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THE ISOTROP CONGRUENCES

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İZMİR

SUMMARY

In this study the necessary geometrical differantial condition which provides the congruance of lines, defined by an analitical function, to be an isotrop congruance has been found. In addition the principal dispersion parameters of the congruance of lines defined by an analitical function has also been investigated.

ÖZET

Bu çalışmada bir analitik fonksiyonla belirtilmiş doğrular kongrüansının izotrop kongrüans olması için gerekli geometrik diferansiyel koşul bulunmuş, ayrıca analitik fonksiyonla belirtilmiş doğrular kongrüansının esas yüzeyleri ve esas dağıtma parametreleri incelenmiştir.

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CHAPTER I

1. INTRODUCTION

1.1. DUAL NEMBERS AND MODUL

Definition 1.1.1. If a, $\bar{a}$ $\in \mathbb{R}$, $\mathbb{D}$ = $\mathbb{R}$ x $\mathbb{R}$ set identified as $\mathbb{D} = \{(a, \bar{a}) : A = a + \varepsilon \bar{a},\ \varepsilon^2 = 0,\ a, \bar{a} \in \mathbb{R}\}$ is named as the set of dual numbers and an element of it is called as a dual number. If adding, multiplication and equality in $\mathbb{D}$ set of dual numbers are as follws

$$+ \quad : \quad \mathbb{D} \times \mathbb{D} \rightarrow \mathbb{D}$$

$$(A, B) \rightarrow (a+b, \bar{a}+\bar{b})$$

$$\bullet \quad : \quad \mathbb{D} \times \mathbb{D} \rightarrow \mathbb{D}$$

$$(A, B) \rightarrow A.B=(a + b,\ a.\bar{b} + \bar{a}b)$$

$$A=B \Leftrightarrow a=b, \quad \bar{a}=\bar{b}$$

the triple of ($\mathbb{D}$, +, •) constitutes a commutative ring with unity With respect to addition and multiplication the identity element of this ring are as follows 0=(0, 0) $\in \mathbb{D}$ and I=(1, 0) $\in \mathbb{D}$. The solution of the equation A.X=X.A=(1,0) will merely be in the form of

$$X = \left(\frac{1}{a}, -\frac{\bar{a}}{a^2}\right)$$

in the ring of dual numbers Provided a≠0, division operation can not be done by the dual numbers in the form of $(0, \bar{a}) \in \mathbb{D}$.

Definition 1.1.2. In a dual number of $A=(a, \bar{a})=a+\varepsilon\bar{a} \in \mathbb{D}$ the term $a\in\mathbb{D}$ is named as the real part, and the term $\bar{a} \in \mathbb{D}$ is called as the dual part of the number. These are written as Rea=a DuA=$\bar{a}$

Definition 1.1.3. Let H be a commutative ring with the unity of 1. An Abelian group on S together with an external operation which has the properities below,

$$H \text{ x} S \rightarrow S$$

$$(a, \alpha) \rightarrow a\alpha$$

is called as a modual on S. Provided a, b∈H and α, β∈S the equations below can be written

$$M_1 : a(\alpha+\beta)= a\alpha + a\beta$$

$$M_2 : (a+b)\,\alpha = a\alpha + b\alpha$$

$$M_3 : (ab)\,\alpha = a\,(b\alpha)$$

$$M_4 : 1.\alpha = a$$

In according to this definition, $\mathbb{D} \times \mathbb{D} \times \mathbb{D} = \mathbb{D}^3$ constitues a modul, in the dual ring.

Definition 1.1.4. The ordered dual triplets, being the elements of $\mathbb{D}$ - modul, are called as dual vectors. Provided $\vec{a}, \vec{\bar{a}} \in \mathbb{R}^3$ by definition, a dual vector of $\vec{A}=(A_1, A_2, A_3)$ on $\mathbb{D}$ - modul is written as follows

$$\vec{A} = (A_1, A_2, A_3)$$

$$= (a_1+\varepsilon \bar{a}_1, a_2+\varepsilon \bar{a}_2, a_3+\varepsilon \bar{a}_3)$$

$$= (a_1, a_2, a_3)+\varepsilon(\bar{a}_1, \bar{a}_2, \bar{a}_3)$$

$$= \vec{a}+\varepsilon\vec{\bar{a}}$$

Defination 1.1.5. The internal product of dual vectors of $\vec{A} = \vec{a}+\varepsilon\vec{\bar{a}}$, $\vec{B}=\vec{b}+\varepsilon\vec{\bar{b}}$ $\varepsilon\mathbb{D}$- modul is a transformation in the term of

$$\langle\ \rangle : \mathbb{D}^3 \to \mathbb{D}^3 \quad \to \mathbb{D}$$

$$(A, B) \quad \to \langle A, B\rangle$$

and is defined as

$$\langle A, B\rangle = \langle \vec{a}, \vec{b}\rangle+\varepsilon[\langle \vec{a}, \vec{\bar{b}}\rangle+\langle \vec{\bar{a}}, \vec{b}\rangle]$$

The properties of the internal product of two vectors of a real vectorial space can strictly be applied on $\mathbb{D}$ -modul.

Definition 1.1.6. The norm of dual vector of $\vec{A} = \vec{a}+\varepsilon\vec{\bar{a}}$ is a dual vector in the form of

$$||\vec{A}|| = (\langle\vec{A}, \vec{A}\rangle)^{½} = ||\vec{a}||+\varepsilon \frac{\langle\vec{a}, \vec{\bar{a}}\rangle}{||\vec{a}||} \quad (\vec{a}\neq 0)$$

Definition 1.1.7. Let the norm of the dual number of $\vec{A}=\vec{a}+\varepsilon\vec{\bar{a}}$ $\in\mathbb{D}$-modul be $||\vec{A}||$ = (1,0). The dual vector of A is called as unit dual vector and in this case $\vec{a}^2 = 1$, $\vec{a}.\vec{\bar{a}}=0$ is written. Because of $\vec{A} = (0, a) \in \mathbb{D}$, the $U = \frac{\vec{A}}{\|\vec{A}\|}$ is the unit dual vector.

$$k = \frac{\langle \vec{a}, \vec{\bar{a}} \rangle}{\|a\|^2}$$

$$\vec{U} = \vec{u} + \varepsilon \vec{\bar{u}} = \frac{\vec{a}}{\|\vec{a}\|} + \varepsilon \frac{\vec{\bar{a}} - k.\vec{a}}{\|\vec{a}\|}$$

is reached.

Definition 1.1.8. Let $\vec{a}(a_1, a_2, a_3)$ and $\vec{\bar{a}}(\bar{a}_1, \bar{a}_2, \bar{a}_3)$ be two vectors and a_1, a_2, a_3, $\bar{a}_1$, $\bar{a}_2$, $\bar{a}_3$, the six element of these two vectors are called as Pluckers coordinates of $\vec{A} = \vec{a} + \varepsilon\vec{\bar{a}} \in \mathbb{D}$ - module vector.

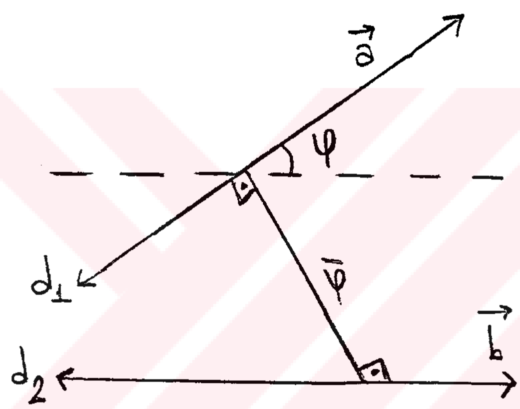
If modul vector of $\vec{A} = \vec{a} + \varepsilon\vec{\bar{a}} \in \mathbb{D}$ - module is dual vector, any four of the Plucker coordinates are independent due to the condition $\vec{a}^2=1$, $\vec{a}.\vec{\bar{a}}= 0$.

Definition 1.1.9. The set of $S=\{\vec{A} = \vec{a}+\varepsilon\vec{\bar{a}}, \|\vec{A}\| =(1,0), \vec{a}, \vec{\bar{a}} \in \mathbb{R}\}$ is called as dual sphere on $\mathbb{D}$ - modul

Theorem 1.1.1. (E. Study Transformation) The dual points of the dual unit sphere having the equation of $\|\vec{A}\| = (1, 0)$ with the condition of $\vec{A} \neq (0, a) \in \mathbb{D}$ are reflected to the lines of $\mathbb{R}^3$ one-to-one. In according to this theorem when the dual unit vectors of $\vec{A} = \vec{a}+\varepsilon\vec{\bar{a}}$ are given, one directed line of $\mathbb{R}^3$ is clearly and completly defined. Here, $\vec{a}$ is the unit real vector on the directed line and the vector $\vec{\bar{a}}$ is the moment of the real vector $\vec{a}$ with respect to the origine.

Difinition 1.1.10. Let $\varphi \in \mathbb{R}$ be the real angle in $\mathbb{R}^3$ between the directed lines that has been demonstrated by the unit dual vectors of $\vec{A}$ and $\vec{B}$ and let $\overline{\varphi} \in \mathbb{R}$ be the shortest distance between directed lines.

The dual number of $\phi = \varphi + \varepsilon\overline{\varphi}$ is called as the the dual angle between dual unit vectors of A and B.



Since the image of the Study mapung of the unit dual vectors will demonstrate the points of A and B of the dual unit sphere in $\mathbb{D}$ - modul, the dual angle of $\phi = \varphi + \varepsilon\overline{\varphi}$ can be thought as the $\widehat{AB}$ arc lenght of the greater dual circle passing through the dual points A and B.

Since rhe Taylor expansion of analitical Cos ϕ and Sin ϕ functions are

$$\text{Cos } \phi = \text{Cos } (\varphi + \varepsilon\overline{\varphi}) = \text{Cos } \varphi - \varepsilon\overline{\varphi} \ \text{Sin } \varphi$$

$$\text{Sin } \phi = \text{Sin } (\varphi + \varepsilon\overline{\varphi}) = \text{Sin } \varphi + \varepsilon\overline{\varphi} \ \text{Cos } \varphi$$

the expression of

$$\langle \vec{A}\vec{B} \rangle = \text{Cos}\ \varphi = \langle \vec{a}\,\vec{b} \rangle + [\langle \vec{a}\,\vec{\bar{b}} \rangle + \langle \vec{\bar{a}}\,\vec{b} \rangle]$$

being the internal product of $\vec{A}$ and $\vec{B}$ given in the equation of 1.1.5. is compared with the expansion Cos ϕ dual valued function

$$\langle \vec{a}\,\vec{b} \rangle = \text{Cos}\ \varphi \quad , \quad \langle \vec{a}\,\vec{\bar{b}} \rangle + \langle \vec{\bar{a}}\,\vec{b} \rangle = -\bar{\varphi}\ \text{Sin}\ \varphi$$

is found.

If the dual vectors of A and B are not unit dual vectors then $\vec{U} = \frac{\vec{A}}{\|\vec{A}\|}$, $\vec{V} = \frac{\vec{B}}{\|\vec{B}\|}$ are unit vectors. And let the dual angle between the dual vectors $\vec{U}$ and $\vec{V}$ be ϕ , since $\vec{U}\ .\ \vec{V} = \text{Cos}\ \phi$, $\langle \vec{A}\vec{B} \rangle = \|\vec{A}\|\ .\ \|\vec{B}\|\ .\ \text{Cos}\ \phi$ is obtained. Using this formula, the relative positions of the directed lines in $\mathbb{R}^3$ with respect to each other, can be studied:

1. If $\langle \vec{A}\vec{B} \rangle$ = merely dual, from Cos $\varphi = 0$, $\varphi = \frac{\pi}{2}$ is found.

Since here $\bar{\varphi} \neq 0$ the directed lines being demonsrated by the unit vectors of $\vec{A}$ and $\vec{B}$ perpendicular but contrary.

2. If $\langle \vec{A}\vec{B} \rangle = 0$ since $\varphi = \frac{\pi}{2}$ and $\bar{\varphi} = 0$ the dual directed lines demonsrated by dual unit vectors of $\vec{A}$ and $\vec{B}$ intersect each other in perpendicular.

3. If $\langle \vec{A}\vec{B} \rangle = 0$ merely real, since $\overline{\varphi} = 0$, the directed lines intersect each other. The the expression of $\langle \vec{a}\vec{\overline{b}} \rangle + \langle \vec{\overline{a}}\vec{b} \rangle = 0$ is the condition of intersection for these two lines.

4. If $\langle \vec{A}\vec{B} \rangle = (1, 0)$ since $\varphi = 0$ and $\overline{\varphi} = 0$ the directed lines are parellel to each other and are in the same direction. But if $\langle \vec{A}, \vec{B} \rangle = -(1, 0)$ since $\varphi = \pi$, $\overline{\varphi} = 0$ these directed lines are parellel but in reverse directions.

Definition 1.1.11. The external product of dual module vectors of $\vec{A}$, $\vec{B}$ $\in \mathbb{D}$ is a function in the form of

$$\Lambda : \mathbb{D}^3 \times \mathbb{D}^3 \to \mathbb{D}^3$$

and is defined as

$$\vec{A} \Lambda \vec{B} = \vec{a} \Lambda \vec{b} + \varepsilon (\vec{a} \Lambda \vec{b} + \vec{a} \Lambda \vec{b})$$

In addition, let N be the dual unit vector and $\varphi = \varphi + \varepsilon\overline{\varphi}$ be the angle between the dual vectors of A and B, then;

$$\vec{A} \Lambda \vec{B} = \|\vec{A}\| . \|\vec{B}\| \operatorname{Sin} \varphi . \vec{N}$$

is written.

Definition 1.1.12. Provided $\lambda_i = \lambda_i + \varepsilon \overline{\lambda}_i \in \mathbb{D}$ - module $(1 \le i \le 3.)$ For the dual vectors of $\vec{A}$, $\vec{B}$, $\vec{C}$ $\in \mathbb{D}$ - modul the equation of $\Lambda_1 \vec{A} + \Lambda_2 \vec{B} + \Lambda_3 \vec{C} = 0$ is satisfied by $\Lambda_1 = \Lambda_2 = \Lambda_3 = 0$ the dual vectors of $\vec{A}$, $\vec{B}$, $\vec{C}$ are called as linear independent, otherwise are called dependent vectors.

Definition 1.1.13. If the directed lines represented by the dual unit vectors of $\vec{A}$, $\vec{B}$, $\vec{C}$ $\in \mathbb{D}$ - module R^3 intersect each other perpendicular. The dual unit vectors of $\vec{A}$, $\vec{B}$, $\vec{C}$ are called as orthonormal dual vectors.

Theorem 1.1.2. If the modul vector of $\vec{A}$, $\vec{B}$, $\vec{C}$ $\in \mathbb{D}$ are orthonormal the set of $S = (\vec{A}, \vec{B}, \vec{C})$ constitutes a base in $\mathbb{D}$ - modul.

Theorem 1.1.3. If S= (A, B) is a linear independent set in $\mathbb{D}$ - modul, a base system can be established in the form of $\vec{E} = (\vec{A}, \vec{B}, \vec{C})$. [1]

1.2. THE RULED SURFACES AND THE CONGRUENCE

Let $u \in \mathbb{R}$ be an arbitrary parameter. The unit vector of

$$\vec{A}(u) = \vec{a}(u) + \varepsilon \vec{a}(u) \quad (1)$$

demonstrates a ruled surface in the Euclidian space of $\mathbb{R}^3$ and a dual curve on unit sphere. By the conditions of $\vec{A}_1 = \vec{A}$, $\vec{A}_2 = \frac{\vec{A}_1'}{\|\vec{A}_1'\|}$, $\vec{A}_3 = \vec{A}_1 \Lambda \vec{A}_2$ it is possible to found the Blachke trihedron of ($\vec{A}_1$, $\vec{A}_2$, $\vec{A}_3$) to every causaltant of a ruled surface given by the equation (1). The derivation formulas of Blachke trihedron will be in the form of

$$\begin{aligned} \vec{A}_1' &= P\vec{A}_2 \\ \vec{A}_2' &= -P\vec{A}_1 + Q\vec{A}_3 \\ \vec{A}_3' &= -Q\vec{A}_2 \end{aligned} \quad (2)$$

Here $P = p + \varepsilon \bar{p} = \|\vec{A}'\|$

$$Q = q + \varepsilon \bar{p} = \frac{(\vec{A}_1^{'}, \vec{A}_1^{''}, \vec{A}_1^{'''})}{\vec{A}_1^{'2}} \tag{3}$$

The quantities of P and Q are called as dual curvative and dual torsion of ruled surface respectively. The dual arc lenght of the ruled surface of A = A (u), from $u=u_o$ to u = u, is

$$S = s + \varepsilon \bar{s} = \int_{u_o}^{u} \left\| \vec{A}_1^{'} \right\| . \, du = \int_{u_o}^{u} P \, . \, du,$$

The dispersion parameter of the ruled surface is in the form of

$$\frac{1}{d} = \lambda = \frac{\vec{a}_1^{'} \, \vec{\bar{a}}_1^{'}}{\vec{a}_1^{'2}} = \frac{\bar{P}}{P} \tag{5}$$

If u and ϑ are real and arbitrary parameters, the dual unit vector of

$$\vec{A}\,(u, \vartheta) = \vec{a}\,(u\,, \vartheta) + \varepsilon\,\vec{\bar{a}}(u\,, \vartheta) \tag{6}$$

demonstrate a congruence of lines in $\mathbb{R}^3$ space and a dual domain on the unit sphere. The dual arc element and the dispersing parameter of ruled surface in the congruences of A = A (u , ϑ) given by the equation (2) are in the form of

$$d\vec{A}^2 = E du^2 + 2\,F\,du\,d\vartheta + G d\vartheta^2$$

$$=(e\,du^2 + 2f\,du\,d\vartheta + g d\vartheta^2) + \varepsilon(\bar{e}\,du^2 + 2\bar{f}\,du\,d\vartheta + \bar{g} d\vartheta^2) \tag{7}$$

$$= I + \varepsilon\,II$$

and

$$\lambda = \frac{1}{d} = \frac{II}{21} = \frac{\bar{e}du^2 + 2\bar{f}du\, d\vartheta + \bar{g}d\vartheta}{2(e\, du^2 + 2f\, du\, d\vartheta + gd\vartheta^2)} \qquad (8)$$

The extremum values of dispersion parameter are called as the principle dispersion parameters and are defined by the equation of,

$$\begin{vmatrix} 2e\lambda - \bar{e} & 2f\lambda - \bar{f} \\ 2f\lambda - \bar{f} & 2g\lambda - \bar{g} \end{vmatrix} = 0 \qquad (9)$$

The directions of $\frac{du}{d\vartheta}$, reflected to the principal dispersing parameter are called as principle directions and the differantial equation of these are in the form of

$$\begin{vmatrix} e\, du + f\, d\vartheta & \bar{e}\, du + \bar{f}\, d\vartheta \\ f\, du + g\, d\vartheta & \bar{f}\, du + \bar{g}\, d\vartheta \end{vmatrix} = 0 \qquad (10)$$

The ruled surfaces of the congruence, satisfying the last equation are named as principle ruled surfaces. There are always two families of principle surfaces of a congruence.

The differential equation (10) of principle ruled surfaces has no meaning in the case below:

$$\frac{\bar{e}}{e} = \frac{\bar{f}}{f} = \frac{\bar{g}}{g} = \lambda \qquad (11)$$

This is characterized by which all ruled surfaces through a line of a congruence has same drall. These types of congruences has been entitled by A Ribaucour as isotrop congruences.

The expressions of

$$h = \frac{1}{2}(\lambda_1 + \lambda_2) = \frac{1}{4}\ \frac{e\bar{g} - 2f\bar{f} + g\bar{e}}{(eg - f^2)^2} \tag{12}$$

$$k = \lambda_1 \bullet \lambda_2 \qquad = \frac{1}{4}\ \frac{\bar{e}\,\bar{g} - \bar{f}^2}{(eg - f^2)^2} \tag{13}$$

are called as the mean and the cumulative dispersion of the congruence, respectively. [2]

Prof. Dr. M. Kula has studied the principle surfaces and a equation between the principle parameters of the line congruence, determined by analitical functions in [3] and [4].

Here we have studied the conditions of being isotrop of this congruence and we also have given a comment about the principle surface of it.

CHAPTER II

THE ISOTROP CONGRUENCE DEFINED BY ANALITICAL FUNCTIONS

In a Cartesian coordinate system of OXYZ, let us sketch two axis from the point of 0 (0 , 0 , h) parallel to the axis of OX and OY, and thus, let's establish a second Cartesian system.

Let $W = f(z) = u(x, y) + i\vartheta(x, y)$ be continious analitical function. With the aid of this function let's arrange the points of the plane of $W = u + i\vartheta$ to the points of $z = x + iy$.

With the condition of

$$\frac{X - x}{x - u} = \frac{Y - y}{y - \vartheta} = \frac{Z - 0}{0 - h} = k \qquad \text{and} \qquad -k = t$$

The expression line D , connecting the two points will be in the form of

$$D : \begin{cases} X = x + (u - x)\, t \\ Y = y + (\vartheta - y)\, t \\ Z = h \, . \, t \end{cases} \tag{14}$$

When the point of $z = x + iy$ scans the definition domain, the line D constitutes a line congruance in space of lines.

Let's tend to express this congruence by unit vector of

$$\vec{A} = \vec{a}(x, y) + \varepsilon \vec{\bar{a}}(x, y) \qquad (15)$$

To do this; provided

$$\ell^2 = (u - x)^2 + (\vartheta - y)^2 + h^2$$

$$\vec{\alpha} = \vec{\alpha}(u - x, \vartheta - y, h)$$

the vectors of

$$\vec{a} = \frac{\vec{\alpha}}{\|\vec{\alpha}\|}, \qquad \vec{\bar{a}} = \vec{r} \wedge \vec{a}$$

are

$$\vec{a} = \vec{a}\left(\frac{u - x}{\ell}, \frac{\vartheta - y}{\ell}, \frac{h}{\ell}\right)$$

$$\vec{\bar{a}} = \vec{\bar{a}}\left(\frac{y h}{\ell}, \frac{-x h}{\ell}, \frac{x\vartheta - u y}{\ell}\right) \qquad (16)$$

Then, from (14) the expression of congruence using dual unit vectors, are in the form of

$$\vec{A} = \vec{A}\left(\frac{u - x}{\ell} + \varepsilon \frac{y h}{\ell}, \frac{\vartheta - y}{\ell} - \varepsilon \frac{x h}{\ell}, \frac{h}{\ell} + \varepsilon \frac{x\vartheta - uy}{\ell}\right)$$

$$\vec{A} = \vec{a}\,(x\,,\,y) + \varepsilon\,\vec{\bar{a}}\,(x\,,\,y) \tag{17}$$

From (7), the dual arc element of congruence in this ruled surface will be in the form of,

$$d\vec{A}^{2} = \left[(\vec{a}_x d_x + \vec{a}_y d_y) + \varepsilon\,(\vec{\bar{a}}_x\, d_x + \vec{\bar{a}}_y\, d_y) \right]^2$$

$$d\vec{A} = E d_x^2 + 2\,F\,d_x d_y + G\,d_y^2$$

When the formulas (16) are partially derivated the quantities of

$$E = \vec{a}_x^{\,2} + 2\,\varepsilon\,\vec{a}_x\,\vec{\bar{a}}_x = e + \varepsilon\,\bar{e}\,;$$

$$e = \vec{a}_x^{\,2} \qquad \bar{e} = 2\,\vec{a}_x\,\vec{\bar{a}}_x$$

$$F = \vec{a}_x\,a_y + \varepsilon\,\vec{a}_x\,\vec{\bar{a}}_y + \varepsilon\,\vec{a}_y\,\vec{\bar{a}}_x = f + \varepsilon\,\bar{f}$$

$$f = \vec{a}_x\,\vec{a}_y$$

$$\bar{f} = \vec{a}_x\,\vec{\bar{a}}_y + \vec{a}_y\,\vec{\bar{a}}_x$$

$$G = \vec{a}_y^{\,2} + \varepsilon\,\vec{a}_y\,\vec{\bar{a}}_y = g + \varepsilon\,\bar{g}\,;$$

$$g = \vec{a}_y^{\,2}, \qquad g = \vec{a}_y\,\vec{\bar{a}}_y$$

will be

$$e = \vec{a}_x^{\,2} = \frac{\vartheta_x^2 + (u_x - 1) - \ell_x^2}{\ell^2}$$

$$\overline{e} = 2\,\vec{a}_x\,\vec{\overline{a}}_x = -\frac{2h\vartheta_x}{\ell^2}$$

$$f = \vec{a}_x\,\vec{a}_y = -\frac{\ell_x\ \ell_y}{\ell^2} \qquad (18)$$

$$\overline{f} = \vec{a}_x\,\vec{\overline{a}}_y + \vec{a}_y\,\vec{\overline{a}}_x = 0$$

$$g = \vec{a}_y^{\,2} = \frac{u_y^2 + (\vartheta_y - 1) - \ell_y^2}{\ell^2}$$

$$\overline{g} = 2\,\vec{a}_y\,\vec{\overline{a}}_y = -\frac{2h\vartheta_y}{\ell^2}$$

Now let's seek the condition of being isotrop of the congruance (15). To do this the equations (18) are substituted in the equation (11) provided λ = constant. An equation system with partial derivatives is obtained in the form of

$$\frac{-2h\vartheta_x}{u_x^2 + (u_x - 1) - \ell_x^2} = \frac{0}{\ell_x\ \ell_y} = \frac{2h\vartheta_y}{u_y^2 + (\vartheta_y - 1) - \ell_y^2} = \lambda(\text{cons})$$

$$\begin{cases} \ell_x\,\ell_y = 0 \\ -2h\vartheta_x = \lambda\left(\vartheta_x^2 + (u_x - 1) - \ell_x^2\right) \\ 2h\vartheta_y = \lambda\left(u_y^2 + (\vartheta_y - 1) - \ell_y^2\right) \end{cases} \qquad (19)$$

Let's investigate the solution of this system, if $\varphi = \varphi(y)$ and $h = h(x)$ are arbitrary functions, the solution of the equation system (19) will be from $\ell_x = 0, \ell = \varphi(y)$ or $\ell_y = 0, \ell = h(x)$

If here $\ell = \varphi(y)$, with the condition of $\mu = \dfrac{\lambda}{2h}$ = constant ($\neq 0$), the equation $(19)_2$ and $(19)_3$ will have the form of

$$\vartheta_x = -\mu\left[\vartheta_x^2 + (u_x - 1)\right]$$
$$\vartheta_y = \mu\left[u_y^2 + (\vartheta_y - 1) + \varphi'^2\right] \qquad (20)$$

If the equations $(20)_1$ and $(20)_2$ are partially derivated in respect to y and x, respectivly

$$\vartheta_{xy}\ (1 + 2\mu\ \vartheta_x) = -\mu\, u_{xy}$$
$$\vartheta_{yx}\ (1 - \mu) = 2\mu u_y \,.\, u_{yx} \qquad (21)$$

are obtained. Since $u = u(x, y)$ and $\vartheta = \vartheta(x, y)$ are analitical functions the condition of $\vartheta_{yx} = \vartheta_{xy} \neq 0$, $u_{yx} = u_{xy} \neq 0$, $u_x = \vartheta_y$, $u_y = -\vartheta_x$ exist. Using these conditions, if we proportionate the equations of (21)

$$\frac{(1 - \mu)}{1 + 2\mu\,\vartheta_x} = 2\,\vartheta_x$$

is found. And from this; with the condition of

$$\frac{1}{2\mu} = a = \text{constant} \quad \text{and} \quad \frac{1-\mu}{4\mu} = b = \text{constant}$$

a differential equation of second order with costant coefficient partial derivative is obtained in the form of

$$\vartheta_x{}^2 + a\,\vartheta_x = b \qquad (22)$$

Here, the relationship of $2a = 4b + 1$ is valid between the constans of a and b.

Now let's study the case of $\ell_y = 0$ and $\ell = h(x)$. In this situation too,

provided $\dfrac{\lambda}{2h} = \mu$ = constant from the equations of $(19)_2$ and $(19)_3$.

$$\begin{aligned} \vartheta_x &= -\mu\left[\vartheta_x^2 + (u_x - 1) - h'^2\right] \\ \vartheta_y &= \mu\left[u_y^2 + (\vartheta_y - 1)\right] \end{aligned} \qquad (23)$$

are obtained. Thus, the equations of (23) are partially derivated with respect to y and x respectively and are proportionated ; the partial derivation of (22) is obtained and the constants a and b are the constans above.

Conclusion 2.1. The isotrop congruance under investigation is free from the lenght of line D and the differantial equation with partial derivative is unique and is in the form of

$$\vartheta_x^2 + a\vartheta_x = b \qquad (24)$$

Here,

$$a = \frac{1}{2\mu} = \text{constant}, \quad b = \frac{1-\mu}{4\mu} \quad \text{and}$$

$$2a = 4b + 1$$

Equation (24) is a first degree non linear differantial equation with partial derivative. From the solution of this equation $\vartheta = \vartheta(x, y)$ and $u_y = -\vartheta_x$ $u_x = \vartheta_y$ are obtained and from the condition of being analytical $u = u(x, y)$ is found; thus using these two; the conditions defining the isotrop congruance are stated.

Now let's study the principle surfaces of the congruance that is defined by the equation (17). To do this, let's carry the equations (18) previously determined; to their places in equations (10) that identify the principle surface of the congruance. The differantial equation below is obtained.

$$\vartheta_x \left\{ \ell_x . \ell_y . d_x^2 + (\ell_y^2 - \ell_x^2)\, d_x\, d_y - \ell_x . \ell_y . d_y^2 \right\} = 0 \quad (25)$$

Here the equation of $\vartheta_x = 0$ requires that $\vartheta_y = 0$; in according to the condition of being analitical. The equation of $u_y = 0$ and $\vartheta_x = 0$ necessitate that the functions of $\vartheta = \vartheta(x)$ and $u = u(x)$ to be in the forms of

$$u(x) = k_1 x + k_2$$
$$\vartheta(x) = k_1 y + k_3 \qquad (26)$$

Here $k_1, k_2, k_3 \in \mathbb{R}$. If the functions of $u = u(x)$ and $\vartheta = \vartheta(x)$ obtained from (26) are carried to their places in the equation (14)

$$x = (1 + tk_1 - t)\, x + tk_2$$

$$y = (1 + tk_2 - t)\, y + tk_3$$

$$z = th$$

are found. Here choosing $t = \dfrac{1}{1 - k_1}$

$$x_0 = \frac{k_2}{1 - k_1}, \quad y_0 = \frac{k_3}{1 - k_1}, \quad z_0 = \frac{h}{1 - k_1} \qquad (27)$$

constant points are obtained.

Conclusion 2.2. Provided $k_1, k_2, k_3 \in \mathbb{R}$. choosing the analitical function in the form of below

$$f(z) = k_1 z + (k_2 + ik_3)$$

The line congruance (1) is the congruance that has been established by all lines passing through the equation (27) and all rulled surfaced belonging to the congruance are the cones that have the top point coordinates of (x_0, y_0, z_0).

Now let's study the condition of $\vartheta_x \neq 0$ from the equation (25)

$$\ell_x \ell_y d_x^2 + (\ell_y^2 - \ell_x^2)\, d_x d_y - \ell_x \ell_y d_y^2 = 0$$

or

$$(\ell_x d_x + \ell_y d_y)(\ell_y d_x - \ell_x d_y) = 0 \qquad (28)$$

is obtained. Firstly from the equation of

$$\ell_x d_x + \ell_y d_y = 0$$

it is seen that a family of principle surfaces is defined by the equation of

$$\ell (x , y) = \text{constant} \qquad (29)$$

Considering that here ℓ is the distance between contrary points of (x , y , o) and (u , ϑ, h). The parts of the same surface rays of this family remaining between the planes $z = 0$; and $z = h$ in other words the parts remaining between the planes that parellel to the plane $z = 0$, are parellel to each other. The opposition of this property is also true. $\ell (x , y) =$ constant , satisfy the equation of $\ell_x d_x + \ell_y d_y = 0$

Conclusion 2.3. In the congruance that has been identified by an analytical function of $W = f (z) = u (x , y) + i\vartheta (x , y)$, the biconditional proposition, making a ruled surface of the congruance belonged to the family of principle surfaces that is defined by the equation of $\ell (x , y) =$ constant, is that the parts of the rays of this ruled surface remaining between the planes that are parellel to the plane $z = 0$, to be equal each other. And secondly let's take hand the equation below of the equation (28)

$$\ell_y d_x - \ell_x d_y = 0 \qquad (30)$$

From $\ell^2 = (u - x)^2 + (\vartheta - y)^2 + h^2$ finding the necessary partial derivatives ℓ_x and ℓ_y and transfering them to their places in the equation (30)

$$\left[(u - x)\, u_y + (\vartheta - y)\,(\vartheta_y - 1)\right] d_x - \left[(u - x)\,(u_x - 1) + (\vartheta - y)\vartheta_x\right] d_y = 0$$

is obtained. Taking the conditions being analytical into consideration $u_x = \vartheta_y$ and $u_y = -\vartheta_x$

$$\left[-(u-x)\vartheta_x + (\vartheta - y)(\vartheta_y - 1\right]d_x - \left[(u-x)(\vartheta_y - 1) - (\vartheta - y)u_y\right]d_y = 0$$

$$-(u-x)\, d\,(\vartheta - y) + (\vartheta - y)\, d\,(u-x) = 0$$

$$\frac{d\,(u-x)}{(u-x)} = \frac{d\,(\vartheta - y)}{(\vartheta - y)}$$

that means,

$$\frac{\vartheta - y}{u - x} = \text{constant} \qquad (31)$$

is found. The equation (31) determines the other family of principle surfaces and this shows that the proportion of the projection coordinates $z = 0$ of the line D given by the equation (14) is constant.

Conclusion 2.4. In a congruance that has been identified by analytical function of $W = f(z) = u(x, y) + i\vartheta\,(x, y)$ the beconditional proposition making a ruled surface of this congruance belonged to the family of ruled surfaces is that the proportion of this projection coordinates of this ruled surfaces rays on any plane to be parellel to the plane $z = 0$

Finally let's study the dispersion parameters of the principle surfaces family of the congruance from the equation below that determines the first principle surface family;

$$\ell_x d_x + \ell_y d_y = 0 , \qquad \ell(x, y) = \text{constant}$$

and taking into consideration that

$$\ell_x d_y + \ell_y d_y = 0$$

$$\ell_x^2 d_x^2 + \ell_y^2 d_y^2 + 2 \ell_x . \ell_y . d_x . d_y = 0$$

also from the equation of (7) and (18) and the conditions of being analitical of the function

$$I = e d_x^2 + 2f d_x d_y + g d_y^2 = \left[\vartheta_x^2 + (\vartheta_y - 1)^2 \right] . \frac{d_x^2 + d_y^2}{\ell^2}$$

$$II = \bar{e} d_x^2 + 2\bar{f} d_x d_y + \bar{g} d_y^2 = 2hu_y . \frac{d_x^2 + d_y^2}{\ell^2}$$

are found. Transfering the equations above to their places in the equations (8), the dispersion parameter of the principle surface is found as

$$\lambda_1 = \frac{II}{2I} = \frac{h . u_y}{\vartheta_x^2 + (\vartheta_y - 1)^2} \qquad (32)$$

And now let's find the dispersion parameter of secondary principle surface family from the equation below

$$\ell_y . d_x - \ell_x d_y = 0$$

$$\ell_y^2 d_x^2 + \ell_x^2 d_y^2 - 2 \ell_x . \ell_y . d_x . d_y = 0$$

and from the equation of (7) and (18), also considering the conditions making the function analytical

$$I = ed_x^2+2fd_xd_y + gd_y^2=\left[\vartheta_x^2 + (\vartheta_y -1)^2 - (\ell_x^2 + \ell_y^2)\right] . \frac{d_x^2 + d_y^2}{\ell^2}$$

$$II = \bar{e}d_x^2 + 2\,\bar{f}\,d_x\,d_y + \bar{g}\,d_y^2 = 2hu_y \, . \frac{d_x^2 + d_y^2}{\ell^2}$$

are found. Transfering these, to their places in the equations (8) the dispersion parameter of secondary principle surface is found as follow

$$\lambda_2 = \frac{II}{2I} = \frac{h \, . \, u_y}{\vartheta_x^2 + (\vartheta_y - 1) - (\ell_x^2 + \ell_y^2)} \qquad (33)$$

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