma\beta)\alpha$.

(iii) for every object C, there is an *identity morphism* $1_C : C \longrightarrow C$ such that $1_C\alpha = \alpha$ and $\beta 1_C = \beta$ for all $\alpha : C' \longrightarrow C$ and $\beta : C \longrightarrow C''$.

The *opposite category* $\mathcal{C}^{op}$ is defined to be a category with Obj($\mathcal{C}^{op}$)= Obj($\mathcal{C}$), with morphisms $\mathrm{Hom}_{\mathcal{C}^{op}}(A,B) = \mathrm{Hom}_{\mathcal{C}}(B,A)$, and with composition the reverse of that in $\mathcal{C}$; that is, $g * f = f \circ g$, where $*$ is a composition in $\mathcal{C}^{op}$ and $\circ$ is a composition in $\mathcal{C}$.

A morphism $\alpha : C \longrightarrow C'$ in a category $\mathcal{C}$ is an *isomorphism* if there exists $\beta : C' \longrightarrow C$ such that $\alpha\beta = 1_{C'}$ and $\beta\alpha = 1_C$.

Definition 2.7.1. If $\mathcal{C}$ and $\mathcal{D}$ are categories, then a *functor* $T : \mathcal{C} \longrightarrow \mathcal{D}$ is a function such that

(i) if $A \in \mathrm{Obj}(\mathcal{C})$, then $T(A) \in \mathrm{Obj}(\mathcal{D})$,

(ii) if $f : A \longrightarrow A'$ in $\mathcal{C}$, then $T(f) : T(A) \longrightarrow T(A')$ in $\mathcal{D}$,

(iii) if $f : A \longrightarrow A'$ and $g : A' \longrightarrow A''$ in $\mathcal{C}$, then $T(gf) = T(g)T(f)$,

(iv) $T(1_A) = 1_{T(A)}$ for every $A \in \mathrm{Obj}(\mathcal{C})$.

Thus, there is a map

$$\mathrm{Hom}_{\mathcal{C}}(C,C') \longrightarrow \mathrm{Hom}_{\mathcal{D}}(T(C),T(C')) \tag{2.7.1}$$

given by $f \mapsto T(f)$ for every pair $C,C' \in \mathrm{Obj}(\mathcal{C})$. The functor T is called *faithful* if these maps are one-to-one and, T is called *full* if they are onto. A functor $T : \mathcal{C}^{op} \longrightarrow \mathcal{D}$ is said to be a *contravariant functor* from $\mathcal{C}$ to $\mathcal{D}$.

Definition 2.7.2. Let S and T be functors $\mathcal{C} \longrightarrow \mathcal{D}$. A *natural transformation* $\eta : S \longrightarrow T$ is obtained by taking for each object C in $\mathcal{C}$ a morphism $\eta_C : S(C) \longrightarrow T(C)$ in $\mathcal{D}$, so that for every morphism $\alpha : C \longrightarrow C'$ in $\mathcal{C}$ the following diagram commutes:

$$\begin{tikzcd} S(C) \arrow[r, "\eta_C"] \arrow[d, "S(\alpha)"'] & T(C) \arrow[d, "T(\alpha)"] \\ S(C') \arrow[r, "\eta_{C'}"'] & T(C') \end{tikzcd} .$$

η is called a *natural equivalence* if each η_C is an isomorphism in $\mathcal{D}$.

A category $\mathcal{C}$ is called a *small category* if the class of objects of $\mathcal{C}$ is a set. If $\mathbf{I}$ is a small category and $\mathcal{C}$ is any category, the *functor category* $\mathbf{Fun}(\mathbf{I}, \mathcal{C})$ can be defined, where the objects are the functors $\mathbf{I} \longrightarrow \mathcal{C}$ and the morphisms are the natural transformations between such functors; see, for example, Stenström (1975, Chap. IV, §7) for details.

A category $\mathcal{C}$ is called a *preadditive category* if each set $\mathrm{Hom}_{\mathcal{C}}(C, C')$ is an abelian group and the composition map $\mathrm{Hom}_{\mathcal{C}}(C', C'') \times \mathrm{Hom}_{\mathcal{C}}(C, C') \longrightarrow \mathrm{Hom}_{\mathcal{C}}(C, C'')$ is bilinear, that is, given morphisms $f, g \in \mathrm{Hom}_{\mathcal{C}}(C, C')$ and $\alpha, \beta \in \mathrm{Hom}_{\mathcal{C}}(C', C'')$, we have $\alpha(f+g) = \alpha f + \alpha g$ and $(\alpha+\beta)f = \alpha f + \beta f$.

Let $\mathcal{C}$ and $\mathcal{D}$ be preadditive categories. A functor $T : \mathcal{C} \longrightarrow \mathcal{D}$ is called *additive* if $T(\alpha + \alpha') = T(\alpha) + T(\alpha')$ for all $\alpha, \alpha' : C \longrightarrow C'$ in $\mathcal{C}$, that is, the map (2.7.1) is a group homomorphism.

Definition 2.7.3. Let $\mathcal{C}$ and $\mathcal{D}$ be two preadditive categories, and let $S : \mathcal{C} \longrightarrow \mathcal{D}$ and $T : \mathcal{D} \longrightarrow \mathcal{C}$ be two additive functors. S is said to be a *left adjoint* of T (symmetrically T is said to be a *right adjoint* of S) if there is a natural equivalence

$$\eta : \mathrm{Hom}_{\mathcal{C}}(-, T(-)) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(S(-), -)$$

of functors $\mathcal{C}^{op} \times D \longrightarrow \mathcal{A}b$, that is, for every pair of objects $C \in \mathrm{Obj}(\mathcal{C})$ and $D \in$

$\mathrm{Obj}(\mathcal{D})$, there is an isomorphism

$$\eta_{C,D} : \mathrm{Hom}_{\mathcal{C}}(C, T(D)) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(S(C), D)$$

which is natural in C and D.

Let $\mathcal{C}$ be a preadditive category. A *zero object* in $\mathcal{C}$ is an object Z of $\mathcal{C}$ such that $\mathrm{Hom}_{\mathcal{C}}(C,Z) = 0$ and $\mathrm{Hom}_{\mathcal{C}}(Z,C) = 0$ (the trivial abelian group) for every object C of $\mathcal{C}$. Any two zero objects are isomorphic, so we denote them by a single zero object 0 of $\mathcal{C}$. A morphism $f : A \longrightarrow B$ in $\mathcal{C}$ is a *monomorphism* if $f\alpha = 0$ implies $\alpha = 0$ for every morphism $\alpha : X \longrightarrow A$. Dually, f is an *epimorphism* if $\beta f = 0$ implies $\beta = 0$ for every morphism $\beta : B \longrightarrow X$. Two monomorphisms $f : A \longrightarrow B$ and $f' : A' \longrightarrow B$ are said to be *equivalent* if there is an isomorphism $h : A \longrightarrow A'$ such that $f'h = f$. An equivalence class of monomorphisms into $C \in \mathrm{Obj}(\mathcal{C})$ is called a *subobject* of C. Dually, quotient objects are defined. Two epimorphisms $f : A \longrightarrow B$ and $g : A \longrightarrow B'$ are said to be *equivalent* if there is an isomorphism $h : B \longrightarrow B'$ such that $hf = g$. An equivalence class of epimorphisms onto $A \in \mathrm{Obj}(\mathcal{C})$ is called a *quotient object* of A. When A is a subobject of B we write $A \subseteq B$, and so we write B/A for the quotient object $\mathrm{Coker}(A \longrightarrow B)$ of B.

Definition 2.7.4. Let $\mathcal{C}$ be a preadditive category with a *zero* object and let $f : A \longrightarrow B$ be a morphism in $\mathcal{C}$. Then a *kernel* of f, denoted by $\ker f$, is a morphism $k : K \longrightarrow A$ such that $fk = 0$, and for every morphism $g : C \longrightarrow A$ with $fg = 0$, there exists a unique morphism $h : C \longrightarrow K$ such that $g = kh$. Note that, K is denoted by $\mathrm{Ker}\, f$ and that $\ker f$ is a unique monomorphism (more precisely, any two kernels of a morphism represent the same subobject). Also, f is a monomorphism if and only if $\mathrm{Ker}\, f = 0$.

Dually, a *cokernel* of f, denoted by $\mathrm{coker}\, f$, is a morphism $p : B \longrightarrow C$ such that $pf = 0$, and for every morphism $g : B \longrightarrow D$ with $gf = 0$, there exists a unique morphism $h : C \longrightarrow D$ such that $hp = g$. Note that, C is denoted by $\mathrm{Coker}\, f$ and that

$\operatorname{coker} f$ is a unique epimorphism, and f is an epimorphism if and only if $\operatorname{Coker} f = 0$.

Definition 2.7.5. Let $\mathcal{C}$ be a preadditive category with a *zero* object. A *product* of a family $(C_i)_{i\in I}$ of objects of $\mathcal{C}$ is an object C together with morphisms $\pi_i : C \longrightarrow C_i \quad (i \in I)$ such that for every object X and morphisms $f_i : X \longrightarrow C_i$, there is a unique morphism $f : X \longrightarrow C$ with $\pi_i f = f_i$ for all $i \in I$. The product C is unique up to isomorphism and is denoted by $\prod_{i\in I} C_i$.

Dually, a *coproduct* of a family $(C_i)_{i\in I}$ of objects of $\mathcal{C}$ is an object C together with morphisms $e_i : C_i \longrightarrow C \quad (i \in I)$, such that for every object X and morphisms $f_i : C_i \longrightarrow X$, there is a unique morphism $f : C \longrightarrow X$ with $fe_i = f_i$ for all $i \in I$. The coproduct C is unique up to isomorphism and is denoted by $\coprod_{i\in I} C_i$. Note that, since $\mathcal{C}$ is a preadditive category, the coproduct is called a *direct sum* and is denoted by $\bigoplus_{i\in I} C_i$.

Let $\mathcal{C}$ be a preadditive category with a zero object such that every morphism has a kernel and a cokernel. For every morphism $\alpha : B \longrightarrow C$, we have the following commutative diagram (see, for example, Stenström (1975, Chap. IV, §4) for details):

$$\begin{tikzcd}
\operatorname{Ker}\alpha \arrow[r, "\ker\alpha"] & B \arrow[r, "\alpha"] \arrow[d, "\lambda"'] & C \arrow[r, "\operatorname{coker}\alpha"] & \operatorname{Coker}\alpha \\
& \operatorname{Coker}(\ker\alpha) \arrow[r, "\widetilde{\alpha}"'] & \operatorname{Ker}(\operatorname{coker}\alpha) \arrow[u, "\mu"'] &
\end{tikzcd} \qquad (2.7.2)$$

Definition 2.7.6. Let $\mathcal{C}$ be a preadditive category with a zero object. $\mathcal{C}$ is said to be an *abelian category* if

(i) every finite family of objects of $\mathcal{C}$ has a product and a coproduct,

(ii) every morphism in $\mathcal{C}$ has a kernel and a cokernel, and

(iii) the morphism $\widetilde{\alpha}$ of (2.7.2) is an isomorphism for every morphism α.

For every morphism $f : A \longrightarrow B$ in an abelian category, the *image* of f is defined as

$\operatorname{Im} f = \operatorname{Ker}(\operatorname{coker} f)$. Thus f has a factorization

$$A \xrightarrow{\alpha} \operatorname{Im} f \xrightarrow{\beta} B$$

where α is an epimorphism and β is a monomorphism.

Let $\mathcal{C}$ be an abelian category. A *sequence*

$$\cdots \longrightarrow C_{n-1} \xrightarrow{\alpha_{n-1}} C_n \xrightarrow{\alpha_n} C_{n+1} \longrightarrow \cdots$$

is *exact* at C_n if $\operatorname{Im} \alpha_{n-1} = \operatorname{Ker} \alpha_n$ (equal as subobjects of C_n).

An additive functor $T : \mathcal{C} \longrightarrow \mathcal{D}$ between abelian categories $\mathcal{C}$ and $\mathcal{D}$ is said to be an *exact functor* if it carries exact sequences in $\mathcal{C}$ into exact sequences in $\mathcal{D}$.

Let $\mathcal{C}$ be an abelian category. A *short exact sequence* in $\mathcal{C}$ is a sequence $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ such that $\operatorname{Im} f = \operatorname{Ker} g$. In this case, f is a monomorphism and g is an epimorphism. Two short exact sequences

$$\mathbb{E} : 0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0 \quad \text{and} \quad \mathbb{E}' : 0 \longrightarrow A \xrightarrow{f'} B' \xrightarrow{g'} C \longrightarrow 0$$

in $\mathcal{C}$ starting with A and ending with C are said to be *equivalent* if we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "1_A"'] & B \arrow[r, "g"] \arrow[d, "\psi"'] & C \arrow[r] \arrow[d, "1_C"'] & 0 \\
0 \arrow[r] & A \arrow[r, "f'"] & B' \arrow[r, "g'"] & C \arrow[r] & 0
\end{tikzcd}$$

with some morphism $\psi : B \longrightarrow B'$, where $1_A : A \longrightarrow A$ and $1_C : C \longrightarrow C$ are identity morphisms. Denote by $\operatorname{Ext}_{\mathcal{C}}(C,A)$ the set of equivalence classes of all short exact sequences in $\mathcal{C}$ starting with A and ending with C.

Let $\mathcal{C}$ be an abelian category. A class $\mathcal{F}$ of objects in $\mathcal{C}$ is said to be *closed under extensions* if for every short exact sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

of objects in $\mathcal{F}$, we have $B \in \mathcal{F}$ whenever $A, C \in \mathcal{F}$. An object Q of $\mathcal{C}$ is said to be *projective* if the functor $\mathrm{Hom}_{\mathcal{C}}(Q, -) : \mathcal{C} \longrightarrow \mathcal{A}b$ is exact, and is said to be *injective* if the functor $\mathrm{Hom}_{\mathcal{C}}(-, Q) : \mathcal{C}^{op} \longrightarrow \mathcal{A}b$ is exact.

The category $\mathcal{C}$ is said to *have enough projectives* if every object of $\mathcal{C}$ is a quotient object of a projective object, and $\mathcal{C}$ is said to *have enough injectives* if every object is a subobject of an injective object. For instance, it is well-known that the category R-$\mathcal{M}od$ has enough projectives and injectives.

Let $\mathcal{C}$ an abelian category. An object C of $\mathcal{C}$ is said to be a *generator* for $\mathcal{C}$ if $\mathrm{Hom}_{\mathcal{C}}(C, -)$ is faithful, and is a *cogenerator* if $\mathrm{Hom}_{\mathcal{C}}(-, C)$ is faithful.

Proposition 2.7.7. *(Stenström, 1975, Propositions IV.6.3, 6.5) A projective object P of an abelian category $\mathcal{C}$ is a generator if and only if there exists a nonzero morphism $P \longrightarrow C$ for every $C \neq 0$ in $\mathcal{C}$, and an injective object E is a cogenerator if and only if there exists a nonzero morphism $C \longrightarrow E$ for every $C \neq 0$ in $\mathcal{C}$.*

Definition 2.7.8. A family $(U_i)_{i \in I}$ of objects of an abelian category $\mathcal{C}$ is said to be a *family of generators* for $\mathcal{C}$ if for every nonzero morphism $\alpha : B \longrightarrow C$ in $\mathcal{C}$, there exists a morphism $\beta : U_i \longrightarrow B$, for some $i \in I$, such that $\alpha\beta \neq 0$. If, moreover, $\mathcal{C}$ has coproducts, then $\bigoplus_{i \in I} U_i$ is a generator for $\mathcal{C}$ (see, for example, Stenström (1975, Chap. IV, Example 3)).

Let $\mathcal{C}$ be a preadditive category, $\mathbf{I}$ be a small category and $F : \mathbf{I} \longrightarrow \mathcal{C}$ be a functor. A set of morphisms $\alpha_i : X \longrightarrow F(i)$ for all $i \in \mathrm{Obj}(\mathbf{I})$ is said to be *compatible* if for every morphism $\lambda : i \longrightarrow j$ in $\mathbf{I}$, the following diagram commutes:

$$\begin{tikzcd} & X \arrow[dl, "\alpha_i"'] \arrow[dr, "\alpha_j"] & \\ F(i) \arrow[rr, "F(\lambda)"] & & F(j) \end{tikzcd} .$$

In category theory, the functor F is often referred to as a *diagram in $\mathcal{C}$ of type* $\mathbf{I}$, and the set α_i of morphisms as a *commutative cone* with vertex X over the diagram F.

A *commutative co-cone* with vertex X over the diagram F, denoted by $\alpha : F \longrightarrow X$, is a set of morphisms $\alpha_i : F(i) \longrightarrow X$ for all $i \in \mathrm{Obj}(\mathbf{I})$ such that for every morphism $\lambda : i \longrightarrow j$ in $\mathbf{I}$ the following diagram commutes:

$$\begin{tikzcd} F(i) \arrow[rr, "F(\lambda)"] \arrow[dr, "\alpha_i"'] & & F(j) \arrow[dl, "\alpha_j"] \\ & X & \end{tikzcd} .$$

Definition 2.7.9. A *limit* (or *projective limit*) of a diagram $F : \mathbf{I} \longrightarrow \mathcal{C}$ is a cone $\pi : \varprojlim F \longrightarrow F$ such that for every cone $\alpha : X \longrightarrow F$ there exists a unique morphism $\xi : X \longrightarrow \varprojlim F$such that $\pi_i \xi = \alpha_i$ for every $i \in \mathrm{Obj}(\mathbf{I})$. This limit is unique up to isomorphism, if it exists. The category $\mathcal{C}$ is called *complete* if the limit exists for every diagram $F : \mathbf{I} \longrightarrow \mathcal{C}$ when $\mathbf{I}$ is small.

A *colimit* (or *inductive limit*) of a diagram $F : \mathbf{I} \longrightarrow \mathcal{C}$ is a co-cone $\iota : F \longrightarrow \varinjlim F$ such that for every co-cone $\alpha : F \longrightarrow X$ there exists a unique morphism $\xi : \varinjlim F \longrightarrow X$ such that $\xi \iota_i = \alpha_i$ for every $i \in \mathrm{Obj}(\mathbf{I})$. This colimit is unique up to isomorphism, if it exists. The category $\mathcal{C}$ is called *co-complete* if the colimit exists for every diagram $F : \mathbf{I} \longrightarrow \mathcal{C}$ when $\mathbf{I}$ is small.

A partially ordered set $\mathbf{I}$ is called a *directed set* if for every $i, j \in \mathbf{I}$, there exists a $k \in \mathbf{I}$ such that $i \leq k$ and $j \leq k$. If $\mathbf{I}$ is a directed set and $\mathcal{C}$ is an arbitrary category, then a functor $\mathbf{I} \longrightarrow \mathcal{C}$ is called a *direct system* in $\mathcal{C}$, and a functor $\mathbf{I}^{op} \longrightarrow \mathcal{C}$ is called an *inverse system* in $\mathcal{C}$. The colimit of a direct system $\mathbf{I} \longrightarrow \mathcal{C}$ is called a *direct limit*, and the limit of an inverse system $\mathbf{I}^{op} \longrightarrow \mathcal{C}$ is called an *inverse limit*.

Note that, for a cofinal subset K of a directed set I, the direct limit (respectively inverse limit) over I is isomorphic to the direct limit (resp. inverse limit) over K (see, for example, Rotman (2009, Exercise 5.22, p. 255)).

Let $\mathcal{C}$ be an abelian category. A well-ordered direct system $\{C_\alpha : \alpha < \lambda\}$ of objects in $\mathcal{C}$ is said to be *continuous* if $C_0 = 0$ and, for every limit ordinal $\omega < \lambda$, we have $C_\omega = \varinjlim C_\alpha$, where the limit is taken over all ordinals $\alpha < \lambda$. A continuous directed system $\{C_\alpha : \alpha < \lambda\}$ is called a *continuous directed union* if all morphisms in the system are monomorphisms.

A cocomplete abelian category (or abelian category with coproducts) $\mathcal{C}$ is called a *Grothendieck category* if direct limits are exact in $\mathcal{C}$ and $\mathcal{C}$ has a generator. For example, the category R-$\mathcal{M}od$ is a Grothendieck category by Stenström (1975, Proposition I.5.3).

For a cocomplete abelian category $\mathcal{C}$, a family $(C_i)_{i\in I}$ of subobjects of an object C of $\mathcal{C}$ is said to be a *direct family* if I is a directed set when one defines $i \leq j$ whenever $C_i \subseteq C_j$.

Let $\mathcal{C}$ be an abelian category and C be an object in $\mathcal{C}$ with a family $\{C_i\}_{i\in I}$ of subobjects. The monomorphisms $C_i \longrightarrow C$ induce a morphism $\alpha : \bigoplus_{i\in I} C_i \longrightarrow C$. The image of α is called the *sum* of the subobjects C_i and is denoted by $\sum\limits_{i\in I} C_i$. Dually, the epimorphisms $C \longrightarrow C/C_i$ induce a morphism $\beta : C \longrightarrow \prod\limits_{i\in I} C/C_i$. The kernel of β is called the *intersection* of the subobjects C_i and is denoted by $\bigcap\limits_{i\in I} C_i$.

In the following proposition, for two monomorphisms $f : C_i \longrightarrow C$ and $g : C_j \longrightarrow C$

(i.e., C_i and C_j are subobjects of C), we write $C_i \leq C_j$ if there is a morphism $h : C_i \longrightarrow C_j$ such that $gh = f$ (h will then be a monomorphism).

Proposition 2.7.10. *(Stenström, 1975, Chap. IV, Proposition 4.2) If $\{C_i\}_{i\in I}$ is a family of subobjects of C in an abelian category $\mathcal{C}$, then $\sum_{i\in I} C_i$ is a least upper bound and $\bigcap_{i\in I} C_i$ is a greatest lower bound for the family.*

Proposition 2.7.11. *(Stenström, 1975, Proposition V.1.1) Let $\mathcal{C}$ be a cocomplete abelian category and let C be an object of $\mathcal{C}$. Then direct limits are exact in $\mathcal{C}$ if and only if for every subobject B of C one has*

$$\left(\sum_{i\in I} C_i\right) \cap B = \sum_{i\in I} (C_i \cap B)$$

where $(C_i)_{i\in I}$ is a direct family of subobjects of C.

Let $\mathcal{C}$ be a Grothendieck category. A subobject B of an object C is said to be *essential* if $B \cap C' \neq 0$ for every nonzero object C' with $C' \subseteq C$. An *injective envelope* of an object in C is an essential monomorphism $C \longrightarrow E$, where E is an injective object.

Proposition 2.7.12. *(Stenström, 1975, Chap. X, Corollary 4.3) Every object in a Grothendieck category is a subobject of an injective object.*

From the previous result we deduce that every Grothendieck category has enough injectives. Moreover, by for example Stenström (1975, Chap. V, Example 1), every Grothendieck category has injective envelopes.

See Section 1.3 for the definition of a locally finitely generated category.

Proposition 2.7.13. *(see, for example, Stenström (1975, Chap. V, Proposition 3.2)) Let $\mathcal{C}$ be a Grothendieck category. An object C of $\mathcal{C}$ is* finitely generated *if and only if the functor* $\operatorname{Hom}_{\mathcal{C}}(C,-)$ *preserves direct unions, that is,*

$$\varinjlim \operatorname{Hom}_{\mathcal{C}}(C, D_i) \cong \operatorname{Hom}_{\mathcal{C}}(C, \sum_i D_i)$$

for every direct family $(D_i)_{i\in I}$ of subobjects of an object D in $\mathcal{C}$.

Proposition 2.7.14. *((see, for example, Stenström (1975, Chap. V, Proposition 3.4))) Let $\mathcal{C}$ be a locally finitely generated Grothendieck category. An object C of $\mathcal{C}$ is* finitely presented *if and only if the functor* $\mathrm{Hom}_{\mathcal{C}}(C,-)$ *preserves direct limits, that is,*

$$\varinjlim \mathrm{Hom}_{\mathcal{C}}(C,D_i) \cong \mathrm{Hom}_{\mathcal{C}}(C,\varinjlim D_i)$$

for every direct system $(D_i)_{i\in I}$ in $\mathcal{C}$.

Proposition 2.7.15. *(Stenström, 1975, Chap. V, Example 2) Every finitely generated projective object is finitely presented.*

2.8 Torsion Theories for Abelian Categories

In this section we give the definition and some useful properties of torsion theories in abelian categories. We refer to Stenström (1975, Chap. VI, §2) for further details for torsion theories in abelian categories.

A category is said to be *locally small* if the class of the subobjects of any given object is a set (recall that subobjects are equivalence classes of monomorphisms).

Torsion theories were introduced in Dickson (1966) for an abelian category $\mathcal{C}$ which was moreover assumed to be *subcomplete*, that is, locally small and for every set $\{C_\lambda\}_{\lambda\in\Lambda}$ of subobjects of $\mathcal{C}$, $\bigoplus_{\lambda\in\Lambda} C_\lambda$ and $\prod_{\lambda\in\Lambda} A_\lambda$ exist. Thus, throughout this section, unless otherwise stated, we assume $\mathcal{C}$ to be a complete, cocomplete and locally small abelian category.

A *preradical* τ of $\mathcal{C}$ is defined to be a subfunctor of the identity functor on $\mathcal{C}$, that is, for every object C, $\tau(C)$ is a subobject of C and every morphism $C \longrightarrow D$ in $\mathcal{C}$ induces a morphism $\tau(C) \longrightarrow \tau(D)$ by restriction.

If τ_1 and τ_2 are preradicals, then the preradicals $\tau_1\tau_2$ and $\tau_1 : \tau_2$ can be defined as follows:

$$\tau_1\tau_2(C) = \tau_1(\tau_2(C)) \quad \text{and} \quad (\tau_1 : \tau_2)(C)/\tau_1(C) = \tau_2(C/\tau_1(C)).$$

A preradical τ is said to be *idempotent* if $\tau(\tau(C)) = \tau(C)$, and is called a *radical* if $\tau : \tau = \tau$, that is, $\tau(C/\tau(C)) = 0$ for every object C of $\mathcal{C}$.

To a preradical τ the following two classes of objects in $\mathcal{C}$ are defined:

$$\mathcal{T}_\tau = \{C \in \mathrm{Obj}(\mathcal{C}) \mid \tau(C) = C\}, \quad \mathcal{F}_\tau = \{C \in \mathrm{Obj}(\mathcal{C}) \mid \tau(C) = 0\}.$$

$\mathcal{T}_\tau$ is called a *pretorsion class* which is closed under quotient objects and coproducts, and $\mathcal{F}_\tau$ is called *pretorsion free class* which is closed under subobjects and products (see, for example, Stenström (1975, Chap. VI, Proposition 1.2)).

Any preradical τ is a functor preserves monomorphisms. In general this functor need not be exact.

Proposition 2.8.1. *(see, for example, Stenström (1975, Chap. VI, Proposition 1.7)) The following are equivalent for a preradical τ for the category $\mathcal{C}$:*

(i) τ *is a* left exact *functor;*

(ii) if $D \subseteq C$, then $\tau(D) = \tau(C) \cap D$;

(iii) τ is idempotent and $\mathcal{T}_\tau$ is closed under subobjects.

If these equivalent conditions are satisfied, then τ is called a hereditary *preradical.*

Torsion theories for abelian categories were introduced in Dickson (1966, p. 224):

Definition 2.8.2. A *torsion theory* for the category $\mathcal{C}$ is a pair $(\mathcal{T},\mathcal{F})$ of classes of objects in $\mathcal{C}$ such that

(i) $\mathrm{Hom}_{\mathcal{C}}(T,F)=0$ for all $T\in\mathcal{T}$, $F\in\mathcal{F}$.

(ii) If $\mathrm{Hom}_{\mathcal{C}}(C,F)=0$ for all $F\in\mathcal{F}$, then $C\in\mathcal{T}$.

(iii) If $\mathrm{Hom}_{\mathcal{C}}(T,C)=0$ for all $T\in\mathcal{T}$, then $C\in\mathcal{F}$.

$\mathcal{T}$ is called a *torsion class* and its objects are *torsion objects*, while $\mathcal{F}$ is called a *torsion free class* and its objects are *torsion free objects*.

Proposition 2.8.3. *(Stenström, 1975, Chap. VI, Proposition 2.1) The following properties of a class $\mathcal{T}$ of objects in the category $\mathcal{C}$ are equivalent:*

(i) $\mathcal{T}$ is a torsion class for some torsion theory;

(ii) $\mathcal{T}$ is closed under quotient objects, coproducts and extensions.

Proposition 2.8.4. *(Stenström, 1975, Chap. VI, Proposition 2.2) The following properties of a class $\mathcal{F}$ of objects in the category $\mathcal{C}$ are equivalent:*

(i) $\mathcal{F}$ is a torsion free class for some torsion theory;

(ii) $\mathcal{F}$ is closed under subobjects, products and extensions.

Proposition 2.8.5. *(Stenström, 1975, Chap. VI, Proposition 2.3) There is a bijective correspondence between torsion theories and idempotent radicals in $\mathcal{C}$.*

A torsion theory $(\mathcal{T},\mathcal{F})$ for the category $\mathcal{C}$ is called *hereditary* if $\mathcal{T}$ is *hereditary*, that is, $\mathcal{T}$ is closed under subobjects.

Proposition 2.8.6. *(Stenström, 1975, Chap. VI, Proposition 3.1) There is a bijective correspondence between hereditary torsion theories and left exact radicals.*

Proposition 2.8.7. *(Stenström, 1975, Chap. VI, Proposition 3.2) Let $\mathcal{C}$ have injective envelopes. A torsion theory $(\mathcal{T},\mathcal{F})$ for $\mathcal{C}$ is hereditary if and only if $\mathcal{F}$ is closed under injective envelopes.*

2.9 Cotorsion Theories

This section contains some definitions and elementary properties of cotorsion theories (or cotorsion pairs). The notion of cotorsion groups has been introduced in Harrison (1959) and independently in Nunke (1959) and Fuchs (1960). The concept of cotorsion theory were introduced by Salce (1979) in the category of abelian groups; however, the definition can be extended to abelian categories. Actually, the definition of cotorsion theory is analogous of the definition of torsion theory, replacing the functor Hom by Ext. Cotorsion theories have been used to study covers and envelopes (see Enochs & Jenda (2000, Chap. 7)), particularly in the proof of the flat cover conjecture which has been open for nearly twenty years (see Bican et al. (2001)).

Throughout this section, the letter $\mathcal{A}$ will denote an abelian category, and all classes considered are closed under isomorphisms.

Given a class $\mathcal{F}$ of objects of $\mathcal{A}$, ${}^{\perp}\mathcal{F}$ (respectively $\mathcal{F}^{\perp}$) is defined as the class of objects C such that $\mathrm{Ext}_{\mathcal{A}}(C,F) = 0$ (resp. $\mathrm{Ext}_{\mathcal{A}}(F,C) = 0$) for all $F \in \mathcal{F}$. ${}^{\perp}\mathcal{F}$ and $\mathcal{F}^{\perp}$ are called *orthogonal classes* of $\mathcal{F}$.

Definition 2.9.1. Let $(\mathcal{F},\mathcal{C})$ be a pair of classes in $\mathcal{A}$. A class $\mathcal{D}$ is said to *generate* the pair $(\mathcal{F},\mathcal{C})$ if ${}^{\perp}\mathcal{D} = \mathcal{F}$ (and so $\mathcal{D} \subseteq \mathcal{F}^{\perp}$) and a class $\mathcal{G}$ is said to *cogenerate* $(\mathcal{F},\mathcal{C})$ if $\mathcal{G}^{\perp} = \mathcal{C}$ (and so $\mathcal{G} \subseteq {}^{\perp}\mathcal{C}$).

A pair $(\mathcal{F},\mathcal{C})$ of classes of objects in $\mathcal{A}$ is called a *cotorsion theory* (or a *cotorsion pair*) if $\mathcal{F}^{\perp} = \mathcal{C}$ and ${}^{\perp}\mathcal{C} = \mathcal{F}$.

The following examples of cotorsion theories can be found, for example, in Enochs & Jenda (2000, §7.1).

Example 2.9.2. The pairs $(R\text{-}\mathcal{M}od, Inj)$ and $(\mathcal{P}roj, R\text{-}\mathcal{M}od)$ are cotorsion theories where $\mathcal{Inj}$ and $\mathcal{Proj}$ denote the classes of injective and projective modules respectively. The cotorsion theory $(R\text{-}\mathcal{Mod}, \mathcal{Inj})$ is cogenerated by the set of modules R/I where I is a left ideal of R (because, an R-module E is injective if and only if $\mathrm{Ext}^1_R(R/I, E) = 0$ for all left ideals I of R), and it is generated by the class of injective modules.

Note that if $(\mathcal{F}, \mathcal{C})$ is a cotorsion theory, then $\mathcal{F}$ and $\mathcal{C}$ are both closed under extensions and direct summands, and if the category $\mathcal{A}$ has projective (respectively injective) objects, then $\mathcal{F}$ (resp. $\mathcal{C}$) contains all the projective (resp. injective) objects. Also, $\mathcal{F}$ is closed under arbitrary direct sums and $\mathcal{C}$ is closed under arbitrary direct products. If the pair $(\mathcal{F}, \mathcal{C})$ is generated by a set X (not just a class), then $(\mathcal{F}, \mathcal{C})$ is generated by the object $\prod\limits_{M \in X} M$, and if $(\mathcal{F}, \mathcal{C})$ is cogenerated by a set X (not just a class), then $(\mathcal{F}, \mathcal{C})$ is cogenerated by the object $\bigoplus\limits_{M \in X} M$.

An abelian group G is called *cotorsion* if $\mathrm{Ext}(T, G) = 0$ for every torsion free group T; see, for example, Fuchs (1970, §54). This notion has been generalized to modules: An R-module M is said to be *cotorsion* if $\mathrm{Ext}^1_R(F, M) = 0$ for all flat modules F; see, for example, Enochs & Jenda (2000, Definition 5.3.22).

Example 2.9.3. (Enochs & Jenda, 2000, Lemma 7.1.4) Let $\mathcal{F}$ be the class of all flat modules. Then $\mathcal{F}^{\perp} = \mathcal{C}$ is the class of cotorsion modules. In this case, $(\mathcal{F}, \mathcal{C})$ will be a cotorsion theory; it is called *flat cotorsion theory*.

Definition 2.9.4. A pair $(\mathcal{F}, \mathcal{C})$ of classes of objects in $\mathcal{A}$ is said to have *enough injectives* if for every object M in $\mathcal{A}$ there is an exact sequence

$$0 \longrightarrow M \longrightarrow C \longrightarrow F \longrightarrow 0$$

with $C \in \mathcal{C}$ and $F \in \mathcal{F}$, and $(\mathcal{F}, \mathcal{C})$ is said to have *enough projectives* if for every object M there is an exact sequence

$$0 \longrightarrow C \longrightarrow F \longrightarrow M \longrightarrow 0$$

with $F \in \mathcal{F}$ and $C \in \mathcal{C}$.

Remark 2.9.5. Eklof and Trlifaj proved that every cotorsion theory cogenerated by a set of modules has enough projectives and injectives (Eklof & Trlifaj, 2001, Theorem 10). Moreover, the *flat cover conjecture* of Enochs (that is, every module has a flat cover) (Enochs, 1981) is equivalent to the conjecture that the flat cotorsion theory $(\mathcal{F}, \mathcal{C})$ of R-modules has enough projectives. Indeed, if $(\mathcal{F}, \mathcal{C})$ has enough projectives, then for every R-module M there is an exact sequence

$$0 \longrightarrow C \longrightarrow F \xrightarrow{\varphi} M \longrightarrow 0$$

with $C \in \mathcal{C}$ and $F \in \mathcal{F}$. So, for every flat module F', we obtain that

$$\mathrm{Hom}_R(F', F) \longrightarrow \mathrm{Hom}_R(F', M) \longrightarrow 0$$

is exact since $\mathrm{Ext}^1_R(F', C) = 0$ (as C is cotorsion). This means that $\varphi : F \longrightarrow M$ is a flat precover (see the next section for the definition of a cover), and it is known that the existence of flat precovers implies the existence of flat covers (by Enochs (1981, Theorem 3.1)). Therefore, Enochs proved the flat cover conjecture by proving that $(\mathcal{F}, \mathcal{C})$ is cogenerated by a set.

2.10 Covers and Envelopes

In this section, we give some needed properties of covers and envelopes for a given class of objects in an abelian category. See Section 1.3 for the definitions and the motivation for the study of covers and envelopes. Throughout this section the letter $\mathcal{A}$ will denote an abelian category, and $\mathcal{F}$ will denote a class of objects in $\mathcal{A}$.

The proofs of the following elementary properties of $\mathcal{F}$-covers and $\mathcal{F}$-envelopes can be found, for example, in Xu (1996, §1.2) for module categories, but the same argument of the proofs carry over to abelian categories. Suppose that $\mathcal{F}$ is closed under isomorphisms, direct summands and under finite direct sums.

If an $\mathcal{F}$-cover exists, then it is unique up to isomorphism:

Proposition 2.10.1. *If $\varphi_1 : F_1 \longrightarrow M$ and $\varphi_2 : F_2 \longrightarrow M$ are two different $\mathcal{F}$-covers of an object M, then $F_1 \cong F_2$.*

Also, $\mathcal{F}$-covers are direct summands of $\mathcal{F}$-precovers:

Proposition 2.10.2. *Suppose that an object M admits an $\mathcal{F}$-cover, and that $\varphi : F \longrightarrow M$ is an $\mathcal{F}$-precover. Then $F = F_1 \oplus K$ for subobjects F_1 and K of F such that the restriction $\varphi\,|_{F_1} : F_1 \longrightarrow M$ is an $\mathcal{F}$-cover of M and $K \subseteq \mathrm{Ker}(\varphi)$.*

We have the dual results for $\mathcal{F}$-envelopes, that is, if an $\mathcal{F}$-envelope exists then it is unique up to isomorphism, and $\mathcal{F}$-envelopes are direct summand of $\mathcal{F}$-preenvelopes.

The following two results are known as Wakamutsu's Lemmas (see, for example, Xu (1996)).

Proposition 2.10.3. *(Xu, 1996, Lemma 2.1.1) Let $\varphi : F \longrightarrow M$ be an $\mathcal{F}$-cover of an object M. If the class $\mathcal{F}$ is closed under extensions, then* $\mathrm{Ker}(\varphi) \in \mathcal{F}^{\perp}$.

Dually we have

Proposition 2.10.4. *(Xu, 1996, Lemma 2.1.2) Let $\varphi : M \longrightarrow F$ be an $\mathcal{F}$-envelope of an object M. If the class $\mathcal{F}$ is closed under extensions, then* $\mathrm{Coker}(\varphi) \in {}^{\perp}\mathcal{F}$.

CHAPTER THREE

RAD-SUPPLEMENTED MODULES

In this chapter, we investigate some properties of Rad-supplemented modules and in general τ-supplemented modules where τ is a radical for $R\text{-}\mathcal{M}od$. In Section 3.5, we answer one of our main questions that when are all left R-modules Rad-supplemented. We investigate some further properties of τ-supplemented modules in Section 3.3. For some rings R, we also determine when all left R-modules are τ-supplemented in Section 3.4. We describe Rad-supplemented modules over Dedekind domains using the structure of supplemented modules over Dedekind domains which was completely determined in Zöschinger (1974) (see Section 3.6). See Wisbauer (1991, §41) and the recent monograph Clark et al. (2006) for the results (and the definitions) related to (*weak*) *supplements* and (*weakly*) *supplemented* modules.

Throughout this chapter we shall follow the terminology and notation as in Clark et al. (2006, §10) and Al-Takhman et al. (2006), since we will mainly refer to these for τ-supplemented modules and Rad-supplemented modules. Unless otherwise stated, τ will be a radical for $R\text{-}\mathcal{M}od$.

3.1 τ-reduced and τ-coatomic Modules

Let τ be a preradical for $R\text{-}\mathcal{M}od$ and let M be a module. By taking τ instead of Rad in the definitions of reduced and coatomic module definitions in Zöschinger (1974, p. 47), we define the following:

(i) M is said to be a τ-*torsion* module if $\tau(M) = M$, that is, M is in the pretorsion class $\mathcal{T}_\tau$.

(ii) M is said to be a τ-*reduced* module if it has *no* nonzero τ-torsion submodule, that is, for every submodule U of M, $\tau(U) = U$ implies $U = 0$ or equivalently, $\tau(U) \neq U$ for every nonzero submodule U of M.

(iii) M is said to be a τ-*coatomic* module if it has *no* nonzero τ-torsion factor module, that is, for every submodule U of M, $\tau(M/U) = M/U$ implies $U = M$ or equivalently, $\tau(M/U) \neq M/U$ for every proper submodule U of M.

For $\tau = \mathrm{Rad}$, a Rad-torsion module will be called a *radical module*, a Rad-reduced module will be called a *reduced module* and a Rad-coatomic module will be called a *coatomic module* following the terminology in Zöschinger (1974). Coatomic modules appear in the theory of supplemented, semiperfect, and perfect modules. See Zöschinger (1974, Lemma 1.5) for some properties of reduced and coatomic modules.

Remark 3.1.1. See Golan (1986, pp. 29,63) for the definitions and properties of τ-dense submodules of a module and τ-cotorsionfree modules for a *hereditary* idempotent preradical τ on R-$\mathcal{M}od$: A submodule N of a module M is said to be τ-*dense* in M if M/N is τ-torsion, that is, $\tau(M/N) = M/N$, and a module M is said to be τ-*cotorsionfree* if it has *no* proper τ-dense submodules. Our definition of τ-coatomic module coincides with τ-cotorsionfree module, but in our case, τ need not be idempotent or hereditary; in particular, Rad is not hereditary. The properties for τ-cotorsionfree modules given in Golan (1986) hold under this *hereditary* assumption. For example, arbitary direct sum of τ-cotorsionfree modules is τ-cotorsionfree when τ is a *hereditary* idempotent preradical, but in our case, for just an (idempotent) preradical τ, arbitrary direct sum of τ-coatomic modules need not be τ-coatomic.

By the results in Stenström (1975, Chap. VI, §2), the properties of τ-torsion and τ-reduced modules in the following Proposition 3.1.2 are obtained.

Proposition 3.1.2. *Let τ be a preradical for R-$\mathcal{M}od$.*

(i) The class of τ-torsion modules is closed under quotients and direct sums.

Moreover, if τ is a radical, then the class of τ-torsion modules is closed under extensions.

(ii) *Every factor module of a τ-coatomic module is τ-coatomic.*

(iii) *The class of τ-reduced (respectively τ-coatomic) modules is closed under extensions, that is, if*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0 \tag{3.1.1}$$

is a short exact sequence of modules such that A and C are τ-reduced (resp. τ-coatomic), then B is also τ-reduced (resp. τ-coatomic).

Proof. (i) Let M be a τ-torsion module, that is, $\tau(M) = M$. Then for every submodule $K \subseteq M$, we have $M/K = (\tau(M) + K)/K \subseteq \tau(M/K)$ by Proposition 2.4.3-(i). Thus $\tau(M)/K = \tau(M/K)$ since $\tau(M/K) \subseteq M/K$ is always true. Now, for a family $(M_i)_I$ of τ-torsion modules (for some index set I) we have, by Proposition 2.4.1, that

$$\tau\left(\bigoplus_{i\in I} M_i\right) = \bigoplus_{i\in I} \tau(M_i) = \bigoplus_{i\in I} M_i.$$

Moreover, if τ is a radical and

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is a short exact sequence such that A and $C \cong B/A$ are τ-torsion modules, then

$$B/A = \tau(B/A) = \tau(B)/A$$

where the last equality holds by Proposition 2.4.3-(ii) (since $A = \tau(A) \subseteq \tau(B)$). This implies that $\tau(B) = B$.

(ii) Let M be a τ-coatomic module and let $U \subseteq M$. Suppose that $\tau[(M/U)/(K/U)] =$

$(M/U)/(K/U)$ for submodules $U \subseteq K \subseteq M$. Since $(M/U)/(K/U) \cong M/K$, we obtain that $\tau(M/K) = M/K$, and so $M = K$ (since M is τ-coatomic). Thus $M/U = K/U$.

(iii) To prove this, in the above short exact sequence (3.1.1) we can assume, without loss of generality, that $A \subseteq B$, $C = B/A$, the map $A \longrightarrow B$ is the inclusion homomorphism and the map $B \longrightarrow C = B/A$ is the canonical epimorphism. Let U be a submodule of B. Suppose firstly that A and $C = B/A$ are τ-reduced, and $\tau(U) = U$. Then by Proposition 2.4.2-(iii), $(U+A)/A = \tau((U+A)/A)$. Since $B/A = C$ is τ-reduced, we obtain that $(U+A)/A = 0$, and so $U \subseteq A$. Therefore since A is τ-reduced, $U = 0$ as required. Now suppose that A and $C = B/A$ are τ-coatomic, and $\tau(B/U) = B/U$. Then by Proposition 2.4.2,

$$\tau[(B/U)/((U+A)/U)] = (B/U)/((U+A)/U).$$

We have the following natural isomorphisms:

$$(B/A)/((U+A)/A) \cong B/(U+A) \cong (B/U)/((U+A)/U).$$

So we also have $\tau[(B/A)/((U+A)/A)] = (B/A)/((U+A)/A)$. Since $B/A = C$ is τ-coatomic, $(U+A)/A = B/A$ and so $U+A = B$. Then we have

$$A/(U\cap A) \cong (U+A)/U = B/U \qquad \text{and} \qquad \tau(B/U) = B/U.$$

Then $\tau(A/(U\cap A)) = A/(U\cap A)$ which implies that $U\cap A = A$ as A is τ-coatomic. Thus $A \subseteq U$, and so $U+A = B$ implies that $U = B$ as required. □

Proposition 3.1.3. *Let τ be a radical for R-$\mathcal{M}od$. If a module M is τ-coatomic, then $\tau(M) \ll M$.*

Proof. Suppose $\tau(M) + L = M$ for some submodule $L \subseteq M$. Since $M/L = (\tau(M) + L)/L \subseteq \tau(M/L)$, we obtain that $M/L = \tau(M/L)$. This gives $L = M$ since M is

τ-coatomic. Hence $\tau(M) \ll M$. □

3.2 The Largest τ-torsion Submodule P_τ(M)

In this section, we define the largest τ-torsion submodule $P_\tau(M)$ of a module M, and give some properties of it which will be used frequently in this chapter.

By $P_\tau(M)$ we denote the sum of *all* τ-torsion submodules of M, that is,

$$P_\tau(M) = \sum\{U \subseteq M \mid \tau(U) = U\}.$$

Note that for $\tau = \text{Rad}$, $P_\tau(M)$ will be denoted by just $P(M)$.

Remark 3.2.1. It can be seen immediately that a module M is τ-reduced if and only if M is P_τ-torsion free, that is, $P_\tau(M) = 0$.

Remark 3.2.2. For a ring R, $P(_RR)$ will be the sum of all *left* ideals I of R such that $\text{Rad} I = I$. In this thesis, by $P(R)$ we will mean $P(_RR)$. Now, define $P(R_R)$ to be the sum of all *right* ideals I of R such that $\text{Rad} I = I$. Thus, the question has been raised whether $P(_RR) = P(R_R)$ or not. For example, $J(_RR) = J(R_R)$ and so the notation $J(R)$ is used for the Jacobson radical of R.

The following theorem contains some useful elementary properties of $P_\tau(M)$:

Theorem 3.2.3. *Let τ be a preradical for R-$\mathcal{M}od$ and let M be an R-module.*

(i) *P_τ is an* idempotent preradical.

(ii) *If $M \subseteq N$ for a module N, then $P_\tau(M) \subseteq \tau(N)$. In particular, $P_\tau(M) \subseteq \tau(M)$.*

(iii) *$\tau(P_\tau(M)) = P_\tau(M)$, that is, $P_\tau(M)$ is a τ-torsion module, and so $P_\tau(M)$ is the* largest *τ-torsion submodule of M (by its definition).*

(iv) If $P_\tau(M) \subseteq V$ for a submodule V of M, then $P_\tau(M) \subseteq \tau(V)$.

(v) $P_\tau(\tau(M)) = P_\tau(M)$

(vi) The pretorsion class of P_τ equals the pretorsion class of τ and the pretorsion free class of P_τ contains the pretorsion free class of τ:

$$\mathcal{T}_{P_\tau} = \mathcal{T}_\tau \qquad \textit{and} \qquad \mathcal{F}_{P_\tau} \supseteq \mathcal{F}_\tau.$$

(vii) Moreover, if τ is a radical, then the factor module $M/P_\tau(M)$ is τ-reduced, that is, $P_\tau\big(M/P_\tau(M)\big) = 0$, and so P_τ is an idempotent radical.

Proof. (i) Clearly, $P_\tau(M) \subseteq M$ for every R-module M. Let $f : M \longrightarrow N$ be a homomorphism of R-modules. If U is a τ-torsion submodule of M, then $f(U)$ is a τ-torsion submodule of N by Proposition 2.4.2-(ii). So

$$\begin{aligned} f(P_\tau(M)) &= f\left(\sum\{U \subseteq M \mid \tau(U) = U\}\right) \\ &= \sum\{f(U) \mid U \subseteq M, \tau(U) = U\} \\ &\subseteq \sum\{V \subseteq N \mid \tau(V) = V\} = P_\tau(N). \end{aligned}$$

This shows that P_τ is a preradical. To show that P_τ is idempotent, we only need to show that $P_\tau(M) \subseteq P_\tau(P_\tau(M))$. For every submodule U of M such that $\tau(U) = U$, we clearly have $U \subseteq P_\tau(M)$, and so $U \subseteq P_\tau(P_\tau(M))$. Thus

$$P_\tau(M) = \sum\{U \subseteq M \mid \tau(U) = U\} \subseteq P_\tau(P_\tau(M)).$$

(ii) For every submodule U of M such that $\tau(U) = U$, since $U \subseteq N$ also, we have $U = \tau(U) \subseteq \tau(N)$ by Proposition 2.4.1-(i). Thus

$$P_\tau(M) = \sum\{U \subseteq M \mid \tau(U) = U\} \subseteq \tau(N).$$

(iii) Clearly, $\tau(P_\tau(M)) \subseteq P_\tau(M)$. Conversely, we have

$$P_\tau(M) = P_\tau(P_\tau(M)) \subseteq \tau(P_\tau(M))$$

by parts (i) and (ii).

(iv) If $P_\tau(M) \subseteq V$, then $\tau(P_\tau(M)) \subseteq \tau(V)$. Since $\tau(P_\tau(M)) = P_\tau(M)$ by part (iii), the result follows.

(v) Since $\tau(M) \subseteq M$, we already have $P_\tau(\tau(M)) \subseteq P_\tau(M)$. Conversely, by part (ii), $P_\tau(M) \subseteq \tau(M)$ and so $P_\tau(P_\tau(M)) \subseteq P_\tau(\tau(M))$. By part (i), $P_\tau(P_\tau(M)) = P_\tau(M)$. Thus $P_\tau(M) \subseteq P_\tau(\tau(M))$.

(vi) Let N be any module. If $\tau(N) = N$, then by definition of P_τ, we obtain $N \subseteq P_\tau(N) \subseteq N$, and so $P_\tau(N) = N$. Conversely, if $P_\tau(N) = N$, then by part (ii), $N = P_\tau(N) \subseteq \tau(N) \subseteq N$, so $\tau(N) = N$. Now if $\tau(N) = 0$, then by part (ii), $P_\tau(N) \subseteq \tau(N) = 0$, and thus $P_\tau(N) = 0$ as desired.

(vii) Suppose $U/P_\tau(M) = \tau(U/P_\tau(M))$, where U is a submodule of M such that $P_\tau(M) \subseteq U$. Then $P_\tau(M) \subseteq \tau(U)$ by part (iv). So

$$U/P_\tau(M) = \tau(U/P_\tau(M)) = \tau(U)/P_\tau(M)$$

by Proposition 2.4.3-(ii) which implies $U = \tau(U)$, and so by definition of $P_\tau(M)$, we obtain $U \subseteq P_\tau(M)$. Thus $U = P_\tau(M)$, that is, $U/P_\tau(M) = 0$. This means that $M/P_\tau(M)$ is τ-reduced, and so $P_\tau(M/P_\tau(M)) = 0$. □

Remark 3.2.4. In general, given any class $\mathbb{A}$ of modules, a *preradical* $\tau^{\mathbb{A}}$ is defined by setting for every module N,

$$\tau^{\mathbb{A}}(N) = \sum\{\operatorname{Im} f \mid f : A \longrightarrow N \text{ in } R\text{-}\mathcal{M}od, A \in \mathbb{A}\}.$$

and if $\mathbb{A}$ is a pretorsion class, then $\tau^{\mathbb{A}}$ is an idempotent preradical (see, for example, Clark et al. (2006, 6.5)). In our case, the preradical P_τ is equal to $\tau^{\mathbb{A}}$ when the pretorsion class $\mathbb{A}$ is equal to the pretorsion class of τ (i.e., $\mathbb{A} = \mathcal{T}_\tau$). See also Stenström (1975, Chap. VI, §1); P_τ is the largest idempotent preradical that is smaller than τ and see Stenström (1975, Chap. VI, Exercise 4) for the parts (iii), (v) of Theorem 3.2.3. Since P_τ is an idempotent radical when τ is a radical, it gives a torsion theory for R-$\mathcal{M}od$ with torsion class $\mathcal{T}_{P_\tau} = \mathcal{T}_\tau$ and torsion free class $\mathcal{F}_{P_\tau}$.

Proposition 3.2.5. *For a preradical τ, the class of τ-reduced modules is closed under submodules, direct products and direct sums.*

Proof. Let M be a τ-reduced module and let $U \subseteq M$. Then $P_\tau(U) \subseteq P_\tau(M) = 0$, that is, U is τ-reduced. For a family $(N_i)_{i\in I}$ of τ-reduced modules,

$$P_\tau\left(\prod_i N_i\right) \subseteq \prod_i P_\tau(N_i) = 0 \qquad \text{(by Proposition 2.4.1-(iv))},$$

that is, the product is τ-reduced. Finally, since the direct sum is a submodule of the direct product the result follows immediately. □

3.3 τ-supplemented Modules

Throughout the rest of this chapter, τ denotes a radical on R-$\mathcal{M}od$. See Al-Takhman et al. (2006) and Clark et al. (2006, §10) for properties of τ-supplements and τ-supplemented modules. In this section we give some further properties of τ-supplemented modules.

A module M is called π-*projective* if for any submodules U, V of M such that $M = U + V$, there exists a homomorphism $f : M \longrightarrow M$ with $\operatorname{Im} f \subseteq U$ and $\operatorname{Im}(1-f) \subseteq V$. See Wisbauer (1991, 41.14) for details.

The following proposition contains some properties from Al-Takhman et al. (2006)

that we shall use frequently:

Proposition 3.3.1. *(Al-Takhman et al., 2006, 2.2, 2.3, 2.6) Let M be a τ-supplemented module. Then*

(i) Every factor module and every direct summand of M is τ-supplemented.

(ii) $M/\tau(M)$ is a semisimple module.

(iii) If N is a τ-supplemented module, then $M+N$ is τ-supplemented.

(iv) If M is π-projective, then M is amply τ-supplemented.

For a submodule V of a module M, it is known that the property

$$\operatorname{Rad} V = V \cap \operatorname{Rad} M$$

holds if V is a supplement in M (Wisbauer, 1991, 41.1) and moreover if V is coclosed in M (Clark et al., 2006, 3.7). We show that this property also holds when V is a Rad-supplement in M. In general:

Theorem 3.3.2. *If V is a τ-supplement in a module M, then $\tau(V) = V \cap \tau(M)$.*

Proof. $\tau(V) \subseteq V \cap \tau(M)$ always holds. To show the converse we only require to show that $(V \cap \tau(M))/\tau(V) = 0$. Since V is a τ-supplement in M, there exists a submodule $U \subseteq M$ such that $U + V = M$ and $U \cap V = \tau(V)$ by Proposition 1.1.1-(ii). Then

$$M/(U \cap V) = (U/(U \cap V)) \oplus ((V/U \cap V)) = (U/\tau(V) \oplus (V/\tau(V)).$$

Since τ is a radical, we obtain

$$\tau(M/\tau(V)) = \tau(U/\tau(V)) \oplus \tau(V/\tau(V)) = \tau(U/\tau(V)) \oplus 0 = \tau(U/\tau(V)).$$

Since $\tau(V) \subseteq \tau(M)$, we have, by Proposition 2.4.3, that

$$\tau(M)/\tau(V) = \tau(M/\tau(V)) = \tau(U/\tau(V)), \quad \textit{and}$$

$$\begin{aligned}
(V \cap \tau(M))/\tau(V) &= (V/\tau(V)) \cap (\tau(M)/\tau(V)) = (V/\tau(V)) \cap \tau(U/\tau(V)) \\
&\subseteq (V/\tau(V)) \cap (U/\tau(V)) \\
&= (U \cap V)/\tau(V) = \tau(V)/\tau(V) = 0.
\end{aligned}$$ □

Corollary 3.3.3. *If V is a Rad-supplement in a module M, then* $\operatorname{Rad} V = V \cap \operatorname{Rad} M$.

We shall also give other proofs of this result after giving some needed properties.

Proposition 3.3.4. *Let M be a module and let V be a Rad-supplement in M. For $K \subseteq M$, if $K \subseteq \operatorname{Rad} M$, then $K \cap V \subseteq \operatorname{Rad} V$.*

Proof. Assume that V is a Rad-supplement of a submodule $U \subseteq M$ in M. Now suppose on the contrary that $K \cap V \nsubseteq \operatorname{Rad} V$. Then there exists a maximal submodule $T \subseteq V$ such that $K \cap V \nsubseteq T$. So there exists an $m \in (K \cap V) \setminus T$. Since T is a maximal submodule of V, $T + Rm = V$. Thus $M = U + V = U + T + Rm$. Since $Rm \subseteq K \subseteq \operatorname{Rad} M$, we obtain $Rm \ll M$ as Rm is a finitely generated submodule of $\operatorname{Rad} M$. So $M = (U + T) + Rm$ implies that $U + T = M$. Then by modular law $V = V \cap M = V \cap (U + T) = (V \cap U) + T = T$ since $V \cap U \subseteq \operatorname{Rad} V \subseteq T$. This contradicts with T being a maximal submodule of V. □

Second proof of Corollary 3.3.3. For a submodule $V \subseteq M$, $\operatorname{Rad} V \subseteq V \cap \operatorname{Rad} M$ always holds. Conversely, let $x \in V \cap \operatorname{Rad} M$. Then $Rx \subseteq V$ and $Rx \subseteq \operatorname{Rad} M$. Then by Proposition 3.3.4, we obtain $Rx = Rx \cap V \subseteq \operatorname{Rad} V$. So $x \in \operatorname{Rad} V$ as required. □

The formula $\operatorname{Rad} V = V \cap \operatorname{Rad} M$ holds for all submodules $V \subseteq M$ only in the case $\operatorname{Rad} M = 0$.

Proposition 3.3.5. *Let M be a module. Then the following are equivalent:*

(i) $\operatorname{Rad} V = V \cap \operatorname{Rad} M$ *for* every *submodule* $V \subseteq M$,

(ii) $\operatorname{Rad} M = 0$.

Proof. $(i) \Rightarrow (ii)$: Let $x \in \operatorname{Rad} M$ and let $V = Rx$. Then $V \subseteq \operatorname{Rad} M$ and, by hypothesis, $\operatorname{Rad} V = V \cap \operatorname{Rad} M = V$. But since $V = Rx$ is cyclic (and so finitely generated), $V \subseteq \operatorname{Rad} V$ implies that $V \ll V$. Thus $Rx = V = 0$, and so $x = 0$ which shows that $\operatorname{Rad} M = 0$.

$(ii) \Rightarrow (i)$: Let V be a submodule of M. Then $\operatorname{Rad} V \subseteq V \cap \operatorname{Rad} M$ always holds. Since $\operatorname{Rad} M = 0$, $\operatorname{Rad} V = 0 = V \cap \operatorname{Rad} M$. □

Corollary 3.3.6. *Let M be a module. Every submodule of M is a Rad-supplement (coneat) in M if and only if M is semisimple.*

Proof. $(\Leftarrow)$ It is obvious, since every submodule of a semisimple module is a direct summand, and so a Rad-supplement in M. Indeed, if V is a direct summand of M, then there exists a submodule $U \subseteq M$ such that $U \oplus V = M$. This means that $U + V = M$ and $U \cap V = 0 \subseteq \operatorname{Rad} V$, that is, V is a Rad-supplement of U in M.

$(\Rightarrow)$ Let V be a submodule of M. Since V is a Rad-supplement in M, there exists a submodule U of M such that $V + U = M$ and $V \cap U \subseteq \operatorname{Rad} V$. We also have $\operatorname{Rad} V = V \cap \operatorname{Rad} M$ (by Corollary 3.3.3). Since $\operatorname{Rad} M = 0$ by Proposition 3.3.5, $\operatorname{Rad} V = 0$ already. Thus $V \cap U \subseteq \operatorname{Rad} V = 0$ implies that V is a direct summand of M, that is, M is semisimple. □

Corollary 3.3.7. *For a module M, if* $\operatorname{Rad} M \neq 0$*, then M has a submodule V_0 such that V_0 is not a Rad-supplement in M and* $\operatorname{Rad} V_0 \neq V_0 \cap \operatorname{Rad} M$*, and so there exists $x \in V_0$ such that for the cyclic submodule Rx of V_0, $Rx \ll M$ but $Rx \not\ll V_0$.*

For further insight into the property $\operatorname{Rad} V = V \cap \operatorname{Rad} M$ when V is a Rad-supplement of a submodule U of M in M, we give another proof that goes through the relation between maximal submodules of M that contains U and maximal submodules of V.

Proposition 3.3.8. *Let V be a submodule of a module M and let T be a maximal submodule of V. Then the following are equivalent:*

(i) There exists a maximal submodule L of M such that $L \cap V = T$.

(ii) V/T is not small in M/T, that is, the inclusion $T \subseteq V$ is not cosmall in M.

Proof. $(i) \Rightarrow (ii)$: $T = L \cap V \subseteq L$ implies $T \subseteq L$. Since T is a maximal submodule of V, there exists an $x \in V \setminus T$. Then $x \notin L$ because if $x \in L$, then we would have $x \in L \cap V = T$, contradicting $x \notin T$. Thus $L + Rx = M$ since L is a maximal submodule of M. Since $Rx \subseteq V$ as $x \in V$, we obtain $L + V = M$. Then since $T \subseteq V$ and $T \subseteq L$,

$$M/T = (L+V)/T = (L/T) + (V/T)$$

and $L/T \neq M/T$ because $L \neq M$ as L is maximal submodule of M. This shows that V/T is not small in M/T.

$(ii) \Rightarrow (i)$: Since V/T is not small in M/T, $(V/T) + (L/T) = M/T$ for some submodule $L \subseteq M$ such that $T \subseteq L \neq M$. Since V/T is simple as T is a maximal submodule of V, $(V/T) \cap (L/T)$ is either 0 or V/T. If $(V/T) \cap (L/T) = V/T$, then $V/T \subseteq L/T$ and $M/T = (V/T) + (L/T) = L/T$, contradicting $L \neq M$. Thus

$$(V \cap L)/T = (V/T) \cap (L/T) = 0,$$

and so we have $V \cap L = T$. Since $(V/T) \cap (L/T) = 0$, we obtain $M/T = (V/T) \oplus (L/T)$. Hence

$$M/L \cong (M/T)/(L/T) \cong V/T$$

is simple which implies that L is maximal in M. □

Proposition 3.3.9. *Let M be a module and V be a Rad-supplement submodule of M. If T is a maximal submodule of V, then there exists a maximal submodule L of M such that $L \cap V = T$.*

Proof. Suppose T is a maximal submodule of V. Then V/T is simple. Since V is a Rad-supplement in M, there exists a submodule $U \subseteq M$ such that $U + V = M$ and $U \cap V \subseteq \operatorname{Rad} V$. Let $L = U + T$. Then since $T \subseteq V$, $L + V = (U + T) + V = U + V = M$.

Since T is maximal in V and $\mathrm{Rad}V$ is the intersection of all maximal submodules of V, we have $U \cap V \subseteq \mathrm{Rad}V \subseteq T$. Thus

$$L \cap V = (U+T) \cap V = (U \cap V) + T = T.$$

Now we have $L \neq M$ because if $L = M$, then $V = M \cap V = L \cap V = T$ contradicting T is maximal in V. Thus

$$M/T = (L+V)/T = (L/T) + (V/T)$$

where $L/T \neq M/T$. This shows that V/T is not small in M/T. Now use Proposition 3.3.8 to end the proof. $\square$

Indeed, the above proofs show the following:

Proposition 3.3.10. *Let M be a module and U, V be submodules of M such that V is a Rad-supplement of U in M. For every maximal submodule T of V, the submodule $U+T$ is maximal in M.*

This gives another proof of Corollary 3.3.3:

Third proof of Corollary 3.3.3. For a submodule $V \subseteq M$, $\mathrm{Rad}V \subseteq V \cap \mathrm{Rad}M$ always holds. Suppose V is a Rad-supplement of a submodule U of M in M, that is, $U+V = M$ and $U \cap V \subseteq \mathrm{Rad}V$. Using the previous proposition, since the radical of a module is the intersection of all its maximal submodules, we obtain:

$$\begin{aligned} V \cap \mathrm{Rad}M \quad \subseteq \quad & V \cap \bigcap \{U+T \mid T \text{ maximal in } V\} \\ & = \bigcap \{V \cap (U+T) \mid T \text{ maximal in } V\} \\ & = \bigcap \{T \mid T \text{ maximal in } V\} = \mathrm{Rad}V \end{aligned}$$

Note that in the second equality we have used the following: when T is a maximal submodule of V, then since $U \cap V \subseteq \operatorname{Rad} V \subseteq T \subseteq V$, we have

$$V \cap (U + T) = (U \cap V) + T = T. \qquad \square$$

Proposition 3.3.11. *Let K, L, M be modules such that $K \subseteq L \subseteq M$.*

(i) If K is a τ-supplement in M, then it is a τ-supplement in L.

(ii) If $K \subseteq \tau(L)$ and L/K is a τ-supplement in M/K, then L is a τ-supplement in M.

(iii) If K is a τ-supplement in L and L is a τ-supplement in M, then K is a τ-supplement in M.

Proof. (i) Since K is a τ-supplement in M, there exists a submodule $U \subseteq M$ such that $U + K = M$ and $U \cap K \subseteq \tau(K)$. So $L = L \cap M = L \cap (U + K) = L \cap U + K$ and $(L \cap U) \cap K = U \cap K \subseteq \tau(K)$.

(ii) Since L/K is a τ-supplement in M/K, there exists a submodule $U \subseteq M$ with $K \subseteq U$ such that $U/K + L/K = M/K$ and $(U/K) \cap (L/K) \subseteq \tau(L/K)$. So we obtain $U + L = M$ and

$$(U \cap L)/K = (U/K) \cap (L/K) \subseteq \tau(L/K) = \tau(L)/K$$

where the last equality holds by Proposition 2.4.3 since $K \subseteq \tau(L)$. Hence $U \cap L \subseteq \tau(L)$, and so L is a τ-supplement (of U) in M.

(iii) Recall that τ-$\mathcal{S}uppl$ is the class of all short exact sequences

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

of modules such that $\operatorname{Im} f$ is a τ-supplement in B. By Proposition 1.1.1, the class τ-$\mathcal{S}uppl$ is the proper class injectively generated by all modules M such that $\tau(M) = 0$. By the definition of proper classes, the composition of two

τ-$\mathcal{S}uppl$-monomorphisms is an τ-$\mathcal{S}uppl$-monomorphism (see Section 2.3). If K is a τ-supplement in L and L is a τ-supplement in M, then the inclusions $K \hookrightarrow L$ and $L \hookrightarrow M$ are τ-$\mathcal{S}uppl$-monomorphisms and so their composition $K \hookrightarrow M$ is also an τ-$\mathcal{S}uppl$-monomorphism, that is, K is a τ-supplement in M. $\square$

Proposition 3.3.12. *Let M be a module and let N, K be submodules of M such that $M = N + K$. If K is τ-supplemented, then K contains a τ-supplement of N in M.*

Proof. Since K is τ-supplemented, the submodule $N \cap K$ of K has a τ-supplement in K, that is, there exists a submodule $L \subseteq K$ such that $(N \cap K) + L = K$ and $(N \cap K) \cap L \subseteq \tau(L)$. Then $M = N + K = N + (N \cap K) + L = N + L$ and $N \cap L = (N \cap K) \cap L \subseteq \tau(L)$. Hence L is a τ-supplement of N in M. $\square$

Proposition 3.3.13.

(i) Every τ-torsion module is τ-supplemented.

(ii) The module $P_\tau(M)$ is τ-supplemented for every module M.

Proof. (i) Let M be a τ-torsion module, that is, $\tau(M) = M$. Then each submodule U of M has a τ-supplement M in M, that is, $U + M = M$ and $U \cap M = U \subseteq M = \tau(M)$.

(ii) Since, by Theorem 3.2.3-(iii), $P_\tau(M)$ is a τ-torsion submodule of M, the result follows by (i).

$\square$

Theorem 3.3.14. *If a module M is τ-reduced and τ-supplemented, then M is τ-coatomic,* $\mathrm{Rad}\, M = \tau(M)$ *and M is weakly supplemented.*

Proof. Let U be a proper submodule of M. Since M is τ-supplemented, there exists a submodule $V \subseteq M$ such that $U + V = M$ and $U \cap V \subseteq \tau(V)$. So we have $\tau(V/(U \cap V)) = \tau(V)/(U \cap V)$ (by Proposition 2.4.3). We also have $\tau(V) \neq V$ since M is τ-reduced, and so $\tau(V)/(U \cap V) \neq V/(U \cap V)$. Therefore, using the fact that $M/U = (U+V)/U \cong$

$V/(U \cap V)$ we obtain

$$\tau(M/U) \cong \tau(V/(U \cap V)) = \tau(V)/(U \cap V) \neq V/(U \cap V),$$

or equivalently, $\tau(M/U) \neq M/U$, that is, M is τ-coatomic. By Proposition 3.1.3, $\tau(M) \ll M$ and thus $\tau(M) \subseteq \operatorname{Rad} M$. By Proposition 3.3.1, $M/\tau(M)$ is semisimple since M is τ-supplemented. Then $\operatorname{Rad}(M/\tau(M)) = 0$, and so $\operatorname{Rad} M \subseteq \tau(M)$. Hence $\operatorname{Rad} M = \tau(M)$. Since $\operatorname{Rad} M = \tau(M) \ll M$ and M is a semilocal module (that is, $M/\operatorname{Rad} M = M/\tau(M)$ is semisimple), we obtain that M is weakly supplemented by Lomp (1999, Theorem 2.7). □

Theorem 3.3.15. *If M is a τ-supplemented module, then $\operatorname{Rad} M \subseteq \tau(M)$, and*

$$\operatorname{Rad}(M/P_\tau(M)) = \tau(M/P_\tau(M)) = \tau(M)/P_\tau(M).$$

Proof. By Proposition 3.3.1, $M/\tau(M)$ is semisimple and so $\operatorname{Rad}(M/\tau(M)) = 0$ which gives $\operatorname{Rad} M \subseteq \tau(M)$. The module $M/P_\tau(M)$ is τ-supplemented as a factor module of the τ-supplemented module M. Since $M/P_\tau(M)$ is τ-reduced, $\operatorname{Rad}(M/P_\tau(M)) = \tau(M/P_\tau(M))$ by Theorem 3.3.14. Finally, since $P_\tau(M) \subseteq \tau(M)$ (by Theorem 3.2.3-(ii)), $\tau(M/P_\tau(M)) = \tau(M)/P_\tau(M)$ (by Proposition 2.4.3). □

Proposition 3.3.16. *The following are equivalent for a module M and a submodule $K \subseteq P_\tau(M)$:*

(i) M is τ-supplemented;

(ii) M/K is τ-supplemented;

(iii) $M/P_\tau(M)$ is τ-supplemented.

Proof. Since every factor module of a τ-supplemented module is τ-supplemented, $(i) \Rightarrow (ii) \Rightarrow (iii)$ are clear. To prove $(iii) \Rightarrow (i)$, take a submodule $U \subseteq M$. By hypothesis, there exists a submodule $V \subseteq M$ with $P_\tau(M) \subseteq V$ such that

$$[(U + P_\tau(M))/P_\tau(M)] + [V/P_\tau(M)] = M/P_\tau(M)$$

and

$$\begin{aligned}(U \cap V + P_\tau(M))/P_\tau(M) &= [(U + P_\tau(M))/P_\tau(M)] \cap [V/P_\tau(M)] \\ &\subseteq \tau(V/P_\tau(M)) = \tau(V)/P_\tau(M).\end{aligned}$$

Note that the last equality holds by Theorem 3.2.3-(iv). So we have $U + V = M$ and $U \cap V \subseteq \tau(V)$, that is, V is a τ-supplement of U in M. □

Corollary 3.3.17. *The following are equivalent for a ring R:*

(i) every R-module is τ-supplemented;

(ii) every free R-module is τ-supplemented;

(iii) every τ-reduced R-module is τ-supplemented.

Proof. $(i) \Rightarrow (ii)$ and $(i) \Rightarrow (iii)$ are clear. $(ii) \Rightarrow (i)$ follows since every module is an epimorphic image of a free R-module and being τ-supplemented is preserved under passage factor modules. To prove $(iii) \Rightarrow (i)$ take an R-module M. Since $M/P_\tau(M)$ is τ-reduced, we obtain that $M/P_\tau(M)$ is τ-supplemented by the hypothesis. So M is τ-supplemented by Proposition 3.3.16. □

Proposition 3.3.18. *If V is a τ-supplement in a module M and V is τ-coatomic, then V is a supplement in M.*

Proof. Since V is a τ-supplement in M, there exists $U \subseteq M$ such that $U + V = M$ and $U \cap V \subseteq \tau(V)$. Since V is τ-coatomic, we have by Proposition 3.1.3 that $\tau(V) \ll V$. Then $U \cap V \subseteq \tau(V) \ll V$, and thus V is a supplement in M. □

Proposition 3.3.19. *If M is a τ-reduced module that is totally τ-supplemented, then M is totally supplemented.*

Proof. Since being τ-reduced is inherited by submodules, it is enough to prove that M is supplemented. Let $U \subseteq M$ and V be a τ-supplement of U in M. Then $U + V = M$ and $U \cap V \subseteq \tau(V)$. By hypothesis, V is τ-supplemented and τ-reduced. So by Theorem 3.3.14, V is τ-coatomic. Then $\tau(V) \ll V$ by Proposition 3.1.3. Therefore $U \cap V \ll V$, and so V is a supplement of U in M. Hence M is supplemented. $\square$

Clearly supplemented modules are Rad-supplemented, thus we obtain the following result:

Corollary 3.3.20. *If M is a reduced module, then M is totally Rad-supplemented if and only if M is totally supplemented.*

3.4 When are all Left R-modules τ-supplemented?

In this section we shall characterize the rings all of whose (left) modules are τ-supplemented for some particular radicals τ including Rad. We start to this section with some basic definitions.

See Section 2.5 for projective covers and perfect rings.

An epimorphism $f : N \to M$ is said to be a τ-*cover* if $\operatorname{Ker} f \subseteq \tau(N)$. If moreover N is projective, then f is called a *projective* τ-*cover*. A ring R is called left τ-*perfect* (τ-*semiperfect*) if every (finitely generated) left R-module has a projective τ-cover.

These rings are studied in Azumaya (1992) and Xue (1996) for the radical $\tau = \text{Rad}$, and in Nakahara (1983) for a larger class of preradicals.

The relation between τ-cover and τ-supplements is the following:

Proposition 3.4.1. *(by Al-Takhman et al. (2006, 2.14)) For a module M and a submodule $U \subseteq M$, the following are equivalent:*

(i) M/U has a projective τ-cover;

(ii) U has a τ-supplement V which has a projective τ-cover.

It is clear from the definitions and Proposition 3.4.1 that if R is a left τ-(semi)perfect ring then every (finitely generated) left R-module is τ-supplemented. But the converse need not be true, for example when $\tau = \mathrm{Rad}$; see Example 3.5.3.

Lemma 3.4.2. *If R is a ring that is a τ-reduced left R-module and if the free left R-module $F = R^{(\mathbb{N})}$ is τ-supplemented, then $\tau(R)$ is left T-nilpotent.*

Proof. Since $P_\tau(R) = 0$ and $P_\tau(F) = (P_\tau(R))^{(\mathbb{N})} = 0$, F is τ-reduced. Then F is τ-coatomic by Theorem 3.3.14, and so by Proposition 3.1.3

$$\tau(R)F = (\tau(R))^{(\mathbb{N})} = \tau(F) \ll F.$$

Therefore $\tau(R)$ is left T-nilpotent by Anderson & Fuller (1992, Lemma 28.3). □

Theorem 3.4.3. *If R is a ring that is a τ-reduced left R-module, then the free left R-module $F = R^{(\mathbb{N})}$ is τ-supplemented if and only if R is left perfect and $\tau(R) = J(R)$.*

Proof. Suppose $F = R^{(\mathbb{N})}$ is τ-supplemented. Then R is τ-supplemented as a direct summand of F. Since R is also τ-reduced by hypothesis, we obtain $\tau(R) = J(R)$ by Theorem 3.3.14. By Lemma 3.4.2, $J(R) = \tau(R)$ is left T-nilpotent. Since R is τ-supplemented, $R/J(R) = R/\tau(R)$ is semisimple by Proposition 3.3.1. Hence R is left perfect by Theorem 2.5.3. Conversely suppose R is left perfect and $\tau(R) = J(R)$. Let $U \subseteq F = R^{(\mathbb{N})}$. Since R is left perfect, every left R-module, and in particular, F/U has a projective cover. Then by 2.5.2-(i), U has a supplement V in the free module F such that V is a direct summand of F. Since F is free, its direct summand V is projective. So $\tau(V) = \tau(R)V$ by properties of radicals. Since V is a supplement of U in M, $U + V = M$

and $U \cap V \ll V$. So $U \cap V \subseteq \mathrm{Rad}(V)$. Since R is a left perfect ring, it is a semilocal ring (see Theorem 2.5.3), and so $\mathrm{Rad}(V) = J(R)V$ (by Proposition 2.5.2-(iii)). Thus $U \cap V \subseteq \mathrm{Rad}(V) = J(R)V = \tau(R)V = \tau(V)$. Hence V is a τ-supplement of U in M. $\square$

Remark 3.4.4. Note that the above proof for the converse implication works for every free left R-module F, not necessarily countably generated. Moreover, since every factor module of a τ-supplemented module is τ-supplemented and every module is isomorphic to a factor module of a free module, we have:

Corollary 3.4.5. *If R is a ring that is a τ-reduced left R-module, then every (free) left R-module is τ-supplemented if and only if R is left perfect and $\tau(R) = J(R)$.*

It is easy to see that a radical τ on R-modules is also a radical on $R/P_\tau(R)$-modules since every $R/P_\tau(R)$-module can be considered as an R-module (with annihilator containing $P_\tau(R)$). We shall use this fact in the proof of the following theorem:

Theorem 3.4.6. *For a ring R with $P_\tau(R) \subseteq J(R)$, the following are equivalent:*

(i) every left R-module is τ-supplemented;

(ii) every free left R-module is τ-supplemented;

(iii) the free left R-module $F = R^{(\mathbb{N})}$ is τ-supplemented;

(iv) the quotient ring $R/P_\tau(R)$ is left perfect and $\tau(R) = J(R)$.

Proof. $(i) \Leftrightarrow (ii)$ follows by Corollary 3.3.17. $(ii) \Rightarrow (iii)$ is clear.

$(iii) \Rightarrow (iv)$: Since F is τ-supplemented, so is its factor module $\overline{F} = F/P_\tau(F) \cong (R/P_\tau(R))^{(\mathbb{N})}$. The R-module $\overline{F}$ can be considered as an $R/P_\tau(R)$-module and τ can be considered also as a radical on $R/P_\tau(R)$-modules. By Theorem 3.4.3, since $R/P_\tau(R)$ is τ-reduced, we obtain that the quotient ring $R/P_\tau(R)$ is left perfect and

$$\tau(R/P_\tau(R)) = J(R/P_\tau(R)).$$

Then by Proposition 2.4.3, $\tau(R/P_\tau(R)) = \tau(R)/P_\tau(R)$ (since $P_\tau(R) \subseteq \tau(R)$ by Theorem 3.2.3-(ii)), and $J(R/P_\tau(R)) = J(R)/P_\tau(R)$ since $P_\tau(R) \subseteq J(R)$ by hypothesis. Hence $\tau(R) = J(R)$.

$(iv) \Rightarrow (ii)$: By properties of radicals, since $P_\tau(R) \subseteq \tau(R) = J(R)$ by hypothesis, we obtain for the left perfect quotient ring $S = R/P_\tau(R)$ that:

$$\tau(S) = \tau(R/P_\tau(R)) = \tau(R)/P_\tau(R) = J(R)/P_\tau(R) = J(R/P_\tau(R)) = J(S).$$

By Corollary 3.4.5, every free S-module is τ-supplemented, where we consider τ also as a radical on S-modules. Let F be a free R-module. Then $F \cong R^{(I)}$ for some index set I. By Proposition 3.3.16, it is enough to prove that $\overline{F} = F/P_\tau(F) \cong S^{(I)}$ is τ-supplemented. But this holds since $\overline{F}$ can be considered as a free S-module. $\square$

3.5 When are all Left R-modules Rad-supplemented?

Using the results of the previous sections for $\tau = \mathrm{Rad}$, we obtain the following characterization of the rings R over which every R-module is Rad-supplemented. Of course, more work still remains to understand $P(R)$ and the condition that $R/P(R)$ is left perfect.

Theorem 3.5.1. *For a ring R, the following are equivalent.*

(i) *every left R-module is Rad-supplemented;*

(ii) *every reduced left R-module is Rad-supplemented;*

(iii) *every reduced left R-module is supplemented;*

(iv) *the free left R-module $R^{(\mathbb{N})}$ is Rad-supplemented;*

(v) *$R/P(R)$ is left perfect.*

Proof. $(i) \Leftrightarrow (iv) \Leftrightarrow (v)$ is obtained by Theorem 3.4.6 since $P(R) \subseteq \mathrm{Rad}(R) = J(R)$. $(i) \Leftrightarrow (ii)$ follows by Corollary 3.3.17. $(iii) \Rightarrow (ii)$ holds since supplemented modules are Rad-supplemented. To prove $(ii) \Rightarrow (iii)$, take any reduced left R-module M. Then every submodule of M is also reduced, and so Rad-supplemented by hypothesis (ii). So M is a reduced module that is totally Rad-supplemented. Thus, by Corollary 3.3.20, M is totally supplemented, and so supplemented. □

A preradical τ for R-$\mathcal{M}od$ is said to be *normal* if $\tau(P) \neq P$ for every nonzero projective module P.

Proposition 3.5.2. *(Nakahara, 1983, Theorem 1.5) Let τ be a* normal *radical for R-$\mathcal{M}od$. Then R is left τ-perfect if and only if $\tau = \mathrm{Rad}$ and R is left perfect.*

The following is an example of a ring R that is not left perfect (and so not left Rad-perfect by Proposition 3.5.2 since Rad is normal by, for example, Kasch (1982, Theorem 9.6.3)) but where all R-modules are Rad-supplemented.

Example 3.5.3. Let k be a field. In the polynomial ring $k[x_1, x_2, \ldots]$ with countably many indeterminates x_n $(n \in \mathbb{N})$, consider the ideal $I = (x_1^2, x_2^2 - x_1, x_3^2 - x_2, \ldots)$ generated by x_1^2 and $x_{n+1}^2 - x_n$ for every $n \in \mathbb{N}$. In the quotient ring $R = k[x_1, x_2, \ldots]/I$, the maximal ideal $M = (x_1, x_2, \ldots)/I$ of R generated by all $\overline{x_n} = x_n + I$, $n \in \mathbb{N}$, is the *unique* maximal ideal of R. This is because, if K is any maximal ideal of R, then $\bar{x}_1^2 = 0 \in K$ and so $\bar{x}_1 \in K$ since K is a prime ideal. Now $\bar{x}_2^2 = \bar{x}_1 \in K$ and so $\bar{x}_2 \in K$. By induction, we obtain $\bar{x}_n^2 = \bar{x}_{n-1} \in K$ and so $\bar{x}_n \in K$ for all $n \in \mathbb{N}$. Therefore $K = M$, as desired. Since $\bar{x}_n = \bar{x}_{n+1}^2$ for every $n \in \mathbb{N}$, we obtain $M = M^2$. So $\mathrm{Rad}\, M = M$, and thus $P(R) = M$. Since the ring $R/P(R) = R/M$ is a field (and so perfect), every R-module is Rad-supplemented (by Theorem 3.5.1). By Anderson & Fuller (1992, Lemma 28.3), $M = J(R)$ is not (left) T-nilpotent (since $J(R)M = M^2 = M$), and so R is not a (left) perfect ring (by Theorem 2.5.3).

In Büyükaşık & Lomp (2008), it is proved that the class of rings that are Rad-supplemented lies properly between the classes of semilocal rings and semiperfect rings.

Recall that a ring R is said to be a *left duo ring* if every left ideal of R is a two-sided ideal.

We shall characterize the left duo rings R that are Rad-supplemented left R-modules, firstly, by proving the following lemma:

Lemma 3.5.4. *If R is a left duo ring and I, A, B are left ideals of R such that $A+B=R$ and $A\cap B = IA\cap IB$, then $A\cap B = I(A\cap B)$.*

Proof. Clearly $I(A\cap B)\subseteq A\cap B$. Conversely let $x\in A\cap B = IA\cap IB$. Since $A+B=R$, we have $a+b=1$ for some $a\in A$ and $b\in B$. Then $x = xa+xb$ and $x=\sum_{i\in I'} s_i a_i = \sum_{i\in I''} t_i b_i$ where I', I'' are finite index sets, $a_i\in A$, $b_i\in B$ and $s_i, t_i\in I$. Now we have,

$$xb = \sum_{i\in I'} s_i a_i b \in I(AB) \text{ and } xa = \sum_{i\in I''} t_i b_i a \in I(BA).$$

Since R is a left duo ring, $AB\subseteq A\cap B$ and $BA\subseteq A\cap B$. So,

$$x = xa+xb \in I(BA)+I(AB)\subseteq I(A\cap B),$$

and thus $A\cap B\subseteq I(A\cap B)$. □

Theorem 3.5.5. *If R is a left duo ring such that $P(R)=0$, then R is a Rad-supplemented left R-module if and only if R is semiperfect.*

Proof. If R is semiperfect, then R is a supplemented, and so a Rad-supplemented, left R-module. Conversely, suppose R is a Rad-supplemented left R-module. Then R is semilocal and R is an amply Rad-supplemented left R-module by Proposition 3.3.1. Let A' be a left ideal of R. Since R is an amply Rad-supplemented left R-module, A' has a Rad-supplement B in R, and B has a Rad-supplement $A\subseteq A'$ in R. So $R=$

$A' + B = A + B$, $A \cap B \subseteq A' \cap B \subseteq \operatorname{Rad} B$ and $A \cap B \subseteq \operatorname{Rad} A$. Thus $A \cap B = (\operatorname{Rad} A) \cap (\operatorname{Rad} B)$. Let $J = J(R)$. Then $A \cap B = JA \cap JB = J(A \cap B)$ by Lemma 3.5.4. Since R is a semilocal ring, $\operatorname{Rad}(A \cap B) = J(A \cap B)$. Then $A \cap B$ is a radical submodule of R, and so $A \cap B \subseteq P(R) = 0$. This gives that $R = A \oplus B$. Therefore $JB \subseteq J \ll R$ implies that $\operatorname{Rad}(B) = JB \ll B$ since B is a direct summand of R. Hence B is a supplement of A' in R. This shows that R is a supplemented left R-module, and thus R is semiperfect by Proposition 2.5.2-(ii). $\square$

Theorem 3.5.6. *For a left duo ring R, the following are equivalent:*

(i) $R/P(R)$ *is semiperfect;*

(ii) the left R-module R is Rad-supplemented;

(iii) every finitely generated free left R-module is Rad-supplemented;

(iv) every finitely generated left R-module is Rad-supplemented.

Proof. $(ii) \Rightarrow (iii)$ follows by Proposition 3.3.1. $(iii) \Rightarrow (iv)$ holds since every finitely generated module is an epimorphic image of a finitely generated free module and Rad-supplemented modules are closed under epimorphic images.

$(iv) \Rightarrow (ii)$ is clear.

$(i) \Rightarrow (ii)$: Since the quotient ring $S = R/P(R)$ is semiperfect, $R/P(R)$ is a Rad-supplemented left S-module, and so a Rad-supplemented left R-module. Then the left R-module R is Rad-supplemented by Proposition 3.3.16.

$(ii) \Rightarrow (i)$: The factor module $R/P(R)$ is also a Rad-supplemented left R-module. So the ring $S = R/P(R)$ is a Rad-supplemented left S-module with $P(S) = 0$, and so $S = R/P(R)$ is semiperfect by Theorem 3.5.5. $\square$

Remark 3.5.7. Note that all implications except $(ii) \Rightarrow (i)$ of Theorem 3.5.6 hold for each ring R, while the implication $(ii) \Rightarrow (i)$ raises the question whether a Rad-supplemented ring R with $P(R) = 0$ is necessarily semiperfect.

3.6 Rad-supplemented Modules over Dedekind Domains

Most of the results which will be given in this section can be found in Özdemir (2007, Chap. 6), but we give the proofs here for completeness.

Following the terminology in abelian groups, an R-module M over a Dedekind domain is said to be *bounded* if $rM = 0$ for some nonzero $r \in R$.

The structure of supplemented modules over Dedekind domains is completely determined in Zöschinger (1974):

Theorem 3.6.1. *(Zöschinger, 1974, Theorems 2.4. and 3.1) Let R be a Dedekind domain with quotient field $K \neq R$. Let M be an R-module.*

(i) Suppose R is a local Dedekind domain, that is, a discrete valuation ring (DVR) with the unique prime element p. Then M is supplemented if and only if $M \cong R^a \oplus K^b \oplus (K/R)^c \oplus B$ for some R-module B, where a, b, c are nonnegative integers and $p^n B = 0$ for some integer $n > 0$.

(ii) Suppose R is non local. Then M is supplemented if and only if M is torsion and every primary component of M is a direct sum of an artinian submodule and a bounded submodule.

Part (i) of the above theorem for Rad-supplemented modules is obtained as follows:

Theorem 3.6.2. *Let R be a DVR with quotient field $K \neq R$, and p be the unique prime element. Then M is Rad-supplemented if and only if $M \cong R^a \oplus K^{(I)} \oplus (K/R)^{(J)} \oplus B$ for some R-module B, where a is a nonnegative integer, I, J are arbitrary index sets and $p^n B = 0$ for some integer $n > 0$.*

Proof. ($\Rightarrow$) If M_1 is the divisible part of M, then there exists a reduced submodule M_2 of M such that $M = M_1 \oplus M_2$. Since M_2 is also Rad-supplemented, it is coatomic by Theorem 3.3.14. Then by Zöschinger (1974, Lemma 2.1), $M_2 = R^a \oplus B$, for some nonnegative integer a and a bounded module B. Since M_1 is divisible, $M_1 \cong K^{(I)} \oplus (K/R)^{(J)}$ for some index sets I and J (see Kaplansky (1952, Theorem 7)).

($\Leftarrow$) The module $N = K^{(I)} \oplus (K/R)^{(J)}$ is divisible, and so $\operatorname{Rad} N = N$. Then N is Rad-supplemented by Proposition 3.3.13. By Theorem 3.6.1, the module $R^a \oplus B$ is supplemented, and hence Rad-supplemented. Therefore the direct sum $R^a \oplus K^{(I)} \oplus (K/R)^{(J)} \oplus B$ is Rad-supplemented. $\square$

For the structure of coatomic modules over commutative Noetherian rings see Zöschinger (1980); the Noetherian assumption in the following proposition is needed to have that every submodule of a coatomic module over a commutative Noetherian ring is coatomic (Zöschinger, 1980, Lemma 1.1).

Proposition 3.6.3. *Let R be a commutative noetherian ring and M be a reduced R-module. Then M is Rad-supplemented if and only if M is supplemented.*

Proof. Suppose M is Rad-supplemented. Then M is coatomic by Theorem 3.3.14, and so every submodule of M is coatomic since R is a commutative noetherian ring. Let U be a submodule of M and V be a Rad-supplement of U in M. Then V is coatomic, and so $U \cap V \subseteq \operatorname{Rad} V \ll V$. Thus V is a supplement of U in M. The converse is clear. $\square$

Since the structure of supplemented modules over Dedekind domains is known by Theorem 3.6.1-(ii), it is enough to characterize Rad-supplemented modules in terms of supplemented modules. Note that, for an R-module M, where R is a Dedekind domain, $P(M)$ equals the *divisible part* of M.

Theorem 3.6.4. *Let R be a Dedekind domain and M be an R-module. Then M is Rad-supplemented if and only if $M/P(M)$ is (Rad)supplemented.*

Proof. Since R is a Dedekind domain, M has a decomposition as $M = P(M) \oplus N$ for some reduced submodule N of M. If M is Rad-supplemented, then $N \cong M/P(M)$ is also Rad-supplemented. Since N is reduced, N is supplemented by Proposition 3.6.3. Conversely, suppose $N \cong M/P(M)$ is Rad-supplemented. By Proposition 3.3.13-(ii), the submodule $P(M)$ is already Rad-supplemented. Therefore $M = P(M) \oplus N$ is Rad-supplemented as a sum of two Rad-supplemented modules. $\square$

These characterizations can be used to provide examples of Rad-supplemented modules which are not supplemented:

Example 3.6.5. Let R be a Dedekind domain with quotient field $K \neq R$. The R-module $M = K^{(I)}$ is Rad-supplemented for every index set I. If R is a local Dedekind domain (i.e. a DVR), then M is supplemented only when I is finite. If R is a non-local Dedekind domain, then M is not supplemented for every index set I, since M is not torsion.

CHAPTER FOUR
ENOCHS' NEAT HOMOMORPHISMS AND MAX-INJECTIVE MODULES

In this chapter, we deal with neat homomorphisms of Enochs which we call E-neat homomorphisms and max-injective modules. For some properties of left C-rings, see Section 4.1. In Section 4.2, we are interested in max-injective modules. We observe that they are nothing but $\mathcal{N}eat$-coinjective modules. See Section 4.3, for E-neat homomorphisms; we collect some useful properties of them taken from Bowe (1972). In Section 4.4, we investigate the homomorphisms which we call Z-neat homomorphisms, due to Zöschinger; in fact, we deal with the question that when E-neat and Z-neat homomorphisms coincide. Zöschinger has showed their equivalence for abelian groups and we prove they are equivalent over Dedekind domains. In Section 4.5, we consider E-neat homomorphisms which are epimorphism, and we show that E-neat epimorphisms do not define a proper class. In Section 4.6, we are interested in Z-coneat homomorphisms of Zöschinger which were introduced by dualizing E-neat homomorphisms. We also study the class of all short exact sequences defined by coclosed submodules.

4.1 *C*-rings of Renault

This section contains some properties of left C-rings of Renault (1964). See Section 1.2 for the definition of a left C-ring. For further details, see, for example, Mermut (2004, §3.3).

A commutative Noetherian ring in which every *nonzero* prime ideal is maximal is a C-ring (see Mermut (2004, Proposition 3.3.6)). So, in particular, a Dedekind domain is also a C-ring. And moreover, Stenström shows that if R is a commutative Noetherian ring in which every *nonzero* prime ideal is maximal (and thus R is a C-ring), then $\mathcal{C}ompl = \mathcal{N}eat$ (Stenström, 1967b, Corollary to Proposition 8). Generalov gives a characterization of this equality in terms of the ring R.

See also Mermut (2004, Theorem 3.3.2) for a proof of the following result:

Proposition 4.1.1. *(Generalov, 1978, Theorem 5) A ring R is a left C-ring if and only if $\mathcal{C}ompl = \mathcal{N}eat$.*

Proposition 4.1.2. *(Renault, 1964, Proposition 1.2) A ring R is a left C-ring if and only if for every essential proper left ideal of R,* $\mathrm{Soc}(R/I) \neq 0$.

Proposition 4.1.3. *(Mermut, 2004, Proposition 3.3.9) A commutative domain R is a C-ring if and only if every nonzero torsion module has a simple submodule.*

4.2 Max-injective Modules are Injective only for C-rings

In this section, we deal with max-injective modules, and we give a proof of the result that all max-injective R-modules are injective over a left C-ring R with our interest in the proper classes $\mathcal{N}eat$ and $\mathcal{C}ompl$; it is indeed already given in Smith (1981).

Of course, all injective R-modules are max-injective for every ring R, but the converse is not always true as the following example shows:

Example 4.2.1. (Crivei, 1998, Example 16) Let $R = F[[X,Y]]$ be the ring of formal power series on the set of commuting indeterminates X,Y over a field F. Then R is a max-injective R-module which is not injective.

Note the following useful diagram lemma:

Lemma 4.2.2. *(see, for example, Fuchs & Salce (2001, Lemma I.8.4)) Suppose*

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] & B \arrow[r, "g"] & C \arrow[r] & 0 \\ 0 \arrow[r] & A_1 \arrow[r, "f_1"'] \arrow[u, "\alpha"] & B_1 \arrow[r] \arrow[u] & C_1 \arrow[r] \arrow[u, "\beta"'] & 0 \end{tikzcd}$$

is a commutative diagram of modules with exact rows. Then β *can be lifted to a homomorphism* $C_1 \longrightarrow B$ *if and only if* α *can be extended to a map* $B_1 \longrightarrow A$*, that is, there exists* $\tilde{\beta} : C_1 \longrightarrow B$ *such that* $g\tilde{\beta} = \beta$ *if and only if there exists* $\tilde{\alpha} : B_1 \longrightarrow A$ *such that* $\tilde{\alpha} f_1 = \alpha$*:*

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] & B \arrow[r, "g"] & C \arrow[r] & 0 \\ 0 \arrow[r] & A_1 \arrow[r, "f_1"'] \arrow[u, "\alpha"] & B_1 \arrow[r] \arrow[u] \arrow[ul, dotted, "\tilde{\alpha}"'] & C_1 \arrow[r] \arrow[u, "\beta"'] \arrow[ul, dotted, "\tilde{\beta}"] & 0 \end{tikzcd}$$

The following result has been given in Crivei (1998); we give its elementary proof here for completeness:

Proposition 4.2.3. *(Crivei, 1998, Theorem 2) For a module M, the following are equivalent:*

(i) M is max-injective;

(ii) $\mathrm{Ext}^1_R(S,M) = 0$ *for every simple module S;*

(iii) $\mathrm{Ext}^1_R(S,M) = 0$ *for every semisimple module S;*

(iv) M is injective with respect to every short exact sequence ending with a simple module, that is,

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "\alpha"'] & B \arrow[r, "g"] \arrow[dl, dashed, "\tilde{\alpha}"] & S \arrow[r] & 0 \\ & M & & & \end{tikzcd}$$

(v) every simple module S is projective with respect to the following short exact sequence:

$$0 \longrightarrow M \xrightarrow{i} \mathrm{E}(M) \xrightarrow{\sigma} \mathrm{E}(M)/M \longrightarrow 0, \qquad (4.2.1)$$

where i is the inclusion monomorphism and σ *is the natural epimorphism.*

Proof. $(i) \Rightarrow (ii)$: Let

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r, "f"] & B \arrow[r, "g"] & S \arrow[r] & 0 \end{tikzcd}$$

be a short exact sequence with S simple. We can assume that $M \subseteq B$ and f is the inclusion monomorphism. We show that this sequence splits. Since R projective, we have the following commutative diagram with exact rows for $S \cong R/P$, where P is a maximal left ideal of R:

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, hook, "f"] & B \arrow[r, "g"] & S \arrow[r] & 0 \\
0 \arrow[r] & P \arrow[r, hook, "i"'] \arrow[u, dashed, "\alpha"] & R \arrow[r, "\sigma"'] \arrow[u, dashed, "h"] \arrow[ul, dotted, "\widetilde{\beta}"] & S \arrow[r] \arrow[u, "1_S"'] \arrow[ul, dotted, "\widetilde{\alpha}"] & 0
\end{tikzcd}$$

Here $h : R \longrightarrow B$ exists since R is projective (i.e. $gh = 1_S\sigma$), and $\alpha = h|_P$. By hypothesis, since M is max-injective, there is a homomorphism $\widetilde{\beta} : R \longrightarrow M$ such that $\widetilde{\beta}i = \alpha$, and so by Lemma 4.2.2, there is also a homomorphism $\widetilde{\alpha} : S \longrightarrow B$ such that $g\widetilde{\alpha} = 1_S$, that is, the first row splits. Hence $\mathrm{Ext}^1_R(S, M) = 0$.

$(ii) \Rightarrow (iv)$: Let

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] & B \arrow[r] & S \arrow[r] & 0 \end{tikzcd}$$

be a short exact sequence with S simple. For every homomorphism $f : A \longrightarrow M$, we obtain the following pushout diagram:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "f"'] & B \arrow[r] \arrow[d, dashed] \arrow[dl, dotted, "\beta"] & S \arrow[r] \arrow[d, dashed, "1_S"] \arrow[dl, dotted, "\alpha"] & 0 \\
0 \arrow[r, dashed] & M \arrow[r, dashed] & B' \arrow[r, dashed, "g"'] & S \arrow[r, dashed] & 0
\end{tikzcd}$$

By hypothesis, the second row splits, and so there is a homomorphism $\alpha : S \longrightarrow B$ such that $g\alpha = 1_S$. Thus by Lemma 4.2.2, f can also be extended to $\beta : B \longrightarrow M$, that is, $\beta|_A = f$. Hence M is injective with respect to the first row as required.

$(iv) \Rightarrow (i)$: Let $f : P \longrightarrow M$ be a homomorphism, where P is a maximal ideal of R.

We have the following diagram with exact row:

$$\begin{tikzcd} 0 \arrow[r] & P \arrow[r, "i"] \arrow[d, "f"'] & R \arrow[r, "g"] \arrow[dl, dashed, "g"] & R/P \arrow[r] & 0 \\ & M & & & \end{tikzcd}$$

Since R/P is simple, f can be extended to g by hypothesis. Hence M is max-injective.

$(ii) \Rightarrow (iii)$: Let S be a semisimple module. Then $S = \bigoplus_{i \in I} S_i$ for simple submodules S_i of S. Thus,

$$\mathrm{Ext}^1_R(S, M) = \mathrm{Ext}^1_R\left(\bigoplus_{i \in I} S_i, M\right) \cong \prod_{i \in I} \mathrm{Ext}^1_R(S_i, M) = 0,$$

where the isomorphism follows by, for example, Rotman (2009, Proposition 7.21) and the last equality by hypothesis.

$(iii) \Rightarrow (ii)$: Clear since a simple module is semisimple.

$(ii) \Leftrightarrow (v)$: For every simple module S, $\mathrm{Ext}^1_R(S, M) = 0$ if and only if the induced sequence

$$0 \longrightarrow \mathrm{Hom}_R(S, M) \longrightarrow \mathrm{Hom}_R(S, \mathrm{E}(M)) \longrightarrow \mathrm{Hom}_R(S, \mathrm{E}(M)/M) \longrightarrow 0$$

is exact by the long exact Hom $-$ Ext sequence, because $\mathrm{E}(M)$ is an injective module. This means that S is projective with respect to the short exact sequence (4.2.1). $\square$

The following result has been observed by Engin Büyükaşık, for the proper class projectively generated by simple modules. It holds in general for every projectively generated proper class as has been shown in Sklyarenko (1978, Proposition9.5).

Proposition 4.2.4. *A module M is $\mathcal{N}$eat-coinjective if and only if* $\mathrm{Ext}^1_R(S, M) = 0$ *for every simple modules S.*

Proof. $(\Rightarrow)$ Let

$$\mathbb{E} \quad : 0 \longrightarrow M \longrightarrow B \longrightarrow S \longrightarrow 0$$

be a short exact sequence from $\mathrm{Ext}^1_R(S,M)$, where S is simple. Since $\mathbb{E}$ starts with M and M is $\mathcal{N}eat$-coinjective by hypothesis, $\mathbb{E} \in \mathcal{N}eat$. Now since $\mathcal{N}eat$ is projectively generated by simples, $\mathbb{E}$ splits.

($\Leftarrow$) Let

$$\mathbb{E} \quad : 0 \longrightarrow M \longrightarrow B \longrightarrow C \longrightarrow 0$$

be a short exact sequence starting with a module M, and $f : S \longrightarrow C$ be homomorphism where S is a simple module. Then we have the following pullback diagram:

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] & B \arrow[r, "h"] & C \arrow[r] & 0 \\
0 \arrow[r, dashed] & M \arrow[u, dashed, "1_M"] \arrow[r, dashed, "g"'] & B' \arrow[u, dashed] \arrow[ul, dotted, "\beta"] \arrow[r, dashed] & S \arrow[u, "f"'] \arrow[ul, dotted, "\alpha"] \arrow[r, dashed] & 0
\end{tikzcd}$$

By hypothesis, the second row splits, that is, there is a homomorphism $\beta : B' \longrightarrow M$ (as M a direct summand) such that $\beta g = 1_M$. Thus by Lemma 4.2.2, there is also a homomorphism $\alpha : S \longrightarrow B$ such that $h\alpha = f$. This means S is projective w.r.t the first row. Thus $\mathbb{E} \in \mathcal{N}eat$, and so M is $\mathcal{N}eat$-coinjective. $\square$

Thus max-injective modules are nothing but just $\mathcal{N}eat$-coinjective modules (by Propositions 4.2.3 and 4.2.4). Note that, a module M is $\mathcal{N}eat$-coinjective if and only if M is neat in every module containing M. Also note that a module M is $\mathcal{N}eat$-coinjective if and only if M is neat in its injective envelope $\mathrm{E}(M)$.

By Proposition 4.2.3, a module M is max-injective if and only if M is injective with respect to every short exact sequence ending with a simple module. Instead of simple modules, semi-artinian modules can be taken. To prove (i) implies (ii) in the following theorem, Crivei have used Loewy series of C; a homomorphism $\psi : A \longrightarrow M$ was extended to B by transfinite induction since $B/A \cong C$ is a Loewy module (by Proposition 2.2.4, as C is semi-artinian).

Theorem 4.2.5. *(Crivei, 1998, Theorem 6) The following are equivalent for a module M:*

(i) M is max-injective;

(ii) M is injective with respect to every short exact sequence of modules

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

where C is a semi-artinian module.

Remark 4.2.6. The following connection between max-injective modules and torsion theories was given in Crivei (2000). Let $\tau_D = (\mathcal{T}, \mathcal{F})$ be the Dickson torsion theory for R-$\mathcal{M}od$; the torsion class $\mathcal{T}$ consists of semi-artinian modules and the torsion free class $\mathcal{F}$ consists of all modules with zero socle (see Example 2.4.11). The corresponding Gabriel filter $F(R)$ consists of all left ideals of R such that R/I is a left semi-artinian module (see Definition 2.4.7). A module M is τ_D-*injective* if any homomorphism from any left ideal $I \in F(R)$ to M can be extended to R or equivalently, if M is injective with respect to every short exact sequence of modules $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$, where C is τ_D-torsion. Thus, by Theorem 4.2.5, a module M is max-injective if and only if it is τ_D-injective.

We know by Proposition 4.1.1 that if R is a left C-ring, then $\mathcal{C}ompl = \mathcal{N}eat$, and so $\mathcal{N}eat$-coinjectives (i.e. max-injectives) are $\mathcal{C}ompl$-coinjectives which are known to be just injectives. So, our goal on this section is to prove the converse.

Lemma 4.2.7. *Let A be a submodule of a module B.*

(i) If $\mathrm{Soc}(B/A) = 0$*, then* B/A *is* $\mathcal{N}eat$*-coprojective and so A is neat in B.*

(ii) Equivalently, if A is not neat in B, then $\mathrm{Soc}(B/A) \neq 0$.

Proof. Clear since $\mathcal{N}eat$ is the proper class projectively generated by all simple modules and there is no nonzero homomorphism from a simple module to B/A when $\mathrm{Soc}(B/A) = 0$. $\square$

The following result is needed for the proof of Lemma 4.2.9:

Proposition 4.2.8. *(Mermut, 2004, Corollary 3.2.5) For a short exact sequence*

$$\mathbb{E}: \qquad 0 \longrightarrow A \xrightarrow{i_A} B \xrightarrow{g} C \longrightarrow 0$$

of modules, where A is a submodule of B and i_A is the inclusion map, the following are equivalent:

(i) $\mathbb{E} \in \mathcal{N}eat = \pi^{-1}(\{R/P | P$ *is a maximal left ideal of* $R\})$*;*

(ii) For every maximal left ideal P of R, and for every $b \in B$, if $Pb \subseteq A$, then there exists $a \in A$ such that $P(b-a) = 0$.

Lemma 4.2.9. *Let A be a submodule of a module B. If A is* essential *and* neat *in B, then* $\mathrm{Soc}(B/A) = 0$.

Proof. Suppose on the contrary that $\mathrm{Soc}(B/A) \neq 0$. Then there exists a simple subdmodule $S = R(b+A) \cong R/P$ of B/A where $b \in B \setminus A$ and P is a maximal left ideal of R such that $P = \{r \in R \mid rb \in A\}$. So $Pb \subseteq A$, then by the characterization of neat submodules in Proposition 4.2.8, there exists $a \in A$ such that $P(b-a) = 0$. Since $b \notin A$, $b-a \notin A$ also. But then for the element $b-a \in B \setminus A$, we obtain a contradiction with A being essential in B, because

$$0 = P(b-a) = [R(b-a)] \cap A \neq 0.$$

Note that the first equality follows since $\{r \in R \mid r(b-a) = 0\} = \{r \in R \mid rb \in A\} = P$ since $a \in A$. □

So, by Lemmas 4.2.7 and 4.2.9, we obtain:

Corollary 4.2.10. *For an* essential *submodule A of a module B, we have A is neat in B if and only if* $\mathrm{Soc}(B/A) = 0$.

Corollary 4.2.11. *Let A be a submodule of a module B. Let K be a complement of A in B and let A' be a complement of K in B such that A' contains A. Then we already know that A is essential in A' and A' is closed (and so neat) in B. In this case, the following are equivalent:*

(i) *A is neat in B;*

(ii) *A is neat in A';*

(iii) $\mathrm{Soc}(A'/A) = 0$.

Theorem 4.2.12. *If all $\mathcal{N}$eat-coinjective modules are injective, then for every module A that is* not injective*, $E(A)/A$ is semi-artinian, and so* $\mathrm{Soc}(E(A)/A)$ *is essential in $E(A)/A$.*

Proof. Let $A \subseteq U \subsetneqq E(A)$. In this case, the injective envelope of the submodule U is also $E(A)$. If U is neat in $E(U) = E(A)$, then U will be $\mathcal{N}$*eat*-coinjective. But then by our hypothesis U will be injective, and so we will have $U = E(U) = E(A)$, contradicting $U \neq E(A)$. So we must have that U is *not* neat in $E(A)$. But then by Corollary 4.2.10, $\mathrm{Soc}(E(A)/U) \neq 0$. Hence $\mathrm{Soc}[(E(A)/A)/(U/A)] \neq 0$ since $(E(A)/A)/(U/A) \cong E(A)/U$. This proves that every nonzero homomorphic image of $E(A)/A$ has a nonzero socle. Thus $E(A)/A$ is semi-artinian and so every nonzero homomorphic image of $E(A)/A$ has essential socle. In particular, $\mathrm{Soc}(E(A)/A)$ is essential in $E(A)/A$. □

Corollary 4.2.13. *If all $\mathcal{N}$eat-coinjective modules are injective, then for every module A that is* not injective*, if $A \subsetneqq B \subseteq E(A)$, then* $\mathrm{Soc}(B/A) \neq 0$.

Proof. By Theorem 4.2.12,

$$\mathrm{Soc}(B/A) = (B/A) \cap \mathrm{Soc}(E(A)/A) \neq 0$$

since $\mathrm{Soc}(E(A)/A)$ is essential in $E(A)/A$ and $B/A \neq 0$ as $B \neq A$ by hypothesis. □

Theorem 4.2.14. *If all $\mathcal{N}$eat-coinjective modules are injective, then R is a left C-ring, that is, for every module B and for every essential proper submodule A of B,* $\mathrm{Soc}(B/A) \neq 0$.

Proof. Let A be an essential proper submodule of a module B. Then we can assume that $A \subsetneq B \subseteq E(A)$. In this case, $A \neq E(A)$, that is, A is not injective and Corollary 4.2.13 gives $\mathrm{Soc}(B/A) \neq 0$. □

Whitehead test modules for projectivity and injectivity have been studied in Trlifaj (1996): A module N is said to be a *Whitehead test module for projectivity* (shortly a *p-test module*) if for every module M, $\mathrm{Ext}^1_R(M,N) = 0$ implies M is projective. Dually, *i*-test modules are defined; a module N is said to be a *Whitehead test module for injectivity* (shortly an *i-test module*) if for every module M, $\mathrm{Ext}^1_R(N,M) = 0$ implies M is injective.

Remark 4.2.15. Note that for $R = \mathbb{Z}$, the question "Is $\mathbb{Z}$ a *p*-test $\mathbb{Z}$-module?" or equivalently, the question "Is there a *Whitehead group* G (that is, $\mathrm{Ext}(G,\mathbb{Z}) = 0$) which is not free?" is the well-known Whitehead problem. Eklof and Shelah have given a full answer to this problem (Eklof & Shelah, 1994): It was proved that for any uncountable cardinal λ, if there is a λ-*free* Whitehead group of cardinality λ (that is, every subgroup of cardinality $< \lambda$ is free) which is not free, then there are many Whitehead groups of cardinality λ which are not free.

J. Trlifaj showed the existence of *i*-test modules for an arbitrary ring, and *p*-test modules for a left perfect ring.

Proposition 4.2.16. *(Trlifaj, 1996, Proposition 1.2) Let R be a ring. Let $\mathcal{E}$ be the set of all proper essential left ideals of R. Put $M = \bigoplus_{I \in \mathcal{E}} R/I$. Then M is an i-test module.*

Proposition 4.2.17. *(Trlifaj, 1996, Proposition 1.4) Let R be a left perfect ring. Denote by $\mathcal{M}$ the class of all maximal left ideals of R. Put $N = \bigoplus_{I \in \mathcal{M}} R/I$. Then N is a p-test module.*

We collect some characterizations of C-rings in the following theorem:

Theorem 4.2.18. *For a ring R, the following are equivalent:*

(i) *R is a left C-ring;*

(ii) $\mathcal{C}ompl = \mathcal{N}eat$;

(iii) *All max-injective (i.e. $\mathcal{N}eat$-coinjective) R-modules are injective;*

(iv) $\mathcal{S}^{\perp} = Inj$, *where $\mathcal{S}$ and Inj denote the classes of all (semi)simple R-modules and injective R-modules respectively;*

(v) $\bigoplus\limits_{\substack{P \subseteq R \\ max.}} R/P$ *is an i-test module, where the direct sum is over all maximal left ideals of R.*

Proof. $(i) \Leftrightarrow (ii)$ is by Proposition 4.1.1, and $(iii) \Leftrightarrow (iv)$ follows by Proposition 4.2.3. $(i) \Rightarrow (iii)$ follows by arguments given just before Lemma 4.2.7. $(iii) \Rightarrow (i)$ is Theorem 4.2.14.

By Proposition 4.2.3, $\mathrm{Ext}^1_R(S,M) = 0$ for every simple module S if and only if $\mathrm{Ext}^1_R(T,M) = 0$ for every semisimple module T. So, $\{\text{all simple modules}\}^{\perp} = \{\text{all semisimple modules}\}^{\perp}$. $(iv) \Rightarrow (v)$: Suppose that $\mathrm{Ext}^1_R\left(\bigoplus\limits_{\substack{P \subseteq R \\ max.}} R/P, M\right) = 0$ for a module M. Then we have by, for example, Rotman (2009, Proposition 7.21) that

$$\prod_{\substack{P \subseteq R \\ max.}} \mathrm{Ext}^1_R(R/P, M) \cong \mathrm{Ext}^1_R\left(\bigoplus_{\substack{P \subseteq R \\ max.}} R/P, M\right) = 0,$$

and so $\mathrm{Ext}^1_R(R/P, M) = 0$ for every left maximal ideal P of R, that is, $M \in \mathcal{S}^{\perp}$ where $\mathcal{S}$ is the class of all simple R-modules. Thus M is injective by hypothesis (iv).

$(v) \Rightarrow (iv)$: $Inj \subseteq \mathcal{S}^{\perp}$ is clear since for an injective module M, $\mathrm{Ext}^1_R(S, M) = 0$ for every module S. Conversely, let $M \in \mathcal{S}^{\perp}$, that is, $\mathrm{Ext}^1_R(S, M) = 0$ for every simple module S. Since for every simple module S, $S \cong R/P$ for a maximal left ideal P of R, we obtain, by Rotman (2009, Proposition 7.21), that

$$\mathrm{Ext}^1_R\left(\bigoplus_{\substack{P \subseteq R \\ max.}} R/P, M\right) \cong \prod_{\substack{P \subseteq R \\ max.}} \mathrm{Ext}^1_R(R/P, M) = 0.$$

Thus, by hypothesis (v), M is an injective module. □

Proposition 4.2.19. *Any direct product of max-injective modules is max-injective.*

Proof. Since $\mathcal{N}eat$ is a projectively generated proper class, it is $\prod$-closed by Proposition 2.3.8, and so any product of $\mathcal{N}eat$-coinjective (i.e. max-injective) modules is also $\mathcal{N}eat$-coinjective by Proposition 2.3.9. This result can also be proved using the standard argument to prove that direct product of injectives is injective.
Second proof: If $\{M_i\}_{i\in I}$ is a family of max-injective modules, then

$$\mathrm{Ext}^1_R\left(S, \prod_{i\in I} M_i\right) \cong \prod_{i\in I} \mathrm{Ext}^1_R(S, M_i) = 0$$

for every simple module S, that is, $\prod_{i\in I} M_i$ is max-injective. Here the isomorphism follows by, for example, Rotman (2009, Proposition 7.22) and the last equality by Proposition 4.2.3. □

Note that if R is a left C-ring, then every direct product of closed (=complement) submodules is closed (see Proposition 4.3.6). But, we do not know if the converse holds.

4.3 E-neat Homomorphisms

In this section, the definition due to Enochs (1971) and Bowe (1972) of neat homomorphisms (we call E-neat homomorphisms) and some properties of them will be given.

Definition 4.3.1. A homomorphism $f : M \longrightarrow N$ is called *E-neat* if given any *proper* submodule H of a module G and any homomorphism $\sigma : H \longrightarrow M$, the homomorphism σ has a *proper* extension in G whenever $f\sigma$ has a *proper* extension in G, that is, a commutative diagram

$$\begin{tikzcd} H \arrow[r] \arrow[d, "\sigma"'] & G' \arrow[r] \arrow[dr, dashed, "\tau"] & G \\ M \arrow[rr, "f"'] & & N \end{tikzcd} \tag{4.3.1}$$

with $H \subsetneqq G' \subseteq G$, always guarantees the existence of a commutative diagram

$$\begin{tikzcd} H \arrow[r] \arrow[d, "\sigma"'] & G'' \arrow[r] \arrow[dl, dashed] & G \\ M \arrow[rr, "f"] & & N \end{tikzcd} \tag{4.3.2}$$

with $H \subsetneqq G'' \subseteq G$.

A submodule T of N is called E*-neat* if the canonical monomorphism $i : T \longrightarrow N$ is E-neat (see Bowe (1972)).

Example 4.3.2. (i) A is a neat subgroup of an abelian group B (i.e., $pA = A \cap pB$ for every prime number p) if and only if the monomorphism $i : A \hookrightarrow B$ is E-neat (Enochs & Jenda, 2000, Example 4.3.8).

(ii) For a commutative domain, if a homomorphism $\varphi : T \longrightarrow M$ is torsion free cover, then φ is E-neat (Enochs & Jenda, 2000, Proposition 4.3.9).

The following equivalent conditions for E-neat homomorphisms (especially (iv)) are very useful:

Theorem 4.3.3. *(by Bowe (1972, Theorem 1.2)) The following are equivalent for a homomorphism* $f : M \longrightarrow N$*:*

(i) *f is an E-neat;*

(ii) *In Definition 4.3.1, it suffices to take $G = R$ and H a left ideal of R;*

(iii) *In Definition 4.3.1, it suffices to take σ a monomorphism and G as an essential extension of H;*

(iv) *There are no proper extensions of f in the injective envelope $\mathrm{E}(M)$ of M.*

We shall frequently use the last equivalent condition in the previous theorem for E-neat homomorphisms. Note that this is a natural concept; by Zorn's lemma, for a given homomorphism $f : A \longrightarrow B$, there exists a maximal extension $\widetilde{f} : \widetilde{A} \longrightarrow B$ in the injective envelope $\mathrm{E}(A)$ of A for some $\widetilde{A}$ such that $A \subseteq \widetilde{A} \subseteq \mathrm{E}(A)$, and that $\widetilde{f}$ is an E-neat homomorphism:

$$\begin{tikzcd}
\mathrm{E}(A) & \\
\widetilde{A} \arrow[u, hook] \arrow[dr, "\widetilde{f}"] & \\
A \arrow[u, hook] \arrow[r, "f"'] & B
\end{tikzcd}$$

and $\widetilde{f}$ is E-neat. The following result can be found in Bowe (1972, §1), without proof, as an example of E-neat homomorphisms; the E-neat monomorphisms are just $\mathcal{C}ompl$-monomorphisms:

Lemma 4.3.4. *For a submodule M of a module N, the inclusion monomorphism $i : M \longrightarrow N$ is E-neat if and only if M has no proper essential extensions in N, that is, M is a closed submodule (*=complement submodule*) of N.*

Proof. ($\Rightarrow$) Suppose that the monomorphism $i : M \hookrightarrow N$ is E-neat and that M is not closed in N, that is, M has a proper essential extension N' in N (i.e. $M \trianglelefteq N' \subseteq N$ and

$M \neq N'$). Now embed N' into the injective envelope $\mathrm{E}(M)$ of M (since $M \trianglelefteq N'$):

$$\begin{tikzcd} M \arrow[r, hook, "\trianglelefteq"] \arrow[d] & N' \arrow[dl, dashed, "\alpha"] \\ \mathrm{E}(M) & \end{tikzcd}$$

We can assume that $N' \subseteq \mathrm{E}(M)$ and α is the inclusion monomorphism. So we have the commutative diagram:

$$\begin{tikzcd} M \arrow[r, hook, "\neq"'] \arrow[d, "1_M"'] & N' \arrow[r, hook] \arrow[d, dashed, hook] & \mathrm{E}(M) \\ M \arrow[r, hook, "i"] & N & \end{tikzcd}$$

Since $i : M \longrightarrow N$ is E-neat, there exists a commutative diagram:

$$\begin{tikzcd} M \arrow[r, hook, "\neq"'] \arrow[d, "1_M"'] & N'' \arrow[r, hook] \arrow[dl, dashed, "\phi"] & \mathrm{E}(M) \\ M \arrow[r, hook, "i"'] & N & \end{tikzcd}$$

with $M \subsetneqq N' \subseteq \mathrm{E}(M)$, that is, $\phi|_M = 1_M$. Let $\psi = i\phi : N'' \longrightarrow N$. Thus ψ is a proper extension of i in the injective envelope of M. This contradicts with the fact that i is an E-neat monomorphism (see Theorem 4.3.3-(iv)). Hence M is closed in N.

($\Leftarrow$) Conversely, suppose that M is closed in N and that $i : M \longrightarrow N$ is not E-neat. Then by Theorem 4.3.3, there is a proper extension $\sigma : M' \longrightarrow N$ for some $M' \supsetneqq M$ of $i : M \longrightarrow N$ in the injective envelope $\mathrm{E}(M)$ of M:

$$\begin{tikzcd} \mathrm{E}(M) & \\ M' \arrow[u, hook] \arrow[dr, dashed, "\sigma"] & \\ M \arrow[u, hook, "j", "\neq"'] \arrow[r, hook, "i"] & N \end{tikzcd}$$

Since $M \trianglelefteq \mathrm{E}(M)$, we have $M \trianglelefteq M'$, and so σ must be a monomorphism. Because, if $\operatorname{Ker}\sigma \neq 0$, then $M \cap \operatorname{Ker}\sigma \neq 0$ since $M \trianglelefteq M'$. But $M \cap \operatorname{Ker}\sigma = \operatorname{Ker} i = 0$ since $\sigma j = i$.

Now identifying $\sigma(M')$ with M', we obtain $M \subseteq M' = \sigma(M') \subseteq N$. Since M is closed in N and $M \trianglelefteq M'$, we must have $M = M'$. But this contradicts the fact that $M' \supsetneqq M$. Hence $i : M \longrightarrow N$ is E-neat. $\square$

Remark 4.3.5. Let R be a left C-ring. Then a submodule $A \subseteq B$ is neat in B if and only if it is closed in B, and so we infer from the previous result that A is neat in B if and only if it is E-neat.

Since a monomorphism is E-neat if and only if it is a closed monomorphism (=$\mathcal{C}ompl$-monomorphism), we have:

Proposition 4.3.6. *Let R be a left C-ring. If $\{f_i : E_i \longrightarrow F_i\}_{i\in A}$ is a family of $\mathcal{C}ompl$-monomorphisms (i.e. closed monomorphisms), then $f = \prod_{i\in A} f_i : \prod_{i\in A} E_i \longrightarrow \prod_{i\in A} F_i$ is also $\mathcal{C}ompl$-monomorphism. So, if every f_i is an inclusion monomorphism and E_i is a closed submodule of F_i for every $i \in A$, then $\prod_{i\in A} E_i$ is a closed submodule of $\prod_{i\in A} F_i$.*

Proof. Since over a left C-ring, $\mathcal{N}eat = \mathcal{C}ompl$ (by Proposition 4.1.1) and $\mathcal{N}eat$ is a projectively generated proper class, it is $\prod$-closed by Proposition 2.3.8. So every direct product of closed (=complement) submodules is closed. $\square$

The following theorem give a characterization for the direct product of E-neat homomorphisms for the Singular Torsion Theory (=Goldie torsion theory) for R-$\mathcal{M}od$ with R nonsingular; see Example 2.4.12):

Theorem 4.3.7. *Bowe (1972, Theorem 2.4) Let $(\mathcal{T}_G, \mathcal{F}_G)$ be the Singular Theory for R-$\mathcal{M}od$ with $Z(_RR) = 0$ (i.e. $R \in \mathcal{F}_G$). Then the following are equivalent:*

(i) For all nonzero proper essential left ideals I of R, R/I contains a simple submodule (i.e. $\mathrm{Soc}(R/I) \neq 0$) [that is, R is a left C-ring];

(ii) every nonzero module $E \in \mathcal{T}_G$ contains a simple submodule;

(iii) $f = \prod_{i\in A} f_i : \prod_{i\in A} E_i \longrightarrow \prod_{i\in A} F_i$ *is an* E*-neat homomorphism for every family* $\{f_i : E_i \longrightarrow F_i\}_{i\in A}$ *of* E*-neat homomorphisms of modules.*

Remark 4.3.8. (see Bowe (1972, Theorem 2.5)) Let $(\mathcal{T}_D, \mathcal{F}_D)$ be the Dickson Torsion Theory for R-$\mathcal{M}od$. Then if $Z({}_RR) = 0$, $\mathrm{Soc}({}_RR) = 0$ and the equivalent conditions of Theorem 4.3.7 are satisfied, then the Singular Torsion Theory and the Dickson Torsion Theory coincide. Indeed,

(i) If R is a left C-ring such that $Z({}_RR) = 0$, then $\mathcal{T}_G \subseteq \mathcal{T}_D$.

(ii) If $\mathrm{Soc}_R R = 0$, then every simple R-module is in $\mathcal{T}_G$.

(iii) If R is a left C-ring such that $Z({}_RR) = 0$ and $\mathrm{Soc}_R R = 0$, then $\mathcal{T}_G = \mathcal{T}_D$.

The following result has been given in Bowe (1972) without proof:

Proposition 4.3.9. *(Bowe, 1972, Remark 1 after Theorem 1.2)*

(i) If $f : M \longrightarrow N$ *and* $g : N \longrightarrow G$ *are* E*-neat homomorphisms, then* gf *is a* E*-neat homomorphism.*

(ii) If $f : M \longrightarrow N$ *is a* E*-neat homomorphism and* N *is injective, then* M *is also injective.*

Proof. (i) Suppose we have the following commutative diagram for modules $H \subsetneqq K' \subseteq K$:

$$\begin{tikzcd} H \arrow[rr, hook, "\neq"'] \arrow[d, "\sigma"'] & & K' \arrow[r] \arrow[d, "\tau"] & K \\ M \arrow[r, "f"'] & N \arrow[r, "g"'] & G & \end{tikzcd}$$

Since g is E-neat and $gf\sigma$ has a proper extension in K, we obtain that $f\sigma$ has a proper extension, that is, there is a module $K'' \supsetneqq H$ and a homomorphism $\phi : K'' \longrightarrow N$ such that $\phi|_H = f\sigma$. Now since f is E-neat, there is a module $K''' \supsetneqq H$ and a homomorphism $\varphi : K''' \longrightarrow M$ such that $\varphi|_H = \sigma$. This means σ has a proper extension in K, that is, gf is E-neat.

(ii) Since N is injective, we have the following commutative diagram:

$$\begin{tikzcd} M \arrow[r, hook] \arrow[d, "f"'] & \mathrm{E}(M) \arrow[dl, dashed, "\widetilde{f}"] \\ N & \end{tikzcd}$$

Now suppose that M is not injective, then $M \neq \mathrm{E}(M)$. But, this means that f has a proper extension $\widetilde{f}$ in the injective envelope $\mathrm{E}(M)$ of M. This contradicts with f being E-neat. Hence M is injective. □

Recall that a ring R is said to be *left hereditary* if every left ideal of R is projective.

The following theorem characterizes left hereditary rings in terms of E-neat homomorphisms:

Theorem 4.3.10. *(Bowe, 1972, Theorem 2.1) For a ring R, the following are equivalent:*

(i) R is left hereditary.

(ii) Quotients of injective R-modules are injective.

(iii) If f is a E-neat homomorphism and $f = ip$ is the natural decomposition of f with p an epimorphism and i a monomorphism, then p and i are E-neat.

(iv) If f is a E-neat homomorphism and $f = hg$, where h and g are epimorphisms, then h and g are E-neat.

Proposition 4.3.11. *If $f : E \longrightarrow F$ and $g : F \longrightarrow G$ are homomorphisms and gf is E-neat, then f is also E-neat.*

Proof. Suppose we have the following diagram for modules $H \subsetneqq K' \subseteq K$:

$$\begin{tikzcd} H \arrow[r, hook, "\neq"'] \arrow[d, "\sigma"'] & K' \arrow[r, hook] \arrow[d, "\tau"] \arrow[dr, dashed, "g\tau"] & K \\ E \arrow[r, "f"'] & F \arrow[r, "g"'] & G \end{tikzcd}$$

We must show that σ has a proper extension in K. Since $gf\sigma = (g\tau)|_H$ and gf is E-neat, there is a module $K'' \supsetneqq H$ and a homomorphism $\phi : K'' \longrightarrow E$ such that $\phi|_H = \sigma$, that is, ϕ is a proper extension of σ in K:

$$\begin{tikzcd} H \arrow[r, hook, "\neq"'] \arrow[d, "\sigma"'] & K'' \arrow[r, hook] \arrow[dl, dashed, "\phi"] & K\,. \\ E \arrow[r, "f"'] & F & \end{tikzcd}$$

$\square$

4.4 Z-neat Homomorphisms

In this section, we firstly give an equivalent definition of E-neat homomorphisms considered by Zöschinger. Also, by considering the first two equivalent conditions of Theorem 4.4.2 given by Zöschinger for abelian groups, we define Z-neat homomorphisms of R-modules using the second condition (ii). We show that E-neat homomorphisms and Z-neat homomorphisms coincide over Dedekind domains as in the case of abelian groups given by Zöschinger.

Zöschinger (1978) gives an equivalent definition for E-neat homomorphisms as follows (without proof):

Proposition 4.4.1. *(Zöschinger, 1978, p. 307) $f : A \longrightarrow A'$ is a E-neat homomorphism if and only if for every decomposition $f = \beta\alpha$ where α is an essential monomorphism, α is an isomorphism.*

Proof. $(\Rightarrow)$ Let $f = \beta\alpha$ where $\alpha : A \longrightarrow B$ is an essential monomorphism and $\beta : B \longrightarrow A'$ is a homomorphism. Since α is a monomorphism, identifying $\alpha(A) = A$, we obtain $A = \alpha(A) \trianglelefteq B$. So there is a monomorphism $g : B \longrightarrow E(A)$ such that $g\alpha = i$ where

$i : A \longrightarrow E(A)$ is inclusion:

$$\begin{tikzcd} A \arrow[r, hook, "\alpha", "\trianglelefteq"'] \arrow[d, "i"'] & B \arrow[dl, dashed, "g"] \\ E(A) & \end{tikzcd}$$

So we can use that $B \subseteq E(A)$ and g is the inclusion monomorphism. Thus f has an extension β in the injective envelope $E(A)$ of A:

$$\begin{tikzcd} E(A) & \\ B \arrow[u, hook, "g"] \arrow[dr, dashed, "\beta"] & \\ A \arrow[u, hook, "\alpha"] \arrow[r, "f"'] & A' \end{tikzcd}$$

Since f is E-neat, this extension cannot be proper, that is, α must be an isomorphism.

($\Leftarrow$) Conversely, to show that f is E-neat, suppose that there is an extension β of f in $E(A)$, that is, suppose we have the following commutative diagram where $A \subseteq B \subseteq E(A)$ and α is the inclusion monomorphism:

$$\begin{tikzcd} E(A) & \\ B \arrow[u, hook] \arrow[dr, dashed, "\beta"] & \\ A \arrow[u, hook, "\alpha"] \arrow[r, "f"'] & A' \end{tikzcd}$$

Since $A \trianglelefteq E(A)$, we obtain $A \trianglelefteq B$, and so by hypothesis $f = \beta\alpha$ implies that α is an isomorphism. Hence f has no proper extension in $E(A)$ which implies f is E-neat. $\square$

Zöschinger has given the following theorem which gives a characterization of E-neat homomorphisms for abelian groups in Zöschinger (1978) without proof. He has explained us how to prove. In the following proof for abelian groups, we should also use the characterization of E-neat epimorphisms of modules over any ring given in Corollary 4.4.5: For a submodule K of a module A, the natural epimorphism $f : A \longrightarrow A/K$ is E-neat if and only if $A/K \trianglelefteq E(A)/K$. Moreover, we shall use the following result for abelian groups in the proof of the following theore: For abelian groups $A \subseteq B$, A is a neat subgroup of B if and only if A is a closed subgroup of B if and only if the inclusion monomorphism $i : A \hookrightarrow B$ is E-neat.

Theorem 4.4.2. *(Zöschinger, 1978, Satz* 2.3**) Let A and* A' *be abelian groups. For a homomorphism* $f : A \longrightarrow A'$ *the following are equivalent:*

(i) f *is* E*-neat.*

(ii) $\operatorname{Im} f$ *is closed in* A' *and* $\operatorname{Ker} f \subseteq \operatorname{Rad} A$.

(iii) $f^{-1}(pA') = pA$ *for all prime numbers* p.

(iv) If the following diagram is a pushout diagram of abelian groups and α *is an essential monomorphism, then* α' *is also an essential monomorphism:*

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & B \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ A' & \xrightarrow[\alpha']{} & B' \end{array}$$

Proof. Let $f = ip : A \longrightarrow A/K \cong \operatorname{Im} f \hookrightarrow A'$ where $K = \operatorname{Ker} f$, $p : A \longrightarrow A/K$ is the natural epimorphism and $i : \operatorname{Im} f \hookrightarrow A'$ is the inclusion monomorphism.

$(i) \Rightarrow (ii)$: Since $R = \mathbb{Z}$ is hereditary, we obtain by Theorem 4.3.10 that i and p are E-neat. Since the monomorphism i is E-neat, $\operatorname{Im} i = \operatorname{Im} f$ is closed in A' by Lemma 4.3.4. Now we have by Corollary 4.4.5 that $A/K \trianglelefteq E(A)/K$, and so $\operatorname{Soc}(E(A)/K) \subseteq A/K$. To show that $K \subseteq \operatorname{Rad} A = \bigcap_{p \text{ prime}} pA$, we shall show that $K \subseteq pA$ for all prime numbers p. Let $x \in K$. Since $K \subseteq E(A)$ and $E(A)$ is divisible, for every prime number p, there exists $y \in E(A)$ such that $x = py$. So, $p(y+K) = x+K = 0$ in $E(A)/K$. Thus $y+K \in \operatorname{Soc}(E(A)/K) \subseteq A/K$, that is, $y \in A$, and so $x = py \in pA$.

$(ii) \Rightarrow (i)$: By Proposition 4.3.9, it suffices to show that the homomorphisms i and

p are E-neat. Since $\operatorname{Im} i = \operatorname{Im} f$ is closed in A', $i : \operatorname{Im} f \hookrightarrow A'$ is E-neat by Lemma 4.3.4. To show the the epimorphism $p : A \longrightarrow A/K$, where $K = \operatorname{Ker} f$, is E-neat we shall show that $A/K \trianglelefteq E(A)/K$ by Corollary 4.4.5. Let $0 \neq x + K \in E(A)/K$ with $x \in E(A)$. If $x \in A$, then we are done, so assume that $x \notin A$. Since $A \trianglelefteq E(A)$ there exists an integer n such that $0 \neq nx \in A$, and so $n(x+K) \in A/K$. If $n(x+K) \neq 0$ in $E(A)/K$, then we are done. Suppose that $n(x+K) = 0$ in $E(A)/K$. In this case, we can assume that n is the order of the element $x+K$ in the abelian group $E(A)/K$. Since $x \neq A$, we have $n > 1$. Firstly, if $n = p$ is a prime number, then $px \in K$. So, $px \in pA$ since $K \subseteq \operatorname{Rad} A = \bigcap_{q \text{ prime}} qA$ by hypothesis. Then $px = pa$ for some $a \in A$, and so $p(x-a) = 0$. Thus $x - a \in \operatorname{Soc}(E(A))$, and so $x - a \in A$ since $\operatorname{Soc}(E(A)) \subseteq A$ as $A \trianglelefteq E(A)$. This shows that $x \in A$, contradicting $x \notin A$. Secondly, suppose that n is not prime. Then there exists a prime number p and $m \in \mathbb{Z}^+$ such that $n = pm$ where $1 < m < n$. Let $y = mx$. Then $y \notin K$ since $m(x+K) \neq 0$ as n is the order of $x+K$. So, $0 \neq y + K \in E(A)/K$ with $y \in E(A)$. If $y \in A$, then $0 \neq m(x+K) \in A/K$ and we are done. Suppose $y \notin A$. Then $0 \neq y + K \in E(A)/K$ with $y \in E(A)$ and $py = pmx = nx \in K$. Now as in the first case in the previous arguments, we obtain that $y \in A$ which is a contradiction.

$(ii) \Rightarrow (iii)$: Since $\operatorname{Im} f = f(A)$ is closed in A', it is a neat subgroup of A', that is, $pf(A) = f(A) \cap pA'$ for every prime number p. Let p be a prime number. Then $f(pA) = pf(A) \subseteq pA'$, and so $pA \subseteq f^{-1}(pA')$. Conversely, take any element $x \in f^{-1}(pA')$. Then $f(x) \in pA'$, and since $x \in A$ already, we obtain $f(x) \in f(A) \cap pA' = pf(A)$. So, $f(x) = pf(a)$ for some $a \in A$ or equivalently, $f(x - pa) = 0$, that is,

$$x - pa \in \operatorname{Ker} f \subseteq \operatorname{Rad} A = \bigcap_{q \text{ prime}} qA$$

by hypothesis. Thus $x - pa \in pA$, and so $x = p(a + a')$ for some $a' \in A$. This implies $x \in pA$, and thus $pA = f^{-1}(pA')$.

$(iii) \Rightarrow (ii)$: We show that $\operatorname{Im} f = f(A)$ is a neat subgroup of A'. Let p be a prime number. Of course, $pf(A) \subseteq f(A) \cap pA'$. For the converse, let $f(a) = pa' \in f(A) \cap pA'$ for some $a \in A$ and $a' \in A'$. Then $a \in f^{-1}(pA') = pA$ by hypothesis. So $a = p\widetilde{a}$ for some $\widetilde{a} \in A$. Thus $f(a) = pf(\widetilde{a}) \in pf(A)$ as required. For every prime number p, $\operatorname{Ker} f = f^{-1}(0) \subseteq f^{-1}(pA') = pA$ by hypothesis. Thus $\operatorname{Ker} f \subseteq \bigcap_{p \text{ prime}} pA \subseteq \operatorname{Rad} A$.

$(i) \Leftrightarrow (iv)$: It follows by Theorem 4.4.4 proved below. $\square$

Remark 4.4.3. In the pushout diagram of Theorem 4.4.2-(iv), it is always true that if α' is an essential monomorphism, then α is an essential monomorphism by, for example, Anderson & Fuller (1992, §5, Exercise 15).

Because of the equivalent conditions (i) and (iv) for abelian groups in the previous theorem, we define Z-neat homomorphisms of modules; see Section 1.2 for the definition of Z-neat homomorphisms.

Zöschinger has explained us how to prove the equivalence of the conditions (i) and (iv) in the previous theorem for an arbitrary ring:

Theorem 4.4.4. *The following are equivalent for a homomorphism $f : A \longrightarrow A'$:*

(i) $f : A \longrightarrow A'$ is E-neat;

(ii) If

$$\begin{tikzcd} A \arrow[r, hook, "\alpha"] \arrow[d, "f"'] & E(A) \arrow[d, "f'"] \\ A' \arrow[r, "\alpha'"'] & B' \end{tikzcd}$$

is a pushout diagram of f and the inclusion monomorphism α, then α' is an essential monomorphism;

(iii) If

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "f"'] & B \arrow[d, "f'"] \\ A' \arrow[r, "\alpha'"'] & B' \end{tikzcd}$$

is a pushout diagram of f and α where α is an essential monomorphism, then α' is also an essential monomorphism.

Proof. $(i) \Rightarrow (ii)$: Suppose that the pushout diagram in (ii) is given with α an essential monomorphism. Of course, α' is monic by the properties of pushout. We show that α' is essential. Without loss of generality, we can assume that $A' \subseteq B'$ and α' is an inclusion monomorphism, and so $f(a) = f'(a)$ for all $a \in A$. Now suppose on the contrary that $A' \ntrianglelefteq B'$. Then there exists a nonzero complement C' of A' in B', and so $A' \oplus C' \trianglelefteq B'$. By pushout of f and α we have the following commutative diagram with exact rows, where ψ is an isomorphism:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "\alpha"] \arrow[d, "f"'] & E(A) \arrow[r, "g"] \arrow[d, dashed, "f'"] & E(A)/A \arrow[r] \arrow[d, dashed, "\psi"] & 0 . \\
0 \arrow[r] & A' \arrow[r, dashed, "\alpha'"] & B' \arrow[r, dashed, "h"] & B'/A' \arrow[r] & 0
\end{tikzcd}$$

Here g and h are natural epimorphisms and $\psi(x+A) = f'(x)+A'$ for every $x \in E(A)$. Let $\psi^{-1}(A' \oplus C'/A') = U/A$ where $A \subseteq U \subseteq E(A)$. Since $(A' \oplus C')/A' \cong C' \neq 0$, $U/A \neq 0$ as ψ is an isomorphism, and so $A \subsetneqq U$. Since, for every $u \in U$, $f'(u)+A' = \psi(u+A) \in (A' \oplus C')/A'$ we obtain $f'(U) \subseteq A' \oplus C'$. Thus we have found a proper extension $\pi\sigma$ of f in $E(A)$, where $\sigma : U \longrightarrow A' \oplus C'$ is the restriction of f' to U defined by $\sigma(u) = f'(u)$ for every $u \in U$, and $\pi : A' \oplus C' \longrightarrow A'$ is the projection onto A':

$$\begin{tikzcd}
E(A) & \\
U \arrow[u, hook] \arrow[r, "\sigma"] & A' \oplus C' \arrow[d, "\pi"] \\
A \arrow[u, hook, "i"] \arrow[u, phantom, "\neq"'] \arrow[r, "f"'] & A'
\end{tikzcd}$$

This contradicts with f being E-neat. Hence α' must be essential.

$(ii) \Rightarrow (iii)$: Consider the pushout diagram given in (iii) with α an essential monomorphism. Without loss of generality, assume $A \subseteq B$ and α is the inclusion monomorphism. Clearly α' is monic by the properties of pushout. Since $A \trianglelefteq B$ we can also assume $B \subseteq E(A)$. Take a pushout of β and f', that is, obtain the second square (2) in the following diagram. This gives us the following pushout diagram of f and $\beta\alpha$ since the squares (1) and (2) are pushout diagrams:

$$\begin{tikzcd}
A \arrow[r, hook, "\alpha"] \arrow[d, "f"'] \arrow[dr, phantom, "(1)"] & B \arrow[r, hook, "\beta"] \arrow[d, "f'"] \arrow[dr, phantom, "(2)"] & E(A) \arrow[d, dashed, "f''"] \\
A' \arrow[r, "\alpha'"'] & B' \arrow[r, dashed, "\beta'"'] & C'
\end{tikzcd} .$$

Thus, by hypothesis (ii), $\beta'\alpha'$ is an essential monomorphism, and so α' is also essential.

$(iii) \Rightarrow (i)$: Suppose that $f = \beta\alpha$ where α is an essential monomorphism:

$$\begin{tikzcd}
A \arrow[r, "\alpha"] \arrow[d, "f"'] & B \arrow[dl, "\beta"] \\
A' &
\end{tikzcd}$$

By pushout we have the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "\alpha"] \arrow[d, "f"'] & B \arrow[r] \arrow[dl, "\beta"] \arrow[d, dashed] & C \arrow[r] \arrow[d, dashed, "1_C"] \arrow[dl, dotted, "\gamma"'] & 0 \\
0 \arrow[r] & A' \arrow[r, dashed, "\alpha'"'] & B' \arrow[r, dashed] & C \arrow[r] & 0
\end{tikzcd} .$$

Since β exists, there exists $\gamma : C \longrightarrow B'$ that makes the indicated triangle commutative by Lemma 4.2.2. That means the second row splits, that is, $\alpha'(A')$ is a direct summand of B'. But by hypothesis α' is essential since α is essential. So $\alpha'(A') = B'$ must hold. Thus α is an isomorphism. □

Corollary 4.4.5. *For a module A and a submodule K, the natural epimorphism f : $A \longrightarrow A/K$ is E-neat if and only $A/K \trianglelefteq E(A)/K$.*

Proof. The result follows immediately from Theorem 4.4.4 by taking the following pushout diagram where α and α' are inclusion monomorphisms, and f' is the natural epimorphism:

$$\begin{tikzcd} A \arrow[r, hook, "\alpha"] \arrow[d, "f"'] & E(A) \arrow[d, "f'"] \\ A/K \arrow[r, "\alpha'"'] & E(A)/K \end{tikzcd} .$$

□

Proposition 4.4.6. *(Generalov, 1983, Proposition 4) Let R be a ring that can be embedded in a module S such that* $\operatorname{Rad} S = R$*. Then:*

(i) For every module A, there exists a module B such that $\operatorname{Rad} B = A$*.*

(ii) If, in addition, the module S/R is semisimple, then an essential extension B of A such that B/A is a semisimple module can be taken as B in (i).

Lemma 4.4.7. *For an abelian group A, there exists an abelian group $B \supseteq A$ such that* $\operatorname{Rad} B = A$*, $A \trianglelefteq B$, and B/A is semisimple.*

Proof. For the submodule $S := \sum_{p \text{ prime}} \mathbb{Z}\frac{1}{p} \subseteq \mathbb{Q}$ of the $\mathbb{Z}$-module $\mathbb{Q}$ of rational numbers, we have $S/\mathbb{Z} = \operatorname{Soc}(\mathbb{Q}/\mathbb{Z})$ is a semisimple $\mathbb{Z}$-module and $\operatorname{Rad} S = \mathbb{Z}$ (see, for example, Mermut (2004, Lemma 4.6.2)). Thus the result follows by Proposition 4.4.6. □

Recall that a ring R is said to be a *left quasi-duo ring* if each maximal ideal is a two-sided ideal.

Proposition 4.4.8. *(Generalov, 1983, Lemma 3) Let R be a left quasi-duo ring. Then for every module M,*

$$\operatorname{Rad} M = \bigcap_{\substack{P \subseteq {}_R R \\ max.}} PM,$$

where the intersection is over all maximal left ideals of R.

Lemma 4.4.9. *(by Generalov (1983, Theorem 7, Proposition 4)) Let R be a commutative domain of which every maximal ideal is invertible and K be the field of fractions of R. Let $S \subseteq K$ be the submodule of the R-module K such that $S/R =$* $\operatorname{Soc}(K/R)$*. Then:*

(i) $\operatorname{Rad} S = R$ *and* S/R *is a semisimple* R*-module,*

(ii) For a free R*-module* $F := \bigoplus_{\lambda\in\Lambda} R$ *for some index set* Λ*, take the* R*-module* $A := \bigoplus_{\lambda\in\Lambda} S$*. Then* $\operatorname{Rad} A = F$ *and* $A/\operatorname{Rad} A$ *is a semisimple* R*-module.*

Proof. The proof in Mermut (2004, Lemma 5.4.1) for Dedekind domains works also in this lemma.

(i) Since $S/R = \operatorname{Soc}(K/R)$, it is clearly semisimple. So $\operatorname{Rad}(S/R) = 0$. Hence $\operatorname{Rad} S \subseteq R$. Let P be a maximal ideal of R. Then P is an invertible ideal by hypothesis, that is, for the submodule $P^{-1} \subseteq K$, $PP^{-1} = R$. Hence P^{-1} is a homogenous semisimple R-module with each simple submodule isomorphic to R/P. So, the quotient P^{-1}/R is also semisimple. Thus $P^{-1}/R \subseteq \operatorname{Soc}(K/R) = S/R$ which implies that $P^{-1} \subseteq S$. So $R = PP^{-1} \subseteq PS$. Then, by Proposition 4.4.8,

$$\operatorname{Rad} S = \bigcap_{\substack{P \,\subseteq_{max.}\, {}_R R}} PS \supseteq R,$$

and thus $\operatorname{Rad} S = R$.

(ii) $\operatorname{Rad} A = \bigoplus_{\lambda\in\Lambda} \operatorname{Rad} S = \bigoplus_{\lambda\in\Lambda} R = F$ and $A/\operatorname{Rad} A = \bigoplus_{\lambda\in\Lambda}(S/R)$ is semisimple.

□

Lemma 4.4.10. *Let* A *and* B *be submodules of a module. If* $A \trianglelefteq (A+B)$ *and* B *is semisimple, then* $B \subseteq A$*.*

Proof. Let $B = \bigoplus_{i\in I} S_i$, where S_i is a simple submodule of B for all $i \in I$ (where I is some indexing set). Let $S \subseteq B$ be one of the S_i's, say $S = Rx$ for some $0 \neq x \in S$. Since $A \trianglelefteq (A+B)$ and $x = 0 + x \in A + B$, there is $0 \neq r \in R$ such that $0 \neq rx \in A$, and so $0 \neq rx \in S \cap A$. This implies that $S \cap A = S$ since S is simple, that is, $S \subseteq A$. Thus every simple submodule S of B belongs to A. Hence $B \subseteq A$. □

Lemma 4.4.11. *Let R be a commutative domain of which every maximal ideal is invertible. Let $K \subseteq A$ be R-modules with* $\operatorname{Rad} A = 0$. *If the natural epimorphism $f : A \longrightarrow A/K$ is E-neat, then $K = 0$.*

Proof. By Lemma 4.4.9, there exists an R-module S such that $\operatorname{Rad} S = R$ and S/R is semisimple. So by Proposition 4.4.6, there exists an R-module $B \supseteq K$ such that $\operatorname{Rad} B = K$ with $K \trianglelefteq B$, and B/K is semisimple. Then we can embed B into the injective envelope $E(A)$ of A:

$$\begin{tikzcd}
K \arrow[r, hook, "\trianglelefteq"] \arrow[d, hook] & B \arrow[dl, dashed] \\
E(A) &
\end{tikzcd}$$

So we can assume that $K \subseteq B \subseteq E(A)$. Now, by Corollary 4.4.5, $A/K \trianglelefteq E(A)/K$, and so $A/K \trianglelefteq (A+B)/K = (A/K) + (B/K)$ since $A + B \subseteq E(A)$. The module B/K is semisimple by hypothesis, and so we obtain $B/K \subseteq A/K$ by Lemma 4.4.10. Thus $B \subseteq A$, and $K = \operatorname{Rad} B \subseteq \operatorname{Rad} A = 0$, that is, $K = 0$. $\square$

Corollary 4.4.12. *Let R be a commutative domain of which every maximal ideal is invertible. If $f : A \longrightarrow A'$ is an E-neat epimorphism and* $\operatorname{Rad} A = 0$, *then* $\operatorname{Ker} f = 0$ *and so f is an isomorphism.*

Proposition 4.4.13. *If R is a left C-ring and if all maximal left ideals of R are projective, then R is a left hereditary ring.*

Proof. Since every maximal left ideal of R is projective, it follows by Crivei (1998, Theorem 10) that every factor module of a max-injective module is max-injective. Since R is aleft C-ring, by Theorem 4.2.14, every max-injective module is injective. Thus, every factor module of an injective module is injective. Hence R is left hereditary by Theorem 4.3.10. $\square$

Recall that a ring R is an *N-domain* if and only if R is a commutative domain whose all maximal ideals are projective (and so all maximal ideals are invertible and finitely generated). So, if R is an N-domain and a C-ring, then R is a commutative hereditary domain, that is, a Dedekind domain (by Proposition 4.4.13).

Theorem 4.4.14. *Let R be Dedekind domain. For R-modules $K \subseteq A$, the natural epimorphism $f : A \longrightarrow A/K$ is E-neat if and only if $K \subseteq \mathrm{Rad}A$.*

Proof. ($\Leftarrow$) Suppose that for $A \subseteq B \subseteq E(A)$, there exists a homomorphism $h : B \longrightarrow A/K$ that extends f:

$$\begin{tikzcd} E(A) & \\ B \arrow[u, hook] \arrow[dr, dashed, "h"] & \\ A \arrow[u, hook] \arrow[r, "f"'] & A/K \end{tikzcd}$$

Then $h(a) = f(a) = a + K$ for every $a \in A$, and $A \trianglelefteq B$ since $A \trianglelefteq E(A)$. Therefore $h(K) = 0$, and so we obtain the following diagram:

$$\begin{tikzcd} A/K \arrow[r, hook] \arrow[d, "1_{A/K}"'] & B/K \arrow[dl, dashed, "h'"] \\ A/K & \end{tikzcd}$$

such that $h'(b+K) = h(b)$ for every $b \in B$. This diagram is also commutative: for every $a \in A$, $h'(a+K) = h(a) = f(a) = a+K = 1_{A/K}(a+K)$. This means that the diagram splits, and so A/K is a direct summand of B/K. Then there is a submodule C with $K \subseteq C \subseteq B$ such that $(A/K) \oplus (C/K) = B/K$. So $A + C = B$ and $A \cap C = K$. Since $K \subseteq \mathrm{Rad}A$ by hypothesis, we obtain that A is a Rad-supplement of C in B or equivalently, A is a coneat submodule of B. Then A is also a neat submodule of B by Mermut (2004, Theorem 5.4.6). So A is closed in B by Proposition 4.1.1 since the Dedekind domain is is a C-ring. But then $A \trianglelefteq B$ implies that $A = B$. Thus $f : A \longrightarrow A/K$ has no proper extension in the injective envelope of A. Hence f is E-neat by Theorem 4.3.3.

($\Rightarrow$) Conversely, suppose that $f : A \longrightarrow A/K$ is an E-neat homomorphism. We have the following commutative diagram:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\sigma"'] & A/K \arrow[dr, "\rho"] & \\
A/(\mathrm{Rad}A) \arrow[rr, "\phi"] & & (A/K)/(\mathrm{Rad}(A/K))
\end{tikzcd}$$

where σ and ρ are natural epimorphisms, and $\phi = \beta\alpha$, where

$$\alpha : A/\mathrm{Rad}A \longrightarrow (A/\mathrm{Rad}A)/((\mathrm{Rad}A+K)/\mathrm{Rad}A)$$

is the natural epimorphism and

$$\beta : (A/\mathrm{Rad}A)/((\mathrm{Rad}A+K)/\mathrm{Rad}A) \longrightarrow (A/K)/(\mathrm{Rad}(A/K))$$

is defined by

$$(a+\mathrm{Rad}A)+[(\mathrm{Rad}A+K)/\mathrm{Rad}A] \mapsto (a+K)+\mathrm{Rad}(A/K).$$

Thus $\phi(a+\mathrm{Rad}A) = (a+K)+\mathrm{Rad}(A/K)$ for every $a \in A$. Since f is E-neat (by hypothesis) and ρ is E-neat (by "($\Leftarrow$)" of this theorem), we obtain $\phi\sigma = \rho f$ is E-neat by Proposition 4.3.9. Since R is left hereditary and ϕ and σ are epimorphisms, ϕ is E-neat by Theorem 4.3.10. So α is E-neat by Proposition 4.3.11 since $\phi = \beta\alpha$ is E-neat. Now since the natural epimorphism α is E-neat and $\mathrm{Rad}(A/\mathrm{Rad}A) = 0$, we obtain by Lemma 4.4.11 that $(\mathrm{Rad}+K)/\mathrm{Rad}A = 0$, that is, $\mathrm{Rad}A+K = \mathrm{Rad}A$. Thus $K \subseteq \mathrm{Rad}A$ as required. $\square$

Corollary 4.4.15. *Let R be a Dedekind domain. For every R-module A, the natural epimorphism $f : A \longrightarrow A/\mathrm{Rad}A$ is E-neat.*

Corollary 4.4.16. *Let R be a Dedekind domain. An epimorphism $f : A \longrightarrow B$ of modules is E-neat if and only if* $\mathrm{Ker} f \subseteq \mathrm{Rad}A$.

Now we give the main result of this section which generalizes the equivalence of (i) and (ii) in Theorem 4.4.2 for abelian groups to Dedekind domains:

Theorem 4.4.17. *Let R be a Dedekind domain. Then a homomorphism $f : A \longrightarrow A'$ of R-modules is E-neat if and only if it is Z-neat, that is,* $\operatorname{Im} f$ *is closed in A' and* $\operatorname{Ker} f \subseteq \operatorname{Rad} A$.

Proof. We have $f = ip$ where $p : A \longrightarrow A/\operatorname{Ker} f \cong \operatorname{Im} f$ is the natural epimorphism and $i : \operatorname{Im} f \longrightarrow A'$ is the inclusion monomorphism.

($\Rightarrow$) Suppose that $f : A \longrightarrow A'$ is E-neat. Since R is hereditary, both of p and i are E-neat by Theorem 4.3.10. Thus $\operatorname{Im} i = \operatorname{Im} f$ is closed in A' by Lemma 4.3.4 since the monomorphism i is E-neat, and $\operatorname{Ker} f \subseteq \operatorname{Rad} A$ by Corollary 4.4.16 since the epimorphism p is E-neat.

($\Leftarrow$) Conversely, suppose that $f : A \longrightarrow A'$ is Z-neat. Then since $\operatorname{Im} f$ is closed in A' the monomorphism i is E-neat by Lemma 4.3.4, and since $\operatorname{Ker} f \subseteq \operatorname{Rad} A$ the epimorphism p is E-neat by Corollary 4.4.16. Hence the composition $f = ip$ is also E-neat by Proposition 4.3.9-(i). □

4.5 The Class of E-neat Epimorphisms

In this section, we deal with properties of E-neat homomorphisms which are epimorphisms. By showing that a splitting epimorphism is not always E-neat, we show that the class of all short exact sequences of modules defined by E-neat epimorphisms does not form a proper class.

Theorem 4.5.1. *A splitting epimorphism $f : M \longrightarrow N$ is E-neat if and only if* $\operatorname{Ker} f$ *is injective. That is, for modules A and B, the splitting (projection) epimorphism $f : A \oplus B \longrightarrow A$ is E-neat if and only if* $\operatorname{Ker} f \cong B$ *is injective.*

Proof. ($\Rightarrow$) Suppose that f is E-neat. Consider the injective envelope $E(B)$ of B. Since

$B \trianglelefteq E(B)$, we have an essential monomorphism

$$\alpha : A \oplus B \hookrightarrow A \oplus E(B) \subseteq E(A) \oplus E(B) \subseteq E(A \oplus B).$$

Indeed, $A+B \subseteq A+E(B)$ and for $a \in A \cap E(B), Ra \cap B \subseteq A \cap B = 0$, and so $B \trianglelefteq E(B)$ implies $Ra = 0$, that is, $a = 0$. Clearly, $A \oplus B \trianglelefteq A \oplus E(B)$. Thus we obtain the following commutative diagram:

$$\begin{tikzcd}
E(A \oplus B) = E(A) \oplus E(B) & \\
A \oplus E(B) \arrow[u, hook] \arrow[dr, dashed, "\sigma"] & \\
A \oplus B \arrow[u, hook, "\alpha"] \arrow[r, "f"'] & A
\end{tikzcd}$$

where σ is a projection. In the above diagram, $\sigma\alpha((a,b)) = \sigma((a,b)) = a = f((a,b))$, that is, $\sigma|_{A \oplus B} = f$. But since f is E-neat, it cannot have a proper extension in the injective envelope of $A \oplus B$ by Theorem 4.3.3. Then $A \oplus B = A \oplus E(B)$, and so $B = E(B)$. Hence $B \cong \operatorname{Ker} f$ is injective.

($\Leftarrow$) Let $M = A \oplus B$, that is, $M = A + B$ and $A \cap B = 0$. Then $\mathrm{E}(M) = \mathrm{E}(A \oplus B) = \mathrm{E}(A) \oplus B$ since $B \cong \operatorname{Ker} f$ is injective by hypothesis. By Theorem 4.3.3-(iv) it suffices to show that f does not have a proper extension in $\mathrm{E}(M)$. Suppose on the contrary that f has an extension $g : C \longrightarrow A$ in $\mathrm{E}(M)$ where $M \subseteq C \subseteq \mathrm{E}(M)$. Then we have the following commutative diagram, where $i : A \hookrightarrow A \oplus B$ is the inclusion monomorphism:

$$\begin{tikzcd}
\mathrm{E}(M) = \mathrm{E}(A) \oplus B & \\
C \arrow[u, hook] \arrow[dr, "g"] & \\
M = A \oplus B \arrow[u, hook, "j"] \arrow[r, "f"'] & A \\
A \arrow[u, hook, "i"] \arrow[ur, "1_A"'] &
\end{tikzcd}$$

Since $ji : A \hookrightarrow C$ splits, there exists a submodule K of C such that $C = A \oplus K$. Therefore, $M = A \oplus B \hookrightarrow C = A \oplus K$, and so $B \cong (A \oplus B)/A \subseteq C/A$. Since B is injective by hypothesis, $(A \oplus B)/A$ is a direct summand in C/A, and so $C/A = (A \oplus B)/A \oplus (L/A)$ for a submodule L of C with $A \subseteq L$. Thus $C = (A \oplus B) + L$ and $(A \oplus B) \cap L = A$. So we have the following commutative diagram:

$$\begin{tikzcd}
A \arrow[r, hook] \arrow[d, "1_A"'] & L \arrow[r, hook] \arrow[dl, "e"] & C \arrow[dll, "g"] \\
A & &
\end{tikzcd}$$

with $g\mid_L = e$. Since A is a direct summand of C, it is also a direct summand of L. So, $L = A \oplus U$ for some $U \subseteq L$. Then $C = (A \oplus B) + L = (A \oplus B) + (A \oplus U) = A \oplus B \oplus U$. So we have $M = A \oplus B \subseteq C \subseteq \mathrm{E}(M)$. Since $M \trianglelefteq \mathrm{E}(M)$, we have $M \trianglelefteq C$. But, then we must have $U = 0$ since $M = A \oplus B$ is a direct summand of $C = (A \oplus B) \oplus U$. Hence $C = M$ as required. $\square$

Theorem 4.5.1 shows that the class $\mathcal{EN}eat$ only contains splitting short exact sequences which starts with an injective module. So it does not contain all splitting short exact sequences if there modules that are not injective (that is, if the ring R is not semisimple). Now let us check whether $\mathcal{EN}eat$ satisfies the other conditions for being a proper class (P1, P3, P4 in Section 2.3):

Theorem 4.5.2. *The class $\mathcal{EN}eat$ satisfies the conditions P1 and P3 for every ring R, and P4-(ii) for left hereditary rings, but it does not satisfy the conditions P2 unless the ring R is semisimple. As a result, the class $\mathcal{EN}eat$ forms a proper class if and only if R is a semisimple ring.*

Proof. **P1.** Let

$$\mathbb{E} : 0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0 \quad \text{and} \quad \mathbb{E}' : 0 \longrightarrow A \xrightarrow{f'} B' \xrightarrow{g'} C \longrightarrow 0$$

be short exact sequences of modules that are isomorphic, that is, we have the following commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "1_A"'] & B \arrow[r, "g"] \arrow[d, "\psi"'] & C \arrow[r] \arrow[d, "1_C"'] & 0 \\ 0 \arrow[r] & A \arrow[r, "f'"] & B' \arrow[r, "g'"] & C \arrow[r] & 0 \end{tikzcd}$$

with some isomorphism $\psi : B \longrightarrow B'$, where $1_A : A \longrightarrow A$ and $1_C : C \longrightarrow C$ are identity maps. Assume that $\mathbb{E} \in \mathcal{EN}eat$, that is, $g : B \longrightarrow C$ is an E-neat epimorphism. We shall to show that $\mathbb{E}' \in \mathcal{EN}eat$, that is, $g' : B' \longrightarrow C$ is an E-neat epimorphism. Suppose we have a commutative diagram for modules $H \subsetneqq G' \subseteq G$:

$$\begin{tikzcd} & H \arrow[r, hook, "\neq"'] \arrow[dl, dashed, "\varphi"'] \arrow[d, "\sigma"] & G' \arrow[r, hook] \arrow[dr, "\tau"] & G \\ B \arrow[r, "\psi"'] & B' \arrow[rr, "g'"'] & & C \end{tikzcd}$$

Since ψ is an isomorphism, the homomorphism $\varphi = \psi^{-1}\sigma : H \longrightarrow B$ exists. Now since $g'\psi : B \longrightarrow C$ is E-neat, there is a submodule $G'' \supsetneqq H$ of G and a homomorphism $\gamma : G'' \longrightarrow B$ such that $\gamma|_H = \varphi$:

$$\begin{tikzcd} H \arrow[r, hook, "\neq"'] \arrow[d, "\varphi"'] & G'' \arrow[r, hook] \arrow[dl, dashed, "\gamma"] \arrow[d, dashed, "\psi\gamma"] & G \\ B \arrow[r, "\psi"'] & B' \arrow[r, "g'"'] & C \end{tikzcd}$$

Since there is $\psi\gamma : G'' \longrightarrow B'$ such that $(\psi\gamma)\,|_H = \psi(\gamma\,|_H) = \psi\varphi = \sigma$, we obtain g' is E-neat.

P2. By Theorem 4.5.1, the splitting short exact sequences which do not start with an injective module are not in $\mathcal{EN}eat$. So P2 holds if and only if every left R-module is injective, that is, if and only if R is a semisimple ring (see, for example, Sharpe & Vámos (1972, Proposition 3.7)).

P3. (i) Let $f : A \longrightarrow B$ and $g : B \longrightarrow C$ be $\mathcal{EN}eat$-monomorphisms, that is, the row containing f and the column containing g in the following diagram are in the class $\mathcal{EN}eat$. We shall show that gf is an $\mathcal{EN}eat$-monomorphism. Consider the following commutative diagram, where we assume that $A \subseteq B \subseteq C$, and f and g are inclusion monomorphisms, and the other maps are naturally defined:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "1_A"'] & B \arrow[r, "\alpha"] \arrow[d, "g"] & B/A \arrow[r] \arrow[d, "\beta"] & 0 \\
0 \arrow[r] & A \arrow[r, "gf"] & C \arrow[r, "h"] \arrow[d, "\gamma"] & C/A \arrow[r] \arrow[d, "\varphi"] & 0 \\
& & C/B \arrow[r, "1_{C/B}"] \arrow[d] & C/B \arrow[d] & \\
& & 0 & 0 &
\end{tikzcd} \tag{4.5.1}$$

According to the diagram, the epimorphisms α and γ are E-neat, and we must show that the epimorphism h is E-neat. Suppose we have a commutative diagram for modules $H \subsetneqq G' \subseteq G$:

$$\begin{tikzcd}
H \arrow[r, hook, "\neq"'] \arrow[d, "\sigma"'] & G' \arrow[r, hook] \arrow[d, "\tau"] \arrow[dr, dashed, "\varphi\tau"] & G \\
C \arrow[r, "h"'] & C/A \arrow[r, "\varphi"'] & C/B
\end{tikzcd}$$

Since $(\varphi\tau)\mid_H = \varphi(\tau\mid_H) = (\varphi h)\sigma$ and $\varphi h : C \longrightarrow C/B$ is E-neat, there is a submodule $G'' \supsetneqq H$ of G and $\phi : G'' \longrightarrow C$ so that $\phi|_H = \sigma$:

$$\begin{tikzcd}
H \arrow[r, hook, "\neq"'] \arrow[d, "\sigma"'] & G'' \arrow[r, hook] \arrow[dl, dashed, "\phi"] & G \\
C \arrow[r, "h"'] & C/A &
\end{tikzcd}$$

This means that $h : C \longrightarrow C/A$ is E-neat.

(ii) Let $\alpha : A \longrightarrow B$ and $\beta : B \longrightarrow C$ be $\mathcal{EN}eat$-epimorphisms. Then α and β are E-neat, and so $\beta\alpha$ is E-neat by Proposition 4.3.9-(i), that is, $\beta\alpha$ is $\mathcal{EN}eat$-epimorphism.

P4. (ii) Let R be a left hereditary ring, and let $\alpha : A \longrightarrow B$ and $\beta : B \longrightarrow C$ be epimorphisms. If $\beta\alpha$ is an $\mathcal{EN}eat$-epimorphism, that is, if $\beta\alpha$ is E-neat, then β is an E-neat by Theorem 4.3.10 (since α and β are epimorphisms).

As a result, if R is semisimple, then $\mathcal{EN}eat = \mathcal{S}plit_R$. Indeed, if R is semisimple, then every left R-module is injective. So, every short exact sequence in $\mathcal{EN}eat$ is splitting. Now, every splitting short exact sequence is in $\mathcal{EN}eat$ by Theorem 4.5.1.

Conversely, if $\mathcal{EN}eat$ forms a proper class, then P2 of being a proper class holds, and so R is semisimple. □

Let $f : A \longrightarrow B$ and $g : B \longrightarrow C$ be monomorphisms and gf be an $\mathcal{EN}eat$-monomorphism. To show that $\mathcal{EN}eat$ satisfies **P4**-(i), we must show that f is an $\mathcal{EN}eat$-monomorphism. Consider the commutative diagram (4.5.1). So we must show that $\alpha : B \longrightarrow B/A$ is E-neat when $h : C \longrightarrow C/A$ is E-neat. But this is not satisfied, for example, for abelian groups as the following example shows:

Example 4.5.3. Let $\mathbb{Z}_{(5)}$ be the localization of the prime ideal $(5) = 5\mathbb{Z}$ of $\mathbb{Z}$. Then $\mathbb{Z}_{(5)}$ consists of all rational numbers with denominators relatively prime to 5, that is,

$$\mathbb{Z}_{(5)} = \left\{ \frac{a}{b} \mid a, b \in \mathbb{Z} \text{ and } (b,5) = 1 \right\}.$$

Obviously, we have $\mathbb{Z} \subseteq \mathbb{Z}_{(5)} \subseteq \mathbb{Q}$. We shall show that the epimorphism $\alpha : \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z}$ is E-neat, but the epimorphism $\beta : \mathbb{Z}_{(5)} \longrightarrow \mathbb{Z}_{(5)}/\mathbb{Z}$ is not E-neat. Since $\mathbb{Q}$ is injective, $\operatorname{Rad}\mathbb{Q} = \mathbb{Q}$, and so $\operatorname{Ker}\alpha = \mathbb{Z} \subseteq \operatorname{Rad}\mathbb{Q}$. This means α is E-neat by Proposition 4.4.14. Since $\mathbb{Z}_{(5)}$ is q-divisible for every prime $q \neq 5$, we obtain

$$\operatorname{Rad}(\mathbb{Z}_{(5)}) = \bigcap_{q \text{ prime}} q\mathbb{Z}_{(5)} = 5\mathbb{Z}_{(5)}.$$

Then $\operatorname{Ker}\beta = \mathbb{Z} \nsubseteq 5\mathbb{Z}_{(5)} = \operatorname{Rad}(\mathbb{Z}_{(5)})$ since, for example, $3 \in \mathbb{Z} \setminus 5\mathbb{Z}_{(5)}$. Indeed, if

$3 \in 5\mathbb{Z}_{(5)}$, then $3 = 5(\frac{a}{b})$ for some $a, b \in \mathbb{Z}$ with b relatively prime to 5. But, then $a = \frac{3b}{5} \in \mathbb{Z}$ which is impossible. Hence β is not E-neat, again by Proposition 4.4.14.

4.6 *Z*-coneat Homomorphisms and the Proper Class $\mathcal{C}oclosed$

Our interest in this section is *coneat* homomorphisms of Zöschinger which have been studied in Zöschinger (1978) (we call them *Z-coneat* homomorphisms) and the class of coclosed monomorphisms. We also study coclosed monomorphisms of modules and we show that the class of all short exact sequences defined by *is coclosed* submodules forms a proper class, denoted by $\mathcal{C}oclosed$. The properties needed to prove this found in Clark et al. (2006) and Zöschinger (2006).

See Section 1.2 for the definition of E-neat homomorphisms and its dual definition of Z-coneat homomorphisms considered by Zöschinger for abelian groups:

Dual to Theorem 4.4.2 for E-neat homomorphisms, the following theorem was given by Zöschinger (1978) for Z-coneat homomorphisms of abelian groups. Note that for an abelian group A and an integer n, $A[n] = \{a \in A \mid na = 0\}$.

Theorem 4.6.1. *(Zöschinger, 1978, Satz 2.3) For a homomorphism $g : C' \to C$ of abelian groups, the following are equivalent:*

(i) *g is Z-coneat;*

(ii) $\operatorname{Ker} g$ *is coclosed in* C' *and* $\operatorname{Soc} C \subseteq \operatorname{Im} g$*;*

(iii) $g(C'[p]) = C[p]$ *for all prime p;*

(iv) *If the diagram below is a pullback diagram and β is a small epimorphism, then β' is also a small epimorphism.*

$$\begin{tikzcd} B' \arrow[r, "\beta'"] \arrow[d, "g'"'] & C' \arrow[d, "g"] \\ B \arrow[r, "\beta"'] & C \end{tikzcd}$$

Now we show that $\mathcal{C}oclosed$ forms a proper class.

The following proposition contains some properties of coclosed submodules, which will be used in this section, from Clark et al. (2006):

Proposition 4.6.2. *(Clark et al., 2006, 3.7) Let $K \subseteq L \subseteq M$ be submodules. Then:*

(i) If $L \overset{cc}{\hookrightarrow} M$, then $L/K \overset{cc}{\hookrightarrow} M/K$.

(ii) If $K \overset{cc}{\hookrightarrow} M$, then $K \overset{cc}{\hookrightarrow} L$ and the converse is true if $L \overset{cc}{\hookrightarrow} M$.

For completeness note the following lemma from Zöschinger (2006) with its proof:

Lemma 4.6.3. *(Zöschinger, 2006, Lemma A.4) Let $U \subseteq V \subseteq M$ be submodules. If $U \overset{cc}{\hookrightarrow} M$ and $V/U \overset{cc}{\hookrightarrow} M/U$, then $V \overset{cc}{\hookrightarrow} M$.*

Proof. Assume that $V/X \ll M/X$ where $X \subseteq V$ for some submodule X of M with $X \subseteq V$. We shall show that $X = V$. Firstly, let us show that $U/(U \cap X) \ll M/(U \cap X)$. Assume that $U/(U \cap X) + W/(U \cap X) = M/(U \cap X)$ for a submodule W of M with $U \cap X \supseteq W$. Clearly, we have $U + W = M$. To show that $W = M$, we shall show that $V/(U + W \cap X) \ll M/(U + W \cap X)$. Suppose that

$$V/(U+W\cap X)+Z/(U+W\cap X)=M/(U+W\cap X)$$

for a submodule Z of M with $(U + W \cap X) \subseteq Z$. Clearly, Clearly we have $V + Z = M$. Since $U \subseteq Z$, by modular law, we have $Z = Z \cap M = Z \cap (U + W) = U + (Z \cap W)$. Then $V + U + (Z \cap W) = Z + V = M$, and so $V + (Z \cap W) = M$ since $U \subseteq V$. Thus,

$$M/X = V/X + [(Z\cap W)+X]/X.$$

Since $V/X \ll M/X$ by hypothesis, we obtain $M/X = [(Z\cap W)+X]/X$, and so $(Z\cap W)+X = M$. Now, by modular law, we have

$$W = W\cap M = W\cap((Z\cap W)+X) = (Z\cap W)+(W\cap X),$$

and so $W \subseteq Z$ since $W\cap X \subseteq Z$ already. So $M = U+W = U+Z = Z$ since $U \subseteq Z$. This shows that $V/(U+W\cap X)+Z/(U+W\cap X) = M/(U+W\cap X)$. But then we have

$$(V/U)/[(U+W\cap X)/U] \cong V/(U+W\cap X) \ll M/(U+W\cap X) \cong (M/U)/[(U+W\cap X)/U].$$

So $V/U = (U+W\cap X)/U$ since $V/U \overset{cc}{\hookrightarrow} M/U$ by hypothesis. This clearly implies $V = U+W\cap X$. Then, by modular law,

$$X = X\cap V = X\cap(U+(W\cap X)) = (X\cap U)+(W\cap X),$$

and so $X \subseteq W$ since $U\cap X \subseteq W$. Since $U \subseteq V$ and $U+W = M$, we have $V+W = M$. So $M/X = V/X+W/X$. This implies that $W/X = M/X$ since $V/X \ll M/X$ (by assumption) or equivalently, $W = M$ which shows that $U/(U\cap X) \ll M/(U\cap X)$. Now since $U \overset{cc}{\hookrightarrow} M$ (by hypothesis), we obtain $U = U\cap X$, and so $U \subseteq$. Hence $X = V$ since $X/U \subseteq V/U$ and $V/X \ll M/X$. Indeed, since

$$(V/U)/(X/U) \cong V/X \ll M/X \cong (M/U)/(X/U)$$

we obtain $V/U = X/U$ (since $V/U \overset{cc}{\hookrightarrow} M/U$ by hypothesis). $\square$

Theorem 4.6.4. *The class $\mathcal{C}$oclosed, that is, the class of all short exact sequences*

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

of modules such that $\operatorname{Im}\alpha$ is coclosed in B *forms a proper class.*

Proof. Let $K \subseteq A \subseteq B$ and C be submodules.

P1. Suppose we have the following commutative diagram of modules and module homomorphisms:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "1_A"'] & B \arrow[r, "g"] \arrow[d, "\psi"'] & C \arrow[r] \arrow[d, "1_C"'] & 0 \\
0 \arrow[r] & A \arrow[r, "f'"] & B' \arrow[r, "g'"] & C \arrow[r] & 0
\end{tikzcd}$$

with some isomorphism $\psi : B \longrightarrow B'$, where $1_A : A \longrightarrow A$ and $1_C : C \longrightarrow C$ are identity maps. We can assume that $A \subseteq B'$ and f' is an inclusion monomorphism. We shall show that, if $A \xhookrightarrow{cc} B$, then $A \xhookrightarrow{cc} B'$. If there is a submodule $K \subseteq A$ such that $A/K \ll B'/K$, then we have $\psi^{-1}(A)/K \ll \psi^{-1}(B')/K$. Since $\psi^{-1}(B') = B$ and $\psi^{-1}(A) = A$ as ψ is an isomorphism, we obtain $A/K \ll B/K$, and so $A = K$ since $A \xhookrightarrow{cc} B$. Hence $A \xhookrightarrow{cc} B'$.

P2. $\mathcal{C}oclosed$ contains all splitting short exact sequences, since every direct summand is coclosed. Indeed, let $A \oplus B = M$ for submodules A, B of a module M. Let us show that $A \xhookrightarrow{cc} M$. Suppose that there is a submodule $K \subseteq A$ such that $A/K \ll M/K$. Since

$$(A/K) \cap ((B+K)/K) = [A \cap (B+K)]/K = (K + A \cap B)/K = 0,$$

we obtain $M/K = (A/K) \oplus (B+K)/K$. Since $A/K \ll M/K$ by hypothesis, we have $A/K = 0$ or $A = K$. Hence $A \xhookrightarrow{cc} M$.

P3. (i) Let $f : A \longrightarrow B$ and $g : B \longrightarrow C$ be $\mathcal{C}oclosed$-monomorphisms. We can assume that $B \subseteq C$ and g is an inclusion monomorphism. Since $A \xhookrightarrow{cc} B$ and $B \xhookrightarrow{cc} C$, we obtain $A \xhookrightarrow{cc} C$ by Proposition 4.6.2-(ii). This means that gf is a $\mathcal{C}oclosed$-monomorphism.

(ii) Let $K \subseteq A \subseteq B$ and assume that the natural epimorphisms $\alpha : B \longrightarrow B/K$ and $\beta : B/K \longrightarrow B/A \cong (B/K)/(A/K)$ be $\mathcal{C}oclosed$-epimorphisms. We shall show that $\beta\alpha$ is a $\mathcal{C}oclosed$-epimorphism, that is, $A \xhookrightarrow{cc} B$ by the following commutative diagram where f, g and h are inclusion monomorphisms, and α and β are natural epimorphisms:

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r, hook, "f"] \arrow[d, "1_A"'] & A \arrow[r] \arrow[d, hook, "g"] & A/K \arrow[r] \arrow[d, hook, "h"] & 0 \\
0 \arrow[r] & K \arrow[r, hook] & B \arrow[r, "\alpha"] \arrow[d, "\beta\alpha"] & B/K \arrow[r] \arrow[d, "\beta"] & 0 \\
 & & B/A \arrow[r, "1_{B/A}"'] \arrow[d] & B/A \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

Then according to the diagram, we have $K \overset{cc}{\hookrightarrow} B$ and $A/K \overset{cc}{\hookrightarrow} B/K$. We shall show that $A \overset{cc}{\hookrightarrow} B$. This follows by Lemma 4.6.3.

P4. (i) Let $f : A \longrightarrow B$ and $g : B \longrightarrow C$ be monomorphisms and let gf be a $\mathcal{C}losed$-monomorphism. We shall show that f is a $\mathcal{C}losed$-monomorphism. We can assume that $A \subseteq B \subseteq C$. We have $A \overset{cc}{\hookrightarrow} C$ and we shall show that $A \overset{cc}{\hookrightarrow} B$. This result follows by Proposition 4.6.2-(ii).

(ii) Let $A \subseteq B \subseteq C$. For the natural epimorphisms, $\alpha : C \longrightarrow C/A$ and $\beta : C/A \longrightarrow C/B \cong (C/A)/(B/A)$, assume that $\beta\alpha$ is a $\mathcal{C}losed$-epimorphism. We shall show that β is a $\mathcal{C}losed$-epimorphism. We have $B \overset{cc}{\hookrightarrow} C$ and we shall show that $B/A \overset{cc}{\hookrightarrow} C/A$. This result follows by Proposition 4.6.2-(i). $\square$

CHAPTER FIVE

TORSION FREE AND COMPONENTWISE FLAT COVERS OF QUIVERS

In this chapter, we study in the category of representations by modules of a quiver Q, denoted by $(Q, R\text{-}\mathcal{M}od)$, which is a Grothendieck category with enough injectives and projectives. In Section 5.1 we give some preliminary notions and explain the category $(Q, R\text{-}\mathcal{M}od)$.

In Enochs et al. (2004a), it was proved that for the existence of $\mathcal{F}$-covers and $\mathcal{F}^{\perp}$-envelopes in the general setting of a Grothendieck category (not necessarily with enough projectives), it suffices to show that the class $\mathcal{F}$ satisfies some standard conditions. In this way, we prove the existence of "componentwise" flat covers in $(Q, R\text{-}\mathcal{M}od)$ for every ring R and any quiver Q in Section 5.3. We also prove the existence of torsion free covers in $(Q, R\text{-}\mathcal{M}od)$ for a wide class of quivers in Section 5.2. In the last section, we compare the "categorical" flat covers and "componentwise" flat covers giving some examples.

Throughout this chapter, all torsion theories considered for $R\text{-}\mathcal{M}od$ will be hereditary, and *faithful* (i.e., R will be torsion free). Also, we will consider the following two properties during this chapter:

(**A**) Any direct sum of torsion free injective R-modules is *injective*.

(**B**) For every vertex v of a quiver Q, the set $\{t(a) \mid s(a) = v\}$ is *finite*.

5.1 The Category $(Q, R\text{-}\mathcal{M}od)$

The notions of quiver and linear representation of a quiver were introduced in Gabriel (1972a). The classical representation theory of quivers involved *finite* quivers (i.e., a quiver with finitely many vertices and edges) and assumed that the ring was an algebraically closed field with the assumption that all vector spaces involved were finite dimensional. But recently, representations by modules over more general quivers have been studied (see Enochs & Herzog (1999), Enochs et al. (2002), Enochs et al. (2003a), Enochs et al. (2004b), Enochs & Estrada (2005a), Enochs et al. (2007), Enochs et al. (2009)).

See Section 1.3 for the definition of quivers and some related notions that needed in this chapter:

Example 5.1.1. The following is an example of a quiver whose vertices are v_0, v_1, v_2 and whose arrows are $\alpha, \beta, \delta, \gamma$:

$$Q \equiv v_0 \underset{\beta}{\overset{\alpha}{\rightrightarrows}} v_1 \underset{\gamma}{\overset{\delta}{\rightrightarrows}} v_2 \quad .$$

A *tree* is a quiver Q having a vertex v such that for every vertex w of Q, there exists a unique path $p : v \longrightarrow w$. Such a vertex is unique and it is called the *root* of the tree.

For a given quiver Q, *the (left) path space* (or *the path tree associated to* Q), denoted by $P(Q)$, is the quiver whose vertices are the paths p of Q and whose arrows are the pairs $(p, ap) : p \longrightarrow ap$, where a is an arrow of Q such that ap is defined (i.e. $s(a) = t(p)$). For every vertex v of Q, denote by $P(Q)_v$ the subtree of $P(Q)$ containing all paths of Q starting at v.

Example 5.1.2. Let Q be the quiver in Example 5.1.1. Then the left path spaces of Q are the following:

$$P(Q)_{v_0} \equiv \begin{tikzcd} & & \delta\alpha\,, \\ & \alpha \arrow[ur, "(\alpha,\delta\alpha)"] \arrow[r, "(\alpha,\gamma\alpha)"'] & \gamma\alpha \\ v_0 \arrow[ur, "(v_0,\alpha)"] \arrow[dr, "(v_0,\beta)"'] & & ; \\ & \beta \arrow[r, "(\beta,\delta\beta)"] \arrow[dr, "(\beta,\gamma\beta)"'] & \delta\beta \\ & & \gamma\beta \end{tikzcd} \quad P(Q)_{v_1} \equiv \begin{tikzcd} & \delta \\ v_1 \arrow[ur, "(v_1,\delta)"] \arrow[dr, "(v_1,\gamma)"'] & ; \\ & \gamma \end{tikzcd} \quad P(Q)_{v_2} \equiv \quad v_2$$

Recall that a *representation by modules* of a given quiver Q is defined as a functor $X : Q \longrightarrow R\text{-}\mathcal{M}od$. Such a representation is determined by giving a module $X(v)$ for every vertex v of Q and a homomorphism $X(a) : X(v_1) \longrightarrow X(v_2)$ for each arrow $a : v_1 \longrightarrow v_2$ of Q.

A morphism η between two representations X and Y is a *natural transformation*, so it will be a family of homomorphisms $\eta_v : X(v) \longrightarrow Y(v)$ such that $Y(a)\eta_{v_1} = \eta_{v_2}X(a)$ for each arrow $a : v_1 \longrightarrow v_2$ of Q, that is, the following diagram commutes:

$$\begin{tikzcd} X(v_1) \arrow[r, "X(a)"] \arrow[d, "\eta_{v_1}"'] & X(v_2) \arrow[d, "\eta_{v_2}"] \\ Y(v_1) \arrow[r, "Y(a)"'] & Y(v_2) \end{tikzcd} .$$

Thus the representations of a quiver Q by modules over a ring R form a category, denoted by $(Q, R\text{-}\mathcal{M}od)$.

Remark 5.1.3. Actually, our category $(Q, R\text{-}\mathcal{M}od)$ is a special case of the functor category $\mathbf{Fun}(\mathbf{I}, \mathcal{C})$ where $\mathbf{I} = Q$ and $\mathcal{C} = R\text{-}\mathcal{M}od$ (see Section 2.7 page 46). So, by Stenström (1975, Chap. IV, Proposition 7.1), we infer that $(Q, R\text{-}\mathcal{M}od)$ is an abelian category, and from the arguments given in the proof we notice that *kernels,* *cokernels* and *products* are constructed "componentwise". From this we deduce that *colimits* and *limits* are computed "componentwise" as well. In particular, this tells us that *intersections* (which are a special case of pullbacks) and *sums* are also computed componentwise.

Therefore, η will be a monomorphisms (respectively, epimorphism, isomorphism) whenever η_v is a monomorphism (resp. epimorphism, isomorphism) of modules for all $v \in V$. In particular, if η_v's are just inclusions, then X is said to be a subfunctor of Y or, by using our terminology, a *subrepresentation* of Y. Similarly, *quotient representations* are also defined componentwise.

Example 5.1.4. Let Q be the quiver in Example 5.1.1. Then a representation of Q is given as follows: $X \equiv \quad X(v_0) \underset{g_0}{\overset{f_0}{\rightrightarrows}} X(v_1) \underset{g_1}{\overset{f_1}{\rightrightarrows}} X(v_2)$

is a representation of Q, where $X(v_0), X(v_1), X(v_2)$ are modules and f_0, g_0, f_1, g_1 are homomorphisms.

Definition 5.1.5. For a given quiver Q and a ring R, *the path ring* of Q over R, denoted by RQ, is the free left R-module whose base are all paths of Q, and where the multiplication is the obvious composition between two paths:

$$q \cdot p = \begin{cases} qp & \text{if } t(p) = s(q) \\ 0 & \text{if } t(p) \neq s(q) \end{cases}$$

The product of basis elements is then extended to arbitrary elements of RQ. In other words, there is a direct sum decomposition

$$RQ = RQ_0 \oplus RQ_1 \oplus \ldots \oplus RQ_n \oplus \ldots$$

of the free left R-module RQ, where, for each $n \geq 0$, RQ_n is the submodule of RQ

generated by the set Q_n of all paths of length n.

A ring R is said to be a ring *with enough idempotents* if there exists a family $\{e_\alpha\}_{\alpha\in A}$ of pairwise *orthogonal* idempotents $e_\alpha \in R$ (that is, $e_\alpha \neq e_\beta$ for $\alpha \neq \beta$ and $e_\alpha^2 = e_\alpha$ for all $\alpha, \beta \in R$) such that

$$R = \bigoplus_{\alpha\in A} e_\alpha R = \bigoplus_{\alpha\in A} Re_\alpha.$$

R is said to be a ring *with local units* if for every finite set $S \subseteq R$, there exists an idempotent $e \in R$ such that $S \subseteq eRe$.

By Wisbauer (1991, Chap.10, §49), we have a ring R with enough idempotents is a ring with local units.

Remark 5.1.6. *RQ* is a ring with *enough idempotents*,. Indeed, the set of vertices of Q is a family of orthogonal idempotents (since $v \cdot w = 0$ for $v \neq w$, and $v^2 = v$ for all $v, w \in V$) such that

$$RQ = \bigoplus_{v\in V} vRQ = \bigoplus_{v\in V} RQv.$$

So, RQ is a ring with local units. Indeed, consider the sets

$$A = \{s(p_i) : \sum_i r_i p_i \in S\} \quad \text{and} \quad B = \{t(p_j) : \sum_j s_j p_j \in S\}$$

where p_k is a path of Q for each k and $r_k, s_k \in R$. Then $e = \sum\limits_{v\in A\cup B} v$ is the required idempotent.

Note that RQ has an identity element if and only if the set of vertices V of Q is finite (see, for example, Assem et al. (2006, chap. II, Lemma 1.4)). In this case,

$$1_{RQ} = v_1 + v_2 + \cdots v_n$$

where $v_1, v_2, \ldots v_n$ are distinct vertices of Q.

Remark 5.1.7. It is known that the categories $(Q, R\text{-}\mathcal{M}od)$ and $RQ\text{-}\mathcal{M}od$, the category of unital RQ-modules or equivalently, the category of left RQ-modules $_{RQ}M$ such that $RQM = M$, are equivalent; see, for example, Estrada (2003, Chap. 2) for details. Indeed, given a representation X of Q, we have an R-module $\bigoplus_{v\in V} X(v)$, and the *action* of a path $p = v_0 \longrightarrow v_1 \longrightarrow \cdots \longrightarrow v_n$ is the composition of homomorphisms

$$\bigoplus_{v\in V} X(v) \longrightarrow X(v_0) \longrightarrow X(v_1) \longrightarrow \cdots \longrightarrow X(v_n) \longrightarrow \bigoplus_{v\in V} X(v),$$

where the first homomorphism is the projection homomorphism and the last is the inclusion homomorphism. Thus, by this action, any representation X of Q can be given an RQ-module structure. Conversely, let M be an RQ-module. Then we can construct a representation X of Q so that $X(v) = vM$ for all $v \in V$, and for each arrow $a : v \longrightarrow w$ of Q, $X(a) : vM \longrightarrow wM$ is just scalar multiplication by a (that is, for each $vm \in vM$, $X(a)(vm) = avm = am = wam \in wM$). For this equality, where Q is finite and R is an algebraically closed field with all vector spaces involved are finite dimensional, see also Assem et al. (2006, Chap. III.1) or Auslander et al. (1995, Chap. III, Theorem 1.5).

Example 5.1.8. Let Q be a quiver consisting of a single point but no arrows. The defining basis of the path ring RQ is $\{v\}$, where v is the identity of the path ring RQ. Thus $RQ \cong R$, where the isomorphism being induced by the R-linear map such that $v \mapsto 1$. So, $(Q, R\text{-}\mathcal{M}od) \cong RQ\text{-}\mathcal{M}od \cong R\text{-}\mathcal{M}od$. This means that a representation of Q is just an R-module.

Example 5.1.9. Let Q be the quiver consisting of a single point and a single loop:

$$Q \equiv \qquad v\bullet \circlearrowright a$$

Then the defining basis of the path ring RQ (i.e. the paths of Q) is $\{v, a, a^2, \ldots\}$, where v is the identity of RQ. Thus $RQ \cong R[x]$, where $R[x]$ is a ring of polynomials with coefficients in R and where the isomorphism being induced by the R-linear map such that $v \mapsto 1$ and $a \mapsto x$. So, $(Q, R\text{-}\mathcal{M}od) \cong R[x]\text{-}\mathcal{M}od$.

Note that RQ is a projective generator of $RQ\text{-}\mathcal{M}od$ (see Wisbauer (1991, Chap. 10, §49.1)). So, it follows that $(Q, R\text{-}\mathcal{M}od)$ is a Grothendieck category with a projective generator, and thus with enough projectives by, for example, Stenström (1975, Chap. IV, §6, Example (3)). Moreover, for the explicit presentation of the method to construct a family of projective generators for the category $(Q, R\text{-}\mathcal{M}od)$, see Enochs et al. (2004b) or Estrada (2003, Chap. 2). Also, $(Q, R\text{-}\mathcal{M}od)$ has enough injectives (see Stenström (1975, Chap. X, Corollary 4.3)).

Remark 5.1.10. For a given quiver Q, one can define a family of projective generators from an adjoint situation as it is shown in Mitchell (1972). The family $\{S_v(R) \mid v \in V\}$ is a family of projective generators for the category of representations $(Q, R\text{-}\mathcal{M}od)$ where for each $v \in V$ the functor $S_v : R\text{-}\mathcal{M}od \longrightarrow (Q, R\text{-}\mathcal{M}od)$ is defined in Mitchell (1972, §28) as follows:

For an R-module M, the representation $S_v(M)$ is defined for all $w \in V$ by

$$S_v(M)(w) = \bigoplus_{p \in Q(v,w)} M$$

where $Q(v,w)$ is the set of all paths of Q such that $s(p) = v$ and $t(p) = w$, and for each arrow $a : w_1 \to w_2$ of Q,

$$S_v(M)(a) : \bigoplus_{q \in Q(v,w_1)} M \longrightarrow \bigoplus_{p \in Q(v,w_2)} M$$

is given by $(m_q)_{Q(v,w_1)} \mapsto (u_p)_{p \in Q(v,w_2)}$, where

$$u_p = \begin{cases} 0 & \text{if } p \notin aQ(v,w_1), \\ m_q & \text{if } p \in aQ(v,w_1) \end{cases}$$

and $aQ(v,w_1) = \{aq \mid q \in Q(v,w_1)\}$. In other words,

$$S_v(M)(a) = \bigoplus_{q\in Q(v,w_1)} M \overset{\bigoplus_q 1_M}{\longrightarrow} \bigoplus_{p\in aQ(v,w_1)} M \hookrightarrow \bigoplus_{p\in Q(v,w_2)} M.$$

Then S_v is a left adjoint functor of the evaluation functor $T_v : (Q, R\text{-}\mathcal{M}od) \longrightarrow R\text{-}\mathcal{M}od$ given by $T_v(X) = X(v)$ for every representation X in $(Q, R\text{-}\mathcal{M}od)$; see, for example, Enochs et al. (2004b, Propositions 3.1 and 3.2). That is, for every $v \in V$, we have

$$\mathrm{Hom}_Q(S_v(M), X) \cong \mathrm{Hom}_R(M, T_v(X)).$$

Since $(Q, R\text{-}\mathcal{M}od)$ has coproducts, we obtain by Definition 2.7.8 that $\bigoplus_{v\in V} S_v(R)$ is a generator for $(Q, R\text{-}\mathcal{M}od)$.

Example 5.1.11. Consider the quiver

$$Q \equiv \quad \begin{tikzcd} \bullet v \arrow[r, "b"'] \arrow[rr, bend left, "c"] & \bullet w_1 \arrow[r, "a"'] & \bullet w_2 \end{tikzcd}.$$

Then $Q(v,w_1) = \{b\}$, $Q(v,w_2) = \{ab, c\}$ and $aQ(v,w_1) = \{ab\}$. For a module M, we have $S_v(M)(w_1) = M_b$, $S_v(M)(w_2) = M_{ab} \oplus M_c$, and $S_v(M)(a) = 1_{M_{ab}} \oplus 0$, where $M_b = M_a = M_{ab} = M$. In other words,

$$S_v(M)(a) : M \longrightarrow M \oplus M \text{ is given by } m \mapsto (m, 0).$$

Proposition 5.1.12. *The category $(Q, R\text{-}\mathcal{M}od)$ is locally finitely presented.*

Proof. We shall show that for every finitely generated module M, $S_v(M)$ is a finitely generated representation in $(Q, R\text{-}\mathcal{M}od)$ for every $v \in V$. Let $\sum_{i\in I} Y_i$ be a direct union of subrepresentations of Y. Since S_v is a left adjoint functor of the restriction functor T_v, we have

$$\mathrm{Hom}_Q(S_v(M), \sum_i Y_i) \cong \mathrm{Hom}_R(M, \sum_i Y_i(v)).$$

Since $\sum_i Y_i$ is computed componentwise and M is finitely generated module, we obtain

$$\mathrm{Hom}_R(M, \sum_i Y_i(v)) \cong \varinjlim_I \mathrm{Hom}_R(M, Y_i(v)) \cong \varinjlim_I \mathrm{Hom}_Q(S_v(M), Y_i)$$

Thus by Proposition 2.7.13, $S_v(M)$ is finitely generated. So, $\{S_v(R) \mid v \in V\}$ is a family of finitely generated projective generators, and so finitely presented generators (by 2.7.15). Hence $(Q, R\text{-}\mathcal{M}od)$ is a locally finitely presented category. $\square$

5.2 Torsion Free Covers in $(Q, R\text{-}\mathcal{M}od)$ Relative to a Torsion Theory

By injective representations of a quiver Q, we mean injective objects in the category $(Q, R\text{-}\mathcal{M}od)$.

Throughout this section, Q will be a *source injective representation quiver* , that is, for every ring R every *injective* representation X in $(Q, R\text{-}\mathcal{M}od)$ is characterized by the following conditions (call shortly *SIRQ*):

(i) $X(v)$ is an injective R-module, for every vertex v of Q.

(ii) For every vertex v, the morphism

$$X(v) \longrightarrow \prod_{s(a)=v} X(t(a))$$

induced by $X(v) \longrightarrow X(t(a))$ is a splitting epimorphism, where the product is over all arrows a in Q with $s(a) = v$; see Enochs et al. (2009, Definition 2.2).

Remark 5.2.1. (Enochs et al., 2009, Proposition 2.1) For each quiver Q, if $X \in (Q, R\text{-}\mathcal{M}od)$ is *injective*, then the above conditions (i) and (ii) of *SIRQ* always hold.

Now let us give some examples of source injective representation quivers:

(i) Each quiver with a finite number of vertices and without oriented cycles is a source injective representation quiver.

(ii) The infinite line quivers:

$$A_\infty \equiv \qquad \cdots \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet\ ,$$

$$A^\infty \equiv \qquad \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \cdots\ ,$$

$$A_\infty^\infty \equiv \qquad \cdots \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \cdots$$

are source injective representation quivers.

(iii) Infinite barren trees are source injective representation quivers: Recall that if T is a tree with the root v, we can divide the set of vertices into "states" in such a way that the first state is $\{v\}$, the second is the set of sons of v (that is, the vertices w such that there is an arrow from v to w), the third is given by the sons of the vertices in the second state, and so on. We say that T is *barren* if the number of vertices n_i of the i'th state of T is finite for every $i \in \mathbb{N}$, and the sequence of positive natural numbers $n_1, n_2, \dots$ stabilizes, that is, there exists $r \in \mathbb{N}$ such that $n_{r+j} = n_r$ for all $j \in \mathbb{N}$ (see Enochs et al. (2009, Corollaries 5.4-5.5)). For example, the tree

$$\begin{tikzcd}
 & \bullet \arrow[r] & \bullet \arrow[r] & \cdots \\
\bullet \arrow[ur] \arrow[r] \arrow[dr] & \bullet \arrow[r] & \bullet \arrow[r] & \cdots \\
 & \bullet \arrow[r] & \bullet \arrow[r] & \cdots
\end{tikzcd}$$

is barren.

(iv) The quiver with two vertices and infinitely many arrows between these two vertices is a source injective representation quiver, but does not satisfy the property (**B**) (at page 129): $\bullet \underset{\vdots}{\overset{\vdots}{\rightrightarrows}} \bullet$

Once we have given some examples of source injective representation quivers, now let us give an example that is not a source injective representation quiver:

Example 5.2.2. The n-loop, that is a loop with n vertices, is not a source injective representation quiver. To see this, let v_i be a vertex and $a_i : v_i \longrightarrow v_{i+1}$ be an arrow of the quiver for all $i = 1,2,\ldots,n$, where $v_{n+1} = v_1$. Now consider the representation X defined as follows: $X(v_i) = E \times \cdots \times E$ (n times) and $X(a_i)(x_1,\ldots x_n) = (x_n, x_1, \ldots, x_{n-1})$, where E is an injective R-module and $x_i \in E$ for all $i = 1,\ldots n$. Then it is clear that X satisfies the conditions (i) and (ii) of *SIRQ*. But, X is not an injective representation since it is not a divisible RQ-module. This is because, there is element $(a_n a_{n-1} \cdots a_1 + a_1 a_n \cdots a_2 + \cdots + a_{n-1} a_{n-2} \cdots a_n) - 1_{RQ}$ of RQ which is not zero divisor such that

$$[(a_n a_{n-1} \cdots a_1 + a_1 a_n \cdots a_2 + \cdots + a_{n-1} a_{n-2} \cdots a_n) - 1_{RQ}] \cdot m = 0$$

for every element $m = (m_1, \ldots, m_n)$ where $m_i \in X(v_i) = E \times \cdots \times E$ for all $i = 1,2,\ldots,n$. Indeed, if $m_i = (m_i^1, m_i^2, \ldots, m_i^n)$, where $m_i^j \in E$, for all $i, j = 1,\ldots,n$, then

$$(a_n a_{n-1} \cdots a_1) \cdot m + (a_1 a_n \cdots a_2) \cdot m + \cdots + (a_{n-1} a_{n-2} \cdots a_n) \cdot m =$$

$$X(a_n)X(a_{n-1}) \cdots X(a_1)m_1 + \cdots + X(a_{n-1})X(a_{n-2}) \cdots X(a_n)m_n =$$

$$X(a_n) \cdots X(a_2)(m_n^1, m_1^1, \ldots, m_{n-1}^1) + \cdots + X(a_{n-1}) \cdots X(a_1)(m_n^n, m_1^n, \ldots, m_{n-1}^n) = \cdots$$

$$((m_1^1, \ldots, m_n^1), 0, \ldots, 0) + \cdots + (0, \ldots, 0, (m_1^n, \ldots, m_n^n)) = (m_1, \ldots, m_n) = m$$

See Remark 5.1.7 for the action of a path used in this example.

Now let $(\mathcal{T}, \mathcal{F})$ be a torsion theory for $R\text{-}\mathcal{M}od$. Then we can define a torsion theory $(\mathcal{T}_{cw}, \mathcal{F}_{cw})$ for $(Q, R\text{-}\mathcal{M}od)$, by defining the torsion class such that

$$\mathcal{T}_{cw} = \{X \in (Q, R\text{-}\mathcal{M}od) \mid X(v) \in \mathcal{T} \text{ for all } v \in V\}.$$

This is because $\mathcal{T}_{cw}$ is closed under quotient representations, direct sums and extensions (as so is $\mathcal{T}$) (see, for example, Stenström (1975, VI, Proposition 2.1)).

Remark 5.2.3. Since the torsion class $\mathcal{T}_{cw}$ is closed under subrepresentations, our torsion theory $(\mathcal{T}_{cw}, \mathcal{F}_{cw})$ is hereditary.

Proposition 5.2.4. *Let $X \in (Q, R\text{-}\mathcal{M}od)$. Then $X \in \mathcal{F}_{cw}$ if and only if $X(v) \in \mathcal{F}$ for all $v \in V$.*

Proof. $(\Rightarrow)$ Let $X \in \mathcal{F}_{cw}$. Then for every $M \in \mathcal{T}$, we have

$$\mathrm{Hom}_R(M, T_v(X)) \cong \mathrm{Hom}_Q(S_v(M), X) = 0$$

since $S_v(M) \in \mathcal{T}_{cw}$ (as $\mathcal{T}$ is closed under direct sums). Thus $X(v) = T_v(X) \in \mathcal{F}$ for all $v \in V$.

$(\Leftarrow)$ Suppose that $X(v) \in \mathcal{F}$ for all $v \in V$. Let $A \in \mathcal{T}_{cw}$. If $\gamma : A \longrightarrow X$ is a morphism of representations, then we have module homomorphisms $\gamma_v : A(v) \longrightarrow X(v)$ for all $v \in V$. Since $A(v) \in \mathcal{T}$, then $\mathrm{Hom}_R(A(v), X(v)) = 0$ and so $\gamma_v = 0$ for all $v \in V$. Thus $\gamma = 0$, that is, $\mathrm{Hom}_Q(A, X) = 0$. This means that $X \in \mathcal{F}_{cw}$. □

Theorem 5.2.5. *Any representation of $\mathcal{F}_{cw}$ can be embedded in a torsion free and injective representation.*

Proof. Let $X \in \mathcal{F}_{cw}$ be any representation of Q. Since $(Q, R\text{-}\mathcal{M}od)$ has enough injectives and $(\mathcal{T}_{cw}, \mathcal{F}_{cw})$ is hereditary, then $\mathcal{F}_{cw}$ is closed under injective envelopes (see Dickson (1966, Theorem 2.9)). Thus X can be embedded in its torsion free injective envelope. □

Lemma 5.2.6. *Let X, X', Y and Z be representations of Q.*

(i) If X has an $\mathcal{F}_{cw}$-precover and $Z \subseteq X$, then Z also has an $\mathcal{F}_{cw}$-precover.

(ii) If X is injective, then $\psi : X' \longrightarrow X$ is an $\mathcal{F}_{cw}$-precover of X if and only if for every morphism $\phi : Y \longrightarrow X$ with $Y \in \mathcal{F}_{cw}$ and Y injective, there exists $f : Y \longrightarrow X'$ such that $\psi f = \phi$.

Proof. (i) Let $\psi : X' \longrightarrow X$ be an $\mathcal{F}_{cw}$-precover. Consider the morphism $\psi_1 : \psi^{-1}(Z) \longrightarrow Z$. Then $\psi^{-1}(Z) \in \mathcal{F}_{cw}$ since $\mathcal{F}_{cw}$ is closed under subrepresentations. Now for every morphism $\phi : Y \longrightarrow Z$ with $Y \in \mathcal{F}_{cw}$, there is a morphism $f : Y \longrightarrow X'$ such that $\psi f = \phi$. Therefore, $f(Y) \subseteq \psi^{-1}(Z)$ and so ϕ can be factored through ψ_1.

(ii) The condition is clearly necessary. Let $\phi_1 : Y_1 \longrightarrow X$ be a morphism with $Y_1 \in \mathcal{F}_{cw}$. Then by Theorem 5.2.5, Y_1 can be embedded in a representation $Y \in \mathcal{F}_{cw}$ which is injective. Now since X is injective, there is a morphism $\phi : Y \longrightarrow X$ such that $\phi\,|_{Y_1} = \phi_1$. So, by hypothesis, there exists a morphism $f : Y \longrightarrow X'$ such that $\psi f = \phi$. It follows that $(\psi f)\,|_{Y_1} = \phi\,|_{Y_1} = \phi_1$. □

Lemma 5.2.7. *Let E be a module and let $\{E_i\}_{i\in I}$ be a direct family of submodules of E. If $\bigoplus_{i\in I} E_i$ is injective, then $\sum_{i\in I} E_i$ is injective.*

Proof. Define homomorphisms $\varphi : \bigoplus_i E_i \longrightarrow \sum_i E_i$ by

$$\varphi((e_{i_1}, e_{i_2}, \ldots, e_{i_m})) = e_{i_1} + e_{i_2} + \cdots + e_{i_m},$$

and $\psi : \sum_i E_i \longrightarrow \bigoplus_i E_i$ by

$$\psi(e_{k_1} + e_{k_2} + \cdots + e_{k_m}) = (0, \ldots, 0, \underbrace{e_{k_1} + e_{k_2} + \cdots + e_{k_m}}_{\in E_t}, 0, \ldots, 0).$$

The second homomorphism is well-defined since I is a directed set. Indeed, since $\{E_i\}_{i\in I}$ is a direct family of submodules, for every $i, j \in I$, there exists a $k \in I$ with $k \geq i, j$ such that $E_i \subseteq E_k$ and $E_j \subseteq E_k$, and so there exists a $t \geq ki$ such that $E_{k_i} \subseteq E_t$ for all $i = 1, 2, \ldots, m$. This implies that $e_{k_1} + e_{k_2} + \cdots + e_{k_m} \in E_t$. Therefore, we have

$$\varphi\psi(e_{k_1} + e_{k_2} + \cdots + e_{k_m}) = \varphi(0, \ldots, 0, e_{k_1} + e_{k_2} + \cdots + e_{k_m}, 0, \ldots, 0) = e_{k_1} + e_{k_2} + \cdots + e_{k_m}.$$

So, $\varphi\psi = 1_{\sum E_i}$. Thus, ψ is a monomorphisms, and so $\sum_i E_i$ is a direct summand of $\oplus_i E_i$ (since $\oplus_i E_i$ is injective by hypothesis). Hence $\sum_i E_i$ is also injective. $\square$

Lemma 5.2.8. *Let $E \in (Q, R\text{-}\mathcal{M}od)$ and let $\{E_i\}_{i\in I}$ be a direct family of injective subrepresentations of E such that $E_i \in \mathcal{F}_{cw}$ for all $i \in I$. If R satisfies ($\boldsymbol{A}$) and if Q satisfies ($\boldsymbol{B}$), then $\sum_{i\in I} E_i \in \mathcal{F}_{cw}$ and it is injective.*

Proof. Since each E_i is an injective representation such that $E_i \in \mathcal{F}_{cw}$, then $E_i(v)$ is an injective module such that $E_i(v) \in \mathcal{F}$, for all $v \in V$ and $i \in I$. So, $\bigoplus_i E_i(v)$ is also an injective module by hypothesis. Then by Lemma 5.2.7, $\sum_i E_i(v)$ is also injective. Then the representation $\sum_i E_i$ satisfies the condition (i) of *SIRQ*. Now taking the union of the splitting epimorphisms $E_i(v) \longrightarrow \prod_{s(a)=v} E_i(t(a))$, we obtain the following splitting epimorphism:

$$\left(\sum_i E_i\right)(v) \longrightarrow \sum_i \prod_{s(a)=v} E_i(t(a)) \cong \prod_{s(a)=v} \left(\sum_{i\in I} E_i\right)(t(a))$$

where the isomorphism follows since the product is finite by hypothesis (as Q satisfies (**B**)). This means $\sum_i E_i$ is also satisfies (ii). Thus it is an injective representation since Q is a source injective representation quiver. Finally, since $E_i(v) \in \mathcal{F}$ then $\sum_i E_i(v) \in \mathcal{F}$ for all $v \in V$, and so $\sum_i E_i \in \mathcal{F}_{cw}$. $\square$

Proposition 5.2.9. *Let Q be any quiver satisfying ($\boldsymbol{B}$). Then R satisfies ($\boldsymbol{A}$) if and only if any direct sum of injective representations of $\mathcal{F}_{cw}$ is injective.*

Proof. ($\Rightarrow$) The proof is the same as the proof of Lemma 5.2.8 by taking $\bigoplus_{i\in I} E_i$ instead of $\sum_{i\in I} E_i$.

($\Leftarrow$) The proof is immediate by considering the quiver $Q \equiv \cdot v$ which trivially satisfies (**B**). This is because $(Q, R\text{-}\mathcal{M}od) \cong R\text{-}\mathcal{M}od$ in this case. $\square$

Note that, in the previous proposition, which will be useful in the proof of the following theorem, we cannot omit the assumption that Q satisfies (**B**) as the following example shows:

Example 5.2.10. Consider the following quiver Q that has infinitely many arrows starting at v_0,

$$Q \equiv \begin{tikzcd} & \vdots & v_1 \\ v_0 \arrow[urr] \arrow[rr] \arrow[drr] & & v_2 \\ & \vdots & v_3 \end{tikzcd}$$

which, of course, does not satisfy (**B**) for the vertex v_0. For the ring of integers, $R = \mathbb{Z}$, consider the category $(Q, \mathbb{Z}\text{-}\mathcal{M}od)$. Then the indecomposable injective and torsion free representations of $(Q, \mathbb{Z}\text{-}\mathcal{M}od)$ (with respect to the usual torsion theory) are as follows:

$$E_0 \equiv \begin{tikzcd} & \vdots & 0 \\ \mathbb{Q} \arrow[urr] \arrow[rr] \arrow[drr] & & 0, \\ & \vdots & 0 \end{tikzcd} \qquad E_1 \equiv \begin{tikzcd} & \vdots & \mathbb{Q} \\ \mathbb{Q} \arrow[urr] \arrow[rr] \arrow[drr] & & 0, \\ & \vdots & 0 \end{tikzcd} \qquad E_2 \equiv \begin{tikzcd} & \vdots & 0 \\ \mathbb{Q} \arrow[urr] \arrow[rr] \arrow[drr] & & \mathbb{Q} \cdots \\ & \vdots & 0 \end{tikzcd}$$

that is, for all $i \in \mathbb{N}$, the representation E_i has a module $\mathbb{Q}$ at the vertices v_0 and v_i, and *zero* otherwise. Therefore, the direct sum of the representations of E_i for $i \geq 1$will be as follows:

$$\bigoplus_{i \geq 1} E_i \equiv \begin{tikzcd} & \vdots & \mathbb{Q} \\ \mathbb{Q}^{(\mathbb{N})} \arrow[urr] \arrow[rr] \arrow[drr] & & \mathbb{Q} \\ & \vdots & \mathbb{Q} \end{tikzcd}$$

If we show that $\bigoplus\limits_{i \geq 1} E_i$ is not an injective representation of $(Q, \mathbb{Z}\text{-}\mathcal{M}od)$, then we will see that the statement of Proposition 5.2.9 does not hold for this Q (since $R = \mathbb{Z}$ satisfies (**A**)). Now suppose on the contrary that $\bigoplus\limits_{i \geq 1} E_i$ is injective. Then, by Remark 5.2.1, we have (ii) of *SIRQ*, that is,

$$\bigoplus_{i\geq 1} E_i(v_0) = \left(\bigoplus_{i\geq 1} E_i\right)(v_0) \longrightarrow \prod_{s(a)=v_0} \left(\bigoplus_{i\geq 1} E_i\right)(t(a)) = \prod_{s(a)=v_0} \bigoplus_{i\geq 1} E_i(t(a))$$

is a splitting epimorphism or equivalently, $\mathbb{Q}^{(\mathbb{N})} \longrightarrow \mathbb{Q}^{\mathbb{N}}$ is a splitting epimorphism. However, this is impossible since $\mathbb{Q}^{(\mathbb{N})}$ has a countable basis, but $\mathbb{Q}^{\mathbb{N}}$ does not have it since $\mathbb{Q}^{\mathbb{N}}$ is uncountable.

Recall that a representation of a quiver Q is said to be *finitely generated* if it is finitely generated as an object of the category of representations of Q.

Theorem 5.2.11. *Let Q be any quiver satisfying (**B**). If R satisfies (**A**), then every injective representation of $\mathcal{F}_{cw}$ is the direct sum of indecomposable injective representations of $\mathcal{F}_{cw}$.*

Proof. Following the proof of Stenström (1975, Proposition 4.5) we argue as follows. Let $E \in \mathcal{F}_{cw}$ be an injective representation of Q. Consider all independent families $(E_i)_{i\in I}$ of indecomposable torsion free and injective subrepresentations of E. Then by Zorn's lemma, there is a maximal such family $(E_i)_{i\in I}$. Since $\oplus_{i\in I} E_i \in \mathcal{F}_{cw}$ and it is injective (by Proposition 5.2.9), we can write $E = (\bigoplus_i E_i) \oplus E'$. To show that $E' = 0$ it is enough to show that every injective representation with $0 \neq E' \in \mathcal{F}_{cw}$ contains a nonzero indecomposable direct summand. Consider the set of all subrepresentations of E' such that

$$\Sigma = \{E'' \subset E' \mid E'' \in \mathcal{F}_{cw} \text{ is injective s.t. } C \nsubseteq E'' \text{ where } 0 \neq C \subseteq E' \text{ is f.g.}\}$$

(In fact, we can take such a nonzero finitely generated representation C, since $(Q, R\text{-}\mathcal{M}od)$ is locally finitely generated). Now take $\overline{E} = \sum_{E''\in\Omega} E''$ where Ω is a chain of Σ. Then by Lemma 5.2.8, $\overline{E} \in \mathcal{F}_{cw}$ and it is injective. Clearly $C \nsubseteq \overline{E}$ since C is finitely generated (indeed, if $C \subseteq \overline{E}$ then $C \subseteq E''$ for some $E'' \in \Omega$ which is impossible). This shows that $\overline{E} \in \Sigma$ and in fact it is an upper bound of Ω. Then by Zorn's lemma Σ has a maximal element, say E''. Now we have $E' = E'' \oplus D$ where $0 \neq D$ is an indecomposable representation. For if $D = D' \oplus D''$ with $D' \neq 0$ and $D'' \neq 0$, then $(E'' + D') \cap (E'' + D'') = E''$, and so either $C \nsubseteq E'' + D'$ or $C \nsubseteq E'' + D''$ which contradicts the maximality of E'' in Σ. Hence, every nonzero E' contains an indecomposable direct summand, which completes the proof. □

Proposition 5.2.12. *Let Q be any quiver satisfying (**B**). If R satisfies (**A**), then $(Q, R\text{-}\mathcal{M}od)$ admits $\mathcal{F}_{cw}$-precovers.*

Proof. Since the category $(Q, R\text{-}\mathcal{M}od)$ has enough injectives, it suffices to show that any *injective* representation X has an $\mathcal{F}_{cw}$-precover (by Lemma 5.2.6-(i)), and so we can take an injective representation $Y \in \mathcal{F}_{cw}$ (by Lemma 5.2.6-(ii)). Let $\{E_\mu \mid \mu \in \Lambda\}$ denote the set of representatives of indecomposable injective representations of $\mathcal{F}_{cw}$. Let $H_\mu = \mathrm{Hom}_Q(E_\mu, X)$ and then define $X' = \bigoplus_{\mu \in \Lambda} E_\mu^{(H_\mu)}$. So there is a morphism $\psi : X' \longrightarrow X$ such that $\psi\,|_{E_\mu} \in H_\mu$. Thus every morphism $\phi : Y \longrightarrow X$ with an injective representation $Y \in \mathcal{F}_{cw}$ factors through the canonical map $\psi : X' \longrightarrow X$, since $Y = \bigoplus_{\mu \in \Lambda'} E_\mu$ (by Theorem 5.2.11) where $\Lambda' \subseteq \Lambda$. □

To prove that $(Q, R\text{-}\mathcal{M}od)$ admits $\mathcal{F}_{cw}$-covers, we first need the following results. See Theorem 2.4.8 for the Gabriel filter $F(R)$.

Lemma 5.2.13. *(by Teply (1969, Proposition 2.1)) Let $F \in \mathcal{F}$. If R satisfies (**A**), then we have $F / \bigcup_{i \in K} F_i \in \mathcal{F}$ for a chain $\{F_i\}_{i \in K}$ of submodules of F with $F/F_i \in \mathcal{F}$.*

Proof. Suppose on the contrary that $F / \bigcup_i F_i \notin \mathcal{F}$. Then there exists $I \in F(R)$ such that $Ix \subseteq \bigcup_i F_i$, where $x \in F \setminus \bigcup_i F_i$. Since $F(R)$ has a cofinal subset of finitely generated left ideals (by Theorem 2.6.2), there exists a finitely generated left ideal $J \subseteq I$ such that $J \in F(R)$. So, $Jx \subseteq \bigcup_i F_i$ and this implies that $Jx \subseteq F_k$ for some $k \in K$ (since J is finitely generated and $\{F_i\}_{i \in K}$ is a chain of submodules). But this contradicts with the fact that $F/F_k \in \mathcal{F}$. □

We also need the following lemmas by the same methods of proofs given in, for example, Xu (1996, Lemmas 1.3.6-1.3.7) for the usual torsion theory for R-$\mathcal{M}od$ over commutative domains.

Lemma 5.2.14. *Let Q be any quiver satisfying (**B**) and let R satisfy (**A**). If $\psi : X' \longrightarrow X$ is an $\mathcal{F}_{cw}$-precover of the representation X, then we can derive an $\mathcal{F}_{cw}$-precover $\phi : Y \longrightarrow X$ such that there is no non-trivial subrepresentation $S \subseteq \ker(\phi)$ with $Y/S \in \mathcal{F}_{cw}$.*

Proof. Let Σ be a set of all subrepresentations $S \subseteq X'$ such that $S \subset \ker(\psi)$ and $X'/S \in \mathcal{F}_{cw}$. Then the union of any chain of elements of Σ, say $T = \bigcup S$, belongs to Σ. Indeed, since $X'/S \in \mathcal{F}_{cw}$ for every $S \in \Sigma$, then $X'(v)/S(v) \in \mathcal{F}$. So, by Lemma 5.2.13, $X'(v)/T(v) \in \mathcal{F}$ for every vertex v of Q. This means that $X'/T \in \mathcal{F}_{cw}$. Clearly $T \subseteq \ker(\psi)$. Let T' be a maximal element of Σ by Zorn's lemma. Thus if we take $Y = X'/T'$, then the induced map $\phi : Y \longrightarrow X$ is the desired torsion free precovering of X; for if there exists a morphism $\phi' : Y' \longrightarrow X$ with $Y' \in \mathcal{F}_{cw}$, then there is a morphism $f' : Y' \longrightarrow X'$ such that $\psi f' = \phi'$ (since ψ is a precovering). Now, taking the morphism $f = \sigma f' : Y' \longrightarrow Y$ (where $\sigma : X' \longrightarrow X'/T'$ is the natural morphism) we obtain $\phi f = (\phi\sigma) f' = \psi f' = \phi'$ as desired. Diagrammatically,

$$\begin{tikzcd}
& & Y' \arrow[dll, bend right, "f'"'] \arrow[dl, dashed, "f"'] \arrow[d, "\phi'"] \\
X' \arrow[r, "\sigma"] \arrow[rr, bend right, "\psi"'] & Y \arrow[r, "\phi"] & X
\end{tikzcd}$$

(here ϕ is the induced map $X'/T' \longrightarrow X'/\ker(\psi) \cong \mathrm{Im}(\psi) \subseteq X$). Moreover, if there is a subrepresentation $L/T' \subseteq Y = X'/T'$ such that $L/T' \subseteq \ker(\phi)$ and $X'/L \cong (X'/T')/(L/T') \in \mathcal{F}_{cw}$, then $L \subseteq \ker(\psi)$ (since $\psi(L) = \phi\sigma(L) = \phi(L/T') = 0$). But then $L \in \Sigma$, and so $L = T'$ since T' is maximal in Σ. $\square$

The *cardinality of a representation M of a quiver Q* is defined as $|M| = |\bigoplus_{v \in V} M(v)|$.

Lemma 5.2.15. *Let Q be any quiver satisfying (**B**) and let R satisfy (**A**). If $\phi : Y \longrightarrow X$ is an $\mathcal{F}_{cw}$-precovering of X with no non-trivial subrepresentation $S \subseteq Y$ such that $S \subseteq$* $\ker(\phi)$ *and $Y/S \in \mathcal{F}_{cw}$, then this $\mathcal{F}_{cw}$-precover is actually an $\mathcal{F}_{cw}$-cover of X.*

Proof. Let $f : Y \longrightarrow Y$ be a morphism such that $\phi = \phi f$, that is, the diagram

$$\begin{tikzcd} & Y \arrow[dl, "f"'] \arrow[d, "\phi"] \\ Y \arrow[r, "\phi"'] & X \end{tikzcd}$$

is commutative. We will show that f is an automorphism. Since $\ker f \subseteq \ker(\phi)$ and $Y/\ker f \in \mathcal{F}_{cw}$ (as $Y/\ker f \cong \operatorname{Im} f \subseteq Y$ and $\mathcal{F}_{cw}$ is closed under subrepresentations), $\ker f = 0$ by hypothesis, that is, f is one-to-one. Now it remains to show that f is onto. Let A be a set such that $Y \subseteq A$ and $|Y| < |A|$. Let Σ be a set of pairs (Y_0, ϕ_0) such that $\phi_0 : Y_0 \longrightarrow X$ is an $\mathcal{F}_{cw}$-precovering of X without non-trivial subrepresentations $S \subseteq \ker(\phi_0)$ with $Y_0/S \in \mathcal{F}_{cw}$, and $Y_0 \subseteq A$ as a subset. $\Sigma \neq 0$, since $(Y, \phi) \in \Sigma$. Partially order Σ by setting $(Y_0, \phi_0) \subseteq (Y_1, \phi_1)$ if and only if $Y_0 \subseteq Y_1$ and $\phi_1 |_{Y_0} = \phi_0$. Now, for every chain $\{(Y_\alpha, \phi_\alpha)\}_{\alpha \in W}$ of Σ, let $Y^* = \cup_{\alpha \in W} Y_\alpha$ and define $\phi^* : Y^* \longrightarrow X$ by $\phi^*(x) = \phi_\beta(x)$ if $x \in Y_\beta$. Then $(Y^*, \phi^*) \in \Sigma$. In fact, $Y^* \in \mathcal{F}_{cw}$ and $Y^* \subseteq A$ is clear, and if there is $\phi' : F \longrightarrow X$ with $F \in \mathcal{F}_{cw}$, then there exists $g' : F \longrightarrow Y_\beta$, for every $\beta \in W$, such that $\phi_\beta g' = \phi'$. So, taking $f' = ig' : F \longrightarrow Y^*$ (where $i : Y_\beta \longrightarrow Y^*$ is inclusion), we obtain, for every $x \in F$,

$$(\phi^* f')(x) = (\phi^* i g')(x) = (\phi^* g')(x) = \phi_\beta g'(x) = \phi'(x).$$

Thus $\phi^* : Y^* \longrightarrow X$ is an $\mathcal{F}_{cw}$-precovering of X. Beside, if there is a subrepresentation $S \subseteq \ker(\phi^*)$ with $Y^*/S \in \mathcal{F}_{cw}$, then

$$\bigcup_{\alpha \in W} (Y_\alpha + S)/S = Y^*/S \in \mathcal{F}_{cw}, \text{ and so } Y_\alpha/(S \cap Y_\alpha) \cong (Y_\alpha + S)/S \in \mathcal{F}_{cw}.$$

Moreover, $S \cap Y_\alpha \subseteq \ker(\phi_\alpha)$ since $S \cap Y_\alpha \subseteq Y_\alpha$ implies that $\phi_\alpha(S \cap Y_\alpha) = \phi^*(S \cap Y_\alpha) = 0$.

Thus $S \cap Y_\alpha = 0$ for all $\alpha \in W$ (since $(Y_\alpha, \phi_\alpha) \in \Sigma$), and so $S = 0$.

Now let (Y^*, ϕ^*) be a maximal element of Σ which exists by Zorn's lemma. Consider the commutative diagram which exists since ϕ is a $\mathcal{F}_{cw}$-precovering and $Y^* \in \mathcal{F}_{cw}$:

$$\begin{tikzcd} & Y^* \arrow[dl, "f_1"'] \arrow[d, "\phi^*"] \\ Y \arrow[r, "\phi"'] & X . \end{tikzcd}$$

Since $\ker(f_1) \subseteq \ker(\phi^*)$ and $Y^*/\ker(f_1) \in \mathcal{F}_{cw}$, $\ker(f_1) = 0$, that is, f_1 is one-to-one. We will show that f_1 is onto. Suppose on the contrary that f_1 is not onto, that is, $f_1(Y*) \subsetneqq Y$. Then $f_{1v}(Y^*(v)) \subsetneqq Y(v)$ for some vertices $v \in V \setminus V'$ of Q (note that, $f_{1w} : Y^*(w) \longrightarrow Y(w)$ is onto for the remain vertices $w \in V'$ of Q). Let $B \subseteq A$ such that $|B| = |Y(v) - f_{1v}(Y^*(v))|$ and such that $Y^*(v) \cap B = 0$. Such a B is available because $|A| > |Y| = |Y^*| > |Y^*(v)|$ (where the equality holds since $Y \subseteq Y^*$ and $f_1 : Y^* \longrightarrow Y$ is one-to-one). Let $Y_0^v = Y^*(v) \cup B$ and let $g^v : Y_0^v \longrightarrow Y(v)$ be the bijection such that $g^v \mid_{Y^*(v)} = f_{1v}$ and $g^v(B) = Y(v) - f_{1v}(Y^*(v))$ for all $v \in V \setminus V'$. Then Y_0^v can be made uniquely into an R-module so that g^v becomes an R-isomorphism. So we can define a representation Y_0 such that $Y_0(v) = Y_0^v$ if $v \in V \setminus V'$ and $Y_0(v) = Y^*(v)$ if $v \in V'$; and for each arrow $a : v_1 \longrightarrow v_2$ of Q, if $v_1, v_2 \in V \setminus V'$ then:

$$Y_0(a) : Y_0(v_1) \longrightarrow Y_0(v_2) \equiv Y_0^{v_1} \xrightarrow{g^{v_1}} Y(v_1) \xrightarrow{Y(a)} Y(v_2) \xrightarrow{g^{v_2 -1}} Y_0^{v_2}$$

and if $v_1, v_2 \in V'$, then $Y_0(a) = Y^*(a) : Y^*(v_1) \longrightarrow Y^*(v_2)$. So there is an isomorphism $g = (g_v)_{v \in V} : Y_0 \longrightarrow Y$ of representations, where $g_v = g^v$ if $v \in V \setminus V'$ and $g_v = f_{1v}$ if $v \in V'$ (note that, if $v \in V'$ then f_{1v} will be an isomorphism). We see that $Y^* \subseteq Y_0$ as representations. Consider the pair $(Y_0, \phi g)$. Then since g is an isomorphism and $\phi : Y \longrightarrow X$ is an $\mathcal{F}_{cw}$-precovering of X, $\phi g : Y_0 \longrightarrow X$ is also an $\mathcal{F}_{cw}$-precovering of X.

Moreover, if $S \subseteq \ker(\phi g)$ with $Y_0/S \in \mathcal{F}_{cw}$, then $g(S) \subseteq \ker(\phi)$ and

$$Y/g(S) = g(Y_0)/g(S) \cong Y_0/S \in \mathcal{F}_{cw}.$$

So, by hypothesis, $g(S) = 0$ and so $S = 0$. This shows that $(Y_0, \phi g) \in \Sigma$. Finally, $\phi g \mid_{Y^*} = \phi f_1 = \phi^*$, and so $(Y_0, \phi g) \gneqq (Y^*, \phi^*)$. But this is a contradiction since (Y^*, ϕ^*) is maximal in Σ. Thus $B = \emptyset$, and so $|Y(v) - f_{1v}(Y^*(v))| = |B| = 0$. Hence f_{1v} is onto for every vertex v of Q, that is, f_1 is onto. So f_1 is an isomorphism. Now, we have $\phi^* = \phi f_1 = \phi(f f_1)$. So, $f f_1$ is an isomorphism by the same argument. Hence f is onto as desired. □

Theorem 5.2.16. *Let Q be any quiver satisfying (**B**) and let R satisfy (**A**). Then every representation in $(Q, R\text{-}\mathcal{M}od)$ has a unique, up to isomorphism, $\mathcal{F}_{cw}$-cover.*

Proof. The existence part of the proof follows by Proposition 5.2.12 and and Lemmas 5.2.14 and 5.2.15, and the uniqueness part follows by Proposition 2.10.1. □

Example 5.2.17. Let R satisfy (**A**). Consider the quiver $Q \equiv \bullet \longrightarrow \bullet$. For every module M, if we take the torsion free cover $\psi : T \longrightarrow M$ of M (this is possible in R-$\mathcal{M}od$, see Corollary 2.6.4), then

$$\begin{array}{ccc} \overline{T} & & \\ \downarrow{\scriptstyle \overline{\psi}} & & \\ \overline{M} & & \end{array} \qquad \begin{tikzcd} \operatorname{Ker}\psi \arrow[r, "i"] \arrow[d] & T \arrow[d, "\psi"] \\ 0 \arrow[r] & M \end{tikzcd}$$

is an $\mathcal{F}_{cw}$-cover of the representation $0 \longrightarrow M$. In fact, if there is a morphism

$$\begin{tikzcd} T_1 \arrow[r, "\alpha"] \arrow[d] & T_2 \arrow[d, "\beta"] \\ 0 \arrow[r] & M \end{tikzcd}$$

where $T_1 \longrightarrow T_2 \in \mathcal{F}_{cw}$, then there exists $f : T_2 \longrightarrow T$ such that $\psi f = \beta$ since ψ is torsion free precover, and so taking $g = f\alpha : T_1 \longrightarrow \operatorname{Ker}\psi$ (it is well-defined since for every $x \in T_1, \psi f\alpha(x) = \beta\alpha(x) = 0$) we see that it is an $\mathcal{F}_{cw}$-precover. And if there is an endomorphism $\overline{f} = (g, f) : \overline{T} \longrightarrow \overline{T}$ such that $\overline{\psi}\overline{f} = \overline{\psi}$, then f is an automorphism (since ψ is a torsion free cover), and so g is a monomorphism. To show that g is epic, take any $y \in \operatorname{Ker}\psi$. Then $y = f(x)$ for some $x \in T$ (since f is epic). Since $\psi(x) = \psi f(x) = 0$, $x \in \operatorname{Ker}\psi$ and thus $y = f(x) = g(x)$ implies that g is epic. Hence $\overline{f}$ is an automorphism, that is, $\overline{\psi}$ is an $\mathcal{F}_{cw}$-cover.

Remark 5.2.18. In Dunkum (2009), the question was raised whether the category $(A^{\infty}, R\text{-}\mathcal{M}od)$ admits torsion free covers, where

$$A^{\infty} \equiv \bullet \longrightarrow \bullet \longrightarrow \cdots .$$

By Theorem 5.2.16, if R satisfies (**A**), then the category $(A_{\infty}, R\text{-}\mathcal{M}od)$ admits torsion free covers (since the quiver A^{∞} satisfies (**B**)).

5.3 Componentwise Flat Covers in $(Q, R\text{-}\mathcal{M}od)$

In Rump (2010), flat covers are shown to exist in locally finitely presented Grothendieck categories. Then the category $(Q, R\text{-}\mathcal{M}od)$ admits flat covers for every quiver Q since it is a locally finitely presented Grothendieck category. This is because $(Q, R\text{-}\mathcal{M}od)$ has a family of finitely generated projective generators, and thus, by Proposition 2.7.15, $(Q, R\text{-}\mathcal{M}od)$ has a family of finitely presented generators. Here by "flat" we mean *categorical flat* representations of Q defined as $\varinjlim P_i$ where each P_i is a projective representation of Q; see Enochs et al. (2004b).

Now we will define flat representations *componentwise* which are different from *categorical* flat representations.

Definition 5.3.1. Let Q be any quiver and let M be a representation of Q. We call M *componentwise flat* if $M(v)$ is a flat R-module for all $v \in V$.

This definition is not the categorical definition of flat representations, but it is the correct one when we consider $(Q, R\text{-}\mathcal{M}od)$ as the category of presheaves over a topological space. From now on, by $\mathscr{F}_{cw}$ we denote the class of all componentwise flat representations.

Also, let us define *pure subrepresentations* componentwise.

Definition 5.3.2. Let M be a representation of Q. We call a subrepresentation $P \subseteq M$ *componentwise pure* if $P(v) \subseteq M(v)$ is pure submodule for all $v \in V$.

An *element* x of a representation X of a quiver Q is defined to be an element of $X(v)$ for some $v \in V$.

In the proof of the following lemma, we can consider the representation generated by an element "x". Let M be a representation of Q and let $x \in M$ (so $x \in M(v)$ for some $v \in V$). Since S_v is a left adjoint of T_v (see Remark 5.1.10), we have

$$\mathrm{Hom}_R(R, M(v)) \cong \mathrm{Hom}_Q(S_v(R), M)$$

for all $v \in V$. So we have a unique morphism $\varphi : S_v(R) \longrightarrow M$ that corresponds to the homomorphism $\varphi_x : R \longrightarrow M(v)$ given by $\varphi_x(1) = x$. Thus $\mathrm{Im}(\varphi)$ is the subrepresentation of M generated by x.

Lemma 5.3.3. *Let $\aleph$ be an infinite cardinal such that $\aleph \geq \sup\{|R|, |V|, |E|\}$. Let M be a representation of Q. Then for every $x \in M$, there exists a componentwise pure subrepresentation P of M such that $|P| \leq \aleph$ and $x \in P$.*

Proof. Let $x \in M(v)$ with $v \in V$. Then consider the subrepresentation $M^0 \subseteq M$ generated by x. Then $|M^0| \leq \aleph$ since

$$|S_v(R)(w)| = |\bigoplus_{Q(v,w)} R| \leq |V| \cdot |E| \cdot |\mathbb{N}| \cdot |R| \leq \aleph \cdot \aleph_0 = \aleph.$$

Since $|M^0(v)| \leq \aleph$ for all $v \in V$, we can apply Xu (1996, Lemma 2.5.2), so there exist pure submodules $M^1(v)$ of $M(v)$ such that $|M^1(v)| \leq \aleph$ and $M^0(v) \subseteq M^1(v)$, $\forall v \in V$. Now consider the subrepresentation M^2 of M generated by $M^1(v)$ such that $M^1(v) \subseteq M^2(v)$ for all $v \in V$. Then $|M^2| = |\bigoplus_{v \in V} M^2(v)| = |V| \cdot |M^2(v)| \leq \aleph$ since $|M^2(v)| \leq \aleph$ as $|M^1(v)| \leq \aleph$ for all $v \in V$. So applying Xu (1996, Lemma 2.5.2) again, there exist pure submodules $M^3(v)$ of $M(v)$ such that $|M^3(v)| \leq \aleph$ and $M^2(v) \subseteq M^3(v)$ for all $v \in V$. Now consider the subrepresentation M^4 of M generated by $M^3(v)$ such that $M^3(v) \subseteq M^4(v)$. Then $|M^4| \leq \aleph$. So proceed by induction to find a chain of subrepresentations of M: $M^0 \subseteq M^1 \subseteq M^2 \subseteq \cdots$ such that $|M^n| \leq \aleph$ for every $n \in \mathbb{N}$. Therefore, by taking $P = \bigcup_{n<\omega} M^n$ we obtain a pure subrepresentation P of M which satisfies the hypothesis of the lemma. Indeed, P is a componentwise pure subrepresentation of M, because for every $v \in V$, the set $\{n \in \mathbb{N} : M^n(v) \text{ is pure in } M(v)\}$ is cofinal, and the set $\{n \in \mathbb{N} : M^n \text{ is a subrepresentation of } M\}$ is also cofinal. Finally, it is clear that $|P| \leq \aleph_0 \cdot \aleph = \aleph$, and $x \in P$ since $x \in M^0(v)$. $\square$

Let $\mathcal{A}$ be an abelian category. Recall that the pair $(\mathscr{F}, \mathscr{F}^{\perp})$ of classes in $\mathcal{A}$ is *cogenerated* by a set if there exists a set $T \subseteq \mathscr{F}$ such that $T^{\perp} = \mathscr{F}^{\perp}$ (see Definition 2.9.1 or, for example, Enochs & Jenda (2000, Chap.7)).

Theorem 5.3.4. *The pair $(\mathscr{F}_{cw}, \mathscr{F}_{cw}^{\perp})$ is cogenerated by a set.*

Proof. Let $F \in \mathscr{F}_{cw}$ and take any element $x_0 \in F$. Then by Lemma 5.3.3, there exists a componentwise pure subrepresentation $F_0 \subseteq F$ such that $x_0 \in F_0$ and $|F_0| \leq \aleph$ for a suitable cardinal number. Since a pure submodule of a flat module is flat, $F_0 \in \mathscr{F}_{cw}$, and so $F/F_0 \in \mathscr{F}_{cw}$ (see Proposition 2.3.6). Then take any element $x_1 \in F/F_0$ and find a componentwise pure (and so componentwise flat) subrepresentation $F_1/F_0 \subseteq F/F_0$ such that $x_1 \in F_1/F_0$ and $|F_1/F_0| \leq \aleph$. Since $F_0, F_1/F_0 \in \mathscr{F}_{cw}$, we have $F_1 \in \mathscr{F}_{cw}$ and so $F/F_1 \in \mathscr{F}_{cw}$. Now take $x_2 \in F/F_1$ and, since $\mathscr{F}_{cw}$ is closed under direct limits, proceed by transfinite induction to find, when α is a successor ordinal, subrepresentations $F_\alpha \subseteq F$ such that $F_\alpha/F_{\alpha-1} \in \mathscr{F}_{cw}$ (and so $F_\alpha \in \mathscr{F}_{cw}$) and that $|F_\alpha/F_{\alpha-1}| \leq \aleph$. When ω is a limit ordinal, define $F_\omega = \bigcup_{\alpha<\omega} F_\alpha$. So $F_\omega \in \mathscr{F}_{cw}$ and $|F_\omega| \leq \aleph$ for every ω. Now there exists an ordinal λ such that F is a direct union of the continuous chain $\{F_\alpha \mid \alpha < \lambda\}$ where by construction $F_0, F_{\alpha+1}/F_\alpha \in \mathscr{F}_{cw}$ and $|F_0| \leq \aleph, |F_{\alpha+1}/F_\alpha| \leq \aleph$. Thus if we choose a set T of representatives of all componentwise flat representations with cardinality less than or equal to $\aleph$, then by Eklof & Trlifaj (2001, Lemma 1), we see that the pair $(\mathscr{F}_{cw}, \mathscr{F}_{cw}^{\perp})$ is cogenerated by T (note that Eklof & Trlifaj (2001, Lemma 1) is for $R\text{-}\mathcal{M}od$, but the same arguments of the proof carry over general Grothendieck categories; indeed, the proof needs only embeddability of each module into an injective one, so the lemma holds in any Grothendieck category). $\square$

To show that the category $(Q, R\text{-}\mathcal{M}od)$ admits $\mathscr{F}_{cw}$-covers and $\mathscr{F}_{cw}^{\perp}$-envelopes we shall use the following theorem.

Theorem 5.3.5. *(Enochs et al., 2004a, Theorem 2.6) Let $\mathscr{F}$ be a class of objects of a Grothendieck category $\mathcal{C}$ which is closed under direct sums, extensions and well ordered direct limits and such that the generator of $\mathcal{C}$ is in $\mathscr{F}$. If $(\mathscr{F}, \mathscr{F}^{\perp})$ is cogenerated by a set, then every object M in $\mathcal{C}$ has an $\mathscr{F}$-cover and an $\mathscr{F}^{\perp}$-envelope.*

Theorem 5.3.6. *For every quiver Q, any representation of Q has an $\mathscr{F}_{cw}$-cover and an $\mathscr{F}_{cw}^{\perp}$-envelope.*

Proof. It is clear that $\mathscr{F}_{cw}$ is closed under direct sums, extensions and well ordered direct limits (as so is the class of all flat modules). Moreover, $S_v(R)(w) = \bigoplus_{Q(v,w)} R$ is a projective (and so flat) module for all $w \in V$. Thus $S_v(R) \in \mathscr{F}_{cw}$, and so the projective generator $\bigoplus_{v \in V} S_v(R)$ of $(Q, R\text{-}\mathcal{M}od)$ is in $\mathscr{F}_{cw}$. Now, apply Theorem 5.3.5 with Theorem 5.3.4 to get the result. $\square$

Over Prüfer domains, a module is flat if and only if it is torsion free (see Rotman (1979) for the details). Combining this fact with the previous result, we have that:

Theorem 5.3.7. *Let R be a Prüfer domain. Then every representation in $(Q,R\text{-}\mathcal{M}od)$ has an $\mathscr{F}_{cw}$-cover agreeing with its $\mathcal{F}_{cw}$-cover.*

Remark 5.3.8. In Example 5.2.10, since Q does not satisfy (**B**) we cannot use Theorem 5.2.16 to determine whether $(Q,\mathbb{Z}\text{-}\mathcal{M}od)$ admits $\mathcal{F}_{cw}$-covers. However, since $R=\mathbb{Z}$ is a Prüfer domain, $(Q,\mathbb{Z}\text{-}\mathcal{M}od)$ admits $\mathcal{F}_{cw}$-covers by Theorem 5.3.7.

5.4 Comparison of Categorical and Componentwise Flat Covers

In this section, we will provide some examples on the different kinds of covers have been studied throughout the chapter.

The *categorical* flat representations are characterized (for rooted quivers) in Enochs et al. (2004b, Theorem 3.7) as follows: a representation F of a quiver Q is *flat* if and only if $F(v)$ is a flat module and the morphism

$$\bigoplus_{t(a)=v} F(s(a)) \longrightarrow F(v)$$

is a pure monomorphism for every vertex $v \in V$. In this case, as we pointed out at the beginning of Section 5.3, it is known that $(Q,R\text{-}\mathcal{M}od)$ admits categorical flat covers for every quiver Q. Moreover, we have proved in Theorem 5.3.6 that $(Q,R\text{-}\mathcal{M}od)$ also admits $\mathscr{F}_{cw}$-covers (=componentwise flat covers). In this section, we will give some examples of categorical flat covers and of $\mathscr{F}_{cw}$-covers showing that these two kinds of covers do not coincide in general.

Since every module has a flat cover (Bican et al., 2001), every module has a cotorsion envelope by Xu (1996, Theorem 3.4.6).

Example 5.4.1. Let Q be the quiver $\bullet \longrightarrow \bullet$. Let us take any module M and the flat cover $\varphi : F \longrightarrow M$ of it. Then:

(i)

$$\begin{array}{ccc} 0 & \longrightarrow & F \\ \downarrow & & \downarrow \varphi \\ 0 & \longrightarrow & M \end{array}$$

is a flat cover of the representation $0 \longrightarrow M$. To see this, let

$$\begin{array}{ccc} F_1 & \xrightarrow{\alpha} & F_2 \\ \downarrow & & \downarrow \beta \\ 0 & \longrightarrow & M \end{array}$$

be a morphism, where α is a pure monomorphism and F_1, F_2 are flat modules. Then F_2/F_1 is also a flat module. Since φ is a flat cover, there exists $\delta : F_2 \longrightarrow F$ such that $\varphi\delta = \beta$. It is clear that $\varphi\delta\alpha = \beta\alpha = 0$, and so there exists a unique $h : F_1 \longrightarrow \operatorname{Ker}\varphi$ such that $\delta\alpha = ih$, where $i : \operatorname{Ker}\varphi \longrightarrow F$ is inclusion. From the short exact sequence $0 \longrightarrow F_1 \longrightarrow F_2 \longrightarrow F_2/F_1 \longrightarrow 0$, we obtain

$$\operatorname{Hom}_R(F_2, \operatorname{Ker}\varphi) \longrightarrow \operatorname{Hom}_R(F_1, \operatorname{Ker}\varphi) \longrightarrow 0,$$

since $\operatorname{Ext}^1_R(F_2/F_1, \operatorname{Ker}\varphi) = 0$ by Wakamutsu's lemma (see Proposition 2.10.3). So there is $z : F_2 \longrightarrow \operatorname{Ker}\varphi$ such that $z\alpha = h$. Now if we consider $\delta - z : F_2 \longrightarrow F$, then clearly $\varphi(\delta - z) = \beta$ and $(\delta - z)\alpha = 0$.

(ii) If we take the flat cover $f : G \longrightarrow \operatorname{Ker}\varphi$ of $\operatorname{Ker}\varphi$ then

$$\begin{array}{ccc} G & \xrightarrow{t} & F \\ \downarrow & & \downarrow \varphi \\ 0 & \longrightarrow & M \end{array}$$

is an $\mathscr{F}_{cw}$-cover of the representation $0 \longrightarrow M$. In fact, if

$$\begin{array}{ccc} F_1 & \xrightarrow{\alpha} & F_2 \\ \downarrow & & \downarrow \beta \\ 0 & \longrightarrow & M \end{array}$$

is a morphism where $F_1 \longrightarrow F_2 \in \mathscr{F}_{cw}$, then clearly there exists $h : F_2 \longrightarrow F$ such that $\varphi h = \beta$, since φ is a flat cover. Since $\varphi h\alpha = \beta\alpha = 0$, the map $h\alpha : F_1 \longrightarrow \operatorname{Ker}\varphi$ is well-defined. Then there exists $h' : F_1 \longrightarrow G$ such that $fh' = h\alpha$ since f is a flat cover, and so $h\alpha = th'$. This shows that $\{0, \varphi\}$ is an $\mathscr{F}_{cw}$-precover. To see that it is a cover, suppose there is an endomorphism

$$\begin{tikzcd} G \arrow[r, "t"] \arrow[d, "g"'] & F \arrow[d, "g'"] \\ G \arrow[r, "t"] & F \end{tikzcd}$$

such that $0g = 0$ and $\varphi g' = \varphi$. Then clearly g' is an automorphism since φ is a flat cover. Now we show that g is also an automorphism. Since $\varphi g' i = \varphi i = 0$, there exists $\psi : \operatorname{Ker}\varphi \longrightarrow \operatorname{Ker}\varphi$ where $i : \operatorname{Ker}\varphi \longrightarrow F$ an inclusion monomorphism (see Definition 2.7.4). Actually, ψ is an automorphism (see the comment of g being an automorphism in Example 5.2.17), and so from the commutative diagram

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d, "g"'] & \operatorname{Ker}\varphi \arrow[d, "\psi"] \\ G \arrow[r, "f"] & \operatorname{Ker}\varphi \end{tikzcd}$$

we obtain that g is also an automorphism (by using the fact that f is a cover).

Remark 5.4.2. Note that in the previous example $0 \longrightarrow F$ cannot be an $\mathscr{F}_{cw}$-precover of $0 \longrightarrow M$. Because by (ii), $G \longrightarrow F$ is an $\mathscr{F}_{cw}$-cover of $0 \longrightarrow M$ with $G \neq 0$ and it is known that covers are direct summand of precovers (see Proposition 2.10.2). So, if $0 \longrightarrow F$ were an $\mathscr{F}_{cw}$-precover of $0 \longrightarrow M$, then we would have

$$(0 \longrightarrow F) = (G \longrightarrow F) \oplus (H_1 \longrightarrow H_2) = (G \oplus H_1 \longrightarrow F \oplus H_2)$$

for some representation $H_1 \longrightarrow H_2$ of Q. This implies that $0 = G \oplus H_1$ which contradicts the fact that $G \neq 0$.

Remark 5.4.3. Comparing with Example 5.2.17; $\mathrm{Ker}\,\varphi \longrightarrow F$ is a torsion free cover but not an $\mathscr{F}_{cw}$-cover of $0 \longrightarrow M$ (unless $\mathrm{Ker}\,\varphi$ is a flat module). Because the class of torsion free modules is closed under submodules, but the class of flat modules is not.

Example 5.4.4. Let Q be the quiver $\bullet \longrightarrow \bullet$. Let us take any module M and the flat cover $\varphi : F \longrightarrow M$ of it. Then,

$$\begin{tikzcd} F \arrow[r, "id"] \arrow[d, "\varphi"'] & F \arrow[d, "\varphi"] \\ M \arrow[r, "id"] & M \end{tikzcd}$$

is both a (categorical) flat cover and an $\mathscr{F}_{cw}$-cover of the representation $M \xrightarrow{id} M$. In fact, if there is a morphism

$$\begin{tikzcd} F_1 \arrow[r, "h"] \arrow[d, "\psi_1"'] & F_2 \arrow[d, "\psi_2"] \\ M \arrow[r, "id"] & M \end{tikzcd}$$

where F_1, F_2 are flat modules and h is a pure monomorphism, then clearly there is $f : F_2 \longrightarrow F$ such that $\varphi f = \psi_2$ (since φ is a flat cover). Taking $fh : F_1 \longrightarrow F$, we see that $\varphi fh = \psi_2 h = \psi_1$. This means that $F \xrightarrow{id} F$ is a flat precover, and clearly it is a flat cover (since id_F is a pure monomorphism). Since we have not used the fact that h is pure, then $F \xrightarrow{id} F$ is also an $\mathscr{F}_{cw}$-cover of $M \xrightarrow{id} M$.

Example 5.4.5. Let Q be the quiver $\bullet \longrightarrow \bullet \longrightarrow \bullet$ and let M be a module. Let us take the flat cover $\varphi : F \longrightarrow M$ of M. Then:

(i) If we take the cotorsion envelope $i : F \longrightarrow C$ of F, then C will be a flat module by Xu (1996, Theorem 3.4.2)). Therefore, we have a (categorical) flat representation $\overline{F} \equiv F \xrightarrow{i} C \xrightarrow{k_1} C \times F$ where k_1 is a canonical inclusion (since $C \times F$ is flat, and k_1 and i are pure monomorphisms). We show that

$$\begin{tikzcd} \overline{F} \arrow[d, "\overline{\varphi}"'] & F \arrow[r, "i"] \arrow[d, "\varphi"] & C \arrow[r, "k_1"] \arrow[d, "0"] & C \times F \arrow[d, "\varphi p_2"] \\ \overline{M} & M \arrow[r] & 0 \arrow[r] & M \end{tikzcd}$$

is a flat cover of the representation $\overline{M}$ of Q, where $p_2 : C \times F \longrightarrow F$ is a projection. In fact, if there is a morphism

$$\begin{tikzcd} F_1 \arrow[r, "\alpha"] \arrow[d, "t_1"] & F_2 \arrow[r, "\beta"] \arrow[d, "0"] & F_3 \arrow[d, "t_3"] \\ M \arrow[r] & 0 \arrow[r] & M \end{tikzcd}$$

with F_1, F_2, F_3 flat modules and α, β pure monomorphisms, then clearly there exists $f : F_1 \longrightarrow F$ such that $\varphi f = t_1$ since F is a flat cover of M. From the short exact sequence $0 \longrightarrow F_1 \longrightarrow F_2 \longrightarrow F_2/F_1 \longrightarrow 0$, we obtain that

$$\operatorname{Hom}_R(F_2, C) \longrightarrow \operatorname{Hom}_R(F_1, C) \longrightarrow 0$$

is exact, since $\operatorname{Ext}^1_R(F_2/F_1, C) = 0$ (as F_2/F_1 is flat and C is cotorsion). Here F_2/F_1 is flat because α is a pure monomorphism. So, there exists $g : F_2 \longrightarrow C$ such that $g\alpha = if$. Now, since F is a flat cover of M, there exists $\tau_2 : F_3 \longrightarrow F$ such that $\varphi\tau_2 = t_3$. Moreover, from the short exact sequence

$$0 \longrightarrow F_2 \longrightarrow F_3 \longrightarrow F_3/F_2 \longrightarrow 0 \tag{5.4.1}$$

we obtain that

$$\operatorname{Hom}_R(F_3, C) \longrightarrow \operatorname{Hom}_R(F_2, C) \longrightarrow 0$$

is exact. Then there exists $\tau_1 : F_3 \longrightarrow C$ such that $\tau_1\beta = g$. Since $\varphi\tau_2\beta = t_3\beta = 0$, there exists a unique $\gamma : F_2 \longrightarrow \operatorname{Ker}\varphi$ such that $\gamma = \tau_2\beta$. Similarly, if we take $\operatorname{Ker}\varphi$ instead of C, by (5.4.1), there exists $z : F_3 \longrightarrow \operatorname{Ker}\varphi$ such that $z\beta = \gamma$. Therefore, by defining $h : F_3 \longrightarrow C \times F$ such that $h(x) = \big(\tau_1(x), (\tau_2 - z)(x)\big)$ for all $x \in F_3$, we see that $\varphi p_2 h = \varphi(\tau_2 - z) = t_3$, and moreover $h\beta = \big(\tau_1\beta, (\tau_2 - z)\beta\big) = (g, 0) = k_1 g$. Thus $\overline{\varphi} : \overline{F} \longrightarrow \overline{M}$ is a flat precover. To see that it is a cover, let $\overline{s} = \{f, g, h\} : \overline{F} \longrightarrow \overline{F}$ be an endomorphism such that $\overline{\varphi}\overline{s} = \overline{\varphi}$. It is clear that f and g are automorphisms. For $h : C \times F \longrightarrow C \times F$, we set $h_{ij} = \pi_i h e_j$ ($i, j = 1, 2$) where π_k is a projection and e_k is an injection. We can write h in a matrix form as

$$\begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}.$$

Since $hk_1 = k_1 g$, we have $h_{11} = g$ and $h_{21} = 0$, and since φ is a cover, $\varphi = \varphi h_{22}$ implies that h_{22} is an automorphism. Hence h is an automorphism and so is $\bar{s}$, that is, $\overline{\varphi}$ is a flat cover of $\overline{M}$.

(ii) If we take the flat cover $f : G \longrightarrow \operatorname{Ker}\varphi$ of $\operatorname{Ker}\varphi$, then it is immediate that

$$\begin{tikzcd} F \arrow[r, "0"] \arrow[d, "\varphi"'] & G \arrow[r, "t"] \arrow[d, "0"'] & F \arrow[d, "\varphi"'] \\ M \arrow[r, "0"] & 0 \arrow[r, "0"] & M \end{tikzcd}$$

is an $\mathscr{F}_{cw}$-cover of the representation $M \longrightarrow 0 \longrightarrow M$. In fact,

$$\begin{tikzcd} G \arrow[r, "t"] \arrow[d, "0"'] & F \arrow[d, "\varphi"] \\ 0 \arrow[r] & M \end{tikzcd}$$

is an $\mathscr{F}_{cw}$-cover of the representation $0 \longrightarrow M$ (by Example 5.4.1-(ii)), and

$$\begin{tikzcd} F \arrow[r] \arrow[d, "\varphi"'] & 0 \arrow[d, "0"] \\ M \arrow[r] & 0 \end{tikzcd}$$

is an $\mathscr{F}_{cw}$-cover of the representation $M \longrightarrow 0$.

CHAPTER SIX

CONCLUSIONS

In the first part of this thesis, we focused on Rad-supplemented modules and in general τ-supplemented modules, where τ is a radical for $R\text{-}\mathcal{M}od$. Our main result is that every left R-module is Rad-supplemented if and only if $R/P(R)$ is a left perfect ring, where $P(R)$ consists of all left ideals I of R such that $\operatorname{Rad} I = I$. We devote the second part to neat homomorphisms of Enochs and max-injective modules. We prove that every max-injective R-module is injective if and only if the ring R is a left C-ring, with our interest in the proper class $\mathcal{C}ompl$ and $\mathcal{N}eat$. In the last part of the thesis, we deal with the category of representations by modules of a quiver, denoted by $(Q, R\text{-}\mathcal{M}od)$. We show the existence of torsion free covers, relative to a torsion theory, for a wide class of quivers under some conditions, in $(Q, R\text{-}\mathcal{M}od)$. We also prove the existence of componentwise flat covers in $(Q, R\text{-}\mathcal{M}od)$ for any ring R and any quiver Q.

REFERENCES

Al-Takhman, K., Lomp, C., & Wisbauer, R. (2006). τ-complemented and τ-supplemented modules. *Algebra and Discrete Mathematics*, (3), 1–15.

Aldrich, S. T., Enochs, E. E., García Rozas, J. R., & Oyonarte, L. (2001). Covers and envelopes in Grothendieck categories: flat covers of complexes with applications. *J. Algebra*, 243(2), 615–630.

Alizade, R., Bilhan, G., & Smith, P. F. (2001). Modules whose maximal submodules have supplements. *Comm. Algebra*, 29(6), 2389–2405.

Alizade, R., & Mermut, E. (2004). The inductive closure of supplements. *Journal of the Faculty of Science Ege University*, 27, 33–48.

Anderson, F. W., & Fuller, K. R. (1992). *Rings and Categories of Modules*. New-York: Springer.

Assem, I., Simson, D., & Skowroński, A. (2006). *Elements of the representation theory of associative algebras. Vol. 1*, vol. 65 of *London Mathematical Society Student Texts*. Cambridge: Cambridge University Press, ISBN 978-0-521-58423-4; 978-0-521-58631-3; 0-521-58631-3. Techniques of representation theory.

Auslander, M., Reiten, I., & O., S. (1995). *Representation theory of Artin algebras*, vol. 36 of *Cambridge Studies in Advanced Mathematics*. Cambridge: Cambridge University Press, ISBN 0-521-41134-3.

Auslander, M., & Smalø, S. O. (1980). Preprojective modules over Artin algebras. *J. Algebra*, 66(1), 61–122.

Azumaya, G. (1992). A cracterization of semi-perfect rings and modules. In *"Ring Theory," Proc. Biennial Ohio-Denison Conf.*, pp. 28–40, World Scientific Publ., Singapore.

Bass, H. (1960). Finitistic dimension and a homological generalization of semi-primary rings. *Trans. Amer. Math. Soc.*, 95, 466–488.

Bican, L., El Bashir, R., & Enochs, E. (2001). All modules have flat covers. *Bull. London Math. Soc.*, 33(4), 385–390.

Bowe, J. J. (1972). Neat homomorphisms. *Pacific J. Math.*, 40, 13–21.

Buschbaum, D. (1959). A note on homology in categories. *Ann. of Math.*, 69(1), 66–74.

Büyükaşık, E., & Lomp, C. (2008). On a recent generalization of semiperfect rings. *Bull. Aust. Math. Soc.*, 78(2), 317–325.

Cartan, H., & Eilenberg, S. (1956). *Homological Algebra*. Princeton Landmarks in Mathematics and Physics series, New Jersey: Princeton Univesity, 13th ed.

Clark, J., Lomp, C., Vanaja, N., & Wisbauer, R. (2006). *Lifting modules*. Frontiers in Mathematics, Basel: Birkhäuser Verlag. Supplements and projectivity in module theory.

Cohn, P. M. (1959). On the free product of associative rings. *Math. Z.*, 71, 380–398.

Crawley-Boevey, W. (1994). Locally finitely presented additive categories. *Comm. Algebra*, 22(5), 1641–1674.

Crivei, S. (1998). *m*-injective modules. *Mathematica*, 40(63)(1), 71–78.

Crivei, S. (2000). Modules injective with respect to maximal ideals. *An. Ştiinţ. Univ. Ovidius Constanţa Ser. Mat.*, 8(2), 13–19. Euroconference on Algebra and Representation Theory (Constantza, 2000).

Crivei, S. (2004). *Injective Modules Relative to Torsion Theories*. Cluj-Napoca, Romania: Editura Fundatiei Pentru Studii Europene, ISBN 973-8254-51-5.

Dickson, S. E. (1966). A torsion theory for Abelian categories. *Trans. Amer. Math. Soc.*, 121, 223–235.

Ding, N. Q., & Chen, J. L. (1993). The flat dimensions of injective modules. *Manuscripta Math.*, 78(2), 165–177.

Dung, N. V., Huynh, D., Smith, P. F., & Wisbauer, R. (1994). *Extending Modules*. No. 313 in Putman Research Notes in Mathematics Series, Harlow: Longman.

Dunkum, M. (2009). A generalization of Baer's lemma. *Czechoslovak Math. J.*, 59(134)(1), 241–247.

Eckmann, B., & Schopf, A. (1953). Über injektive Moduln. *Arch. Math. (Basel)*, 4, 75–78.

Eklof, P. C., & Shelah, S. (1994). A combinatorial principle equivalent to the existence of non-free Whitehead groups. In *Abelian group theory and related topics (Oberwolfach, 1993)*, vol. 171 of *Contemp. Math.*, pp. 79–98, Providence, RI: Amer. Math. Soc.

Eklof, P. C., & Trlifaj, J. (2001). How to make Ext vanish. *Bull. London Math. Soc.*, 33(1), 41–51.

Enochs, E. (1963). Torsion free covering modules. *Proc. Amer. Math. Soc.*, 14, 884–889.

Enochs, E., & Estrada, S. (2005a). Projective representations of quivers. *Comm. Algebra*, 33(10), 3467–3478.

Enochs, E., & Estrada, S. (2005b). Relative homological algebra in the category of quasi-coherent sheaves. *Adv. Math.*, 194(2), 284–295.

Enochs, E., Estrada, S., & García Rozas, J. R. (2009). Injective representations of infinite quivers. applications. *Canad. J. Math.*, 61, 315–335.

Enochs, E., Estrada, S., García Rozas, J. R., & Iacob, A. (2007). Gorenstein quivers. *Arch. Math. (Basel)*, 88(3), 199–206.

Enochs, E., Estrada, S., García Rozas, J. R., & Oyonarte, L. (2003a). Flat covers of representations of the quiver a_∞. *Int. J. Math. Math.Sci.*, 70, 4409–4419.

Enochs, E., Estrada, S., García Rozas, J. R., & Oyonarte, L. (2003b). Generalized quasi-coherent sheaves. *J. Algebra Appl.*, 2(1), 63–83.

Enochs, E., Estrada, S., García Rozas, J. R., & Oyonarte, L. (2004a). Flat covers in the category of quasi-coherent sheaves over the projective line. *Comm. Algebra*, 32(4), 1497–1508.

Enochs, E., García Rozas, J. R., Oyonarte, L., & Park, S. (2002). Noetherian quivers. *Quaest. Math.*, 25(4), 531–538.

Enochs, E., Oyonarte, L., & Torrecillas, B. (2004b). Flat covers and flat representations of quivers. *Comm. Algebra*, 32(4), 1319–1338.

Enochs, E. E. (1971). Torsion free covering modules. II. *Arch. Math. (Basel)*, 22, 37–52.

Enochs, E. E. (1981). Injective and flat covers, envelopes and resolvents. *Israel J. Math.*, 39(3), 189–209.

Enochs, E. E., & Herzog, I. (1999). A homotopy of quiver morphisms with applications to representations. *Canad. J. Math.*, 51, 294–308.

Enochs, E. E., & Jenda, O. M. G. (2000). *Relative homological algebra*, vol. 30 of *de Gruyter Expositions in Mathematics*. Berlin: Walter de Gruyter & Co., ISBN 3-11-016633-X.

Erdoğan, E. (2004). *Absolutely supplement and absolutely complement modules*. M.

Sc. Thesis, İzmir Institute of Technology, İzmir.

Estrada, S. (2003). *Cubiertas y Envolventes en Categorias de Representatciones*. Ph.D. thesis, Universidad de Almería, Departamento de Álgebra y Analisis Matemático, Almería, SPAIN.

Facchini, A. (1998). *Module theory*, vol. 167 of *Progress in Mathematics*. Basel: Birkhäuser Verlag, ISBN 3-7643-5908-0. Endomorphism rings and direct sum decompositions in some classes of modules.

Freyd, P. (1964). *Abelian categories. An introduction to the theory of functors*. Harper's Series in Modern Mathematics, New York: Harper & Row Publishers.

Fuchs, L. (1960). Notes on abelian groups. II. *Acta Math. Acad. Sci. Hungar.*, 11, 117–125.

Fuchs, L. (1970). *Infinite Abelian Groups*, vol. 1. New York: Academic Press.

Fuchs, L. (2010). Neat submodules over integral domains. *In preparation.*

Fuchs, L., & Salce, L. (2001). *Modules over non-Noetherian domains*, vol. 84 of *Mathematical Surveys and Monographs*. Providence, RI: American Mathematical Society.

Gabriel, P. (1972a). Indecomposable representations i. *Manuscripta Math.*, 6, 71–103.

Gabriel, P. (1972b). Unzerlegbare Darstellungen. I. *Manuscripta Math.*, 6, 71–103;

correction, ibid. 6 (1972), 309.

Generalov, A. I. (1972). On the definition of purity of modules. *Math. Notes*, 11, 232–235. Translated from Russian from *Mat. Zametki* 11(4), 375-380 (1972).

Generalov, A. I. (1978). On weak and *w*-high purity in the category of modules. *Math. USSR, Sb.*, 34, 345–356. Translated from Russian from *Mat. Sb., N. Ser.* 105(147), 389-402 (1978).

Generalov, A. I. (1983). The *w*-cohigh purity in a category of modules. *Math. Notes*, 33(5-6), 402–408. Translated from Russian from *Mat. Zametki* 33(5), 785-796 (1983).

Golan, J. S. (1986). *Torsion theories*, vol. 29 of *Pitman Monographs and Surveys in Pure and Applied Mathematics*. Harlow: Longman Scientific & Technical, ISBN 0-582-99808-5.

Golan, J. S., & Teply, M. L. (1973). Torsion-free covers. *Israel J. Math.*, 15, 237–256.

Harrison, D., Irwin, J. M., Peercy, C. L., & Walker, E. A. (1963). High extensions of abelian groups. *Acta Math. Acad. Sci. Hunger.*, 14, 319–330.

Harrison, D. K. (1959). Infinite abelian groups and homological methods. *Ann. of Math. (2)*, 69, 366–391.

Hartshorne, R. (1977). *Algebraic geometry*. New York: Springer-Verlag, ISBN 0-387-90244-9. Graduate Texts in Mathematics, No. 52.

Honda, K. (1956). Realism in the theory of abelian groups I. *Comment. Math. Univ. St. Paul*, 5, 37–75.

Kaplansky, I. (1952). Modules over Dedekind rings and valuation rings. *Trans. Amer. Math. Soc.*, 72, 327–340.

Kasch, F. (1982). *Modules and rings*, vol. 17 of *London Mathematical Society Monographs*. London: Academic Press Inc. [Harcourt Brace Jovanovich Publishers]. Translated from the German and with a preface by D. A. R. Wallace.

Kiełpiński, R. (1967). On Γ-pure injective modules. *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.*, 15, 127–131.

Koşan, M. T. (2007). τ-supplemented modules and τ-weakly supplemented modules. *Arch. Math. (Brno)*, 43(4), 251–257.

Koşan, T., & Harmanci, A. (2004). Modules supplemented relative to a torsion theory. *Turkish J. Math.*, 28(2), 177–184.

Lam, T. Y. (1999). *Lectures on modules and rings*, vol. 189 of *Graduate Texts in Mathematics*. New York: Springer-Verlag.

Lam, T. Y. (2001). *A first course in noncommutative rings*, vol. 131 of *Graduate Texts in Mathematics*. New York: Springer-Verlag, 2nd ed.

Lang, S. (2002). *Algebra*, vol. 211 of *Graduate Texts in Mathematics*. New York: Springer-Verlag, 3rd ed., ISBN 0-387-95385-X.

Lomp, C. (1999). On semilocal modules and rings. *Comm. Algebra*, 27(4), 1921–1935.

Maclane, S. (1963). *Homology*. Berlin-Göttingen-Heidelberg: Springer-Verlag.

Maranda, J.-M. (1964). Injective structures. *Trans. Amer. Math. Soc.*, 110, 98–135.

Mermut, E. (2004). *Homological Approach to Complements and Supplements*. Ph.D. thesis, Dokuz Eylül University, The Graduate School of Natural and Applied Sciences, İzmir/Turkey.

Mishina, A. P., & Skornyakov, L. A. (1976). *Abelian groups and modules*, vol. 107 of *American Mathematical Society Translations. Ser. 2*. Providence, R. I.: American Mathematical Society. Translated from Russian from *Abelevy gruppy i moduli*, Izdat. Nauka, Moscow (1969).

Mitchell, B. (1972). Rings with several objects. *Advances in Math.*, 8, 1–161.

Nakahara, S. (1983). On a generalization of semiperfect rings. *Osaka J. Math.*, 20(1), 43–50.

Nunke, R. J. (1959). Modules of extensions over dedekind rings. *Illunois J. of Math.*, 3, 222–241.

Özdemir, S. (2007). *Neat and Coneat Subgroups*. M. Sc. Thesis, Dokuz Eylül University, The Graduate School of Natural and Applied Sciences, İzmir/Turkey.

Renault, G. (1964). Étude de certains anneaux a liés aux sous-modules compléments dun a-module. *C. R. Acad. Sci. Paris*, 259, 4203–4205.

Rotman, J. (1979). *An Introduction to Homological Algebra*. New York: Academic Press.

Rotman, J. J. (2009). *An introduction to homological algebra*. Universitext, New York: Springer, 2nd ed., ISBN 978-0-387-24527-0.

Rump, W. (2010). Flat covers in abelian and in non-abelian categories. *Adv. Math.*, 225(3), 1589–1615.

Salce, L. (1979). Cotorsion theories for abelian groups. In *Symposia Mathematica, Vol. XXIII (Conf. Abelian Groups and their Relationship to the Theory of Modules, INDAM, Rome, 1977)*, pp. 11–32, London: Academic Press.

Sharpe, D. W., & Vámos, P. (1972). *Injective modules*. London: Cambridge University Press. Cambridge Tracts in Mathematics and Mathematical Physics, No. 62.

Sklyarenko, E. G. (1978). Relative homological algebra in categories of modules. *Russian Math. Surveys*, 33(3), 97–137. Traslated from Russian from *Uspehi Mat. Nauk* 33, no. 3(201), 85-120 (1978).

Smith, P. F. (1981). Injective modules and prime ideals. *Comm. Algebra*, 9(9), 989–999.

Stenström, B. T. (1967a). Pure submodules. *Arkiv för Matematik*, 7(10), 159–171.

Stenström, B. T. (1967b). High submodules and purity. *Arkiv för Matematik*, 7(11), 173–176.

Stenström, B. (1968). Purity in functor categories. *J. Algebra*, 8, 352–361.

Stenström, B. (1975). *Rings of quotients*. New York: Springer-Verlag. Die Grundlehren der Mathematischen Wissenschaften, Band 217, An introduction to methods of ring theory.

Teply, M. L. (1969). Torsionfree injective modules. *Pacific J. Math.*, 28, 441–453.

Teply, M. L. (1976). Torsion-free covers. II. *Israel J. Math.*, 23(2), 132–136.

Trlifaj, J. (1996). Whitehead test modules. *Trans. Amer. Math. Soc.*, 348(4), 1521–1554.

Wang, M. Y., & Zhao, G. (2005). On maximal injectivity. *Acta Math. Sin. (Engl. Ser.)*, 21(6), 1451–1458.

Wang, Y., & Ding, N. (2006). Generalized supplemented modules. *Taiwanese J. Math.*, 10(6), 1589–1601.

Warfield, R. B. (1969). Purity and algebraic compactness for modules. *Pac. J. Math.*, 28, 699–719.

Wisbauer, R. (1991). *Foundations of Module and Ring Theory*. Reading: Gordon and Breach.

Xu, J. (1996). *Flat covers of modules*, vol. 1634 of *Lecture Notes in Mathematics*. Berlin: Springer-Verlag, ISBN 3-540-61640-3.

Xue, W. (1996). Characterization of semiperfect and perfect rings. *Publicacions Matematiques*, 40(1), 115–125.

Zöschinger, H. (1974). Komplementierte Moduln über Dedekindringen. *J. Algebra*, 29, 42–56.

Zöschinger, H. (1978). Über Torsions- und κ-Elemente von Ext(C,A). *J. Algebra*, 50(2), 299–336.

Zöschinger, H. (1980). Koatomare Moduln. *Math. Z.*, 170(3), 221–232.

Zöschinger, H. (2006). Schwach-injektive Moduln. *Period. Math. Hungar.*, 52(2), 105–128.

NOTATION

R	an associative ring with unit unless otherwise stated
$R[x]$	the ring of polynomials with indeterminate x
$\mathbb{Z}$, $\mathbb{N}$	the ring of integers, the set of all positive integers
$\mathbb{Q}$	the field of rational numbers
R-module or module	a *left* R-module
R-$\mathcal{M}od$, $\mathcal{M}od$-R	the categories of *left* R-modules, *right* R-modules
$\mathcal{A}b = \mathbb{Z}$-$\mathcal{M}od$	the categeory of abelian groups ($\mathbb{Z}$-modules)
$\cong$	isomorphic
$\subseteq$, $\underset{max.}{\subseteq}$	submodule, maximal submodule
$\ll$	small (=superfluous) submodule
$\trianglelefteq$	essential submodule
$K \underset{M}{\overset{cs}{\hookrightarrow}} L$	cosmall inclusion in M
$M \overset{cc}{\hookrightarrow} N$	coclosed submodule
$M \otimes_R N$	the tensor product of the *right* R-module M and the *left* R-module N
$\mathrm{Hom}_R(M,N)$	all R-module homomorphisms from M to N
$\mathrm{Ext}^1_R(C,A)$	the equivalence classes of short exact sequences of R-modules starting with A and ending with C
$\mathrm{Ker}\, f$	the kernel of the homomorphism f
$\mathrm{Coker}\, f$	the cokernel of the homomorphism f
$\mathrm{Im}\, f$	the image of the homomorphism f
$\mathrm{E}(M)$	the injective envelope (hull) of a module M
$\mathrm{Soc}\, M$	the socle of the R-module M
$\mathrm{Rad}\, M$	the radical of the R-module M
$Z(M)$	the singular submodule of the R-module M

$M^\flat$	the character module, that is, the *right* R-module $\mathrm{Hom}_{\mathbb{Z}}(M,\mathbb{Q}/\mathbb{Z})$ where M is a left R-module
$\mathrm{Jac}\,R$ or $J(R)$	the Jacobson radical of the ring R
$\mathrm{Ann}_R(X)$	$(0:X)=\{r\in R \mid rX=0\}=$ the *left* annihilator of a subset X of a *left* R-module M
$\mathrm{Ann}_R(x)$	$(0:x)=\{r\in R \mid rx=0\}=$ the *left* annihilator of an element x of a *left* R-module M
τ	a preradical for the category R-$\mathcal{M}od$
$P_\tau(M)$	the largest τ-torsion submodule of the R-module M, that is, $P_\tau(M)=\sum\{U\subseteq M \mid \tau(U)=U\}$
$(\mathcal{T},\mathcal{F})$	a torsion theory with the torsion class $\mathcal{T}$ and the torsion free class $\mathcal{F}$
$F(R)$	a Gabriel filter of left ideals of the ring R
$\mathrm{Obj}(\mathcal{C})$	the class of objects of the category $\mathcal{C}$
$\mathcal{F}^\perp$, ${}^\perp\mathcal{F}$	the right orthogonal class, the left orthogonal class of objects of an abelian category $\mathcal{A}$, that is, $\mathcal{F}^\perp=\{C\in\mathrm{Obj}(\mathcal{A}) \mid \mathrm{Ext}^1_R(F,C)=0, \forall F\in\mathcal{F}\}$, ${}^\perp\mathcal{F}=\{C\in\mathrm{Obj}(\mathcal{A}) \mid \mathrm{Ext}^1_R(C,F)=0, \forall F\in\mathcal{F}\}$
$(\mathcal{F},\mathcal{C})$	a cotorsion theory (or cotorsion pair) in an abelian category, that is, $\mathcal{F}^\perp=\mathcal{C}$ and ${}^\perp\mathcal{C}=\mathcal{F}$
$\mathcal{P}$	a proper class of R-modules
$\pi^{-1}(\mathcal{M})$	the proper class of R-modules projectively generated by a class $\mathcal{M}$ of R-modules
$\iota^{-1}(\mathcal{M})$	the proper class of R-modules injectively generated by a class $\mathcal{M}$ of R-modules
$\tau^{-1}(\mathcal{M})$	the proper class of R-modules flatly generated by a class $\mathcal{M}$ of *right* R-modules

$\mathcal{S}plit_R$	the smallest proper class of R-modules consisting of *only splitting* short exact sequences of R-modules
$\mathcal{A}bs_R$	the largest proper class of R-modules consisting of *all* short exact sequences of R-modules (absolute purity)
$\mathcal{P}ure_{\mathbb{Z}}$	the proper class of pure-exact sequences of abelian groups
$\mathcal{N}eat_{\mathbb{Z}}$	the proper class of neat-exact sequences of abelian groups
$\mathcal{P}ure_R$	the proper class of pure-exact sequences of R-modules (Cohn's purity)
$\mathcal{C}ompl$	the proper class of R-modules determined by complement submodules
$\mathcal{S}uppl$	the proper class of R-modules determined by supplement submodules
$\mathcal{N}eat$	the proper class of R-modules determined by neat submodules
$\mathcal{C}o\text{-}\mathcal{N}eat$	the proper class of R-modules determined by coneat submodules
$\mathcal{C}oclosed$	the proper class of R-modules determined by coclosed submodules
$\mathcal{E}\mathcal{N}eat$	the class of all short exact sequences of R-modules defined by E-neat epimorphisms
$\mathcal{I}nj$	the class of all injective modules
$\mathcal{P}roj$	the class of all projective modules
$\bigoplus_{i\in I} C_i$	coproduct (or direct sum) in a preadditive category
$\prod_{i\in I} C_i$	product in a preadditive category
$Q=(V,E)$	a quiver with the set of vertices V and the set of arrows E
$v \xrightarrow{a} w$	an arrow of a quiver with starting vertex $s(a)=v$ and terminal vertex $t(a)=w$

$X(v)$	a module assigned to the vertex v of Q in a representation X of Q
$(Q,R\text{-}\mathcal{M}od)$	the category of representations by modules of Q over R
the property **(A)**	**(A)**: for a torsion theory for $R\text{-}\mathcal{M}od$, a direct sum of torsion free injective modules is injective
the property **(B)**	**(B)**: $\{t(a) \mid a \in E \text{ and } s(a) = v\} < \infty$, for every vertex v of a quiver $Q = (V,E)$
$SIRQ$	the conditions for being a source injective representation quiver
RQ	the path ring of Q, that is, a free left R-module whose base are the paths of Q (it may not have an identity element). It is a ring with enough idempotents and it has an identity element if and only if the set of vertices of Q is finite.
$RQ\text{-}\mathcal{M}od$	the category of unital RQ-modules (i.e. ${}_{RQ}M$ such that $RQM = M$)
$Q(v,w)$	the set of all paths of Q starting at v and ending at w
S_v	the Mitchell's functor $S_v : R\text{-}\mathcal{M}od \longrightarrow (Q,R\text{-}\mathcal{M}od)$, defined as $S_v(M)(w) = \bigoplus_{p \in Q(v,w)} M$; for each arrow $a : w_1 \to w_2$, $S_v(M)(a) : \bigoplus_{q \in Q(v,w_1)} M \longrightarrow \bigoplus_{p \in Q(v,w_2)} M$ given by $(m_q)_q \mapsto (u_p)_p$, where $u_p = m_q$ if $p \in aQ(v,w_1)$ and $u_p = 0$ otherwise
T_v	the restriction functor $T_v : (Q,R\text{-}\mathcal{M}od) \longrightarrow R\text{-}\mathcal{M}od$, $T_v(X) = X(v)$ for each $v \in V$, and for each morphism $\eta : X \longrightarrow Y$ in $(Q,R\text{-}\mathcal{M}od)$, $T_v(\eta) = \eta_v : X(v) \longrightarrow Y(v)$
$\mathrm{Hom}_{\mathcal{C}}(A,B)$	the set of all morphisms from A to B in the category $\mathcal{C}$
$\mathrm{Hom}_Q(X,Y)$	the set of all morphisms between the representations X and Y in the category $(Q,R\text{-}\mathcal{M}od)$
$\mathrm{Ext}_{\mathcal{C}}(C,A)$	the equivalence classes of short exact sequences in an abelian category $\mathcal{C}$ starting with A and ending with C

$\ker f$	the kernel of a morphism $f : A \to B$ in an abelian category; it is a monomorphism $\ker f : \operatorname{Ker} f \to A$, where $\operatorname{Ker} f$ is a subobject of A
$\operatorname{coker} f$	the cokernel of a morphism $f : A \to B$ in an abelian category; it is an epimorphism $\operatorname{coker} f : B \to \operatorname{Coker} f$, where $\operatorname{Coker} f$
$\varinjlim$	colimit or direct limit
$\varprojlim$	limit or inverse limit
$\|X\|$	cardinality of a set X

INDEX