ON THE
TOTAL TORSION

A Thesis Submitted to the
Graduate School of Natural and Applied Sciences of
Dokuz Eylül University
In Partial Fulfillment of the Requirements for
the Degree of Master of Sciences in Mathematics

by
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June, 1997
İZMİR

M.Sc THESIS EXAMINATION RESULT FORM

We certify that we have read this thesis and that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ACKNOWLEDGMENTS

I would like to thank my advisor Prof. Dr. Ömer KÖSE for his generous guidance and enthusiastic encouragment and Prof. Dr. Refail ALİZADE for his great assistance.

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ABSTRACT

It is well known that the total torsion of a closed spherical curve is zero. In this thesis the paper of [3] is discussed and by using [6] a new proof of the above fact is given.In addition it is shown that the total torsion of regularly homotopic spherical curves are equal.

ÖZET

Kapalı küresel bir eğrinin toplam burulmasının sıfır olduğu bilinmektedir. Bu çalışmada [3]'deki makale tartışılmış ve [6]'daki makale kullanılarak toplam burulmaya ait verilen özellik yeniden kanıtlanmıştır. Bunlara ek olarak, regüler homotopik küresel eğrilerin toplam burulmalarının eşit olduğu gösterilmiştir.

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CHAPTER ONE

INTRODUCTION

It is a good approach to think the torsion of a curve at a point as the directed angle between the binormals at that point and its sufficiently close (with the increasing parameter manner) neighbour point. This result comes from the directed angle between the oscullating planes, pp 59-61 of [1].

The torsion on a line segment of a polygonal curve is constant. So defining the total torsion of a curve with its polygonal approximation curve is easier than studying on a regular curve. It is proved at p. 33 of [1] and [3] that an approximation can be done with the definition of arc-length and geometrical approach (defining a polygonal curve with gluing the cutting points on the curve of a sphere which is used for covering the curve).Also it is proved by [3] that an approximation can be done and its total torsion is approximately the same of the original curve.

As a result of polygonal approximation to a curve, the total torsion of a regular closed curve on S^2-sphere is zero by [3].By [6] spherical curves satisfy some conditions, with these conditions it is proved that the total torsion of a regular closed spherical curve is zero.

Using the above conclusions and the definition of regular homotopy , p.505 of [5],it is proved that the total torsions of regularly homotopic two curves are the same.

(1.1) THE TORSION OF A CURVE:

Let P and Q denote neighbour points on a curve of class C^3, such that; P=α(s) and Q=α(s+h) where h>0. And also note that P and Q are not rectification points, that is $\chi \neq 0$.

It is clear that the binormals at P and Q, b(s) and b(s+h), are normals to the osculating planes at P and Q respectively.

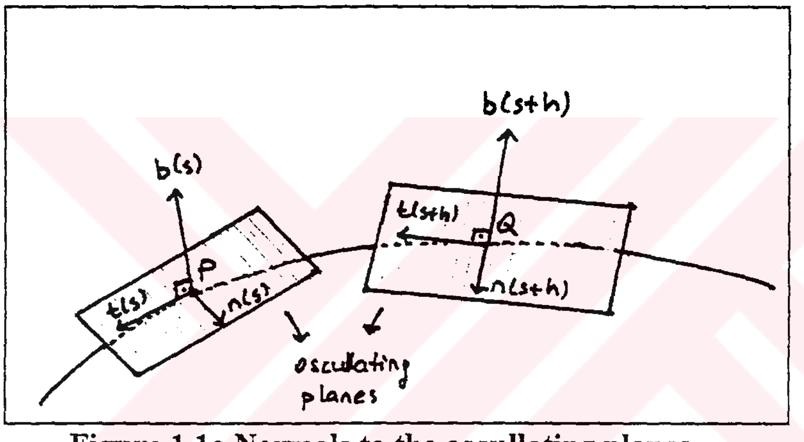

Figure 1.1a Normals to the osculating planes

If a rotation superposes the oscullating plane at P onto the oscullating plane at Q then b(s) will be send into b(s+h). Thus the angle between these oscullating planes is also the angle between the binormals, let it is denoted by θ.

And the limit of ratio θ/h as $h \to 0$ is called the torsion of the curve at P;

$$\tau(P) = \lim_{h \to 0} \frac{\theta}{h}$$

$\tau(P)$ measures the extent to which α fails to lie in its oscullating plane at s.

Another important point is the sign of the angle θ. It is obvious that the normal plane at point P divides R^3 into two half-spaces. If b(s)xb(s+h) and t(s) (tangent vector at s) point into the same half-space then we give (+) sign to the angle θ, otherwise θ has (-) sign. Since,

$$\lim_{h\to 0} \frac{\theta}{h} = \tau,$$

θ and τ have the same sign. This can be proved by a simple method; if we get the Taylor Expansion of b(s+h) at the point s,

$$b(s+h)= b(s) + h\, b'(s) + o(h)$$
$$b(s+h)= b(s) - h.\tau.n(s)+ o(h)$$

Consequently,

$$b(s)xb(s+h) = -h.\tau.b(s)xn(s) + o(h)$$

or

$$b(s)xb(s+h) = h.\tau.t(s) + o(h)$$

thus

$$(b(s)xb(s+h)).t(s) = h.\tau + o(h)$$

Since h is sufficiently small chosen o(h) cannot affect on $h\tau$. So $h.\tau>0$ implies (b(s)xb(s+h)).t(s)>0 the angle between (b(s)xb(s+h)) and t(s) is acute angle.Similarly , $h.\tau<0$ implies (b(s)xb(s+h)).t(s)<0 and the angle between them is obtuse angle.

This is why (b(s)xb(s+h)) and t(s) are pointing the same half-space determined by normal plane at P gives (+) sign to θ (depending on τ) , otherwise (-) sign.

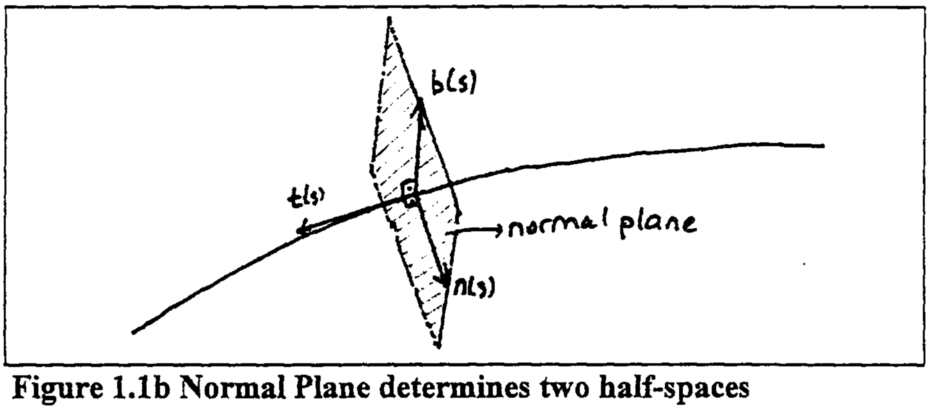

Figure 1.1b Normal Plane determines two half-spaces

(1.2) THE TORSION OF A POLYGONAL CURVE:

Now let α be a polygonal curve in R^3 with vertices v_i.For simplicity we will assume that α is closed and the number of vertices is finite. With above approximation to smooth torsion , we will define the polygonal torsion of a curve.

Smooth torsion was a function of points , now polygonal torsion be a function of segments;

$$\sigma_i = \{ (1\text{-}t).v_i + t.v_{i+1} : t \in [0,1] \}$$

so

$$\tau(\sigma_i) = \tau_i .$$

(1.2.1) Definition : If σ_{i-1} , σ_i , and σ_{i+1} are coplanar then $\tau(\sigma_i) = 0$. If σ_{i-1} , σ_i , and σ_{i+1} are not coplanar , it is clear that the normal plane to v_{i+1} - v_i divides R^3 into two half-spaces , and v_{i+1} - v_i lies in exactly one of these half-spaces ; let θ_i denotes the angle between $-\pi$ and π whose magnitude is the (undirected) angle between the binormals

$$b_i = \frac{(v_i - v_{i-1})x(v_{i+1} - v_i)}{|(v_i - v_{i-1})x(v_{i+1} - v_i)|}$$

and ,

$$b_{i+1} = \frac{(v_{i+1} - v_i)x(v_{i+2} - v_{i+1})}{|(v_{i+1} - v_i)x(v_{i+2} - v_{i+1})|}$$

And same as in the smooth torsion case sign of θ changes $\pm$ as $\pm$ $(b_i x b_{i+1})$ points into the same half-space determined by v_{i+1} - v_i. Then,

$$\tau(\sigma_i) = \frac{\theta_i}{|v_{i+1} - v_i|}$$

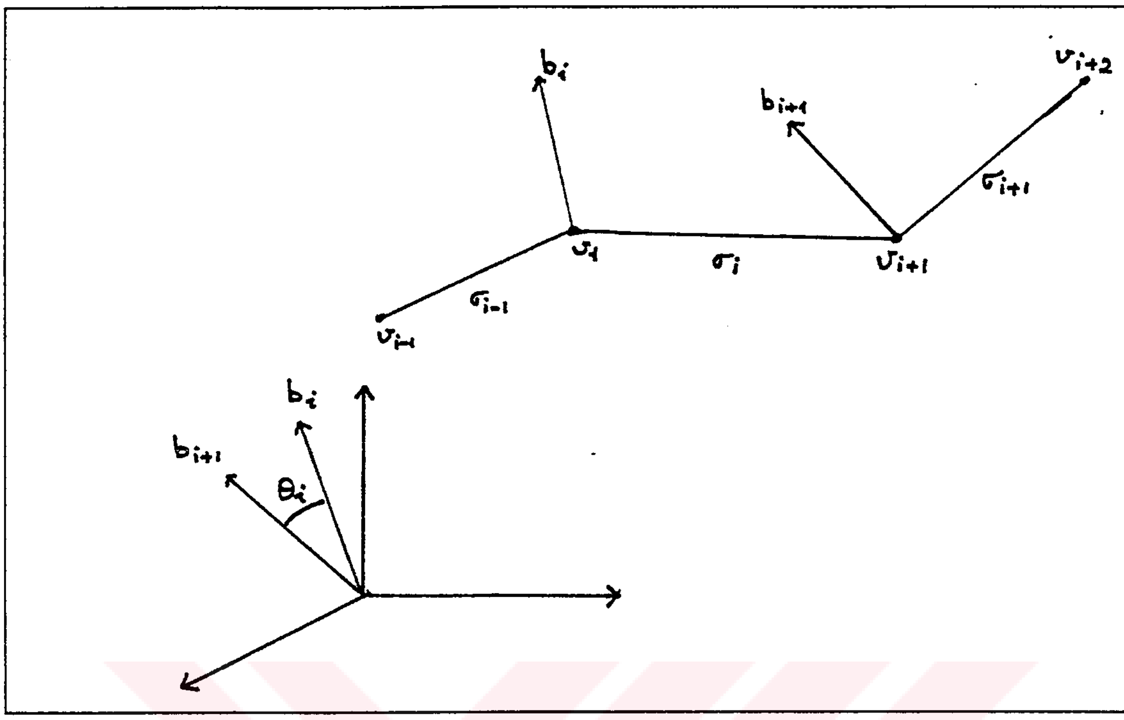

Figure 1.2 Binormals at the polygonal case.

(1.3) POLYGONAL SECANT APPROXIMATION OF A CURVE:

Now we obtain the polygonal secant approximation of a curve by using the definition of arc-length of the curve and also with the aid of a geometric approach.

(1.3.1) The arc-length of a curve:

The length of a path $r = r(t)$ in a closed interval $[a,b]$ is defined as follows.

Divide the segment $[a,b]$ into a finite number of nonoverlapping intervals using division points

$$a = t_0 \leq t_1 \leq \dots\dots\dots\dots \leq t_n = b$$

and form the sum of lengths of the segments connecting pairs of points corresponding to two consecutive values t_i and t_{i+1} of the parameter in this subdivision :

$$\sum_{k=1}^{n} \left| r(t_k) - r(t_{k-1}) \right|$$

The least upper bound (l.u.b.) of these sums of all subdivisions is called the arc-length of the path. It is seen that the approximation of arc-length of a curve can help us to make a polygonal secant approximation by using the same subdivisions. Also the subdivision which corresponds to the l.u.b. case will be the best approximation to that curve.

(1.3.2) COMPACTNESS OF A CURVE:

Let us make a geometric approach to a unit speed regular curve $\alpha : R \to R^3$ to have a polygonal secant approximation γ. And for every $\varepsilon > 0$ there is an integer N and the length of γ is within ε of the length of α. And γ is composed of precisely N segments , all of which have the same length:

For a given $\varepsilon > 0$ and any polygonal secant approximation β to α whose length is within ε of the length of α ; let δ denotes the length of the shortest segment of β. Since the graph of α is compact , it can be covered by , say , N open spherical balls of radius δ.

Now pick an arbitrary parameter value s_0 , and let r be a positive number much smaller than δ. Construct a 2-sphere with center $\alpha(s_0)$ and radius r ; this sphere will intersect the graph of α at a unique point $\alpha(s_1)$ where $s_1 > s_0$. Construct a 2-sphere with center $\alpha(s_1)$ and radius r; this sphere will intersect the graph of α at a unique point $\alpha(s_2)$ where $s_2 > s_1$. Continue in this manner until N 2-spheres have been constructed.

By connecting $\alpha(s_0), \alpha(s_1), \ldots\ldots\ldots, \alpha(s_N)$ with line segments we obtain a polygonal secant approximation to that part of α starting at $\alpha(s_0)$ and ending at $\alpha(s_N)$ whose segments are all of equal length.

Lets define the arc-length function $h(r)$ from $\alpha(s_N)$ to $\alpha(s_0)$ (that is the forward direction) will clearly be positive. Since the maximum length between the origins of two neghbour balls is 2δ it is clear that if r gets close to 2δ the directed arc-length h(r) will be negative.

As a function $h : [0,2\delta] \rightarrow R$ is continuous and $h(0) \geq 0$ and $h(2\delta) \leq 0$ since the image $h([0,2\delta])$ of the interval by h must be connected , it must contain 0. This means that $h(r) = 0$ for some $r \in [0,2\delta]$; perfoming our construction with $r = r_0$ we obtain the desired approximation γ .

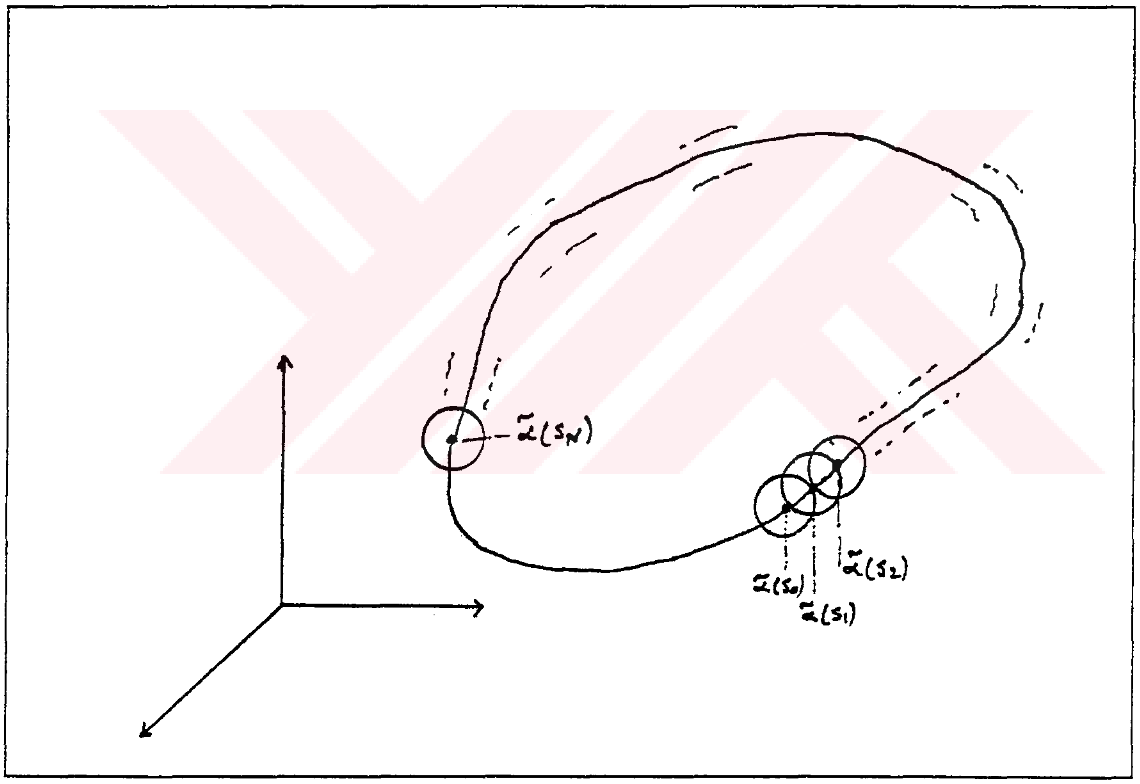

Figure 1.3 Spherical covers of a curve.

CHAPTER TWO
THE TOTAL TORSION OF A CURVE

Now we will give a theorem which is several times proved by different methods. But polygonal secant approximation method is the focus of our study. Following theorem and its proof will help us to get our aim.

(2.1) THEOREM: The total torsion $\int \tilde{\tau} ds$ of a closed unit speed regular curve $\tilde{\alpha}: R \to S^2$ on the unit 2-sphere S^2 is zero. (Tilda over an alphabet is used for the regular case.)

By definition of polygonal torsion , $\tau(\sigma_i) = \dfrac{\theta_i}{|v_{i+1} - v_i|}$, we have forced the following result.

(2.1.1) Proposition: Let $\{\alpha_i\}$ be a sequence of polygonal secant approximations to the closed regular curve $\tilde{\alpha}$ such that the vertices of α_i are vertices of α_j for $i < j$, and such that α_i approaches $\tilde{\alpha}$ uniformly as i tends to ∞. Then for each s_0 the torsion $\tilde{\tau}(s_0) = \lim_{i \to \infty} \tau(\sigma_i)$ where, for each i, σ_i is the segment of α_i for which $\tilde{\alpha}(s_0)$ lies between v_i and v_{i+1}. Now if α is the sufficiently close polygonal secant approximation to the closed unit speed regular curve $\tilde{\alpha}$ then we may approximate the total torsion of $\tilde{\alpha}$:

$$\int \tilde{\tau}\, ds \cong \sum_i \tau_i |v_{i+1} - v_i| = \sum_i \theta_i .$$

The summation taken over all segments σ_i of α_0. The following proposition is the key to the proof of theorem 2.1.

(2.1.2) Proposition: Let α be a closed polygonal curve in R^3 whose vertices all lie on S^2 and for which the lengths of all segments σ_i are equal (this latter condition is the polygonal analog of unit speed), then $\sum_i \theta_i$,

the summation taken over all segments σ_i of α, is an integral multiple of 2π.

Proof: For each segment σ_i of α we can define a map T_i from the unit circle $S^1 \subset R^2 \subset R^3$ to S^2 as follows : Let Π_i denotes the perpendicular bisecting plane of σ_i , and let p_i denotes the point of intersection of $\Pi_i \cap S^2$ and the ray emanating from the origin which passes through the midpoint σ_i. Since two points of a rotation of R^3 is known then we can define the third point uniquely , then the rotation of R^3 taking e_1 to p_i and e_3 to $(v_{i+1} - v_i)/|v_{i+1} - v_i|$ is unique.And T_i is the restriction of this map to S_1.

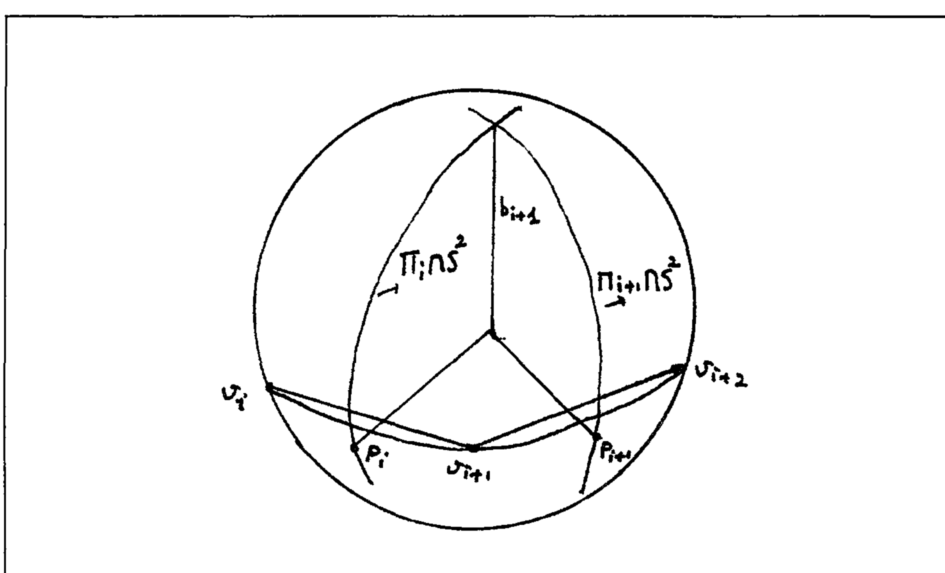

Figure 2.1 The rotation of binormals.

Since b_i and b_{i+1} are in $\Pi_i \cap S^2$, and by construction, θ_i is the directed angle from $T_i^{-1}(b_i)$ to $T_i^{-1}(b_{i+1})$.Furthermore $T_i^{-1}(b_{i+1}) = T_{i+1}^{-1}(b_{i+1})$; this is simply the fact that $T_{i+1} \circ T_i^{-1} = \Pi_i \to \Pi_{i+1}$ is rotation about the line containing b_{i+1}.

We can prove this by thinking a rotation $\gamma_i : \Pi_i \to \Pi_{i+1}$ as taking the the unit vector b_{i+1} constant.Since a rotation of this type can be done and uniqueness of T_i show that $T_{i+1} \circ T_i^{-1} : \Pi_i \to \Pi_{i+1}$ is the rotation about the line containing b_{i+1}.Then,

$$T_{i+1} \circ T_i^{-1}(b_{i+1}) = b_{i+1}$$

or

$$T_i^{-1}(b_{i+1}) = T_{i+1}^{-1}(b_{i+1})$$

The rotation $T_{i+1} \circ T_i^{-1}$ about the line containing b_{i+1} is satisfied by the necessity of the lengths of σ_i are equal, otherwise $T_{i+1} \circ T_i^{-1}$ is not this rotation.

Thus θ_0 is the angle from $T_0^{-1}(b_0)$ to $T_0^{-1}(b_1) = T_1^{-1}(b_1)$, θ_1 is the angle from $T_1^{-1}(b_1)$ to $T_1^{-1}(b_2) = T_2^{-1}(b_2)$, etc. Since the terminal side of the last angle θ_i is again $T_0^{-1}(b_0)$, it shows that $\sum_i \theta_i$ is an integral multiple of 2π.

(2.1.3) COROLLARY: If α_u , $u \in [0,1]$ is a continuous deformation from α_0 to α_1 , and at each stage of the deformation, α_u satisfies the assumptions of proposition(2.1.2) (the common length of the segments is allowed to vary from one stage of the deformation to another) , then $\sum_i \theta_i$ is the same for both α_0 and α_1.

PROOF: Let $f:[0,1] \to Z$ be the function for the curves α_u , $u \in [0,1]$, which gives the integer that is obtained from the total torsion (integer from the multiple of 2π) for that curve. For example: let the total torsion of α_i $i \in [0,1]$ is $\int \tau ds = 2\pi n$, then f(i)=n.

Since f is continuous f maps [0,1] , the connected set, to a connected set in Z. So;

$f[0,1] \subseteq Z$ must be connected.Then, $f[0,1]$ must be an integer.

(See also [4] for connected spaces).

As a result, $f(0) = f(1) = m$ (m is any integer). $f(0) = f(1)$ implies $\sum_i \theta_i$ is the same for both α_0 and α_1.

It is obvious that we can get a planer curve by the use of this continuous deformation. Since the total torsion of a planer curve is zero then the integral multiple of 2π arising in Proposition is zero.

We have sketched the main ideas here for rigoruous proofs see [3].

(2.2) AN OTHER PROOF OF THEOREM(2.1):

(2.2.1) LEMMA : If f, g are C^∞ functions on a proper interval L such that;

$$f(s)^2 + g(s)^2 = 1 \qquad \text{for all } s \in L,$$

then there exists a C^∞ function θ on L such that,

$$\cos\theta(s) = f(s), \sin\theta(s) = g(s). \qquad (2.2.1a)$$

Furthermore , if s_0 is any point on L and θ_0 is the unique constant satisfying the conditions;

$$\cos\theta_0 = f(s_0), \sin\theta_0 = g(s_0), 0 \le \theta_0 \le 2\pi,$$

then the function θ is given explicitly by,

$$\theta(s) = \int_{s_0}^{s} (fg' - gf')ds + \theta_0 . \qquad (2.2.1b)$$

(By a proper interval we mean an interval with non-empty interior).

Proof: Let C be the unit circle in the plane with rectangular coordinates (x,y) and the origin O. Then the point P(s): (f(s),g(s)) moves continuously on C as s moves continuously on L. Let A be the point on C such that $A\hat{O}P(s_0)=\theta_0$, where s_0 and θ_0 are as defined in lemma(2.2.1).

If s_1 is any point on L, we can reach it by allowing s to increase or decrease continuously from s_0. When s, starting from s_0, moves continuously towards s_1 and finally reaches it, the point P(s), starting from $P(s_0)$ moves continuously along C until it finally stops at the position $P(s_1)$. (The point P(s) may go back and forth or around C one or more times before finally stopping at the position $P(s_1)$). We put $\theta(s_1)$ equal to the sum of θ_0 and the 'algebraic' angle described by the radius vector OP(s). This defines a single valued function
θ on L such that ; $\theta(s_0)=\theta_0$.

We assert that θ is the function satisfying the required conditions. Obviously , θ satisfies (2.2.1a). We now prove that θ is given by (2.2.1b) so that it is C^∞.

Since θ is continuous, we have by (2.2.1a) that

$$f'(s) = -\sin\theta(s).\theta'(s)$$

$$g'(s) = \cos\theta(s).\theta'(s)$$

Since $\cos\theta$ and $\sin\theta$ can not be both zero at the same point, the above two equations show that θ' exists and is continuous.

Let us now put

$$\phi(s) = \int_{s_0}^{s}(fg' - gf')ds + \theta_0$$

it is clear that $\theta' = fg' - gf' = \phi'$ This together with $\theta(s_0) = \theta_0 = \phi(s_0)$, proves that $\theta = \phi$.

(2.2.2) THEOREM : If a C^∞ Frenet curve $\beta: x(s)$, $s \in L$, lies on a sphere of radius R then there is a C^∞ function $\phi(s)$ on L such that;

$$k_1 \sin\phi = \frac{1}{R} \quad , \quad k_2 + \phi' = 0$$

Proof: Let β lies on a sphere with center at c and radius equal to R. Then $|x-c|^2 = R^2$ and (x-c). e_1 = 0. Therefore by **lemma (2.2.1)** there exists a C^∞ function $\phi(s)$ on L such that;

$\phi(s)$ is the angle between $(\overrightarrow{x-c})$ and $\overrightarrow{e_3}$. And also remember that;

$$x' = e_1$$
$$e_1' = k_1 e_2$$
$$e_2' = -k_1 e_1 + k_2 e_3$$
$$e_3' = -k_2 e_2$$

so,

$$(x\text{-}c) = R\cos\phi e_3 - R\sin\phi e_2$$

differentiation gives,

$$e_1 = -\sin\phi\phi' R e_3 + R\cos\phi e_3' - R\cos\phi\phi' e_2 - R\sin\phi e_2'$$

$$e_1 = -R\sin\phi\phi' e_3 - k_2 R\cos\phi e_2 - R\cos\phi\phi' e_2 + k_1 R\sin\phi e_1 - k_2 R\sin\phi e_3$$

$$e_1 = k_1 R\sin\phi e_1 - (k_2 + \phi')R\cos\phi e_2 - (k_2 + \phi')R\sin\phi e_3$$

from which it follows $\quad k_1 R\sin\phi = 1$

and

(since $\cos\phi$ and $\sin\phi$ can not be zero at the same time so) $k_2 + \phi' = 0$.

If we take the integral of the second result $k_2 + \phi' = 0$ at the interval $[0, 2\pi]$ (because we want to find the total torsion of a closed curve) we can get;

$$\int_\beta k_2 ds = -\int_\beta \phi' ds$$

since,

$$-\int_\beta \phi' ds = -\int_0^{2\pi} \phi' ds = -(\phi(s)\Big|_0^{2\pi}) \quad \text{and} \quad \phi(2\pi) = \phi(0)$$

then

$$\int_\beta k_2 ds = 0.$$

(2.3) TORSION ON REGULARLY HOMOTOPIC SPHERICAL CURVES:

After now we are going to deal with the total torsion of special kind of homotopic unit speed regular curves on the unit 2-sphere S^2.

(2.3.1) Definition: Two regular curves on a manifold M are said to be regularly homotopic and the homotopy $g_v : I \to M$ can be chosen such that for each $v \in I, g_v$ is a regular curve, $g'_v(0) = g'_0(0), g'_v(1) = g'_0(1)$.

Also a new curve, by gluing the end point of a curve and beginning point of another curve, may be defined as follows.

(2.3.2) Definition: If α, β are curves on S^2 with $\alpha(1) = \beta(0)$. The product of α and β is the curve $\alpha.\beta$, defined by:

$$(\alpha.\beta)(t) = \begin{cases} \alpha(2t) \ldots 0 \le t \le 1/2 \\ \beta(2t-1) \ldots 1/2 < t \le 1 \end{cases}$$

(2.3.3) Definition: If α is a curve, the same curve with opposite direction, α^{-1}, can be defined as:

$$\alpha^{-1}(t) = \alpha(1-t).$$

By this definition it is clear that if two curves are homotopic, that is $\alpha_0 \approx \alpha_1$, then by defining α_1^{-1} as $\alpha_1^{-1}(t) = \alpha_1(1-t)$ (opposite direction of $\alpha_1(t)$) we can get $\alpha_1(1) = \alpha_1^{-1}(0)$. And also $\alpha_0(1) = \alpha_1^{-1}(0)$ implies the product $\alpha_0.\alpha_1^{-1}$ is a closed curve, with $\alpha_0(0) = \alpha_1^{-1}(1)$.

If we take these two homotopic curves α_0 and α_1 as regularly homotopic and if they are defined as unit speed on S^2, then a regular unit speed closed curve can be obtained by the product $\alpha_0.\alpha_1^{-1}$.

Let β be the curve of the product $\alpha_0.\alpha_1^{-1}$, that is $\beta=\alpha_0.\alpha_1^{-1}$. Then the total torsion of β is

$$\int_\beta \tau ds$$

Since,

$$\int_\beta \tau ds \cong \sum_i \theta_i .$$

We can define,

$$\sum_i \theta_i = \sum_j \theta_j + \sum_k \theta_k$$

where $\sum_j \theta_j$ is the summation of the directed angles between the binormals of the curve α_0 and $\sum_k \theta_k$ the same for the curve α_1^{-1}.

It is clear that the directed angles between the binormals of the curve α_1 have opposite directions of the angles between the binormals of the curve α_1^{-1}. Then we can easily represent the angles between the binormals of the curve α_1 as $-\sum_k \theta_k$.

Since we have formed a closed unit speed regular curve β on the unit 2-sphere S^2 by the product $\alpha_0.\alpha_1^{-1}$ then we can say that,

$$\int_\beta \tau ds = 0 .$$

Hence,

$$\int_\beta \tau ds \cong \sum_i \theta_i$$

which is given by the linear secant approximation.

So

$$\sum_i \theta_i = 0 \quad \text{(by } \textbf{Theorem 2.1}\text{)}$$

This implies:

$$\sum_i \theta_i = \sum_j \theta_j + \sum_k \theta_k = 0$$

or

$$\sum_j \theta_j = -\sum_k \theta_k$$

where the right hand side represents the total torsion approximation of the curve α_1,that is:

$$-\sum_k \theta_k \cong \int_{\alpha_1} \tau ds.$$

And also

$$\sum_j \theta_j \cong \int_{\alpha_0} \tau ds.$$

Then our conclusion occurs here,

$$\int_{\alpha_0} \tau ds = \int_{\alpha_1} \tau ds.$$

In other words: The total torsion of regularly homotopic unit speed curves on unit 2-sphere S^2 are equal.

REFERENCES

[1] Abraham Goetz (1970).Introduction to Differential Geometry. Addison-Wesley.

[2] Joseph J.Rotman (1988).An Introduction to Algebraic Topology. Springer-Verlag.

[3] Michael A. Penna (1980). Total torsion. Ph.D. thesis at the University of Illinois, Urbana, under H.Osborn.

[4] Michael C.Gemignani (1972). Elementary Topology (2nd ed.). Addison-Wesley.

[5] S.Smale (1958). Regular Curves on Riemannian Manifolds. Trans. Amer. Math. Soc., 81, 492-512.

[6] Wong,Y.C. & Lai, H.F. (1967). A critical examination of the theory of curves in three dimensional differential geometry. Tohoku Mathematical Journal, 19, 1-31.