DOKUZ EYLÜL UNIVERSITY

GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

OPTIMIZATION OF LOSS PROBABILITY IN THE *GI*/*M*/*n*/0 QUEUEING MODEL WITH HETEROGENEOUS SERVERS

by

Hanifi Okan İŞGÜDER

March, 2013

İZMİR

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PH.D. THESIS EXAMINATION RESULT FORM

We have read the thesis entitled **"OPTIMIZATION OF LOSS PROBABILITY IN THE *GI/M/n/0* QUEUEING MODEL WITH HETEROGENEOUS SERVERS"** completed by **HANİFİ OKAN İŞGÜDER** under supervision of **PROF. DR. CAN CENGİZ ÇELİKOĞLU** and we certify that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Doctor of Philosophy.

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OPTIMIZATION OF LOSS PROBABILITY IN THE *GI/M/n/0* QUEUEING MODEL WITH HETEROGENEOUS SERVERS

ABSTRACT

This study is mainly concerned with the finite-capacity queueing system with recurrent input, n heterogeneous servers, and no waiting line represented by $GI/M/n/0$. The service discipline is addressed in two different ways. Firstly, customers choose only one server from the empty servers with equal probability. Secondly, customers choose the server with the lowest index number among the empty servers with probability 1. In both cases, when all servers are busy, customers depart from the system without taking any service. These customers are called ‘lost customers’ and the flows of lost customers are called ‘stream of overflows’.

The queueing model $GI/M/n/0$ with heterogeneous servers is analyzed using semi-Markov process. The semi-Markov process representation of the system is described and the kernel functions of semi-Markov process are derived. An implementation of this formula is performed for the queueing model $GI/M/3/0$ with heterogeneous servers. Using the kernels of semi-Markov process, one-step transition probabilities, and steady-state probabilities are obtained for the related queueing model.

The stream of overflows is analyzed for the queueing model $GI/M/n/0$ with heterogeneous servers, the Laplace-Stieltjes transform of the distribution of the time between overflows is obtained and the loss probability of customers is formulated. An implementation of this formula is performed for the queueing model $GI/M/2/0$ with heterogeneous servers, and the loss probability of customers is computed.

It becomes computationally intractable to compute the exact solution of loss probability, besides it is impossible to minimize the loss probability according to distribution of arrival process as the number of servers increases. In this respect a quite extensive simulation study is performed and the loss probability is computed for different distributions of interarrival times and different service disciplines. The

conditions in which the loss probability is minimum are determined by simulation optimization.

Keywords: Semi-Markov process, Laplace-Stieltjes transform, loss probability, stream of overflows, queueing, simulation.

HETEROJEN KANALLI *GI*/*M*/*n*/0 KUYRUK MODELİNDE KAYBOLMA OLASILIĞININ OPTİMİZASYONU

ÖZ

Bu çalışmada rekurent girişli, sınırlı kapasiteli, bekleme hattının olmadığı, n heterojen kanallı $GI/M/n/0$ kuyruk modeli incelenir. Hizmet disiplini iki farklı şekilde ele alınır. Birincisinde, müşteriler boş olan kanallardan herhangi birinden eşit olasılıkla hizmet alır. İkincisinde, müşteriler boş olan kanallar arasından index numarası en düşük olan kanalda 1'e eşit olasılıkla hizmet alır. Her iki durumda da, bütün kanallar dolu ise, müşteriler hiç bir hizmet almadan sistemden ayrılır. Bu müşteriler 'kayıp müşteriler', kayıp müşterilerin akımı ise 'kaybolan müşteri akımı' olarak adlandırılır.

Heterojen kanallı $GI/M/n/0$ kuyruk modelinin analizi yarı-Markov süreci kullanılarak yapılır. Sistemi temsil eden yarı-Markov süreci tanımlanır ve yarı-Markov sürecinin çekirdek fonksiyonları türetilir. Bu formülün bir uygulaması heterojen kanallı $GI/M/3/0$ kuyruk modeli için gösterilir. Yarı-Markov sürecinin çekirdekleri kullanılarak, bir-adım geçiş olasılıkları ve durağan durum olasılıkları ilgili kuyruk modeli için elde edilir.

Heterojen kanallı $GI/M/n/0$ kuyruk modeli için kaybolan müşteri akımının analizi yapılır, kaybolma anları arasındaki sürenin dağılımının Laplace-Stieltjes dönüşümü elde edilir ve müşterinin kaybolma olasılığı formüle edilir. Bu formülün bir uygulaması heterojen kanallı $GI/M/2/0$ kuyruk modeli için gösterilir ve müşterinin kaybolma olasılığı hesaplanır.

Kanal sayısı artarken kaybolma olasılığının tam çözümünün bulunması sayısal olarak zorlaşır, ayrıca geliş süreci dağılımına göre kaybolama olasılığının minimize edilmesi imkansız hale gelir. Bu açıdan oldukça geniş bir simülasyon çalışması yapılır ve kaybolma olasılığı, gelişlerarası sürelerin farklı dağılımları için ve farklı hizmet disiplinleri için hesaplanır. Kaybolma olasılığının minimum olduğu koşullar simülasyon optimizasyonuyla belirlenir.

Anahtar sözcükler: Yarı-Markov süreci, Laplace-Stieltjes dönüşümü, kaybolma olasılığı, kaybolan müşteri akımı, kuyruk, simülasyon.

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CHAPTER ONE
INTRODUCTION

Queuing theory which is founded by Danish scientist Agner Krarup Erlang in 1917 has become one of the most important elements of the science and the technology, recently. Thanks to the studies of many valuable scientists such as Palm (1943), Takacs (1956, 1957, 1962), Bhat (1965, 1968), Çinlar (1967a, 1967b), Whitt (1972), Gnedenko & Kovalenko (1989) and Atkinson (1995, 2000, 2009), the theory has been enriched by presenting important results and various application areas.

During the early years the fundamental problems handled had been the determination and the calculation of performance measures such as mean number of customers in the queue, mean waiting time in the queue, and mean service time. On the other hand, in the subsequent years, the theory made progress in analyzing the problems such as minimizing the time and work loss and determination of the uninterrupted working time. In other words optimizing the system performance by increasing the service quality and attaining the outstanding service has become one of the most important problems, recently. In addition, queuing models closer to new and real systems have been introduced and examined related to the development of the production, communication and computer systems.

The queuing systems without waiting line have been analyzed extensively. In this kind of systems, since some of arriving customers left without taking any service, a very important problem called the analysis of *"stream of overflows"* appeared. The stream of overflows in queuing systems without waiting line was first studied by Palm (1993). Palm (1943) proved that in $GI/M/n/0$ queuing system, the stream of overflows is a renewal process and found the Laplace-Stieljes transform of the interoverflow time distribution and obtained the loss probability by using difference equations. This problem presented by Palm was also examined in subsequent years by the scientists such as Khintchine (1960), Takacs (1956), and Çinlar & Disney (1967). Çinlar & Disney (1967) obtained the generating function of the stream of overflows in the $M/G/1/n{-}1$ system.

The models related to queuing systems without waiting line in the literature can be classified into two groups in general:

a) *M*/*M*/*n*/0 queuing model: Since there is no waiting line in the system, a customer arriving in the system when all servers are busy leaves without taking any service. This model is analyzed by means of Markov process since the interarrival times and the service times have exponential distribution.

b) *GI*/*M*/*n*/0 and *M*/*G*/*n*/0 queueing models: Since there is no waiting line in both systems, a customer arriving in the system when all servers are busy leaves without taking any service. However interarrival times are independent of each other and have an arbitrary distribution in the former, whereas in the latter, the service times are independent of each other and have an arbitrary distribution. Since these models cannot be analyzed by Markov process, methods such as supplementary variable, embedded Markov chain, and semi-Markov process were developed. The fundamental problem in this kind of models is the calculation of loss probability and the minimization of this probability.

A/*B*/*n*/*m*/*d* notation given by Kendall (1953), facilitates the definition of the models in the analysis of the queuing systems. *A* represents the distribution function of interarrival times, *B* represents the distribution function of the service time, *n* represents the number of servers, *m* represents the number of customers waiting in line, and finally *d* represents the service discipline. Specially, the letter *M* stands for the exponential distribution whereas *G* represents an arbitrary distribution; *GI* indicates that interarrival times are independent of each other and have an arbitrary distribution function.

1.1 Problem Statement

Conny Palm (1943) studied the queuing model *GI*/*M*/*n*/0 with identical servers and no waiting line in his study named "*Intensitätsschwankungen im Fernsprechverkehr*". In this model, interarrival times are independent of each other and have distribution function $F(t)$, and their expected value is finite. There are *n*-identical servers in the system. The service time of each customer in server k $(k=1,2,\ldots n)$ is a random variable represented by η and has an exponential distribution with parameter μ, i.e. $P(\eta \leq t) = 1 - e^{-\mu t}$, $t \geq 0$. The customer, who arrives in the system, chooses the server with the lowest index number among the empty servers with probability 1. Since the servers in this model are identical, such an assumption in terms of service discipline does not affect the traffic flow. In other words, the assigned index number of the server to the arriving customer at any time t is not important in Palm's model.

In real life, it is obvious that the servers may not be identical. In this kind of systems, it is more realistic to suppose that the servers are heterogeneous and to model the system accordingly, however the analysis of the model becomes relatively difficult.

The service discipline gains a great importance when servers are assumed to be heterogeneous in the model examined by Palm (1943). Namely, from which server an arriving customer in the system at any time *t* receives the service is very important and directly affects the analysis of the model. In other words, depending on the service discipline, the calculation of the functions representing the system and therefore the calculation of performance measures of the system differ significantly. This is the only reason for the difficulty of this kind of systems.

In this thesis, the model of Palm (1943) is generalized by assuming the servers heterogeneous, namely, the queuing model *GI*/*M*/*n*/0 with heterogeneous servers and no waiting line is analyzed. In this model, interarrival times are independent of each other and have distribution function $F(t)$ and their expected value is finite. There are

n heterogeneous servers in the system. That is, their mean service times are different from each other. The service time of each customer in server k is a random variable represented by η_k and has an exponential distribution with parameter μ_k, $k = 1, 2, \ldots, n$.

The service discipline is addressed in two different ways. Firstly, the customer arriving in the system starts the service in any of the empty servers with equal probability. This discipline is called as 'Random Selection Discipline' or briefly 'Random Entry' by the author. In the second case, the customer arriving in the system chooses the server with the lowest index number among the empty servers with probability 1 introduced that was introduced by Palm (1943). This discipline is briefly known as "Ordered Entry" in the literature.

Since there is no waiting line in the addressed model, when all servers are busy, an arriving customer leaves without taking any service. In this respect, many problems such as the stream of overflows, the distribution of the stream of overflows, loss probability of a customer, and the optimization of loss probability arise.

In terms of the optimization of loss probability, depending on arrival flow and the service discipline, the loss probability can be minimized in two different manners. In some cases the conditions where the system is optimal cannot be determined theoretically. In such cases, the determination of optimal conditions by simulation design appears as a different problem.

The aim of this thesis is to solve abovementioned problems, to generalize the queueing model $GI/M/n/0$ with homogeneous servers first addressed by Palm (1943), to analyze a queuing model closer to real systems, to calculate the loss probability of an arriving customer, and to minimize this probability.

1.2 Thesis Outline

The queuing model $GI/M/n/0$ with heterogeneous servers introduced in this thesis is analyzed by means of semi-Markov process that is one of the most important subjects of the stochastic process theory. The model addressed in this sense is a perfect application of semi-Markov process. Additionally, overflow times of the customer in the model forms a delayed renewal process. Therefore, some concepts, definitions, theorems, and proofs of those theorems related to the renewal theory that is one of the most important subjects of the stochastic process theory are given in Chapter Two for better understanding and easier interpretation of this thesis. The fundamental concepts of renewal theory are briefly explained in Section 2.1. Some applications of the renewal processes related to the queuing theory and the reliability theory are explained with examples. Moreover, some theorems such as Abel and Tauber related to Laplace-Stieltjes transforms frequently used in the thesis are examined. The renewal function, limit theorems for renewal processes, delayed renewal process, Markov renewal process, and semi-Markov process are other subjects that are explained in Chapter Two.

In Chapter Three, a comprehensive literature review on especially related to queuing models without waiting line has been presented. Afterwards, "the model $GI/M/n/0$ with heterogeneous servers and no waiting line" addressed in this thesis is explained with its assumptions. Kernel functions of the process are obtained by defining the semi-Markov process representing the model. An implementation of loss formula is performed for the queuing model $GI/M/3/0$ with heterogeneous servers. The condition in which the loss probability is minimum is explained with a theorem by optimizing the loss probability depending on the arrival flow. Additionally, the distribution of the time between overflows is obtained by analyzing the stream of overflows. Also, Palm's recurrence formula and an extension of Palm's recurrence formula are examined in detail. For the queuing model $GI/M/n/0$ with ordered entry, it is revealed by a numeric example that, the loss probability obtained by Yao (1986, 1987) as a function of the extension of Palm's recurrence Formula, is not correct for n=3.

In Chapter Four, simulation models are defined for both random entry and ordered entry service disciplines of the queuing model *GI*/*M*/*n*/0 with heterogeneous servers and no waiting line. The variation in the loss probability is experimentally observed for different interarrival time distributions. Theoretical studies carried out in the literature related to the minimization of the loss probability are supported by simulation optimization.

Finally in Chapter Five, concluding remarks and a discussion of the future research which can be followed as extensions of this thesis are presented.

1.3 Contributions

The main contributions of this thesis are summarized as follows:

1) '*A generalization of Takacs's Formula*' for 'the queueing model *GI*/*M*/*n*/0 with heterogeneous servers' is obtained by deriving kernel probabilities of the semi-Markov process. Thus an embedded Markov chain of semi-Markov process for the queuing model *GI*/*M*/*n*/0 with heterogeneous servers is obtained (Section 3.2).

2) By defining the overflow times of the customers and showing that the time until the first loss epoch and successive interoverflow times are independent from each other and have a different distribution, it is shown that overflow times in the system are delayed renewal process (Section 3.3).

3) The Laplace-Stieltjes transform of the distribution of the stream of overflows is derived for the *GI*/*M*/*n*/0 queuing model with heterogeneous servers. An implementation of the Laplace-Stieljes transform of the distribution of the stream of overflows is performed for the queuing model *GI*/*M*/2/0 with heterogeneous servers (Section 3.3).

4) It is shown that how a generalization of Takacs's formula is applied for the queuing model *GI*/*M*/3/0 with heterogeneous servers and also the loss probability is obtained for the above mentioned model (Subsections 3.2.1 and 3.2.2).

5) The loss probability obtained for the queuing model *GI*/*M*/3/0 with heterogeneous servers is minimized according to the arrival process (Subsection 3.2.3).

6) Steady-state probabilities are obtained as a solution of the determinant of the embedded Markov chain (Section 3.4).

7) '*An Extension of Palm's Loss Formula*' is derived for 'the queueing model *GI*/*M*/*n*/0 with heterogeneous servers'. An implementation of this formula was performed for the queuing model *GI*/*M*/2/0 with heterogeneous servers and the loss probability of customers was computed (Section 3.4).

8) It was explained that an extension of Palm's recurrence formula addressed by Yao (1986, 1987) is a heuristic formula and does not guarantee the exact solution. (Subsection 3.5.1)

9) The contradiction between the main theorem, given by Yao (1987) related to the optimization of the loss probability, and the loss probability formula, again given by Yao (1986, 1987), is proved with a numerical example (Subsection 3.5.2).

10) It is explained with a numerical example that 'an extension of Palm's Loss Formula' that we obtained in this thesis is compatible with the main theorem of Yao (1987) (Subsection 3.5.2).

11) Studies available in the literature related to the optimization of the loss probability are supported by a simulation study. For the situations in which it

is not theoretically possible to minimize the loss probability according to the interarrival time distribution, the simulation optimization approach is proposed and designed. As a result of simulation optimization, the optimal conditions for the system are determined. (Chapter 4).

1.4 Publications

The followings are a complete list of publications (2010, 2011, and 2012) and a submission due to the work presented in this thesis.

1) Isguder, H. O., & Celikoglu, C. C. (2010). Sonlu kapasiteli heterojen kuyruk modeli için geçiş olasılıklarının elde edilmesi. *7. İstatistik Günleri Sempozyumu*, Ankara, Türkiye, 51-52.

2) Isguder, H. O. & Uzunoglu-Kocer, U. (2010). Optimization of loss probability for *GI*/*M*/3/0 queuing system with heterogeneous server. *Anadolu University Journal of Science and Technology B – Theoretical Sciences 1*(1), 73-89.

3) Isguder, H. O., Uzunoglu-Kocer, U., & Celikoglu, C. C. (2011). Generalization of the Takacs's formula for *GI*/*M*/*n*/0 queuing system with heterogeneous servers. Lecture Notes in Engineering and Computer Science 1, 45-47.

4) Isguder, H. O., & Celikoglu, C. C. (2012). Minimizing the loss probability in *GI*/*M*/3/0 queueing system with ordered entry. *Scientific Research and Essays 7*(8), 963-968.

5) Isguder, H. O. (2012). An extension of Palm's recurrence formula. *8th World Congress in Probability and Statistics*, Istanbul, Turkey, 45.

6) Isguder, H. O., & Celikoglu, C. C. (2010). Computation of loss probability in *GI*/*M*/*n*/0 queueing model. *8th International Symposium of Statistics*, Eskişehir, Turkey.

7) Isguder, H. O., & Uzunoglu-Kocer, U. (2012), Analysis of the *GI*/*M*/*n*/0 Queuing System with Ordered Entry, *submitted*.

CHAPTER TWO
RENEWAL THEORY

In this chapter, renewal process, renewal function, limit theorems for renewal processes, delayed renewal process, Markov renewal process, and semi-Markov process matters among the most important matters of the stochastic processes theory are briefly explained. Definitions, theorems, and examples taking place in this chapter will facilitate the comprehension of Chapter Three. This section has been prepared by the help of the studies carried out by Pyke & Schaufele (1964), Feller (1966), Çinlar (1969, 1975) and Ross (1996). For more information about in this chapter, the mentioned references may be consulted.

2.1 Renewal Process

The renewal theory arose from the need for analyzing the problems related to breakdown and renewal (repair) of a machine in random times. This theory extended its application area (mathematical analysis, physics, economy, engineering, holding line models, reliability analysis, etc.) and now became one of the most important tools used by millions of researchers. Many problems solved by using difficult methods can be easily solved by means of the renewal theory. In this section, information will be presented about basic concepts of the mentioned theory.

2.1.1 Basic Concepts

Assume that $X_1, X_2, \dots$ are independent, positive random variables having identical distribution function F and that expected value of each is finite:

$$\mu = E[X_k] = \int_0^\infty [1 - F(x)]dx < \infty \ , \ k \geq 1 . \tag{2.1}$$

In this case, the sequence

$$S_0 = 0\ ,\ S_n = X_1 + \cdots + X_n\ ,\ n \geq 1, \tag{2.2}$$

is called as *renewal process* or recurrent process. Each S_n is called as *n*th *renewal time* and $X_n = S_n - S_{n-1}$ as *n*th *renewal period.*

Let's consider the following function defined by means of $(S_n)_0^\infty$

$$N(t) = \max\{n : S_n \leq t\} = \sum_{n=1}^{\infty} I(S_n \leq t)\ . \tag{2.3}$$

If each $S_n < t$, then $N(t) = \infty$. The function (2.3) is also called as *renewal process* in the literature. $N(t)$ represents the number of renewal times settled in the range $(0,t]$. Therefore, $N(t)$ is a random variable and it is the number of the last term smaller than and equal to *t* in the sequence (S_n). From the definition (2.3), following requirements are obtained:

$$N(t) < n \iff S_n > t\ , \tag{2.4}$$

$$N(t) = n \iff S_n \leq t < S_{n+1}\ . \tag{2.5}$$

Thus, $S_{N(t)}$ is the last renewal time coming before the *t* and $S_{N(t)+1}$ is the first renewal time coming before *t* (Figure 2.1).

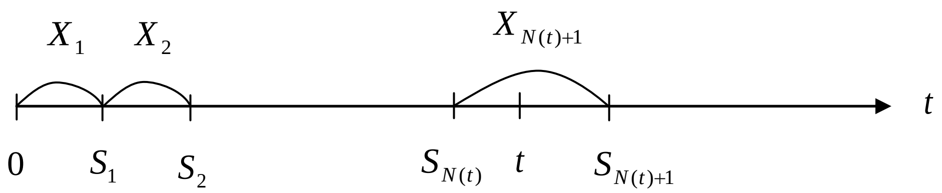

Figure 2.1 Renewal times.

In addition, a trajectory of $N(t)$ process is given in the Figure 2.2.

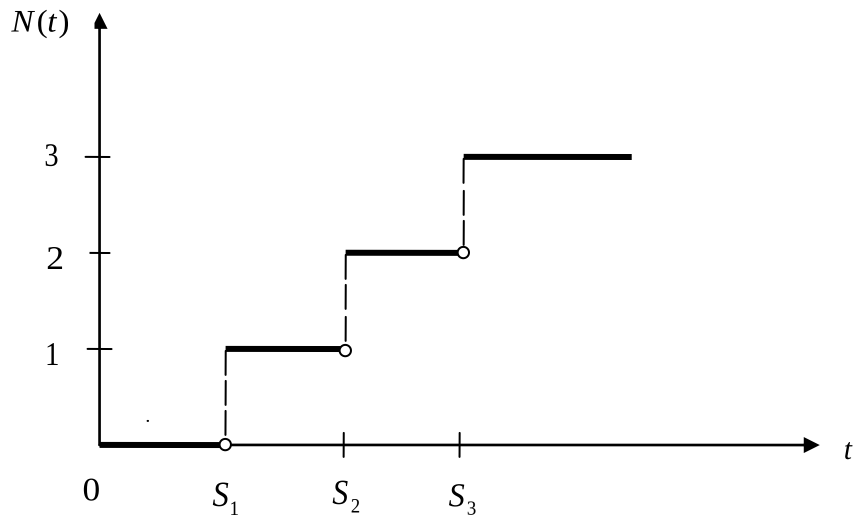

Figure 2.2 A trajectory of $N(t)$ process.

As it can be seen, each S_n is at jumping point of the process $N(t)$, and the size of the jumps is equal to one.

From the requirement (2.4) or (2.5), following equation is obtained for $N(t)$ process:

$$P\{N(t)=n\}=F_n(t)-F_{n+1}(t)\ ,\ n\geq 0\,, \tag{2.6}$$

where $F_0(t)=1$, $F_n(t)$ is the distribution function of the S_n:

$$F_n(t)=P(S_n<t)\ ,\ n\geq 1. \tag{2.7}$$

Since $X_1, X_2, \ldots$ have an independent distribution function *F;* F_n is *n*-tuple convolution of the *F*. Convolution formula is explained by Definition (2.1) by means of the Theorem 2.1 given below.

Theorem 2.1 (see, Feller, 1966)**.** *Suppose that X and Y are two continuous random variables, and* f *is their joint density function. In this case, the density function of the sum X+Y is given by the formula below:*

$$f_{X+Y}(t) = \int_{-\infty}^{\infty} f(t-y, y)dy\,. \tag{2.8}$$

This formula is as follows for independent *X* and *Y*:

$$f_{X+Y}(t) = \int_{-\infty}^{\infty} f_X(t-y) f_Y(y)dy\,. \tag{2.9}$$

Definition 2.1 (see, Feller, 1966)**.** The integral presented in the formula (2.9) is called as the convolution of the functions f_X and f_Y and shown as $f_X(t) * f_Y(t)$. This formula is also called as convolution formula. The density function of the sum of two independent continuous random variables according to the theorem above is obtained as follows by means of the convolution formula:

$$f_{X+Y}(t) = f_X(t) * f_Y(t)\,. \tag{2.10}$$

If $P(X_k < t) = 1 - e^{-\lambda t}$, $t \geq 0$, then the renewal process $N(t)$ is called as *Poisson process*, because in this case the $N(t)$ has a Poisson distribution with parameter λt. In fact, since the distribution function of the S_n is

$$F_n(t) = 1 - \sum_{k=0}^{n-1} (\lambda t)^k e^{-\lambda t} / k!\,,\ n \geq 1, \tag{2.11}$$

it becomes $P(N(t) = k) = (\lambda t)^k e^{-\lambda t} / k!$ according to the formula (2.6).

Renewal processes are used in various fields of the science. Some of them are illustrated below:

a) Suppose that Z_n, $n \geq 0$ is recurrent Markov chain and $Z_0 = i$. In this case, successive transition times to state *i* from a renewal process:

$$S_1 = \min\{n \geq 1 : Z_n = i\}, \quad S_k = \min\{n > S_{k-1} : Z_n = i\}, \; k \geq 2. \qquad (2.12)$$

b) In *M*/*G*/1 queueing system, the arrival times of the customers in the system form a Poisson process and passage times of a server from busy condition to idle condition form a renewal process; starting times of uninterrupted operation durations of a server in the queueing system *G*/*M*/1 form a renewal process.

c) In the *reliability theory*, average lifetime of the systems with changeable elements is discussed. For example, if a unit starts working at the starting time $S_0 = 0$ and breaks down at time $S_1 = X_1$, it is replaced by a new unit. The new unit breaks down at time $S_2 = X_1 + X_2$ and is replaced by another one, and this process is continued in indicated manner. Thus, *n*th renewal time is represented by S_n .

Following theorem represents basic characteristics of the $N(t)$.

Theorem 2.2 (see, Ross, 1996)**.** $N(t)$ *function provides following characteristics:*

a) *For each* $t \geq 0$, $P(N(t) < \infty) = 1$.

b) $N(t) \to \infty$ $(t \to \infty)$, *with probability* 1.

c) $\dfrac{N(t)}{t} \to \dfrac{1}{\mu}$ $(t \to \infty)$, *with probability* 1.

Proof. (a) According to the law of large numbers, $(S_n / n) \to \mu$ with probability 1. Since $\mu > 0$ is follows that $S_n \to \infty$, accordingly the inequality $S_n \leq t$ is possible for at least finite number of values of *n*. From this fact and (2.2), $N(t) < \infty$ is obtained.

This characteristic can also be proved by using the Chebyshev inequality: We can write for each $\alpha \in R$ as:

$$P(S_n < \alpha) = P(e^{-S_n} > e^{-\alpha}) \leq e^{\alpha} E[e^{-S_n}] = e^{\alpha} (E[e^{-X_1}])^n . \qquad (2.13)$$

From (2.13) and $E[e^{-X_1}]<1$, $\lim_{n\to\infty} P(S_n < \alpha) = 0$ is found, namely $S_n \to \infty$ with probability 1.

(b) As $t \to \infty$ for each large number *n*, since $P(N(t) \le n) = P(S_n \ge t) = 1 - F_n(t) \to 0$, $P(\lim_{t\to\infty} N(t) = \infty) = 1$ is obtained.

(c) According to (2.2), the inequality $S_{N(t)} \le t < S_{N(t)+1}$, and from there the following relation is found

$$\frac{S_{N(t)}}{N(t)} \le \frac{t}{N(t)} < \frac{S_{N(t)+1}}{N(t)} . \tag{2.14}$$

As $t \to \infty$, $N(t) \to \infty$. Here from and from the law of large numbers, as $t \to \infty$, $S_{N(t)} / N(t) \to \mu$ is obtained. Theorem is proven.

2.1.2 Laplace-Stieltjes Transform

Suppose that the F is a monotonously increasing in the range $[0,\infty)$ and is a non-negative function. In this case:

$$\widetilde{F}(s) = \int_0^\infty e^{-sx} dF(x) . \tag{2.15}$$

Stieltjes integral is called as *Laplace-Stieltjes* (LS) *transform* of the *F*, where the *s* is a complex variable. The function (2.15) is analytical in the zone $\{s : \operatorname{Re} s > S_0\}$ for the *F* satisfying the condition $F(x) < M\, e^{-S_0 x}$, $x \ge 0$.

Now suppose that the X is a random variable that is non-negative, and the F is the distribution function of the X. In this case, the Laplace-Stieltjes transform $\tilde{F}(s)$ can be shown as the expected value of the X:

$$\tilde{F}(s) = E[e^{-sX}]\,. \tag{2.16}$$

This function exists for each $s \geq 0$.

The following relation exists between the function (2.15) and Laplace transform of the F, $F_L(s) = \int_0^\infty e^{-sx} F(x)\,dx$

$$\tilde{F}(s) = s\,F_L(s)\,. \tag{2.17}$$

Some characteristics of Laplace-Stieltjes transform are given below.

a) If $F = aF_1 + bF_2$, $\tilde{F} = a\tilde{F}_1 + b\tilde{F}_2$.

b) If $H(x) = \int\limits_0^x e^{-\lambda t} dt$, $\tilde{H}(s) = \tilde{F}(s)/s$.

c) If $H(x) = \int\limits_0^x e^{-\lambda t} dF(t)$, $\tilde{H}(s) = \tilde{F}(s+\lambda)$.

d) $f(x) = F'(x)$, $x \geq 0$, if its derivative exists and is a monotonously increasing function, $\tilde{f}(s) = s\tilde{F}(s) - sF(0)$.

e) If $H(x) = F_1(x) * F_2(x)$, $\tilde{H}(s) = \tilde{F}_1(s)\,\tilde{F}_2(s)$.

As $t \to \infty$ $(t \to 0)$, from the behavior of the $F(t)$, its Laplace-Stieltjes transform, the problem for finding the behavior of the $\tilde{F}(s)$ as $s \to 0$ $(s \to \infty)$ is called as *Abelian Theorem* and conversely the problem for determining the behavior of the $F(t)$ as $t \to \infty$ according to the behavior of the $\tilde{F}(s)$ as $s \to 0$ is called as *Tauberian Theorem*.

Theorem 2.3 (Abelian theorem)**.** *If* $\lim_{x\to\infty} F(x)$ *is finite, then*

$$\lim_{s\to 0}\tilde{F}(s) = \lim_{x\to\infty} F(x). \tag{2.18}$$

If $\lim_{n\to\infty} a_n$ *is finite* and $a(s) = \sum_{n=0}^{\infty} a_n s^n$ *, then*

$$\lim_{s\to 1}(1-s)\,a(s) = \lim_{n\to\infty} a_n\,. \tag{2.19}$$

The inverse of this theorem is not correct. However, the following theorem can be used:

Theorem 2.4 (Tauberian theorem)**. a)** *If* $F(x)\geq 0$ *and if the following limit is exist:*

$$\lim_{s\to 0} s^{\alpha}\tilde{F}(s)\ ,\ \ \alpha > 0\,, \tag{2.20}$$

then,

$$\lim_{T\to\infty} T^{-\alpha}\int_0^T F(x)\,dx = \frac{1}{\Gamma(\alpha+1)}\lim_{s\to\infty} s^{\alpha}\tilde{F}(s)\,. \tag{2.21}$$

b) *If* $\lim_{s\to 1}(1-s)\,a(s) < \infty$ and $\lim_{n\to\infty} n(a_n - a_{n-1}) = 0$ *, then* (2.19) *is correct.*

The Tauberian theorem gives information about the average of the *F* but not about the *F* itself.

2.2 Renewal Function

The renewal function plays an important role in the analysis of renewal processes. In fact, the basic characteristics of the renewal processes are expressed by this function. It is defined as the expected value of renewal times occurring in the range $(0,t]$, namely

$$m(t) = E[N(t)]\,,\ t \ge 0\,. \tag{2.22}$$

There is a one-to-one correspondence between $m(t)$ and $F(t)$, therefore the $m(t)$ uniquely determines the renewal process. Certain characteristics of the renewal function $m(t)$ are explained below:

a) For each $t \ge 0$, $m(t) < \infty$.

Proof. Since $X_k > 0$, there is such $\delta > 0$ that $P(X_k \ge \delta) > 0$. Now suppose

$$\overline{X}_k = \begin{cases} 0\,, & X_k < \delta \\ 1\,, & X_k \ge \delta\ . \end{cases} \tag{2.23}$$

In this case, the following sequence becomes a renewal process:

$$\overline{S}_n = \overline{X}_1 + \cdots + \overline{X}_n\ ,\ n \ge 1. \tag{2.24}$$

Let $\overline{N}(t)$ corresponds to the number of renewal times until the time *t* of this process. In this case, it becomes $E[N(t)] \le E[\overline{N}(t)]$. It can be seen from (2.23) and (2.24) that the sum of each $\overline{S}_n$ takes values as $0,1,2,\ldots$ and $\overline{X}_1, \overline{X}_2, \ldots$ are independent random variables, each of them takes the value 1 with the probability $\beta = P(X_i \ge \delta)$, $i = 1,2,\ldots$ and the value 0 with the probability $1-\beta$. Accordingly, $\overline{S}_n$, has binominal distribution with parameters (n,β), namely the following can be written:

$$P(\overline{S}_n = k) = C(n,k)\beta^k (1-\beta)^{n-k}\,. \tag{2.25}$$

By using the total probability formula, the following is obtained:

$$
\begin{aligned}
P(\overline{S}_{n+1} < t) &= P(\overline{S}_n + \overline{X}_{n+1} < t) \\
&= (1-\beta)P(\overline{S}_n < t) + \beta P(\overline{S}_n < t-1).
\end{aligned} \tag{2.26}
$$

And herefrom, the following can be written:

$$P(\overline{S}_n < t) - P(\overline{S}_{n+1} < t) = \beta[P(\overline{S}_n < t) - P(\overline{S}_n < t-1)]. \tag{2.27}$$

Herefrom and from the formula (2.3), the following is found:

$$P([\overline{N}(t) = n) = \beta P(t-1 \le \overline{S}_n < t) = \beta(\overline{S}_n = [t]), \tag{2.28}$$

where $[t]$ and integer part of the t are shown. Herefrom and from (2.25) with $k = [t]$, the following is found:

$$P[\overline{N}(t) = n] = C(n,k)\beta^{k+1}(1-\beta)^{n-k} \;, \; n \ge k. \tag{2.29}$$

Since the expected value of negative binominal distribution with parameters (k, β) is $E[\overline{N}(t)] = [t]/\beta$, $m(t) = E[N(t)] \le E[\overline{N}(t)] = [t]/\beta$, namely the $m(t)$ is finite.

b) The renewal function can be shown in the following form:

$$m(t) = \sum_{n=1}^{\infty} F_n(t), \tag{2.30}$$

where n-tuple convolution of F is shown with F_n.

Proof. The formula (2.30) is obtained from the equation (2.3) as follows:

$$m(t) = \sum_{n=1}^{\infty} nP(m(t) = n) = \sum_{n=1}^{\infty} n[F_n(t) - F_{n+1}(t)]$$
$$= \sum_{n=1}^{\infty} nF_n(t) - \sum_{n=2}^{\infty} (n-1)F_n(t) = \sum_{n=1}^{\infty} F_n(t). \tag{2.31}$$

The $m(t)$ is the first moment of the $N(t)$. rth moment of the $N(t)$, $m_r(t) = E[N(t)^r]$, is found as follows:

$$m_r(t) = \sum_{n=1}^{\infty} n^r [F_n(t) - F_{n+1}(t)]. \tag{2.32}$$

Herefrom and from the partial sums formula, the following is found

$$m_r(t) = \sum_{n=1}^{\infty} [n^r - (n-1)^r] F_n(t). \tag{2.33}$$

Herefrom, the second moment of the $N(t)$ is obtained:

$$m_2(t) = \sum_{n=1}^{\infty} (2n-1) F_n(t)$$
$$= m(t) + 2\sum_{n=2}^{\infty} (n-1) F_n(t). \tag{2.34}$$

Herefrom, Laplace-Stieltjes transform of the $m_2(t)$ is found:

$$\tilde{m}_2(s) = \tilde{m}(s) + 2\sum_{n=2}^{\infty} (n-1) \tilde{F}(s)^n$$
$$= \tilde{m}(s) + 2\tilde{F}(s)^2 \sum_{n=1}^{\infty} n \tilde{F}(s)^{n-1} \tag{2.35}$$
$$= \tilde{m}(s) + 2[\frac{\tilde{F}(s)}{1-\tilde{F}(s)}]^2 = \tilde{m}(s) + 2\tilde{m}(s)^2 .$$

From this equation, $m_2(t)$ is obtained:

$$m_2(t) = m(t) + 2\int_0^t m(t-y)\,dm(y)\,. \tag{2.36}$$

For example, since $m(t) = \lambda(t)$ for a Poisson process with parameter λ, $m_2(t)$ is found as follows:

$$m_2(t) = \lambda t + 2\int_0^t \lambda(t-y)\,d(\lambda y) = \lambda t + (\lambda t)^2\,. \tag{2.37}$$

c) The renewal function is the unique solution of the following integral equation:

$$m(t) = F(t) + \int_0^t m(t-x)dF(x)\,. \tag{2.38}$$

Proof. Actually we can write the convolution of the functions *a*, and *b* by representing with $a * b$:

$$\begin{aligned} m(t) &= F(t) + \sum_{n=1}^{\infty} F_{n+1}(t) = F(t) + \sum_{n=1}^{\infty} F(t) * F_n(t) \\ &= F(t) + F(t) * \sum_{n=1}^{\infty} F_n(t) = F(t) + F(t) * m(t)\,. \end{aligned} \tag{2.39}$$

Thus, the following equation equivalent to (4) is obtained

$$m(t) = F(t) + F(t) * m(t)\,. \tag{2.40}$$

Now suppose that the $M(t)$ is the second solution of the equation (2.30) . In this case, the function $h(t) = m(t) - M(t)$ will be the solution of the equation $h(t) = F(t) * h(t)$. Here from $h(t) = F_n(t) * h(t)$ is obtained. Since $m(t) < \infty$ for each t, the sequence (2.30) is convergent, accordingly while $n \to \infty$, $F_n(t) \to 0$. From there and the previous equation, $h(t) = 0$ is found, namely $M(t) = m(t)$.

Alternative proof. We can write it by using the expected value formula:

$$m(t) = E[N(t)] = \int_0^{\infty} E[N(t)/X_1 = x]\, dF(x)$$

$$= \int_0^{t} [1 + EN(t-x)]\, dF(x) \tag{2.41}$$

$$= F(x) + \int_0^{t} m(t-x)\, dF(x)\,.$$

The equation (2.38) is called the *renewal equation*. This equation can be written as follows:

$$F(t) = \int_0^{t} \tilde{F}(t-y)\, dm(y)\,, \tag{2.42}$$

where $\tilde{F} = 1 - F$.

The following formula is obtained from (2.40) for $\tilde{m}(s) = \int_0^{\infty} e^{-st} dm(t)$, Laplace-Stieltjes transform of the $m(t)$:

$$\tilde{m}(s) = \frac{\tilde{F}(s)}{1 - \tilde{F}(s)}\,, \tag{2.43}$$

where the Laplace-Stieltjes transform of the F is represented by $\tilde{F}(s)$. This formula is obtained by applying the Laplace-Stieltjes transform to the equation (2.40) and by using the theorem ‘Laplace-Stieltjes transform of the convolution of two functions is equal to the multiplication of their Laplace-Stieltjes transforms’. The formula (2.43) is obtained from the equation (2.30).

From the formula (2.43), the following result is obtained:

$$\lim_{n\to\infty} m(t) = \lim_{s\to 0} \tilde{m}(s) = \frac{1}{1-1} = \infty . \tag{2.44}$$

It is seen from the formula (2.43) that there is a one-to-one correspondence between the functions $F(t)$ and $m(t)$. Each of the formulas (2.30), (2.38), and (2.43) can be used for finding the $m(t)$. In the example addressed below, the $m(t)$ is found for $F(t) = 1 - e^{-\lambda t}$, $t \geq 0$.

Example 2.1 $F(t) = 1 - e^{-\lambda t}$, $t \geq 0$. In this case, since the density function of $S_n = X_1 + \cdots + X_n$ is $f_n(t) = \lambda(\lambda t)^{n-1} e^{-\lambda t} /(n-1)!$, $F_n(t)$ becomes the integral of this function in the range $(0,t)$ the $m(t)$ function that we desire to find obtain as follows as required by the formula (2.30):

$$m(t) = \lambda \int_0^t \sum_{n=1}^{\infty} \frac{(\lambda t)^{n-1}}{(n-1)!} e^{-\lambda t} dt = \lambda \int_0^t e^{\lambda t} \cdot e^{\lambda t} dt = \lambda t . \tag{2.45}$$

The same result can be obtained by using the formula (2.43). Since the Laplace-Stieltjes transform of the $F(t)$ is $\tilde{F}(s) = \dfrac{\lambda}{s+\lambda}$, $\tilde{m}(s) = \dfrac{\lambda/(s+\lambda)}{1-\lambda/(s+\lambda)} = \dfrac{\lambda}{s}$, from there $m(t) = \lambda t$ is found.

Thus $m(t)$ is a linear function for a Poisson process with parameter λ . The inverse of this statement is also correct: Renewal process whose renewal function is $m(t) = at$ is a Poisson process with parameter *a*. Indeed, since the Laplace-Stieltjes transform of $m(t) = at$ is a/s, the equation (2.43) takes the following form:

$$\frac{\tilde{F}(s)}{1-\tilde{F}(s)} = \frac{a}{s} , \tag{2.46}$$

and here from $\tilde{F}(s) = \dfrac{a}{s+a}$, namely $F(t) = 1 - e^{-at}$ is found.

2.3 Limit Theorems for Renewal Processes

Asymptotic analysis of the renewal function $N(t)$ as $t \to \infty$ is a very important subject in the application of the renewal theory. The proof of a few theorems related to the subject mentioned in this section will be given.

Theorem 2.5 (The elementary renewal theorem)**.** *For the renewal function* $m(t)$

$$\lim_{t\to\infty} \frac{m(t)}{t} = \frac{1}{\mu}, \tag{2.47}$$

asymptotic equation is correct, where, if $\mu = \infty$, $1/\mu = 0$ *is accepted.*

Proof. According to Tauberian theorem, for each monotonously increasing function $u(t) \geq 0$ the following equation exists:

$$\lim_{s\to 0} \tilde{u}(s) = \lim_{t\to\infty} \frac{u(t)}{t}. \tag{2.48}$$

In this equation, the equation (2.47) is obtained by taking $u(t) = m(t)$ and considering that $\tilde{m}(s) = [1 - \tilde{F}(s)]^{-1} F(s)$:

$$\lim_{t\to\infty} \frac{m(t)}{t} = \lim_{s\to 0} \frac{s\tilde{F}(s)}{1 - \tilde{F}(s)} = \frac{-1}{\tilde{F}'(0)} = \frac{1}{\mu}. \tag{2.49}$$

According to the equation (2.47), the average number of renewal within a time unit for large *t* is equal to the inverse of the average time between these renewals.

Theorem 2.6 (The key renewal theorem, Smith, 1954)**.** *Suppose that* F *is a non-lattice distribution function. If* $Q(x)$*, it is a function monotonously decreasing in the range* $[0,\infty)$ *and satisfying the condition* $\int_0^\infty Q(x)\,dx < \infty$ *. In this case, the following asymptotic equation is correct:*

$$\lim_{t\to\infty}\int_0^t Q(t-x)\,dm(x) = \frac{1}{\mu}\int_0^\infty Q(x)\,dx\,. \tag{2.50}$$

This theorem belongs to Smith and he has called it as key of renewal theorem. Different limit results are obtained for renewal process by selecting the function $Q(x)$ for which the equation (2.50) is found.

Theorem 2.7 (Blackwell's theorem, Blackwell, 1948)**.** If the F *is a non-lattice distribution function, for each* $h > 0$:

$$\lim_{t\to\infty}[m(t+h) - m(t)] = h/\mu\,. \tag{2.51}$$

Theorem 2.8 (Smith, 1958)**.** *The key renewal theorem and Blackwell theorem are equivalent, namely* (2.50)$\Leftrightarrow$(2.51).

Proof. For proving the requirement (2.50)$\Rightarrow$(2.51) let's select the function $Q(x)$ present in (2.50) as follows:

$$Q(x) = \begin{cases} 1\ , & 0 \le t \le h \\ 0\ , & t > h. \end{cases} \tag{2.52}$$

In this case, the left side of (2.50) is equivalent to the following integral

$$Q * m(x) = \int_{t-h}^{t} dH(x) = H(t) - H(t-h)\,. \tag{2.53}$$

And its right side is equivalent to (h/a), from there (2.51) is obtained. Thus the proposition (2.50)$\Rightarrow$(2.51) is correct.

For proving the requirement (2.51) $\Rightarrow$(2.50), let's show the integral in the left side of (2.50)

$$y_1(t)\int_0^{t/2} Q(t-x)dm(x),\quad y_2(t)=\int_{t/2}^{t} Q(t-x)\,dm(x)\,, \tag{2.54}$$

as the sum of the integrals above, and let's prove (2.54) can be written as follows as $t\to\infty$

$$y_1(t)\to 0,\quad y_2(t)\to Q/\mu\,,\quad Q=\int_0^{\infty} Q(t)\,dt\,. \tag{2.55}$$

Since the $Q(t)$ is monotonously decreasing, it is $0\le y_1(t)\le Q(t/2)\,m(t/2)$ is written. From this fact and as $t\to\infty$, since

$$tQ(t)\to 0,\quad m(t)/t\to 1/\mu\,, \tag{2.56}$$

we find $y_1(t)\to 0$. Now let's select it in a manner that it will be $h>0$, $hQ(0)<\varepsilon$ according to given number of $\varepsilon>0$. In this case, the following equation is correct:

$$0<Q-T_h<\varepsilon\,,\ T_h=h\sum_{n=1}^{\infty} Q(nh)\,. \tag{2.57}$$

Let's choose such a large t that we can obtain the following:

$$h\sum_{n=[t/2h]}^{\infty} Q(nh)<\varepsilon\,. \tag{2.58}$$

From the equation (2.51), the following is found for $u \geq t/2$

$$\left|\frac{m(u+h)-m(u)}{h}-\frac{1}{\mu}\right|<\varepsilon\,. \tag{2.59}$$

In the light of this information, the following is obtained for $y_2(t)$

$$(\frac{1}{\mu}-\varepsilon)(T_h-\varepsilon)<y_2<(\frac{1}{\mu}+\varepsilon)T_h\,. \tag{2.60}$$

From (2.60) and (2.57) for large enough *t*, the following is found:

$$(\frac{1}{\mu}-\varepsilon)(Q-2\varepsilon)<y_2<(\frac{1}{\mu}+\varepsilon)(Q+\varepsilon)\,. \tag{2.61}$$

Since $\varepsilon > 0$ is arbitrary, it becomes $y_2(t) \to Q/\mu$. Thus, while $t \to \infty$, (2.50) is obtained:

$$Q(t)*m(t)=y_1(t)+y_2(t)\to Q/\mu\,. \tag{2.62}$$

2.4 Delayed Renewal Process

Suppose that $X_1, X_2, \ldots$ are independent positive random variables and that $P(X_1 < t) = F_1(t)$, $P(X_k < t) = F(t)$, $k \geq 2$. In this case, the sequence $S_n = X_1 + \cdots + X_n$, $n \geq 1$ is called as *delayed renewal process*. The renewal function of this process

$$m_1(t)=EN(t)\,,\;\; N_1(t)=\max\{n:S_n\leq t\}\,, \tag{2.63}$$

provides following characteristics:

$$m_1(t) = F_1(t) * \sum_{n=2}^{\infty} F_n(t) = F_1(t) * m(t), \tag{2.64}$$

$$\begin{aligned} m_1(t) &= F_1(t) + \int_0^t m_1(t-x)\, dF(x) \\ &= F_1(t) + \int_0^t F(t-x)\, dm_1(x), \end{aligned} \tag{2.65}$$

$$\tilde{m}_1(s) = \frac{\tilde{F}_1(s)}{1-\tilde{F}(s)} \quad . \tag{2.66}$$

Overflow times of the customers in the queueing system $GI/M/n/0$ form a delayed renewal process. This system is analyzed for two different service disciplines in Chapter Three.

2.5 Markov Renewal Process

Suppose that $(\Omega, \Im, P)$ is a probability space, X_n and T_n are random variables defined in this space and respectively taking the values $E = \{0,1,\ldots\}$ and $R^+ = [0,\infty)$ for each $n \in Z^+$, if the sequence $0 = T_0 \le T_1 \le T_2 \cdots$. $(X_n, T_n \; ; \; n \ge 0)$ satisfy the following characteristic, it is called as Markov renewal process with state space E:

$$\begin{aligned} &P(X_{n+1} = j, T_{n+1} - T_n \le t \mid X_0 = i_0, \ldots, X_{n-1} = i_{n-1}, X_n = i; T_0 = t_0, \ldots, T_n = t_n) \\ &\qquad = P(X_{n+1} = j, T_{n+1} - T_n \le t \mid X_n = i), \end{aligned} \tag{2.67}$$

for all $n \in Z^+$, $i, j \in E$, and $t \in R^+$.

Suppose that $(X_n, T_n \; ; \; n \ge 0)$ is time-homogeneous: that is, for any $i, j \in E$, and $t \in R^+$,

$$P(X_{n+1} = j, T_{n+1} - T_n \le t \mid X_n = i) = Q_{ij}(t), \tag{2.68}$$

independent of n. The family of probabilities $Q_{ij}(t)$, $(i, j \in E, t \in R^{+})$ is called as a semi-Markov transition kernel over E. For each pair (i, j), the following equation is obtained with $t \to \infty$:

$$p_{ij} = \lim_{t \to \infty} Q_{ij}(t) \,. \tag{2.69}$$

It is easy to see from (2.67) that

$$p_{ij} \geq 0, \quad \sum_{j \in E} p_{ij} = 1, \tag{2.70}$$

namely, p_{ij} are the transition probabilities for certain Markov chains with state space E. This implies that $(X_n, n \geq 0)$ is a Markov chain with a state space E and a transition matrix P. On the other hand, the increments $T_1 - T_0, T_2 - T_1, \ldots$ are conditionally independent considering the Markov chain $X_0, X_1, \ldots$. If the state space E consists of a single point, then the increments are independent and identically distributed, namely $(T_n, n \geq 0)$ is a renewal process. Finally, the term Markov renewal process is a generalization of Markov chains and renewal processes.

2.6 Semi-Markov Process

Semi-Markov process was introduced independently and almost simultaneously by Levy (1954), and Smith (1955). Essential developments of semi-Markov process theory were proposed by Pyke (1961a, 1961b), and Çinlar (1969). Semi-Markov processes are connected to the Markov renewal process. Theory of semi-Markov process allows the establishment and the resolution of many models in queueing theory. The queueing model *GI*/*M*/*n*/0 with heterogeneous servers to be addressed in Chapter Three will be modeled by means of semi-Markov process.

A stochastic process $(Y_t, t \geq 0)$ given by the following relation

$$Y_t = X_n \ , \ t \in [T_n, T_{n+1}) \ , \tag{2.71}$$

is called as a semi-Markov process generated by the Markov renewal process related to the kernel $Q_{ij}(t)$, $(i, j \in E, t \in R^+)$.

The length of a sojourn interval $[T_n, T_{n+1})$ is a random variable whose distribution depends both on the state X_n being visited and the state X_{n+1} to be entered next. The successive states visited form a Markov chain and, conditional on that sequence, the successive sojourn times are independent. These form a Markov chain called an *embedded* Markov chain of semi-Markov process. The semi-Markov process is irreducible if the embedded Markov chain is irreducible too.

CHAPTER THREE
AN EXTENSION OF PALM'S LOSS FORMULA

Conny Palm (1943) analyzed the queueing model $GI/M/n/0$ consisting of identical servers without waiting line and obtained the loss probability of the customer in the system. In this model, the customer arriving in the system gets service with 'Ordered Entry' service discipline. Namely, the customer starts the service in the server with the lowest index number among the empty servers with probability 1. Takacs (1959) mentions from the ordered entry discipline in his article titled 'On the limiting distribution of the number of coincidences concerning telephone exchange' as follows: "*C. Palm (1943), let us suppose that the channels are numbered by* $1,2,\ldots,r,\ldots,$ *and that an incoming call realizes a connection through that idle channel which has the lowest serial number. This assumption does not restrict the generality since* $\{\eta(t)\}$ *is independent of the system of the handling of traffic*". Herein $\eta(t)$ is the number of customers present in the system at time t. Namely, since the servers are identical in Palm's model, the index number of the server in which the customer is available at any time t is not relevant. Therefore, in the queueing model $GI/M/n/0$ with homogeneous server, there is no difference between services taken by customers arriving in the system with 'Ordered Entry', 'Random Entry', or an another service discipline. When the servers are heterogeneous, the number of the customers present in the system depends on the system of the handling of traffic, and in this case, the service discipline gains a great importance.

In this section, the queueing model $GI/M/n/0$ with heterogeneous servers without waiting line is examined. The mentioned model is separately analyzed for both 'Random Entry' and 'Ordered Entry' service disciplines and the formula for the loss probability of the customer is obtained. This formula is called as '*An Extension of Palm's Loss Formula'*.

3.1 Literature Review

The queueing models with identical servers and no waiting line have been examined and analyzed extensively. Since these models have been applied in many areas like telecommunication networks, design of call centers, wireless networks, computer communication systems, and emergency service systems, they have been taking on great importance. The classical model with no waiting line is the $M/M/n/0$ queueing system which was first examined by Erlang (1917). Erlang (1917) obtained the probability of being state k for the $M/M/n/0$ model as follows:

$$P_k = \frac{\rho^k / k!}{\sum_{k=0}^{n} (\rho^k / k!)} \quad , \ 0 \le k \le n \, , \tag{3.1}$$

where $\rho = \lambda / \mu$ is the offered load, λ^{-1} and μ^{-1} are the means of the interarrival times and service times, respectively. Formula (3.1) is known as Erlang's loss formula for $k = n$. This formula is of great importance for the mathematical modeling of communication systems and has been a source of inspiration to analyze more complicated systems.

Konig & Matthes (1963) generalized Erlang's formula for dependent service times. Takacs (1969) analyzed the model, suggested by Erlang (1917), using discrete-parameter stochastic process considering the arrival and departure times of the customers in the system. Brumelle (1978) generalized Erlang's formula for dependent arrivals and dependent service rates and obtained the mean system waiting time of a customer.

Palm (1943) extended the model suggested by Erlang, for the state of having independent interarrival times with a general distribution and examined the $GI/M/n/0$ queueing model. Palm (1943) analyzed the stream of overflows in the $GI/M/n/0$ queueing model and computed the loss probability of customers in the system as follows:

$$\frac{1}{P_n} = \sum_{k=0}^{n} \binom{n}{k} c_k \,, \tag{3.2}$$

where, with f being the Laplace-Stieltjes transform of distribution of the interarrival time, c_k are

$$c_0 = 1, \quad c_k = \prod_{k=1}^{n} \frac{1 - f(k\mu)}{f(k\mu)} \ (1 \le k \le n)\,. \tag{3.3}$$

Takacs (1956) proved that limit distribution of being in any state was independent of the initial state. At the same time, he obtained similar results also when the number of servers was infinite. Takacs (1957) obtained Palm's loss formula (given by Eq. 3.2) in a simpler way by using the method of finite difference equations. Takacs (1958) demonstrated that the sequence of random variables $\{\eta_n\}$ $(n = 1,2,\ldots)$, which is the number of customers staying in the system immediately before the arrival of the *n*th customer in the system, forms a Markov chain and obtained its one-step transition probabilities $p_{ij} = P[\eta_{n+1} = j \mid \eta_n = i]$ as follows:

$$p_{ij} = \binom{i+1}{j} \int_0^{\infty} e^{-j\mu t} (1 - e^{-\mu t})^{i+1-j} \, dF(t)\,, \tag{3.4}$$

for $j = 1, 2, \ldots, n-1$ $p_{n,j} = p_{n-1,j}$, and $F(t)$ is distribution of interarrival times.

There are several studies which assume both the interarrival and service times have general distribution. In the *GI*/*G*/1 queueing model with no waiting line, Halfin (1981) obtained the distribution function of the interoverflow times of customers. By making a discrete-time analysis of the *GI*/*G*/2 loss system, Atkinson (1995) presented an alternative to Erlang's loss model when the arrival process did not well approximate the Poisson process. Again in another study by Atkinson (2000), the $C_2/G/1$ queueing model and the $C_2/G/1$ loss system were examined. Atkinson (2000) showed that, with c_X being the coefficient of variation of interarrival time, when $c_X^2 < 1$, the probability of delay and the probability of loss are both increasing in

$\beta(s)$ for the above-mentioned models, respectively. Herein $\beta(s)$ is the Laplace-Stieltjes transform of the service time distribution.

The assumption of identical servers is mostly invalid in real life. The literature on Markovian queueing systems with heterogeneous servers is mature. Gumbel (1960) obtained the limit distribution of the number of customers in the system for the *M*/*M*/*n* model with infinite waiting line and heterogeneous servers. Singh (1970) examined the Markovian queueing system with two heterogeneous servers. Singh (1970) computed the performance measures of the system and compared these results with the homogeneous Markovian two-server model. Singh (1971) obtained the steady-state probabilities, the mean number of customers waiting in the queue, and the mean system waiting time of a customer for the queueing model with infinite waiting line and three heterogeneous servers. Lin & Elsayed (1978) developed a computer program to numerically solve multichannel Markovian ordered entry queueing system with heterogeneous servers and storage. Fakinos (1980) gave a generalization of the Erlang's loss formula for the case of non-identical servers. Kaufman (1980) analyzed the model *M*/G/*n*/0 with heterogeneous servers and random selection discipline. Elsayed (1983) developed two computer programs to determine the optimal allocation of storage spaces among three heterogeneous servers in a finite source queueing system. Alpaslan & Shahbazov (1996) proved that EW_q and EW get minimum values under the condition that $\mu_1 + \cdots + \mu_n = \mu$ for the *M*/*M*/*n* model with heterogeneous servers when $\mu_1 = \mu_2 = \cdots = \mu_n = c/n$. Kumar, Madheswari, & Venkatakrishnan (2007) examined Markovian queueing model *M*/*M*/2 with heterogeneous servers and infinite waiting line also considering the fact that, catastrophes fitting the Poisson distribution with a rate of γ might take place. Alves et al. (2011) derived upper bounds for the average number in queue L_q and the average waiting in queue W_q of heterogeneous multi-server Markovian queues, $M/M_i/c$. Nath & Enns (1981) proved that the loss probability is minimum under the fastest service rule for the queueing model *M*/*M*/*n*/0 with heterogeneous servers.

There are not many studies on non-Markovian queueing systems with heterogeneity. Nawijn (1984) considered the two-server queueing model with ordered entry and finite waiting rooms. In this model, it was assumed that the service time was exponential and the arrival process was deterministic. Nawijn (1984) calculated the overflow probability for the defined queueing model by implementing the matrix solution. Alpaslan (2002) obtained the distribution function of the stream of overflows for the *GI*/*M*/2/0 system with heterogeneous servers. In this study, Alpaslan (2002) assume that an arriving customer takes service in the first server with a probability π_1 and in the second server with a probability π_2, as $\pi_1 + \pi_2 = 1$. Isguder & Uzunoglu-Kocer (2010) minimized the loss probability according to the distribution of interarrival times for the *GI*/*M*/3/0 queueing model with heterogeneous servers and random entry. Gontijo, Atuncar, Cruz, & Kerbache (2011) evaluated algorithms using kernel estimator methods to estimate the performance measures of the non-Markovian $GI^X/M/c/N$ queueing system with bulk arrivals, and they compared simulation results for some theoretical distributions. Isguder, Uzunoglu-Kocer & Celikoglu (2011) examined a *GI*/*M*/*n*/0 queueing system with random entry and heterogeneous servers, and they obtained the kernels of semi-Markov process representing the system.

Queueing systems with no waiting lines are also frequently used in the studies on the modeling of emergency service systems, such as fire department, the police, and ambulances. Mendonça & Morabito (2001) analyzed the working system of the ambulances positioned on the superhighway between Sao Paulo and Rio de Janeiro in Brazil by means of the balance equations they built for $n = 6$ bases and 10 different atoms, and they computed the loss probability of customers in the system. Atkinson, Kovalenko, Kuznetsov & Mykhalevych (2006, 2008) generalized the results obtained by Mendonça & Morabito (2001) for *n* bases and 2*n* atoms and obtained the analytical solution of the loss probability. Nevertheless, it is very difficult to make an exact solution as the number of equations extremely increases with increasing number of bases. Therefore, two heuristic methods were proposed to approximate stationary loss probability (Atkinson, Kovalenko, Kuznetsov & Mykhalevych, 2006, 2008). Moreover, the simulation approach was proposed to approximate the stationary loss probability (Atkinson, Kovalenko, Kuznetsov & Mykhalevych, 2008).

In the following section, the assumptions of the queueing model *GI*/*M*/*n*/0 with heterogeneous servers are explained. The kernels of the semi-Markov process representing the model are derived.

3.2 Analyzing the *GI*/*M*/*n*/0 Queueing Model with Heterogeneous Servers Using Semi-Markov Process

‘*The GI*/*M*/*n*/0 *queueing system with finite capacity and heterogeneous servers*’ is analyzed in this section. In this model, interarrival times are independent of each other and have distribution function $F(t)$ and their expected value is finite $(a = \int_0^\infty [1 - F(t)]dt < \infty)$. There are n non-identical servers in the system. That is, their mean service times are different from each other. The service time of each customer in server k is a random variable represented by η_k and has an exponential distribution with parameter μ_k $(k = 1, 2, \ldots, n)$, i.e. $P(\eta_k \le t) = 1 - e^{-\mu_k t}$, $t \ge 0$. The service time is independent of the arrival process.

The service discipline is addressed in two different ways. Firstly, the service discipline takes place with the ‘*Random Entry*’ principle. That is, the customer, who arrives in the system, starts the service in any of the empty servers with probability $1/l$, $l = 1, 2, \ldots, n$, where l is the number of empty servers at the arrival time of the customer. In the second case, however, the service discipline takes place with the ‘*Ordered Entry*’ principle. That is, the customer who arrives in the system starts the service in the server with the lowest index number among the empty servers with probability 1.

If all servers are busy, the customer who arrives in the system leaves the system without taking any service. Such customers are called ‘*lost customers*’. The main problem herein is the computation of the probability of lost customers.

Let $t_0, t_1, t_2, \ldots$ be the arrival times of the customers, where $0 = t_0 < t_1 < \cdots$. Let the random variable T_n represent the interarrival time between two consecutive customers; that is $T_n = t_{n+1} - t_n$ for $n \geq 0$, and $T_0 = 0$. Let $S(t)$ be the number of customers in the system at time *t* and $S_n = S(t_n - 0)$, $n \geq 0$, where S_n is the number of customers staying in the system immediately before the arrival of the *n*th customer. The semi-Markov process representing the system can be defined as $\{X(t),\, t \geq 0\}$, $X(t) = S_n$ if and only if $t_n \leq t < t_{n+1}$. Suppose that $Q(x)$ is a square matrix consisting of the elements $Q_{ij}(x)$, where $Q_{ij}(x)$ is the kernels of the semi-Markov process.

$$Q_{ij}(x) = P[(S_{n+1} = j\,,\, T_n \leq x) \big| S_n = i\,]. \tag{3.5}$$

According to the semi-Markov process and the total probability formula, functions (3.5) are computed individually for ‘random entry’ and ‘ordered entry’ disciplines using equations (3.6) and (3.7) given as follows.

For Random Entry Discipline:

Considering $i = 0,1,\ldots,n-1$, and $Q_{n,j}(x) = Q_{n-1,j}(x)$,

$$Q_{ij}(x) = \frac{1}{\binom{n}{i+1}} \int_0^x \sum p_{k_1} p_{k_2} \cdots p_{k_u} q_{l_1} q_{l_2} \cdots q_{l_v} \, dF(t)\,, \tag{3.6}$$

where $p_{k_u} = e^{-\mu_{k_u} t}$, $q_{l_v} = 1 - e^{-\mu_{l_v} t}$. In addition, the summation under the equation given by (3.6) extends over all k_u’s and l_v’s such that $1 \leq k_1 < k_2 < \cdots < k_u \leq n$ and $1 \leq l_1 < l_2 < \cdots < l_v \leq n$ with $k_u \neq l_v$, where (u, v) pair takes the values $(j, i+1-j)$ for $0 \leq j \leq i+1$. Note that an empty product of probabilities denotes 1. Furthermore, as only one customer arrives in the system within any interarrival time, $Q_{ij}(x) = 0$ for $j > i+1$.

For Ordered Entry Discipline:

Let $r_1, r_2, \ldots, r_i$ $(i = 1,2,\ldots,n-1)$ be the index numbers of the busy servers. Let *m* be the index number of the server with the lowest index number among the empty servers at the arrival time of the *n*th customer. Considering $i = 0,1,\ldots,n-1$, and $Q_{n,j}(x) = Q_{n-1,j}(x)$,

$$Q_{ij}(x) = \frac{1}{\binom{n}{i}} \int_0^x \sum g(r_1, r_2, \ldots, r_i, m)\, dF(t)\,. \tag{3.7}$$

The summation under the equation given by (3.7) extends over all r_i 's with $1 \le r_1 < r_2 < \cdots < r_i \le n$ and $m = \min\left\{\{1,2,\ldots,n\} \middle| \{r_1, r_2, \ldots, r_i\}\right\}$. Note that $g(r_0, m) = g(m)$. On the right the function *g* is given by

$$g(r_1, r_2, \ldots, r_i, m) = \sum p_{k_1} p_{k_2} \cdots p_{k_u} q_{l_1} q_{l_2} \cdots q_{l_v}\ , \tag{3.8}$$

where the summation of the right of equation (3.8) extends over all k_u 's and l_v 's in such a way that $k_1 < k_2 < \cdots < k_u$ and $l_1 < l_2 < \cdots < l_v$ from the set $\{r_1, r_2, \ldots, r_i, m\}$ with $k_u \ne l_v$, where (u,v) pair takes the values $(j, i+1-j)$ for $0 \le j \le i+1$ and $p_{k_u} = e^{-\mu_{k_u} t}$, $q_{l_v} = 1 - e^{-\mu_{l_v} t}$. Note that an empty product of probabilities denotes 1. In addition, as only one customer arrives in the system within any interarrival time, $Q_{ij}(x) = 0$ for $j > i+1$. The summation under the integral in formula (3.7) allows assignment to the server with the lowest index number among the empty servers. In this way, the ordered entry service discipline is realized.

Square matrix $[q(s)]_0^n$ resulting from the Laplace-Stieltjes transform of functions $Q_{ij}(x),\ i, j = 0,1,\ldots,n$ is as follows:

$$q(s)=\begin{bmatrix} q_{00}(s) & q_{01}(s) & 0 & \cdots & 0 \\ q_{10}(s) & q_{11}(s) & q_{12}(s) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ q_{n-1,0}(s) & q_{n-1,1}(s) & q_{n-1,2}(s) & \cdots & q_{n-1,n}(s) \\ q_{n-1,0}(s) & q_{n-1,1}(s) & q_{n-1,2}(s) & \cdots & q_{n-1,n}(s) \end{bmatrix}, \tag{3.9}$$

where

$$q_{ij}(s)=\int_0^{\infty} e^{-st}dQ_{ij}(x),\ \operatorname{Re}\{s\}\geq 0\ (i,j=0,1,\ldots,n)\,. \tag{3.10}$$

$\{S_n, n\geq 0\}$ is an embedded Markov chain with probabilities p_{ij} of the semi-Markov process $\{X(t), t\geq 0\}$ with the state space $D=(0,1,\ldots,n)$. This Markov chain is *irreducible* and *aperiodic*. In addition, when *x* adequately approximates infinity, $\lim_{x\to\infty} Q_{ij}(x)=p_{ij}$, and $P=[p_{ij}]_0^n$ is a stochastic matrix (Pyke, 1961a). On the other hand, according to the Tauberian theorem 2.4(12) (see, Widder, 1946), it is written as $P=q(0)$.

Theorem 3.1 *When assumed that the mean service times of servers are equal* $(\mu_1=\mu_2=\cdots=\mu_n=\mu)$ *and for* $x\to+\infty$ *, formulae* (3.6) *and* (3.7) *yield formula* (3.4).

Proof. Depending on the the assumption $\mu_1=\mu_2=\cdots=\mu_n=\mu$, formulae (3.6) and (3.7) are written as follows for $i=1,2,\ldots,n-1$ and $0\leq j\leq i+1$, respectively:

$$Q_{ij}(x)=\frac{1}{\binom{n}{i+1}}\int_0^x \binom{n}{j} e^{-j\mu t}\binom{n-j}{i+1-j}(1-e^{-\mu t})^{i+1-j}\,dF(t)\,. \tag{3.11}$$

$$Q_{ij}(x) = \frac{1}{\binom{n}{i}} \int_0^x [\underbrace{g(\cdot) + g(\cdot) + \cdots g(\cdot)}_{\binom{n}{i} \text{ terms}}] dF(t)$$
$$= \frac{1}{\binom{n}{i}} \int_0^x \binom{n}{i} \cdot \binom{i+1}{j} \cdot e^{-j\mu t} (1 - e^{-\mu t})^{i+1-j} \, dF(t). \qquad (3.12)$$

In the last two equations obtained above, after some algebraic operations have been made and when $x \to +\infty$

$$\lim_{x \to \infty} Q_{ij}(x) = p_{ij} = \binom{i+1}{j} \int_0^\infty e^{-j\mu t} (1 - e^{-\mu t})^{i+1-j} \, dF(t) \, . \qquad (3.13)$$

The proof has been completed.

Corollary 3.1 Formulae (3.6) and (3.7) are the generalizations of Takacs's formula (3.4) for 'the *GI*/*M*/*n*/0 queueing model with heterogeneous servers' for random entry and ordered entry disciplines, respectively.

Theorem 3.2 (see, Çinlar, 1975)**.** *Let X Markov chain with state space* $D = (0,1,\ldots,n)$ *and transition matrix P. Suppose X is irreducible and aperiodic. Then all states of the Markov chain X are recurrent non-null, and steady-state probabilities* π_j *are the unique nonnegative solution of following linear equations*

$$\pi_j = \sum_{i=0}^{\infty} \pi_i \, p_{ij} \,, \quad j \in D \,, \qquad (3.14)$$

$$\sum_{j=0}^{\infty} \pi_j = 1 \, . \qquad (3.15)$$

The queueing model *GI*/*M*/3/0 with heterogeneous servers is examined for random entry and ordered entry disciplines in Subsection 3.2.1 and Subsection 3.2.2

respectively. Equations (3.14) and (3.15) given in Theorem 3.2 are solved for addressed queueing models, and also steady-state probabilities and the loss probability are computed.

*3.2.1 The Model GI/M/***3/0** *with Random Entry*

In this subsection, the way of computing formula (3.6) is explained in detail. The loss probability of the customer in the system is computed for $n=3$ by means of Laplace-Stieltjes transforms of the kernel functions of the semi-Markov process.

Model assumptions are the same as explained in 3.2. The service discipline is random entry and the number of servers is limited to 3. Kernel functions $(Q_{ij}(x);\, i,j=0,1,2,3)$ of the semi-Markov process representing the system are easily obtained by using the formula (3.6). Kernel functions and Laplace-Stieltjes transforms of these functions for the model *GI*/*M*/3/0 with random entry are obtained as follows.

For $Q_{00}(x)$, considering $(u,v)=(j,i+1-j)=(0,0+1-0)\Rightarrow(u,v)=(0,1)$ in the formula (3.6) , it is written as $Q_{00}(x)=\frac{1}{\binom{3}{1}}\int_0^x \sum_{1\le l_1\le 3} q_{l_1}\, dF(t)=\frac{1}{3}\int_0^x (q_1+q_2+q_3)dF(t)$.

Herefrom, the following is obtained:

$$Q_{00}(x)=\frac{1}{3}\int_0^x [(1-e^{-\mu_1 t})+(1-e^{-\mu_2 t})+(1-e^{-\mu_3 t})]dF(t)\,. \tag{3.16}$$

Thus, the Laplace-Stieltjes transform of $Q_{00}(x)$ represented by $q_{00}(s)$ is obtained as follows:

$$q_{00}(s)=f(s)-\frac{1}{3}[f(s+\mu_1)+f(s+\mu_2)+f(s+\mu_3)]\,. \tag{3.17}$$

For $Q_{01}(x)$, considering $(u,v)=(j,i+1-j)=(1,0+1-1)\Rightarrow(u,v)=(1,0)$ in the formula (3.6), it is written as $Q_{01}(x)=\frac{1}{\binom{3}{1}}\int_0^x\sum_{1\le k_1\le 3}p_{k_1}dF(t)=\frac{1}{3}\int_0^x(p_1+p_2+p_3)dF(t)$. Herefrom, the following is obtained:

$$Q_{01}(x)=\frac{1}{3}\int_0^x(e^{-\mu_1 t}+e^{-\mu_2 t}+e^{-\mu_3 t})dF(t)\,. \tag{3.18}$$

Thus, the Laplace-Stieltjes transform of $Q_{01}(x)$, represented by $q_{01}(s)$ is obtained as follows:

$$q_{01}(s)=\frac{1}{3}[f(s+\mu_1)+f(s+\mu_2)+f(s+\mu_3)]. \tag{3.19}$$

For $Q_{10}(x)$, considering $(u,v)=(j,i+1-j)=(0,1+1-0)\Rightarrow(u,v)=(0,2)$ in the formula (3.6), it is written as $Q_{10}(x)=\frac{1}{\binom{3}{2}}\int_0^x\sum_{1\le l_1<l_2\le 3}q_{l_1}q_{l_2}dF(t)$. More clearly, it is written as $Q_{10}=\frac{1}{3}\int_0^x(q_1q_2+q_1q_3+q_2q_3)dF(t)$. Herefrom, the following is obtained:

$$\begin{aligned}Q_{10}(x)=\frac{1}{3}\int_0^x\ &[(1-e^{-\mu_1 t})(1-e^{-\mu_2 t})+(1-e^{-\mu_1 t})(1-e^{-\mu_3 t})\\&+(1-e^{-\mu_2 t})(1-e^{-\mu_3 t})]dF(t).\end{aligned} \tag{3.20}$$

Thus, the Laplace-Stieltjes transform of $Q_{10}(x)$ represented by $q_{10}(s)$ is obtained as follows:

$$\begin{aligned}q_{10}(s)=f(s)&-\frac{2}{3}[f(s+\mu_1)+f(s+\mu_2)+f(s+\mu_3)]\\&+\frac{1}{3}[f(s+\mu_1+\mu_2)+f(s+\mu_1+\mu_3)+f(s+\mu_2+\mu_3)].\end{aligned} \tag{3.21}$$

For $Q_{11}(x)$, considering $(u,v)=(j,i+1-j)=(1,1+1-1)\Rightarrow(u,v)=(1,1)$ in the formula (3.6), it is written as $Q_{11}(x)=\frac{1}{\binom{3}{2}}\int_0^x \sum_{\substack{1\le k_1\le 3\\ 1\le l_1\le 3\\ k_1\ne l_1}} p_{k_1}q_{l_1}dF(t)$. More clearly, it is written as $Q_{11}=\frac{1}{3}\int_0^x (p_1q_2+p_1q_3+p_2q_1+p_2q_3+p_3q_1+p_3q_2)dF(t)$. Herefrom, the following is obtained:

$$Q_{11}(x)=\frac{1}{3}\int_0^x \begin{array}{l}[e^{-\mu_1 t}(1-e^{-\mu_2 t})+e^{-\mu_1 t}(1-e^{-\mu_3 t})+e^{-\mu_2 t}(1-e^{-\mu_1 t})\\ \quad +e^{-\mu_2 t}(1-e^{-\mu_3 t})+e^{-\mu_3 t}(1-e^{-\mu_1 t})+e^{-\mu_3 t}(1-e^{-\mu_2 t})]dF(t).\end{array} \tag{3.22}$$

Thus, the Laplace-Stieltjes transform of $Q_{11}(x)$ represented by $q_{11}(s)$ is obtained as follows:

$$\begin{aligned} q_{11}(s)=\frac{2}{3}[&f(s+\mu_1)+f(s+\mu_2)+f(s+\mu_3)\\ &-f(s+\mu_1+\mu_2)+f(s+\mu_1+\mu_3)+f(s+\mu_2+\mu_3)]. \end{aligned} \tag{3.23}$$

For $Q_{12}(x)$, considering $(u,v)=(j,i+1-j)=(2,1+1-2)\Rightarrow(u,v)=(2,0)$ in the formula (3.6), it is written as $Q_{12}(x)=\frac{1}{\binom{3}{2}}\int_0^x \sum_{1\le k_1<k_2\le 3} p_{k_1}p_{k_2}dF(t)$. More clearly, it is written as $Q_{12}=\frac{1}{3}\int_0^x (p_1p_2+p_1p_3+p_2p_3)dF(t)$. Herefrom, the following is obtained:

$$Q_{12}(x)=\frac{1}{3}\int_0^x [e^{-\mu_1 t}e^{-\mu_2 t}+e^{-\mu_1 t}e^{-\mu_3 t}+e^{-\mu_2 t}e^{-\mu_3 t})]dF(t). \tag{3.24}$$

Thus, the Laplace-Stieltjes transform of $Q_{12}(x)$ represented by $q_{12}(s)$ is obtained as follows:

$$q_{12}(s)=\frac{1}{3}[f(s+\mu_1+\mu_2)+f(s+\mu_1+\mu_3)+f(s+\mu_2+\mu_3)]. \qquad (3.25)$$

For $Q_{20}(x)$, considering $(u,v)=(j,i+1-j)=(0,2+1-0)\Rightarrow(u,v)=(0,3)$ in the formula (3.6), it is written as $Q_{20}(x)=\frac{1}{\binom{3}{3}}\int_0^x \sum_{1\le l_1<l_2<l_3\le 3} q_{l_1}q_{l_2}q_{l_3}dF(t)=\int_0^x q_1q_2q_3dF(t)$. Herefrom, the following is obtained:

$$Q_{20}(x)=\int_0^x (1-e^{-\mu_1 t})(1-e^{-\mu_2 t})(1-e^{-\mu_3 t})dF(t). \qquad (3.26)$$

Thus, the Laplace-Stieltjes transform of $Q_{20}(x)$ represented by $q_{20}(s)$ is obtained as follows:

$$\begin{aligned} q_{20}(s)= {} & f(s)-f(s+\mu_1)-f(s+\mu_2)-f(s+\mu_3)+f(s+\mu_1+\mu_2) \\ & +f(s+\mu_1+\mu_3)+f(s+\mu_2+\mu_3)-f(s+\mu_1+\mu_2+\mu_3). \end{aligned} \qquad (3.27)$$

For $Q_{21}(x)$, considering $(u,v)=(j,i+1-j)=(1,2+1-1)\Rightarrow(u,v)=(1,2)$ in the formula (3.6), it is written as $Q_{21}(x)=\frac{1}{\binom{3}{3}}\int_0^x \sum_{\substack{1\le k_1\le 3\\ 1\le l_1<l_2\le 3\\ k_u\ne l_v}} p_{k_1}q_{l_1}q_{l_2}dF(t)$. More clearly, it is written as $Q_{21}=\int_0^x (p_1q_2q_3+p_2q_1q_3+p_3q_1q_2)dF(t)$. Herefrom, the following is obtained:

$$\begin{aligned} Q_{21}(x)=\int_0^x & [e^{-\mu_1 t}(1-e^{-\mu_2 t})(1-e^{-\mu_3 t}) \\ & +e^{-\mu_2 t}(1-e^{-\mu_1 t})(1-e^{-\mu_3 t})+e^{-\mu_3 t}(1-e^{-\mu_1 t})(1-e^{-\mu_2 t})]dF(t). \end{aligned} \qquad (3.28)$$

Thus, the Laplace-Stieltjes transform of $Q_{21}(x)$ represented by $q_{21}(s)$ is obtained as follows:

$$q_{21}(s) = f(s+\mu_1) + f(s+\mu_2) + f(s+\mu_3) - 2[f(s+\mu_1+\mu_2) + f(s+\mu_1+\mu_3) + f(s+\mu_2+\mu_3)] + 3f(s+\mu_1+\mu_2+\mu_3). \tag{3.29}$$

For $Q_{22}(x)$, considering $(u,v) = (j, i+1-j) = (2, 2+1-2) \Rightarrow (u,v) = (2,1)$ in the formula (3.6), it is written as $Q_{22}(x) = \frac{1}{\binom{3}{3}} \int_0^x \sum_{\substack{1\le k_1<k_2\le 3 \\ 1\le l_1\le 3 \\ k_u \ne l_v}} p_{k_1} p_{k_2} q_{l_1} dF(t)$. More clearly, it is written as $Q_{22} = \int_0^x (p_1 p_2 q_3 + p_1 p_2 q_2 + p_2 p_3 q_1) dF(t)$. Herefrom, the following is obtained:

$$Q_{22}(x) = \int_0^x [e^{-\mu_1 t} e^{-\mu_2 t}(1-e^{-\mu_3 t}) + e^{-\mu_1 t} e^{-\mu_3 t}(1-e^{-\mu_2 t}) + e^{-\mu_2 t} e^{-\mu_3 t}(1-e^{-\mu_1 t})] dF(t). \tag{3.30}$$

Thus, the Laplace-Stieltjes transform of $Q_{22}(x)$ represented by $q_{22}(s)$ is obtained as follows:

$$q_{22}(s) = f(s+\mu_1+\mu_2) + f(s+\mu_1+\mu_3) + f(s+\mu_2+\mu_3) + 3f(s+\mu_1+\mu_2+\mu_3). \tag{3.31}$$

For $Q_{23}(x)$, considering $(u,v) = (j, i+1-j) = (3, 2+1-3) \Rightarrow (u,v) = (3,0)$ in the formula (3.6), it is written as $Q_{23}(x) = \frac{1}{\binom{3}{3}} \int_0^x \sum_{1\le k_1<k_2<k_3\le 3} p_{k_1} p_{k_2} p_{k_3} dF(t) = \int_0^x p_1 p_2 p_3 dF(t)$. Herefrom, the following is obtained:

$$Q_{23}(x) = \int_0^x e^{-\mu_1 t} e^{-\mu_2 t} e^{-\mu_3 t} dF(t). \tag{3.32}$$

Thus, the Laplace-Stieltjes transform of $Q_{23}(x)$ represented by $q_{23}(s)$ is obtained as follows:

$$q_{23}(s) = f(s+\mu_1+\mu_2+\mu_3). \tag{3.33}$$

Since only one customer arrives in the system in any interarrival time, considering $Q_{ij}(x)=0$ for $j>i+1$, $Q_{02}(x)=0$, $Q_{03}(x)=0$, and $Q_{13}(x)=0$ are obtained. Additionally, as required by the formula (3.6), kernel functions of $Q_{3,j}(x)$ are equal to $Q_{2,j}(x)$, namely, $Q_{3,j}(x)=Q_{2,j}(x)$ can be written. Thus, for $j=0,1,2,3$, the Laplace-Stieltjes transforms of kernel functions of the semi-Markov process $Q_{2,j}(x)$ are as follows:

$$q_{3j}(s) = q_{2j}(s)\ ,\ \ j=0,1,2,3. \tag{3.34}$$

One-step transition probabilities for the queueing model *GI*/*M*/3/0 with random entry are computed by means of kernel functions of the semi-Markov process formulated above or Laplace-Stieltjes transforms of kernel functions. According to Theorem 2.4 (Tauberian theorem); considering $\lim\limits_{x\to\infty} Q_{ij}(x) = \lim\limits_{s\to 0} q_{ij}(s)$, one-step transition probabilities $p_{ij} = \lim\limits_{s\to 0} q_{ij}(s)\ (i,j=0,1,2,3)$ for the related model are obtained as follows:

$$p_{00} = 1-\frac{1}{3}[f(\mu_1)+f(\mu_2)+f(\mu_3)], \tag{3.35}$$

$$p_{01} = \frac{1}{3}[f(\mu_1)+f(\mu_2)+f(\mu_3)], \tag{3.36}$$

$$\begin{aligned} p_{10} = 1 &- \frac{2}{3}[f(\mu_1)+f(\mu_2)+f(\mu_3)] \\ &+ \frac{1}{3}[f(\mu_1+\mu_2)+f(\mu_1+\mu_3)+f(\mu_2+\mu_3)], \end{aligned} \tag{3.37}$$

$$p_{11} = \frac{2}{3}[f(\mu_1) + f(\mu_2) + f(\mu_3) \\ - f(\mu_1+\mu_2) - f(\mu_1+\mu_3) - f(\mu_2+\mu_3)], \tag{3.38}$$

$$p_{12} = \frac{1}{3}[f(\mu_1+\mu_2) + f(\mu_1+\mu_3) + f(\mu_2+\mu_3)], \tag{3.39}$$

$$p_{20} = 1 - f(\mu_1) - f(\mu_2) - f(\mu_3) + f(\mu_1+\mu_2) \\ + f(\mu_1+\mu_3) + f(\mu_2+\mu_3) - f(\mu_1+\mu_2+\mu_3), \tag{3.40}$$

$$p_{21} = f(\mu_1) + f(\mu_2) + f(\mu_3) - 2[f(\mu_1+\mu_2) + f(\mu_1+\mu_3) \\ + f(\mu_2+\mu_3)] + 3f(\mu_1+\mu_2+\mu_3), \tag{3.41}$$

$$p_{22} = f(\mu_1+\mu_2) + f(\mu_1+\mu_3) + f(\mu_2+\mu_3) - 3f(\mu_1+\mu_2+\mu_3), \tag{3.42}$$

$$p_{23} = f(\mu_1+\mu_2+\mu_3), \tag{3.43}$$

$$p_{02} = 0\ ,\ p_{03} = 0,\ p_{13} = 0, \tag{3.44}$$

$$p_{3j} = p_{2j}\ ,\ j = 0,1,2,3. \tag{3.45}$$

Using the facts given by Theorem 3.2, by means of one-step transition probabilities explained above $(p_{ij}\ ,\ 0 \le i, j \le 3)$, steady-state probabilities $\pi_j\ \ (j = 0,1,2,3)$ for the queueing model *GI*/*M*/3/0 with random entry are obtained as follows as the solution of linear equation system given by (3.14):

$$\pi_0 = \frac{(1 - \frac{2}{3}f_1 + \frac{2}{3}f_2)(1 - f_2 + 3f_3) - \frac{1}{3}f_2(f_1 - 2f_2 + 3f_3)}{(1 - f_2 + 2f_3)(1 - \frac{1}{3}f_1 + \frac{1}{3}f_2) + \frac{1}{3}f_2(1 - \frac{2}{3}f_1 + f_2 - f_3)}\ , \tag{3.46}$$

$$\pi_1 = \frac{\frac{1}{3} f_1 (1 - f_2 + 2 f_3)}{(1 - f_2 + 2 f_3)(1 - \frac{1}{3} f_1 + \frac{1}{3} f_2) + \frac{1}{3} f_2 (1 - \frac{2}{3} f_1 + f_2 - f_3)}, \qquad (3.47)$$

$$\pi_2 = \frac{f_1 f_2 (1 - f_3)/9}{(1 - f_2 + 2 f_3)(1 - \frac{1}{3} f_1 + \frac{1}{3} f_2) + \frac{1}{3} f_2 (1 - \frac{2}{3} f_1 + f_2 - f_3)}, \qquad (3.48)$$

$$\pi_3 = \frac{f_1 f_2 f_3 /9}{(1 - f_2 + 2 f_3)(1 - \frac{1}{3} f_1 + \frac{1}{3} f_2) + \frac{1}{3} f_2 (1 - \frac{2}{3} f_1 + f_2 - f_3)}, \qquad (3.49)$$

where $f_1 = f(\mu_1) + f(\mu_2) + f(\mu_3)$, $f_2 = f(\mu_1 + \mu_2) + f(\mu_1 + \mu_3) + f(\mu_2 + \mu_3)$, and $f_3 = f(\mu_1 + \mu_2 + \mu_3)$.

Probabilities π_0, π_1, π_2 and π_3 denote the probability of being idle, the probability that only one server is busy in the system, the probability that two servers are busy in the system, and the probability that all servers are busy, respectively. As no waiting line is available in the system, the probability that all servers are busy is equivalent to the probability of loss of customers in the system. That is, formula (3.49) is equal to the loss probability.

Under the condition $\mu_1 = \mu_2 = \mu_3 = \mu$, the formula of loss probability given by (3.49) satisfies Palm's loss formula (3.2) for *n*=3.

It must be noted that the formula (3.49) is obtained by Isguder & Uzunoglu-Kocer (2010).

3.2.2 The Model GI/M/3/0 with Ordered Entry

In this subsection, the computation procedure of the formula (3.7) is explained in detail. The loss probability of the customer in the system is computed for $n=3$ by means of Laplace-Stieltjes transforms of kernel functions of the semi-Markov process.

Model assumptions are as explained in Section 3.2. The service discipline is an ordered entry and the number of servers is limited to 3. Kernel functions $(Q_{ij}(x);\, i, j = 0,1,2,3)$ of the semi-Markov process representing the system are easily obtained by using the formula (3.7) together with (3.8). Kernel functions and their Laplace-Stieltjes transforms for the model *GI/M*/3/0 with ordered entry are obtained as follows.

$Q_{00}(x)$ is written as $Q_{00}(x)=\frac{1}{\binom{3}{0}}\int_0^x g(r_0,m)dF(t)=\int_0^x g(m)dF(t)$ by using the formula (3.7). By using the equation (3.8), it is obvious that $g(m)=q_1$. Considering $(u,v)=(j,i+1-j)=(0,0+1-0)\Rightarrow(u,v)=(0,1)$, the following is obtained:

$$Q_{00}(x)=\int_0^x q_1 dF(t)=\int_0^x (1-e^{-\mu_1 t})dF(t)\,. \tag{3.50}$$

Herefrom, the Laplace-Stieltjes transform of $Q_{00}(x)$ represented by $q_{00}(s)$ is obtained as follows:

$$q_{00}(s)=f(s)-f(s+\mu_1)\,. \tag{3.51}$$

$Q_{01}(x)$ is written as $Q_{00}(x)=\frac{1}{\binom{3}{0}}\int_0^x g(r_0,m)dF(t)=\int_0^x g(m)dF(t)$ by using the formula (3.7). By using the equation (3.8), it is obvious that $g(m)=p_1$. Considering $(u,v)=(1,0+1-1)\Rightarrow(u,v)=(1,0)$, the following is obtained:

$$Q_{01}(x) = \int_0^x p_1 dF(t) = \int_0^x e^{-\mu_1 t} dF(t)\,. \tag{3.52}$$

Herefrom, the Laplace-Stieltjes transform $q_{01}(s)$ of $Q_{01}(x)$ is obtained as follows:

$$q_{01}(s) = f(s+\mu_1)\,. \tag{3.53}$$

$Q_{10}(x)$ is written as $Q_{10}(x) = \frac{1}{\binom{3}{1}} \int_0^x g(r_1, m) dF(t) = \frac{1}{3} \int_0^x g(r_1, m) dF(t)$ by using the formula (3.7). By using the equation (3.8), it is obvious that $\sum_{\substack{1 \le r_1 \le 3 \\ m = \min\{1,2,3\} \\ m \ne r_1}} g(r_1, m) = g(1,2) + g(2,1) + g(3,1)$.

Considering $(u,v) = (0,1+1-0) \Rightarrow (u,v) = (0,2)$, it can be written as follows: $Q_{10}(x) = \frac{1}{3} \int_0^x (q_1 q_2 + q_2 q_1 + q_3 q_1) dF(t) = \frac{1}{3} \int_0^x (2 q_1 q_2 + q_3 q_1) dF(t)$. Herefrom, the following is obtained:

$$Q_{10}(x) = \frac{1}{3} \int_0^x [2(1-e^{-\mu_1 t})(1-e^{-\mu_2 t}) + (1-e^{-\mu_1 t})(1-e^{-\mu_3 t})] dF(t)\,. \tag{3.54}$$

Herefrom, the Laplace-Stieltjes transform $q_{10}(s)$ of $Q_{10}(x)$ ’in LS is obtained as follows:

$$\begin{aligned} q_{10}(s) = {} & f(s) - f(s+\mu_1) - \frac{2}{3} f(s+\mu_2) \\ & - \frac{1}{3} f(s+\mu_3) + \frac{2}{3} f(s+\mu_1+\mu_2) + \frac{1}{3} f(s+\mu_1+\mu_3). \end{aligned} \tag{3.55}$$

$Q_{11}(x)$ is written as $Q_{11}(x)=\frac{1}{\binom{3}{1}}\int_0^x g(r_1,m)dF(t)=\frac{1}{3}\int_0^x g(r_1,m)dF(t)$ by using the formula (3.7). By using the equation (3.8), it is obvious that $\sum_{\substack{1\le r_1\le 3\\ m=\min\{1,2,3\}\\ m\neq r_1}} g(r_1,m)=g(1,2)+g(2,1)+g(3,1)$.

Considering $(u,v)=(1,1+1-1)\Rightarrow(u,v)=(1,1)$, it is computed as $g(1,2)=\sum_{\substack{k_1,l_1\in\{1,2\}\\ k_1\neq l_1}} p_{k_1}q_{l_1}=p_1q_2+p_2q_1$. $g(2,1)$ and $g(3,1)$ are computed similarly. Thus, it can be written as $Q_{11}(x)=\frac{1}{3}\int_0^x (2p_1q_2+2p_2q_1+p_1q_3+p_3q_1)dF(t)$. Herefrom, the following is obtained:

$$Q_{11}(x)=\frac{1}{3}\int_0^x \begin{aligned}[t]&[2e^{-\mu_1 t}(1-e^{-\mu_2 t})+2e^{-\mu_2 t}(1-e^{-\mu_1 t})\\ &+e^{-\mu_1 t}(1-e^{-\mu_3 t})+e^{-\mu_3 t}(1-e^{-\mu_1 t})]dF(t).\end{aligned} \tag{3.56}$$

Herefrom, the Laplace-Stieltjes transform $q_{11}(s)$ of $Q_{11}(x)$'in LS is obtained as follows:

$$\begin{aligned}q_{11}(s)=&\ f(s+\mu_1)+\frac{2}{3}f(s+\mu_2)+\frac{1}{3}f(s+\mu_3)\\ &-\frac{4}{3}f(s+\mu_1+\mu_2)-\frac{2}{3}f(s+\mu_1+\mu_3).\end{aligned} \tag{3.57}$$

$Q_{12}(x)$ is written as $Q_{12}(x)=\frac{1}{\binom{3}{1}}\int_0^x g(r_1,m)dF(t)=\frac{1}{3}\int_0^x g(r_1,m)dF(t)$ by using the formula (3.7). By using the equation (3.8), it is obvious that $\sum_{\substack{1\le r_1\le 3\\ m=\min\{1,2,3\}\\ m\neq r_1}} g(r_1,m)=g(1,2)+g(2,1)+g(3,1)$.

Considering $(u,v)=(2,1+1-2)\Rightarrow(u,v)=(2,0)$, it is computed as $g(1,2)=\sum_{k_1<k_2\in\{1,2\}} p_{k_1}p_{k_2}\cdot 1 = p_1p_2$. $g(2,1)$ and $g(3,1)$ are computed similarly. Herefrom, it can be written as $Q_{12}(x)=\frac{1}{3}\int_0^x (2p_1p_2+p_1p_3)dF(t)$. Herefrom, the following is obtained:

$$Q_{12}(x)=\frac{1}{3}\int_0^x \quad (2e^{-\mu_1 t}e^{-\mu_2 t}+e^{-\mu_1 t}e^{-\mu_3 t})dF(t)\,. \tag{3.58}$$

Herefrom, the Laplace-Stieltjes transform $q_{12}(s)$ of $Q_{12}(x)$ is obtained as follows:

$$q_{12}(s)=\frac{2}{3}f(s+\mu_1+\mu_2)+\frac{1}{3}f(s+\mu_1+\mu_3)\,. \tag{3.59}$$

$Q_{20}(x)$ is written as $Q_{20}(x)=\frac{1}{\binom{3}{2}}\int_0^x g(r_1,r_2,m)dF(t)=\frac{1}{3}\int_0^x g(r_1,r_2,m)dF(t)$ by using the formula (3.7). By using the equation (3.8), $\sum_{\substack{1\le r_1<r_2\le 3\\ m=\min\{1,2,3\}\\ m\ne r_i}} g(r_1,r_2,m)=g(1,2,3)+g(1,3,2)+g(2,3,1)$ is found. Considering $(u,v)=(0,2+1-0)\Rightarrow(u,v)=(0,3)$, it is computed as $g(1,2,3)=\sum_{l_1<l_2<l_3\in\{1,2,3\}} 1\cdot q_{l_1}q_{l_2}q_{l_3}=q_1q_2q_3$. $g(1,3,2)$ and $g(2,3,1)$ are computed similarly. Thus, $Q_{20}(x)=\frac{1}{3}\int_0^x 3q_1q_2q_3dF(t)$ can be written. Herefrom, the following is obtained:

$$Q_{20}(x)=\int_0^x \quad (1-e^{-\mu_1 t})(1-e^{-\mu_2 t})(1-e^{-\mu_3 t})dF(t)\,. \tag{3.60}$$

Herefrom, the Laplace-Stieltjes transform $q_{20}(s)$ of $Q_{20}(x)$ is obtained as follows:

$$q_{20}(s) = f(s) - f(s+\mu_1) - f(s+\mu_2) - f(s+\mu_3) + f(s+\mu_1+\mu_2) \\ + f(s+\mu_1+\mu_3) + f(s+\mu_2+\mu_3) - f(s+\mu_1+\mu_2+\mu_3). \tag{3.61}$$

$Q_{21}(x)$ is written as $Q_{21}(x) = \frac{1}{3}\int_0^x g(r_1, r_2, m)dF(t)$ by using the formula (3.7). By using the equation (3.8), $\sum_{\substack{1\le r_1<r_2\le 3 \\ m=\min\{1,2,3\} \\ m\ne r_i}} g(r_1, r_2, m) = g(1,2,3) + g(1,3,2) + g(2,3,1)$ is found. Considering $(u,v) = (1, 2+1-1) \Rightarrow (u,v) = (1,2)$, it is computed as $g(1,2,3) = \sum_{\substack{k_1\in\{1,2,3\} \\ l_1<l_2\in\{1,2,3\} \\ k_u\ne l_v}} p_{k_1}\cdot q_{l_1}q_{l_2} = p_1q_2q_3 + p_2q_1q_3 + p_3q_1q_2$. $g(1,3,2)$ and $g(2,3,1)$ are computed similarly. Thus, $Q_{21}(x) = \frac{1}{3}\int_0^x 3(p_1q_2q_3 + p_2q_1q_3 + p_3q_1q_2)dF(t)$ can be written. Herefrom, the following is obtained:

$$Q_{21}(x) = \int_0^x \begin{aligned}[t] &[e^{-\mu_1 t}(1-e^{-\mu_2 t})(1-e^{-\mu_3 t}) + e^{-\mu_2 t}(1-e^{-\mu_1 t})(1-e^{-\mu_3 t}) \\ &+ e^{-\mu_3 t}(1-e^{-\mu_1 t})(1-e^{-\mu_2 t})]dF(t). \end{aligned} \tag{3.62}$$

Herefrom, the Laplace-Stieltjes transform $q_{21}(s)$ of $Q_{21}(x)$ is obtained as follows:

$$q_{21}(s) = f(s+\mu_1) + f(s+\mu_2) + f(s+\mu_3) - 2[f(s+\mu_1+\mu_2) \\ + f(s+\mu_1+\mu_3) + f(s+\mu_2+\mu_3)] + 3f(s+\mu_1+\mu_2+\mu_3). \tag{3.63}$$

$Q_{22}(x)$ is written as $Q_{22}(x) = \frac{1}{3}\int_0^x g(r_1, r_2, m)dF(t)$ by using the formula (3.7). By using the equation (3.8), $\sum_{\substack{1\le r_1<r_2\le 3 \\ m=\min\{1,2,3\} \\ m\ne r_i}} g(r_1, r_2, m) = g(1,2,3) + g(1,3,2) + g(2,3,1)$ is found. Considering $(u,v) = (2, 2+1-2) \Rightarrow (u,v) = (2,1)$, it is computed as $g(1,2,3) = \sum_{\substack{k_1<k_2\in\{1,2,3\} \\ l_1\in\{1,2,3\} \\ k_u\ne l_v}} p_{k_1}p_{k_2}\cdot q_{l_1} = p_1p_2q_3 + p_1p_3q_2 + p_2p_3q_1$. $g(1,3,2)$ and $g(2,3,1)$ are

computed similarly. Thus, $Q_{22}(x)=\frac{1}{3}\int_0^x 3(p_1p_2q_3+p_1p_3q_2+p_2p_3q_1)dF(t)$ can be written. Herefrom, the following is obtained:

$$Q_{22}(x)=\int_0^x \begin{aligned}&[e^{-\mu_1 t}e^{-\mu_2 t}(1-e^{-\mu_3 t})+e^{-\mu_1 t}e^{-\mu_3 t}(1-e^{-\mu_2 t})\\ &+e^{-\mu_2 t}e^{-\mu_3 t}(1-e^{-\mu_1 t})]dF(t).\end{aligned} \tag{3.64}$$

Herefrom, the Laplace-Stieltjes transform $q_{22}(s)$ of $Q_{22}(x)$ is obtained as follows:

$$\begin{aligned}q_{22}(s)=&f(s+\mu_1+\mu_2)+f(s+\mu_1+\mu_3)+f(s+\mu_2+\mu_3)\\ &-3f(s+\mu_1+\mu_2+\mu_3).\end{aligned} \tag{3.65}$$

It is written as $Q_{23}(x)=\frac{1}{3}\int_0^x g(r_1,r_2,m)dF(t)$. By using the equation (3.8), $\sum_{\substack{1\le r_1<r_2\le 3\\ m=\min\{1,2,3\}\\ m\ne r_i}} g(r_1,r_2,m)=g(1,2,3)+g(1,3,2)+g(2,3,1)$ is found. Considering $(u,v)=(3,2+1-3)\Rightarrow(u,v)=(3,0)$, it is computed as $g(1,2,3)=\sum_{k_1<k_2<k_3\in\{1,2,3\}} p_{k_1}p_{k_2}p_{k_3}\cdot 1=p_1p_2p_3$. $g(1,3,2)$ and $g(2,3,1)$ are computed similarly. Thus, $Q_{23}(x)=\frac{1}{3}\int_0^x 3p_1p_2p_3dF(t)$ can be written. Herefrom, the following is obtained:

$$Q_{23}(x)=\int_0^x e^{-\mu_1 t}e^{-\mu_2 t}e^{-\mu_3 t}dF(t)\,. \tag{3.66}$$

Herefrom, the Laplace-Stieltjes transform $q_{23}(s)$ of $Q_{23}(x)$ is obtained as follows:

$$q_{23}(s)=f(s+\mu_1+\mu_2+\mu_3). \tag{3.67}$$

Since only one customer arriving in the system within any interarrival time, considering $Q_{ij}(x)=0$ for $j>i+1$, $Q_{02}(x)=0$, $Q_{03}(x)=0$ and $Q_{13}(x)=0$ are obtained. Additionally, as required by the formula (3.7), kernel functions of $Q_{3,j}(x)$ are equal to $Q_{2,j}(x)$, namely $Q_{3,j}(x)=Q_{2,j}(x)$ can be written. Thus, the Laplace-Stieljes transform of kernel functions of the semi-Markov process of $Q_{2,j}(x)$ for $j=0,1,2,3$ are as follows:

$$q_{3j}(s)=q_{2j}(s)\ ,\ \ j=0,1,2,3. \tag{3.68}$$

One-step transition probabilities for the queueing model *GI*/*M*/3/0 with ordered entry are computed by means of kernel functions of the semi-Markov process formulated above or Laplace-Stieltjes transforms of kernel functions. According to Theorem 2.4'e (Tauberian theorem); considering $\lim_{x\to\infty} Q_{ij}(x)=\lim_{s\to 0} q_{ij}(s)$, one-step transition probabilities $p_{ij}=\lim_{s\to 0} q_{ij}(s)\ (i,j=0,1,2,3)$ for the corresponding model are obtained as follows:

$$p_{00}=1-f(\mu_1), \tag{3.69}$$

$$p_{01}=f(\mu_1), \tag{3.70}$$

$$p_{10}=1-f(\mu_1)-\frac{2}{3}f(\mu_2)-\frac{1}{3}f(\mu_3)+\frac{2}{3}f(\mu_1+\mu_2)+\frac{1}{3}f(\mu_1+\mu_3), \tag{3.71}$$

$$p_{11}=f(\mu_1)+\frac{2}{3}f(\mu_2)+\frac{1}{3}f(\mu_3)-\frac{4}{3}f(\mu_1+\mu_2)-\frac{2}{3}f(\mu_1+\mu_3), \tag{3.72}$$

$$p_{12}=\frac{2}{3}f(\mu_1+\mu_2)+\frac{1}{3}f(\mu_1+\mu_3), \tag{3.73}$$

$$p_{20} = 1 - f(\mu_1) - f(\mu_2) - f(\mu_3) + f(\mu_1+\mu_2) \\ + f(\mu_1+\mu_3) + f(\mu_2+\mu_3) - f(\mu_1+\mu_2+\mu_3), \tag{3.74}$$

$$p_{21} = f(\mu_1) + f(\mu_2) + f(\mu_3) - 2[f(\mu_1+\mu_2) + f(\mu_1+\mu_3) \\ + f(\mu_2+\mu_3)] + 3f(\mu_1+\mu_2+\mu_3), \tag{3.75}$$

$$p_{22} = f(\mu_1+\mu_2) + f(\mu_1+\mu_3) + f(\mu_2+\mu_3) - 3f(\mu_1+\mu_2+\mu_3), \tag{3.76}$$

$$p_{23} = f(\mu_1+\mu_2+\mu_3), \tag{3.77}$$

$$p_{02} = 0\ ,\ p_{03} = 0,\ p_{13} = 0, \tag{3.78}$$

$$p_{3j} = p_{2j}\ ,\ j = 0,1,2,3. \tag{3.79}$$

Using the facts given by Theorem 3.2, by means of one-step transition probabilities $(p_{ij}\ ,\ 0 \le i, j \le 3)$ computed above, steady-state probabilities $\pi_j\ \ (j = 0,1,2,3)$ for the queueing model *GI*/*M*/3/0 with ordered entry are obtained as follows as the solution of linear equation system given by (3.14):

$$\pi_0 = \frac{\begin{array}{l} 3 - 3f(\mu_2+\mu_3) + f(\mu_1+\mu_2+\mu_3)[6 + 2f(\mu_1+\mu_2) + f(\mu_1+\mu_3)] \\ -[3f(\mu_1) + 2f(\mu_2) + f(\mu_3)][1 + 2f(\mu_1+\mu_2+\mu_3) - f(\mu_2+\mu_3)] \\ - f(\mu_1+\mu_3)[1 - 2f(\mu_1) - f(\mu_2)] + f(\mu_1+\mu_2)[1 + f(\mu_1) - f(\mu_3)] \end{array}}{\begin{array}{l} f(\mu_1+\mu_2+\mu_3)[2f(\mu_1+\mu_2) + f(\mu_1+\mu_3)] \\ - f(\mu_1+\mu_3)[1 - f(\mu_2)] + f(\mu_1+\mu_2)[1 - f(\mu_3)] \\ +[3 - 2f(\mu_2) - f(\mu_3)][1 - f(\mu_2+\mu_3) + 2f(\mu_1+\mu_2+\mu_3)] \end{array}}, \tag{3.80}$$

$$\pi_1 = \frac{3f(\mu_1)[1-f(\mu_1+\mu_2)-f(\mu_1+\mu_3)-f(\mu_1+\mu_2)+2f(\mu_1+\mu_2+\mu_3)]}{\begin{array}{l} f(\mu_1+\mu_2+\mu_3)[2f(\mu_1+\mu_2)+f(\mu_1+\mu_3)] \\ \quad -f(\mu_1+\mu_3)[1-f(\mu_2)]+f(\mu_1+\mu_2)[1-f(\mu_3)] \\ \quad +[3-2f(\mu_2)-f(\mu_3)][1-f(\mu_2+\mu_3)+2f(\mu_1+\mu_2+\mu_3)] \end{array}}, \tag{3.81}$$

$$\pi_2 = \frac{f(\mu_1)[2f(\mu_1+\mu_2)+f(\mu_1+\mu_3)][1-f(\mu_1+\mu_2+\mu_3)]}{\begin{array}{l} f(\mu_1+\mu_2+\mu_3)[2f(\mu_1+\mu_2)+f(\mu_1+\mu_3)] \\ \quad -f(\mu_1+\mu_3)[1-f(\mu_2)]+f(\mu_1+\mu_2)[1-f(\mu_3)] \\ \quad +[3-2f(\mu_2)-f(\mu_3)][1-f(\mu_2+\mu_3)+2f(\mu_1+\mu_2+\mu_3)] \end{array}}, \tag{3.82}$$

$$\pi_3 = \frac{f(\mu_1)f(\mu_1+\mu_2+\mu_3)[2f(\mu_1+\mu_2)+f(\mu_1+\mu_3)]}{\begin{array}{l} f(\mu_1+\mu_2+\mu_3)[2f(\mu_1+\mu_2)+f(\mu_1+\mu_3)] \\ \quad -f(\mu_1+\mu_3)[1-f(\mu_2)]+f(\mu_1+\mu_2)[1-f(\mu_3)] \\ \quad +[3-2f(\mu_2)-f(\mu_3)][1-f(\mu_2+\mu_3)+2f(\mu_1+\mu_2+\mu_3)] \end{array}}. \tag{3.83}$$

Probabilities π_0, π_1, π_2 and π_3 denote the probability of being idle, the probability that only one server is busy in the system, the probability that two servers are busy in the system, and the probability that all servers are busy, respectively. As no waiting line is available in the system, the probability that all servers are busy is equivalent to the probability of loss of customers in the system. That is, formula (3.83) is equal to the loss probability.

Under the condition $\mu_1 = \mu_2 = \mu_3 = \mu$, the formula of loss probability given by (3.83) satisfies Palm's loss formula (3.2) for *n*=3.

It must be noted that the formula (3.83) is obtained by Isguder & Celikoglu (2010).

The most important problem is the minimization of the loss probability of customer in the queueing system addressed in this study and similar queueing systems. Alpaslan (1996), Saglam & Shahbazov (2007) minimized the loss probability of the customer in the system for 'the queueing model *GI*/*M*/2/0 with heterogeneous servers'. Isguder & Uzunoglu-Kocer (2010) minimized the loss probability for 'the queueing model *GI*/*M*/3/0 with random entry' according to arrival flow. Isguder & Celikoglu (2012) minimized the loss probability for 'the queueing model *GI*/*M*/3/0 with ordered entry' according to the arrival flow. In the mentioned studies, it was proven by using the inequality $f(s) \geq e^{-as}$ obtained from Jensen equation that the loss probability is minimum when interarrival time distribution is selected as deterministic among the distributions which has the same mean.

In the following subsequent section, the theorem given by Isguder & Celikoglu (2012) related to the minimization of the loss probability for 'the queueing model *GI*/*M*/3/0 with ordered entry' is explained in detail.

3.2.3 Optimization of Loss Probability According to Arrival Process

Let H_a be a class of distribution functions F of the interarrival times, the mean of which is constant *a*. Let $P_{loss}(F)$ be the loss probability for the *GI*/*M*/3/0 queueing system with heterogeneous servers and ordered entry, and $F \in H_a$. Assume that $D(t)$ is the deterministic distribution, in which $D(t) = 1$ for $t \leq a$ and $D(t) = 0$ for $t > a$. It is clearly seen here that $D \in H_a$ and e^{-as} are the Laplace-Stieljes transforms of $D(t)$.

Theorem 3.3 (Isguder & Celikoglu, 2012)**.** *When the distribution of interarrival times fits the deterministic distribution* $(D \in H_a)$ *among all distribution functions included in class* H_a, *loss probability* $P_{loss}(F)$ *becomes minimum, that is,* $\min_{F \in H_a} P_{loss}(F) = P_{loss}(D)$.

Proof. To minimize the loss probability, let formula (3.83) be arranged in the following way:

$$P_{loss}(F) = \frac{f_1 f_{123}(2f_{12} + f_{13})}{\begin{array}{c}(-f_{123})(-2f_{12} - f_{13}) + (-f_{13})(1 - f_2) - (-f_{12})(1 - f_3) \\ +(3 - 2f_2 - f_3)(1 - f_{23}) - (3 - 2f_2 - f_3)(-2f_{123})\end{array}} . \tag{3.84}$$

where $f_1 = f(\mu_1)$, $f_2 = f(\mu_2)$, $f_3 = f(\mu_3)$, $f_{12} = f(\mu_1 + \mu_2)$, $f_{13} = f(\mu_1 + \mu_3)$, $f_{23} = f(\mu_2 + \mu_3)$ and $f_{123} = f(\mu_1 + \mu_2 + \mu_3)$.

The numerator of formula (3.84), $f_1 f_{123}(2f_{12} + f_{13})$, is written as follows by means of inequality $f(s) \geq e^{-as}$ obtained from the Jensen's inequality (See, Shahbazov, 2005):

$$\begin{aligned} f_1 f_{123}(2f_{12} + f_{13}) &= f(\mu_1) f(\mu_1 + \mu_2 + \mu_3)[2f(\mu_1 + \mu_2) + f(\mu_1 + \mu_3)] \\ &\geq e^{-a\mu_1} e^{-a(\mu_1+\mu_2+\mu_3)}(2e^{-a(\mu_1+\mu_2)} + e^{-a(\mu_1+\mu_3)}). \end{aligned} \tag{3.85}$$

The following inequality is obtained by applying inequality $f(s) \geq e^{-as}$ to the Laplace-Stieljes transforms included in the denominator of formula (3.84), respectively:

$$\begin{aligned} &(-f_{123})(-2f_{12} - f_{13}) + (-f_{13})(1 - f_2) \\ &\quad -(-f_{12})(1 - f_3) + (3 - 2f_2 - f_3)(1 - f_{23}) \\ &\quad -(3 - 2f_2 - f_3)(-2f_{123}) \leq e^{-a(\mu_1+\mu_2+\mu_3)}(2e^{-a(\mu_1+\mu_2)} + e^{-a(\mu_1+\mu_3)}) \\ &\qquad - e^{-a(\mu_1+\mu_3)}(1 - e^{-a\mu_2}) + e^{-a(\mu_1+\mu_2)}(1 - e^{-a\mu_3}) \\ &\qquad + (3 - 2e^{-a\mu_2} - e^{-a\mu_3})(1 - e^{-a(\mu_1+\mu_2)} + 2e^{-a(\mu_1+\mu_2+\mu_3)}) \end{aligned} \tag{3.86}$$

where

$$- f_2 = -f(\mu_2) \leq -e^{-a\mu_2} , \tag{3.87}$$

$$- f_3 = -f(\mu_3) \leq -e^{-a\mu_3} , \tag{3.88}$$

$$- f_{12} = f(\mu_1 + \mu_2) \leq -e^{-a(\mu_1+\mu_2)} , \tag{3.89}$$

$$- f_{13} = f(\mu_1 + \mu_3) \leq -e^{-a(\mu_1+\mu_3)}, \tag{3.90}$$

$$- f_{23} = f(\mu_2 + \mu_3) \leq -e^{-a(\mu_2+\mu_3)}, \tag{3.91}$$

$$- f_{123} = f(\mu_1 + \mu_2 + \mu_3) \leq -e^{-a(\mu_1+\mu_2+\mu_3)}. \tag{3.92}$$

If inequalities (3.85) and (3.86) are inserted into their appropriate places in the numerator and denominator of formula (3.84) respectively, the following inequality is obtained:

$$P_{loss}(F) \geq \frac{e^{-a\mu_1} e^{-a(\mu_1+\mu_2+\mu_3)}(2e^{-a(\mu_1+\mu_2)} + e^{-a(\mu_1+\mu_3)})}{\begin{array}{l} e^{-a(\mu_1+\mu_2+\mu_3)}(2e^{-a(\mu_1+\mu_2)} + e^{-a(\mu_1+\mu_3)}) \\ \quad - e^{-a(\mu_1+\mu_3)}(1 - e^{-a\mu_2}) + e^{-a(\mu_1+\mu_2)}(1 - e^{-a\mu_3}) \\ \quad + (3 - 2e^{-a\mu_2} - e^{-a\mu_3})(1 - e^{-a(\mu_1+\mu_2)} + 2e^{-a(\mu_1+\mu_2+\mu_3)}) \end{array}}. \tag{3.93}$$

As the Laplace-Stieljes transform of $D(t)$ is e^{-as}, the right side of the last inequality obtained above has the value of $P_{loss}(D)$. Based on this, it is obtained that $\min\limits_{F \in H_a} P_{loss}(F) = P_{loss}(D)$. The proof has been completed.

Corollary 3.2 The loss probability becomes minimum with probability 1 when a deterministic distribution is selected among the interarrival distributions with the same mean for 'the queueing model *GI*/*M*/3/0 with ordered entry'.

It is not possible to minimize the loss probability with the method addressed above as the number of servers is increased. The results obtained by Theorem 3.3 will be supported with simulation study and it will be proven by simulation optimization in Chapter Four that, according the arrival input, the optimal condition is reached again by deterministic distribution when the number of servers increases.

In the subsequent section, the Laplace-Stieljes transform of the distribution of the stream of overflows is obtained by analyzing the stream of overflows. Also, the loss probability is formulated directly depending on determinant without a need for the solution of the equation system taking place in Theorem 3.2 and given by (3.14).

3.3 Analyzing the Stream of Overflows from *GI*/*M*/*n*/0 with Heterogeneous Servers

Let the instants of overflows be $\tau_0, \tau_1, \tau_2, \ldots,$ where $0 = \tau_0 < \tau_1 < \cdots$ and $W_k = \tau_k - \tau_{k-1}$ for $k \geq 1$. Sequence $\{\tau_k\,,\, k \geq 1\}$ is called 'stream of overflows'. Interoverflow times W_1 and $W_k\,,\, k \geq 2$, are independent and nonnegative random variables and equal to the first passage time from 0 to *n* and the recurrence time to *n* in the semi-Markov process $\{X(t),\, t \geq 0\}$, respectively. Therefore, sequence $\{W_n\,,\, n \geq 1\}$ denotes the interarrival times of the delayed renewal process. For ease, they are written as $T_{0n} = W_1$ and $T_{nn} = W_k\,,\; k \geq 2$, where T_{0n} and T_{nn} are the first passage time from 0 to n and the recurrence time to n, respectively. $\varphi_{0n}(s)$ and φ_{nn} are the Laplace-Stieltjes transforms of T_{0n} and T_{nn}, respectively. Çinlar & Disney (1967) analyzed the stream of overflows for a finite queueing model with a recurrent arrival process and a single exponential server and obtained the Laplace-Stieltjes transforms of the interoverflow times that were independent and had an identical distribution.

Pyke (1961b) proved that the inverse of matrix $I - q(s) = [\delta_{ij} - q_{ij}(s)]$ was present under $\operatorname{Re}\{s\} > 0$ and obtained the results given by (3.94) and (3.95) for the Laplace-Stieljes transform of the distribution of the first passage times and the Laplace-Stieljes transform of the distribution of recurrence times:

$$\varphi_{0n}(s) = r_{0n}(s) / r_{nn}(s)\,, \tag{3.94}$$

$$1 - \varphi_{nn}(s) = 1 / r_{nn}(s)\,, \tag{3.95}$$

where r_{0n} and r_{nn} are the (n,0)th and (n,n)th entries of matrix $[I-q(s)]^{-1}$, respectively. Note that δ_{ij} is the well-known Kronecker delta. Using the formula of the inverse of the matrix, we obtain the equations for Laplace-Stieljes transform of the first passage time distribution and Laplace-Stieljes transform of the recurrence time distribution as given in (3.96) and (3.97), respectively:

$$\varphi_{0n}(s) = D_{0n}(s)/D_{nn}(s), \tag{3.96}$$

$$1-\varphi_{nn}(s) = |1-q(s)|/D_{nn}(s), \tag{3.97}$$

where D_{0n} and D_{nn} are the cofactors of the (n,0)th and (n,n)th entries of matrix $I-q(s)$, respectively. On the other hand, the mean recurrence time to n is found as follows by means of (3.97):

$$E[T_{nn}] = D(m_0, m_1, \ldots, m_n)/D_{nn}(0), \tag{3.98}$$

where, for $i=1,2,\ldots n$, m_i is the expected value of the sojourn time in state i and $D(m_0, m_1, \ldots, m_n)$ is the determinant of matrix $[I-q(0)]$, the 0th column of which is vector $(m_0, m_1, \ldots, m_n)'$. On the other hand, if the semi-Markov process is irreducible and if T_{ii} has a non-lattice distribution with a finite mean, then P_i exists and is independent of the initial state (see, Ross, 1996). Furthermore,

$$P_i = m_i / E[T_{ii}]. \tag{3.99}$$

Note that P_i is equal to the long-run proportion of time where the process is in state i.

Using matrix (3.9) and the determinant properties of the matrix, we can write $|I-q(s)|$ and determinants D_{0n} and D_{nn} as follows, respectively:

$$\left|I - q(s)\right| = \begin{vmatrix} 1-f & -q_{01} & 0 & \cdots & 0 \\ 1-f & 1-q_{11} & -q_{12} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1-f & -q_{n-1,1} & -q_{n-1,2} & \cdots & -q_{n-1,n} \\ 1-f & -q_{n-1,1} & -q_{n-1,2} & \cdots & 1-q_{n-1,n} \end{vmatrix}, \tag{3.100}$$

$$D_{0n}(s) = (-1)^n \cdot (-q_{01})(-q_{12})(-q_{23}) \cdots (-q_{n-1,n}), \tag{3.101}$$

$$D_{nn}(s) = \begin{vmatrix} 1-q_{00} & -q_{01} & 0 & \cdots & 0 \\ -q_{10} & 1-q_{11} & -q_{12} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -q_{n-2,0} & -q_{n-2,1} & -q_{n-2,2} & \cdots & -q_{n-2,n-1} \\ -q_{n-1,0} & -q_{n-1,1} & -q_{n-1,2} & \cdots & 1-q_{n-1,n-1} \end{vmatrix}, \tag{3.102}$$

where $f = f(s)$ and $q_{ij} = q_{ij}(s)$, $0 < i, j < n$. Elements of the above determinants are obtained by applying the Laplace-Stieljes transform to formulae (3.6) and (3.7) for random entry and ordered entry disciplines, respectively.

Definition 3.1 Provided that (3.101), (3.102) and (3.100), (3.102) are inserted into their appropriate places in formulae (3.96) and (3.97), respectively, the obtained formulae are defined as the Laplace-Stieljes transform of the distribution of the stream of overflows from 'the *GI*/*M*/*n*/0 queueing model with heterogeneous servers'.

Example 3.1 Consider the *GI*/*M*/2/0 queueing model with heterogeneous servers. The assumptions of the system are as explained in Section 3.2. Using Definition 3.1, formulae (3.6) and (3.7), after some algebraic operations have been made, the Laplace-Stieljes transform of the distribution of the stream of overflows in the queueing model concerned is obtained as follows for random entry and ordered entry disciplines, respectively.

For Random Entry Discipline:

$$\varphi_{02}(s) = \frac{[f(s+\mu_1) + f(s+\mu_2)]\, f(s+\mu_1+\mu_2)}{[1-f(s)][2-f(s+\mu_1)-f(s+\mu_2)] + f(s+\mu_1+\mu_2)[f(s+\mu_1)+f(s+\mu_2)]}, \qquad (3.103)$$

$$1-\varphi_{22}(s) = \frac{[1-f(s)][2-f(s+\mu_1)-f(s+\mu_2)+2f(s+\mu_1+\mu_2)]}{[1-f(s)][2-f(s+\mu_1)-f(s+\mu_2)] + f(s+\mu_1+\mu_2)[f(s+\mu_1)+f(s+\mu_2)]}\ . \qquad (3.104)$$

For Ordered Entry Discipline:

$$\varphi_{02}(s) = \frac{f(s+\mu_1)\, f(s+\mu_1+\mu_2)}{[1-f(s)][1-f(s+\mu_2)+2f(s+\mu_1+\mu_2)] + f(s+\mu_1)\, f(s+\mu_1+\mu_2)}, \qquad (3.105)$$

$$1-\varphi_{22}(s) = \frac{[1-f(s)][1-f(s+\mu_2)+f(s+\mu_1+\mu_2)]}{[1-f(s)][1-f(s+\mu_2)+2f(s+\mu_1+\mu_2)] + f(s+\mu_1)\, f(s+\mu_1+\mu_2)}, \qquad (3.106)$$

where $f(s) = \int_0^\infty e^{-st} dF(t)$, $f(s+\mu_1) = \int_0^\infty e^{-st} e^{-\mu_1 t} dF(t)$, $f(s+\mu_2) = \int_0^\infty e^{-st} e^{-\mu_2 t} dF(t)$, and $f(s+\mu_1+\mu_2) = \int_0^\infty e^{-st} e^{-(\mu_1+\mu_2)t} dF(t)$.

3.4 Steady-State Probabilities and Loss Probability from *GI*/*M*/*n*/0 with Heterogeneous Servers

Using (3.98), the fact that $m_0 = m_1 = \cdots = m_n = a$, the following is obtained:

$$E[T_{ii}] = \frac{\Delta}{D_{ii}(0)},\ \ i = 0,1,\ldots,n\,, \qquad (3.107)$$

where $a = \int_0^\infty [1 - F(t)]dt$, $D_{ii}(0)$ are the cofactors of the (i,i)th entries of matrix $[I - q(0)]$ and

$$\Delta = \begin{vmatrix} a & -p_{01} & 0 & \cdots & 0 \\ a & 1-p_{11} & -p_{12} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a & -p_{n-1,1} & -p_{n-1,2} & \cdots & -p_{n-1,n} \\ a & -p_{n-1,1} & -p_{n-1,2} & \cdots & 1-p_{n-1,n} \end{vmatrix}, \tag{3.108}$$

$$\Delta = a \cdot \begin{vmatrix} 1 & -p_{01} & 0 & \cdots & 0 \\ 1 & 1-p_{11} & -p_{12} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & -p_{n-1,1} & -p_{n-1,2} & \cdots & -p_{n-1,n} \\ 1 & -p_{n-1,1} & -p_{n-1,2} & \cdots & 1-p_{n-1,n} \end{vmatrix} = a \cdot D(1,1,\ldots 1). \tag{3.109}$$

Depending on this and formula (3.99), the steady-state probabilities of the system are obtained as follows:

$$\pi_i = \frac{a}{E[T_{ii}]} = \frac{D_{ii}(0)}{D(1,1,\ldots,1)}, \quad i = 0,1,\ldots,n. \tag{3.110}$$

For the *GI*/*M*/*n*/0 queuing model with heterogeneous servers, the probability that all servers are busy (π_n) is obtained using formula (3.110) as follows:

$$\pi_n = \frac{D_{nn}(0)}{D(1,1,\ldots,1)}, \tag{3.111}$$

where $D_{nn}(0)$ is easily obtained by writing 0 instead of s in determinant (3.102). Using the determinant properties of the matrix, it can also be written as $D_{nn}(0) = (-1)^n (-p_{01})(-p_{12})(-p_{23}) \cdots (-p_{n-1,n})$. Since no waiting line is available in the system, the loss probability of a customer, is equal to the probability that all servers are busy. In this way, the loss probability of the *GI*/*M*/*n*/0 queuing model with heterogeneous servers is obtained as follows:

$$\pi_n = \frac{(-1)^n(-p_{01})(-p_{12})(-p_{23})\cdots(-p_{n-1,n})}{D(1,1,\ldots,1)} \; . \tag{3.112}$$

Corollary 3.3 Provided that transition probabilities $p_{ij}\ (i, j = 1, 2, \ldots, n)$ in the last equation obtained above are computed by means of formula (3.6) or (3.7), under the reality of Theorem 3.1, formula (3.112) is an extension of Palm's loss formula (3.2) for 'the *GI*/*M*/*n*/0 queueing model with heterogeneous servers'.

Example 3.2 Let us reconsider Example 3.1. After some algebraic operations using equations (3.6) and (3.7) and formula (3.112), the loss probabilities of customers for the *GI*/*M*/2/0 queueing model with heterogeneous servers are provided through the following equations (3.113) and (3.114) for random entry and ordered entry disciplines, respectively:

$$\pi_2 = \frac{[f(\mu_1) + f(\mu_2)]f(\mu_1 + \mu_2)/2}{1 - [f(\mu_1) + f(\mu_2)]/2 + f(\mu_1 + \mu_2)}, \tag{3.113}$$

$$\pi_2 = \frac{f(\mu_1)f(\mu_1 + \mu_2)}{1 - f(\mu_2) + f(\mu_1 + \mu_2)}, \tag{3.114}$$

where, for $k = 1, 2$, $f(\mu_k) = \lim_{s\to 0}\int_0^\infty e^{-st}e^{-\mu_k t}dF(t)$ and $f(\mu_1 + \mu_2) = \lim_{s\to 0}\int_0^\infty e^{-st}e^{-(\mu_1+\mu_2)t}dF(t)$. Loss probabilities (3.113) and (3.114) obtained above yield Palm's loss formula (3.2) with $n = 2$ when $\mu_1 = \mu_2$.

Let's compute loss probabilities computed for both random entry and ordered entry disciplines in Subsection 3.2.1 and Subsection 3.2.2 for the queueing model

GI/*M*/3/0 by means of the extension of Palm's loss formula (3.112) obtained by analyzing the stream of overflows. The formula (3.112) is written as follows for *n*=3:

$$\pi_3 = \frac{(-1)^3(-p_{01})(-p_{12})(-p_{23})}{D(1,1,1,1)} \,, \tag{3.115}$$

where,

$$D(1,1,1,1) = \begin{vmatrix} 1 & -p_{01} & 0 & 0 \\ 1 & 1-p_{11} & -p_{12} & 0 \\ 1 & -p_{21} & 1-p_{22} & -p_{23} \\ 1 & -p_{21} & -p_{22} & 1-p_{23} \end{vmatrix} . \tag{3.116}$$

If the determinant $D(1,1,1,1)$ is calculated, the following is obtained:

$$D(1,1,1,1) = p_{10}(1-p_{22}-p_{23}) + p_{01}(1-p_{22}-p_{23}+p_{12}) + p_{12}p_{20} . \tag{3.117}$$

If the equation (3.118) is written in its place in the equation (3.115), the loss probability for the queueing model *GI*/*M*/3/0 is obtained as follows:

$$\pi_3 = \frac{(-1)^3(-p_{01})(-p_{12})(-p_{23})}{p_{10}(1-p_{22}-p_{23}) + p_{01}(1-p_{22}-p_{23}+p_{12}) + p_{12}p_{20}} \,. \tag{3.118}$$

For the queueing model *GI*/*M*/3/0 with heterogeneous servers and random entry the loss probability given by (3.49) is obtained by writing the one-step transition probabilities given by (3.36), (3.37), (3.39), (3.40), (3.42), and (3.43) in its place in (3.118). Similarly, for the queueing model *GI*/*M*/3/0 with heterogeneous servers and ordered entry, the loss probability given by (3.83) is obtained by writing one-step transition probabilities given by (3.70), (3.71), (3.73), (3.74), (3.76), and (3.77) in its place in (3.118).

The loss probability of the customer in the system is more easily computed by using the extension of Palm's loss formula (3.112) obtained by means of the stream of overflows without solving the linear equation system (3.14). Because, computing the determinant given by (3.109) is a more practical and rapid method rather than solving the linear equation system (3.14). Therefore, the extension of Palm's loss formula (3.112) proposed in this thesis is an effective and important formula in terms of the direct calculation of the loss probability of the customer in the system without need for calculating the steady-state probabilities in the system.

'The queueing model $GI/M/n/0$ with ordered entry' was analyzed by means of finite difference equations in the literature and the loss probability was obtained as a function of an extension of Palm's recurrence formula (Yao 1986, 1987). In the subsequent section, Palm's recurrence formula and an extension of Palm's recurrence formula are examined in detail. Details of the studies that take place in the literature related to 'the queueing model $GI/M/n/0$ with ordered entry' are explained and it is revealed that the results obtained about the Laplace-Stieljes transform of the distribution of the stream of overflows and the loss probability in this thesis are more superior than those of other studies.

3.5 Palm's Recurrence Formula

In this Section[1], Palm's recurrence formula and an extension of Palm's recurrence formula are investigated. The relationship of these formulas with the loss probability -- theorems related to the optimization of the loss probability according to the service discipline -- and the results of these theorems are explained in detail. The contradiction between the loss probability obtained by Yao (1986, 1987) as a function of the extension of Palm's recurrence formula advanced in the literature and again his main theorem (Yao, 1987) is revealed by a numerical example. Also, it is showed that the results obtained in this thesis are not controversial with the main theorem of Yao (1987) by means of a numerical example.

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[1] It must be noted that the studies explained in the Section 3.5 of this chapter have been presented by Isguder (2012) at 8th World Congress in Probability and Statistics organized by the Bernoulli Society and the Institute of Mathematical Statistics.

Palm (1943) proved that the distribution function of interoverflow times $G_k(t)$ satisfy the following integral equations system:

$$G_k(t) = G_{k-1}(t) - \int_0^t (1-e^{-\mu y})[1-G_k(t-y)]dG_{k-1}(y),\ k=1,2\ldots, \tag{3.119}$$

where $G_0(t) \equiv F(t)$. Taking the Laplace-Stieljes transform of (3.119), Takacs (1958, 1959) obtained the Palm's recurrence formula as follows:

$$f_k(s) = \frac{f_{k-1}(s+\mu)}{1-f_{k-1}(s)+f_{k-1}(s+\mu)}\ ,\ k=1,2\ldots, \tag{3.120}$$

where $f_0(s) = f(s)$ is the Laplace-Stieljes transform of interarrival time distribution $F(t)$. Herefrom, the loss probability for the queueing model *GI*/*M*/*n*/0 with identical servers is obtained as a function of Palm's recurrence formula as follows:

$$p_n = f(\mu) f_1(\mu) \cdots f_{n-1}(\mu)\,. \tag{3.121}$$

The formula (3.121) is equivalent to the formula (3.2).

Since the servers are identical in the queueing model *GI*/*M*/*n*/0 examined by Takacs (1958, 1959), Palm (1943) indicated that the number of customers in the system is independent from the traffic flow. Namely, in Palm's (1943) model, there is no difference between assignment to any of the empty servers (Random Entry), assignment to the server with the lowest index number among the empty servers (Ordered Entry), assignment to the server giving the fastest service from the empty servers (The Fastest Service-Rule), or taking service with any other service principle. However, once the servers are assumed to be heterogeneous, the service discipline must be examined very carefully. Because, the service discipline in the models with heterogeneous servers affects the analysis of the model.

Many researchers such as Cooper (1976), Matsui & Fukuta (1977), Nath & Enns (1981), Nawijn (1983, 1984), Pourbabai & Sonderman (1986), Pourbabai (1987), Yao (1986, 1987), Alpaslan (1996), Saglam & Shahbazov (2007), Isguder & Uzunoglu-Kocer (2010) and Isguder, Uzunoglu-Kocer & Celikoglu (2011), and Isguder & Celikoglu (2012) realizing this condition have modeled and analyzed the queueing systems with heterogonous servers.

Cooper (1976) examined the Markovian queue with heterogeneous servers and states that if the servers work at different rates, then the birth-and-death process representing the system will be a multi-dimensional birth-and-death process. Cooper (1976) also states that the solution of such models is difficult and stresses that the method he proposed permits the solution of the problems with ordered heterogeneous servers, without requiring a detailed solution of multi-dimensional birth-and-death equations. Many researchers such as Matsui & Fukuta (1977), Pourbabai & Sonderman (1986), Pourbabai (1987), Alpaslan (1996), Isguder & Uzunoglu-Kocer (2010), and Isguder & Celikoglu (2012), either studied the limited number of servers such as 2 or 3 or presented approximate solutions for the loss probability.

On the other hand, Nath & Enns (1981) analyzed the $M/M/n/0$ queueing system with ordered entry and computed the loss probability. Once the fastest service rule is applied, they proved that the loss probability is minimum. Besides, Yao (1986, 1987) analyzed the queueing model $GI/M/n/0$ with ordered entry and computed the loss probability of the customer. Yao (1987) proved the loss probability is minimum under the fastest service rule by optimizing the system according to the service discipline. All of the analyses carried out by Yao (1986, 1987) were directly performed by generalization of Palm’s recurrence formula for heterogeneous servers.

The author’s claim is that none of these studies adequately address the computation of loss probability for the $GI/M/n/0$ queueing system with ordered entry.

In Subsection 3.5.1 and Subsection 3.5.2, the studies carried out by Yao (1986, 1987) on Palm’s recurrence formula will be explained and the contradiction between the main result of Yao's (1987) main theorem and the loss probability will be revealed with a numerical example.

3.5.1 An Extension of Palm's Recurrence Formula

An extension of Palm's recurrence formula was first introduced by Cooper (1976) during the analysis of 'the model *M*/*M/n* queue with ordered entry'. Later, Nath & Enns (1981) used this formula for being able to analyze 'the queueing model *M*/*M*/*n*/0 with ordered entry'. Yao (1986, 1987) examined the queueing model *GI*/*M*/*n*/0 introduced by Palm under the hypothesis that the servers are heterogeneous.

Yao (1986, 1987) extended the formula (3.120) for the queueing model *GI*/*M*/*n*/0 with heterogeneous servers and ordered entry as follows:

$$f_k(s) = \frac{f_{k-1}(s+\mu_k)}{1 - f_{k-1}(s) + f_{k-1}(s+\mu_k)} \quad (1 \le k \le n)\,, \tag{3.122}$$

where $f_0(s) = f(s)$ is the Laplace-Stieltjes transform of interarrival time distribution $F(t)$. Equation (3.122) was denoted the Laplace-Stieltjes transform of the interoverflow times distribution from the first *k* servers for $k = 1,2,\ldots,n$ for the model *GI*/*M*/*n*/0 with ordered entry by Yao (1986). It must be noted that interarrival times in the models of Cooper (1976) and Nath & Enns (1981) are distributed exponentially and its Laplace-Stieltjes transform is $f(s) = \lambda/(s+\lambda)$. Namely, Cooper (1976) and Nath & Enns (1981) derived the formula (3.122) as a function of the exponential distribution by assuming that the initial case is distributed exponentially.

Palm's recurrence formula (3.120) was obtained by taking the Laplace-Stieltjes transform of the system of integral equations (3.119). It must be noted that the equation (3.122) called as an extension of Palm's recurrence formula has not been obtained by taking any system of integral equations or Laplace-Stieltjes transform of any function in the studies taking place in the literature. The formula (3.122) was obtained by writing directly μ_k instead of μ in Palm's recurrence formula (3.120).

Based on all these explanations, it is claimed that the extension of Palm's recurrence formula (3.122) is obtained heuristically and doesn't guarantee the exact solution of the loss probability. It is also claimed that the equation (3.122) is not the Laplace-Stieljes transform of the interoverflow times distribution for the model *GI*/*M*/*n*/0 with ordered entry.

The loss probability in the heterogeneous system *GI*/*M*/*n*/0 with ordered entry that is a function of the equation (3.122) was obtained by Yao (1986, 1987) as follows:

$$p_n = f(\mu_1) f_1(\mu_2) \cdots f_{n-1}(\mu_n) . \tag{3.123}$$

The formula (3.123) gives a randomly correct result for $n=1$ and $n=2$ However, this formula is not correct for $n=3$. The fact that this formula is not correct for $n=3$ is explained step by step.

For $n=3$, namely for the queueing model *GI*/*M*/3/0 with ordered entry, the loss probability is obtained as follows by means of the formula (3.123):

$$p_3 = \frac{f(\mu_1) f(\mu_1+\mu_2) f(\mu_1+\mu_2+\mu_3)[1-f(\mu_3)+f(\mu_2+\mu_3)]}{\begin{array}{r}[1-f(\mu_2)+f(\mu_1+\mu_2)]\cdot\{[1-f(\mu_3)][1-f(\mu_2+\mu_3)] \\ +f(\mu_1+\mu_2+\mu_3)[2-2f(\mu_3)+f(\mu_1+\mu_3)]\}\end{array}} . \tag{3.124}$$

Loss formula (3.124) must give the same results with the loss formula (3.83) (obtained for the same model in Subsection 3.2.2). However, if formulas (3.83) and (3.124) are examined carefully, it is clearly seen that numerators and denominators of these formulas are different from each other. This difference is also shown numerically with the numerical example 3.3 to be given in subsequent section. This difference stems from the fact that the equation (3.122) is not a Laplace-Stieltjes transform of the interoverflow times distribution for the queueing model *GI*/*M*/*n*/0 with ordered entry.

The validity of all of these claims will be revealed by means of Example 3.3 in subsequent subsection 3.5.2. By using the main theorem given by Yao (1987) related to the optimization of the loss probability, the validity of the claims suggested by the author will be proven.

3.5.2 Optimization of Loss Probability According to Service Discipline

Optimization of the loss probability according to the service discipline will be emphasized in this section. Yao (1987) proved that the loss probability for the queueing model *GI*/*M*/*n*/0 with ordered entry would take the minimum value under the fastest service rule. Here the fastest service rule is realized by assigning the customer arriving in the system to the fastest server among the empty servers rather than assigning to the server with the lowest index number among the empty servers.

In this section, definitions, theorems, and results given by Yao (1987) are explained related to the minimization of the loss probability. Claims laid in Section 3.5.1 are proven by using the theorems and the results again obtained by Yao (1987). The contradiction of the formula (given by Eq. 3.123) obtained by Yao (1987) for the loss probability with his own theorem is shown with a numerical example.

The following Definition 3.2 is given by Yao (1987).

Definition 3.2 (Yao, 1987)**.** For any two permutation vectors $\mathbf{x}^1$ and $\mathbf{x}^2$ of $\mathbf{x} = (x_i)$, $\mathbf{x}^1 \geq_{sa} \mathbf{x}^2$ if $\mathbf{x}^1$ can be obtained from $\mathbf{x}^2$ through successive pairwise interchange of neighboring complements, with each interchange correcting an inversion of the decreasing order of complements.

The following Theorem 3.4(ii), and Corollary 3.4(ii) proved that in the queueing model *GI*/*M*/*n*/0 with ordered entry, the loss probability was minimum with probability 1, under the fastest-service rule.

Theorem 3.4 (Yao, 1987)**.** *Consider a system of n servers. Let* $\boldsymbol{\mu}^1$ *and* $\boldsymbol{\mu}^2$ *be two server arrangements, and use the superscripts* 1 *and* 2 *to index quantities corresponding to the two arrangements. If* $\boldsymbol{\mu}^1 \geq_{sa} \boldsymbol{\mu}^2$ *, then*

(i) $T_k^1 \geq_L T_k^2$ *, for all* $k = 1, \ldots, n$,

(ii) $\mathbf{p}^1 \leq \mathbf{p}^2$,

(iii) $\mathbf{b}^1 \leq_{wm} \mathbf{b}^2$.

Corollary 3.4 (Yao, 1987)**.** For any server arrangement $\boldsymbol{\mu}$,

(i) $T_k(\boldsymbol{\mu}_{\downarrow}) \geq_L T_k(\boldsymbol{\mu}) \geq_L T_k(\boldsymbol{\mu}_{\uparrow})$, for all $k = 1, \ldots, n$,

(ii) $\mathbf{p}(\boldsymbol{\mu}_{\downarrow}) \leq \mathbf{p}(\boldsymbol{\mu}) \leq \mathbf{p}(\boldsymbol{\mu}_{\uparrow})$,

(iii) $\mathbf{b}(\boldsymbol{\mu}_{\downarrow}) \leq_{wm} \mathbf{b}(\boldsymbol{\mu}) \leq_{wm} \mathbf{b}(\boldsymbol{\mu}_{\uparrow})$.

In the following example, the loss probability of customers is computed numerically for $n = 2$ and $n = 3$ by using both formula (3.112) proposed in this thesis and formula (3.123) obtained by Yao (1986, 1987).

Example 3.3 Consider the *M*/*M*/2/0 and the *M*/*M*/3/0 queueing models with heterogeneous servers and ordered entry, respectively. The arrival rates and the service rates for the models addressed are summarized in Table 3.1. The numerical results are provided in Table 3.1; first by applying the fastest-service rule and then for an arbitrary permutation of service rates:

Table 3.1 A Numerical Example for *M*/*M*/2/0 and *M*/*M*/3/0 with Ordered Entry

***M*/*M*/2/0 with Ordered Entry**			***M*/*M*/3/0 with Ordered Entry**		
Parameters	The Fastest-Service Rule	Arbitrary Permutation	Parameters	The Fastest-Service Rule	Arbitrary Permutation
λ	90	90	λ	90	90
μ_1	60	45	μ_1	60	60
μ_2	45	60	μ_2	45	10
			μ_3	10	45
Loss probabilities	Calculations	Calculations	Loss probabilities	Calculations	Calculations
π_2	0.34839	0.35714	π_3	0.26533	0.27705
p_3	0.34839	0.35714	p_3	**0.29988**	**0.19856**

In Table 3.1, π_n, $(n = 2,3)$ represents the loss probability obtained by formula (3.112), which is proposed in this thesis; whereas p_n, $(n = 2,3)$ represents the loss probability obtained by using formula (3.123).

When $n = 2$, formula (3.112) and (3.123) yield the same result (see, Formula 3.114). Loss probabilities π_2 and p_2 are easily calculated by writing $\lambda/(\lambda+\mu_1)$ instead of $f(\mu_1)$, $\lambda/(\lambda+\mu_2)$ instead of $f(\mu_2)$, and $\lambda/(\lambda+\mu_1+\mu_2)$ instead of $f(\mu_1+\mu_2)$ respectively in formula (3.114). Values of λ, μ_1, and μ_2 are given in Table 3.1. It is obviously seen from Table 3.1 that the numerical values of π_2 and p_2 are the same.

On the other hand, when $n = 3$, the formula (3.112) yields the formula (3.83) and the formula (3.123) yields the formula (3.124). The values of the loss probabilities π_3 and p_3 in Table 3.1 are computed by placing $\lambda/(\lambda+\mu_k)$ in lieu of $f(\mu_k)$ for $k = 1,2,3$; $\lambda/(\lambda+\mu_k+\mu_r)$ in lieu of $f(\mu_k+\mu_r)$ for $1 \le k < r \le 3$; and $\lambda/(\lambda+\mu_1+\mu_2+\mu_3)$ in lieu of $f(\mu_1+\mu_2+\mu_3)$ in the formula (3.83) and the formula (3.124) respectively. Where, the values of λ, μ_1, μ_2 and μ_3 are given in Table 3.1. It can be also seen from Table 3.1 that numerical values of loss probabilities π_3 and p_3 are different from each other.

According to Theorem 3.4(ii), and Corollary 3.4(ii), the p_3 value should be minimum with probability 1 under the fastest-service rule. Nevertheless, when Table 3.1 is carefully examined, it is clearly seen that the p_3 value is not minimum under the fastest-service rule. This abnormal situation shows that the formula obtained by Yao (1987) for the loss probability contradicts his own theorem. This unexpected case results from the fact that the formula (3.122) does not fully satisfy the ordered entry service discipline.

On the other hand, π_3 computed with formula (3.83) proposed in this thesis takes its minimum value under the fastest-service rule. That is, an extension of Palm's loss formula (3.112) satisfies Theorem 3.4(ii), and Corollary 3.4(ii).

Corollary 3.5 An extension of Palm's recurrence formula (3.122), examined by Yao (1986, 1987) not satisfies Palm's recurrence formula (3.2) for the queueing model *GI*/*M*/*n*/0 with ordered entry when number of servers is more than 2.

Corollary 3.6 Loss formula (3.123) examined by Yao (1986, 1987) is valid for only when the number of the servers is 1 or 2.

According to Corollary 3.5 and Corollary 3.6, it is obvious that Yao (1986, 1987) couldn't completely overcome the analysis of the model queueing *GI*/*M*/*n*/0 with ordered entry.

In the subsequent chapter, since the calculation of the extension of Palm's loss formula (3.112) becomes increasingly difficult as the number of servers increases, the loss probability is calculated with a simulation approach. Theoretical studies carried out related to the minimization of the loss probability in Subsection 3.2.3 and Subsection 3.5.2 are supported with a considerably comprehensive simulation design.

CHAPTER FOUR
SIMULATION DESIGN

It becomes computationally intractable to compute the loss probability given by formula (3.112) as the number of servers increases. For the cases with more than one server, the loss probability can be obtained easily with the simulation approach. The simulation model developed and the findings obtained are presented in this chapter.

To obtain the point estimate and confidence interval of the loss probability for the finite-capacity $GI/M/n/0$ queueing system with heterogeneous n servers defined in Section 3.2, the discrete-event simulation model is used. More detailed information on statistical estimation in simulation and discrete-event simulation can be obtained from Alexopoulos (2006), Law & Kelton (2000), Banks, Carson, & Nelson (1996), and Fishman (2001).

The simulation study is examined under two main headings depending on the service principles random entry and ordered entry disciplines in this Chapter[2]. Furthermore, for the ordered entry discipline, simulation results are also given under the fastest-service rule.

For ease, the tabulated models have been expressed by being encoded. In this encoding, the first character symbolizes the distribution of the interarrival times, while the second one symbolizes the service discipline, and the last one symbolizes the traffic intensity. For instance, W-R-080 represents the queueing system where the interarrival time fits the Weibull distribution, the service discipline is random entry, and the traffic intensity is 0.80. Similarly M-OE-095 represents the queueing system where the interarrival time fits the exponential distribution, the service discipline is ordered entry, and the traffic intensity is 0.95.

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[2] It must be noted that the studies explained in this chapter have been presented by Isguder and Celikoglu (2012) at 8th International Symposium of Statistics organized by the Anadolu University.

4.1 The Simulation Model

The times between successive arrivals $\{AT_i, i \geq 1\}$ are independent and identically distributed random variables with an arbitrary probability distribution function, with finite mean $E(AT) = a$. In the simulation study, four different probability distributions, i.e. exponential (M), gamma (Ga), Weibull (W) and deterministic (D) distributions, were used for the interarrival time distribution. The service times $\{ST_{ij}, i \geq 1\}$ for each server *j* ($j = 1,2,...,n$) are independent and identically distributed random variables from the exponential distribution with finite mean $E(ST_{ij}) = 1/\mu_j$, where μ_j is the service rate for server *j*. Hence, $\mu = \sum_{k=1}^{n} \mu_k$ is the total service rate for the system. Clearly, $a\mu > 1$ must be achieved for steady state. Both the arrival process and the service process are independent of each other and the servers work independently of each other.

In the simulation model, the service process is randomly derived from the exponential distribution according to the given service rates. By considering the total service rate for a given traffic intensity, the mean of interarrival times is obtained. Random data with this mean are derived from the distribution stated for the interarrival time. In this way, even if the interarrival time distribution is different for each number of servers, it is ensured that the data with an identical mean are used. This is essential to make a comparison.

The model was considered a finite horizon simulation model. That is, the system is evaluated at a specific time interval and the system is empty at the initial time. One of the important points here is the determination of the replication number, while the other one is how long a replication will be run. To decide how long the system would work, first of all the loss probability in the event that it worked for 500 hours in each replication was obtained and then the working duration of the system was increased fivefold and, at the end of 2,000 hours, the loss probability was computed again. In conclusion, it was observed that the loss probability only increased by 0.004 units. Since the loss probability is not growing as the simulation proceeds, the system is stable (Henderson & Nelson, 2006). In this study, the estimation of loss probability is found depending on the finite sample obtained by running the simulation program for 1,000 times. The system was run for 500 hours in each replication.

4.1.1 The Algorithm

There are two processes in each replication, i.e. arrival and departure processes. The algorithm can be summarized as follows:

- The interarrival times are derived depending on the service rate $\mu = \sum_{k=1}^{n} \mu_k$ and the system utilization factor (ρ) in such a way that their average will be a.

- The last arrival time (AT) and the last departure time (DT) are compared and the next event is determined. If $\mathrm{AT} < \mathrm{DT}$, an arrival takes place and the arrival process is run; otherwise, because the next event will be departure, the departure process is run.

- Arrival process: The number of empty servers (nes) is determined. If all servers are busy (nes=0), arrival is recorded as the lost customer. DT is updated and AT is determined for the new arrival. AT and DT are compared again. If there is more than one empty server, it is determined which servers are empty and arrival is assigned to one of the empty servers according to the service discipline (Random Entry or Ordered Entry). The busy state of the servers (SS) and AT are updated. Considering which of the busy servers will first become empty, DT is updated.

- Departure process: If the next event is departure, it is determined which server will first become empty and SS is updated. DT is updated for the server that has become empty and the previous ones are maintained.

- After the time predicted for simulation has been completed, the total number of lost customers for the steady-state system is determined. The loss probability is computed by dividing the number of lost customers by the total number of customers in the system.

The simulation algorithm is presented in Figure 4.1. The arrival and departure processes are configured with the algorithm in Figure 4.2 and Figure 4.3, respectively.

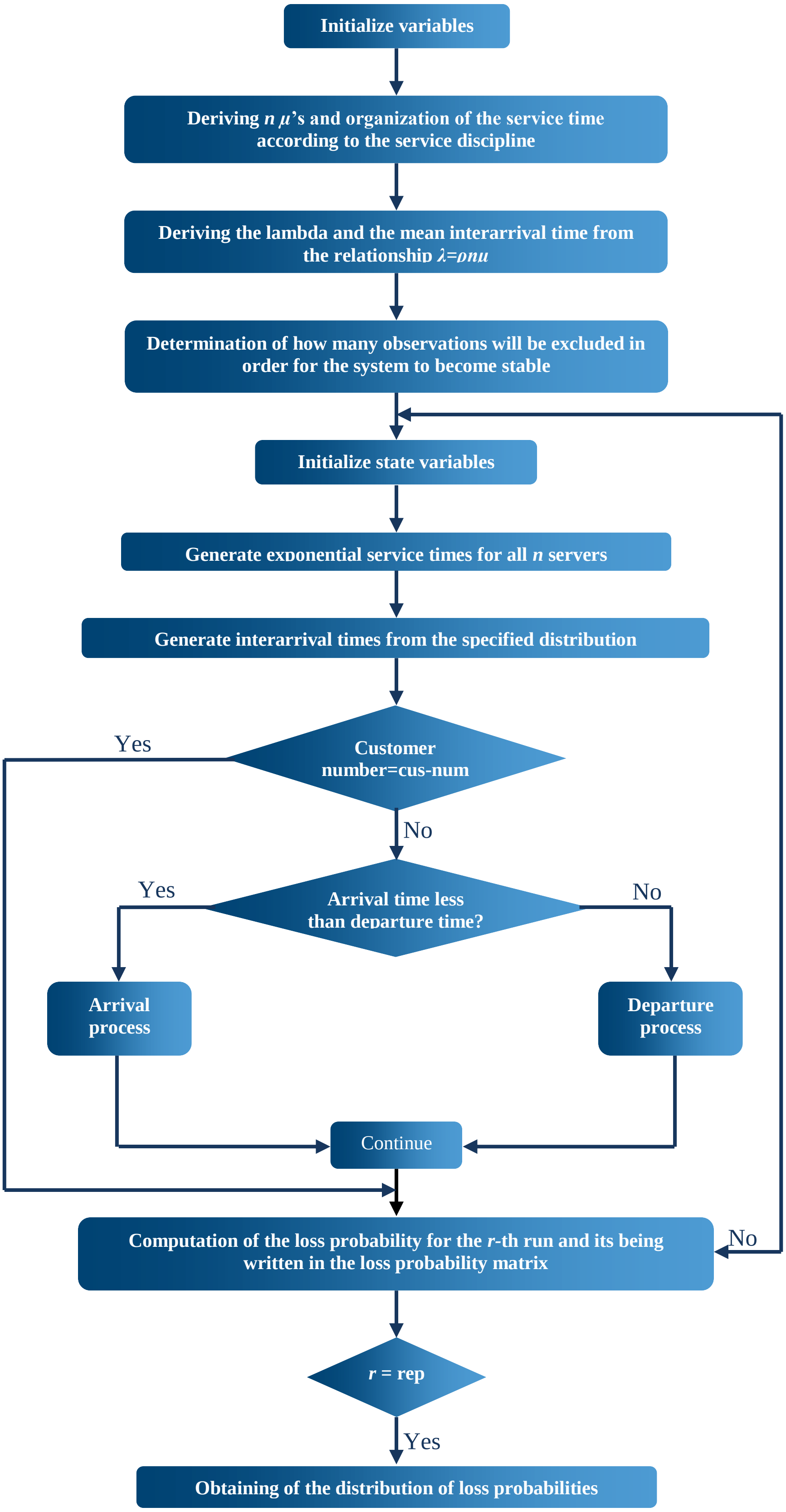

Figure 4.1 Simulation algorithm.

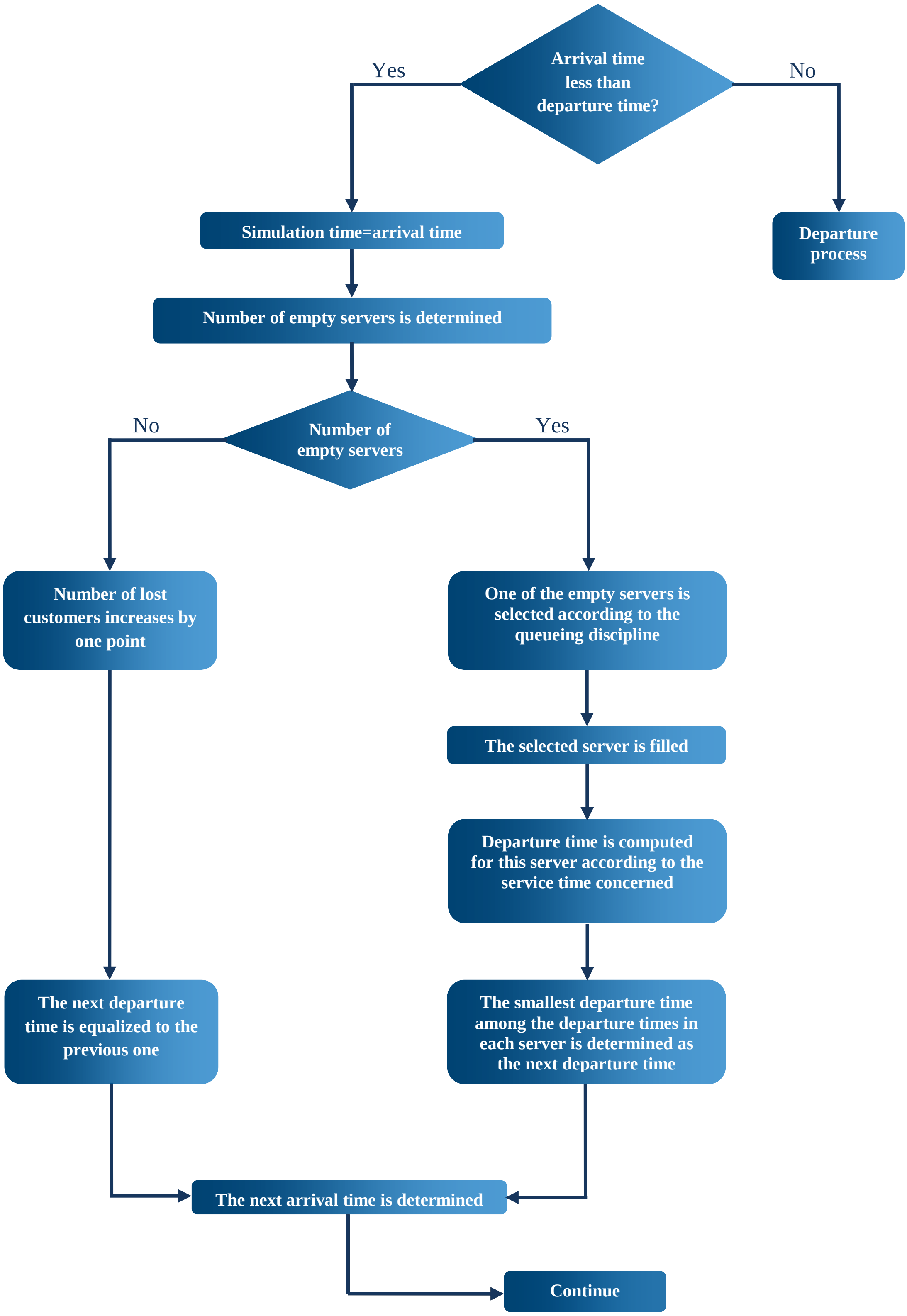

Figure 4.2 Algorithm of the arrival process.

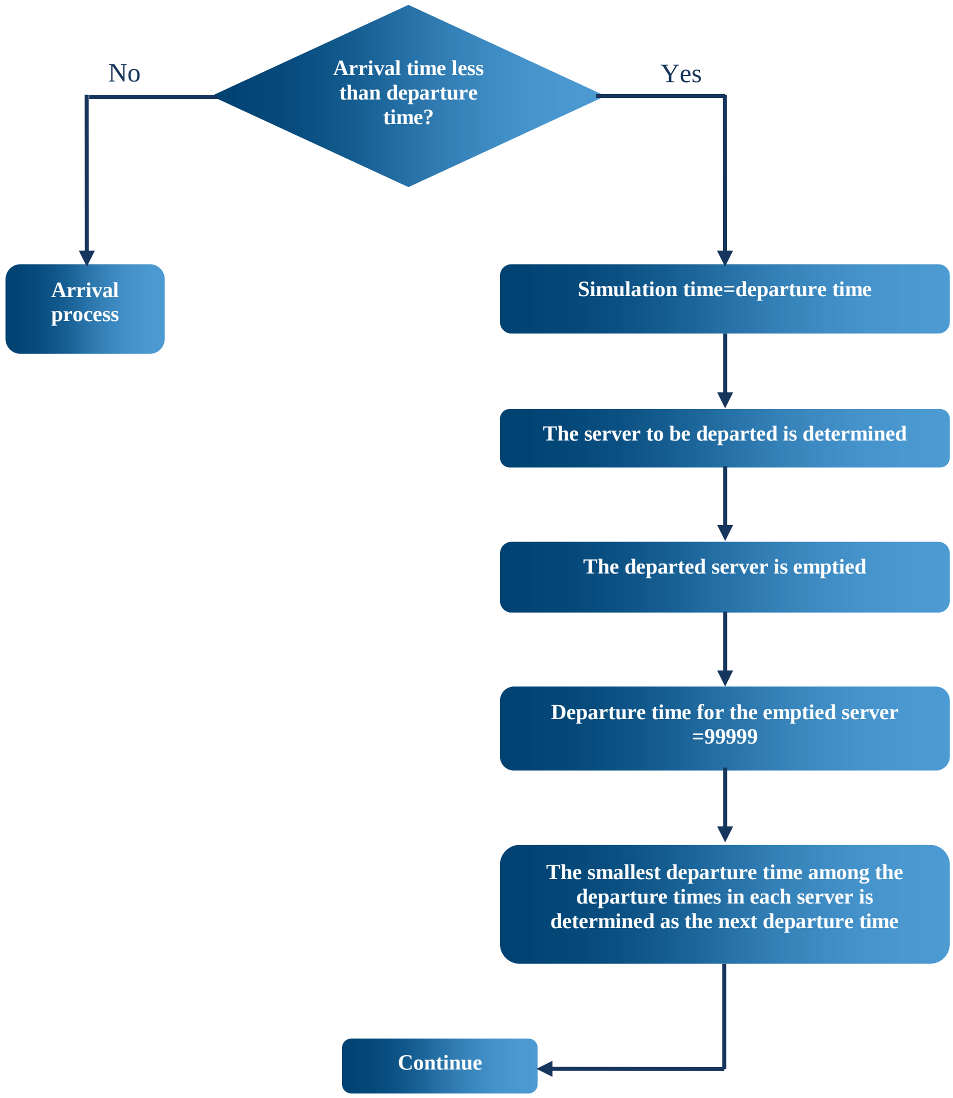

Figure 4.3 Algorithm for the departure process.

4.1.2. Assessment of the Loss Probability

In each run, the system works for 500 hours. Under the steady-state condition, the number of lost customers is determined for each run and it is divided by the total number of customers served in the system to obtain the loss probability (P_{loss}) for a single run. By repeating this procedure for 1,000 times, the probability distribution for the loss probability is obtained and, from this, the estimation of the average number of lost customers $(\hat{P}_{loss})$ and the standard error of the estimate $(\hat{\sigma}_{P_{loss}})$ are computed, and the 95% confidence interval (95% CI) is constructed as

$\hat{P}_{loss} - z_{0.025}\,\hat{\sigma}_{\hat{P}_{loss}} / \sqrt{r} \le P_{loss} \le \hat{P}_{loss} + z_{0.025}\,\hat{\sigma}_{\hat{P}_{loss}} / \sqrt{r}$, where r indicates the number of replications.

4.2 Computational Experiments

This section, where the computations will be presented, is examined in two parts. In the first part which is the verification part, it is checked whether the implementation of the simulation program corresponds to the model. Verification is the process of comparing the computer code with the model to ensure that the code is a correct implementation of the model. In the second part, the results obtained by the implementation of the verified simulation model will be presented.

4.2.1 Verification of the Simulation Model

To verify the simulation model, the finite-capacity $M/M/2/0$, $D/M/2/0$, $M/M/3/0$ and $D/M/3/0$ queueing models with heterogeneous servers are considered. By attaining the analytical solutions of these models, exact results are obtained for the loss probabilities. Later on, the simulation program is run for the same models and the loss probabilities are approximated. The obtained results are presented in Tables 4.1 and 4.2 for random entry and ordered entry disciplines, respectively.

Table 4.1 Loss probabilities under random entry.

Model	n	Arrival rate (λ)	Service rates	Loss probability		Error %
				Exact solution	Simulation	
M-R-080	2	4	μ_1=1, μ_2=4	0.36364	0.36307	-0.16
D-R-080	2	4	μ_1=1, μ_2=4	0.23033	0.23012	-0.09
M-R-095	2	4.75	μ_1=1, μ_2=4	0.41542	0.40491	-2.53
D-R-095	2	4.75	μ_1=1, μ_2=4	0.29725	0.29118	-2.04
M-R-080	3	9.6	μ_1=7, μ_2=1, μ_3=4	0.29729	0.30419	2.32
D-R-080	3	9.6	μ_1=7, μ_2=1, μ_3=4	0.17967	0.18736	4.28
M-R-095	3	11.4	μ_1=7, μ_2=1, μ_3=4	0.35547	0.35336	-0.59
D-R-095	3	11.4	μ_1=7, μ_2=1, μ_3=4	0.25009	0.25236	0.91

Table 4.2 Loss probabilities under ordered entry.

Model	n	Arrival rate (λ)	Service rates	Loss probability: Exact solution	Loss probability: Simulation	Error %
M-OE-080	2	4	μ_1=1, μ_2=4	0.37647	0.37599	-0.13
D-OE-080	2	4	μ_1=1, μ_2=4	0.24290	0.24270	-0.08
M-OE-095	2	4.75	μ_1=1, μ_2=4	0.42618	0.41442	-2.76
D-OE-095	2	4.75	μ_1=1, μ_2=4	0.30794	0.30167	-2.04
M-OE-080	3	9.6	μ_1=7, μ_2=1, μ_3=4	0.28524	0.29246	2.53
D-OE-080	3	9.6	μ_1=7, μ_2=1, μ_3=4	0.16670	0.17429	4.55
M-OE-095	3	11.4	μ_1=7, μ_2=1, μ_3=4	0.34566	0.34351	-0.62
D-OE-095	3	11.4	μ_1=7, μ_2=1, μ_3=4	0.23952	0.24162	0.88

The percentage error (Error%) is given by

$$\text{Error\%} = \frac{(\text{obtained}\, P_{loss} - \text{exact}\ P_{loss})}{\text{exact}\ P_{loss}} * 100\%. \tag{4.1}$$

The simulation model is verified by the fact that the maximum percentage errors in Tables 1 and 2 are 4.28% and 4.55%, respectively. That is, for the cases that are difficult to find with analytical solutions, the simulation approach presented might be used to approximate the loss probability.

4.2.2 Computational Results

In this section, how the loss probabilities vary when we increase the number of servers under the assumption that the interarrival time follows different distributions is investigated with the simulation approach and the results are presented. Loss probabilities under random entry and ordered entry disciplines are approximated for the *GI*/*M*/*n*/0 queueing system with heterogeneous servers, respectively. For these estimates, standard error and 95% confidence interval are also given. Note that for all cases, the interarrival times are examined individually for exponential distribution, gamma distribution, Weibull distribution and deterministic distribution, respectively. Moreover, the cases, where the numbers of servers are 5, 10, 50 and 100 for all above-mentioned distributions, respectively, are individually examined.

The loss probabilities for the model concerned when the traffic intensities are 0.80 and 0.95 under random entry discipline are given in Tables 4.3 and 4.4, respectively.

Table 4.3 Loss probabilities for the model concerned when the traffic intensity is 0.80.

Model	n	Arrival rate (λ)	Service rate ($\sum_{k=1}^{n}\mu_k$)	Loss probability	95% CI Lower bound	95% CI Upper bound	Standard error
M-R-080	5	0.232	0.290	0.19954	0.19907	0.20001	0.00024
	10	0.544	0.680	0.12317	0.12269	0.12364	0.00024
	50	5.920	7.400	0.02248	0.02215	0.02282	0.00017
	100	19.840	24.800	0.00595	0.00577	0.00613	0.00009
Ga-R-080	5	0.232	0.290	0.16013	0.15972	0.16054	0.00021
	10	0.544	0.680	0.09440	0.09399	0.09480	0.00021
	50	5.920	7.400	0.01324	0.01301	0.01347	0.00012
	100	19.840	24.800	0.00265	0.00254	0.00277	0.00006
W-R-080	5	0.232	0.290	0.14065	0.14026	0.14104	0.00020
	10	0.544	0.680	0.08112	0.08077	0.08147	0.00018
	50	5.920	7.400	0.00927	0.00910	0.00947	0.00009
	100	19.840	24.800	0.00147	0.00139	0.00154	0.00004
D-R-080	5	0.232	0.290	0.11201	0.11168	0.11234	0.00017
	10	0.544	0.680	0.06067	0.06034	0.06099	0.00017
	50	5.920	7.400	0.00484	0.00472	0.00496	0.00006
	100	19.840	24.800	0.00054	0.00050	0.00059	0.00002

Table 4.4 Loss probabilities for the model concerned when the traffic intensity is 0.95.

Model	n	Arrival rates (λ)	Service rates ($\sum_{k=1}^{n}\mu_k$)	Loss probability	95% CI Lower bound	95% CI Upper bound	Standard error
M-R-095	5	0.2755	0.2900	0.25992	0.25929	0.26056	0.00032
	10	0.6460	0.6800	0.19019	0.18957	0.19081	0.00032
	50	7.0300	7.4000	0.08364	0.08301	0.08428	0.00032
	100	23.5600	24.8000	0.05515	0.05451	0.05578	0.00032
Ga-R-095	5	0.2755	0.290	0.22528	0.22471	0.22585	0.00029
	10	0.6460	0.680	0.16368	0.16314	0.16421	0.00027
	50	7.0300	7.400	0.06929	0.06873	0.06984	0.00028
	100	23.5600	24.800	0.04432	0.04379	0.04484	0.00027
W-R-095	5	0.2755	0.290	0.20785	0.20733	0.20837	0.00027
	10	0.6460	0.680	0.14957	0.14906	0.15009	0.00026
	50	7.0300	7.400	0.06181	0.06129	0.06234	0.00027
	100	23.5600	24.800	0.03902	0.03856	0.03949	0.00024
D-R-095	5	0.2755	0.290	0.18121	0.18073	0.18168	0.00024
	10	0.6460	0.680	0.12950	0.12904	0.12996	0.00023
	50	7.0300	7.400	0.05197	0.05152	0.05241	0.00023
	100	23.5600	24.800	0.03221	0.03179	0.03264	0.00022

The loss probabilities for the model concerned when the traffic intensities are 0.80 and 0.95 under ordered entry discipline are given in Tables 4.5 and 4.6, respectively.

Table 4.5 Loss probabilities for the model concerned when the traffic intensity is 0.80.

Model	n	Arrival rate (λ)	Service rate ($\sum_{k=1}^{n}\mu_k$)	Loss probability	95% CI Lower bound	95% CI Upper bound	Standard error
M-OE-080	5	0.232	0.290	0.20252	0.20205	0.20298	0.00024
	10	0.544	0.680	0.12209	0.12160	0.12258	0.00025
	50	5.920	7.400	0.02407	0.02375	0.02439	0.00016
	100	19.840	24.800	0.00510	0.00492	0.00528	0.00009
Ga-OE-080	5	0.232	0.290	0.15897	0.15854	0.15939	0.00022
	10	0.544	0.680	0.09877	0.09836	0.09917	0.00021
	50	5.920	7.400	0.01460	0.01436	0.01484	0.00012
	100	19.840	24.800	0.00164	0.00155	0.00173	0.00005
W-OE-080	5	0.232	0.290	0.13882	0.13841	0.13922	0.00021
	10	0.544	0.680	0.08074	0.08038	0.08110	0.00018
	50	5.920	7.400	0.01085	0.01066	0.01104	0.00010
	100	19.840	24.800	0.00099	0.00093	0.00106	0.00003
D-OE-080	5	0.232	0.290	0.11021	0.10988	0.11055	0.00017
	10	0.544	0.680	0.06302	0.06272	0.06332	0.00015
	50	5.920	7.400	0.00397	0.00386	0.00408	0.00006
	100	19.840	24.800	0.00034	0.00031	0.00037	0.00001

Table 4.6 Loss probabilities for the model concerned when the traffic intensity is 0.95.

Model	n	Arrival rate (λ)	Service rate ($\sum_{k=1}^{n}\mu_k$)	Loss probability	95% CI Lower bound	95% CI Upper bound	Standard error
M-OE-095	5	0.275	0.290	0.26189	0.26123	0.26255	0.00034
	10	0.646	0.680	0.19133	0.19071	0.19194	0.00031
	50	7.030	7.400	0.08846	0.08784	0.08907	0.00031
	100	23.560	24.800	0.05225	0.05160	0.05289	0.00033
Ga-OE-095	5	0.275	0.290	0.22770	0.22714	0.22824	0.00028
	10	0.646	0.680	0.16149	0.16092	0.16205	0.00029
	50	7.030	7.400	0.06853	0.06796	0.06909	0.00028
	100	23.560	24.800	0.04262	0.04209	0.04316	0.00027
W-OE-095	5	0.275	0.290	0.21007	0.20954	0.21060	0.00027
	10	0.646	0.680	0.15191	0.15142	0.15240	0.00025
	50	7.030	7.400	0.06298	0.06246	0.06349	0.00026
	100	23.560	24.800	0.03966	0.03918	0.04014	0.00024
D-OE-095	5	0.275	0.290	0.17751	0.17703	0.17799	0.00024
	10	0.646	0.680	0.12945	0.12899	0.12991	0.00023
	50	7.030	7.400	0.04555	0.04510	0.04601	0.00023
	100	23.560	24.800	0.03205	0.03165	0.03245	0.00021

Both under random entry discipline (Tables 4.3 and 4.4) and ordered entry discipline (Tables 4.5 and 4.6), it is observed that in the models concerned, the loss probability decreases, as expected, when the number of servers increases for all distributions of interarrival times. On the other hand, it is observed that in models D-R-080, D-OE-080, D-R-095 and D-OE-095 the loss probability takes a much smaller value as compared to the other models. The results given in Tables 4.4 and 4.6 are summarized in Figures 4.4 and 4.5.

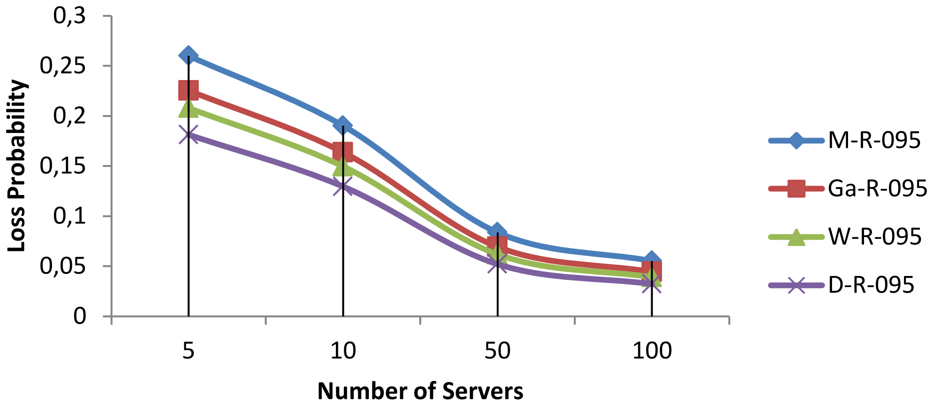

Figure 4.4 Loss probabilities for the queueing model *GI*/*M*/*n*/0 with random entry.

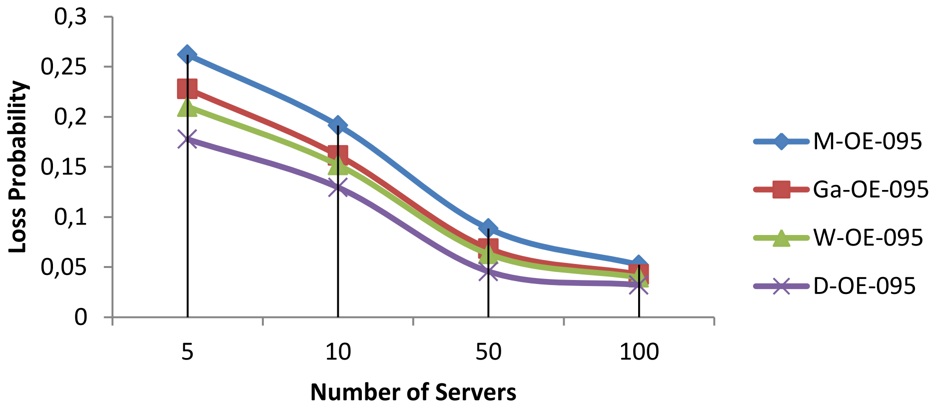

Figure 4.5 Loss probabilities for the queueing model *GI*/*M*/*n*/0 with ordered entry.

For the ordered entry discipline, the customer, who arrives in the system, starts the service in the server with the smallest mean service time instead of starting the service in the server with the lowest index number among the empty servers. In this way, the fastest-service rule is implemented. Let the fastest-service rule be symbolized with OE1.

The loss probabilities under the fastest-service rule when the traffic intensities are 0.80 and 0.95 are given in Tables 4.7 and 4.8, respectively.

Table 4.7 Loss probabilities for the model concerned when the traffic intensity is 0.80.

Model	n	Arrival rate (λ)	Service rate ($\sum_{k=1}^{n}\mu_k$)	Loss probability	95% CI Lower bound	95% CI Upper bound	Standard error
M-OE1-080	5	0.232	0.290	0.19461	0.19415	0.19507	0.00024
	10	0.544	0.680	0.11361	0.11315	0.11408	0.00024
	50	5.920	7.400	0.00862	0.00839	0.00886	0.00012
	100	19.840	24.800	0.00032	0.00028	0.00037	0.00002
Ga-OE1-080	5	0.232	0.290	0.15523	0.15481	0.15564	0.00021
	10	0.544	0.680	0.08515	0.08475	0.08556	0.00021
	50	5.920	7.400	0.00337	0.08475	0.08556	0.00021
	100	19.840	24.800	0.00005	0.00003	0.00006	0.00001
W-OE1-080	5	0.232	0.290	0.13567	0.13527	0.13607	0.00020
	10	0.544	0.680	0.07182	0.07146	0.07218	0.00018
	50	5.920	7.400	0.00181	0.00172	0.00190	0.00004
	100	19.840	24.800	0.00001	0.00001	0.00002	0.00000
D-OE1-080	5	0.232	0.290	0.10707	0.10674	0.10741	0.00017
	10	0.544	0.680	0.05239	0.05209	0.05270	0.00016
	50	5.920	7.400	0.00055	0.00051	0.00059	0.00002
	100	19.840	24.800	0.00000	0.00000	0.00000	0.00000

Table 4.8 Loss probabilities for the model concerned when the traffic intensity is 0.95.

Model	n	Arrival rate (λ)	Service rate ($\sum_{k=1}^{n}\mu_k$)	Loss probability	95% CI Lower bound	95% CI Upper bound	Standard error
M-OE1-095	5	0.275	0.290	0.25608	0.25543	0.25672	0.00033
	10	0.646	0.680	0.18226	0.18164	0.18289	0.00032
	50	7.030	7.400	0.06509	0.06444	0.06574	0.00033
	100	23.560	24.800	0.03480	0.03414	0.03546	0.00034
Ga-OE1-095	5	0.275	0.290	0.22118	0.22061	0.22174	0.00029
	10	0.646	0.680	0.15544	0.15489	0.15599	0.00028
	50	7.030	7.400	0.05140	0.05085	0.05195	0.00028
	100	23.560	24.800	0.02561	0.02510	0.02612	0.00026
W-OE1-095	5	0.275	0.290	0.20339	0.20286	0.20393	0.00027
	10	0.646	0.680	0.14195	0.14143	0.14247	0.00027
	50	7.030	7.400	0.04486	0.04434	0.04538	0.00026
	100	23.560	24.800	0.02141	0.02094	0.02187	0.00024
D-OE1-095	5	0.275	0.290	0.17663	0.17615	0.17712	0.00025
	10	0.646	0.680	0.12230	0.12185	0.12276	0.00023
	50	7.030	7.400	0.03579	0.03533	0.03624	0.00023
	100	23.560	24.800	0.01558	0.01520	0.01596	0.00020

When the fastest-service rule is implemented, it is observed that the loss probabilities become much smaller for all the models considered, as compared to the other disciplines (Tables 4.7 and 4.8). The results given in Table 4.8 are summarized in Figure 4.6.

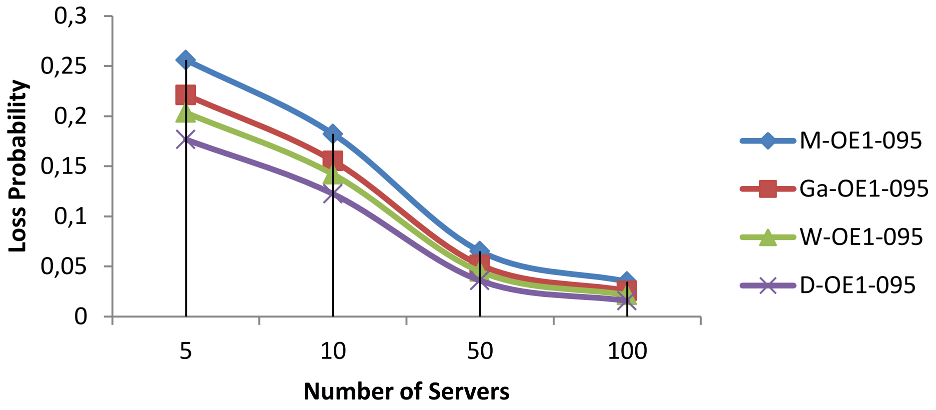

Figure 4.6 Loss probabilities for the the queueing model *GI*/*M*/*n*/0 with OE1-discipline.

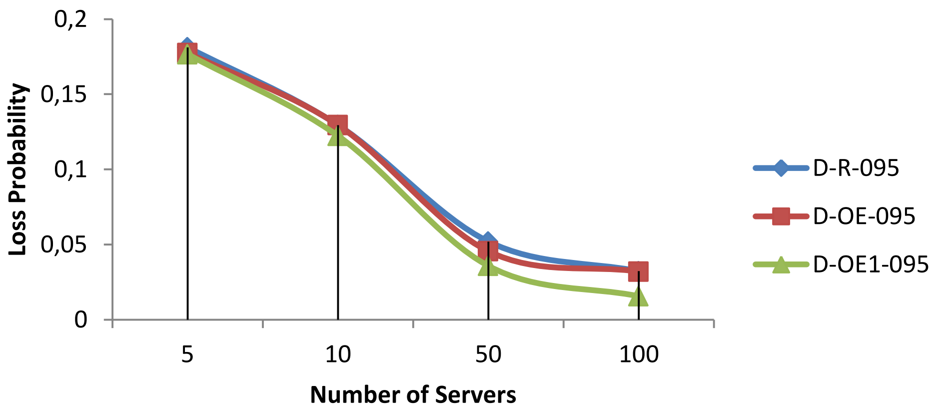

Figure 4.7 Loss probabilities for the queueing model *D*/*M*/*n*/0 with heterogeneous servers.

In conclusion, for the *GI*/*M*/*n*/0 queueing model with heterogeneous servers, the loss probability takes its lowest value both when the interarrival times are deterministically distributed and the fastest-service rule is implemented. This result is clearly seen from Figure 4.7.

CHAPTER FIVE
CONCLUSIONS

In this thesis, the finite-capacity $GI/M/n/0$ queueing system with recurrent input and heterogeneous servers has been studied. The semi-Markov process representing the system has been formulated and the Takacs's formula given by (3.4) has been generalized both for random entry and ordered entry service disciplines. An implementation of a generalization of Takacs's formula is performed for the queueing model $GI/M/3/0$ with heterogeneous servers. It has been proved that the loss probability for the queueing model $GI/M/3/0$ with ordered entry is minimum when interarrival times fit the deterministic distribution. By analyzing the stream of overflows in the system, the Laplace-Stieltjes transform of the distribution of the stream of overflows and loss probability (3.112), which is an extension of well-known Palm's formula (given by Eq. 3.2), have been obtained. An implementation of this formula is performed for the queueing model $GI/M/2/0$ with heterogeneous servers and the loss probability of customers in the system is computed. It is proven that the extension of Palm's recurrence formula (given by Eq. 3.122) addressed by Yao (1986, 1987) is a heuristic formula and doesn't guarantee the exact solution. Furthermore the conditions in which the loss probability is minimum is determined by simulation optimization.

5.1 Concluding Remarks

Even though there have been many studies on the queueing models with heterogeneous servers since Gumbel (1960), most of these studies have only solved this problem for a limited number of servers or proposed a solution for n servers by generalizing Palm's recurrence formula. This thesis differs from the others in that formulae (3.6) and (3.7), which are the generalizations of Takacs's formula (3.4), are proposed for the $GI/M/n/0$ queueing model under both random entry and ordered entry disciplines. These formulae, obtained by means of the semi-Markov process representing the system, enable the attainment of the efficient and exact solution in practice. In this context, the analysis of the queueing model $GI/M/n/0$ with heterogeneous servers handled in this thesis is an excelent implementation of semi-Markov process.

It is shown that overflow times of the customers in the *GI*/*M*/*n*/0 queueing model with heterogeneous servers are delayed renewal process. By analyzing the stream of overflows, steady-state probabilities and loss probability as a solution of the determinant of embedded Markov chain of semi-Markov process are derived. On the other hand, computability of the loss probability of a customer in the system by using the extension of Palm's loss formula (3.112) without a need for solving the linear equation system (3.14) provides an important contribution to the literature. Calculating the determinant given by the formula (3.109) rather than solving the linear equation system (3.14) is a more practical and rapid method. Therefore, an extension of Palm's loss formula (3.112) proposed in this thesis is an effective and important formula in terms of direct calculation of the loss probability of the customer in the system without a need for calculating steady-state probabilities in the system.

The *GI*/*M*/*n*/0 queueing model with ordered entry was examined by Yao (1986, 1987) before this thesis. However, it has proven that Yao (1986, 1987) could not overcome the problem obtaining the Laplace-Stieltjes transform of the distribution of the stream of overflows and formulating the loss probability. The contradiction between the main theorem of Yao (1987) concerning the optimization of the loss probability and the formula of loss probability obtained by Yao (1986, 1987) is proven with Example 3.3. On the other hand, it is explained by Example 3.3 that, an extension of Palm's Loss Formula (given by Eq. 3.112) we obtained in this thesis doesn't contradict the main theorem of Yao (1987).

When the numbers of servers are 5, 10, 50 and 100 and the interarrival time distributions are exponential, gamma, Weibull and deterministic in the simulation model, the loss probabilities are computed for both random entry and ordered entry service disciplines. The loss probability is minimized in two different ways according to the service discipline and according to the distribution of interarrival times. It is observed that the loss probability obtained when the interarrival time is deterministically distributed is smaller than the loss probability obtained under the assumption that the interarrival time fits the other (exponential, gamma and Weibull) distributions. On the other hand, it is observed that the loss probability is decreased when the arriving customer who arrives in the system is assigned to the fastest-working server instead of entering the server with the lowest index number among the empty servers according to the service discipline (the fastest-service rule). Both when the fastest-service rule is applied and the interarrival time distribution is deterministic, the loss probability takes its minimum value within all of these combinations.

5.2 Future Research

Kaufman (1980) analysis the queueing model $M/G/n/0$ with heterogeneous servers and random selection discipline. In the model addressed by Kaufman (1980), if the service discipline is selected as ordered entry rather than random entry, since the service servers are heterogeneous, the analysis of the model will be completely changed. Therefore, the analysis of the queueing model $M/G/n/0$ with heterogeneous servers and ordered entry can be considered as a future research. The main problem is obtaining the distribution of the time between overflows and formulating the loss probability in this proposed model. Also, it is obvious in this model that the loss probability will be minimized under the fastest-service rule. Mathematical proof of this problem can also be considered as a future research.

An extension of Palm's recurrence formula (3.122) used in the analysis of the $GI/M/n/0$ queueing model with ordered entry by Yao (1986) first proposed by Cooper (1976) and was used in the analysis of the model $M/M/n$ queue with ordered entry. In this thesis, it has been proven that an extension of Palm's recurrence formula (3.122) given by Yao (1986, 1987) was obtained completely heuristically. Therefore, revisiting the queueing model that Cooper (1976) was addressed and verifying the results and proposing new results if necessary would be considered as future research.

When obtaining the exact solution is either difficult or impossible, the use of approximate solution methods such as Markov chain Monte Carlo simulations, heuristic and meta-heuristic for the numerical analysis of the $GI/M/n/0$ model with heterogeneous servers may be considered for future research. In addition, for the cases in which the interarrival times are phase-type distributed such as Coxian, hyper-exponential and matrix-exponential, the loss probability can be computed approximately by developing new heuristic methods. For example, Atkinson (2009) developed two new heuristics, which are called the *GM* heuristic and the *MG* heuristic, for the $GI/G/n/0$ queueing model. In summary, it is an important challenge to efficiently estimate the loss probability with heuristic approach methods. The formulae (3.6, 3.7 and 3.112) proposed in this thesis will facilitate the finding of exact solutions for phase-type distributions, the development of new heuristic methods, and the estimation of the loss probability.

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