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NEAT EXACT SEQUENCES OF ABELIAN GROUPS

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CONTENTS

ABSTRACT

A subgroup of an abelian group is said to be neat if divisibility of it's element by every prime implies divisibility in this subgroup. Neatness is a generalization of pureness. First we establish fundamental properties of neat subgroups, then prove that the class of all short neat-exact sequences determined by neat subgroups is proper in Buchsbaum's sense. Neat projective groups are direct sums of cyclic groups of prime order and free group. Neat injective groups are direct sums of cyclic groups of prime order and divisible groups. Neat exact sequences are completely determined by projective (injective) property of cyclic groups of finite order. The subgroup of the group of extensions given by neat exact sequences is the frattini subgroup.

ÖZ

Değişmeli bir grubun alt grubunun her elemanının herhangi bir asal sayı tarafından bölünebilirliği, elemanın bu asal sayı tarafından alt grupta da bölünebilirliği sonucunu veriyorsa bu alt gruba neat altgrup denir. Neat olma durumu pure olma durumunun genelleştirilmesidir. İlk olarak neat altgrupların temel özellikleri verildi. Sonra neat altgruplar tarafından oluşturulan kısa neat tam dizilerin sınıfının Buchsbaum'un anlamında düzgün sınıf oluşturduğu ispatlandı. Neat projektif gruplar asal mertebeli devirli grupların direkt toplamıdır ve serbest grupturlar. Neat injektif gruplar ise asal mertebeli devirli grupların direkt toplamıdır ve bölünebilir grupturlar. Neat tam diziler sonlu devirli grupların projektif (injektif) özelliği ile saptanır. Neat tam diziler tarafından oluşturulan grup genişlemelerinin altgrubu Frattini altgrubudur.

INTRODUCTION

Though homological algebra appeared as an algebraic instrument for the solution of topological problems. Relative homological algebra originated from two algebraic notions: pure subgroups of abelian groups and change of rings. We study neat subgroups of abelian groups which is a generalization of pureness.

In the first chapter we give some necassary definitions and we prove the fundamental properties of neat subgroups. Properties are similar to those of pure subgroups, but in some cases there are essential distinctions.

In the second chapter we give the definition of a proper class of short exact sequences and prove that the class of all short neat exact sequences is proper.

In the following chapter we study relative homological algebra determined by neatness. We describe completely neat projective and neat injective groups and prove that a short exact sequence is neat if and only if every cyclic group of prime order is projective (injective) with respect to this sequence.

It is well known thatthe pure exact sequences give the ulm's subgroup of the group of extensions. We prove in fourth chapter that the subgroupof the group extensions determined by neat-exact sequences is the Frattini subgroups.

All the groups mentioned in that text are abelian groups . And □ will be used at the end of a proof or at the end of a statement that not neading a proof.

1. SOME FACTS ABOUT NEAT SUBGROUPS.

Definitions.

Let G be a group and n be a non-negative integer (from now on n will always be a non-negative integer), let g be an element of G, we say; **"n divides g in G "** and denoted by (n|g) if there exists g' in G such that $g = ng'$. G is called a **Divisible Group**, if every nonzero integer n, n divides every element g in G. That is the equation $g = nx$ is solvable in G, or equivalently $G = nG$.

Let A be a group and G be a subgroup of A. If for every non-negative integer n, $nG = G \cap nA$, Or equivalently, for every n and for every g from G, if n divides g in A implies n divides g in G, then G is called **pure subgroup** of A. If $p^k G = G \cap p^k A$ for k=1,2,3, ... and for some prime p ,then G is called **p- pure subgroup** of A.

Definition. For a subgroup G of A, if **p divides g in A implies p divides g in G** for every prime p and for every element g of G, or in other words $pG = G \cap pA$, then G is called **neat subgroup** of A.

If for some prime p, $pG = G \cap pA$. Then G is called **p-neat subgroup** of A.

Definitions. Let A be a group, then $T(A) = \{a \in A \mid na = o, \text{ for some } n \in Z^+\}$ is called the **torsion part** of A. That is elements of A having finite order are torsion elements of A.

If $A = T(A)$ then A is called **torsion group**.

If A consists both elements of finite order and elements of infinite order, then it is called a **mixed group**.

If every element in A has infinite order then A is called **torsion free**.

$T_p = \{ a \in A \mid p^k a = 0, \text{ for some } k = 1,2,3,.... \}$ is called **p-subgroup** of A.

1) Every direct summand is neat subgroup

Proof. Let $A=B\oplus C$, where B and C are subgroups of A. We want to show B is neat in A. Let's take any b from B, if p divides b in $B\oplus C$ then, there exists b' from B and c' from C such that $b = p(b'+c')$.That is $b=pb' + pc'$. And we can also write $b = b + 0$, therefore since direct sum is uniquely determined, the equality $pb'+pc' = b + 0$ implies $pb' = b$ .That means B is neat in A. □

2) Torsion part of a mixed group is a neat subgroup.

Proof. $T(A) = \{ a \in A \mid na = 0$ for some $n \in Z^+ \}$. $T(A)$ is a subgroup of A, because if $a_1, a_2 \in T(A)$ then $n_1 a_1 = 0$ and $n_2 a_2 = 0$ for some n_1, n_2 from Z^+. But also $n_2(n_1 a_1) = 0$ and $n_1(n_2 a_2) = 0$ impliying $n_2 n_1 (a_1 - a_2) = 0$, so $a_1 - a_2$ becomes an element of T(A).

For neatness of T(A) in A, let 's take an element a from T(A), assume that $a = pa'$ for some a' from A and there exists a non-negative integer m, such that $ma = 0$ thus $ma = m(pa') = 0$ impliying $(mp)a' = 0$. Since mp is also a non-negative integer so a' is an element of T(A). Therefore $a = pa'$ tells us T(A) is neat in A. □

3) Let A be a group, G be a subgroup of A. if A/G is torsion-free then G is neat in A.

Proof. Let g be an element of G. Let $g = pa$ for some a from A. Then $G = g + G = p(a+G)$. Since A/G is torsion-free, $a + G = G$ i.e. $a \in G$. So G is neat in A. □

4) A p-neat, p-subgroup is neat in A.

Proof. Let G be a p-neat p-subgroup of A for some prime p. Let 's take an element g from G, if p divides g in A then p divides g in G. Now, for any prime $q \neq p$ $\gcd(q, p^k) = 1$. But since $0(g) = p^k$ for some integer "k", q divides every g from G. Because if $(q, p^k) = 1$ then there exists non-negative integers u and v such that $uq + vp^k = 1$ impliying
$uqg + vp^k g = g$ so, $qug = g$ $(p^k g = 0)$ where $ug \in G$. So q divides g in G. Therefore G is neat in A. □

5) In torsion - free groups, intersections of neat subgroups is neat.

Proof. Let $\cap G_i$ be intersection of torsion - free neat subgroups, where i is an element of some index set I. Let 's take $g = pa$ from $\cap G_i$ with $a \in A$, then g is an element of every subgroup G_i. Since every G_i is neat in A then there exists g_i from G_i such that $g = pg_i$ for every $i \in I$. Then that implies $pg_i = pa$, so $p(g_i - a) = 0$. Since groups are torsion free, $g_i - a = 0$ impliying $g_i = a$. So, a is an element of G_i for any $i \in I$, that is $a \in \cap G_i$, therefore $\cap G_i$ is neat in A.

□

6) Let A and B be groups and $f: A \to B$ be an isomorphism, let C be a subgroup of A. If C is neat in A then $f(C)$ is also neat in B.

Proof. Let $f(c) = pb$ for some b from B and for any prime p. Then since f is isomorphism there exists a' from A such that $b = f(a')$. So, $f(c) = p\,f(a') = f(pa')$ which implies $c = pa'$ because f is isomorphism. Since C is neat in A, $c = pc'$ for some $c' \in C$. So $f(c) = f(pc') = p\,f(c')$. Therefore $f(c) = p\,f(c')$ that means $f(C)$ is neat in B. ▯

7) Let A, B, C be groups satisfiying; $A \leq B \leq C$. If A is neat in C, then A is neat in B.

Proof. We know that $a = pc$ with $c \in C$ implies $a = pa'$ for some $a' \in A$. Assume that $a = pb$ for some b from B. Then since B is a subgroup of C, b is also an element of C. But then $a = pa''$ because A is neat in C. Therefore A is neat in B. ▯

8) Let A, B and C be groups. If A is a subgroup of B and $f(A)$ is neat in $f(B)$ for any monomorphism $f: B \to C$, then A is neat in B.

Proof. Assume that $a = pb$ then $f(a) = f(pb) = p\,f(b)$. Since $f(A)$ is neat in $f(B)$ then $f(a) = p\,f(b)$ implies $f(a) = f(pa')$, so ; $a = pa'$. Hence A is neat in B. ▯

__**Lemma 1.**__ Let C be a subgroup of B and B be a subgroup of A. Then,

(a) - If C is neat in B and B is neat in A then C is neat in A.

(b) - If B is neat in A then B/C is neat in A/C.

(c) - If C is neat in A and B/C is neat in A/C then B is neat in A.

Proof. (a) Assume that C is neat in B. Then,

(1).....For any c from C ; $c = pb'$ with b' from B implies $c = pc'$ for some c' from C.

Assume that B is neat in A. Then,

(2)....... For any b from B ; $b = pa$ with a from A implies $b = pb''$ for some b'' from B.

Assume that $c = pa'$ for any c from C and a' from A. But since C is a subgroup of B, then c is an element of B then by (2) $c = pb'''$ for some b''' from B. Then that implies by (1) $c = pc''$ for some c'' from C. Therefore $c = pa'$ where $a' \in A$ implies $c = pc''$ for some c'' from C. Hence C is neat in A.

(b) Assume that B is neat in A. Then

(3)....... $b = pa$ for a from A implies $b = pb'$ for some b' from B. Assume that $b_1 + C = p(a_1 + C)$, with $b_1 \in B$ and $a_1 \in A$, then $b_1 + C = pa_1 + C$ gives us $b_1 - pa_1 \in C$ so $b_1 - pa_1 = c_1$ for some $c_1 \in C$ is an element of C. And it becomes an element of B. Therefore $pa_1 = b_1 - c_1$ is an element of B, so we can write $pa_1 = b_2$ for some $b_2 \in B$, and by (3) $b_2 = pb_3$ so, $b_1 + C = pb_3 + C$ and that implies $b_1 + C = p(b_3 + C)$ Therefore B/C is neat in A/C.

(c) If C is neat in A then,

$c = pa$ implies $c = pc_1$.....(4)

If B/C is neat in A/C then $b + C = p(a_1 + C)$ implies $b + C = p(b_1 + C)$ We want to show B is neat in A. That's why assume that $b = pa_2$ for any b from B and for some a_2 from A. Since $b + C = pa_2 + C$ implies $b + C = pb_1 + C$ and from here we can write $pb_1 = b + c'$ for some c' in C so $pb_1 - b = c'$. Therefore $pb_1 - pa_2 = c'$, impliying $p(b_1 - a_2) = c'$ and we know $b_1 - a_2$ is an element A. So let's say $b_1 - a_2 = a_3$ thus $pa_3 = c'$ and by (4) $c' = pc''$ for some c'' from C but then $pb_1 = b + c'$ will imply $pb_1 = b + pc''$ so $p(b_1 - c'') = b$ and $b_1 - c''$ is in B. Let's say $b_1 - c'' = b_{111}$ therefore $pb_{111} = b$. So B is neat in A. □

Theorem 2. Let B be a subgroup of A such that $pB = 0$ for every prime p. Then the following statements are equivalent.

(a) B is neat in A

(b) B satisfies $B \cap pA = 0$

(c) B is a direct summand of A

Proof: (a) ⇒ (b) Assume that B is neat in A then $pB = B \cap pA$ and since $pB = 0$, then we get $B \cap pA = 0$.

(b) ⇒ (c) Assume that $B \cap pA = 0$, let C be a B - high subgroup of A such that $pA \leq C$. Existence of such a subgroup C is guaranteed by Zorn's lemma. Because consider the set

$$\wp = \{ K \mid K \leq A \ , \ pA \subseteq K \ , \ K \cap B = 0 \}$$

$\wp$ is partially ordered with "$\subseteq$" $\wp$ is non - empty because at least $pA \in \wp$. If we take a chain $\Im$ from $\wp$ then it has an upper bound in $\wp$ because $\cup K_i$ ($K_i \in \Im$) is an upper bound in $\wp$. So every chain in $\wp$ has an upper bound, thus $\wp$ is inductive. Therefore by zorn's lemma $\wp$ has a maximal element call as C.

$pA \leq C$ imply that pa is an element of C for every a from A and for every prime p. Now using lemma (9.8) in [F], we conclude that every a from A is an element of $B \oplus C$, A becomes a subgroup of $B \oplus C$. Also we know $B \oplus C \leq A$. Therefore $A = B \oplus C$.

(c) ⇒ (b) If B is a direct summand of A then by fact (1) B is neat in A □

Corollary3. For a prime p any pA - high subgroup of A is a direct summand of A.

Proof. Let B be a pA - high subgroup of A then $B \cap pA = 0$. So by theorem (2), B is a direct summand of A. ▯

Theorem 4. For a subgroup B of A, the following are equivalent

(a) B is neat in A.

(b) B/pB is a direct summand of A/pB for every prime p.

(c) If $C \leq B$ and $pB \leq C$ then B/C is a direct summand of A/C.

Proof. (a) ⇒ (b) Assume that B is neat in A then we know $pB \leq B \leq A$ then by lemma1 B/pB is also neat in A/pB for any prime p, then we can say p (B/pB) = pB. Therefore by theorem (2) part (c) B/pB is a direct summand of A/pB.

(b) ⇒ (c) Let B/pB be a direct summand of A/pB, and assume that $C \leq B$ such that $pB \leq C$ then there exists projection map $f : A/pB \to B/pB$ such that $f(b + pB) = b + pB$, where $b \in B$.

Let's define $F : A/C \to B/C$ as $F(a + C) = (gof)(a+pB)$ where $pB \leq C \leq B$ and $g : B/pB \to B/C$ such that $g(b+pB) = b + C$

Well definiteness of g is easy. Let's take $b + pB = b' + pB$ and from here $b - b' \in pB$, pB is a subgroup of C. Therefore $b + C = b' + C$. And we know $g(b + pB) = b + C$, $g(b' + pB) = b' + C$. Since $b + C = b' + C$, g is well defined. Similarly, F is well defined. Because, Take $a + C = a' + C$ and also we can write $F(a + C) = (gof)(a + pB) = g(f(a + pB)) = g(a + pB) = a + C$ and $F(a' + C) = (gof)(a' + pB) = g(f(a' + pB)) = g(a' + pB) = a' + C$. Therefore F is well defined. And since F is a projection map B/C is a direct summand of A/C

. **(c) ⇒(a)** Let's assume if $C \leq B$ and $pB \leq C$ then B/C is a direct summand of A/C. Let $b = pa$ for any b from B and for some a from A. If we choose $C = pB$, then by above assumption there exists a projection $f: A/pB \to B/pB$ such that $f(pa+pB) = f(b + pB) = b + pB$. Let $f(a + pB) = b' + pB$. Then $pb' + pB = p(b' + pB) = p f(a' + pB) = f(pa' + pB) = f(b + pB) = b + pB$ for some b' from B .and a' from A. Thus we get $pb' + pB = b + pB$. So, $b - pb' \in pB$. Hence we can write $b - pb' = pb''$. Therefore $b = p(b' + b'')$. Since $b' + b'' \in B$, we can say B is neat in A. ▯

2.PROPER CLASS OF NEAT EXACT SEQUENCES.

Let $\{ A_i \}$ $i \in I$ be a set of abelian groups, $\{ \alpha_i \}$ $i \in I$ be a set of homomorphisms with some index set I. A sequence

$$\ldots \ldots \to A_{i-1} \to A_i \to A_{i+1} \to A_{i+2} \ldots \ldots$$

Such a sequence of abelian groups and homomorphisms is called an exact sequence, if $\operatorname{Im} \alpha_i = \operatorname{Ker} \alpha_{i+1}$ for every $i \in I$

An exact sequence $O \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow O$ is called a short exact sequence, and by definition of an exact sequence we can say α is a monomorphism and β is an epimorphism.

Let $\mathfrak{R}$ be a class of short exact sequences, we write $\alpha \mathfrak{R} \beta$ to mean that a short exact sequence $O \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow O$ (let's denote the short exact sequence with (α,β)) is one of the short exact sequences of $\mathfrak{R}$.

$\mathfrak{R}_m$ is the class of all monomorphisms of elements of $\mathfrak{R}$. It means that there exists an epimorphism β such that $\alpha \mathfrak{R} \beta$

$\mathfrak{R}_e$ is the class of all epimorphisms of elements of $\mathfrak{R}$. It means that there exists a monomorphism α such that $\alpha \mathfrak{R} \beta$

Definition: $\mathfrak{R}$ is called **proper class** (in Buchshaum's sense) and any one of it's elements is called a proper short exact sequence, if the following holds :

1) If $\alpha \mathfrak{R} \beta$ then any short exact sequence which is isomorphic to (α,β) is also in $\mathfrak{R}$.

2) for any abelian groups A and C, $O \to A \to A \oplus C \to C \to O$ is proper exact sequence.

3) (i) If $\alpha o \alpha'$ is defined with $\alpha \in \mathfrak{R}_m$, $\alpha' \in \mathfrak{R}_m$ then $\alpha o \alpha' \in \mathfrak{R}_m$

(ii) If $\beta o \beta'$ is defined with $\beta \in \mathfrak{R}_e$, $\beta' \in \mathfrak{R}_e$ then $\beta o \beta' \in \mathfrak{R}_e$

4) (i) If α , α' are monomorphisms with $\alpha o \alpha' \in \mathfrak{R}m$ then $\alpha' \in \mathfrak{R}m$

(ii) If β , β' are epimorphisms with $\beta o \beta' \in \mathfrak{R}_e$ then $\beta \in \mathfrak{R}_e$.

Definition. An exact sequence $O \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow O$ is said to be **neat exact**, if $\operatorname{Im}\alpha$ is neat in B.

Theorem 5. Neat exact sequences construct a proper class.

Proof. We will check proper class axioms

1) Let E: $O \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow O$ be a neat exact sequence. Let $E' : 0 \longrightarrow A' \xrightarrow{\alpha'} B' \xrightarrow{\beta'} C' \longrightarrow 0$ be another exact sequence which is isomorphic to E. We will show that the exact sequence E' is also neat exact. Then we have a diagram

$$\begin{array}{ccccccccc} O & \longrightarrow & A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C & \longrightarrow & O \\ & & \downarrow{\scriptstyle \delta_1} & & \downarrow{\scriptstyle \delta_2} & & \downarrow{\scriptstyle \delta_3} & & \\ 0 & \longrightarrow & A' & \xrightarrow{\alpha'} & B' & \xrightarrow{\beta'} & C' & \longrightarrow & 0 \end{array}$$

where δ_1 , δ_2 , δ_3 are isomorphisms.

And by commutativity of the diagram $\alpha' \circ \delta_1 = \delta_2 \circ \alpha$, and also we can say $\alpha'(\delta(A)) = \alpha'(A')$ because δ is isomorphism.That's why ;$\delta(\alpha(A)) = \alpha'(A')$ and so $\delta(\alpha(A))$ is a subgroup of B' . Now using the fact (6) , since $\alpha(A)$ is neat in B and δ is isomorphism $\delta(\alpha(A))$ is neat in B' . Therefore $\alpha'(A')$ is neat in B' .

2) Let's take two objects A,C. We want to show $O \to A \to A\oplus C \to C \to O$ is neat exact . We know A is neat in $A\oplus C$, because it is a direct summand . And $\alpha(A)$ is also neat in $A\oplus C$, because α is inclusion function . So our sequence is neat exact .

3) (i) If α and α' are neat monomorphisms then we will show $\alpha' o \alpha$ is also neat monomorphism.

We know that $\alpha(A)$ is neat in B and $\alpha'(B)$ is neat in B'. And we may say $\alpha'(\alpha(A))$ is a subgroup of $\alpha'(B)$. Hence $\alpha'(\alpha(A))$ is neat in $\alpha'(B)$ by fact 8 . since $\alpha'(B)$ is a neat subgroup of B' $\alpha'(\alpha(A))$ is neat in B'. by lemma 1(a). So $\alpha' o \alpha$ is neat epimorphism.

Let 's see above work in the diagram :

$$\begin{array}{ccccccccc} O & \longrightarrow & A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C & \longrightarrow & O \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle 1_B} & & \downarrow{\scriptstyle \delta} & & \\ O & \longrightarrow & \alpha(A) & \xrightarrow{i} & B & \xrightarrow{\pi} & B/\alpha(A) & \longrightarrow & O \\ & & \downarrow{\scriptstyle \alpha'|\alpha(A)} & & \downarrow{\scriptstyle \alpha'} & & \downarrow{\scriptstyle \delta'} & & \\ O & \longrightarrow & \alpha'(\alpha(A)) & \longrightarrow & \alpha'(B) & \longrightarrow & \alpha'(B)/\alpha'(\alpha(A)) & \longrightarrow & O \\ & & \downarrow{\scriptstyle 1} & & \downarrow & & \downarrow & & \\ O & \longrightarrow & \alpha'(\alpha(A)) & \xrightarrow{i} & B' & \longrightarrow & B'/\alpha'(\alpha(A)) & \longrightarrow & O \end{array}$$

(ii) If $\beta' o \beta$ is defined with $\beta: B \to C$ and $\beta': C \to C'$ are neat epimorphisms then we will show that $\beta' o \beta$ is also neat epimorphism. We will define $D = \text{Ker } \beta'$ and $B' = \text{Ker } \beta' o \beta$. So B' will be a subgroup of B. Since Ker β' is a subgroup of C, we have an inclusion map $\alpha' : \text{Ker } \beta' \to C$. We know that $D = \text{Ker } \beta'$ is neat in C, and also $B' = \text{Ker } \beta' o \beta$ becomes a subgroup of B. So we have the following diagram with exact rows and columns. (In view of 3x3 lemma).

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \xrightarrow{\alpha} & B' & \xrightarrow{\beta/B'} & D & \longrightarrow & 0 \\
 & & \downarrow 1_A & & \downarrow i & & \downarrow \alpha' & & \\
0 & \longrightarrow & A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow \beta' o \beta & & \downarrow \beta' & & \\
0 & \longrightarrow & 0 & \longrightarrow & C' & \xrightarrow{1_{C'}} & C' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Where i is an inclusion map, and $\alpha : A \to B'$ is the same homomorphism $\alpha : A \to B$ with changed range.

Now, let $b' = pb$ for any b' from B', for any prime p and for some b from B. Then, we can write $\beta(b') = \beta(pb)$ which implies $\beta(b') = p\beta(b)$. Since $\beta(b')$ is an element of D, then there exists d from D such that $\beta(b') = pd$. $\beta|_{B'}$ is epimorphism, that's why there exists x from B' such that $\beta(x) = d$. So,

$$\beta(b') = pd = p\beta(x) = \beta(px)$$

Therefore $\beta(b') = \beta(px)$, so $b' - px$ is an element of $Ker\beta$, since $Ker\beta = Im\alpha$ we can say $b' - px = \alpha(a)$. But since $b' = pb$,then $pb - px = \alpha(a)$ implies $p(b - x) = \alpha(a)$. We know that $\alpha(A)$ is neat in B, so $\alpha(a) = p\alpha(a')$ for some a' from A . So,

$$\alpha(a) = p\alpha(a') = \alpha(pa')$$

implies $a = pa'$, because α is monomorphism.

Therefore $b' = px + \alpha(a) = px + p\alpha(a') = p(x + \alpha(a'))$ where $x + \alpha(a')$ is an element of B'. Hence, B' is neat in B, that means i(B') is neat in B. So $\beta' o \beta$ is neat epimorphism.

4) (i). If $\alpha : B \to B'$ and $\alpha' : A \to B$ are given monomorphisms with groups A, B, B' . If $\alpha o \alpha' : A \to B'$ is a neat monomorphism then we will show that α' is neat monomorphism.

We know that $(\alpha o \alpha')(A)$ is neat in B'. Since $\alpha'(A)$ is a subgroup of B, we can say $\alpha(\alpha'(A))$ is a subgroup of $\alpha(B)$ and also $\alpha(B)$ is a subgroup of B'. Therefore we can write

$$\alpha(\alpha'(A)) \leq \alpha(B) \leq B'$$

By fact (7) $\alpha(\alpha'(A))$ is neat in $\alpha(B)$. And so, by fact (8) $\alpha'(A)$ is neat in B.
We can see the above work in the following diagram.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \xrightarrow{\alpha'} & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow 1_A & & \downarrow \alpha & & \downarrow & & \\
0 & \longrightarrow & A & \xrightarrow{\alpha o \alpha'} & B' & \longrightarrow & C' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

(ii). Let $\beta: B \to B'$ and $\beta': B' \to C$ be given epimorphisms with groups B, B', C If $\beta' o \beta: B \to C$ is neat epimorphism. Then, we will show that β' is also neat epimorphism. So, $Ker\beta' o \beta$ is a subgroup of $Ker\beta'$. Let $A'' = Ker\beta$ Thus, we can define maps $\beta|_{Ker\beta' o\beta} : Ker\beta' o\beta \to Ker\beta'$, $\alpha': Ker\beta' \to B'$,$\alpha'|_{Ker\beta' o\beta}: Ker\beta' o\beta \to B$ become inclusion maps. And let $\alpha: A'' \to B$, $\alpha: A'' \to Ker\beta' o\beta$ be monomorphisms. That is same monomorphism α is defined with different range.

We can put them on to a diagram as follows.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A'' & \xrightarrow{1_{A''}} & A'' & \longrightarrow & X & \longrightarrow & 0 \\
 & & \downarrow \alpha & & \downarrow \alpha & & \downarrow & & \\
0 & \longrightarrow & Ker\beta' o\beta & \xrightarrow{\alpha'/ Ker\beta' o \beta} & B & \xrightarrow{\beta' o \beta} & C & \longrightarrow & 0 \\
 & & \downarrow \beta|_{Ker\beta' o\beta} & & \downarrow \beta & & \downarrow 1_C & & \\
0 & \longrightarrow & Ker\beta' & \xrightarrow{\alpha'} & B' & \xrightarrow{\beta'} & C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Let's take any x from $Ker\beta'$, and assume that $x = pb'$ for some b' from B'. Since β is epimorphism, also $\beta|_{Ker\beta' o\beta}$ is epimorphism. Therefore there exists y in $Ker\beta' o\beta$ such that $x = \beta(y)$, and there exists b in B such that $b' = \beta(b)$. So,

$$\beta(y) = x = pb' = p\beta(b) = \beta(pb)$$

implies $y - pb$ is in $Ker\beta$, and since $Ker\beta = Im\alpha = A''$, we can write $y - pb = \alpha(a)$ for some a from A''. Let's say $y' = y - \alpha(a) = pb$ then,

$$\beta(y') = \beta(y) - \beta(\alpha(a)) = \beta(y) = x$$

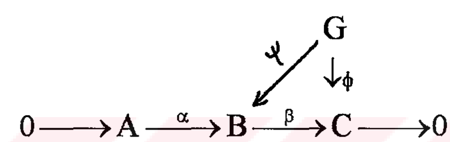
because, $\beta o\alpha = 0$. (Sequence is exact.), since $(\beta' o\beta)(y') = \beta'(x) = 0$, y' is also contained in $\mathrm{Ker}\beta' o\beta$. Now, since $\mathrm{Ker}\beta' o\beta$ is neat in B, then there exists z in $\mathrm{Ker}\beta' o\beta$ such that $y' = pz$. Then,

$$x = \beta(y') = \beta(pz) = p\beta(z)$$

And $\beta(z)$ is in $\mathrm{Ker}\beta'$. Hence, $\mathrm{Ker}\beta'$ is neat in B' means that β' is neat epimorphism. □

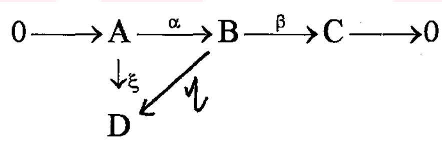
3. NEAT PROJECTIVITY AND NEAT INJECTIVITY

Definition. Let $E: 0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$ be an exact sequence and G be a group.If every diagram



can be completed by a suitable homomorphism $\psi: G \to B$, That is $\beta o\psi = \phi$. For any given homomorphism $\phi: G \to C$, then G is called **projective group** .

Definition. Let $0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$ be an exact sequence and D be a group. If every diagram



can be completed by a suitable homomorphism $\eta: B \to D$, that is $\eta o\alpha = \xi$. For any given homomorphism $\xi: A \to D$, then D is called **İnjective group.**

Definition. If the sequences on the above diagrams are neat exact sequences. Then projective group G is called **neat projective** and the injective group D is called **neat injective.**

Lemma 5. Let A and B be groups, if B is a subgroup of A such that A/B is a cyclic group of order p, then B is a direct summand of A.

Proof. A/B is a cyclic group of order p , say $A/B = \langle a + B \rangle$, $o(a + B) = p$. Let's consider the natural projection $\pi: A \to A/B$. Then $\pi(a) = a + B$ and since $o(a + B) = p$ we can write $\pi(pa) = p\pi(a) = p(a + B) = B$. So pa is an element of

$pa = px$ impliying $p(a-x) = 0$. Then $\pi(a-x) = \pi(a) - \pi(x) = \pi(a)$ because x is an element of B.

Now we will show $\langle a - x \rangle \cap B = 0$. Let y be an element of $\langle a - x \rangle \cap B$ then $y = n(a-x)$ and since y is an element of B. $\pi(y) = B$ that is $n(a + B) = n\pi(a) = n\pi(a-x) = \pi(n(a-x)) = \pi(y) = B$. Since $o(a + B) = p$, then $p|n$ that is $n = n'p$. Hence $y = n(a-x) = n'p(a-x) = 0$

Let's show also $A = \langle a - x \rangle + B$. Let's take an element z from A, since A/B is cyclic we can write it in the form $A = \langle a + B \rangle$, so $z + B = k(a + B)$ for some nonnegative integer k. Then $z - ka \in B$ that is $z - ka = b$ for some b from B. So, $z = ka + b = k(a - x) + (kx + b)$ where $k(a-x) \in \langle a - x \rangle$ and $(kx+b) \in B$. Therefore $k(a - x) + (kx + b) \in \langle a - x \rangle + B$. That means $A \subseteq \langle a - x \rangle + B$. And since $\langle a - x \rangle + B \subseteq A$ at the same time, $\langle a - x \rangle + B = A$.

Hence, $A = B \oplus \langle a - x \rangle$. Which means B is a direct summand of A. □

Theorem 6. An exact sequence E: $0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$ is neat exact if and only if for every prime p the group Z(p) has the projective property relative to E.

Proof. ($\Rightarrow$) Let E be a neat exact sequence, and $\phi: Z(p) \to C$ be any homomorphism. Since Z(p) is simple group, then either $\text{Ker}\phi = 0$ or $\text{Ker}\phi = Z(p)$. But if $\text{Ker}\phi = Z(p)$ then ϕ becomes zero homomorphism and can be trivially lifted to $\psi: Z(p) \to B$ by $\psi(x) = 0$. Now let $\text{Ker}\phi = 0$, that is ϕ be a monomorphism. $\pi: C \to C/\text{Im}\phi$ is the natural homomorphism and let's define $\pi o\beta: B \to C/\text{Im}\phi$. So $i: \text{Ker}(\pi o\beta) \to B$ is the inclusion map. ıf we take x from $\beta(\text{Ker}(\pi o\beta))$ then $(\pi o\beta)(x) = 0$ So, $\beta(x)$ is an element of $\text{Ker}\pi$ and because of the sequence is exact, $\beta(x)$ is also an element of $\text{Im}\phi$, thus $\beta(\text{Ker}(\pi o\beta) \subseteq \text{Im}\phi$. Since ϕ is monomorphism $\phi^{-1}: \text{Im}\phi \to Z(p)$ is well defined. And we can define

$$\beta': \text{Ker}(\pi o\beta) \to Z(p) \text{ as } \beta' = \phi^{-1} o\beta|_{\text{Ker}\,\pi o\beta}.$$

__claim__: $\text{Ker}\beta = \text{Ker}\beta'$.

To show that the claim is true, let's take an element a from $\text{Ker}\beta$, then $\beta(a) = 0$, so $(\pi o\beta)(a) = \pi(\beta(a)) = \pi(o)$. Thus $\beta'(a) = \phi^{-1}(\beta|_{\text{Ker}\pi o\beta})(a) = \phi^{-1}(0) = 0$ Hence $\text{Ker}\beta \subseteq \text{Ker}\beta'$. Conversely let's take an element a from $\text{Ker}\beta'$. Then $\beta'(a) = \phi^{-1} o\beta|_{\text{Ker}\,\pi o\beta}(a) = \phi^{-1}(\beta(a)) = 0$. But since $\phi^{-1}: \text{Im}\phi \to Z(p)$ is an isomorphism $\beta(a) = 0$. That is a is in $\text{Ker}\beta$. So the claim is true. And since E is neat exact $\text{Ker}\beta = \text{Im}\alpha = \text{Ker}\beta'$ therefore we can define $\alpha^{-1}: \text{Ker}\beta' \to A$ and it is an isomorphism here. Hence E' is isomorphic to E, so E' is also neat exact. Since $\text{Ker}\pi o\beta / \text{Ker}\beta'$ is isomorphic to Z(p), appliying lemma5 we can say $\text{Ker}\beta'$ is a direct summand of $\text{Ker}\pi o\beta$. And since $\alpha': \text{Ker}\beta' \to \text{Ker}\pi o\beta$ is inclusion map $\alpha'(\text{Ker}\beta')$ is also a direct summand of $\text{Ker}\pi o\beta$.

Hence

$$0 \longrightarrow \mathrm{Ker}\beta' \longrightarrow Ker(\pi o \beta) \xrightarrow{\beta} Z(p) \longrightarrow 0$$

is a splitting sequence, so by a theorem in [R], there exists $\gamma: Z_p \to \mathrm{Ker}\pi o\beta$ such that $\gamma o\beta' = 1_{Z(p)}$ .Let's define $\psi: Z(p) \to B$ by $\psi = io\gamma = \gamma$, so

$$\beta o\psi = \beta oio\gamma = \gamma o\beta' o\phi = \phi.$$

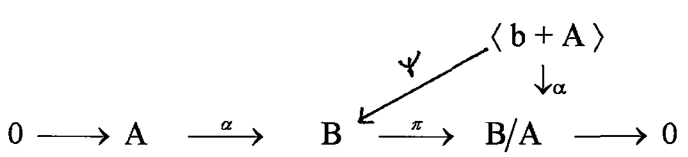
Therefore, Z(p) has the projective property relative to neat exact sequence E. We can see the above work in the following diagram.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Ker}\beta' & \longrightarrow & Ker(\pi o\beta) & \xrightarrow{\beta} & Z(p) & \longrightarrow & 0 \\
 & & \downarrow_{\alpha^{-1}} & & \downarrow_{i} & & \downarrow_{\phi} & & \\
0 & \longrightarrow & A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow_{\pi o\beta} & & \downarrow_{\pi} & & \\
0 & \longrightarrow & 0 & \longrightarrow & C/\mathrm{Im}\phi & \longrightarrow & C/\mathrm{Im}\phi & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

($\Leftarrow$) Conversely, assume that every Z(p) has projective property relative to neat exact sequences. So, let's take a neat exact sequence

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\pi} B/A \longrightarrow 0$$

For any a from A, let $a = pb$ for some b from B, then $p(b + A) = A$. That is the order of $\langle b + A \rangle$ is p. Therefore we can say $\langle b + A \rangle$ is isomorphic to Z(p). And since Z(p) has projective property, we can write the following diagram.



Here α becomes the inclusion map.We know that there exists $\psi:\langle b + A \rangle \to B$ such that $\pi o\psi = \alpha$ and $\psi(b + A) = b'$ for some b' from B. So,

$$pb' = p\psi(b + A) = \psi(p(b + A)) = \psi(0) = 0.$$

Now, $b' + A = \pi(b') = \pi o\psi(b + A) = \alpha(b + A) = b + A$. That is $b' - b$ is an element of A. Therefore $p(b' - b) = pb' - pb = pb = a$ and since $b' - b$ is in A, A is neat in B. That's why $\alpha(A)$ is also neat in B. Because α is inclusion function. □

Theorem 7. Let E: $0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$ be an exact sequence. E is neat exact if and only if every Z(p) has injective property relative to exact sequence E.

Proof. ($\Rightarrow$) Let E be a neat exact sequence and $\phi: A \rightarrow Z(p)$ be any homomorphism. Since Z(p) has no proper subgroups, either $\phi(A)=0$ or $\phi(A) = Z(p)$. If $\phi(A) = 0$ then ϕ becomes zero homomorphism and in that case it can be trivially extended to $\psi: B \rightarrow Z(p)$ with $\psi(x) = 0$. That's why we will admit ϕ as epimorphism. Let's define $\alpha': Z(p) \rightarrow B/\alpha(\text{Ker}\phi)$ such that $\alpha'(x) = (\pi o \alpha)(y)$ where y is taken from $\phi^{-1}(x)$, and $\pi: B \rightarrow B/\alpha(\text{Ker}\phi)$ is natural homomorphism. α' is well defined, because if $y \in \phi^{-1}(x)$ and $y' \in \phi^{-1}(x)$ then $\phi(y) = \phi(y')$, so $y - y'$ is in $\text{Ker}\phi$ that is $\alpha(y - y') \in \alpha(\text{Ker}\phi)$. And clearly $\alpha(\text{Ker}\phi)$ is a subgroup of B, so we can write the following exact sequence;

$$0 \longrightarrow \alpha(\text{Ker}\phi) \xrightarrow{i} B \xrightarrow{\pi} B/\alpha(\text{Ker}\phi) \longrightarrow 0$$

Hence $\alpha(\text{Ker}\phi) = \text{Im}i = \text{Ker}\pi$, so $\pi(\alpha(y - y')) = 0$ implies $(\pi o \alpha)(y) = (\pi o \alpha)(y')$. And since π, α, ϕ are homomorphisms, α' is also a homomorphism. Let's take an element a from $\phi^{-1}(x)$ then $(\alpha' o \phi)(a) = \alpha'(x) = (\pi o \alpha)(a)$. So α' makes the diagram commutative.

α' is monomorphism, because take an element x from $\text{Ker}\alpha'$, that is $\alpha'(x) = 0$. That is $(\pi o \alpha)(y) = 0$ implies $\alpha(y) \in \text{Ker}\pi$, and since $\text{Ker}\pi = \alpha(\text{Ker}\phi)$ then $\alpha(y) = \alpha(z)$ where z is from $\text{Ker}\phi$, so $y = z$ and then $\phi(y) = \phi(z) = 0$, since $x = \phi(y)$ then $x = 0$

Homomorphisms ϕ and π induce a homomorphism $\delta: C \rightarrow C'$ where $C' = (B/\alpha(\text{Ker}\phi)) / \text{Im}\alpha'$. So we have the following diagram with exact rows.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \text{Ker}\phi & \xrightarrow{\alpha} & \alpha(\text{Ker}\phi) & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow i & & \downarrow i & & \downarrow & & \\
0 & \longrightarrow & A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C & \longrightarrow & 0 \\
 & & \downarrow \phi & & \downarrow \pi & & \downarrow \delta & & \\
0 & \longrightarrow & Z(p) & \xrightarrow{\alpha'} & B/\alpha(\text{Ker}\phi) & \xrightarrow{\beta'} & C' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

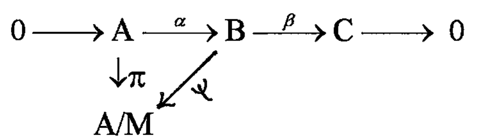
Since first two columns in this diagram are exact, the third one is also exact by 3x3 lemma in [F] that is δ is isomorphism and by theorem5-(3(ii)) $\delta o \beta$ is also neat epimorphism, and by commutativity of the square $\delta o \beta = \beta' o \pi$, so $\beta' o \pi$ is also neat epimorphism, so by theorem5-(4(ii)) β' is neat epimorphism that is α' is neat monomorphism. So $\alpha'(Z(p))$ is neat in $B/\alpha(\mathrm{Ker}\phi)$ and we can easily say $p\alpha'(Z(p)) = 0$ because order of $Z(p)$ is p. Hence by theorem2 $\alpha'(Z(p))$ is a direct summand of $B/\alpha(\mathrm{Ker}\phi)$, that's why there exists a projection $\theta: B/\alpha(\mathrm{Ker}\phi) \to \alpha'(Z(p))$, and consider the isomorphism $\alpha': Z(p) \to \alpha'(Z(p))$. Then define $\gamma: B/\alpha(\mathrm{Ker}\phi) \to Z(p)$ by $\gamma = (\alpha')^{-1} o \theta$. Therefore, $(\gamma o \alpha')(x) = ((\alpha')^{-1} o \theta o \alpha')(x) = ((\alpha')^{-1} o \alpha')(x) = 1_{Z(p)}(x) = x$. So, $\gamma o \alpha' = 1_{Z(p)}$. (here $(\alpha')^{-1} o \theta o \alpha')(x) = ((\alpha')^{-1} o \alpha')(x)$ because $\alpha'(x) \in \alpha'(Z(p))$ implies $\theta o(\alpha'(x)) = \alpha'(x)$.). Hence let's define $\psi: B \to Z(p)$, that is $\psi = \gamma o \pi$. So,

$$\gamma o \pi o \alpha = \gamma o \alpha' o \phi = 1_{Z(p)} o \phi = \phi \text{ implies } \psi o \alpha = \phi$$

Therefore diagram is commutative.

($\Leftarrow$) Conversely assume that every $Z(p)$ has injective property relative to exact sequence $E: 0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$. We can assume that $A \le B$ with regarding α as inclusion map. Assume that E is not neat exact, then $\alpha(A)$ is not neat in B and since α is inclusion map, A is not neat in B. Hence $A \cap pB \neq pA$. So we can take an element a from $(A \cap pB) - pA$ and define $\Gamma = \{ D \mid pA \le D \le A, a \notin D\}$. Clearly every chain $\{D_i\}$ has an upper bound, namely $\cup D_i$. Therefore by Zorn's lemma Γ has a maximal element say M

Consider A/M, then $p(A/M) = 0$ (because $pA \le M$) so A/M is isomorphic to $Z(p)$ by maximality of M. Therefore A/M is injective with respect to E, hence there is $\psi: B \to A/M$ such that the following diagram is commutative.



that is $\psi o \alpha = \pi$, π being the natural projection . Since α is inclusion map $\alpha(a) = a$. $\psi(a)$ is an element of $\psi(pB)$, and we know $\psi(pB)$ is a subgroup of $p(A/M)$ then $\psi(a)$ becomes an element of $p(A/M)$ and since $p(A/M)=0$ then $\psi(a)=0$. But since $a \notin M$, $0 \neq \pi(a) = \psi(\alpha(a)) = \psi(a)$ which gives us a contradiction.

□

Theorem 8. P is neat projective if and only if $P = P' \oplus F$, where P' is a direct sum of cyclic groups of prime order and F is free.

Proof. ($\Rightarrow$) Let P be neat-projective. For each homomorphism $\alpha: Z(p) \to P$ take a group $A_{p,\alpha} = Z(p)$ and let $\beta: F' \to P$ be an epimorphism from a free group F'. Let $A = (\oplus A_{p,\alpha}) \oplus F'$ where the first sum is taken over all prime p and possible homomorphisms α, and let $\gamma: A \to P$ be such an homomorphism that $\gamma|A_{p,\alpha} = \alpha$ and $\gamma|_{F'} = \beta$. (γ exists by property of direct sum) then γ is an epimorphism and it can be easily seen that every Z(p) is projective with respect to γ, therefore γ is a neat epimorphism by theorem 6. Since P is neat projective, γ is splitting, so P is isomorphic to a direct summand of F'. Then p-component $T_p(p)$ of P is isomorphic to a subgroup of $\oplus A_{p,\alpha}$ and therefore is a direct sum of cyclic groups of order p. So torsion part $T(P) = P'$ is a direct sum of cyclic groups of prime order. Since torsion part of A is a direct summand, P' also is a direct summand of P. Let $P = P' \oplus F$, now clearly F is isomorphic to a subgroup of the free group F', hence is free itself.

($\Leftarrow$) Since every cyclic group of prime order is neat-projective, by theorem6 their direct sum also is neat-projective. Therefore every group $P = P' \oplus F$ with P' which is a direct sum of cyclic groups of prime order and with a free group F, is neat-projective. □

Theorem 9. I is neat-projective if and only if $I = I' \oplus D$, where I' is a direct summand of a direct product of cyclic groups of prime order and D is divisible.

Proof. ($\Rightarrow$) Let I be neat-injective, dual to theorem (8) let $B = (\Pi B_{p,\alpha}) \oplus D'$ where $B_{p,\alpha} = Z(p)$, and α all possible homomorphisms $\alpha: I \to Z(p)$, D' is a divisible group with a monomorphism $\beta: I \to D'$. Again by the main property of direct product there is a unique homomorphism $\gamma: I \to B$ such that $\pi_{p,\alpha} o \gamma = \alpha$ and $\pi_{D'} o \gamma = \beta$. It can be seen that γ is a neat-monomorphism, therefore I is a direct summand of B. Then $I = I' \oplus D$ where D is divisible and I' is a direct summand of $\Pi B_{p,\alpha}$.

($\Leftarrow$) Each $B_{p,\alpha}$ is neat-injective by theorem (7), their direct product and it's direct summand I' also is neat-injective. Since divisible group D is neat-injective, $I = I' \oplus D$ is neat injective. □

4. NEAT EXACT SEQUENCES FORM THE FRATTİNİ SUBGROUP OF EXT(C,A)

We can visualize the extension B of A by C as an exact sequence;

$$0 \longrightarrow A \xrightarrow{\upsilon} B \xrightarrow{\nu} C \longrightarrow 0$$

We can construct a category ε such that objects are short exact sequences and morphisms are triples as (α,β,γ) of homomorphisms such that the following diagram has commutative squares:

$$\begin{array}{lccccccccc}
E & 0 \longrightarrow & A & \xrightarrow{\upsilon} & B & \xrightarrow{\nu} & C & \longrightarrow 0 \\
 & & \downarrow\alpha & & \downarrow\beta & & \downarrow\gamma & \\
E': & 0 \longrightarrow & A' & \xrightarrow{\upsilon'} & B' & \xrightarrow{\nu'} & C' & \longrightarrow 0
\end{array}$$

E and E' are said to be equivalent with A=A' and C=C' (denoted by $E \approx E'$) if there is a morphism $(1_A, \beta, 1_C)$

$$\begin{array}{lccccccccc}
E: & 0 \longrightarrow & A & \xrightarrow{\upsilon} & B & \xrightarrow{\nu} & C & \longrightarrow 0 \\
 & & \downarrow 1 & & \downarrow\beta & & \downarrow 1 & \\
E': & 0 \longrightarrow & A' & \xrightarrow{\upsilon'} & B' & \xrightarrow{\nu'} & C' & \longrightarrow 0
\end{array}$$

β becomes isomorphism by 5-lemma

Let $E: 0 \longrightarrow A \xrightarrow{\upsilon} B \xrightarrow{\nu} C \longrightarrow 0$ be an exact sequence. Define a homomorphism $\alpha: A \to A'$ as in the following diagram

$$\begin{array}{ccccccc}
0 \longrightarrow & A & \xrightarrow{\upsilon} & B & \xrightarrow{\nu} & C & \longrightarrow 0 \\
 & \downarrow\alpha & & & & & \\
 & A' & & & & &
\end{array}$$

then we can construct pushout diagram by theorem 10.1 in [F]:

$$\begin{array}{lccccccc}
E: & 0 \longrightarrow & A & \xrightarrow{\upsilon} & B & \xrightarrow{\nu} & C & \longrightarrow 0 \\
 & & & & \downarrow\alpha & & \downarrow\beta & \\
 & & & & A' & \rightarrow & B' &
\end{array}$$

with $B' = A' \oplus B / H$ where $H = \{ (\mu(a), -\alpha(a)) \mid a \in A \}$. Then ν is a monomorphism and $B'/\mathrm{Im}\nu$ is isomorphic to C (see [A]). Therefore we can define a short exact sequence in a natural way such that

$\alpha E: 0 \longrightarrow A' \xrightarrow{\upsilon'} B' \xrightarrow{\nu'} C' \longrightarrow 0$ and the following diagram

$$
\begin{array}{l}
E:\ 0 \longrightarrow A \xrightarrow{\upsilon} B \xrightarrow{\nu} C \longrightarrow 0 \\
\qquad\qquad \downarrow\alpha \qquad \downarrow\beta \qquad \downarrow 1_C \\
\alpha E: 0 \longrightarrow A' \xrightarrow{\upsilon'} B' \xrightarrow{\nu'} C' \longrightarrow 0
\end{array}
$$

is commutative.

Let's apply theorem 10.2 in [F] with C, then $\nu': B' \to C$ as

$$\nu'((a',b)+H) = (1_C o\nu)(b) + \nu' o\mu'(a')$$

since αE is exact $\nu'((a',b)+H)$ and by 10.2 in [F] the diagram is commutative. So αE is the extension of A' by C. Here $\alpha_* = (\alpha,\beta,1_C)$ is a morphism $E \to \alpha E$ in ε

It is well known (see e.g. [F]) that all abelian extensions of the group A by C form an abelian group under baer's addition which is called group of extensions of A by C and is denoted by Ext(C,A). If $\mathfrak{R}$ is a proper class then all $\mathfrak{R}$-extensions of A by C form the subgroup $Ext_{\mathfrak{R}}(C,A)$ of Ext (C,A). In particular, if $\mathfrak{R}$ is the class $\mathfrak{I}$ of all pure exact sequences then

$$Ext_{\mathfrak{I}}(C,A) = Pext\,(C,A) = \cap\, n\, Ext\,(C,A) = (Ext\,(C,A))^1$$

where n is a positive integer. So it gives us Ulm's subgroup of Ext (C;A) (see theorem 53.3 in [F] and the remark next to it). We are going to prove that the subgroup $Ext_{\aleph}(C,A) = Next\,(C;A)$ of Ext (C,A) given by neat exact sequences is the frattini subgroup of Ext(C,A).

Theorem 10. A short exact sequence $E: 0 \longrightarrow A \xrightarrow{\upsilon} B \xrightarrow{\nu} C \longrightarrow 0$ is neat if and only if E is an element of pExt(C,A) for every prime p.

Proof. ($\Rightarrow$) Let E be neat and p be prime. A multiplication homomorphism $p: A \to A$ defined by $p(a) = pa$ is the composition $p = \theta o\gamma$ of an epimorphism $\gamma: A \to pA$ defined by $\gamma(a) = pa$ and an inclusion $\theta: pA \to A$. Then we have the following short exact sequences

$$0 \longrightarrow A[p] \xrightarrow{\delta} A \xrightarrow{\gamma} pA \longrightarrow 0 \quad ....(1)$$

$$0 \longrightarrow pA \xrightarrow{\theta} A \longrightarrow A/pA \longrightarrow 0 \quad ....(2)$$

where δ is an inclusion and π is natural homomorphism. Appliying the functor Ext(C,.) to (2) we shall have the following exact sequence:

$$........ \longrightarrow Ext(C,pA) \xrightarrow{\theta_*} Ext(C,A) \xrightarrow{\pi_*} Ext(C, A/pA) \longrightarrow 0$$

Since E is neat $\pi_*(E)$ is also neat. $p(A/pA) = 0$, hence A/pA is neat injective by theorem? and therefore $\pi_*(E)$ is splitting. Then E becomes an element of $Ker\pi_*$ and since the sequence is exact $Ker\pi_* = Im\theta_*$, so $E \in Im\theta_*$ that is $E = \theta_*(E')$ for some E' from Ext(C,pA). Now appliying Ext(C,.) to (1) we have the following exact sequence.

$$\ldots\ldots\ldots\ldots\ldots\ldots . \longrightarrow Ext(C,A) \xrightarrow{\gamma_*} Ext(C,pA) \longrightarrow 0$$

that is γ_* is epimorphism. Then $E' = \gamma_*(E'')$ for some E" from Ext(C,A). Since $p_* = p$ is the multiplication by p in Ext(C,A) then by lemma 52.1 in [F], $E = \theta_* \gamma_*(E'') = (\theta\gamma)_*(E'') = p_*(E'') = pE''$ and pE" is an element of pExt(C,A) therefore E is also an element of pExt(C,A).

($\Leftarrow$)Conversely let E be divisible by every prime p. Then $E = p_*(E') = \theta_*\gamma_*(E')$ and so $\theta_*\gamma_*(E')$ is an element of $Im\theta_*$ which is equivalent to $Ker\pi_*$ because of the sequence is exact. That is $\pi_*(E)$ is splitting for every prime p.Constructing $\pi_*(E)$ by means of pushout diagram (see p.123 in [F]) we shall have the following commutative diagram with exact rows.

$$\begin{array}{lccccccccc}
E: & 0 \longrightarrow & A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C & \longrightarrow & 0 \\
 & & \downarrow\pi & & \downarrow\beta & & \downarrow 1 & & \\
\pi_*(E): & 0 \longrightarrow & A/pA & \xrightarrow{\delta} & B' & \longrightarrow & C & \longrightarrow & 0
\end{array}$$

$\pi_*(E)$ is splitting, that is there is a homomorphism $\xi: B' \to A/pA$ such that $\xi o \delta = 1_{A/pA}$

To show that E is neat let $\alpha(a) = pb$, where $a \in A$, $b \in B$; p be prime . Then by commutativity of the diagram

$$\delta(a + pA) = \delta o \pi(a) = \beta o \alpha(a) = \beta(pb) = p\beta(b).$$

Appliying ξ to this equality we shall have

$$a + pA = \xi o \delta(a + pA) = \xi(p\beta(b)) = p\xi(\beta(b)).$$

Let $\xi(\beta(b)) = a' + pA$ where $a' \in A$. Then $a + pA = pa' + pA$; hence $a - pa' \in pA$, that is $a - pa' = pa''$ for some a" from A. Therefore $a = p(a' + a'')$ and $\alpha(a) = p\alpha(a'+a'')$, so $\alpha(a)$ is an element of $p\alpha(A)$. It means that $\alpha(A)$ is a neat subgroup in B, that is E is a neat exact sequence.

It is well known that $\cap pA$ is the Frattini subgroup of A for every A, so we have the following corollary.

<u>**Corollary 11.**</u> $Ext_N(C,A)$ is the frattini subgroup of Ext(C,A).

References

[F] . Fuchs, L, Infinite abelian groups vol 1, Academic Press 1970.

[M] Mac Lane, S, Homology, Springer-Verlag 1963

[R] Alizade, R, Lecture notes in Relative Homological Algebra 1994

[K] Kaplansky, I, İnfinite Abelian Groups, The University of Michigan press 1954

[S] Sklyarenko, E; G, Relative homological algebra In categories of Modules, Uspekhi Mat. Nauk. 33:3 (1978) 85-120