EXACTNESS OF ČECH HOMOLOGY THEORY

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by
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İZMİR

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This thesis has been written using LaTeX, which is standard for mathematical texts in the world. Our institute has approved using LaTeX in writing our thesis just this term. I started to learn LaTeX with the help of Halil Oruç from our department whose books were so helpful and with the talk given by Andrew J. Vince to the research assistants on LaTeX(he was visitor in our department this year from University of Florida supported by Fulbright); A. J. Vince has also given to the research assistants a LaTeX book which helped me in putting the thesis in the format that the institute asks.

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ABSTRACT

It is known that the Čech homology sequence of a compact pair (X, A) over a *compact* coefficient group is exact. It is proved that the Čech homology sequence of a compact pair (X, A) over an *algebraically compact* coefficient group is exact, too. To show this it is proved that if the Čech homology sequence of a pair (X, A) (not necessarily a compact pair) over a coefficient group G is exact and H is a *direct summand* of the group G, then the Čech homology sequence of the pair (X, A) over the group H is exact, too. In showing this we work in the category $\mathbf{Inv}_M(\mathbf{Comp})$ of inverse systems of chain complexes over a fixed directed set M which is shown to form a category.

ÖZET

Kompakt bir (X, A) ikilisinin *kompakt* katsayı gruplu Čech homoloji dizisinin tam olduğu bilinmektedir. Kompakt bir (X, A) ikilisinin *cebirsel kompakt* katsayı gruplu Čech homoloji dizisinin de tam olduğu ispatlanmaktadır. Bunun için şu gösterilir: Bir (X, A) ikilisin(kompakt bir ikili olması gerekli değildir) bir G grubu üzerindeki Čech homoloji dizisi tam ise ve H grubu G grubunun bir *direkt toplam terimi* ise, (X, A) ikilisinin H katsayı gruplu Čech homoloji dizisi de tamdır. Bunu kanıtlarken zincir komplekslerinin sabitlenmis bir M yönlendirilmiş kümesi üzerindeki ters sistemlerinden oluşan $\mathbf{Inv}_M(\mathbf{Comp})$ kategorisinde(bir kategori olduğu gösterilir) çalışılır.

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Chapter Three
INVERSE LIMITS OF INVERSE SYSTEMS

Chapter Four
THE ČECH HOMOLOGY THEORY: A PARTIALLY EXACT HOMOLOGY THEORY

CHAPTER ONE
INTRODUCTION

In this thesis it is shown that the Čech homology sequence of a compact pair (X, A) over an algebraically compact coefficient group is exact.

Firstly, we describe axioms for a homology theory we require in the following sections. Some categorical terminology to deal with homology theories is introduced in Section 1.3 of this chapter. With this terminology it becomes easier to describe the homology theory with a coefficient group G on the category **Comp** of chain complexes of abelian groups and using this to define formal homology theory of simplicial complexes in Chapter 2. These and the inverse limit of inverse systems(studied in Chapter 3) suffice to define the Čech homology theory in Chapter 4. In the first two chapters and Sections 3.1, 3.2 we give a summary of the basic theories and terminology we use in the proofs at Section 3.3 and Section 4.2; it is mainly from Eilenberg & Steenrod (1952). The definition of the Čech homology in Section 4.1 is also from Eilenberg & Steenrod (1952). For the proofs of the results stated in these sections, see Eilenberg & Steenrod (1952); we give proofs only for the results we obtained.

Although we say homology theory, Čech homology lacks the exactness axiom required in the Eilenberg-Steenrod axioms, although it is useful for some other reasons(see Eilenberg & Steenrod, 1952; Eda & Kawamura, 1997; Guri, 1993; Watanabe, 1987). Čech homology theory forms a distinguished example of so called a partially exact homology theory.

The Čech homology sequence of a pair (X, A) over a coefficient group G is known to be exact under some restrictions on the pair (X, A) and on the group

G. One such is that the pair (X, A) be a compact pair and the group G be a compact abelian group. In this thesis we show that if the pair (X, A) is a compact pair and the group G is an algebraically compact group, i.e. algebraically a direct summand of a compact group, then exactness is again obtained as proved in Chapter 4. In showing this, we prove that if the Čech homology sequence of a pair (X, A) over a group G is exact and H is a direct summand of the group G, i.e. $G = H \oplus H'$ for some subgroup H' of G, then the Čech homology sequence of the pair (X, A) over the group H is exact, too. To obtain this result we work in the category $\mathbf{Inv}_M(\mathbf{Comp})$ of inverse systems of chain complexes over a fixed directed set M; this category is described in Section 3.3.

Notes about the topic compact abelian groups and algebraically compact abelian groups of infinite abelian groups are given at Section 1.4. Divisible groups, bounded groups, cocyclic groups, a direct product of copies of the additive group of p-adic numbers, linearly compact groups are examples of algebraically compact abelian groups. The definitions of such kind of groups and more about algebraically compact groups like some structure theorems, cardinal invariants can be found in Fuchs (1970).

1.1 Terminology, notation and conventions

We will use only abelian groups and compact abelian groups, not general R-modules for a ring R, so we will give all definitions in terms of abelian groups and compact abelian groups. Unless otherwise stated, by a group we will mean an abelian group and by a compact group, we will mean a compact abelian group, although we generally stress that the groups are abelian; groups will be written additively with zero element 0. In the definition of compactness we include also the Hausdorff condition, i.e. compact spaces are always assumed to be Hausdorff. A compact group is a *topological group* whose topology is compact; a topological group G is a group with a topology such that the functions $G \times G \longrightarrow G$, $(a, b) \mapsto a + b$ and $G \longrightarrow G$, $a \mapsto -a$ are continuous, where $G \times G$ is equipped with the product topology using the topology of G.

Categorical language will be used generally in the following sense. By a map of one object into another object in the category we are working in, we want to mean a map belonging to that category, i.e. a morphism in this category, that is a meaningful map in the structure we are considering. For example by a map $f : A \longrightarrow B$ in the category $\mathcal{Ab}$ of abelian groups, we want to mean a group homomorphism. By a map $f : X \longrightarrow Y$ in the category of topological spaces, we want to mean a continuous function. By a map $f : A \longrightarrow B$ in the category $\mathcal{Ab}_C$ of compact abelian groups, we want to mean a group homomorphism which is also a continuous function, i.e. a continuous homomorphism. By a map $f : A \longrightarrow B$ in the category **Comp** of chain complexes of abelian groups we want to mean a chain homomorphism from the chain complex A to the chain complex B as we will see.

In general, instead of writing the cases for abelian groups and compact abelian groups always separately, we will mean by a group either an abelian group or compact abelian group, and by a homomorphism we will mean either an ordinary group homomorphism or a group homomorphism which is continuous in the corresponding cases. These assumptions will be clear from the context.

In stating the axioms for a homology theory in the next section we need the following definitions and conventions.

Definition 1.1.1. A *pair of sets* (X, A) is defined to be a set X and a subset A of X. In case $A = \emptyset$, the symbol $(X, \emptyset)$ is usually abbreviated by (X) or, simply, by X. A *map* f of (X, A) into (Y, B), in symbols,

$$f : (X, A) \longrightarrow (Y, B)$$

is a single valued function from X to Y such that $f(A) \subset B$. If also $f : (Y, B) \longrightarrow (Z, C)$ is a map of pairs, then the composition of the functions is a map $gf : (X, A) \longrightarrow (Z, C)$ given by $(gf)x = g(fx)$ for each $x \in X$. The relation $(X', A') \subset (X, A)$ means $X' \subset X$ and $A' \subset A$. The map $i : (X', A') \longrightarrow (X, A)$

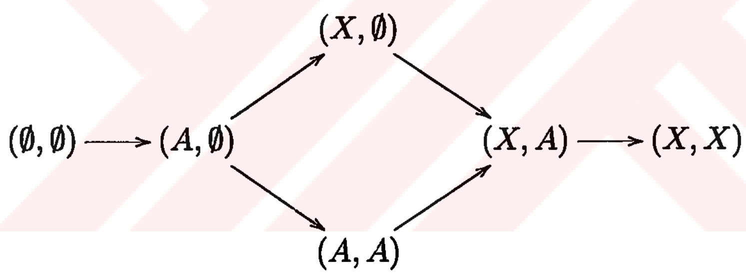
defined by $ix = x$ for each $x \in X'$ is called the *inclusion* map and is denoted by

$$i : (X', A') \hookrightarrow (X, A).$$

If $(X', A') = (X, A)$, then the inclusion map i is called the *identity* map of (X, A). In case when we consider inclusion map of the set A into B and identity maps of the same set A, then we will write $in_A : A \hookrightarrow B$ or simply $in : A \hookrightarrow B$ for the inclusion map and $i_A : A \hookrightarrow A$ or simply $i : A \hookrightarrow A$ for the identity map of A. A function is distinguished from those obtained from it by trivial modifications of the domain or range. Let $f : (X, A) \longrightarrow (Y, B)$ be given, and let $(X', A'), (Y', B')$ be pairs such that $X' \subset X$, $Y' \subset Y$, $f(X') \subset Y'$, and $f(A') \subset B'$. Then the unique map $f' : (X', A') \longrightarrow (Y', B')$ such that $f'(x) = fx$ for each $x \in X'$ is called *the map defined by* f, and f is said to define f'. If $f : (X, A) \longrightarrow (Y, B)$, the map of A into B defined by f is denoted by

$$f_{|A} : A \longrightarrow B.$$

The *lattice* of a pair (X, A) consists of the pairs



all their identity maps, the inclusion maps indicated by arrows, and all their compositions. If $f : (X, A) \longrightarrow (Y, B)$, then f defines a map of every pair of the lattice of (X, A) into the corresponding pair of the lattice of (Y, B).

Definition 1.1.2. A *pair of topological spaces* or, briefly a *pair* is a pair (X, A) where X is a topological space and A is a subspace of X. A map of pairs $f : (X, A) \longrightarrow (Y, B)$ is continuous if the map $X \longrightarrow Y$ defined by f is a continuous function. Note that identity and inclusion maps in topological spaces are always continuous. A pair (X, A) of topological spaces is called *compact* if X is compact and A is a closed(and therefore compact) subset of X; remember that we include the Hausdorff condition in the definition of compactness. We will write briefly *space* for a topological space.

A family $\mathfrak{a}$ of pairs of spaces and maps of such pairs which satisfies the conditions (1) to (5) below is called an *admissible category for homology theory.* The pairs and maps of $\mathfrak{a}$ are called *admissible.*

1. If $(X, A) \in \mathfrak{a}$, then all pairs and inclusion maps of the lattice of (X, A) are in $\mathfrak{a}$.

2. If $f : (X, A) \longrightarrow (Y, B)$ is in $\mathfrak{a}$, then (X, A) and (Y, B) are in $\mathfrak{a}$ together with all maps that f defines of members of the lattice of (X, A) into the corresponding lattice of (Y, B).

3. If f_1 and f_2 are in $\mathfrak{a}$, and their composition $f_1 f_2$ is defined, then $f_1 f_2 \in \mathfrak{a}$.

4. If $I = [0, 1]$ is the closed unit interval, and $(X, A) \in \mathfrak{a}$, then the cartesian product
$$(X, A) \times I = (X \times I, A \times I)$$
is in $\mathfrak{a}$ and the maps
$$g_0, g_1 : (X, A) \longrightarrow (X, A) \times I$$
given by
$$g_0(x) = (x, 0), \qquad g_1(x) = (x, 1)$$
are in $\mathfrak{a}$.

5. There is in $\mathfrak{a}$ a space P_0 consisting of a single point. If X, P are in $\mathfrak{a}$, if $f : P \longrightarrow X$, and if P is a single point, then $f \in \mathfrak{a}$.

The following are examples of admissible categories for homology theory:

$\mathfrak{a}_1 =$ the set of all pairs (X, A) and all maps of such pairs. This is the largest admissible category.

$\mathfrak{a}_C =$ the set of all compact pairs and all maps of such pairs.

$\mathfrak{a}_{LC} =$ the set of pairs (X, A) where X is a locally compact Hausdorff space, A is closed in X, and all maps of such pairs having the property that the inverse images of compact sets are compact sets.

Definition 1.1.3. Two maps $f_0, f_1 : (X, A) \longrightarrow (Y, B)$ in the admissible category $\mathfrak{a}$ are said to be *homotopic* in $\mathfrak{a}$ if there is a map

$$h : (X, A) \times I \longrightarrow (Y, B)$$

in $\mathfrak{a}$ such that

$$f_0 = hg_0, \qquad f_1 = hg_1$$

or, explicitly,

$$f_0(x) = h(x, 0), \qquad f_1(x) = h(x, 1).$$

The map h is called a *homotopy.*

Definition 1.1.4. Let G be a group and L a subgroup. G/L denotes the *factor(quotient)* group, i.e. the group whose elements are the cosets of L in G. The *natural homomorphism*

$$\eta : G \longrightarrow G/L$$

is the function which attaches to each element of G the coset of L which contains it: $\eta(g) = g + L$ for each $g \in L$. When the group G is a compact group, a topology on G/L is introduced as follows: a subset U of G/L is open if and only if $\eta^{-1}(U)$ is open in G. With this *quotient topology* on G/L, G/L is a compact abelian group and η is continuous.

Definition 1.1.5. If $\phi : G \longrightarrow G'$ and $L \subset G$, $L' \subset G'$ are subgroups such that $\phi(L) \subset L'$ then the homomorphism $\tilde{\phi} : G/L \longrightarrow G'/L'$ *induced* by ϕ attaches to each coset of L in G the coset of L' in G'which contains its image under ϕ. The natural maps $\eta : G \longrightarrow G/L$, $\eta' : G' \longrightarrow G'/L'$ and the homomorphisms ϕ, $\tilde{\phi}$ satisfy the commutativity relation $\tilde{\phi}\eta = \eta'\phi$:

$$\begin{array}{ccc} G & \xrightarrow{\ \phi\ } & G' \\ {\scriptstyle \eta}\downarrow & & \downarrow{\scriptstyle \eta'} \\ G/L & \xrightarrow[\tilde{\phi}]{} & G'/L' \end{array}$$

We say that the above diagram is *commutative.*

We will be using commutative diagrams so often.

Definition 1.1.6. A diagram of groups and homomorphisms is said to be *commutative* if we get the same composite homomorphisms whenever we follow directed arrows along different paths from one group to another group in the diagram.

We will use a special diagram pattern in Chapters 3 and 4 repeatedly and say that a diagram of the form

$$\begin{array}{ccc} A_2 & \xleftarrow{\;p\;} & A_1 \\ g_2 \downarrow \; \uparrow f_2 & & g_1 \downarrow \; \uparrow f_1 \\ A_2' & \xleftarrow[\;q\;]{} & A_1' \end{array}$$

is commutative to mean that the diagram obtained by taking f_1, f_2 and leaving out g_1 and g_2 is commutative, and the diagram obtained by taking g_1, g_2 and leaving out f_1 and f_2 is commutative, i.e. the following diagrams

$$\begin{array}{ccc} A_2 & \xleftarrow{\;p\;} & A_1 \\ f_2 \uparrow & & \uparrow f_1 \\ A_2' & \xleftarrow[\;q\;]{} & A_1' \end{array} \qquad \begin{array}{ccc} A_2 & \xleftarrow{\;p\;} & A_1 \\ g_2 \downarrow & & \downarrow g_1 \\ A_2' & \xleftarrow[\;q\;]{} & A_1' \end{array}$$

are commutative, so that we do not draw this pair of diagrams repeatedly.

Definition 1.1.7. A subgroup H of a group G is a said to be a *direct summand* of G if there exists a subgroup A of G such that $G = H \oplus A$(internal direct sum). Not to think over internal or external direct sums, we can say that a group H is a direct summand of a group G if there exist subgroups H' and A of G such that $G = H' \oplus A$ and $H \cong H'$ where $\cong$ means isomorphic as groups; in that case we can identify H and H', so consider H as a subgroup of G.

For a homomorphism $f : A \longrightarrow B$ between groups, $\text{Ker}(f)$ denotes the kernel of f and $\text{Im}(f)$ denotes the image of f.

The following easily obtained fact gives a characterization of a group being a direct summand of another group in terms of mappings which is needed for categorical arguments:

Proposition 1.1.8. *For abelian groups A and B, if there exist group homomorphisms $f : A \longrightarrow B$ and $g : B \longrightarrow A$ such that $g \circ f = i_A$ where $i_A : A \longrightarrow A$ is the identity map of A, then $B = \mathrm{Im}(f) \oplus \mathrm{Ker}(g)$ and as f is a monomorphism $B \cong A \oplus \mathrm{Ker}(g)$, so we can say A is a direct summand of B with the identification of A and $\mathrm{Im}(f)$. Conversely, if A is a direct summand of the group B, then clearly there exists such homomorphisms(Take $f : A \longrightarrow B$ as the inclusion of A in B and $g : B \longrightarrow A$ as the projection of B onto its direct summand A, for these $g \circ f = i_A$).*

$\mathbb{Z}$ denotes the set of all integers; by a sequence we will usually mean one indexed by $\mathbb{Z}$.

Definition 1.1.9. A *lower sequence* G of groups is a collection $\{G_q, \phi_q\}_{q \in \mathbb{Z}}$ or shortly $\{G_q, \phi_q\}$ where for each integer q(positive, negative or zero), G_q is a group, and $\phi_q : G_q \longrightarrow G_{q-1}$ is a homomorphism:

$$G : \qquad \cdots \longrightarrow G_{q+1} \xrightarrow{\phi_{q+1}} G_q \xrightarrow{\phi_q} G_{q-1} \longrightarrow \cdots$$

It is said to be *exact* if $\mathrm{Ker}(\phi_q) = \mathrm{Im}(\phi_{q+1})$ for all $q \in \mathbb{Z}$. A lower sequence $G' = \{G'_q, \phi'_q\}$ is said to be a *subsequence* of the lower sequence $G = \{G_q, \phi_q\}$ if for each q, $G'_q \subset G_q$ and $\phi'_q = \phi_{q|G'_q}$. A subsequence is determined by any set of subgroups $\{G'_q\}$ provided $\phi_q(G'_q) \subset G'_{q-1}$ for each q. The word subsequence is used here in a sense different from the usual one—no terms of the original sequence are discarded. If $G = \{G_q, \phi_q\}, G' = \{G'_q, \phi'_q\}$ are two lower sequences, a *chain homomorphism* or simply *homomorphism* $\psi : G \longrightarrow G'$ is a sequence $\{\psi_q\}_{q \in \mathbb{Z}}$ such that, for each integer q, $\psi_q : G_q \longrightarrow G'_q$ is a homomorphism and the following commutativity relations hold:

$$\phi'_q \psi_q = \psi_{q-1} \phi_q,$$

that is the diagram

$$\begin{array}{ccc} G_{q-1} & \xleftarrow{\phi_q} & G_q \\ \downarrow{\scriptstyle \psi_{q-1}} & & \downarrow{\scriptstyle \psi_q} \\ G'_{q-1} & \xleftarrow[\phi'_q]{} & G'_q \end{array}$$

is commutative. We denote a chain homomorphism by

$$\begin{array}{ccccccc} G: & \cdots \longrightarrow & G_q & \xrightarrow{\phi_q} & G_{q-1} & \longrightarrow \cdots \\ \downarrow \psi & & \downarrow \psi_q & & \downarrow \psi_{q-1} & \\ G': & \cdots \longrightarrow & G'_q & \xrightarrow[\phi'_q]{} & G'_{q-1} & \longrightarrow \cdots \end{array}$$

The subgroups $\{\operatorname{Ker}\psi_q\}$ form a subsequence of G called the *kernel* of ψ, and $\operatorname{Ker}\psi = 0$ means $\operatorname{Ker}\psi_q = 0$ for each q. Likewise $\operatorname{Im}\psi = \{\operatorname{Im}\psi_q\}$ is a subsequence of G'; and when $G' = \operatorname{Im}\psi$, we say that G is *onto*. If each ψ_q is an isomorphism, then ψ is said to be an *isomorphism.*

Definition 1.1.10. If L is a subsequence of the lower sequence G, the *factor sequence* G/L of G by L is the lower sequence composed of the factor groups G_q/L_q and the homomorphisms $\tilde{\phi}_q : G_q/L_q \longrightarrow G_{q-1}/L_{q-1}$ induced by the ϕ_q. Let $\eta_q : G_q \longrightarrow G_q/L_q$ be the natural homomorphism. Since $\tilde{\phi}_q\eta_q = \eta_{q-1}\phi_q$ it follows that $\eta = \{\eta_q\} : G \longrightarrow G/L$. It is called the *natural* homomorphism of G onto G/L.

Definition 1.1.11. Let L be a subsequence of a lower sequence $G = \{G_q, \phi_q\}$. L is said to be a *direct summand* of G if there exists a subsequence L' of G such that for each $q \in \mathbb{Z}$, $G_q = L_q \oplus L'_q$. As in Definition 1.1.7, we will say that a lower sequence L is a *direct summand* of a lower sequence G, if G has a subsequence $\tilde{L}$ which is a direct summand of G in the sense just defined and which is isomorphic to L.

Proposition 1.1.12. *Let $G' = \{G'_q, \phi'_q\}$ and $G = \{G_q, \phi_q\}$ be lower sequences and $f : G' \longrightarrow G$ and $g : G \longrightarrow G'$ be chain homomorphisms such that $g \circ f = i_{G'}$, where $i_{G'} : G' \longrightarrow G'$ is the identity chain homomorphism. Then G' is a direct summand of G and so G'_q is a direct summand of G_q for each $q \in \mathbb{Z}$.*

Proof. $g \circ f = i_{G'}$ implies $g_q \circ f_q = i_{G'_q}$ for each $q \in \mathbb{Z}$, where $i_{G'_q} : G'_q \longrightarrow G'_q$ is the identity map of G'_q. Then by Proposition 1.1.8, $G_q = \operatorname{Im}(f_q) \oplus \operatorname{Ker}(g_q)$ for each $q \in \mathbb{Z}$. $\operatorname{Im}(f) = \{\operatorname{Im}(f_q)\}$ is a subsequence of G and $\operatorname{Ker}(g) = \{\operatorname{Ker}(g_q)\}$ is a subsequence of G, so by Definition 1.1.11, $\operatorname{Im}(f) = \{\operatorname{Im}(f_q)\}$ is a direct summand of G. Since f_q is monomorphism for each q, $\operatorname{Im}(f_q) \cong G'_q$ for each $q \in \mathbb{Z}$. Thus

G' is isomorphic to the subsequence $\text{Im}(f) = \{\text{Im}(f_q)\}$ of G(via f) which implies that G' is a direct summand of G by Definition 1.1.11. □

The conventions and notations introduced for the pairs of sets (X, A) and their maps will also be used for pairs of groups and their homomorphisms.

1.2 Eilenberg-Steenrod Axioms for a Homology Theory

A homology theory H on an admissible category $\mathfrak{a}$ is a collection of three functions as follows: The first is a function $H_q(X, A)$ defined for each pair (X, A) in $\mathfrak{a}$ and each integer q. The value of the function is a group. It is called the *q-dimensional relative homology group of X modulo A.*

The second function is defined for each map

$$f : (X, A) \longrightarrow (Y, B)$$

in $\mathfrak{a}$ and each integer q, and attaches to such a pair a homomorphism

$$f_{*q} : H_q(X, A) \longrightarrow H_q(Y, B).$$

It is called the homomorphism *induced* by f.

The third function $\partial(q, X, A)$ is defined for each (X, A) in $\mathfrak{a}$ and each integer q. Its value is a homomorphism

$$\partial(q, X, A) : H_q(X, A) \longrightarrow H_{q-1}(A)$$

called the *boundary operator*. In $\partial(q, X, A)$, the symbol (q, X, A) is redundant so it will be omitted.

According to the convention of the preceding section, $H_q(X, A)$ is either always an abelian group, or always a compact abelian group. The corresponding conventions govern the homomorphisms ∂ and f_*.

In addition, the three functions are required to have the following properties(seven axioms below):

Axiom 1. If $f = \textit{identity}$, then $f_* = \textit{identity}$.

Explicitly, if f is the identity map of (X, A) in $\mathfrak{a}$ on itself, then, for each q, f_* is the identity map of $H_q(X, A)$ on itself.

Axiom 2. $(gf)_* = g_* f_*$.

Explicitly, if $f : (X, A) \longrightarrow (Y, B)$ and $g : (Y, B) \longrightarrow (Z, C)$ are admissible, then the composition of the induced homomorphisms $f_* : H_q(X, A) \longrightarrow H_q(Y, B)$ and $g_* : H_q(Y, B) \longrightarrow (Z, C)$ is the induced homomorphism $(gf)_* : H_q(X, A) \longrightarrow H_q(Z, C)$.

Axiom 3. $\partial f_* = (f_{|A})_* \partial$.

Explicitly, if $f : (X, A) \longrightarrow (Y, B)$ is admissible and $f_{|A} : A \longrightarrow B$ is the map defined by f, then there are two ways of mapping $H_q(X, A)$ into $H_{q-1}(B)$. As shown in the diagram,

$$\begin{array}{ccc} H_q(X,A) & \xrightarrow{f_*} & H_q(Y,B) \\ \partial \downarrow & & \downarrow \partial \\ H_{q-1}(A) & \xrightarrow[(f_{|A})_*]{} & H_{q-1}(B) \end{array}$$

the composition ∂f_* is obtained by moving over and then down, the composition $(f_{|A})_* \partial$ by moving down and over. The axiom requires that the two homomorphisms have the same value on each element of $H_q(X, A)$. With the commutative diagram terminology, the above diagram is commutative.

Axiom 4. (EXACTNESS AXIOM). If (X, A) is admissible and $i : A \longrightarrow X$, $j : X \longrightarrow (X, A)$ are inclusion maps, then the lower sequence of groups and homomorphisms

$$\ldots \xrightarrow{\partial} H_q(A) \xrightarrow{i_*} H_q(X) \xrightarrow{j_*} H_q(X, A) \xrightarrow{\partial} H_{q-1}(A) \xrightarrow{i_*} \ldots$$

is *exact*. This lower sequence is called the *homology sequence* of (X, A).

To make the above statement more precise the groups and homomorphisms of the lower sequence must be indexed by integers. We choose $H_0(X, A)$ as the 0^{th} group, i.e. $G_{3q} = H_q(X, A)$, $\phi_{3q} = \partial$, $G_{3q+1} = H_q(X)$, etc.

Axiom 5. (HOMOTOPY AXIOM). If the admissible maps $f_0, f_1 : (X, A) \longrightarrow (Y, B)$ are homotopic in $\mathfrak{a}$, then, for each q, the homomorphisms f_{0*}, f_{1*} of $H_q(X, A)$ into $H_q(Y, B)$ coincide.

Axiom 6. (EXCISION AXIOM). If U is an open subset of X whose closure $\bar{U}$ is contained in the interior of A(i.e. $\bar{U} \subset V \subset A$ for some open set V of X), and if the inclusion map $(X - U, A - U) \hookrightarrow (X, A)$ is admissible, then it induces isomorphisms $H_q(X - U, A - U) \longrightarrow H_q(X, A)$ for each q.

An inclusion map $i : (X - U, A - U) \hookrightarrow (X, A)$ where U is open in X and $\bar{U}$ is in the interior of A is called an *excision map* or just an *excision.*

Axiom 7. (DIMENSION AXIOM). If P is an admissible space consisting of a single point, then $H_q(P) = 0$ for all $q \neq 0$.

The consitency of the axioms is easily verified by choosing each $H_q(X, A) = 0$. The interest, naturally, lies in the existence of nontrivial homology theories. We know existence of such. One example is the *singular homology theory* on the largest admissible category $\mathfrak{a}_1$ of all pairs of spaces and their maps(see any of Bredon (1993), Rotman (1988), Munkres (1984), Eilenberg & Steenrod (1952)). Another example on the category $\mathfrak{a}_C$ of compact pairs is the Čech homology theory over a *compact coefficient group* given in Chapter 4. The singular homology theory is essentially the one derived from mappings of triangulable spaces into general spaces, and the Čech theory from the mappings of general spaces into triangulable spaces; both of these homology theories have some extremal properties among all homology theories.

We show in this thesis that on $\mathfrak{a}_C$ we have also the Čech homology theory over an *algebraically compact coefficient group.*

The homotopy and excision axioms can also be formulated in the following forms:

Axiom 5$'$. (HOMOTOPY AXIOM). If (X, A) is admissible and

$g_0, g_1 : (X, A) \longrightarrow (X, A) \times I$ are defined by $g_0(x) = (x, 0), g_1(x) = (x, 1)$, then $g_{0*} = g_{1*}$.

Axiom 6$'$. (EXCISION AXIOM). Let X_1 and X_2 be subsets of a space X such that X_1 is closed and $X = \operatorname{Int} X_1 \cup \operatorname{Int} X_2$. If the inclusion map $i : (X_1, X_1 \cap X_2) \hookrightarrow (X_1 \cup X_2, X_2)$ is admissible, then it induces isomorphisms $i_* : H_q(X_1, X_1 \cap X_2) \longrightarrow H_q(X_1 \cup X_2, X_2)$ for each q.

Here Int denotes the interior of a subset of a topological space.

Theorem 1.2.1. *A map $f : (X, A) \longrightarrow (Y, B)$ defines maps*

$$f_1 : X \longrightarrow Y \qquad f_2 : A \longrightarrow B.$$

The collection of homomorphisms f_*, f_{1*}, and f_{2*} form a chain homomorphism of the homology sequence of (X, A) into that of (Y, B). It will be denoted by f_{**}:

$$\begin{array}{ccccccccccc}
\text{H.S. of } (X,A): \ldots & \longrightarrow & H_{q+1}(X,A) & \xrightarrow{\partial} & H_q(A) & \xrightarrow{i_*} & H_q(X) & \xrightarrow{j_*} & H_q(X,A) & \longrightarrow & \cdots \\
\downarrow {\scriptstyle f_{**}} & & \downarrow {\scriptstyle f_*} & & \downarrow {\scriptstyle f_{2*}} & & \downarrow {\scriptstyle f_{1*}} & & \downarrow {\scriptstyle f_*} & & \\
\text{H.S. of } (Y,B): \ldots & \longrightarrow & H_{q+1}(Y,B) & \xrightarrow{\partial} & H_q(B) & \xrightarrow{i'_*} & H_q(Y) & \xrightarrow{j'_*} & H_q(Y,B) & \longrightarrow & \cdots
\end{array}$$

where H.S. denotes homology sequence and i, j, i', j' are appropriate inclusions.

1.3 c-categories, ∂-functors, h-categories and h-functors

Categorical language is useful when speaking about homology theories. We will introduce in this section more general categories than admissible categories for homology. We will not give the usual definitions of category, functor and subcategory.

The category of prime importance in the axiomatic treatment of homology theory is the category $\mathfrak{a}_1$ of all pairs and continuous maps of this pairs. The admissible categories defined in Definition 1.1.2 are subcategories of $\mathfrak{a}_1$.

The category $\mathcal{Ab}$ consists of abelian groups and their homomorphisms. The category $\mathcal{Ab}_C$ consists of compact abelian groups and their continuous homomorphisms. We may consider the categories $\mathbb{S}_l\mathcal{Ab}$, $\mathbb{S}_l\mathcal{Ab}_C$ whose objects are lower sequences of groups in $\mathcal{Ab}$ or $\mathcal{Ab}_C$ and whose mappings are chain homomorphisms of one such lower sequence into another where the chain homomorphism consists of homomorphisms or continuous homomorphisms in $\mathcal{Ab}$ or $\mathcal{Ab}_C$ respectively. The exact lower sequences form subcategories $\mathbb{E}_l\mathcal{Ab}$, $\mathbb{E}_l\mathcal{Ab}_C$.

We will see in the next chapter the category $\mathbf{K}_s$ of simplicial pairs and their simplicial maps. The triangulable pairs and their continuous maps form a full subcategory of $\mathfrak{a}_1$ which is an admissible category for homology theory.

In terms of functors, homology theory is seen better. Let $\mathfrak{a}$ be an admissible category on which a homology theory is given. Let q be a fixed integer, and define for an admissible map $f : (X, A) \longrightarrow (Y, B)$,

$$H_q(f) = f_* : H_q(X, A) \longrightarrow H_q(Y, B).$$

Then axioms 1 and 2 for a homology theory assert that the pair of functions $H_q(X, A)$ and $H_q(f)$ is a covariant functor H_q on the category $\mathfrak{a}$ with values in the category $\mathcal{Ab}$ or $\mathcal{Ab}_C$. Instead of using the category $\mathcal{Ab}$ we may use the category $\mathbb{E}_l\mathcal{Ab}$ of exact lower sequences in $\mathcal{Ab}$. We then define $H(X, A)$ to be the homology sequence of (X, A), and $H(f)$ to be the chain homomorphism f_{**} of the homology sequence of (X, A) into that of (Y, B) induced by f as in Theorem 1.2.1. Then H is a covariant functor on $\mathfrak{a}$ to $\mathbb{E}_l\mathcal{Ab}$ or $\mathbb{E}_l\mathcal{Ab}_C$. This functor is briefly referred as the *homology functor.*

Definition 1.3.1. A *category with couples*(briefly a *c-category*) is category $\mathfrak{C}$ in which certain pairs (α, β) of maps, called *couples*, are distinguished, subject to the sole condition that the composition $\beta\alpha$ is defined; that is in the category $\mathfrak{C}$ we define some of the pairs (α, β) of maps such that $\beta\alpha$ is defined, to be a *couple* and the remaining *not* to be a couple. If $\alpha : A \longrightarrow B$, $\beta : B \longrightarrow C$ is such a couple, we write

$$(\alpha, \beta) : A \longrightarrow B \longrightarrow C.$$

A covariant[contravariant] functor $T : \mathfrak{C} \longrightarrow \mathfrak{D}$ of a c-category into a c-category, is called a *c-functor* if for each couple (α, β) in $\mathfrak{C}$, the pair of maps $(T\alpha, T\beta)$ $[(T\beta, T\alpha)]$ is a couple in $\mathfrak{D}$.

Example 1.3.2. Consider an admissible category $\mathfrak{a}$. For each pair (X, A) in $\mathfrak{a}$ take only the inclusion maps $i : A \hookrightarrow X$, $j : X \hookrightarrow (X, A)$ as forming a couple

$$(i, j) : A \longrightarrow X \longrightarrow (X, A).$$

Another example of a c-category is obtained from the category $\mathcal{Ab}$(or $\mathcal{Ab}_C$) of groups by definining couples

$$(\phi, \psi) : G_1 \longrightarrow G_2 \longrightarrow G_3$$

whenever $\phi : G_1 \longrightarrow G_2$ has kernel zero, $\psi : G_2 \longrightarrow G_3$ is onto, and $\operatorname{Ker} \psi = \operatorname{Im} \phi$, i.e. whenever the sequence

$$0 \longrightarrow G_1 \xrightarrow{\phi} G_2 \xrightarrow{\psi} G_3 \longrightarrow 0$$

is exact.

Definition 1.3.3. Let $(\alpha, \beta) : A \longrightarrow B \longrightarrow C$ and $(\alpha_1, \beta_1) : A_1 \longrightarrow B_1 \longrightarrow C_1$ be couples in a c-category $\mathfrak{C}$. We define a *map* $(\alpha, \beta) \longrightarrow (\alpha_1, \beta_1)$ *of couples* to be a triple of maps

$$\gamma_1 : A \longrightarrow A_1, \quad \gamma_2 : B \longrightarrow B_1, \quad \gamma_3 : C \longrightarrow C_1$$

in $\mathfrak{C}$ such that commutativity holds in the two squares of the diagram

$$\begin{array}{ccccc} A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C \\ \downarrow{\scriptstyle \gamma_1} & & \downarrow{\scriptstyle \gamma_2} & & \downarrow{\scriptstyle \gamma_3} \\ A_1 & \xrightarrow[\alpha_1]{} & B_1 & \xrightarrow[\beta_1]{} & C_1 \end{array}$$

With the maps thus defined, the couples (α, β) in $\mathfrak{C}$ form a category of their own. A c-functor $T : \mathfrak{C} \longrightarrow \mathfrak{D}$ induces a functor on the category of couples of $\mathfrak{C}$ into that of $\mathfrak{D}$.

Definition 1.3.4. Let $\mathfrak{C}$ be a c-category. We shall consider systems $H = \{H_q(A), \alpha_*, \partial_{(\alpha,\beta)}\}$ where

1. For each object A in $\mathfrak{C}$ and each integer q, $H_q(A)$ is a group.

2. For each map $\alpha : A \longrightarrow B$ and each integer q, $\alpha_* : H_q(A) \longrightarrow H_q(B)$ is a homomorphism.

3. For each couple $(\alpha, \beta) : A \longrightarrow B \longrightarrow C$ and each integer q, $\partial_{(\alpha,\beta)} : H_q(C) \longrightarrow H_{q-1}(A)$ is a homomorphism.

The groups and homomorphisms belong to just one of the categories $\mathcal{A}b$ or $\mathcal{A}b_C$. Such a system H will be called a *covariant ∂-functor* on the c-category $\mathfrak{C}$ provided the following four axioms hold:

Axiom 1. If $\alpha = identity$, then $\alpha_* = identity$.

Axiom 2. $(\beta\alpha)_* = \beta_*\alpha_*$

Axiom 3. If $\gamma_1, \gamma_2, \gamma_3$ form a map of the couple $(\alpha, \beta) : A \longrightarrow B \longrightarrow C$ into the couple $(\alpha_1, \beta_1) : A_1 \longrightarrow B_1 \longrightarrow C_1$, then commutativity holds in the diagram

$$\begin{array}{ccc} H_q(C) & \xrightarrow{\gamma_{3*}} & H_q(C_1) \\ \downarrow{\scriptstyle \partial_{(\alpha,\beta)}} & & \downarrow{\scriptstyle \partial_{(\alpha_1,\beta_1)}} \\ H_{q-1}(A) & \xrightarrow[\gamma_{1*}]{} & H_{q-1}(A_1) \end{array}$$

Axiom 4. For every couple $(\alpha, \beta) : A \longrightarrow B \longrightarrow C$ the sequence

$$\ldots \longrightarrow H_q(A) \xrightarrow{\alpha_*} H_q(B) \xrightarrow{\beta_*} H_q(C) \xrightarrow{\partial} H_{q-1}(A) \longrightarrow \ldots$$

is exact.

Proposition 1.3.5. *Let H be a covariant ∂-functor on a c-category $\mathfrak{D}$ and let $T : \mathfrak{C} \longrightarrow \mathfrak{D}$ be a covariant c-functor. The composition*

$$HT = \{H_q(TA), (T\alpha)_*, \partial_{(T\alpha, T\beta)}\}$$

is then a covariant ∂-functor on $\mathfrak{C}$.

Definition 1.3.6. An *h-category* $\mathfrak{C}$ is a c-category in which

i. A binary relation $\alpha \simeq \beta$(α *homotopic* to β) is given for maps $\alpha, \beta : A \longrightarrow B$ in $\mathfrak{C}$.

ii. Certain maps $\alpha : A \longrightarrow B$ in $\mathfrak{C}$ are singled out and are called *excisions.*

iii. Certain objects of $\mathfrak{C}$ are singled out and are called *points.*

A covariant ∂-functor on $\mathfrak{C}$ which satisfies the anologs of the Homotopy, Excision and Dimession Axioms in Section 1.2 will be called a *homology theory* on the h-category $\mathfrak{C}$.

Let $\alpha : A \longrightarrow B$, $\beta : B \longrightarrow A$ be two maps in an h-category $\mathfrak{C}$. If $\beta\alpha : A \longrightarrow A$ and $\alpha\beta : B \longrightarrow B$ are both homotopic to identity maps, then α and β are both called *homotopy equivalences*, β is called a *homotopy inverse* of α and vice versa. A map $\alpha : A \longrightarrow B$ in $\mathfrak{C}$ which is a composition of a finite number of excisions and homotopy equivalences is called a *generalized excision.*

Proposition 1.3.7. *If H is a homology theory on an h-category $\mathfrak{C}$ and $\alpha : A \longrightarrow B$ is a generalized excision, then $\alpha_* : H_q(A) \longrightarrow H_q(B)$ is an isomorphism.*

Definition 1.3.8. A covariant c-functor $T : \mathfrak{C} \longrightarrow \mathfrak{D}$ on the h-category $\mathfrak{C}$ with values in the h-category $\mathfrak{D}$ is called an *h-functor* if T preserves homotopies, generalized excisions, and points. Explicitly: if $\alpha \simeq \beta$ in $\mathfrak{C}$, then $T\alpha \simeq T\beta$ in $\mathfrak{D}$; if α is a generalized excision in $\mathfrak{C}$, then $T\alpha$ is a generalized excision in $\mathfrak{D}$; and if A is a point in $\mathfrak{C}$, then TA is a point in $\mathfrak{D}$.

Proposition 1.3.9. *Let $T : \mathfrak{C} \longrightarrow \mathfrak{D}$ be a covariant h-functor and H a homology theory on $\mathfrak{D}$. Then the composition HT is a homology theory on $\mathfrak{C}$.*

1.4 Algebraically compact groups

This section is not complete in definitions since this will take us too much into the topic of infinite abelian groups, instead we will give some examples and refer to Kaplansky (1954) and Fuchs (1970) for these examples and further details.

For an example of a compact abelian group we firstly remind the circle group S^1 of the unit circle in the complex plane $\mathbb{C}$; its topology is the topology that it inherets as a subspace of $\mathbb{C}$ and its operation is the ordinary multiplication of complex numbers. This group S^1 is isomorphic to the additive quotient group $\mathbb{R}/\mathbb{Z}$ of real numbers modulo 1 ($\mathbb{R}$ denotes the real numbers). We remind once more that we are dealing with abelian groups, so we do not consider nonabelian compact matrix groups, like orthogonal matrices.

The Pontryagin duality describes the relation between a (locally) compact abelian group G and its character group $G^* = \text{Hom}(G, S^1)$ of all continuous homomorphisms from G to S^1 equipped with compact-open topology. The character groups of discrete abelian groups(with the suitable topology) are just the compact abelian groups. If G is a compact abelian group and H its character subgroup then G is in turn the character group of the discrete group H. Particularly, this leads a duality between discrete torsion groups and 0-dimensional(totally disconnected) compact abelian groups. Using duality theory and the structure theorem for complete modules, some cardinal number invariants are obtained for the structure of compact abelian groups.

The method of describing the algebraic structure of a compact abelian group led to the discovery of algebraically compact groups(Kaplansky, 1954). Any group A satisfying any of the equivalent conditions in the theorem that follows is called an *algebraically compact group*:

Theorem 1.4.1. *(Fuchs, 1970)The following conditions on an abelian group A are equivalent:*

1. *A is algebraically a direct summand of a group that admits a compact topology(i.e. a compact group).*
2. *A is a direct summand in every group G that contains A as a pure subgroup.*
3. *A has the form $C \oplus D$ where C is a divisible group and D is of the form $\prod_p D_p$(the product being extended over all distict primes) where each prime D_p is complete in its p-adic topology.(Kaplansky, 1954)*

4. *A is a direct summand of a direct product of cocyclic groups.*

5. *A is pure-injective.*

6. *If every finite subsystem of a system of equations over A has a solution in A then the whole system is solvable in A.*

Example 1.4.2. The following are all algebraically compact groups:

1. Every divisible group, like the additive group $\mathbb{Q}$ of rational numbers, the quasicyclic group $\mathbb{Z}(p^\infty)$, p prime number, and direct sums of such groups.

2. Every bounded group(i.e. a group A such that $nA = 0$ for some positive integer n).

3. Every finite group.

4. Every cocyclic group(i.e. cyclic groups of order p^k or the quasicyclic group $\mathbb{Z}(p^\infty)$ where p is a prime and k is a positive integer).

5. Every direct summand of an algebraically compact group.

6. Every group whose reduced part is algebraically compact.(A group decomposes into a direct sum of a maximal divisible subgroup and a reduced part with no nonzero divisible subgroup.)

7. A direct product of algebraically compact groups.(A direct product of algebraically compact groups is algebraically compact if and only if each component is algebraically compact.)

8. A direct summand of a direct product of cyclic p-groups.(A reduced algebraically compact group is necessarily of this form.)

9. The additive group of p-adic integers, a direct product of copies of p-adic integers.

10. Linearly compact abelian groups(this class of groups is between algebraically compact groups and compact groups).

11. $\text{Hom}(A, C)$ where A is an arbitrary group and C is an algebraically compact group.

12. $\mathrm{Hom}(A, C)$ where A is a torsion group and C is an arbitrary group.

The additive group $\mathbb{Z}$ of integers is not algebraically compact. More generally, (nonzero) free groups are never algebraically compact.

CHAPTER TWO

HOMOLOGY THEORY OF CHAIN COMPLEXES AND SIMPLICIAL COMPLEXES

In this chapter, we develop the formal homology theory of simplicial complexes which we will use when defining the Čech homology in Chapter 4. Formal homology theory of simplicial complexes is obtained by using the homology theory of chain complexes which is described in the first three sections; simplicial complexes are defined in Section 2.4. We give all the essential steps in the definitions of these homology theories and use the clarifying language of Section 1.3 because in the proofs in Chapter 4, we turn back to these definitions and use the basic properties of tensor product defined in Section 2.2 without reference.

2.1 Homology Theory on the h-category [$\mathbf{Comp}_C$] Comp of Chain Complexes of [Compact] Abelian Groups with Values in the Category [$\mathcal{Ab}_C$] $\mathcal{Ab}$

Definition 2.1.1. A *chain complex* K is a lower sequence $\{C_q(K), \partial_q\}$ of groups and homomorphisms $\partial_q : C_q(K) \longrightarrow C_{q-1}(K)$ such that $\partial_{q-1}\partial_q = 0$ for each integer q. $C_q(K)$ is called the group of *q-chains* of K, and ∂_q is called the *boundary* homomorphism. A map $f : K \longrightarrow K'$ of one chain complex into another is a chain homomorphism of lower sequences as defined in Definition 1.1.9, that is it is a sequence of homomorphisms $f_q : C_q(K) \longrightarrow C_q(K')$ defined for each integer q such that $f_{q-1}\partial_q = \partial'_q f_q$.

Definition 2.1.2. Let $K = \{C_q(K), \partial_q\}$ be a chain complex. The kernel, $Z_q(K)$ of ∂_q is called the group of *q-cycles* of K. The image, $B_q(K)$ of ∂_{q+1} is called the group of *q-boundaries* of K. Since $\partial_q\partial_{q+1} = 0$, $B_q(K)$ is a subgroup of $Z_q(K)$, and the factor group $H_q(K) = Z_q(K)/B_q(K)$ is called the *q-dimensional homology group* of K. If $f : K \longrightarrow K'$ is a map of chain complexes, then f_q sends $Z_q(K)$ into $Z_q(K')$ and $B_q(K)$ into $B_q(K')$, thereby inducing homomorphisms $f_* : H_q(K) \longrightarrow H_q(K')$.

Proposition 2.1.3. *If $f : K \longrightarrow K$ is the identity map, then f_* is the identity homomorphism. If $f : K \longrightarrow K'$ and $g : K' \longrightarrow K''$ are maps of chain complexes, then $(gf)_* = g_* f_*$.*

Chain complexes K and their maps f constitute a category denoted by **Comp** or **Comp**$_C$ according as the groups $C_q(K)$ are in the category $\mathcal{A}b$ or $\mathcal{A}b_C$. Then $H_q(K), f_*$ is a covariant functor from **Comp** to $\mathcal{A}b$ [or **Comp**$_C$ to $\mathcal{A}b_C$]. We shall convert the categories of chain complexes into c-categories and we shall extend $H_q(K), f_*$ to a covariant ∂-functor.

If K is a chain complex and L is a subsequence of K(as in Definition 1.1.9), then both L and K/L are again chain complexes called the *subcomplex* and *factor complex* respectively. Moreover the inclusion map $i : L \longrightarrow K$, and the natural map $\eta : K \longrightarrow K/L$ yield an exact sequence

$$0 \longrightarrow L \xrightarrow{i} K \xrightarrow{\eta} K/L \longrightarrow 0$$

That is $\operatorname{Ker} i = 0$, $\operatorname{Im}(i) = \operatorname{Ker}\eta$ and $\operatorname{Im}\eta = K/L$, so for each $q \in \mathbb{Z}$,

$$0 \longrightarrow L_q \xrightarrow{i_q} K_q \xrightarrow{\eta_q} K_q/L_q \longrightarrow 0$$

is exact. This suggests the following definition:

Definition 2.1.4. Let L, K, M be chain complexes. The maps $\phi : L \longrightarrow K$, $\psi : K \longrightarrow M$ are said to form a *couple* $(\phi, \psi) : L \longrightarrow K \longrightarrow M$, provided the sequence

$$0 \longrightarrow L \xrightarrow{\phi} K \xrightarrow{\psi} M \longrightarrow 0$$

is exact. If, further, the image of ϕ is a direct summand of K(i.e. $\phi(C_q(L))$ is a direct summand of $C_q(K)$ for all q), then the couple (ϕ, ψ) is called *direct.*

With couples defined as above, the categories **Comp** and **Comp**$_C$ become c-categories. One obtains different c-categories by taking *direct couples* only. We will see that it will be necessary to consider direct couples rather than all couples when we consider tensor product in the next section.

Lemma 2.1.5. *Let* $0 \longrightarrow L \xrightarrow{\phi} K \xrightarrow{\psi} M \longrightarrow 0$ *be an exact sequence of chain complexes.*

i. Let $\bar{Z}_q(M) = \psi^{-1}(Z_q(M))$, $\bar{B}_q(M) = \psi^{-1}(B_q(M))$, *and* $\bar{H}_q(M) = \bar{Z}_q(M)/\bar{B}_q(M)$. *Then*

$$\bar{Z}_q(M) = \partial^{-1}(\phi C_{q-1}(L)), \quad \bar{B}_q(M) = B_q(K) + \phi(C_q(L)),$$

and ψ *induces isomorphisms*

$$\bar{\psi} : \bar{H}_q(M) \longrightarrow H_q(M).$$

ii. The boundary homomorphism of the chain complex K *defines homomorphisms*

$$\bar{Z}_q(M) \longrightarrow \phi[Z_{q-1}(L)], \qquad \bar{B}_q(M) \longrightarrow \phi[B_{q-1}(L)]$$

Since the kernel of ϕ *is zero,* $\phi^{-1}\partial$ *defines homomorphisms*

$$\bar{Z}_q(M) \longrightarrow Z_{q-1}(L), \qquad \bar{B}_q(M) \longrightarrow B_{q-1}(L),$$

and induces a homomorphism

$$\Delta : \bar{H}_q(M) \longrightarrow H_{q-1}(L).$$

Definition 2.1.6. The homomorphism

$$\partial_* : H_q(M) \longrightarrow H_{q-1}(L)$$

defined as the composition $\partial_* = \Delta\bar{\psi}^{-1}$ is called the *boundary homomorphism* of the couple $(\phi, \psi) : L \longrightarrow K \longrightarrow M$.

The star in ∂_* has been inserted to distinguish it from the boundary operator within the chain complexes; it will be omitted later.

Theorem 2.1.7. *The system $H = \{H_q(K), f_*, \partial_*\}$ is a covariant ∂-functor on the c-category* **Comp**/**Comp**$_C$] *of chain complexes with values in the category* $\mathcal{A}b[\mathcal{A}b_C]$.

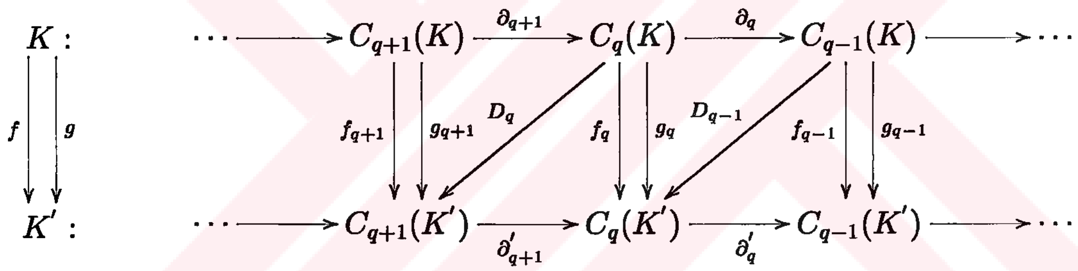
We will now extend the system into a homology theory.

Definition 2.1.8. Let $K = \{C_q(K), \partial_q\}$ and $K' = \{C_q(K'), \partial'_q\}$ be chain complexes and let f, g be two maps of K into K'. A *chain homotopy* D of f into g(notation: $D : f \simeq g$) is a sequence of homomorphisms

$$D_q : C_q(K) \longrightarrow C_{q+1}(K')$$

such that

$$\partial'_{q+1} D_q + D_{q-1}\partial_q = g_q - f_q.$$



If such a homotopy D exists, f and g are called *homotopic* and we write $f \simeq g$.

A map $f : K \longrightarrow L$ of chain complexes is called an *excision* if and only if f maps K isomorphically onto L.

A chain complex $K = \{C_q(K), \partial_q\}$ is called *pointlike* if

$$\partial_q : C_q(K) \longrightarrow C_{q-1}(K)$$

is an isomorphism for q even and > 0, and also for q odd and < 0.

With the above definitions of homotopies, excisions and points, **Comp** and **Comp**$_C$ becomes h-categories and:

Theorem 2.1.9. *The system* $H = \{Hq(K), f_*, \partial_*\}$ *is a homology theory on the h-category* **Comp**/**Comp**$_C$/ *of chain complexes, with values in the category* $\mathcal{A}b$/$\mathcal{A}b_C$/*. (The h-category* **Comp**/**Comp**$_C$/ *can be considered with all couples or with direct couples only).*

A definition needed for later use(in Section 2.5) is the following:

Definition 2.1.10. A chain complex K is said to be *finite*, if, for each q, $C_q(K)$ is a free group on a finite base. Then $H_q(K)$ also has a finite set of generators.

2.2 Tensor Product

We will firstly define tensor product of two abelian groups in the category $\mathcal{A}b$ which will be an abelian group. Then we will define tensor product of a finitely generated abelian group and a compact abelian group which will be a compact abelian group, that is the tensor product of such two groups will be given a topology that will make it compact.

Definition 2.2.1. The *tensor product* $C \otimes G$ of two groups C and G is the group generated by the set of all pairs $(c, g), c \in C, g \in G$ with relations

$$(c_1 + c_2, g) - (c_1, g) - (c_2, g) = 0,$$

$$(c, g_1 + g_2) - (c, g_1) - (c, g_2) = 0,$$

That is, $C \otimes G$ is obtained as follows : Let $R(C, G)$ be the free abelian group generated by the set of pairs (c, g) and let $Y(C, G)$ be the least subgroup of $R(C, G)$ containing all the elements of the form

$$(c_1 + c_2, g) - (c_1, g) - (c_2, g), \qquad (c, g_1, g_2) - (c, g_1) - (c, g_2),$$

then

$$C \otimes G = R(C, G)/Y(C, G).$$

The element of $C \otimes G$ which is the image of the generator (c, g) of $R(C, G)$ will be denoted by $c \otimes g$. These elements generate the group $C \otimes G$ and the relations are

$$(c_1 + c_2) \otimes g = c_1 \otimes g + c_2 \otimes g,$$

$$c \otimes (g_1 + g_2) = c \otimes g_1 + c \otimes g_2.$$

We clearly have for finitely many c_i's in C, g_j's in G, $c \in C$ and $g \in G$:

$$\left(\sum c_i\right) \otimes g = \sum (c_i \otimes g), \qquad c \otimes \sum (g_j) = \sum (c \otimes g_j).$$

Lemma 2.2.2. *For an abelian group G, the function $f : \mathbb{Z} \otimes G \longrightarrow G$ defined by $g(n \otimes g) = ng$ is an isomorphism. Similarly, $G \otimes \mathbb{Z} \cong G$. Both $\mathbb{Z} \otimes G$ and $G \otimes \mathbb{Z}$ will be identified with G by these isomorphisms.*

Definition 2.2.3. If $f : C \longrightarrow C'$ and $h : G \longrightarrow G'$ are group homomorphisms, then the correspondence $c \otimes g \mapsto (fc) \otimes (hg)$ defines a homomorphism

$$f \otimes h : C \otimes G \longrightarrow C' \otimes G'$$

called the homomorphism of the tensor product *induced* by the homomorphisms f and h. In case $G = G'$ and h is the identity, we shall speak of $f \otimes h$ as the homomorphism $C \otimes G \longrightarrow C' \otimes G'$ induced by f and will denote it by a symbol such as f'.

Proposition 2.2.4. *If $i : C \longrightarrow C$, $j : G \longrightarrow G$ are identity maps of groups, then $i \otimes j : C \otimes G \longrightarrow C \otimes G$ is the identity. If $f : C \longrightarrow C'$, $f' : C' \longrightarrow C''$, $h : G \longrightarrow G'$, $h' : G' \longrightarrow G''$ are homomorphisms of groups, then $(f'f) \otimes (h'h) = (f' \otimes h')(f \otimes h)$. That means, $\otimes$ is a covariant functor of two variables, in the category $\mathcal{Ab}$ with values in $\mathcal{Ab}$.*

Proposition 2.2.5. *Let C and G be represented as the direct sums*

$$C = \bigoplus_{\alpha \in M} C_\alpha, \qquad G = \bigoplus_{\beta \in M} G_\beta$$

Then

$$C \otimes G \cong \bigoplus_{(\alpha,\beta) \in M \times N} C_\alpha \otimes G_\beta.$$

Proposition 2.2.6. *If C is a free group with base X, then $C \otimes G$ is generated by elements $x \otimes g$ with relations $x \otimes (g_1 + g_2) = x \otimes g_1 + x \otimes g_2$. If G is also a free group with base Y, then $C \otimes G$ is a free group with base $\{x \otimes y | x \in X, y \in Y\}$.*

Lemma 2.2.7. *If f is a homomorphism of the group B onto the group C, then the induced homomorphism $f' : B \otimes G \longrightarrow C \otimes G$ is also onto.*

Theorem 2.2.8. *If*

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{h} C \longrightarrow 0$$

is an exact sequence of groups and homomorphisms, then the induced sequence

$$A \otimes G \xrightarrow{f'} B \otimes G \xrightarrow{h'} C \otimes G \longrightarrow 0$$

is also exact. (That means, $\cdot \otimes G$ is a right-exact functor from $\mathcal{Ab}$ to $\mathcal{Ab}$ with the terminology of homological algebra.)

It is not always true that f' is a monomorphism like f, i.e. $\operatorname{Ker} f'$ may be nonzero(that is $\cdot \otimes G$ is not an exact functor); but if, further, the image of f is a direct summand of B(i.e. the short exact sequence $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{h} C \longrightarrow 0$ is a splitting one), then the sequence

$$0 \longrightarrow A \otimes G \xrightarrow{f'} B \otimes G \xrightarrow{h'} C \otimes G \longrightarrow 0$$

is exact, and the image of f' is a direct summand of $B \otimes G$(i.e. it is a splitting short exact sequence).

Let's denote by $\mathcal{Ab}'$ the subcategory of $\mathcal{Ab}$ consisting of finitely generated abelian groups and their homomorphisms. We shall generalize the tensor product to a functor on $\mathcal{Ab}'$ and $\mathcal{Ab}_C$ with values in $\mathcal{Ab}_C$.

Lemma 2.2.9. *Let C be a free group on a finite base $c_1, \dots, c_n$. Each element of $C \otimes G$ can be written uniquely in the form $\sum_1^n c_i \otimes g_i$. The function defined by*

$$f(\sum_1^n c_i \otimes g_i) = (g_1, \dots, g_n)$$

is an isomorphism of $C \otimes G$ with the direct product G^n of n factors equal to G.

Definition 2.2.10. Let G be a compact group and C a free group on the base $c_1, \dots, c_n$. The direct product G^n of n factors G is a compact group. The isomorphism $C \otimes G \longrightarrow G^n$ of the preceding Lemma 2.2.9 is now used to carry over the topology of G^n into a *topology* for $C \otimes G$. Then $C \otimes G$ is a compact group and f is continuous.

Lemma 2.2.11. *i. The topology of $C \otimes G$ is independent of the choice of the base in C.*

ii. Let C, D be free groups on finite bases, and $f : C \longrightarrow D$ a homomorphism. Let G, H be compact groups and $h : G \longrightarrow H$ a homomorphism of compact groups, i.e. a continuous homomorphism. Then $f \otimes h : C \otimes G \longrightarrow D \otimes H$ is continuous.

Definition 2.2.12. Let C be a free group with a finite set of generators, and let

$$R \xrightarrow{i} F \xrightarrow{\eta} C$$

be a representation of C as a factor group of a free group F on a finite base(i is the inclusion map, and η is the natural homomorphism). Let G be a compact group. In the induced diagram

$$R \otimes G \xrightarrow{i'} F \otimes G \xrightarrow{\eta'} C \otimes G$$

$R \otimes G$ and $F \otimes G$ are compact, i' is continuous, and by Lemma 2.2.7, η' is onto and $\ker \eta' = \operatorname{Im} i'$. In this way $C \otimes G$ is isomorphic to the compact group $F \otimes G / \operatorname{Im} i'$. Using this isomorphism we carry over the topology of the factor group to provide a *topology* in $C \otimes G$. This is equivalent to defining a set $U \subset C \otimes G$ to be *open* if $\eta^{-1}(U)$ is open in $F \otimes G$.

Lemma 2.2.13. *i. If C has a finite set of generators and G is compact, the topology of $C \otimes G$ is independent of the choice of the representation $C = F/R$ in the Definition 2.2.12.*

ii. If C, D have finite sets of generators, and G, H are compact, and $f : C \longrightarrow D$, $h : G \longrightarrow H$ are homomorphisms(h is continuous), then $f \otimes h : C \otimes G \longrightarrow D \otimes H$ is continuous.

With some appropriate modifications, the previous results carry on the compact groups case, that is the resulting homomorphisms become also continuous; the result for direct sums holds true if it is restricted to finite sums and they are topological direct sums, that is not just algebraic direct sums(the inclusion and projection maps for the terms of the direct sum are required to be continuous).

Summarizing, we have:

Theorem 2.2.14. *$\mathcal{A}b$ denotes the category of abelian groups and their homomorphisms. $\mathcal{A}b'$ denotes the subcategory of $\mathcal{A}b$ consisting of finitely generated groups and their homomorphisms. $\mathcal{A}b_C$ denotes the category of compact abelian groups and their continuous homomorphisms. Then the tensor product is defined in the following cases:*

i. $C \in \mathcal{A}b$, $\quad G \in \mathcal{A}b$, $\quad$ *then* $\quad C \otimes G \in \mathcal{A}b$

ii. $C \in \mathcal{A}b'$, $\quad G \in \mathcal{A}b_C$, $\quad$ *then* $\quad C \otimes G \in \mathcal{A}b_C$

with the previous results valid in the first case and valid for the second case with some appropriate modifications.

2.3 Homology Theory with [compact] coefficient group G on the h-category [Comp$'$]Comp of Chain Complexes of [Finitely Generated] Abelian Groups with Values in the Category [$\mathcal{A}b_C$] $\mathcal{A}b$

The tensor product operation extends to chain complexes as follows:

Definition 2.3.1. If $K = \{C_q(K), \partial_q\}$ is a chain complex and G is a group, define $K \otimes G$ to be the chain complex $\{C_q(K) \otimes G, \partial_q'\}$ where ∂_q' is induced by ∂_q, i.e. $\partial_q' = \partial_q \otimes i$ where i is the identity map of G. If $f : K \longrightarrow K'$ is a map of chain complexes, let $f' : K \otimes G \longrightarrow K' \otimes G$ be the map induced by f, i.e. $f'_q = f_q \otimes i$. The resulting functor from chain complexes to chain complexes is denoted by again $\cdot \otimes G$. There are two cases. **Comp** denotes the category of chain complexes of groups in $\mathcal{Ab}$. Let **Comp**$'$ denote the category of chain complexes of finitely generated groups, i.e. groups from the category $\mathcal{Ab}'$. **Comp**$_C$ denotes the category of chain complexes of compact abelian groups and their continuous homomorphisms.

i. $G \in \mathcal{Ab}$, then $\cdot \otimes G : \mathbf{Comp} \longrightarrow \mathbf{Comp}$

ii. $G \in \mathcal{Ab}_C$, then $\cdot \otimes G : \mathbf{Comp}' \longrightarrow \mathbf{Comp}_C$

Theorem 2.3.2. *If* **Comp**, **Comp**$'$ *are regarded as h-categories in the sense of direct couples(see 2.1.4), then, in both of the two cases,* $\cdot \otimes G$ *is a covariant h-functor.*

Definition 2.3.3. According to Proposition 1.3.9, the composition of the h-functor $\cdot \otimes G$ with the homology theory H on **Comp**[**Comp**$_C$](see Section 2.1) is a new homology theory defined on the domain of $\cdot \otimes G$. It is called the *homology theory with coefficient group* G . For any chain complex K, the group $H_q(K \otimes G)$ is customarily written $H_q(K; G)$ and is called the *q-dimensional homology group of* K *with coefficients in* G, or *the* q^{th} *homology group of* K *over* G. There are two cases:

i. $G \in \mathcal{Ab}$, $K \in \mathbf{Comp}$ then $H_q(K; G) \in \mathcal{Ab}$

ii. $G \in \mathcal{Ab}_C$, $K \in \mathbf{Comp}'$ then $H_q(K; G) \in \mathcal{Ab}_C$

The group of chains, cycles, and boundaries of K with coefficients in G are written $C_q(K; G)$, $Z_q(K; G)$, and $B_q(K; G)$ rather than $C_q(K \otimes G)$, etc. In keeping with this notation the chain $c \otimes g$, where $c \in C_q(K)$ and $g \in G$, will be written gc. Thus, any element of $C_q(K; G)$ is a linear combination $\sum g_i c_i$ of elements of $C_q(K)$ with coefficients in G, and any relation is a consequence of relations of the form

$$(g_1 + g_2)c = g_1 c + g_2 c, \qquad g(c_1 + c_2) = gc_1 + gc_2.$$

Likewise the boundary operator of $K \otimes G$ is given by

$$\partial'_q(\sum g_i c_i) = \sum g_i(\partial_q c_i),$$

and, if $f : K \longrightarrow K'$ is a chain homomorphism, then $f' : K \otimes G \longrightarrow K' \otimes G$ is given by

$$f'_q(\sum g_i c_i) = \sum g_i(f_q c_i).$$

In the case when $G = \mathbb{Z}$ is the group of integers, then using the identification of $C \otimes \mathbb{Z}$ with C for any group C, we have $K \otimes \mathbb{Z} = K$ and $H_q(K; \mathbb{Z}) = H_q(K)$. Thus ordinary homology groups are regarded as those based on integer coefficients.

2.4 Simplicial Complexes

Definition 2.4.1. An *n-simplex* s is a set of $n+1$ objects called vertices, usually denoted by $\{A | A$ a vertice of $s\}$, or shortly by vertices $\{A\}$, together with the set of all real-calued functions α defined on $\{A\}$ satisfying

$$\sum_A \alpha(A) = 1, \qquad \alpha(A) \geq 0.$$

A single function α is called a point of s. The values of α on the vertices of s are called the *barycentric coordinates* of the point α. The distance $\rho(\alpha, \beta)$ of two points α, β of s is defined by

$$\rho(\alpha, \beta) = [\sum_A (\alpha(A) - \beta(A))^2]^{1/2}.$$

The topological space thus defined is denoted by $|s|$. Clearly the barycentric coordinates are continuous functions on $|s|$.

Definition 2.4.2. A simplex s together with a simple ordering $A^0 < \cdots A^q$ of its vertices is called an *ordered simplex.* The correspondence $\alpha \mapsto (\alpha(A^0), \cdots, \alpha(A^q))$ is an isometric map(i.e. a distance preserving map) $s \longrightarrow \mathbb{R}^{n+1}$, called the *canonical embedding* of s in $\mathbb{R}^{n+1}$ ($\mathbb{R}$ denotes the set of all real numbers). The image of s in $\mathbb{R}^{n+1}$ is denoted by Δ^n and is called the *unit simplex* of $\mathbb{R}^{n+1}$.

Δ^n is compact. Hence s is compact for any simplex s.

Definition 2.4.3. A *q-face* s' of an n-simplex s is a q-simplex whose vertices form a subset of the vertices of s.

A point of α' of s' is a function defined over a subset of the vertices of s. It can be extended to a function α on all vertices defined by setting $\alpha(A) = \alpha'(A)$ if A is in s', and $\alpha(A) = 0$ otherwise. Then α is clearly a point of s. The extension α of α' is necessarily unique. The map $\alpha' \longrightarrow \alpha$ imbeds $|s'|$ isometrically in $|s|$. We identify α' with α so that $|s'|$ is a subset of $|s|$, a closed subset.

A 0-simplex has just one vertex A, and just one point $\alpha(A) = 1$. It is customary to identify the vertex with the point and to denote either by A. With this convention the vertices A of s are the 0-faces of s, and are points of s. As a result, $A(B)$ is defined for any two vertices, and $A(B) = 0$ if $A \neq B$, and $A(A) = 1$ for each A. In the unit simplex Δ^n in $\mathbb{R}^{n+1}$ the vertices appear as the unit points on the coordinate axes. In addition Δ^n is the smallest convex set in $\mathbb{R}^{n+1}$ containing these unit points. For this reason the simplex is said to *span* its vertices.

Definition 2.4.4. A *simplicial complex* K is a collection of faces of a simplex s satisfying the condition that every face of a simplex in the collection is likewise in the collection. The *space* $|K|$ of K is the subset of $|s|$ consisting of those points which belong to simplexes of K.

The same simplicial complex K may lie in two different simplexes s_1 and s_2. In such a case K also lies in the simplex s spanning the vertices common to s_1 and s_2. Since the topology of s is the subspace topology of both s_1 and s_2, it follows that the topology of K is independent of the particular simplex s in terms of which it is defined.

The collection of all faces of s including s itself is a simplicial complex. This complex is also denoted by s. The collection of all faces of s excluding s itself, is a simplicial complex and is denoted by $\dot{s}$.

Definition 2.4.5. A simplicial complex K is said to be *n-dimensional*, briefly an *n-complex* provided K contains an n-simplex but no $(n+1)$-simplex(and, therefore, no simplex of dimension $> n$).

Definition 2.4.6. If K is a simplicial complex, a *subcomplex* L of K is a collection of the simplexes of K such that each face of a simplex in L is also in L. Clearly, L is a simplicial complex.

Definition 2.4.7. If K, K' are simplicial complexes and $f : |K| \longrightarrow |K'|$ is a map, then we say that f is *linear* if f is linear in terms of the barycentric coordinates. Precisely, if $\alpha, \alpha^0, \cdots, \alpha^n$ are points of $|K|$ and

$$\alpha = w_0\alpha^0 + \cdots + w_n\alpha^n, \qquad \sum_{i=0}^{n} w_i = 1, \qquad w_i \geq 0,$$

then

$$f(\alpha) = w_0 f(\alpha^0) + \cdots + w_n f(\alpha^n).$$

A linear map which carries vertices into vertices is called *simplicial.*

Let L, L' be subcomplexes of K, K' respectively. By a *linear*[*simplicial*] map $f : (K, L) \longrightarrow (K', L')$ is meant a map of $(|K|, |L|)$ into $(|K'|, |L'|)$ which defines a linear[simplicial] map of K into K'.

Proposition 2.4.8. *i. The identity map* $(K, L) \longrightarrow (K, L)$ *is simplicial.*

ii. If $f : (K, L) \longrightarrow (K', L')$ *and* $g : (K', L') \longrightarrow (K'', L'')$ *are both linear [simplicial], then* $gf : (K, L) \longrightarrow (K'', L'')$ *is linear[simplicial].*

Theorem 2.4.9. *A linear map* $f : (K, L) \longrightarrow (K', L')$ *is uniquely determined by its values on the vertices. A map* ϕ *of the vertices of* K *into points of* K' *can be extended to a linear map* $f : (K, L) \longrightarrow (K', L')$ *if and only if the* ϕ*-image of the set of vertices of any simplex of* K *or* L *is contained in a simplex of* K' *or* L' *respectively. If* ϕ *maps vertices into vertices, then* f *is simplicial.*

Theorem 2.4.10. *If* $f : (K, L) \longrightarrow (K', L')$ *is linear[simplicial], and if* $(\tilde{K}, \tilde{L})$, $(\tilde{K}', \tilde{L}')$ *are subcomplexes of* (K, L),(K', L'), *respectively, such that* f *maps* $|\tilde{K}|$ *into* $|\tilde{K}'|$ *and* $|\tilde{L}|$ *into* $|\tilde{L}'|$, *then the map of* $(\tilde{K}, \tilde{L})$ *into* $(\tilde{K}', \tilde{L}')$ *defined by* f *is linear[simplicial].*

Definition 2.4.11. Given a pair (X, A), a *triangulation* $T = \{t, (K, L)\}$ of (X, A) consists of a simplicial pair (K, L) and a homemomorphic map

$$t : (|K|, |L|) \longrightarrow (X, A).$$

The pair (X, A) together with a triangulation T is called a *triangulated* pair. If a triangulation of a pair (X, A) exists, the pair is called *triangulable.*

We do not study properties of simplicial maps, triangulated spaces, barycentric subdivision or simplicial approximation since for the results we state these definitions suffice althpugh they are certainly needed in proving some of the theorems stated.

2.5 Formal Homology Theory of Simplicial Complexes

Formal homology theory of simplicial complexes is achieved by associating a chain complex to each simplicial complex and then using the results of Section 2.3. The homology theory of simplicial complexes is constructed in two ways. The classical procedure attaches to each simplicial complex K a chain complex K_a which is called the *alternating* chain complex of K. The other procedure attaches to each K the *ordered* chain complex K_o. There is a natural mapping $K_o \longrightarrow K_a$ which induces isomorphisms of their homology groups. The first one is good for computations. We will use only the second one so we will only define that one; it suffices for our purpose.

For the formal homology theory that will be dealt the assumption that the simplicial complex be finite may be dropped:

Definition 2.5.1. Let W be an infinite set of objects called vertices. A *complex K with vertices in W* is a collection of (finite dimensional) simplexes whose vertices are in W, subject to the condition that a face of a simplex in the collection is also in the collection.

The concepts of 'subcomplex' and 'simplicial map' are introduced in the obvious way. The category of pairs of infinite complexes and simplicial maps will be denoted by $\mathbf{K_s}$. The term 'infinite' is always used in the sense of 'finite or infinite', so that the finite complexes form a subcategory of $\mathbf{K_s}$ denoted by $\mathbf{K_s}'$.

Definition 2.5.2. If K is a simplicial complex, an array $A^0 \dots A^q (q \geq 0)$ of vertices of K, included among the vertices of some simplex of K, is called an *elementary q-chain* of K. Precisely an elementary q-chain is a function which, to each integer $i = 0, \dots, q$, assigns a vertex A^i of K such that $A^0 \dots A^q$ all lie in a simplex of K. The free group generated by this set of elementary q-chains of K will be denoted by $C_q(K_o)$. By definition, $C_q(K_o) = 0$ for $q < 0$.

For each elementary q-chain $A^0 \dots A^q (q > 0)$, define

$$\partial_q(A^0 \dots A^q) = \sum_{i=0}^{q} (-1)^i A^0 \dots \hat{A}^i \dots A^q$$

where the circumflex over a vertex indicates that the vertex is omitted. Having defined ∂_q for the generators of $C_q(K_o)$, a homomorphism

$$\partial_q : C_q(K_o) \longrightarrow C_{q-1}(K_o)$$

is uniquely determined. If $q \leq 0$, then $\partial_q = 0$, by definition.

It is checked that $\partial_{q-1}\partial_q = 0$, so that $\{C_q(K_o), \partial_q\}$ is a chain complex. This chain complex will be denoted by K_o. If L is a subcomplex of K, then $C_q(L_o)$ is generated by a subset of the set generating $C_q(K_o)$, and ∂_q on L_o agrees with ∂_q on K_o. Given $c \in C_q(K_o)$ we shall write $c \subset L$ to denote that $c \in C_q(L_o)$.

Definition 2.5.3. For each simplicial pair (K, L), the chain complex K_o/L_o is called the *ordered* chain complex of the pair (K, L). The groups $C_q(K_o/L_o)$ are free groups, and if K is finite, then K_o/L_o is a finite chain complex in the sense of Definition 2.1.10.

Lemma 2.5.4. *If $f : (K, L) \longrightarrow (K', L')$ is simplicial, the homomorphisms $f_q : C_q(K_o) \longrightarrow C_q(K'_o)$ and $f_q : C_q(L_o) \longrightarrow C_q(L'_o)$ defined by*

$$f_q(A^0 \ldots A^q) = f(A^0) \ldots f(A^q)$$

define a map

$$f_o : K_o/L_o \longrightarrow K'_o/L'_o.$$

Moreover, if $f : (K, L) \longrightarrow (K, L)$ is the identity, then f_o is the identity, and if $f : (K, L) \longrightarrow (K', L')$, $g : (K', L') \longrightarrow (K'', L'')$ are simplicial maps, then $(gf)_o = g_o f_o$.

The lemma states that K_o/L_o and f_o form a covariant functor O on the category $\mathbf{K_s}$ of simplicial pairs and simplicial maps with values in the category **Comp** of chain complexes of abelian groups. We now convert $\mathbf{K_s}$ into a c-category by defining couples (i, j) to consist of the inclusion maps $i : L \longrightarrow K$, $j : K \longrightarrow (K, L)$ for each pair (K, L) in $\mathbf{K_s}$. Since i_o is the inclusion map $L_o \longrightarrow K_o$, it follows that the sequence

$$0 \longrightarrow L_o \xrightarrow{i_o} K_o \xrightarrow{j_o} K_o/L_o \longrightarrow 0$$

is exact, thus (i_o, j_o) is a couple on the category **Comp**. Since L_o is a direct summand of K_o, it follows that the couple is direct in the sense of Definition 2.1.4.

Summarizing we have:

Theorem 2.5.5. *The pair K_o/L_o, f_o forms a covariant c-functor O on the category $\mathbf{K_s}$ of simplicial complexes and simplicial maps with values in the c-category* **Comp** *of chain complexes(with only direct couples considered).*

Definition 2.5.6. Two simplicial maps $f, g : (K, L) \longrightarrow (K', L')$ are called *contiguous* if, for every simplex s of K[of L], the simplexes $f(|s|)$ and $g(|s|)$ are faces of a single simplex K'[of L']. This relation will play the role of homotopy in the category $\mathbf{K_s}$.

The term 'contiguity' is used instead of homotopy to avoid confusion with the homotopy of the maps $f, g : (|K|, |L|) \longrightarrow (|K'|, |L'|)$ of the associated topological spaces(when K and K' are finite complexes). Indeed if f and g are contiguous, then they are also homotopic but the converse is generally false.

Theorem 2.5.7. *If the simplicial maps $f, g : (K, L) \longrightarrow (K', L')$ are contiguous, then the induced chain maps $f_o, g_o : K_o/L_o \longrightarrow K'_o/L'_o$ are chain homotopic.*

Definition 2.5.8. If K' and K'' are subcomplexes of a chain complex K, we denote by $K' \cap K''$ and $K' + K''$ the subcomplexes of K defined by

$$C_q(K', K'') = C_q(K') \cap C_q(K''), \qquad C_q(K' + K'') = C_q(K') + C_q(K'')$$

Then clearly,

$$(K' \cap K'')_o = K'_o \cap K''_o, \qquad (K' + K'')_o = K'_o + K''_o$$

Definition 2.5.9. Let K' and K'' be subcomplexes of a simplicial complex K. The inclusion map

$$i : (K', K' \cap K'') \longrightarrow (K' + K'', K'')$$

is called an *excision.*

Definition 2.5.10. *Points* in the category $\mathbf{K_s}$ are defined just to be simplicial complexes consisting of a single vertex.

Having defined the concepts of homotopy, excision, and point in the c-category $\mathbf{K_s}$, it becomes an h-category and the following theorem holds:

Theorem 2.5.11. *If* **Comp** *is regarded as an h-category in the sense of direct couples, then $O : \mathbf{K_s} \longrightarrow$* **Comp** *is a covariant h-functor.*

Definition 2.5.12. According to Proposition 1.3.9, the composition of the h-functor O with the homology theory of **Comp** with coefficient group G yields an homology theory on $\mathbf{K_s}$ called the *homology theory of* $\mathbf{K_s}$ *with coefficient group* G. For any simplicial pair (K, L) the homology groups $H_q(K_o/L_o; G)$ will be written $H_q(K, L; G)$ and will be called the q^{th} *homology group of* (K, L) *with coefficient group* G. We have the following two cases:

i. $G \in \mathcal{A}b$, $(K, L) \in \mathbf{K_s}$, then $H_q(K, L; G) \in \mathcal{A}b$,

ii. $G \in \mathcal{A}b_C, \qquad (K, L) \in \mathbf{K}_{\mathbf{s}}', \qquad \text{then} \qquad H_q(K, L; G) \in \mathcal{A}b_C.$

(When K is a finite complex, i.e. $(K, L) \in \mathbf{K}_{s}'$, K_o/L_o is a finite chain complex, hence $K_o/L_o \in \mathbf{Comp}'$ so we get the second case by Definition 2.3.3.)

CHAPTER THREE
INVERSE LIMITS OF INVERSE SYSTEMS

We describe in this chapter the concept of inverse limits of inverse systems of sets, groups, topological spaces, topological groups, lowers sequences of groups, chain complexes, etc. which are used in defining the Čech homology theory in the next chapter and which form the algebraic part used in this theory. When studying exactness of the Čech homology sequence of a pair, we will use the results of Sections 3.2 and 3.3 of this chapter. Section 3.3 forms the algebraic part dealing with limits of inverse systems of exact sequences used in the proof of Theorem 4.2.2 at Section 4.2 where the exactness of the Čech homology sequence of a compact pair (X, A) over an algebraically compact coefficient group is proved.

3.1 The Category Inv($\mathfrak{a}$) of inverse systems of elements of $\mathfrak{a}$ and limit functor Inv($\mathfrak{a}$)$\to\mathfrak{a}$ for some categories $\mathfrak{a}$ whose objects are sets

Definition 3.1.1. A binary relation $\leq$ in a set M is called a *quasi-order* if it is reflexive and transitive, that is for every $\alpha, \beta, \gamma \in M$,

i. $\alpha \leq \alpha$.

ii. $\alpha \leq \beta$ and $\beta \leq \gamma$ implies $\alpha \leq \gamma$

In general, $\alpha \leq \beta$ and $\beta \leq \alpha$ does *not* imply that $\alpha = \beta$(i.e. we do not take the anti-symmetry condition as in a a partial order).

A *directed set* M is a quasi-ordered set, with $\leq$ denoting quasi-order, such that for each pair $\alpha, \beta \in M$, there exists a $\gamma \in M$ for which $\alpha \leq \gamma$ and $\beta \leq \gamma$. A directed set M' is said to be a *subdirected set of* a directed set M if $M' \subset M$ and $\alpha \leq \beta$ in M' implies $\alpha \leq \beta$ in M. A subdirected set M' of M is *cofinal* in M if, for each $\alpha \in M'$ there exists a β in M such that $\alpha \leq \beta$. If M and N are directed sets, a *map* $\phi : M \longrightarrow N$ *of directed sets* is an order-preserving function from M to N, i.e. $\alpha \leq \beta$ in M implies $\phi\alpha \leq \phi\beta$ in N.

Definition 3.1.2. An *inverse system of sets* $\{X_\alpha(\alpha \in M); \pi_\alpha^\beta\}$(or shortly $\{X, \pi\}$) *over a directed set* M is a function which attaches to each $\alpha \in M$ a set X_α, and to each pair α, β such that $\alpha \leq \beta$ in M, a map

$$\pi_\alpha^\beta : X_\beta \longrightarrow X_\alpha$$

such that

$$\pi_\alpha^\alpha = identity, \qquad \alpha \in M$$

$$\pi_\alpha^\beta \pi_\beta^\gamma = \pi_\alpha^\gamma, \qquad \alpha \leq \beta \leq \gamma \in M.$$

We denote $\{X_\alpha(\alpha \in M); \pi_\alpha^\beta\}$ also shortly by $\{X, \pi\}$ with the meaning that the appropriate indices are put when they are used; in the same sense we shortly write π for π_α^β.
The maps π_α^β are called *projections* of the system. If each X_α is a topological space, or an abelian group, or a topological group, and each projection is respectively, continuous, or group homomorphism, or a continuous homomorphism then $\{X_\alpha(\alpha \in M); \pi_\alpha^\beta\}$(shortly $\{X, \pi\}$) is called an *inverse system of*, respectively, *topological spaces, groups, or topological groups.*

A directed set M becomes a category if each relation $\alpha \leq \beta$ is regarded as a map $\alpha \longrightarrow \beta$. Then an inverse system over M is simply a covariant functor from M to the category of sets and maps, or to the category of groups and homomorphisms, or to the category of topological groups and continuous homomorphisms.

Definition 3.1.3. Let $\{X_\alpha(\alpha \in M); \pi_\alpha^\beta\}$ and $\{X'_{\alpha'}(\alpha' \in M'); \pi'^{\beta'}_{\alpha'}\}$ be inverse systems over M and M' respectively. Then a *map*

$$\Phi : \{X_\alpha(\alpha \in M); \pi_\alpha^\beta\} \longrightarrow \{X'_{\alpha'}(\alpha' \in M'); \pi'^{\beta'}_{\alpha'}\}$$

of inverse systems (shortly $\Phi : \{X, \pi\} \longrightarrow \{X', \pi'\}$) consists of a map

$$\phi : M' \longrightarrow M$$

of directed sets, and, for each $\alpha' \in M'$, a map

$$\phi_{\alpha'} : X_{\phi\alpha'} \longrightarrow X'_{\alpha'}, \qquad \alpha' \in M'$$

such that, if $\alpha' \leq \beta'$ in M', then commutativity holds in the diagram

$$\begin{array}{ccc} X_{\phi\alpha'} & \xleftarrow{\ \pi\ } & X_{\phi\beta'} \\ \phi \downarrow & & \downarrow \phi \\ X'_{\alpha'} & \xleftarrow[\ \pi'\ ]{} & X'_{\beta'} \end{array}$$

where π, π' and ϕ have appropriate indices.
Whenever both of the inverse systems are inverse systems of topological spaces, groups, or topological groups, the components $\phi_{\alpha'}$ of the map Φ are regarded to be continuous, or homomorphic, or continuously homomorphic.
In case

$$\Phi : \{X_\alpha(\alpha \in M); \pi_\alpha^\beta\} \longrightarrow \{X'_{\alpha'}(\alpha' \in M'); \pi'^{\beta'}_{\alpha'}\},$$

$$\Phi' : \{X'_{\alpha'}(\alpha' \in M'); \pi'^{\beta'}_{\alpha'}\} \longrightarrow \{X''_{\alpha''}(\alpha'' \in M''); \pi''^{\beta''}_{\alpha''}\}$$

are two maps of inverse systems, their *composition*

$$\Phi'\Phi : \{X_\alpha(\alpha \in M); \pi_\alpha^\beta\} \longrightarrow \{X''_{\alpha''}(\alpha'' \in M''); \pi''^{\beta''}_{\alpha''}\}$$

is defined to consist of the compositions

$$\phi\phi' \text{ and } \phi'_{\alpha''}\phi_{\phi'\alpha''}, \quad \alpha'' \in M''.$$

The identity map Φ of $\{X_\alpha(\alpha \in M); \pi_\alpha^\beta\}$ is composed of the identities $\phi : M \longrightarrow M$ and $\phi_\alpha : X_\alpha \longrightarrow X_\alpha$, $\alpha \in M$.

Proposition 3.1.4. *Let* $\mathfrak{a}$ *be any of the following categories: sets and maps, spaces and continuous maps, compact spaces and continuous maps, groups and homomorphisms, topological groups and continuous homomorphisms, compact groups and continuous homomorphisms. Inverse systems and maps of such, all of whose elements belong to* $\mathfrak{a}$ *forms a category denoted by* $\mathbf{Inv}(\mathfrak{a})$.

Definition 3.1.5. Let $\{X_\alpha(\alpha \in M); \pi_\alpha^\beta\}$ be an inverse system of sets over a directed set M. The *inverse limit* X_∞(briefly *limit*) of $\{X_\alpha(\alpha \in M); \pi_\alpha^\beta\}$ is the subset of the product

$$\prod_{\alpha \in M} X_\alpha$$

consisting of those functions $x = (x_\alpha)_{\alpha \in M}$(shortly $x = (x_\alpha)$) such that, for each relation $\alpha \leq \beta$ in M,

$$\pi_\alpha^\beta(x_\beta) = x_\alpha.$$

Define the projection

$$\pi_\beta : X_\infty \longrightarrow X_\beta \quad \text{by}$$

$$\pi_\beta(x) = x_\beta \quad \text{for each } x = (x_\alpha)_{\alpha \in M} \in \prod_{\alpha \in M} X_\alpha.$$

If $\{X, \pi\}$ is an inverse system of topological spaces, then X_∞ is assigned the topology it has as a subspace of $\prod_\alpha X_\alpha$. If $\{X, \pi\}$ is an inverse system of abelian groups, then it is easily seen that X_∞ is a subgroup of $\prod_\alpha X_\alpha$ and X_∞ is assigned this structure of a group. Similarly, an inverse limit of an inverse system of topological groups is a topological group.

Definition 3.1.6. Let $\Phi : \{X_\alpha(\alpha \in M); \pi_\alpha^\beta\} \longrightarrow \{X'_{\alpha'}(\alpha' \in M'); \pi'^{\beta'}_{\alpha'}\}$ be a map of one inverse system into another. The *inverse limit* ϕ_∞ of Φ is a map

$$\phi_\infty : X_\infty \longrightarrow X'_\infty$$

defined as follows: If $x \in X_\infty$ and $\alpha \in M'$, set $x'_\alpha = \phi_\alpha(x_{\phi(\alpha)})$. If $\alpha \leq \beta$ in M', it follows by the definition of a map of inverse systems that $\pi'^{\beta}_{\alpha}(x'_\beta) = x'_\alpha$. Therefore $x' = (x'_\alpha)_{\alpha \in M'}$ is an element of X'_∞. Define $\phi_\infty(x) = x'$.

Theorem 3.1.7. *(Limit functor* $\mathbf{Inv}(\mathfrak{a}) \longrightarrow \mathfrak{a}$*) Let* $\mathfrak{a}$ *be any of the following categories:*

sets and maps,

spaces and continuous maps,

compact spaces and continuous maps,

groups and homomorphisms,

topological groups and continuous homomorphisms,

compact groups and continuous homomorphisms.

Let $\mathbf{Inv}(\mathfrak{a})$ *denote the category of inverse systems and maps of such, all of whose elements belong to* $\mathfrak{a}$. *Then the operations of assigning an inverse limit* X_∞ *to each inverse system* $\{X_\alpha(\alpha \in M); \pi_\alpha^\beta\}$ *(shortly* $\{X, \pi\}$*) in* $\mathbf{Inv}(\mathfrak{a})$ *and an inverse limit* $\phi_\infty : X_\infty \longrightarrow X'_\infty$ *to each map*

$$\Phi : \{X_\alpha(\alpha \in M); \pi_\alpha^\beta\} \longrightarrow \{X'_{\alpha'}(\alpha' \in M'); \pi'^{\beta'}_{\alpha'}\}$$

(shortly $\Phi : \{X, \pi\} \longrightarrow \{X', \pi'\}$*) of inverse systems in* $\mathbf{Inv}(\mathfrak{a})$, *form a covariant functor from* $\mathbf{Inv}(\mathfrak{a})$ *to* $\mathfrak{a}$.

3.2 Inverse Systems of Lower Sequences of Abelian Groups and Their Inverse Limits

Definition 3.2.1. An *inverse system of lower sequences* $\{S_\alpha(\alpha \in M); \pi_\alpha^\beta\}$ (shortly $\{S, \pi\}$) over a directed set M is a function which attaches to each $\alpha \in M$ a lower sequence

$$S_\alpha = \{S_{\alpha,q}, \phi_{\alpha,q}\}_{q \in \mathbb{Z}} : \quad \ldots \longrightarrow S_{\alpha,q} \xrightarrow{\phi_{\alpha,q}} S_{\alpha,q-1} \xrightarrow{\phi_{\alpha,q-1}} \ldots$$

and to each relation $\alpha \leq \beta$ in M, a homomorphism $\pi_\alpha^\beta : S_\beta \longrightarrow S_\alpha$ of lower sequences

$$\begin{array}{cccccc}
S_\beta : & \cdots \longrightarrow & S_{\beta,q} & \xrightarrow{\phi_{\beta,q}} & S_{\beta,q-1} & \longrightarrow \cdots \\
\downarrow \pi_\alpha^\beta & & \downarrow \pi_{\alpha,q}^\beta & & \downarrow \pi_{\alpha,q-1}^\beta & \\
S_\alpha : & \cdots \longrightarrow & S_{\alpha,q} & \xrightarrow[\phi_{\alpha,q}]{} & S_{\alpha,q-1} & \longrightarrow \cdots
\end{array}$$

i.e. homomorphisms $\pi_{\alpha,q}^\beta : S_{\beta,q} \longrightarrow S_{\alpha,q}$ for each $q \in \mathbb{Z}$ such that

$$\phi_{\alpha,q} \circ \pi_{\alpha,q}^\beta = \pi_{\alpha,q-1}^\beta \circ \phi_{\beta,q}$$

satisfying

$$\pi_\alpha^\alpha = identity, \qquad \text{and}$$

$$\pi_\alpha^\beta \pi_\beta^\gamma = \pi_\alpha^\gamma \qquad \text{if } \alpha \leq \beta \leq \gamma \text{ in } M$$

Then for any fixed q, the groups and homomorphisms $\{S_{\alpha,q}(\alpha \in M); \pi_{\alpha,q}^\beta\}$ form an inverse system of groups over M; denote its limit group by $S_{\infty,q}$. Again for a fixed q, the homomorphisms $\phi_{\alpha,q}$, $\alpha \in M$ together with identity map of M form a map

$$\Phi_q : \{S_{\alpha,q}(\alpha \in M); \pi_{\alpha,q}^\beta\} \longrightarrow \{S_{\alpha,q-1}(\alpha \in M); \pi_{\alpha,q-1}^\beta\}$$

of inverse systems. Denote the limit of Φ_q by

$$\phi_{\infty,q} : S_{\infty,q} \longrightarrow S_{\infty,q-1}.$$

The lower sequence $S_\infty = \{S_{\infty,q}, \phi_{\infty,q}\}_{q\in\mathbb{Z}}$ so obtained is called the *inverse limit* of the system $\{S_\alpha(\alpha \in M); \pi_\alpha^\beta\}$(shortly $\{S, \pi\}$). In this definition it is to be understood that all groups and homomorphisms in an inverse system of lower sequences belong to the category $\mathcal{Ab}$ of abelian groups or to the category $\mathcal{Ab}_C$ of compact abelian groups; then the limit sequence is also of the same type. Notation of this definition will be used when we work with lower sequences of groups in the next section.

Definition 3.2.2. A lower sequence $\{G_q, \phi_q\}_{q\in\mathbb{Z}}$ is said to be of *order 2* if the composition of any two successive homomorphisms of the sequence is zero, i.e. $\phi_q \circ \phi_{q+1} = 0$ for every $q \in \mathbb{Z}$ so $\mathrm{Ker}(\phi_q) \supset \mathrm{Im}(\phi_{q+1})$. Although this notion coincides with that of chain complex, the 'order 2' language is preferred whenever the lower sequence in question is not treated as a chain complex.

It is not true that the limit sequence of an inverse system of exact lower sequences is exact but we have:

Theorem 3.2.3. *If each sequence of an inverse system of lower sequences is of order 2, then the limit sequence is also of order 2.*

Theorem 3.2.4. *Let $\{S_\alpha(\alpha \in M); \pi_\alpha^\beta\}$(shortly $\{S, \pi\}$) be an inverse system of exact lower sequences over a directed set M where all groups and homomorphisms* *of $\{S,\pi\}$ belong to the category $\mathcal{A}b_C$ of compact abelian groups(i.e. the groups are compact topological groups and homomorphisms are continuous homomorphisms). Then the limit sequence S_∞ of $\{S,\pi\}$ is also exact.*

3.3 The category $\mathbf{Inv}_M(\mathbf{Comp})$ and limit functor $\mathbf{Inv}_M(\mathbf{Comp}) \to \mathbf{Comp}$

Remember that by **Comp** we denoted the category of chain complexes of abelian groups and their chain homomorphisms. Lower sequences of order 2 are chain complexes; they are the same according to their definitions but the terminology 'chain complex' is used generally if from it homology groups are obtained. The lower sequences we will work in the next section when discussing exactness of Čech homology will themselves be homology sequences(of a simplicial complex) and will not be treated as a chain complex, but we will not create a different notation for the lower sequences of order 2; we will again use **Comp** to denote the category of lower sequences of order 2(=chain complexes) and their chain homomorphisms. We will not care much about this now and use chain complex and lower sequence of order 2 interchangeably.

Definition 3.3.1. Fix a directed set M. By $\mathbf{Inv}_M(\mathbf{Comp})$ we will denote the 'category' of inverse systems of chain complexes over the directed set M. The verification that it is a category with the definition of its objects and maps given below is done in the proof of the theorem that follows.

The objects of $\mathbf{Inv}_M(\mathbf{Comp})$ are inverse systems $\{S_\alpha(\alpha \in M);\pi_\alpha^\beta\}$(shortly $\{S,\pi\}$) of lower sequences of order 2(we have defined in Definition 3.2.1 what an inverse system of lower sequences is). Maps(morphisms) of $\mathbf{Inv}_M(\mathbf{Comp})$ are defined as follows:For $\{S_\alpha(\alpha \in M);\pi_\alpha^\beta\}$(shortly $\{S,\pi\}$) and $\{S'_\alpha(\alpha \in M);\pi'^\beta_\alpha\}$ (shortly $\{S',\pi'\}$), a *map*

$$\Psi : \{S,\pi\} \longrightarrow \{S',\pi'\}$$

consists of chain homomorphisms

$$\psi_\alpha : S_\alpha \longrightarrow S'_\alpha, \qquad \alpha \in M$$

such that if $\alpha \leq \beta$ in M, then commutativity holds in the diagram

$$\begin{array}{ccc} S_\alpha & \xleftarrow{\pi_\alpha^\beta} & S_\beta \\ {\scriptstyle \psi_\alpha}\downarrow & & \downarrow{\scriptstyle \psi_\beta} \\ S'_\alpha & \xleftarrow[\pi'^{\beta}_{\alpha}]{} & S'_\beta \end{array}$$

We denote a map in $\mathbf{Inv}_M(\mathbf{Comp})$ by capital greek letters(like Ψ) and its 'components' by small letters(like ψ_α as above or by Ψ_α or by $(\Psi)_\alpha$). *Composition* of two maps

$$\Psi : \{S, \pi\} \longrightarrow \{S', \pi'\}, \qquad \Psi' : \{S', \pi'\} \longrightarrow \{S'', \pi''\}$$

in $\mathbf{Inv}_M(\mathbf{Comp})$ is defined as the map

$$\Psi'\Psi : \{S, \pi\} \longrightarrow \{S'', \pi''\}$$

given by

$$(\Psi'\Psi)_\alpha = \psi'_\alpha \psi_\alpha, \qquad \alpha \in M.$$

Identity map of $\{S, \pi\}$ in $\mathbf{Inv}_M(\mathbf{Comp})$ is the map

$$i_{\{S,\pi\}} : \{S, \pi\} \longrightarrow \{S, \pi\}$$

given by

$$(i_{\{S,\pi\}})_\alpha = i_{S_\alpha}, \qquad \alpha \in M$$

where $i_{S_\alpha} : S_\alpha \longrightarrow S_\alpha$ is the identity chain homomorphism of S_α. The notation of this definiton will be used throughout when working in the category $\mathbf{Inv}_M(\mathbf{Comp})$.

Theorem 3.3.2. $\mathbf{Inv}_M(\mathbf{Comp})$ *forms a category.*

Proof. Firstly, let us verify that the composition $\Psi'\Psi$ in the definition is a map in $\mathbf{Inv}_M(\mathbf{Comp})$. Since Ψ' and Ψ are maps in $\mathbf{Inv}_M(\mathbf{Comp})$, we have the following

Composition of chain homomorphisms is a chain homomorphism(in the category **Comp**), so

$$\psi'_\alpha\psi_\alpha : S_\alpha \longrightarrow S''_\alpha$$

is a chain homomorphism for every $\alpha \in M$. The diagram

$$\begin{array}{ccc} S_\alpha & \xleftarrow{\pi_\alpha^\beta} & S_\beta \\ {\scriptstyle \psi'_\alpha\psi_\alpha}\downarrow & & \downarrow{\scriptstyle \psi'_\beta\psi_\beta} \\ S''_\alpha & \xleftarrow[\pi''^\beta_\alpha]{} & S''_\beta \end{array}$$

is commutative as

$$(\psi'_\alpha\psi_\alpha) \circ \pi_\alpha^\beta = \psi'_\alpha \circ (\psi_\alpha \circ \pi_\alpha^\beta) = (\psi'_\alpha \circ \pi'^\beta_\alpha) \circ \psi_\beta = \pi''^\beta_\alpha \circ (\psi'_\beta\psi_\beta)$$

because by commutativity of the two previous diagrams

$$\psi_\alpha \circ \pi_\alpha^\beta = \pi'^\beta_\alpha \circ \psi_\beta \qquad \text{and} \qquad \psi'_\alpha \circ \pi'^\beta_\alpha = \pi''^\beta_\alpha \circ \psi'_\beta$$

Composition of maps in $\mathbf{Inv}_M(\mathbf{Comp})$ is 'associative' because composition of chain homomorphisms is associative: for maps

$$\Psi : \{S,\pi\} \longrightarrow \{S',\pi'\},\ \Psi' : \{S',\pi'\} \longrightarrow \{S'',\pi''\},\ \Psi'' : \{S'',\pi''\} \longrightarrow \{S''',\pi'''\}$$

in $\mathbf{Inv}_M(\mathbf{Comp})$, we have for every $\alpha \in M$

$$[\Psi''(\Psi'\Psi)]_\alpha = \psi''_\alpha(\psi'_\alpha\psi_\alpha) = (\psi''_\alpha\psi'_\alpha)\psi_\alpha) = [(\Psi''\Psi')\Psi]_\alpha$$

which implies $\Psi''(\Psi'\Psi) = (\Psi''\Psi')\Psi$. For $\{S,\pi\}$ in $\mathbf{Inv}_M(\mathbf{Comp})$, the identity 'map'

$$i_{\{S,\pi\}} : \{S,\pi\} \longrightarrow \{S,\pi\}$$

is really a map in $\mathbf{Inv}_M(\mathbf{Comp})$, because the diagram

$$\begin{array}{ccc} S_\alpha & \xleftarrow{\pi_\alpha^\beta} & S_\beta \\ {\scriptstyle i_{S_\alpha}}\downarrow & & \downarrow{\scriptstyle i_{S_\beta}} \\ S_\alpha & \xleftarrow[\pi_\alpha^\beta]{} & S_\beta \end{array}$$

is clearly commutative as i_{S_α} and i_{S_β} are identity maps of chain complexes. That the map $i_{\{S,\pi\}}$ is an 'identity map' in the category $\mathbf{Inv}_M(\mathbf{Comp})$ follows because for maps

$$\Psi : \{S,\pi\} \longrightarrow \{S',\pi'\}, \qquad \Psi' : \{S',\pi'\} \longrightarrow \{S,\pi\}$$

is clearly commutative as i_{S_α} and i_{S_β} are identity maps of chain complexes. That the map $i_{\{S,\pi\}}$ is an 'identity map' in the category $\mathbf{Inv}_M(\mathbf{Comp})$ follows because for maps

$$\Psi : \{S,\pi\} \longrightarrow \{S',\pi'\}, \qquad \Psi' : \{S',\pi'\} \longrightarrow \{S,\pi\}$$

in $\mathbf{Inv}_M(\mathbf{Comp})$, we have for each $\alpha \in M$

$$(\Psi \circ i_{\{S,\pi\}})_\alpha = \psi_\alpha i_{S_\alpha} = \psi_\alpha$$

and

$$(i_{\{S,\pi\}} \circ \psi')_\alpha = i_{S_\alpha} \psi'_\alpha = \psi'_\alpha$$

which give

$$\Psi \circ i_{\{S,\pi\}} = \Psi \qquad \text{and} \qquad i_{\{S,\pi\}} \circ \Psi' = \Psi'.$$

All these show that $\mathbf{Inv}_M(\mathbf{Comp})$ forms a category with the Definition 3.3.1 □

Definition 3.3.3. Let $\Psi : \{S,\pi\} \longrightarrow \{S',\pi'\}$ be a map in $\mathbf{Inv}_M(\mathbf{Comp})$. We will follow the notation given in the Definitions 3.3.1 and 3.2.1. For every $q \in \mathbb{Z}$ and $\alpha \leq \beta$ in M, the diagram

$$\begin{array}{ccc} S_{\alpha,q} & \xleftarrow{\pi^{\beta}_{\alpha,q}} & S_{\beta,q} \\ {\scriptstyle \psi_{\alpha,q}}\downarrow & & \downarrow{\scriptstyle \psi_{\beta,q}} \\ S'_{\alpha,q} & \xleftarrow[\pi'^{\beta}_{\alpha,q}]{} & S'_{\beta,q} \end{array}$$

is commutative(because the diagram without the q's is commutative by the definition of a map in $\mathbf{Inv}_M(\mathbf{Comp})$) so the map

$$\Psi_q : \{S_{\alpha,q}(\alpha \in M); \pi^{\beta}_{\alpha,q}\} \longrightarrow \{S'_{\alpha,q}(\alpha \in M); \pi'^{\beta}_{\alpha,q}\}$$

which is defined to consist of the identity map of M and $(\Psi_q)_\alpha = \psi_{\alpha,q}, \alpha \in M$ is a map of inverse systems of abelian groups, i.e. a map in the category $\mathbf{Inv}(\mathcal{Ab})$. Hence we can pass to limit by Theorem 3.1.7 to obtain a homomorphism

$$\psi_{\infty,q} : S_{\infty,q} \longrightarrow S'_{\infty,q}.$$

of groups for each $q \in \mathbb{Z}$. The map

$$\psi_\infty : S_\infty \longrightarrow S'_\infty$$

of lower sequences defined by those $\psi_{\infty,q}, q \in \mathbb{Z}$ is called the *inverse limit* of the map $\Psi : \{S,\pi\} \longrightarrow \{S',\pi'\}$.

Proof. For each $q \in \mathbb{Z}$, the following diagram in $\mathbf{Inv}(\mathcal{Ab})$ is commutative:

$$\begin{array}{ccc}
\{S_{\alpha,q}(\alpha \in M); \pi^{\beta}_{\alpha,q}\} & \xrightarrow{\Psi_q} & \{S'_{\alpha,q}(\alpha \in M); \pi'^{\beta}_{\alpha,q}\} \\
\Phi_q \downarrow & & \downarrow \Phi'_q \\
\{S_{\alpha,q-1}(\alpha \in M); \pi^{\beta}_{\alpha,q-1}\} & \xrightarrow[\Psi_{q-1}]{} & \{S'_{\alpha,q-1}(\alpha \in M); \pi'^{\beta}_{\alpha,q-1}\}
\end{array}$$

because for every $\alpha \in M$

$$\psi_{\alpha,q-1} \circ \phi_{\alpha,q} = \phi'_{\alpha,q} \circ \psi_{\alpha,q}$$

as $\psi_\alpha : S_\alpha \longrightarrow S'_\alpha$ is a chain homomorphism. Thus applying the limit functor $\mathbf{Inv}(\mathcal{Ab}) \longrightarrow \mathcal{Ab}$ by Theorem 3.1.7, we get the commutative diagram

$$\begin{array}{ccc}
S_{\infty,q} & \xrightarrow{\psi_{\infty,q}} & S'_{\infty,q} \\
\phi_{\infty,q} \downarrow & & \downarrow \phi'_{\infty,q} \\
S_{\infty,q-1} & \xrightarrow[\psi_{\infty,q-1}]{} & S'_{\infty,q-1}
\end{array}$$

that is, for every $q \in \mathbb{Z}$

$$\psi_{\infty,q-1} \circ \phi_{\infty,q} = \phi'_{\infty,q} \circ \psi_{\infty,q}.$$

But this means that $\psi_\infty : S_\infty \longrightarrow S'_\infty$ is a chain homomorphism. □

Theorem 3.3.5. *Let* **Comp** *denote the category of chain complexes(=lower sequences of order 2) of abelian groups(i.e. groups in* $\mathcal{Ab}$*) and their chain homomorphisms, and* $\mathbf{Inv}_M(\mathbf{Comp})$ *denote the category of inverse systems of chain complexes of abelian groups over a directed set* M*. Then the operation of assigning an inverse limit* S_∞ *to each inverse system* $\{S_\alpha(\alpha \in M); \pi^{\beta}_{\alpha}\}$ *(shortly* $\{S, \pi\}$*) in* $\mathbf{Inv}_M(\mathbf{Comp})$ *and an inverse limit map* $\psi_\infty : S_\infty \longrightarrow S'_\infty$ *in* **Comp** *to each map* $\Psi : \{S, \pi\} \longrightarrow \{S', \pi'\}$ *in* $\mathbf{Inv}_M(\mathbf{Comp})$ *forms a covariant functor from* $\mathbf{Inv}_M(\mathbf{Comp})$ *to* **Comp***, which we call as the limit functor*
$\mathbf{Inv}_M(\mathbf{Comp}) \longrightarrow \mathbf{Comp}$.

Proof. By Theorem 3.2.3, it follows that the limit sequence S_∞ of an inverse system $\{S, \pi\}$ in $\mathbf{Inv}_M(\mathbf{Comp})$ is also a chain complex, thus in **Comp**, because a chain complex is just a lower sequence of order 2.

Proof. By Theorem 3.2.3, it follows that the limit sequence S_∞ of an inverse system $\{S,\pi\}$ in $\mathbf{Inv}_M(\mathbf{Comp})$ is also a chain complex, thus in **Comp**, because a chain complex is just a lower sequence of order 2.

For the identity map $i_{\{S,\pi\}} : \{S,\pi\} \longrightarrow \{S,\pi\}$ of $\{S,\pi\}$ in $\mathbf{Inv}_M(\mathbf{Comp})$,

$$(i_{\{S,\pi\}})_{\infty,q} : S_{\infty,q} \longrightarrow S_{\infty,q}$$

is the identity map for every $q \in \mathbb{Z}$ because it is obtained by applying the limit functor $\mathbf{Inv}(\mathcal{Ab}) \longrightarrow \mathcal{Ab}$ (by Theorem 3.1.7) to the identity map

$$(i_{\{S,\pi\}})_q : \{S_{\alpha,q}(\alpha \in M); \pi^\beta_{\alpha,q}\} \longrightarrow \{S_{\alpha,q}(\alpha \in M); \pi^\beta_{\alpha,q}\}$$

in $\mathbf{Inv}(\mathcal{Ab})$. Thus

$$(i_{\{S,\pi\}})_\infty : S_\infty \longrightarrow S_\infty$$

is the identity map of S_∞ in **Comp**.

For maps

$$\Psi : \{S,\pi\} \longrightarrow \{S',\pi'\}, \qquad \Psi' : \{S',\pi'\} \longrightarrow \{S'',\pi''\}$$

in $\mathbf{Inv}_M(\mathbf{Comp})$, we must check that

$$(\Psi'\Psi)_\infty = \Psi'_\infty \Psi_\infty.$$

Let $q \in \mathbb{Z}$.

$$(\Psi'\Psi)_{\infty,q} : S_\infty \longrightarrow S''_\infty$$

is the limit of the map

$$(\Psi'\Psi)_q : \{S_{\alpha,q}(\alpha \in M); \pi^\beta_{\alpha,q}\} \longrightarrow \{S''_{\alpha,q}(\alpha \in M); \pi''^\beta_{\alpha,q}\}$$

in $\mathbf{Inv}(\mathcal{Ab})$. But that map in $\mathbf{Inv}(\mathcal{Ab})$ is the composition of the maps

$$\Psi'_q : \{S'_{\alpha,q}(\alpha \in M); \pi'^\beta_{\alpha,q}\} \longrightarrow \{S''_{\alpha,q}(\alpha \in M); \pi''^\beta_{\alpha,q}\}$$

and

$$\Psi_q : \{S_{\alpha,q}(\alpha \in M); \pi^\beta_{\alpha,q}\} \longrightarrow \{S'_{\alpha,q}(\alpha \in M); \pi'^\beta_{\alpha,q}\}$$

Theorem 3.3.6. *Let $\{S_\alpha(\alpha \in M); \pi_\alpha^\beta\}$ (shortly $\{S, \pi\}$) and $\{S'_\alpha(\alpha \in M); {\pi'}_\alpha^\beta\}$ (shortly $\{S', \pi'\}$) be inverse systems of exact lower sequences of abelian groups over a directed set M. Assume that we have maps*

$$f : \{S', \pi'\} \longrightarrow \{S, \pi\} \qquad \textit{and} \qquad g : \{S, \pi\} \longrightarrow \{S', \pi'\}$$

in the category $\mathbf{Inv}_M(\mathbf{Comp})$ of inverse systems of chain complexes(of groups in $\mathcal{A}b$) over the directed set M such that

$$g \circ f = i_{\{S', \pi'\}}$$

where $i_{\{S',\pi'\}}$ is the identity map of $\{S', \pi'\}$ in $\mathbf{Inv}_M(\mathbf{Comp})$[so for every $\alpha \in M$ the chain homomorphisms $f_\alpha : S'_\alpha \longrightarrow S_\alpha$ and $g_\alpha : S_\alpha \longrightarrow S'_\alpha$ satisfy $g_\alpha \circ f_\alpha = i_{S'_\alpha}$ so each S'_α is a direct summand of S_α, hence each $S'_{\alpha,q}$ is a direct summand of $S_{\alpha,q}$ for each integer q]. Then

i. *S'_∞ is a direct summand of S_∞.*

ii. *If S_∞ is exact, then S'_∞ is also exact.*

Proof. i. Applying the limit functor $\mathbf{Inv}_M(\mathbf{Comp}) \longrightarrow \mathbf{Comp}$ of the previous Theorem(3.3.5) to

$$g \circ f = i_{\{S', \pi'\}}$$

in $\mathbf{Inv}_M(\mathbf{Comp})$, we get

$$g_\infty \circ f_\infty = i_{S'_\infty}.$$

So by Proposition 1.1.12, S'_∞ is a direct summand of S_∞ hence, $S'_{\infty,q}$ is a direct summand of $S_{\infty,q}$ for each $q \in \mathbb{Z}$.

ii. Since $f_\infty : S'_\infty \longrightarrow S_\infty$ and $g_\infty : S_\infty \longrightarrow S'_\infty$ are chain homomorphisms, we get the following 'commutative' diagram for every $q \in \mathbb{Z}$, that is each square in the following diagram is commutative in the sense described in Definition 1.1.6:

$$\begin{array}{ccccc}
S'_{\infty,q+1} & \xrightarrow{\phi'_{\infty,q+1}} & S'_{\infty,q} & \xrightarrow{\phi'_{\infty,q}} & S'_{\infty,q-1} \\
f_{\infty,q+1} \downarrow \uparrow g_{\infty,q+1} & & f_{\infty,q} \downarrow \uparrow g_{\infty,q} & & f_{\infty,q-1} \downarrow \uparrow g_{\infty,q-1} \\
S_{\infty,q+1} & \xrightarrow[\phi_{\infty,q+1}]{} & S_{\infty,q} & \xrightarrow[\phi_{\infty,q}]{} & S_{\infty,q-1}
\end{array}$$

We already know that S'_∞ is a chain complex, i.e. image⊂kernel. So to show exactness, we must show for each $q \in \mathbb{Z}$ that

$$\operatorname{Ker} \phi'_{\infty,q} \subset \operatorname{Im} \phi'_{\infty,q+1}.$$

Take $x \in \operatorname{Ker} \phi'_{\infty,q}$, so $\phi'_{\infty,q}(x) = 0$. Then

$$f_{\infty,q-1} \circ \phi'_{\infty,q}(x) = 0.$$

By commutativity of the previous diagram,

$$f_{\infty,q-1} \circ \phi'_{\infty,q} = \phi_{\infty,q} \circ f_{\infty,q}.$$

So we get,

$$\phi_{\infty,q}(f_{\infty,q}(x)) = 0,$$

that is,

$$f_{\infty,q}(x) \in \operatorname{Ker} \phi_{\infty,q}.$$

By exactness of S_∞,

$$\operatorname{Ker} \phi_{\infty,q} = \operatorname{Im} \phi_{\infty,q+1}.$$

Thus,

$$f_{\infty,q}(x) = \phi_{\infty,q+1}(y)$$

for some $y \in S_{\infty,q+1}$. Then,

$$g_{\infty,q} \circ f_{\infty,q}(x) = g_{\infty,q} \circ \phi_{\infty,q+1}(y)$$

By commutativity of the previous diagram,

$$g_{\infty,q} \circ \phi_{\infty,q+1} = \phi'_{\infty,q+1} \circ f_{\infty,q+1}$$

and since $g_\infty \circ f_\infty = i_{S'_\infty}$,

$$g_{\infty,q} \circ f_{\infty,q} = i_{S'_{\infty,q}}$$

where $i_{S'_{\infty,q}}$ is the identity map of $S'_{\infty,q}$. So we obtain

$$i_{S'_{\infty,q}}(x) = \phi'_{\infty,q+1} \circ f_{\infty,q+1}(x)$$

that is,

$$x = \phi'_{\infty,q+1}(f_{\infty,q+1}(y)) \in \operatorname{Im} \phi'_{\infty,q+1}$$

as required.

□

Corollary 3.3.7. *If in the previous Theorem 3.3.6, we assume further that $\{S, \pi\}$ is an inverse system of exact lower sequences over M where all groups and homomorphisms of $\{S, \pi\}$ belong to the category $\mathcal{Ab}_C$ of compact abelian groups(so also belongs to the category $\mathcal{Ab}$ of abelian groups), then it follows that S'_∞ is exact as well as S_∞ and is a direct summand of S_∞, so S'_∞ is an exact lower sequence of algebraically compact groups and homomorphisms(just ordinary group homomorphisms).*

Proof. By Theorem 3.2.4, S_∞ is an exact lower sequence. So by Theorem 3.3.6, S'_∞ is a direct summand of S_∞ and it is exact. Since for each $q \in \mathbb{Z}$, $S'_{\infty,q}$ is a direct summand of $S_{\infty,q}$, which is a compact abelian group, each $S'_{\infty,q}$ is algebraically compact by definition. □

CHAPTER FOUR

THE ČECH HOMOLOGY THEORY:A PARTIALLY EXACT HOMOLOGY THEORY

The Čech homology theory is defined on the category $\mathfrak{a}_1$ of arbitrary pairs (X, A) and their maps; the coefficient group is taken in $\mathcal{A}b$. Further, if (X, A) is a compact pair, then the Čech homology group $H_q(X, A)$ is also defined when the coefficient group is taken in $\mathcal{A}b_C$, and is itself in $\mathcal{A}b_C$. The Eilenberg-Steenrod axioms are valid except for the Exactness axiom, which is valid only with some restrictions. The Čech homology sequence of any pair is defined, and it is known that the composition of any two successive homomorphisms in this homology sequence is zero but this homology sequence may not be exact. It is known that when the pair (X, A) is restricted to be a *compact pair* and the group G is restricted to be a *compact group*, then the full exactness axiom is obtained. We prove that it suffices also to restrict (X, A) to be a *compact pair* and G to be an *algebraically compact group*, i.e. a direct summand of a compact group. In case, (X, A) is a triangulable pair, exactness holds without any restricition on the coefficient group. To include theories like Čech homology, the exactness axiom is modified to a *partial exactness axiom* and the resulting such theories are called *partially exact homology theories.* The Čech homology theory has distinguishing features among all partially exact homology theories as described in Eilenberg & Steenrod (1952). For further properties of the Čech homology theory and its relation, comparison with other homology theories, we refer also to Eda & Kawamura (1997), Guri (1993), Watanabe (1987).

4.1 The Čech Homology Theory

Definition 4.1.1. An *indexed family of sets in a space X* is a function α defined on a set V_α of indices such that, for each $v \in V_\alpha, \alpha_v$(the value of α on v) is a subset of X. If

$$X = \bigcup_{v \in V_\alpha} \alpha_v,$$

then α is called a *covering* of X. It is called an *open(closed)* covering of X if each α_v is open(closed) in X. The set of all open coverings of X is denoted by Cov(X). If A is a subset of X, and V_α^A is a subset of V_α such that

$$A \subset \bigcup_{v \in V_\alpha^A} \alpha_v,$$

then we say that α is a *covering* of the pair (X, A) with (V_α, V_α^A) as indexing pair. The set of all open coverings of (X, A) is denoted by Cov(X, A).

Definition 4.1.2. Let α be an indexed family of sets in a space X. Let s_α be the simplicial complex consisting of all simplexes whose vertices are elements of V_α(if V_α is finite, then s_α is itself a simplex). If s is a simplex of s_α, the *carrier* of s, denoted by Car$_\alpha(s)$, is the intersection of those sets α_v which correspond to vertices v of s. The *nerve* of α, denoted by X_α, is the subcomplex of s_α consisting of all simplexes with nonempty carriers. If α is a covering of the pair (X, A) indexed by (V_α, V_α^A), then we denote by A_α the subcomplex of X_α consisting of all simplexes s with vertices in V_α^A such that $A \cap \text{Car}_\alpha(s) \neq \emptyset$. The pair (X_α, A_α) is then called the *nerve* of α. Note that α_v is the carrier of the vertex v.

Lemma 4.1.3. *If s' is a face of s, then $Car_\alpha(s') \supset Car_\alpha(s)$.*

Definition 4.1.4. If $f : (X, A) \longrightarrow (Y, B)$ is a continuous map and β is a covering of (Y, B), then $f^{-1}\beta$ is a covering α of (X, A) with the same indexing pair, $(V_\alpha, V_\alpha^A) = (V_\beta, V_\beta^B)$, and defined by $\alpha_v = f^{-1}(\beta_v)$ for each $v \in V_\beta$. It follows from the continuity of f that, if β is an open(closed) covering, so also is $f^{-1}\beta$.

Lemma 4.1.5. *i. If $f : (X, A) \longrightarrow (Y, B)$ is a continuous map and $\alpha = f^{-1}\beta$ where β is a covering of (Y, B), then the nerve X_α is a subcomplex of*

Y_β and A_α is a subcomplex of B_β. The inclusion map $(X_\alpha, A_\alpha) \hookrightarrow (Y_\beta, B_\beta)$ is denoted by f_β.

ii. *If $f : (X, A) \longrightarrow (X, A)$ is the identity, and α is a covering of (X, A), then $f^{-1}\alpha = \alpha$ and f_α is the identity map of (X_α, A_α).*

iii. *If $f : (X, A) \longrightarrow (Y, B)$ and $g : (Y, B) \longrightarrow (Z, C)$ are continuous maps, γ is a covering of (Z, C), and $\beta = g^{-1}\gamma$, then $f^{-1}g^{-1}\gamma = (gf)^{-1}\gamma$ and $(gf)_\gamma = g_\gamma f_\beta$.*

Definition 4.1.6. Let α and β be two coverings of the pair (X, A). The covering β is called a *refinement* of α, denoted by $\alpha \leq \beta$, if every set of β is contained in some set of α, and every set of β indexed by an element of V_β^A is contained in some set of α indexed by V_α^A. If $\alpha \leq \beta$, a function $p : (V_\beta, V_\beta^A) \longrightarrow (V_\alpha, V_\alpha^A)$ is called a *projection* if $\alpha_{pv} \supset \beta_v$ for each $v \in V_\beta$. The vertex mapping p extends uniquely to a simplicial map $s_\beta \longrightarrow s_\alpha$ which is also denoted by p.

Lemma 4.1.7. i. *The relation $\leq$ of the previous definition is a quasi-order, i.e. $\alpha \leq \alpha$, and $\alpha \leq \beta \leq \gamma$ implies $\alpha \leq \gamma$ for every α, β, $\gamma \in \mathrm{Cov}(X, A)$.*

ii. *The set $\mathrm{Cov}(X, A)$ of open coverings of (X, A) is a directed set with respect to the relation $\leq$.*

Lemma 4.1.8. i. *For any α, the identity map $s_\alpha \longrightarrow s_\alpha$ is a projection. If $p : s_\beta \longrightarrow s_\alpha$, $p' : s_\gamma \longrightarrow s_\beta$ are projections, then their composition $pp' : s_\gamma \longrightarrow s_\alpha$ is a projection.*

ii. *If $\alpha \leq \beta$ are coverings of (X, A), then a projection $p : s_\beta \longrightarrow s_\alpha$ maps (X_β, A_β) into (X_α, A_α). This simplicial map of the nerve of β into that of α is also called a projection and is denoted by the same symbol p.*

Theorem 4.1.9. *If $\alpha \leq \beta$ are two coverings of (X, A), then any two projections $p, p' : (X_\beta, A_\beta) \longrightarrow (X_\alpha, A_\alpha)$ are contiguous simplicial maps.*

Corollary 4.1.10. *Using homology groups of simplicial complexes in the sense of Definition 2.5.12, we have, for any coefficient group G, that the homomorphisms*

$$H_q(X_\beta, A_\beta; G) \longrightarrow H_q(X_\alpha, A_\alpha; G)$$

induced by a projection $(X_\beta, A_\beta) \longrightarrow (X_\alpha, A_\alpha)$ *are independent of the choice of the projection and are therefore uniquely associated with the relation* $\alpha \leq \beta$.

Lemma 4.1.11. *If* $f : (X, A) \longrightarrow (Y, B)$ *is a continuous map, and* $\alpha \leq \beta$ *are coverings of* (Y, B), *then, if* $\alpha' = f^{-1}\alpha$, $\beta' = f^{-1}\beta$, *we have* $\alpha' \leq \beta'$. *If* $p : (Y_\beta, B_\beta) \longrightarrow (Y_\alpha, B_\alpha)$ *is a projection, then p maps* $(X_{\beta'}, A_{\beta'})$ *into* $(X_{\alpha'}, A_{\alpha'})$. *If* p' *is the map so defined by p, then* p' *is a projection and commutativity holds in the diagram*

$$\begin{array}{ccc} (X_{\alpha'}, A_{\alpha'}) & \xleftarrow{\;p'\;} & (X_{\beta'}, A_{\beta'}) \\ \downarrow{\scriptstyle f_\alpha} & & \downarrow{\scriptstyle f_\beta} \\ (Y_\alpha, B_\alpha) & \xleftarrow[\;p\;]{} & (Y_\beta, B_\beta) \end{array}$$

Definition 4.1.12. Let (X, A) be a pair of topological spaces and G an abelian group(in the category $\mathcal{A}b$). Let $\mathrm{Cov}(X, A)$ be the directed set of all open coverings of (X, A). For each $\alpha \in \mathrm{Cov}(X, A)$, let (X_α, A_α) be its nerve, and let

$$H_{q,\alpha} = H_q(X_\alpha, A_\alpha; G).$$

For each relation $\alpha \leq \beta$ in $\mathrm{Cov}(X, A)$, let

$$\pi_\alpha^\beta : H_{q,\beta} \longrightarrow H_{q,\alpha}$$

be the homomorphisms induced by any projection $(X_\beta, A_\beta) \longrightarrow (X_\alpha, A_\alpha)$. The collection $\{H_{q,\alpha}(\alpha \in \mathrm{Cov}(X, A)); \pi_\alpha^\beta\}$ is called the q^{th} *Čech homology system of* (X, A) *over* G.

Theorem 4.1.13. *The* q^{th} *Čech homology system of* (X, A) *over* G *is an inverse system of groups defined on the directed set* $\mathrm{Cov}(X, A)$.

Definition 4.1.14. The inverse limit of the q^{th} Čech homology system of (X, A) over G is denoted by $H_q(X, A; G)$ and is called *the* q^{th} *Čech homology group of* (X, A) *over* G. The group G may belong to the category $\mathcal{A}b$ of abelian groups.

When G is a compact abelian group, i.e. in the category $\mathcal{A}b_C$, the situation is as follows: The Čech system is an inverse system and the passage to the limit is permissible. However the groups $H_q(X_\alpha, A_\alpha; G)$ themselves are not defined for $G \in \mathcal{A}b_C$ since in general the complex X_α is infinite. One could try to avoid this difficulty by replacing the directed set $\mathrm{Cov}(X,A)$ by its subset $\mathrm{Cov}^f(X,A)$ consisting only of *finite coverings*(i.e. coverings α with V_α finite). The previous definitions and results remain valid for the directed set $\mathrm{Cov}^f(X,A)$ when it replaces $\mathrm{Cov}(X,A)$. Of course, the resulting limiting group $H_q^f(X,A;G)$ may not be isomorphic with $H_q(X,A;G)$. However, if the pair (X,A) is compact, then $\mathrm{Cov}^f(X,A)$ is a cofinal subset of $\mathrm{Cov}(X,A)$ and therefore the limits H_q^f and H_q are isomorphic. Thus, for compact pairs, we may limit our attention to *finite* coverings, and thereby define the homology groups $H_q(X,A;G)$ with $G \in \mathcal{A}b_C$. Then the group $H_q(X,A;G)$ is also in $\mathcal{A}b_C$. The situation resembles the one for simplicial complexes; the group $H_q(K,L;G)$ when $G \in \mathcal{A}b_C$ was defined only when the simplicial pair is finite, i.e. in $\mathbf{K_s}'$ of *finite* simplicial complexes(not for any pair in the category $\mathbf{K_s}$ of infinite or finite simplicial complexes). To justify the above discussion we used the following:

Lemma 4.1.15. *If the pair (X,A) is compact, then the set $\mathrm{Cov}^f(X,A)$ consisting of finite coverings is a cofinal subset of $\mathrm{Cov}(X,A)$.*

We will now state the results showing that the Čech homology satisfies the Eilenberg-Steenrod axioms except the exactness axiom.

Theorem 4.1.16. *(Dimension axiom). If P is a single point, then $H_q(P;G) = 0$ for $q \neq 0$ and $H_0(P;G) \cong G$*

Theorem 4.1.17. *Let $f : (X,A) \longrightarrow (Y,B)$ be a continuous map, let $f^{-1} : \mathrm{Cov}(Y,B) \longrightarrow \mathrm{Cov}(X,A)$ be the associated map of the coverings, and, for each $\alpha \in \mathrm{Cov}(Y,B)$, let $f_\alpha : (X_{\alpha'}, A_{\alpha'}) \longrightarrow (Y_\alpha, B_\alpha)$ be the inclusion map of the nerve of $\alpha' = f^{-1}\alpha$ into that of α. Then the induced homomorphisms*

$$f_{\alpha *} : H_q(X_{\alpha'}, A_{\alpha'}; G) \longrightarrow H_q(Y_\alpha, B_\alpha; G),$$

for all $\alpha \in \mathrm{Cov}(Y,B)$ together with f^{-1} form a map $\Phi(f)$ of inverse systems from the q^{th} Čech homology system of (X,A) over G into that of (Y,B).

Definition 4.1.18. The limit of the map $\Phi(f)$ of the q^{th} Čech homology system of (X,A) into that of (Y,B) is denoted by

$$f_* : H_q(X,A;G) \longrightarrow H_q(Y,B;G),$$

and is called *the homomorphism induced by* f. The coefficient group G belongs to the category $\mathcal{Ab}$. If the pair is compact, then we replace Cov by Cov^f throughout, and as above derive the definition of f_* for G in the category $\mathcal{Ab}_C$.

Theorem 4.1.19. *(Axiom 1). If $f : (X, A) \longrightarrow (Y, B)$ is the identity map of the pair (X, A), then f_* is also the identity.*

Theorem 4.1.20. *(Axiom 2). If $f : (X, A) \longrightarrow (Y, B)$ and $g : (Y, B) \longrightarrow (Z, C)$ are continuous maps, then $(gf)_* = g_* f_*$.*

Theorem 4.1.21. *(Homotopy Axiom). Let $g_0, g_1 : (X, A) \longrightarrow (X, A) \times I$ be defined by $g_0(x) = (x, 0)$ $g_1(x) = (x, 1)$; then $g_{0*} = g_{1*}$ for any coefficient group G for which the appropriate Čech groups are defined.*

Theorem 4.1.22. *(Excision Axiom). If U is open in the space X and its closure $\bar{U}$ is contained in the interior of $A \subset X$, then the inclusion map $f : (X - U, A - U) \longrightarrow (X, A)$ induces isomorphisms*

$$f_* : H_q(X - U, A - U) \longrightarrow H_q(X, A)$$

for any coefficient group G for which the respective Čech groups are defined.

The Čech homology groups of (X, A), A, and X are defined as limits of suitable systems of groups defined over the directed sets $\mathrm{Cov}(X, A)$, $\mathrm{Cov}(A, \emptyset)$, and $\mathrm{Cov}(X, \emptyset)$ respectively. In order to define the boundary operator and discuss exactness, it will be convenient to have equivalent definitions in which all these systems are defined over the same directed set. It appears that the directed set $\mathrm{Cov}(X, A)$ is most suitable for this purpose.

Definition 4.1.23. If $\alpha \in \mathrm{Cov}(X, A)$, let S_α be the homology sequence of (X_α, A_α) over G. If $\alpha \leq \beta$ in $\mathrm{Cov}(X, A)$, let $\pi^\beta_\alpha : S_\beta \longrightarrow S_\alpha$ be a map induced by a projection $(X_\beta, A_\beta) \longrightarrow (X_\alpha, A_\alpha)$. The resulting limit sequence is called the *adjusted homology sequence of* (X, A). The groups and homomorphisms of the adjusted sequence is written

$$\ldots \longrightarrow H_{q+1}(X, A) \xrightarrow{\partial'} H_q(A)_{(X,A)} \xrightarrow{i'_*} H_q(X)_{(X,A)} \xrightarrow{j'_*} H_q(X, A) \longrightarrow \ldots$$

To compare the groups with the subscript (X, A) with the groups without this subscript, we introduce two maps,

$$\phi : \mathrm{Cov}(X, A) \longrightarrow \mathrm{Cov}(A, \emptyset),$$

$$\psi : \mathrm{Cov}(X, A) \longrightarrow \mathrm{Cov}(X, \emptyset),$$

Let $\alpha \in \mathrm{Cov}(X, A)$ be indexed by the pair (V_α, V_α^A). Then $\phi\alpha$ is indexed by $(V_\alpha^A, \emptyset)$ and $(\phi\alpha)_v = A \cap \alpha_v$ for $v \in V_\alpha^A$. The covering $\psi\alpha$ is indexed by $(V_\alpha, \emptyset)$ and satisfies $(\psi\alpha)_v = \alpha_v$. Observe that

$$A_\alpha = A_{\phi\alpha}, \qquad X_\alpha = X_{\psi\alpha}.$$

The maps ϕ and ψ and the appropriate identity maps of the homology groups yield maps of inverse systems

$$\Phi\{H_q(A_\alpha; G)(\alpha \in \mathrm{Cov}(A, \emptyset)); \pi_\alpha^\beta\} \longrightarrow \{H_q(A_\alpha; G)(\alpha \in \mathrm{Cov}(X, A)); \pi_\alpha^\beta\}$$

$$\Psi\{H_q(X_\alpha; G)(\alpha \in \mathrm{Cov}(X, \emptyset)); \pi_\alpha^\beta\} \longrightarrow \{H_q(X_\alpha; G)(\alpha \in \mathrm{Cov}(X, A)); \pi_\alpha^\beta\}$$

The inverse limits of the maps Φ and Ψ are homomorphisms

$$\phi_\infty : H_q(A; G) \longrightarrow H_q(A; G)_{(X,A)}, \qquad \psi_\infty : H_q(X; G) \longrightarrow H_q(X; G)_{(X,A)}.$$

Lemma 4.1.24. *The homomorphisms ϕ_∞ and ψ_∞ are isomorphisms.*

Definition 4.1.25. The homomorphism

$$\partial : H_q(X, A; G) \longrightarrow H_{q-1}(A; G)$$

is defined from the diagram

$$H_q(X, A; G) \xrightarrow{\partial'} H_{q-1}(A; G)_{(X,A)} \xleftarrow{\phi_\infty} H_{q-1}(A; G)$$

as $\partial = \phi_\infty^{-1}\partial'$.

Theorem 4.1.26. *(Axiom 3). Let $f : (X, A) \longrightarrow (Y, B)$. Then $(f_{|A})_*\partial = \partial f_*$.*

Theorem 4.1.27. *The Čech homology sequence of a pair (X, A) over any coefficient group G is isomorphic with the adjusted homology sequence. The isomorphism is given by the maps ϕ_∞, ψ_∞ and the identity map of $H_q(X, A; G)$ onto itself.*

With this theorem at hand, the question of the exactness of the Čech homology sequence of a pair (X, A) is replaced by the question of the exactness of the adjusted sequences. The adjusted sequences are however limits of systems of exact sequences defined over the directed set $\mathrm{Cov}(X, A)$. Thus the results of Section 3.2 and Section 3.3 may be applied.

Theorem 3.2.3 yields:

Theorem 4.1.28. *For any pair (X, A) and any abelian group $G \in \mathcal{A}b$, the Čech homology sequence is a sequence of order 2.*

If (X, A) is a compact pair, then, in defining the groups occuring in the homology sequence, we may limit our attention to finite coverings. If G is a compact abelian group, then for each finite covering α, the homology sequence of (X_α, A_α) over G is composed of compact groups and therefore by Theorem 3.2.4 the limit sequence is exact, that is we have:

Theorem 4.1.29. *If (X, A) is a compact pair and G is a compact abelian group, i.e. $G \in \mathcal{A}b_C$, then the Čech homology sequence of (X, A) over G is exact.*

But it is not true that the full exactness axiom holds for any group G even for compact pairs. In Eilenberg & Steenrod (1952) a compact pair is constructed such that the Čech homology sequence with coefficient group $\mathbb{Z}$(the integers) is not exact. But still we can enlarge the class of coefficient groups which produce exact Čech homology sequences for all compact pairs, and that is the content of the next section which uses the results obtained in Chapter 3.

The Čech homology sequence of a triangulable pair is exact without restriction on the coefficient group(see Eilenberg & Steenrod, 1952).

The Čech homology theory on the category $\mathfrak{a}_1$ of all pairs of topological spaces is a typical and a distinguished example of a *partially exact homology theory.* In defining a partially exact homology theory, definition of an admissible category $\mathfrak{a}$ and exactness axiom are modified as follows. In the definition of admissible

category(Definition 1.1.2) condition (5) is replaced by
(5$'$) If (X, A) is triangulable, then $(X, A) \in \mathfrak{a}$. If $f : (X, A) \longrightarrow (Y, B)$ is a map of pairs and (X, A), (Y, B) are triangulable, then $f \in \mathfrak{a}$.

Thus with this definition an admissible category must contain the category of triangulable pairs as a subcategory. The exactness axiom is replaced by a weaker one:

Axiom 4$'$ (PARTIAL EXACTNESS AXIOM). If (X, A) is admissible, the homology sequence of (X, A) is a sequence of order 2. If (X, A) is triangulable, then the sequence is exact.

Then a system $\{H_q(X, A), f_*, \partial\}$ satisfying Eilenberg-Steenrod axioms 1-3,5-7 and 4$'$ is called a *partially exact homology theory.*

4.2 Exactness of the Čech Homology Sequence of a Compact Pair over an Algebraically Compact Coefficient Group

Lemma 4.2.1. *Let H be a direct summand of a group G, say*

$$G = H \oplus H'$$

for some group H'. To each simplicial pair (K, L), we associate a pair of chain homomorphisms $f_{K,L} : S'_{K,L} \longrightarrow S_{K,L}$ and $g_{K,L} : S_{K,L} \longrightarrow S'_{K,L}$ where $S'_{K,L}$ is the homology sequence of the simplicial pair (K, L) over the coefficient group H and $S_{K,L}$ is the homology sequence of the simplicial pair (K, L) over the coefficient group G(in the formal homology theory of simplicial complexes):

$$\begin{array}{ll} S'_{K,L} & = \text{the homology sequence of } (K, L) \text{ over the coefficient group } H \\ f_{K,L} \downarrow \; \uparrow g_{K,L} & \\ S_{K,L} & = \text{the homology sequence of } (K, L) \text{ over the coefficient group } G \end{array}$$

such that

$$g_{K,L} \circ f_{K,L} = i_{S'_{K,L}}$$

where $i_{S'_{K,L}}$ *is the identity map of the lower sequence* $S'_{K,L}$.

This assignment of a pair of chain homomorphisms to each simplicial pair is such that: If

$$p : (K, L) \longrightarrow (K', L')$$

is a simplicial map of a simplicial pair (K, L) *to a simplicial pair* (K', L')*, then the following diagram is commutative:*

$$\begin{array}{ccc} S_{K',L'} & \xleftarrow{\;p_*\;} & S_{K,L} \\ g_{K',L'} \Big\downarrow \Big\uparrow f_{K',L'} & & g_{K,L} \Big\downarrow \Big\uparrow f_{K,L} \\ S'_{K',L'} & \xleftarrow[\;p'_*\;]{} & S'_{K,L} \end{array}$$

where p_* *and* p'_* *denotes the chain homomorphisms of corresponding homology sequences induced by the simplicial map* p.

Proof. The homology groups $H_q(K, L; G)$ of a simplicial pair (K, L) over a coefficient group G is obtained through the following functors(see Chapter 2):

$$(K, L) \xrightarrow{O} K_o/L_o \xrightarrow{\cdot \otimes G} (K_o/L_o) \otimes G \xrightarrow{H_q} H_q((K_o/L_o) \otimes G)$$

For the couple (i, j), where $i : L \longrightarrow K$ and $j : K \longrightarrow (K, L)$ are inclusion maps,

$$L \xrightarrow{i} K \xrightarrow{j} (K, L)$$

in the h-category $\mathbf{K_s}$ of simplicial pairs, applying the covariant h-functor $O : \mathbf{K_s} \longrightarrow \mathbf{Comp}$ gives the direct couple, (i_o, j_o) in **Comp**, i.e. the (split) short exact sequence

$$0 \longrightarrow L_o \xrightarrow{i_o} K_o \xrightarrow{j_o} K_o/L_o \longrightarrow 0$$

in **Comp**. Since $\cdot \otimes G$ and $\cdot \otimes H$ are covariant h-functors from **Comp** to **Comp**, we get by tensoring with H and G the following (split) short exact sequences:

$$0 \longrightarrow L_o \otimes H \xrightarrow{i_o \otimes i_H} K_o \otimes H \xrightarrow{j_o \otimes i_H} (K_o/L_o) \otimes H \longrightarrow 0$$

$$0 \longrightarrow L_o \otimes G \xrightarrow{i_o \otimes i_G} K_o \otimes G \xrightarrow{j_o \otimes i_G} (K_o/L_o) \otimes G \longrightarrow 0$$

in **Comp**, where $i_H : H \longrightarrow H$ and $i_G : G \longrightarrow G$ are identity maps and by the map $i_o \otimes i_H : L_o \otimes H \longrightarrow K_o \otimes H$ we mean the chain homomorphism defined by

$$(i_o \otimes i_H)_q : C_q(L_o \otimes H) = C_q(L_o) \otimes H \longrightarrow C_q(K_o \otimes H) = C_q(K_o) \otimes H$$

$$(i_o \otimes i_H)_q = (i_o)_q \otimes i_H \qquad \text{for each } q \in \mathbb{Z},$$

and the maps $j_o \otimes i_H$, $i_o \otimes i_G$, $j_o \otimes i_G$ are similarly defined. We define two maps between direct couples $(i_o \otimes i_H, j_o \otimes i_H)$ and $(i_o \otimes i_G, j_o \otimes i_G)$ in the c-category **Comp** as follows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & L_o \otimes H & \xrightarrow{i_o \otimes i_H} & K_o \otimes H & \xrightarrow{j_o \otimes i_H} & (K_o/L_o) \otimes H & \longrightarrow & 0 \\
 & & f_1 \downarrow \uparrow g_1 & & f_2 \downarrow \uparrow g_2 & & f_3 \downarrow \uparrow g_3 & & \\
0 & \longrightarrow & L_o \otimes G & \xrightarrow[i_o \otimes i_G]{} & K_o \otimes G & \xrightarrow[j_o \otimes i_G]{} & (K_o/L_o) \otimes G & \longrightarrow & 0
\end{array}$$

where we set

$$f_1 = i_{L_o} \otimes in_H, \qquad g_1 = i_{L_o} \otimes p_H$$

$$f_2 = i_{K_o} \otimes in_H, \qquad g_1 = i_{K_o} \otimes p_H$$

$$f_3 = i_{K_o/L_o} \otimes in_H, \qquad g_1 = i_{K_o/L_o} \otimes p_H$$

where $in_H : H \hookrightarrow G = H \oplus H'$ is the inclusion map of H in G and
$p_H : G = H \oplus H' \longrightarrow H$ is the projection of G onto its direct summand H; i_{L_O} is the identity map of the chain complex L_o, similarly i_{K_o} and i_{K_o/L_o} are identity maps of corresponding chain complexes.

f_1, f_2, f_3, g_1, g_2 and g_3 are chain homomorphisms; let's verify this for f_1, the other cases are just similar. The map $f_1 = i_{L_o} \otimes in_H$ consists of the homomorphisms

$$(f_1)_q = C_q(L_o \otimes H) = C_q(L_o) \otimes H \longrightarrow C_q(L_o \otimes G) = C_q(L_o) \otimes G$$

$$(f_1)_q = i_{C_q(L_o)} \otimes in_H, \qquad q \in \mathbb{Z}$$

where $i_{C_q(L_o)} : C_q(L_o) \longrightarrow C_q(L_o)$ is the identity map of $C_q(L_o)$. f_1 acts only on the H part, keeping the other part fixed by the identity map so that as we show it gives a chain homomorphism. Let $q \in \mathbb{Z}$. Denote by

$$\partial_q : C_q(L_o) \longrightarrow C_{q-1}(L_o)$$

the boundary homomorphism of the chain complex L_o. We must show commutativity of the square in the diagram

$$\begin{array}{llcccccl}
L_o \otimes H : & \cdots \longrightarrow & C_q(L_o) \otimes H & \xrightarrow{\partial_q \otimes i_H} & C_{q-1}(L_o) \otimes H & \longrightarrow \cdots \\
\downarrow f_1 & & \downarrow {\scriptstyle i_{C_q(L_o)} \otimes in_H} & & \downarrow {\scriptstyle i_{C_{q-1}(L_o)} \otimes in_H} & \\
L_o \otimes G : & \cdots \longrightarrow & C_q(L_o) \otimes G & \xrightarrow[\partial_q \otimes i_G]{} & C_{q-1}(L_o) \otimes G & \longrightarrow \cdots
\end{array}$$

to conclude that $f_1 : L_o \otimes H \longrightarrow L_o \otimes G$ is a chain homomorphism, that is we must verify the following equality:

$$(\partial_q \otimes i_G) \circ (i_{C_q(L_o)} \otimes in_H) = (i_{C_{q-1}(L_o)} \otimes in_H) \circ (\partial_q \otimes i_H).$$

This equality is obtained by checking it on the generators $x \otimes h$, $x \in C_q(L_o)$, $h \in H$ of $C_q(L_o) \otimes H$ as follows::

$$(\partial_q \otimes i_G) \circ (i_{C_q(L_o)} \otimes in_H)(x \otimes h) = (\partial_q \otimes i_G)(x \otimes h) = (\partial_q(x)) \otimes h$$

$$(i_{C_{q-1}(L_o)} \otimes in_H) \circ (\partial_q \otimes i_H)(x \otimes h) = (i_{C_{q-1}(L_o)} \otimes in_H)((\partial_q(x)) \otimes h) = (\partial_q(x)) \otimes h.$$

This shows that f_1 is a chain homomorphism. Similarly we get that f_2, f_3, g_1, g_2 and g_3 are chain homomorphisms.

To verify that f_1, f_2, f_3 form a map of the direct couple $(i_o \otimes i_H, j_o \otimes i_H)$ to the direct couple $(j_o \otimes i_G, j_o \otimes i_G)$ and g_1, g_2, g_3 form a map of the couple $(j_o \otimes i_G, j_o \otimes i_G)$ to the couple $(i_o \otimes i_H, j_o \otimes i_H)$, we must show that each square in the diagram

$$\begin{array}{ccccc}
L_o \otimes H & \xrightarrow{i_o \otimes i_H} & K_o \otimes H & \xrightarrow{j_o \otimes i_H} & (K_o/L_o) \otimes H \\
f_1 \downarrow \uparrow g_1 & & f_2 \downarrow \uparrow g_2 & & f_3 \downarrow \uparrow g_3 \\
L_o \otimes G & \xrightarrow[i_o \otimes i_G]{} & K_o \otimes G & \xrightarrow[j_o \otimes i_G]{} & (K_o/L_o) \otimes G
\end{array}$$

is commutative(in the sense described in Definition1.1.6). Commutativity of the diagram with only f's is obtained from the following equalities:

$$f_2 \circ (i_o \otimes i_H) = (i_{K_o} \otimes in_H) \circ (i_o \otimes i_H) = (i_{K_o} \circ i_o) \otimes (in_H \circ i_H) = i_o \otimes in_H$$

$$(i_o \otimes i_G) \circ f_1 = (i_o \otimes i_G) \circ (i_{L_o} \otimes in_H) = (i_o \circ i_{L_o}) \otimes (i_G \circ in_H) = i_o \otimes in_H$$

$$f_3 \circ (j_o \otimes i_H) = (i_{K_o/L_o} \otimes in_H) \circ (j_o \otimes i_H) = (i_{K_o/L_o} \circ j_o) \otimes (in_H \circ i_H) = j_o \otimes in_H$$

$$(j_o \otimes i_G) \circ f_2 = (j_o \otimes i_G) \circ (i_{K_o} \otimes in_H) = (j_o \circ i_{K_o}) \otimes (i_G \circ in_H) = j_o \otimes in_H$$

This shows that f_1, f_2, f_3 form a map the couple $(i_o \otimes i_H, j_o \otimes i_H)$ to the couple $(j_o \otimes i_G, j_o \otimes i_G)$. Similarly g_1, g_2, g_3 form a map of the couple $(j_o \otimes i_G, j_o \otimes i_G)$ to the couple $(i_o \otimes i_H, j_o \otimes i_H)$.

Let $q \in \mathbb{Z}$. Since H_q is a covariant ∂-functor on the h-category **Comp** of chain complexes, it must satisfy axiom 3 for the mapping of couples(see Definition 1.3.4); so we have the following commutative diagram:

$$\begin{array}{ccc}
H_q((K_o/L_o) \otimes H) & \underset{g_{3*}}{\overset{f_{3*}}{\rightleftarrows}} & H_q((K_o/L_o) \otimes G) \\
\downarrow \partial_* & & \downarrow \partial_* \\
H_{q-1}(L_o \otimes H) & \underset{g_{1*}}{\overset{f_{1*}}{\rightleftarrows}} & H_{q-1}(L_o \otimes G)
\end{array}$$

This is the main part to check in obtaining chain maps $f_{K,L}$ and $g_{K,L}$ of the homology sequences:

$$\begin{array}{ccccccccc}
S'_{K,L}: \ldots \to & H_q(L_o \otimes H) & \to & H_q(K_o \otimes H) & \to & H_q((K_o/L_o) \otimes H) & \xrightarrow{\partial_*} & H_{q-1}(L_o \otimes H) & \to \cdots \\
f_{K,L} \downarrow \uparrow g_{K,L} & f_{1*} \downarrow \uparrow g_{1*} & & f_{2*} \downarrow \uparrow g_{2*} & & f_{3*} \downarrow \uparrow g_{3*} & & f_{1*} \downarrow \uparrow g_{1*} & \\
S_{K,L}: \ldots \to & H_q(L_o \otimes G) & \to & H_q(K_o \otimes G) & \to & H_q((K_o/L_o) \otimes G) & \xrightarrow[\partial_*]{} & H_{q-1}(L_o \otimes G) & \to \cdots
\end{array}$$

So we must show commutativity of the three squares in the following diagram:

$$\begin{array}{ccccccc}
H_q(L_o \otimes H) & \xrightarrow{(i_o \otimes i_H)_*} & H_q(K_o \otimes H) & \xrightarrow{(j_o \otimes i_H)_*} & H_q((K_o/L_o) \otimes H) & \xrightarrow{\partial_*} & H_{q-1}(L_o \otimes H) \\
f_{1*} \downarrow \uparrow g_{1*} & & f_{2*} \downarrow \uparrow g_{2*} & & f_{3*} \downarrow \uparrow g_{3*} & & f_{1*} \downarrow \uparrow g_{1*} \\
H_q(L_o \otimes G) & \xrightarrow[(i_o \otimes i_G)_*]{} & H_q(K_o \otimes G) & \xrightarrow[(j_o \otimes i_G)_*]{} & H_q((K_o/L_o) \otimes G) & \xrightarrow[\partial_*]{} & H_{q-1}(L_o \otimes G)
\end{array}$$

Commutativity of the first two squares follows from the commutativity of this part when the H_q functor has not been applied.

So through the above procedure described, we associate to each simplicial pair (K,L) a pair of chain homomorphisms $f_{K,L}$ and $g_{K,L}$:

$$S'_{K,L} \underset{f_{K,L}}{\overset{g_{K,L}}{\rightleftarrows}} S_{K,L}$$

For each $i = 1,2,3$, $g_i \circ f_i = identity$ because $p_H \circ in_H = i_H$ where $i_H : H \longrightarrow H$ is the identity map of H. For example, for f_1,

$$g_1 \circ f_1 = (i_{L_o} \otimes p_H) \circ (i_{L_o} \otimes in_H) = (i_{L_o} \circ i_{L_o}) \otimes (p_H \circ in_H) = i_{L_o} \otimes i_H = i_{L_o \otimes H}$$

where $i_{L_o \otimes H} : L_o \otimes H \longrightarrow L_o \otimes H$ is the identity map $L_o \otimes H$.

Since $g_i \circ f_i = identity$, it follows by applying the functor H_q that,

$$g_{i*} \circ f_{i*} = identity.$$

Thus,

$$g_{K,L} \circ f_{K,L} = i_{S'_{K,L}}.$$

It remains to verify the commutativity of the following diagram for a simplicial map $p : (K,L) \longrightarrow (K',L')$.

$$\begin{array}{ccc} S_{K',L'} & \xleftarrow{\;p_*\;} & S_{K,L} \\ {\scriptstyle g_{K',L'}}\downarrow\uparrow{\scriptstyle f_{K',L'}} & & {\scriptstyle g_{K,L}}\downarrow\uparrow{\scriptstyle f_{K,L}} \\ S'_{K',L'} & \xleftarrow[\;p'_*\;]{} & S'_{K,L} \end{array}$$

We just use a primed notation for the simplicial pair (K',L') in the above procedure of assigning a pair of chain homomorphisms

$$S'_{K',L'} \underset{f_{K',L'}}{\overset{g_{K',L'}}{\rightleftarrows}} S_{K',L'},$$

that is in constructing these homomorphisms we use chain complexes L'_o, K'_o and maps f'_i, g'_i for $i = 1,2,3$. To obtain this result, we must show commutativity of the following three diagrams:

$$\begin{array}{ccc} H_q(L'_o \otimes G) & \xleftarrow{\;p_*\;} & H_q(L_o \otimes G) \\ g'_{1*}\big\downarrow\big\uparrow f'_{1*} & & g_{1*}\big\downarrow\big\uparrow f_{1*} \\ H_q(L'_o \otimes H) & \xleftarrow[\;p'_*\;]{} & H_q(L_o \otimes H) \end{array}$$

$$\begin{array}{ccc} H_q(K'_o \otimes G) & \xleftarrow{\;p_*\;} & H_q(K_o \otimes G) \\ g'_{2*}\big\downarrow\big\uparrow f'_{2*} & & g_{2*}\big\downarrow\big\uparrow f_{2*} \\ H_q(K'_o \otimes H) & \xleftarrow[\;p'_*\;]{} & H_q(K_o \otimes H) \end{array}$$

$$\begin{array}{ccc} H_q((K'_o/L'_o) \otimes G) & \xleftarrow{\;p_*\;} & H_q((K_o/L_o) \otimes G) \\ g'_{3*}\big\downarrow\big\uparrow f'_{3*} & & g_{3*}\big\downarrow\big\uparrow f_{3*} \\ H_q((K'_o/L'_o) \otimes H) & \xleftarrow[\;p'_*\;]{} & H_q((K_o/L_o) \otimes H) \end{array}$$

All these three diagrams are commutative since they are obtained by applying the functor H_q to the following three commutative diagrams in **Comp**:

$$\begin{array}{ccc} L'_o \otimes G & \xleftarrow{\;\tilde{p}_o \otimes i_G\;} & L_o \otimes G \\ g'_1\big\downarrow\big\uparrow f'_1 & & g_1\big\downarrow\big\uparrow f_1 \\ L'_o \otimes H & \xleftarrow[\;\tilde{p}_o \otimes i_H\;]{} & L_o \otimes H \end{array}$$

$$\begin{array}{ccc} K'_o \otimes G & \xleftarrow{\;\tilde{\tilde{p}}_o \otimes i_G\;} & K_o \otimes G \\ g'_2\big\downarrow\big\uparrow f'_2 & & g_2\big\downarrow\big\uparrow f_2 \\ K'_o \otimes H & \xleftarrow[\;\tilde{\tilde{p}}_o \otimes i_H\;]{} & K_o \otimes H \end{array}$$

$$\begin{array}{ccc} (K'_o/L'_o) \otimes G & \xleftarrow{\;p_o \otimes i_G\;} & (K_o/L_o) \otimes G \\ g'_3\big\downarrow\big\uparrow f'_3 & & g_3\big\downarrow\big\uparrow f_3 \\ (K'_o/L'_o) \otimes H & \xleftarrow[\;p_o \otimes i_H\;]{} & (K_o/L_o) \otimes H \end{array}$$

Here $i_H : H \longrightarrow H$ and $i_G : G \longrightarrow G$ are identity maps, $\tilde{p}$ and $\tilde{\tilde{p}}$ are the simplicial maps

$$\tilde{p} : L \longrightarrow L', \qquad \tilde{\tilde{p}} : K \longrightarrow K',$$

which are induced by the simplicial map $p : (K, L) \longrightarrow (K', L')$, $\tilde{p}_o$, $\tilde{\tilde{p}}_o$ and p_o are obtained by applying the functor $O : \mathbf{K_s} \longrightarrow \mathbf{Comp}$ to $\tilde{p}$, $\tilde{\tilde{p}}$ and p.

Commutativity of these three diagrams are easily obtained. Let's check the first one for f_1' and f_1:

$$(\tilde{p}_o \otimes i_G) \circ f_1 = (\tilde{p}_o \otimes i_G) \circ (i_{L_o} \otimes in_H) = (\tilde{p}_o \circ i_{L_o}) \otimes (i_G \circ in_H) = \tilde{p}_o \otimes in_H,$$

and

$$f_1' \circ (\tilde{p}_o \otimes i_H) = (i_{L_o'} \otimes in_H) \circ (\tilde{p}_o \otimes i_H) = (i_{L_o'} \circ \tilde{p}_o) \otimes (in_H \circ i_H) = \tilde{p}_o \otimes in_H.$$

So,

$$(\tilde{p}_o \otimes i_G) \circ f_1 = f_1' \circ (\tilde{p}_o \otimes i_H)$$

which means commutativity of the first diagram with f_1 and f_1'. The other cases are similarly obtained; they hold simply because f_i's and g_i's act only on the H and G parts and p does not effect these parts.

This ends the proof the lemma. □

Theorem 4.2.2. *Let (X, A) be a pair of topological spaces and G be an abelian group such that the Čech homology sequence of (X, A) over the coefficient group G is exact. If H is a direct summand of G, then the Čech homology sequence of (X, A) over H is also exact.*

Proof. Say $G = H \oplus H'$ for some subgroup H' of G. Denote by M, the directed set $\mathrm{Cov}(X, A)$. For each $\alpha \in M$, let

S_α' = the homology sequence of the simplicial pair (X_α, A_α) over the coefficient group H,

S_α = the homology sequence of the simplicial pair (X_α, A_α) over the coefficient group G.

If $\alpha \leq \beta$ in $M = \text{Cov}(X, A)$, let

$$p : (X_\beta, A_\beta) \longrightarrow (X_\alpha, A_\alpha)$$

be a projection. Let

$$\pi_\alpha^\beta : S_\beta \longrightarrow S_\alpha$$

be the chain homomorphism from the homology sequence S_β of (X_β, A_β) over G to the homology sequence S_α of (X_α, A_α) over G induced by the projection p. Let

$$\pi'^\beta_\alpha : S'_\beta \longrightarrow S'_\alpha$$

be the chain homomorphism from the homology sequence S'_β of (X_β, A_β) over H to the homology sequence S'_α of (X_α, A_α) over H induced by the projection p. These give us naturally inverse systems $\{S_\alpha(\alpha \in M); \pi_\alpha^\beta\}$(shortly $\{S', \pi'\}$) and $\{S'_\alpha(\alpha \in M); \pi'^\beta_\alpha\}$(shortly $\{S, \pi\}$) of lower exact sequences over $M = \text{Cov}(X, A)$. The Čech homology sequence of (X, A) is isomorphic to the adjusted homology sequence which is the limit S_∞ of $\{S_\alpha(\alpha \in M); \pi_\alpha^\beta\}$ and which is assumed to be exact by our hypothesis.

By the Lemma 4.2.1, for each $\alpha \in M$, we have chain homomorphisms

$$f_\alpha : S'_\alpha \longrightarrow S_\alpha \qquad \text{and} \qquad g_\alpha : S_\alpha \longrightarrow S'_\alpha$$

such that

$$g_\alpha \circ f_\alpha = i_{S'_\alpha}$$

where $i_{S'_\alpha}$ is the identity map of the lower sequence S'_α. To obtain maps

$$f : \{S', \pi'\} \longrightarrow \{S, \pi\} \qquad \text{and} \qquad g : \{S, \pi\} \longrightarrow \{S', \pi'\}$$

in $\mathbf{Inv}_M(\mathbf{Comp})$ using these f_α's and g_α's, we need to show that for $\alpha \leq \beta$ in M the following diagram is commutative:

$$\begin{array}{ccc}
S_\alpha & \xleftarrow{\pi_\alpha^\beta} & S_\beta \\
g_\alpha \downarrow \uparrow f_\alpha & & g_\beta \downarrow \uparrow f_\beta \\
S'_\alpha & \xleftarrow[\pi'^\beta_\alpha]{} & S'_\beta
\end{array}$$

This follows from Lemma 4.2.1 as π_α^β and ${\pi'}_\alpha^\beta$ are the chain homomorphisms of corresponding homology sequences induced by the same simplicial map $p : (X_\beta, A_\beta) \longrightarrow (X_\alpha, A_\alpha)$.

Thus we get maps

$$f : \{S', \pi'\} \longrightarrow \{S, \pi\} \qquad \text{and} \qquad g : \{S, \pi\} \longrightarrow \{S', \pi'\}$$

in the category in $\mathbf{Inv}_M(\mathbf{Comp})$ such that

$$g \circ f = i_{\{S', \pi'\}}$$

as $g_\alpha \circ f_\alpha = i_{S'_\alpha}$ for every $\alpha \in M$. Then by Theorem 3.3.6 we get our result that S'_∞ is exact as S_∞ is exact. Since S'_∞ is the adjusted Čech homology sequence of (X, A) over the coefficient group H and since this adjusted sequence is isomorphic to the Čech homology sequence of (X, A) over H, we get that the Čech homology sequence of (X, A) over the coefficient group H is exact. □

Corollary 4.2.3. *Let (X, A) be a compact pair and H be an algebraically compact group. Then the Čech homology sequence of (X, A) over the coefficient group H is exact.*

Proof. Since H is algebraically compact, H is a direct summand of a compact group G. Since G is compact, the Čech homology sequence of the compact pair (X, A) over the coefficient group G is exact by Theorem 4.1.29. By Theorem 4.2.2, the Čech homology sequence of the compact pair (X, A) over the group H, which is a direct summand of G, is also exact. □

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