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THE APPLICATIONS OF LIE ALGEBRA TO THE SPATIAL MOTIONS

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ABSTRACT

The oriented lines in Euclidean space (R^3) are in one-to-one correspondence with the points of the dual unit sphere in dual space (D^3). Lie Algebra and module over the dual number ring endowed with a dual valued inner product . We use the Campbell-Hausdorf formula of Lie's general theory. The Campbell-Hausdorf formula is used for the consecutive motion.

In this work, the motion of a point on dual sphere is given by Olinde-Rodrigues formula and also infinitesimal motion of this motion and high degree of acceleration of velocity are calculated . Moreover it is defined with respect to lie bracket.

ÖZ

R^3 Öklid uzayında yönlendirilmiş doğrular, D^3 dual uzayındaki dual birim kürenin noktalarına birebir karşılık gelir . Lie cebri ve modül dual sayılar halkası üzerinde dual değerli iç çarpım oluştururlar. Hareket için genel Lie teorisinin Campbell-Hausdorf formülünü kullanırız. Bu formül ardışık hareketler için kullanılır.

Bu çalışmada dual küre üzerindeki bir noktanın hareketi Olinde-Rodrigues formülü ile verilmiştir. Bu hareketin sonsuz küçük hareketi de gözönünde bulundurulmuş , ayrıca hızın yüksek mertebeden ivmeleri lie parantezi cinsinden ifade edilmiştir.

APPLICATIONS OF LIE ALGEBRA TO THE SPATIAL MOTIONS

1. INTRODUCTION

Kinematics and dynamics of interconnected rigid bodies systems may lead to very complex computations and it is important to perform these computations in the most compact form and to search for their most rational organization. This target motivates a great deal of research on the fundamental operations and the algebraic structures lying behind kinematics methods.

In [1] two opposite approaches may be distinguished: the purely geometrical methods (linear complex and so on [4]) and the matrix methods operating on coordinates.

The main objective of [1] is to develop a unified algebraic approach to mathematical methods in kinematics , including a complete system of basic operations and computation laws. This system synthesizes the essential contributions of linear algebra, screw theory dual numbers method and Lie groups to the subject.

Since the publication of the basic papers [5,6,7,8] many investigations have made use of dual numbers, dual numbers matrix and dual quaternions in kinematics of multi-rigid-body chains, including basic expositions, of Refs. [9,10] including applications to spatial mechanisms and industrial manipulators.

Coordinate-free methods, based on intrinsical computations in groups or Lie Algebras have been introduced in Refs. [11,12,13,14] , for the purpose of mechanism theory , and of dynamics of multibody systems [15,16,17].

In this article Lie's groups are defined by analytical transformations depending on a finite number of real parameters.The motion of a point which is on the dual sphere, is defined by Olinde-Rodrigues formula. Moreover infinitesimal motion of this motion and the high degree of acceleration of velocity can be calculated with aid of Lie bracket. It provides a great number of the advantages of pure geometry.

2. SOME MATHEMATICAL PRELIMINARIES

2.1 DUAL NUMBERS

Dual numbers are "numbers" expressed by $\hat{x} = x + \varepsilon x^*$ with x, x^* in R and ε satisfies $\varepsilon^2 = 0$. The composition rules for the dual numbers result from the definitions:

i) Equality : $\hat{x} = \hat{y}$ iff $x=y, x^* = y^*$

ii) Addition : $\hat{x}+\hat{y} = x + y + \varepsilon\,(x^* + y^*)$

iii) Multiplication : $\hat{x}\hat{y} = x.y + \varepsilon\,(xy^* + yx^*)$

The set of all dual numbers make an abelian ring having the numbers εc^* (c^* real) as divisors of zero denoted by Δ ; in an obvious meaning R is a subring of Δ.

The division $\frac{\hat{x}}{\hat{y}}$ is possible and unambiguous if $y \neq 0$ and it is easily seen that

$$\frac{\hat{x}}{\hat{y}} = \frac{x}{y} + \varepsilon \frac{x^* y - y^* x}{y^2} \qquad (y \neq 0) \qquad (2.1.1)$$

In all other cases division is either impossible or ambiguous. We define for a differentiable funtion f :

$f(\hat{x}) = f(x + \varepsilon x^*) = f(x) + \varepsilon x^* f'(x)$, where f' is the derivative of f. Therefore

$$\text{Sin}(x + \varepsilon x^*) = \text{Sin}\, x + \varepsilon x^* \text{Cos}\, x \qquad (2.1.2)$$

$$\text{Cos}(x + \varepsilon x^*) = \text{Cos}\, x - \varepsilon x^* \text{Sin}\, x \qquad (2.1.3)$$

$$\sqrt{x + \varepsilon x^*} = \sqrt{x} + \varepsilon \frac{x^*}{2\sqrt{x}} \qquad (x > 0) \qquad (2.1.4)$$

We define $|\hat{x}| = \sqrt{\hat{x}^2} = \sqrt{x^2 + 2\varepsilon x x^*}$.

We have therefore in view of (2.1.4):

$$|\hat{x}| = \sqrt{x^2} + \varepsilon \frac{xx^*}{\sqrt{x^2}} = |x| + \varepsilon \frac{xx^*}{|x|}$$

and consequently:

$|\hat{x}| = \hat{x}$ (x>0) ; $|\hat{x}| = -\hat{x}$ (x<0) .

Evidently, $|\hat{x}| = 0$ if $\hat{x} = 0$.

2.2 DUAL VECTORS

Let $O\hat{x}_1\hat{x}_2\hat{x}_3$ be right- handed orthonormal frame of reference in a three dimensional Euclidean space E. The unit vector indicating the positive sense on the x_k-axis will be denoted by i_k (k=1,2,3). The set of all ordered set of dual numbers denoted by Δ-module make a module over abelian ring endowed an abelian group operation (+) and external law $(\hat{\lambda},\hat{x}) \rightarrow \hat{\lambda}\hat{x} = (\hat{\lambda}\hat{x}_1, \hat{\lambda}\hat{x}_2, \hat{\lambda}\hat{x}_3)$ with $\hat{\lambda} \in \Delta$ and $\hat{x} \in \Delta$-module. The axioms of a Δ-module are similar to these of a vector space, except scalars are selected in a ring Δ and not in a field.

A Δ-module is always a real vector space if the Δ ring is reduced to R. Starting from a real vector space E to leads to a Δ-module $\hat{E}$ such that E is a real vector subspace of $\hat{E}$ and $\hat{E} = E \oplus \varepsilon E$ (direct sum of real vector spaces), εE is the set of $\varepsilon x^* = \varepsilon(x^*_1, x^*_2, x^*_3)$.

If (i_1, i_2, i_3) is a basis of E , it is also a basis of $\hat{E}$ over Δ , that is to say ; every $\hat{x}$ in $\hat{E}$ express as $\hat{x} = \hat{x}_1 i_1 + \hat{x}_2 i_2 + \hat{x}_3 i_3 = x_1 i_1 + x_2 i_2 + x_3 i_3 + \varepsilon(x_1^* i_1 + x_2^* i_2 + x_3^* i_3)$, $\vec{x} + \varepsilon \vec{x}^*$ where the $\hat{x}_i$ (i=1,2,3) are uniquely defined dual numbers. Thus a vector in $\hat{E}$ may be thought of as a vector with dual coordinates and called a dual vector.

Scalar and vector products of ordinary vectors extend in a natural way from E to $\hat{E}$ and provide Δ- bilinear operations in $\hat{E}$:

$$\vec{\hat{x}}\vec{\hat{y}} = \sum_{i=1}^{3} \hat{x}_i \hat{y}_i \in \Delta \qquad (2.2.1)$$

$$\vec{\hat{x}} \times \vec{\hat{y}} = (\hat{x}_2 \hat{y}_3 - \hat{x}_3 \hat{y}_2) i_1 + (\hat{x}_3 \hat{y}_1 - \hat{x}_1 \hat{y}_3) i_2 + (\hat{x}_1 \hat{y}_2 - \hat{x}_2 \hat{y}_1) i_3 \qquad (2.2.2)$$

If x≠0 then the norm $\left\|\vec{\hat{x}}\right\|$ of $\vec{\hat{x}}$ is defined by $(\vec{\hat{x}}\vec{\hat{x}})^{1/2}$. From (2.1.4) , we obtain

$$\left\|\vec{\hat{x}}\right\| = \left\|\vec{x}\right\| + \varepsilon \frac{\vec{x}\vec{x}^*}{\left\|\vec{\hat{x}}\right\|} = \left\|\vec{\hat{x}}\right\|\left(1 + \varepsilon \frac{\vec{x}\vec{x}^*}{\left\|\vec{\hat{x}}\right\|^2}\right) \qquad (2.2.3.)$$

A dual vector $\vec{\hat{x}}$ with norm 1 is called a dual unit vector.It follows from (2.2.3) that $\hat{x}$ is a dual unit vector iff the relations

$$\vec{x}\vec{x}=1 \quad , \quad \vec{x}\vec{x}^{*}=0 \qquad (2.2.4)$$

hold simultaneously. It is easy to verify that for any vector $\vec{\hat{x}}=\vec{x}+\varepsilon\vec{x}^{*}$ with $\vec{x}\neq\vec{0}$

the relation $\vec{\hat{x}}_0=\frac{\vec{\hat{x}}}{\left\|\vec{\hat{x}}\right\|}=\frac{\vec{x}+\varepsilon\vec{x}^{*}}{\left\|\vec{x}\right\|(1+\varepsilon\frac{\vec{x}\vec{x}^{*}}{\left\|\vec{x}\right\|^2})}$

$$=\frac{1}{\left\|\vec{x}\right\|}(\vec{x}+\varepsilon\vec{x}^{*})(1-\varepsilon\frac{\vec{x}\vec{x}^{*}}{\left\|\vec{x}\right\|^2})=\frac{\vec{x}}{\left\|\vec{x}\right\|}+\frac{1}{\left\|\vec{x}\right\|}\varepsilon(\vec{x}^{*}-\frac{\vec{x}\vec{x}^{*}}{\left\|\vec{x}\right\|^2}\vec{x})$$

$$=\frac{\vec{x}}{\left\|\vec{x}\right\|}+\varepsilon(\frac{\vec{x}^{*}}{\left\|\vec{x}\right\|}-\frac{\vec{x}\vec{x}^{*}}{\left\|\vec{x}\right\|^3}\vec{x})=\frac{\vec{x}}{\left\|\vec{x}\right\|}+\varepsilon\frac{(\vec{x}\times\vec{x}^{*})\times\vec{x}}{\left\|\vec{x}\right\|^3}$$

$$\vec{\hat{x}}_0=\vec{x}_0+\varepsilon\vec{x}^{*}_0=\frac{\vec{x}}{\left\|\vec{x}\right\|}+\varepsilon\frac{(\vec{x}\times\vec{x}^{*})\times\vec{x}}{\left\|\vec{x}\right\|^3} \qquad (2.2.5)$$

or $\hat{x}=\left\|\hat{x}\right\|\hat{x}_0$ holds.

It is clear that $\vec{\hat{x}}_0$ is a unit dual vector with the same sense as $\vec{\hat{x}}$ and called the axis of $\vec{\hat{x}}$.

3. LIE ALGEBRA AND LIE GROUP

3.1 ALGEBRA:

Let V be an infinite or finite dimensional vector space and be defined $V \times V \to V \cdot$ If (.) operation is satisfied bilinear conditions then V is called an algebra over F.

1) x(y+z)=xy+xz for $\forall$x,y,z $\in$ V

2) λ(xy)=(λx)y=x(λy) for $\forall$x,y $\in$V, $\lambda \in$ F

If x(yz)=(xy)z for $\forall$x,y,z $\in$ V then V is associative algebra.

3.2 LIE ALGEBRA

If the following conditions are satisfied instead of associative law:

1) xx=$x^2 = 0$
2) x(yz)+y(zx)+z(xy)=0 for $\forall$x,y,z $\in$ V

then V is a lie algebra over F.

Let V be a nonassocitive algebra over F , and $\{e_1, e_2, e_3, \ldots\ldots, e_n\}$ be base of V. V is associative iff

$(e_i e_j)e_k = e_i(e_j e_k)$ i,j,k = 1,2,3,.....,n.

If $e_i e_j = \sum_r \gamma_{ijr} e_r$ is defined

$(e_i e_j)e_k = e_i(e_j e_k)$ condition is equal to $\sum_r \gamma_{ijr}\gamma_{rks} = \sum_r \gamma_{irs}\gamma_{jkr}$

V is a lie algebra iff

i) $e^2{}_i = 0 \Leftrightarrow e_i e_j = -e_j e_i$

ii) $(e_i e_j)e_k + (e_j e_k)e_i + (e_k e_i)e_j = 0$ for i,j,k=1,2,3,....,n

These conditions are equivalent to

$\gamma_{iik} = 0$, $\gamma_{ijk} = -\gamma_{jik}$,

$$\sum_r (\gamma_{ijr}\gamma_{rks} + \gamma_{jkr}\gamma_{ris} + \gamma_{kir}\gamma_{rjs}) = 0$$

3.3 LIE BRACKET:

Let V be a vector space over field F and [,] : V×V→V

1) It is 2-linear

[x,by+cz] = b[x,y]+c[x,z]

[ax+by,z] = a[x,z]+b[y,z] for ∀x,y,z ∈ V a,b,c ∈ F

2) [x,y]= -[y,x] for ∀x,y∈ V

3) [x,[y,z]]+[y,[z,x]]+[z,[x,y]] = 0 for ∀x,y,z ∈ V .

[,] transformation is called a lie operation over V.

3.4 LIE GROUP:

A lie group consists of a smooth manifold G which has a group structure.

G×G→G
(x ,y)→xy

We suppose that this group operation , which may be considered as a mapping is smooth and also that the map G→G is smooth.
x→x^{-1}

Example:3.4.1

$(R^n,+)$ is a group and R^n is a manifold . It is lie group.

3.5 RELATION BETWEEN LIE BRACKET AND LIE DERIVATIVE:

If V and W are vector fields on M , then their lie bracket [V,W] is the unique vector field satisfying [V,W] (f) = V(W(f))-W(V(f)) for all smooth functions , f: M→R .

[V,W] is indeed a vector field. In local coordinates if

$$V=\sum \xi^i(x)\frac{\partial}{\partial x^i} \quad , \quad W=\sum \eta^i(x)\frac{\partial}{\partial x^i} \quad \text{then}$$

$$[\,V,W\,]=\sum_{i=1}^{m} \{V(\eta^i)-W(\xi^i)\}\frac{\partial}{\partial x^i}=\sum_{i=1}^{m}\sum_{j=1}^{m} \{\xi^j\frac{\partial\eta^i}{\partial x^j}-\eta^j\frac{\partial\xi^i}{\partial x^j}\}\frac{\partial}{\partial x^i}$$

Example:3.5.1

If $V=y\frac{\partial}{\partial x}$ and $W=x^2\frac{\partial}{\partial x}+xy\frac{\partial}{\partial y}$, then

$$[V,W]=V(x^2)\frac{\partial}{\partial x}+V(xy)\frac{\partial}{\partial y}-W(y)\frac{\partial}{\partial x}$$

$$=y2x\frac{\partial}{\partial x}+yy\frac{\partial}{\partial y}-xy\frac{\partial}{\partial x}=xy\frac{\partial}{\partial x}+y^2\frac{\partial}{\partial y}$$

4 MATRIX GROUPS

4.1 GENERAL LINEAR GROUPS

Definition:

$M_n(k)$ is the set of all n×n matrices with elements from k for k∈{R,⊄,H}.The group of units in the algebra $M_n(R)$ is denoted by GL(n,R), in $M_n(⊄)$ by GL(n,⊄) and in $M_n(H)$ by GL(n,H). These are the general linear groups.

GL(n,R)={A∈M_n(R) | detA≠0}

GL(n,⊄)={A∈M_n(⊄) | detA≠0}

Definition:

Let k∈{R,⊄,H}

O(n,k)={A∈M_n(R) | <xA,yA>=<x,y> for ∀x,y ∈k^n }.

For k=R we write O(n,k) as O(n) and call it the orthogonal group.

For k=$\mathbb{C}$ we write it as U(n) and call it the unitary group.

If $A \in O(n)$ then $\det A \in \{-1,1\}$. We define $SO(n)=\{A \in O(n) \mid \det A=1\}$ and call this the special orthogonal group (the rotation group).

$A \in M_n(R)$ is said to be skew-symmetric if $A+A^t=0$.

4.2 EXPONENTIAL:

Definition:

Let A be a real n×n matrix and set

$$e^A = I + A + A^2/2! + \ldots + A^n/n!$$

where A^2 means the matrix product AA. This sequence converges if each of the n^2 real number sequences

$(I)_{ij} + (A)_{ij} + (A^2/2!)_{ij} + \ldots + (A^n/n!)_{ij}$ converges.

Scalar and vector product of two vectors $\vec{\hat{x}}$ and $\vec{\hat{y}}$ in $\hat{E}$ may be written in the form respectively.

$$\vec{\hat{x}}\,\vec{\hat{y}} = \vec{x}\,\vec{y} + \varepsilon(\vec{x}^*\,\vec{y} + \vec{y}^*\,\vec{x}) = \vec{x}\,\vec{y} + \varepsilon[\hat{x} \mid \hat{y}]$$

$$\vec{\hat{x}} \times \vec{\hat{y}} = \vec{x} \times \vec{y} + \varepsilon(\vec{x}^* \times \vec{y} + \vec{x} \times \vec{y}^*) = \vec{x} \times \vec{y} + \varepsilon(\vec{x}^* \times \vec{y} - \vec{y}^* \times \vec{x}) = [\hat{x}, \hat{y}]$$

These products provide Δ- bilinear operations in $\hat{E}$. Endowed with its Δ- module structure and this vector product, $\hat{E}$ is a Lie algebra over Δ.

5. DUAL STRUCTURES

5.1 DUAL UNIT VECTORS AND ORIENTED LINES

Let - with regard to the frame of reference introduced at the beginning of the preceding section $\vec{x}$ and $\vec{x}^*$ be two free vectors in E satisfying $\vec{x}\,\vec{x}=1$, $\vec{x}\,\vec{x}^*=0$. It is well-known that $\vec{x}$ and $\vec{x}^*$ may be interpreted as the Plücker-vectors of an unambiguously determined line L having $\vec{x}$ as its direction vector and passing through the point $\vec{p} = \vec{x} \times \vec{x}^*$. Providing L with a positive sense according with the sense indicating by $\vec{x}$, this line becomes an oriented line or spear. We call L the carrier of the spear and $\vec{x}$ the spear vector. The vector $\vec{x}^*$ is usually called the moment of the spear (with respect to 0) . It follows from the above that there exists a one-to-one correspondence between the set of all oriented lines in three dimensional space and the set of all dual unit vectors:

$$L \leftrightarrow \vec{x} + \varepsilon \vec{x}^{*} = \vec{\hat{x}} \qquad (5.1.1)$$

5.2 DUAL MATRICES

A matrix, the elements of which are dual numbers , is called dual matrix we restrict our selves to 3×3 matrices. The matrix having $\hat{\alpha}_{ik} = \alpha_{ik} + \varepsilon \alpha^{*}_{ik}$ in its i-th row and k-th column will be denoted by

$$\hat{A} = (\hat{\alpha}_{ik}) = (\alpha_{ik} + \varepsilon \alpha^{*}_{ik}) \qquad (5.2.1)$$

We adapt for dual matrices the same composition rules as for real matrices and we use more over the same nomenclature.As a consequence (5.2.1) may be written :

$$\hat{A} = (\alpha_{ik}) + \varepsilon(\alpha^{*}_{ik}) = A + \varepsilon A^{*}$$

An orthogonal dual matrix is a matrix for which $\hat{A}\hat{A}^{t} = \hat{A}^{t}\hat{A} = I$ where I stands for the unit matrix. The row vectors (column vectors) of an orthogonal dual matrix are mutually dual unit vectors , that is , the scalar product of two different row (column) vectors is zero, other wise 1.

5.3 THE DUAL UNIT SPHERE

For a real vector $\vec{x} = (x_1, x_2, x_3) = x_1 i_1 + x_2 i_2 + x_3 i_3$ we may consider $\vec{x}$ as the position vector with respect to $Ox_1x_2x_3$ of a real point with coordinates (x_1, x_2, x_3). The set of all points $\vec{x}$ with $\vec{x}\vec{x} = 1$ is the real unit sphere. If the dual vector $\vec{\hat{x}} = (\hat{x}_1, \hat{x}_2, \hat{x}_3)$ is not real, we call $\hat{x}_1, \hat{x}_2, \hat{x}_3$ the coordinates of a dual point. The set of all dual points $\vec{\hat{x}}\vec{\hat{x}} = 1$ that is

$$\hat{x}_1^{2} + \hat{x}_2^{2} + \hat{x}_3^{2} = 1$$

$$\hat{x}_1 \hat{x}_1^{*} + \hat{x}_2 \hat{x}_2^{*} + \hat{x}_3 \hat{x}_3^{*} = 0$$

is called the dual unit sphere with 0 as its center. The real unit sphere is a subset of dual unit sphere. The mapping (5.1.1) induces an one-to-one correspondence between the points of dual unit sphere and the oriented lines of three dimensional space.

5.4 DUAL FUNCTIONS OF A REAL PARAMETER

If x and x^* are real functions of a real parameter t , we call $\hat{x}$=x+εx^* is a (dual) function of t. We define

$$\lim_{t\to t_o} \hat{x}(t)= \lim_{t\to t_o} x(t)+ \lim_{t\to t_o} x^*(t),$$

provided that the right-handed member has a well-defined meaning. The function $\hat{x}$(t) is said to be continuous at t_0 if $\lim_{t\to t_o} \hat{x}(t)=\hat{x}(t_o)$

It is called differentiable at t_0 if $\lim_{h\to 0} \frac{\hat{x}(t_0+h)-\hat{x}(t_0)}{h}$.

does exist. In this case the value of this limit is denoted by $\hat{x}'(t_0)$ or sometimes by $\frac{d\hat{x}}{dt}(t_0)$. Obviously $\hat{x}'=x'+\varepsilon x^{*\prime}$.

A dual unit vector $\vec{\hat{x}}$ which is a function of t can be considered as a point $\hat{c}$ on the dual unit sphere the position of which depends on t. The set of these points is called a curve on the dual unit sphere or a dual curve. Under the mapping (5.1.1) a dual curve corresponds to a set of oriented lines. These lines therefore a set of rulers of a ruled surface. The some ruled surface corresponds to the antipodal curve -$\hat{x}$(t) .

6. THE TRANSITION MATRIX

We assume that we are given two triples of points on the dual unit sphere by means of two orthonormal trihedra $\{\hat{i}_1,\hat{i}_2,\hat{i}_3\}$ and $\{\hat{e}_1,\hat{e}_2,\hat{e}_3\}$.

Then any point on the dual unit sphere can be written unambiguously as a linear combination of $\hat{i}_1,\hat{i}_2,\hat{i}_3$ as well as of $\hat{e}_1,\hat{e}_2,\hat{e}_3$. We have therefore for a point $\vec{\hat{x}}$ on the dual unit sphere:

$$\hat{X}_1\hat{i}_1+\hat{X}_2\hat{i}_2+\hat{X}_3\hat{i}_3=\hat{x}_1\hat{e}_1+\hat{x}_2\hat{e}_2+\hat{x}_3\hat{e}_3 \qquad (6.1)$$

The column vectors

$$\vec{\hat{X}} = \begin{bmatrix} \hat{X}_1 \\ \hat{X}_2 \\ \hat{X}_3 \end{bmatrix} \quad \text{and} \quad \vec{\hat{x}} = \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{bmatrix}$$

are the position vectors of $\vec{\hat{X}}$ with respect to $\{\hat{i}_1, \hat{i}_2, \hat{i}_3\}$ and $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$ respectively. We derive from (6.1):

$$\vec{\hat{X}}_i = (\hat{i}_i \hat{e}_1)\hat{x}_1 + (\hat{i}_i \hat{e}_2)\hat{x}_2 + (\hat{i}_i \hat{e}_3)\hat{x}_3 \qquad (\text{i}=1,2,3) \qquad (6.2)$$

putting $(\hat{e}_i \hat{e}_k) = \hat{\alpha}_{ik}$ and introducing the dual matrix $\hat{A} = (\hat{\alpha}_{ik})$, we see that (6.2) expresses that

$$\vec{\hat{X}} = \hat{A}\vec{\hat{x}} \qquad (6.3)$$

Since $\hat{e}_k$ is a linear combination of $\hat{i}_1, \hat{i}_2, \hat{i}_3$ we put $\hat{e}_k = \sum_{j=1}^{3} \beta_{jk} \hat{i}_j$, then $\hat{\alpha}_{ik} = \hat{i}_i \hat{e}_k = \beta_{ik}$..

Therefore $\hat{e}_k = \sum_{j=1}^{3} \hat{\alpha}_{jk} \hat{i}_j$ (k= 1,2,3).

This shows that $\hat{A}$ is an orthogonal dual matrix. We suppose from now on that $\{\hat{i}_1, \hat{i}_2, \hat{i}_3\}$ and $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$ are right-handed trihedra , i.e , $\hat{i}_1 \times \hat{i}_2 = \hat{i}_3$ and $\hat{e}_1 \times \hat{e}_2 = \hat{e}_3$.

We call $\hat{A}$ the transition matrix from the trihedron $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$ onto $\{\hat{i}_1, \hat{i}_2, \hat{i}_3\}$.

7. DUAL SPECIAL MOTION

Using the notations of the preceding section, we suppose that the trihedron $\{\hat{i}_1, \hat{i}_2, \hat{i}_3\}$ is fixed, whereas the vectors $\hat{e}_1, \hat{e}_2, \hat{e}_3$ of the trihedron $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$ are functions of a real parameter t (the time).Then we say that $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$ moves with respect to $\{\hat{i}_1, \hat{i}_2, \hat{i}_3\}$.We may interpret this as follows:

The dual unit sphere K_2 rigidly connected with $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$moves over the dual unit sphere K_1 rigidly connected with $\{\hat{i}_1, \hat{i}_2, \hat{i}_3\}$. We call for shortness sake K_1 the fixed and K_2 the moving sphere.

The motion is called a dual spherical motion and will be denoted by $K_2|K_1$. If $\vec{\hat{x}}$ is a point on K_2 coinciding at the instant t with the point $\vec{\hat{X}}$ on K_1, we have $\vec{\hat{X}} = \hat{A}\vec{\hat{x}}$ where $\hat{A} = (\hat{\alpha}_{ik}) = A + \varepsilon A^*$ is the transition matrix at the said instant from $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$ onto $\{\hat{i}_1, \hat{i}_2, \hat{i}_3\}$. This matrix is the function of t. Since $\hat{A}$ is are an orthogonal dual matrix we have $\hat{A}\hat{A}^T = I$ and therefore:

$$\frac{d\hat{A}}{dt}\hat{A}^t + \hat{A}\frac{d\hat{A}^t}{dt} = 0 \qquad (7.1)$$

where 0 is the 3x3 - zero matrix. This shows that $\frac{d\hat{A}}{dt}\hat{A}^t$ is a skew-symmetric dual matrix. We put :

$$\frac{dA}{dt}A^t = \begin{bmatrix} 0 & -\hat{w}_3 & \hat{w}_2 \\ \hat{w}_3 & 0 & -\hat{w}_1 \\ -\hat{w}_2 & \hat{w}_1 & 0 \end{bmatrix} = \Omega = W + \varepsilon W^* \qquad (7.2)$$

Here $W = \frac{dA}{dt}A^t$ and $W^* = \frac{dA^*}{dt}A^t + A\frac{dA^t}{dt}$. It is clear that the matrices W and W^* are the skew-symmetric.

The dual velocity of the point $\vec{\hat{x}}$ on K_2 is defined as

$$\vec{\hat{v}} = \frac{d\vec{\hat{X}}}{dt} = \frac{d\hat{A}}{dt}\vec{\hat{x}}.$$

Therefore

$$\vec{\hat{v}} = \frac{d\hat{A}}{dt}\vec{\hat{x}} = \frac{d\hat{A}}{dt}\hat{A}^t\hat{A}\vec{\hat{x}} = \frac{d\hat{A}}{dt}\hat{A}^t\vec{\hat{X}} = \Omega\vec{\hat{X}} \qquad (7.3)$$

Introducing the dual vector $\vec{\hat{w}}$ is given by

$\vec{\hat{w}}^t = (\hat{w}_1, \hat{w}_2, \hat{w}_3)$

we may write

$$\vec{\hat{v}} = \frac{d\vec{\hat{X}}}{dt} = \vec{\hat{w}} \times \vec{\hat{X}} \qquad (7.4)$$

The vector $\vec{\hat{w}}$ is called the dual angular velocity or instantaneous screw axis of the spherical (or helicoidal) motion $K_2|K_1$.

The dual unit vector $\vec{\hat{w}}_0 = \dfrac{\vec{\hat{w}}}{\left\|\vec{\hat{w}}\right\|}$ with the same sense as $\vec{\hat{w}}$ corresponds to a point $\vec{\hat{p}}$ of K_1. The point on K_2 coinciding with $\vec{\hat{p}}$ at the instant t has the dual velocity zero; this point also denoted by $\vec{\hat{p}}$. There exists apart from $\vec{\hat{p}}$ one other point of K_2 with dual velocity zero, namely the antipodal point of $\vec{\hat{p}}$ we shall call $\vec{\hat{p}}$ the pole point of the motion $K_2|K_1$. From now on we write $\vec{\hat{p}} = \dfrac{\vec{\hat{w}}}{\left\|\vec{\hat{w}}\right\|}$ and $\left\|\vec{\hat{w}}\right\| = w + \varepsilon w^* = \hat{w}$

8. RELATIVE MOTION

Let K_1, K_2 and K_3 be dual unit spheres rigidly connected with the bases $\{\hat{i}_1, \hat{i}_2, \hat{i}_3\}$ and $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$ respectively. If $\vec{\hat{x}}$ is a point on K_3 coinciding at the instant t with the points $\vec{\hat{X}}$ and $\vec{\hat{\xi}}$ on the dual unit sphere K_1 and K_2 respectively we may write

$$\vec{\hat{X}} = \hat{X}_1 \hat{i}_1 + \hat{X}_2 \hat{i}_2 + \hat{X}_3 \hat{i}_3 = \hat{\xi}_1 \hat{e}_1 + \hat{\xi}_2 \hat{e}_2 + \hat{\xi}_1 \hat{e}_3 = \hat{x}_1 \hat{r}_1 + \hat{x}_2 \hat{r}_2 + \hat{x}_1 \hat{r}_3$$

or

$$\vec{\hat{X}} = \vec{\hat{X}}^t \hat{i} = \vec{\hat{\xi}}^t \hat{e} = \vec{\hat{x}}^t \hat{r}$$

where

$$\vec{\hat{X}} = \begin{bmatrix} \hat{X}_1 \\ \hat{X}_2 \\ \hat{X}_3 \end{bmatrix}, \quad \vec{\hat{\xi}} = \begin{bmatrix} \hat{\xi}_1 \\ \hat{\xi}_2 \\ \hat{\xi}_3 \end{bmatrix}, \quad \vec{\hat{x}} = \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{bmatrix}$$

and

$$\hat{i} = \begin{bmatrix} \hat{i}_1 \\ \hat{i}_2 \\ \hat{i}_3 \end{bmatrix} \quad , \quad \hat{e} = \begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \\ \hat{e}_3 \end{bmatrix} \quad , \quad \hat{r} = \begin{bmatrix} \hat{r}_1 \\ \hat{r}_2 \\ \hat{r}_3 \end{bmatrix}$$

Since we may write

$$\hat{e}_i = \sum_{j=1}^{3} \alpha_{ij} \hat{i}_j$$

$$\hat{r}_i = \sum_{j=1}^{3} \beta_{ij} \hat{i}_j$$

and

$$\hat{r}_i = \sum_{j=1}^{3} \gamma_{ij} \hat{e}_j$$

We have dual orthogonal matrices respectively $\hat{A}$, $\hat{B}$ and $\hat{C}$.

Then we have

$\hat{e} = \hat{A}\hat{i}$, $\hat{r} = \hat{B}\hat{i}$ and $\hat{r} = \hat{C}\hat{e}$

We suppose that K_1 is fixed , i.e. the trihedron $\{\hat{i}_1, \hat{i}_2, \hat{i}_3\}$ is fixed. We may represent the motions $K_2|K_1$, $K_3|K_1$ and $K_3|K_2$ by

$$\hat{X} = A^T \hat{\xi} \;,\; \hat{X} = B^T \hat{x} \text{ and } \hat{\xi} = \hat{C}\hat{x} \qquad (8.1)$$

The rate of changes $\hat{r}$ with respect to the sphere K_1 and K_2 respectively, are

$$\frac{d\hat{r}}{dt} = \Omega_B \hat{r} \;,\; \frac{\delta\hat{r}}{\delta t} = \Omega_C \hat{r} \qquad (8.2)$$

where $\Omega_B = \dfrac{d\hat{B}}{dt}\hat{B}^t$ and $\Omega_C = \dfrac{d\hat{C}}{dt}\hat{C}^t$.

The vectorial expression of the point $\vec{\hat{x}}$ on K_3 can be given by

$$\vec{\hat{x}} = \vec{\hat{x}}^t \hat{r}$$

The dual velocities of $\vec{\hat{x}}$ with respect to K_1 and K_2 are

$$\frac{d\vec{\hat{x}}}{dt}=\frac{d\vec{\hat{x}}^t}{dt}\hat{r}+\vec{\hat{x}}^t\frac{d\hat{r}}{dt}=\left(\frac{d\vec{\hat{x}}^t}{dt}+\vec{\hat{x}}^t\Omega_B\right)\hat{r} \qquad (8.3)$$

$$\frac{\delta\vec{\hat{x}}}{dt}=\frac{d\vec{\hat{x}}^t}{dt}\hat{r}+\vec{\hat{x}}^t\frac{\delta\hat{r}}{dt}=\left(\frac{d\vec{\hat{x}}^t}{dt}+\vec{\hat{x}}^t\Omega_C\right)\hat{r} \qquad (8.4)$$

If $\vec{\hat{x}}$ is fixed on K_2, then we have $\frac{d\vec{\hat{x}}}{dt}=0$. This implies that $\frac{d\vec{\hat{x}}}{dt}=\Omega_B\vec{\hat{x}}$

Substituting this value of $\frac{d\vec{\hat{x}}}{dt}$ into (8.4) we obtain the dual velocity of $\vec{\hat{x}}$ fixed on K_2 with respect to K_1.

$$\frac{\delta\vec{\hat{x}}}{dt}=\vec{\hat{x}}^t(\Omega_C-\Omega_B)\hat{r}$$

We denote this dual velocity vector by $d_f\vec{\hat{x}}$. Introducing the vector $\vec{\hat{\psi}}=(\hat{\psi}_1,\hat{\psi}_2,\hat{\psi}_3)$, $\vec{\hat{\psi}}_i=\vec{\hat{W}}_{Ci}-\vec{\hat{W}}_{Bi}$, (i=1,2,3).

We may write

$$\frac{d_f\vec{\hat{x}}}{dt}=\vec{\hat{\psi}}\times\vec{\hat{x}} \qquad (8.5)$$

If $\frac{d_f\vec{\hat{x}}}{dt}=0$ then $\vec{\hat{x}}$ is a pole point of K_1.

Now by using the theory of SO(n) groups . We define the matrix $\hat{B} = e^{-\Omega}$. Hence we have $\hat{B}^t = e^{\Omega}$. Then we may write the motion $K_3|K_1$ as

$$\vec{\hat{X}} = e^{\Omega} \vec{\hat{x}} \qquad (8.6)$$

Here $\Omega = -(\frac{d\hat{A}}{dt}\hat{A}^t)$.

On the other hand since can be written

$$\hat{r} = \hat{C}\hat{A}\hat{i} = \hat{B}\hat{i}$$

and the trihedron $\hat{i}$ is fixed we have

$$\hat{C}\hat{A} = \hat{B}$$

This equality gives that $\hat{C} = \hat{B}\hat{A}^T$.

Hence we compute all the transition matrix in terms of $\hat{A}$ and its derivatives.

9. THE FORMULAS OF THE MOTION OF A POINT ON THE DUAL SPHERE

The exponential of Ω can be written with the aid of velocity of x.

$$\Omega \vec{\hat{x}} = \vec{\hat{w}} \times \vec{\hat{x}} = \left\|\vec{\hat{w}}\right\| \vec{\hat{p}} \vec{\hat{x}} \quad , \qquad \vec{\hat{w}} = \left\|\vec{\hat{w}}\right\| \vec{\hat{p}} \quad , \qquad \left\|\vec{\hat{w}}\right\| = \hat{w}$$

$$\Omega^2 \vec{\hat{x}} = \hat{w}^2 (\vec{\hat{p}}(\vec{\hat{x}}\,\vec{\hat{p}}) - \vec{\hat{x}})$$

$$\Omega^3 \vec{\hat{x}} = -\hat{w}^2\, \Omega \vec{\hat{x}}$$

•

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•

$$\Omega^{2n+1} \vec{\hat{x}} = (-1)^n \hat{w}^{2n}\, \Omega \vec{\hat{x}} = (-1)^n \hat{w}^{2n+1}\, \vec{\hat{p}} \times \vec{\hat{x}}$$

$$\Omega^{2n+2} \vec{\hat{x}} = (-1)^{n+1} \hat{w}^{2n+2}\, \vec{\hat{x}} + (-1)^n \hat{w}^{2n+2}\, (\vec{\hat{p}}(\vec{\hat{x}}\,\vec{\hat{p}}))$$

thus

$$e^{\Omega} \vec{\hat{x}} = (I + \Omega + \frac{\Omega^2}{2!} + \ldots\ldots + \frac{\Omega^n}{n!} + \ldots\ldots) \vec{\hat{x}}$$

$$=(1-\frac{\hat{w}^2}{2!}+\frac{\hat{w}^4}{4!}-\frac{\hat{w}^6}{6!}+...)\vec{\hat{x}}+(\hat{w}-\frac{\hat{w}^3}{3!}+\frac{\hat{w}^5}{5!}-...)\vec{\hat{p}}\times\vec{\hat{x}}+(\frac{\hat{w}^2}{2!}-\frac{\hat{w}^4}{4!}+\frac{\hat{w}^6}{6!}-...)(\vec{\hat{p}}\vec{\hat{x}})\vec{\hat{p}}$$

We obtain

$$e^{\Omega}\vec{\hat{x}}=Cos\hat{w}\,\vec{\hat{x}}+Sin\hat{w}\,\vec{\hat{p}}\times\vec{\hat{x}}+(1-Cos\hat{w})(\vec{\hat{p}}\vec{\hat{x}})\vec{\hat{p}} \qquad (9.1)$$

It is analogy of Olinde Rodrigues Formula.

If $\vec{\hat{p}}=\vec{\hat{x}}$, i.e , $\vec{\hat{x}}$ is pole point then $e^{\Omega}\vec{\hat{p}}=Cos\hat{w}\,\vec{\hat{p}}+(1-Cos\hat{w})\vec{\hat{p}}=\vec{\hat{p}}$

That is , operator e^{Ω} remains $\vec{\hat{x}}$ fixed.
We know that

$$(\vec{\hat{p}}\times\vec{\hat{x}})\times\vec{\hat{p}}=(\vec{\hat{p}}\vec{\hat{p}})\vec{\hat{x}}-(\vec{\hat{p}}\vec{\hat{x}})\vec{\hat{p}}=\vec{\hat{x}}-(\vec{\hat{p}}\vec{\hat{x}})\vec{\hat{p}}$$

We have

$(\vec{\hat{p}}\vec{\hat{x}})\vec{\hat{p}}=\vec{\hat{x}}-(\vec{\hat{p}}\times\vec{\hat{x}})\times\vec{\hat{p}}$ then

$$e^{\Omega}\vec{\hat{x}}=Cos\hat{w}\,\vec{\hat{x}}+Sin\hat{w}\,\vec{\hat{p}}\times\vec{\hat{x}}+(1-Cos\hat{w})(\vec{\hat{x}}-(\vec{\hat{p}}\times\vec{\hat{x}})\times\vec{\hat{p}}) \qquad (9.2)$$

This is Olinde Rodrigues Formula for a rotation around the axis $\vec{\hat{w}}$ of angle $\hat{w}$.

This represents the rotation of $\vec{\hat{x}}$ around the axis $\vec{\hat{w}}$ of angle $\hat{w}$ and translation along $\vec{\hat{w}}$ w^* times.

The last equation may be expressed in the form.

We have

$\hat{X}(t+dt)=e^{\Omega(t+dt)}\hat{x}=e^{\Omega(t)+\Omega' dt}\hat{x}$

Here we omitted the terms of degree two and higher of dt.

$$e^{\Omega}\vec{\hat{x}}=\vec{\hat{x}}+\vec{\hat{p}}\times\vec{\hat{x}}\,Sin\hat{w}+((\vec{\hat{x}}\vec{\hat{p}})\vec{\hat{p}}-\vec{\hat{x}})(1-Cos\hat{w}) \qquad (9.1.3)$$

$$e^{\Omega}\vec{\hat{x}}=\vec{\hat{x}}+\vec{\hat{p}}\times\vec{\hat{x}}\,Sin\hat{w}+((\vec{\hat{p}}\times(\vec{\hat{p}}\times\vec{\hat{x}}))(1-Cos\hat{w}) \qquad (9.1.4)$$

$$e^{\Omega}\vec{\hat{x}}=\vec{\hat{x}}+[\vec{\hat{p}},\vec{\hat{x}}]Sin\hat{w}+[\vec{\hat{p}},[\vec{\hat{p}},\vec{\hat{x}}]](1-Cos\hat{w}) \qquad (9.1.5)$$

We can use the Campbell-Hausdorf formula of Lie's general theory.Denoting X and Y the differential operators of a certain algebra of Lie:

$$e^X e^Y=(I+X+\frac{X^2}{2!}+....+\frac{X^n}{n!})(I+Y+\frac{Y^2}{2!}+....+\frac{Y^n}{n!})$$

$$= I + X + Y + \frac{(X+Y)^2}{2!} + \frac{XY - YX}{2} + ...$$

$$\cong e^{(X+Y)} + \frac{[X,Y]}{2}$$

Neglecting the terms of degree three or higher, the product is the sum of $e^{(X+Y)}$ and half the Lie bracket of the operators:

$$e^X e^Y \cong e^{(X+Y)} + \frac{[X,Y]}{2}$$

Using this formula we have

$$e^{\Omega} e^{\Omega' dt} = e^{\Omega + \Omega' dt} + [\Omega, \Omega']/2dt \qquad (9.6)$$

$$e^{\Omega+\Omega' dt} = e^{\Omega} e^{\Omega' dt} - \frac{[\Omega,\Omega']}{2} dt = e^{\Omega} e^{\Omega' dt} + \frac{[\Omega',\Omega]}{2} dt$$

$$= e^{\Omega} e^{\Omega' dt} + \frac{\Omega'\Omega - \Omega\Omega'}{2}$$

$$e^{\Omega} e^{\Omega' dt} = e^{\Omega}(I + \Omega' dt) \qquad (9.7)$$

Applying this operator to $\vec{\hat{x}}$ then we have ,

$$e^{\Omega} e^{\Omega' dt} \vec{\hat{x}} = e^{\Omega} \vec{\hat{x}} + e^{\Omega}(\Omega' \vec{\hat{x}}) dt$$

since $\Omega' \vec{\hat{x}} = \vec{\hat{w}}' \times \vec{\hat{x}}$

We get

$$e^{\Omega+\Omega' dt} \vec{\hat{x}} = e^{\Omega} \vec{\hat{x}} + e^{\Omega}(\vec{\hat{w}}' \times \vec{\hat{x}}) dt + \frac{[\Omega',\Omega]}{2} dt$$

$$= e^{\Omega} \vec{\hat{x}} + (I + \Omega + \frac{\Omega^2}{2!} + ... + \frac{\Omega^n}{n!}) \vec{\hat{w}}' \times x dt + \frac{[\Omega',\Omega]}{2} \mathrm{dt}$$

$$= e^{\Omega} \vec{\hat{x}} + (\vec{\hat{w}}' \times \vec{\hat{x}} + \Omega \vec{\hat{w}}' \times \vec{\hat{x}} + \frac{\Omega^2}{2!} \vec{\hat{w}}' \times \vec{\hat{x}} + ...) dt + \frac{[\Omega',\Omega]}{2} dt \qquad (9.8)$$

Since $e^{\Omega+\Omega' dt}$

$$\Omega \vec{\hat{w}}' \times \vec{\hat{x}} = \vec{\hat{w}} \times (\vec{\hat{w}}' \times \vec{\hat{x}}) = -(\vec{\hat{w}}' \times \vec{\hat{x}}) \times \vec{\hat{w}} = -(\vec{\hat{w}}' \vec{\hat{w}})\vec{\hat{x}} + (\vec{\hat{w}} \vec{\hat{x}})\vec{\hat{w}}'$$

$$\Omega^2 \vec{\hat{w}}' \times \vec{\hat{x}} = (\vec{\hat{w}} \vec{\hat{x}})\vec{\hat{w}} \times \vec{\hat{w}}' - (\vec{\hat{w}}' \vec{\hat{w}})\vec{\hat{w}} \times \vec{\hat{x}}$$

$$\Omega^3 \vec{\hat{w}}' \times \vec{\hat{x}} = (\vec{\hat{w}}' \vec{\hat{w}})w^2 \vec{\hat{x}} - (\vec{\hat{w}} \vec{\hat{x}})w^2 \vec{\hat{w}}'$$

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$$\Omega^{2m+1} \vec{\hat{w}}' \times \vec{\hat{x}} = (-1)^{m+1}(\vec{\hat{w}}' \vec{\hat{w}})w^{2m} \vec{\hat{x}} + (-1)^m (\vec{\hat{w}} \vec{\hat{x}})w^{2m} \vec{\hat{w}}' \qquad m=0,1....$$

$$\Omega^{2n} \vec{\hat{w}}' \times \vec{\hat{x}} = (-1)^{n+1} \vec{\hat{w}}^{2n-2} (\vec{\hat{w}} \vec{\hat{x}})\vec{\hat{w}} \times \vec{\hat{w}}' + (-1)^n \vec{\hat{w}}^{2n-2} (\vec{\hat{w}}' \vec{\hat{w}})\vec{\hat{w}} \times \vec{\hat{x}} \qquad n=1,2,....$$

$$e^{\Omega+\Omega' dt} = e^{\Omega} e^{\Omega' dt} + \frac{\Omega'\Omega - \Omega\Omega'}{2}$$

$$= e^{\Omega} \vec{\hat{x}} + \vec{\hat{w}}' \times \vec{\hat{x}} - (\vec{\hat{w}}' \vec{\hat{w}})\vec{\hat{x}} + (\vec{\hat{w}} \vec{\hat{x}})\vec{\hat{w}}' + \frac{(\vec{\hat{w}} \vec{\hat{x}})\vec{\hat{w}} \times \vec{\hat{w}}'}{2!} - \frac{(\vec{\hat{w}}' \vec{\hat{w}})\vec{\hat{w}} \times \vec{\hat{x}}}{2!}$$

$$+ \frac{(\vec{\hat{w}}' \vec{\hat{w}})w^2 \vec{\hat{x}}}{3!} - \frac{(\vec{\hat{w}} \vec{\hat{x}})w^2 \vec{\hat{w}}'}{3!} + ... + (\frac{\Omega'\Omega - \Omega\Omega'}{2})\vec{\hat{x}}$$

$$= e^{\Omega} \vec{\hat{x}} + \vec{\hat{w}}' \times \vec{\hat{x}} + (\vec{\hat{w}}' \vec{\hat{w}})(-1 + \frac{\vec{\hat{w}}^2}{3!} - \frac{\vec{\hat{w}}^4}{5!} + ...)\vec{\hat{x}} + (\vec{\hat{w}} \vec{\hat{x}})(1 - \frac{\vec{\hat{w}}^2}{3!} + \frac{\vec{\hat{w}}^4}{5!} - ....)\vec{\hat{w}}'$$

$$+(\vec{\hat{w}}' \vec{\hat{w}})(-\frac{1}{2!} + \frac{\vec{\hat{w}}^2}{4!} - ...)\vec{\hat{w}} \times \vec{\hat{x}} + (\vec{\hat{w}} \vec{\hat{x}})(\frac{1}{2!} - \frac{\vec{\hat{w}}^2}{4!} + ..)(\vec{\hat{w}} \times \vec{\hat{w}}') + ... + (\frac{\Omega'\Omega - \Omega\Omega'}{2})\vec{\hat{x}}$$

$$= e^{\Omega} \vec{\hat{x}} + \vec{\hat{w}}' \times \vec{\hat{x}} - (\vec{\hat{w}}' \vec{\hat{p}})(w - \frac{\vec{\hat{w}}^3}{3!} + \frac{\vec{\hat{w}}^5}{5!} - ....)\vec{\hat{x}} + (\vec{\hat{p}} \vec{\hat{x}})(w - \frac{\vec{\hat{w}}^3}{3!} + \frac{\vec{\hat{w}}^5}{5!} - ....)\vec{\hat{w}}'$$

$$+(\vec{\hat{p}} \vec{\hat{w}}')(-\frac{\vec{\hat{w}}^2}{2!} + \frac{\vec{\hat{w}}^4}{4!} - \frac{\vec{\hat{w}}^6}{6!} + ...)\vec{\hat{p}} \times \vec{\hat{x}} - (\vec{\hat{p}} \vec{\hat{x}})(-\frac{\vec{\hat{w}}^2}{2!} + \frac{\vec{\hat{w}}^4}{4!} - \frac{\vec{\hat{w}}^6}{6!} + ...)(\vec{\hat{p}} \times \vec{\hat{w}}') + \frac{1}{2}(\vec{\hat{w}}' \times (\vec{\hat{w}} \times \vec{\hat{x}}) - \vec{\hat{w}} \times (\vec{\hat{w}}' \times \vec{\hat{x}}))dt$$

by using triple vector product expansion we get:

$$= e^{\Omega}\vec{\hat{x}} + \vec{\hat{w}}' \times \vec{\hat{x}} - (\vec{\hat{w}}'\,\vec{\hat{p}})Sin\,\hat{w}\,\vec{\hat{x}} + (\vec{\hat{p}}\,\vec{\hat{x}})Sin\,\hat{w}\,\vec{\hat{w}}' + (\vec{\hat{p}}\,\vec{\hat{w}}')(Cos\,\hat{w}-1)(\vec{\hat{p}} \times \vec{\hat{x}})$$

$$-(\vec{\hat{p}}\,\vec{\hat{x}})(Cos\,\hat{w}-1)(\vec{\hat{p}} \times \vec{\hat{w}}') + (\frac{\hat{w}}{2}(\vec{\hat{x}}\,\vec{\hat{w}}')\vec{\hat{p}} - \frac{\hat{w}}{2}(\vec{\hat{x}}\,\vec{\hat{p}})\,\vec{\hat{w}}')dt$$

$$= e^{\Omega}\vec{\hat{x}} + ([\vec{\hat{w}}', \vec{\hat{x}}] - (\vec{\hat{w}}'\,\vec{\hat{p}})Sin\,\hat{w}\,\vec{\hat{x}} + \vec{\hat{p}}\,\vec{\hat{x}}(Sin\,\hat{w} - \frac{\hat{w}}{2}dt)\vec{\hat{w}}' +$$

$$(1-Cos\,\hat{w})(-(\vec{\hat{p}}\,\vec{\hat{w}}')\vec{\hat{p}} \times \vec{\hat{x}} + (\vec{\hat{p}}\,\vec{\hat{x}})\vec{\hat{p}} \times \vec{\hat{w}}') - \frac{1}{2}\hat{w}(\vec{\hat{x}}\,\vec{\hat{w}}')\vec{\hat{p}})dt$$

$$= e^{\Omega}\vec{\hat{x}} + ([\vec{\hat{w}}, \vec{\hat{x}}] - (\vec{\hat{p}}\,\vec{\hat{w}}')Sin\,\hat{w}\,\vec{\hat{x}} + \vec{\hat{p}}\,\vec{\hat{x}}(Sin\,\hat{w} - \frac{\hat{w}}{2}dt)\vec{\hat{w}}' +$$

$$(1-Cos\,\hat{w})(\vec{\hat{p}} \times (\vec{\hat{p}} \times (\vec{\hat{w}}' \times \vec{\hat{x}})) + \frac{1}{2}\hat{w}(\vec{\hat{x}}\,\vec{\hat{w}}')\vec{\hat{p}})\,dt \qquad (9.9)$$

This is the infinitesimal motion of $\vec{\hat{x}}$.
By separating the equation

$$e^{\Omega}\vec{\hat{x}} = \vec{\hat{x}} + \vec{\hat{p}} \times \vec{\hat{x}}\,Sin\,\hat{w} + \vec{\hat{p}} \times (\vec{\hat{p}} \times \vec{\hat{x}})(1-Cos\,\hat{w})$$

in to real and dual parts where

$$\vec{\hat{x}} = \vec{x} + \varepsilon \vec{x}^{*} \quad , \quad \hat{w} = \vec{w} + \varepsilon \vec{w}^{*} \quad , \quad \vec{\hat{p}} = \vec{p} + \varepsilon \vec{p}^{*}$$

$$\vec{p} = \frac{\vec{\hat{w}}}{\left\|\vec{\hat{w}}\right\|} = \frac{\vec{w} + \varepsilon \vec{w}^{*}}{\left\|\vec{\hat{w}}\right\|}$$

then

$$e^{\Omega}\vec{\hat{x}} = x + \varepsilon x^{*} + (p+\varepsilon p^{*}) \times (x + \varepsilon x^{*})(Sinw + \varepsilon w^{*}Cosw) + (p+\varepsilon p^{*}) \times ((p+\varepsilon p^{*}) \times (x+\varepsilon x^{*}))(1-Cosw+\varepsilon w^{*}Sinw)$$

$$= x + \varepsilon x^{*} + (p \times x + \varepsilon(p^{*} \times x + p \times x^{*}))(Sinw + \varepsilon w^{*}Cosw) +$$

$$(\vec{p} + \varepsilon \vec{p}^{*}) \times (\vec{p} \times \vec{x} + \varepsilon(\vec{p} \times \vec{x}^{*} + \vec{p}^{*} \times \vec{x}))(1-Cosw^{*} + \varepsilon w^{*}Sinw)$$

$$= x + \varepsilon x^* + (p \times x)Sinw + \varepsilon(p \times x^* + p^* \times x)Sinw + \varepsilon w^*(p \times x)Cosw +$$
$$((p \times (p \times x) + \varepsilon(p \times (p \times x^*) + \varepsilon p \times (p^* \times x) + \varepsilon p^* \times (p \times x))(1 - Cosw + \varepsilon w^* Sinw)$$
$$= x + (p \times x)Sinw + p \times (p \times x) - p \times (p \times x)Cosw^* + \varepsilon(x^* + (p^* \times x + p \times x^*)Sinw$$
$$w^* p \times xCosw + p \times (p \times x)w^* Sinw + p \times (p \times x^*) + p \times (p^* \times x) + p^* \times (p \times x)$$
$$-Cosw(p \times (p \times x^*)) - Cosw(p \times (p^* \times x)) - Cosw(p^* \times (p \times x))$$

$$= x + [p,x]Sinw + [p,[p,x]] - [p,[p,x]]Cosw + \varepsilon(x^* + ([p,x^*] + [p^*,x])Sinw + w^*[p,x]Cosw + [p,[p,x]]w^* Sinw +$$

$$[p,[p,x^*]] + [p,[p^*,x]] + [p^*,[p,x]] - Cosw[p,[p,x^*]] - Cosw[p,[p^*,x]] - Cosw[p^*,[p,x]]$$

$$= x + [p,x]Sinw + [p,[p,x]] - [p,[p,x]]Cosw + \varepsilon(x^* + ([p,x^*] + [p^*,x])Sinw + [p,[p,x^*]] + [p,[p^*,x]]$$
$$[p^*,[p,x]] - Cosw[p,[p,x^*]] - Cosw[p,[p^*,x]] - Cosw[p^*,[p,x]]$$
$$+([p,x]Cosw + [p,[p,x]]Sinw)w^*) \qquad (9.10)$$

the real part represents a rotation around instantaneous screw axis with angle $\hat{w}$ of point $\vec{\hat{x}}$ and the dual part represents as a translation along this axis.

The acceleration of $\vec{\hat{x}}$ is :($\vec{\psi}$ is dual)

$$d_f \vec{\hat{x}} = \vec{\psi} \times \vec{\hat{x}}$$

We define lie bracket to be

$$[\psi, x] = \vec{\psi} \times \vec{\hat{x}}$$

Hence we have

$$d_f x = [\psi, x]$$

$$\frac{d}{dt}[\psi, x] = \frac{d}{dt}(\psi \times \vec{\hat{x}}) = \psi' \times \vec{\hat{x}} + \psi \times \vec{\hat{x}}'$$

or

$$\frac{d}{dt}[\psi, x] = [\psi', x] + [\psi, x']$$

Now we compute high order acceleration of $\vec{\hat{x}}$ interms of Lie bracket.

$$\frac{d}{dt}d_f x = \frac{d}{dt}[\psi,x] = [\psi',x]+[\psi,x']$$

$$\frac{d^2}{dt^2}d_f x = \frac{d}{dt}[\psi',x]+\frac{d}{dt}[\psi,x']$$

$$=[\psi'',x]+2[\psi',x]+[\psi,x'']$$

$$\frac{d^3}{dt^3}d_f x = \frac{d}{dt}[\psi'',x]+2\frac{d}{dt}[\psi',x]+\frac{d}{dt}[\psi,x'']$$

$$=[\psi''',x]+[\psi'',x']+2[\psi'',x']+2[\psi',x'']+[\psi',x'']+[\psi,x''']$$

$$=[\psi''',x]+3[\psi'',x']+3[\psi',x'']+[\psi,x''']$$

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$$\frac{d^n}{dt^n}d_f x = \frac{d^n}{dt}[\psi,x] = \sum_{j=0}^{n}\binom{n}{j}[\psi^{n-j},x^j] \qquad (9.11)$$

The vectorial product of the velocities of the points $\vec{\hat{x}}$ and $\vec{\hat{y}}$ on K_3 in the direction of instantaneous screw axis $\vec{\hat{\psi}}$,that is,

$$[d_f x, d_f y] = d_f x \times d_f y = (d_f \vec{\hat{x}}\ \vec{\hat{y}})\vec{\hat{\psi}}$$

or

$$d_f x \times d_f y = [d_f x, d_f y] = [[\psi,x][\psi,y]]$$

Hence we get

$$\frac{d}{dt}d_f x \times d_f y = \{[[\psi,x]'[\psi,y]]+[[\psi,x][\psi,y]']\}$$

$$\frac{d_f^{\,2}}{dt^2}d_f x \times d_f y = \{[[\psi,x]''[\psi,y]]+2[[\psi,x]'[\psi,y]']+[[\psi,x][\psi,y]'']\}$$

$$\frac{d_f^{\,3}}{dt^3}d_f x \times d_f y = \{[[\psi,x]'''[\psi,y]]+3[[\psi,x]''[\psi,y]']+3[[\psi,x]'[\psi,y]'']+$$

$$[[\psi,x][\psi,y]''']\}$$

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$$\frac{d_f{}^n}{dt^n} d_f x \times d_f y = \sum_{j=0}^{n} \binom{n}{j} [\psi, x]^{n-j} [\psi, y]^{j}] \quad (9.12)$$

where

$$[\psi, x]^{n-j} = \sum_{k=0}^{n} \binom{n-j}{k} [\psi^{n-j-k}, x^k] \quad (9.13)$$

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