DOKUZ EYLÜL UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

ON THE SOLUTIONS OF NON-LINEAR BOUNDARY VALUE PROBLEMS

by
HATİCE YARCI

August, 2008
İZMİR

ON THE SOLUTIONS OF NON-LINEAR

BOUNDARY VALUE PROBLEMS

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We have read the thesis entitled **“ON THE SOLUTIONS OF NONLINEAR BOUNDARY VALUE PROBLEMS”** completed by **HATİCE YARCI** under supervision of **PROF. DR. GÜZİN GÖKMEN** and we certify that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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Finally, I dedicate this thesis to the memory of my mother who always believed in me and who always knew I would succeed. And I did.

Hatice YARCI

ON THE SOLUTIONS OF NON-LINEAR BOUNDARY VALUE PROBLEMS

ABSTRACT

In this thesis, we present the differential transformation method for the solution of non-linear two point boundary value problems. The method has been discussed with some examples which are presented to show the ability of the method for linear and non-linear equations. The results obtained are in good agreement with the exact solution , Belmann and Kalaba approaches. This result show that the technique introduced here is accurate and easy to apply.

Keywords: Nonlinear Boundary Value Problems, Differential Transform Method

DOĞRUSAL OLMAYAN SINIR DEĞER PROBLEMLERİNİN ÇÖZÜMLERİ ÜZERİNE

ÖZ

Bu tezde, doğrusal olmayan iki nokta sınır değer problemlerinin yaklaşık çözümünü diferansiyel dönüşüm yöntemi ile verdik. Bu yöntemin lineer ve lineer olmayan denklemlerin çözümüne uygulanabilirliğini bazı örneklerle tartıştık. Elde edilen sonuçlar tam çözüme, Belmann ve Kalaba yaklaşımına iyi bir uyum göstermiştir. Bu sonuçlar verilen tekniğin doğruluğunu ve uygulaması kolay bir yöntem olduğunu ortaya koymuştur.

Anahtar Sözcükler: Doğrusal Olmayan Sınır Değer Problemleri, Diferansiyel Dönüşüm Yöntemi.

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CHAPTER ONE

INTRODUCTION

1.1 Previous Studies

In this study we introduce two point boundary value problem formulated by the second order non-linear ordinary differential equations with boundary conditions as follow;

$$y'' = f(x, y, y') \quad , \qquad a < x < b \quad ,$$
$$y(a) = \alpha \ ,$$
$$y(b) = \beta \ .$$

Two point boundary value problems (TPBVPs) arise quite frequently in engineering and scientific applications, in particular, they occur in the process of solving differential equation. Solution to TPBVPs have to satisfy both the initial the final boundary conditions and in many cases this can be a difficult task. There are many computational methods that have been developed for solving TPBVP which have been proven to be of considerable effectiveness, such as the shooting method, the finite-element method, the method of weighted residuals.

In these problems, a new technique called differential transformation method is applied to solve TPBVPs. The concept of differential transformation was first proposed by Zhou in 1986 and was applied to solve linear and non-linear initial value problems in electric circuit analysis. West and Mafi have obtained the eigenvalues for free vibration of column-beam systems on elastic soil using an initial-value numerical method (West and Mafi, 1984). Chen and Ho have proposed a method to solve eigenvalue problems for the free and transverse vibration problems of a rotating twisted Timeshenko beam under axial loading by using differential transform technique (Chen and Ho, 1996). Differential transform has the inherent ability to deal with non-linear problems, and consequently Chiou applied the Taylor transform to solve non-linear vibration problems (Chiou, 1996). Lien-Tsai Yu, and Cha'o-Kuang Chen, applied the differential Taylor transformation to optimize rectangular fins[1] with variable thermal parameters (Yu, L-T, 1998 and Chen, C-K, 1996). DTM has been applied to solve a second-order non-linear differential equation that describes the under damped and over damped motion of a system subject to external excitations by Jang and Chen (Jang and Chen, 1997). Chen and Liu have considered first order both linear and non-linear two point boundary value problems using the differential transformation method (Chen and Liu, 1998). Furthermore, the method maybe employed for the solution of both ordinary and partial differential equations Jang et al, applied the two-dimensional differential transform method to the solution of partial differential equations (Jang et al, 2001). Çatal, has obtained the free vibration circular frequencies of the piles partially embedded in the soil due to supporting conditions of top and bottom ends of the pile using separation of variables (Çatal, 2002). DTM has applied to eigenvalue problems and Sturm-Liouville eigenvalue problems by Hassan (Hassan, 2002). Köksal and Herdem have investigated the first-order non-linear electrical circuits by using differential Taylor Transformation (Köksal and Herdem, 2002). Abbasov et-al used the method of differential transform to obtain approximate solutions of the linear and non-linear equations related to engineering problems and observed that the numerical results are in good agreement with the analytical solutions (Abbasov, 2005). The differential transform method (DTM) has been used to find the non-dimensional natural frequencies of the tapered cantilever Bernoulli-Euler beam by Özdemir and Kaya (Ozdemir and Kaya, 2006).

This approach can be considered as an extended Taylor Series Method of order k. Using the proposed approach an m^{th} order Taylor Series expansion of the analytic of the TPBVP can be obtained throughout the prescribed range. Numerical examples are used to illustrate the effectiveness of the proposed approach.

1.2 Existence and Uniqueness Theorem

The existence, uniqueness theory for boundary value problems is considerably more complicated and less developed than that for initial-value problems. When the boundary conditions are imposed at only two points, which is usual case in

[1] Fin: Maximum heat dissipation for a fixed profile area at a given volume.

applications, a simple theory can be developed for many special cases of equations and systems of equations. This existence and uniqueness theory plays a role in devising and analyzing numerical methods for solving boundary-value problems.

Let us consider first and important class of boundary-value problems in which the solution, $y(x)$, of a second-order equation

$$y'' = f(x, y, y') \tag{1.2.1.a}$$

is required to satisfy at two distinct points relations of the form

$$\begin{aligned} a_0 y(a) - a_1 y'(a) &= \alpha, \qquad |a_0| + |a_1| \neq 0; \\ b_0 y(b) + b_1 y'(b) &= \beta, \qquad |b_0| + |b_1| \neq 0. \end{aligned} \tag{1.2.1.b}$$

The solution is sought on the interval $[a, b] \equiv \{x | a \leq x \leq b\}$.

A formal approach to the solution of this problem is obtained by considering a related *initial*-value problem, say

$$u'' = f(x, u, u'), \tag{1.2.2.a}$$

$$a_0 u(a) - a_1 u'(a) = \alpha, \qquad c_0 u(a) - c_1 u'(a) = s, \tag{1.2.2.b}$$

The second initial condition is to be independent of the first. This is assured if $a_1 c_0 - a_0 c_1 \neq 0$. Without loss in generality we require that c_0 and c_1 be chosen such that

$$a_1 c_0 - a_0 c_1 = 1 \tag{1.2.3}$$

With c_0 and c_1 fixed in this manner, we denote the solution of (1.2.2) and (1.2.3) by

$$u = u(x; s)$$

to focus attention on its dependence on s. Evaluating the solution at $x = b$, we seek a value of s for which

$$\emptyset(s) \equiv b_0 u(b; s) + b_1 u'(b; s) - \beta = 0. \tag{1.2.4}$$

If $s = s^*$ is a root of this equation, we than expect the function

$$y(x) \equiv u(x; s^*)$$

to be solution of the boundary-value problem (1.2.1). This value is true in many cases, and in fact all solutions (1.2.1) can frequently be determined in this way. To be precise, we have the following. (Keller, H.B., 1968)

Theorem 1.2.1. *Let the function $f(x, u_1, u_2)$ be continuous on*

$$R: a \leq x \leq b, u_1^2 + u_2^2 < \infty,$$

and satisfy there a uniform Lipschitz condition in u_1 and u_2. Then the boundary-value problem (1.2.1) has as many solutions as there are distinct roots, $s = s^{(v)}$, of Equation (1.2.1). The solutions of (1.2.1) are

$$y(x) = y^{(v)}(x) \equiv u(x; s^v);$$

that is, the solutions of the initial-value problem (1.2.2) with the initial data $s = s^{(v)}$.

Proof. Introducing the new dependent variables $u_1(x) \equiv u(x)$ and $u_2(x) \equiv u'(x)$, the initial-value problem (1.2.1) can be written as

$$\begin{aligned} u_1' &= u_2, & u_1(a) &= a_1 s - c_1 a, \\ u_2'(x) &= f(x, u_1, u_2), & u_2(a) &= a_0 s - c_0 a. \end{aligned} \tag{1.2.5}$$

Now with the notation

$$u \equiv \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \qquad f(x;u) \equiv \begin{pmatrix} u_2 \\ f(x, u_1, u_2) \end{pmatrix}, \qquad a \begin{pmatrix} a_1 s - c_1 a \\ a_0 s - c_0 a \end{pmatrix},$$

we can apply Theorem1.3, since each component of **f**, and hence **f** itself, is Lipschitz-continuous on R. Thus the initial-value problem (1.2.2), has a unique solution, $u_1 x = u(x;s)$, which exists on $a \leq x \leq b$.
But clearly, if $\emptyset(s) = 0$ for some s, then this solution is also a solution of the boundary-value problem (1.2.1). If $s^{(i)}$and $s^{(j)}$ are distinct roots of Equation (1.2.4) then $u(x;s^{(i)}) \neq u(x;s^{(j)})$, by the uniqueness, so that each distinct root of Equation (1.2.4) yields a distinct solution of (1.2.1).

Now suppose $y(x)$ is a solution of (1.2.1). Then it is also a solution of the initial-value problem (1.2.1) with the parameter value $s = c_0 y(a) - c_1 y'(a)$. But this value of *s* must be satisfying Equation (1.2.4). Thus every solution of the boundary-value problem yields a root of Equation (1.2.4).

By means of this theorem the problem of solving a boundary-value problem is "reduced" to that of finding the root, or roots, of an equation.

There is an important class of problems for which we can be assured that Equation (1.2.4) has a unique root. The existence and uniqueness theory for the corresponding boundary-value problem is then settled. □

Theorem 1.2.2. *Let the function $f(x;u)$ be continuous on the infinite strip*

$$R\colon a \leq x \leq b, \qquad |u| < \infty$$

and satisfy there a Lipschitz condition in u with constant K, uniformly in x;
that is,

$$|f(x;u) - f(x;v)| \leq K|u - v| \textit{ for all } (x;u) \textit{ and } (x;v) \in R.$$

Then

(1) the initial-value problem

$$u' = f(x;u), \quad u(a) = \alpha \tag{1.2.6}$$

has a unique solution $u = u(x;\alpha)$ *defined on the interval*

$$[a,b] \equiv \{x|a \le x \le b\};$$

(2) this solution is Lipschitz-continuous in α *,uniformly in* x *; in fact we have*

$$|u(x;a) - u(x;\beta)| \le e^{K(x-a)}|\alpha - \beta| \quad \text{for} \quad \text{all} \quad (x;\alpha) \text{ and } (x;\beta) \in R.$$

(Keller, H.B., 1968) □

Theorem 1.2.3. *Let the function* $f(x,u_1,u_2)$ *in* (1.2.2) *satisfy the hypothesis of Theorem1.2.1 and have continuous derivatives on R which satisfy, for some positive constant M,*

$$\frac{\partial f}{\partial u_1} > 0, \qquad \left|\frac{\partial f}{\partial u_1}\right| \le M.$$

Let the coefficients in (1.2.1.b) satisfy

$$a_0 a_1 \ge 0, \qquad b_0 b_1 \ge 0. \qquad |a_0| + |b_0| \ne 0.$$

Then the boundary-value problem (1.2.1) has a unique solution

Proof. Since Theorem1.2.1 is applicable, we need only show that Equation (1.2.4) has a unique root. By the assumed continuity of the derivatives of f it easily follows, from the formulation (1.2.5), that Theorem1.2.4 is also applicable. Thus let $u(x;s)$ be the solution of the initial-value problem (1.2.2) and (1.2.3) and define $\xi \equiv \partial u(x;s)/\partial s$. Then $\xi(s)$ is the solution, on$[a,b]$, of the variational equation

$$\xi'' = p(x)\xi' + q(x)\xi,$$

subject to the initial conditions

$$\xi(a) = a_1 \qquad \xi'(a) = a_0,$$

Here we have introduced

$$q(x) \equiv \frac{\partial f\big(x, u(x;s), u'(x;s)\big)}{\partial u_1}, \qquad p(x) \equiv \frac{\partial f\big(x, u(x;s), u'(x;s)\big)}{\partial u_2}.$$

The solution $\xi(x)$ of the above variational problem has a continuous second derivative and is nonzero in some arbitrarily small interval $a < x \leq a + \varepsilon$, by virtue of the initial conditions. We shall show that $\xi(x)$ is also non zero in $a < x \leq b$. With no loss in generality, we may assume that $a_0 a_1 \geq 0$ and the variational problem is linear. Then we will show that $\xi(x)$ is positive in $a < x \leq b$. If this is not true, say $\xi(x) \leq 0$ for some x_* in $a < x < b$, then $\xi(x)$ must have a positive maximum at some point x_0 in $a \leq x_0 < x_*$. However, the maximum cannot be at $x_0 = a$ if $a_0 \neq 0$, since then $\xi'(x) > 0$. For $a_0 = 0$, the variational equation and $\xi(a) = a_1$ imply $\xi''(a) = q(a)a_1 > 0$ and so the maximum is not at $x_0 = a$ in the case either. Hence we have $a < x_0 < x_*$ and at such an interior maximum by the continuity properties of $\xi(x)$.

$$\xi(x_0) > 0, \quad \xi'(x_0) = 0, \quad \xi''(x_0) \leq 0.$$

But the variational equation at this point yields, since $q > 0$,

$$\xi''(x_0) = q(x_0)\xi(x_0) > 0.$$

The contradiction implies that $\xi(x_0) > 0$ on $a < x \leq b$.

From this result it follows that $q(x)\xi(x_0) > 0$ on $a < x < b$, and the variational equation yields the differential inequality

$$\xi''(x) > p(x)\xi'(x), \qquad a < x \leq b.$$

We may "solve" this inequality by the same procedure used in the proof of Theorem1.2.2. Thus we multiply through by integrating factor

$$exp\left[-\int_a^x p(t)dt\right]$$

(which, since $|p| \leq M$, must be positive), to get

$$\left\{exp\left[-\int_a^x p(t)dt\right]\xi'(x)\right\}' > 0.$$

Now integrating the above inequality over $[a, x]$ we get,on recalling that $\xi'(a) = a_0$.

$$\xi'(x) > a_0 exp\left[\int_a^x p(t)dt\right].$$

Another integration and application of $\xi(a) = a_1$ gives

$$\xi(x) > a_1 + a_0 \int_a^x exp\left[\int_a^t p(t')dt'\right]dt. \qquad a < z \leq b.$$

But, since $p \geq -M$, it follows that

$$exp\left[\int_a^t p(t')dt'\right] > e^{-M(t-a)}$$

and, using this, we obtain finally

$$\xi'(x) = \frac{\partial u'(x;s)}{\partial s} > a_0 e^{-M(x-a)} \geq 0,$$

$$\xi(\mathrm{x}) = \frac{\partial \mathrm{u}(\mathrm{x};\mathrm{s})}{\partial \mathrm{s}} > \mathrm{a}_1 + \mathrm{a}_0\left(\frac{1-\mathrm{e}^{-\mathrm{M}(\mathrm{x}-\mathrm{a})}}{M}\right) > 0 \qquad a < x \leq b. \qquad (1.2.7)$$

With no difficulty it can be seen that for the case in which $a_0 \leq 0$ and $a_1 \leq 0$, the inequality signs in Equation(1.2.7) need only be reversed.

In particular, then, setting $x = b$, the function $u(b; s)$ is a monotone function of s with derivative bounded away from zero for any value of a_0.

The same is true of $u'(b; s)$ if $a_0 \neq 0$; if $a_0 = 0$, it is not bounded away from zero. But since b_0 and b_1 do not both vanish or have opposite signs, and $b_0 \neq 0$ if $a_0 = 0$, the function $\emptyset(s)$ in Equation (1.2.4) must have a derivative of one sing which is bounded away from zero for all a_0. Such a function takes on each real value once, and hence $\emptyset(s) = 0$ has a unique root.$\square$

Theorem 1.2.4. *In addition to the hypotheses of Theorem 1.2.2 let the Jacobian of f with respect to u have continuous elements on R; that is, the nth-order matrix*

$$F(x; u) \equiv \frac{\partial f(x; u)}{\partial u} = \left(\frac{\partial f_i(x; u)}{\partial u_j}\right),$$

is continuous on R. Then for any α *the solution* $u(x; \alpha)$ *of (1.2.6) is continuously differentiable with respect to* α_k, $k = 1,2, \dots, n$. *In fact, the derivative* $\partial u(x; \alpha)/\partial \alpha_k \equiv \xi^{(k)}(x)$ *is the solution, on [a,b], of the linear system*

$$\frac{d}{dx}\xi(x) = F\big(x; (u; a)\big)\xi(x), \qquad (1.2.8.a)$$

$$\xi(a) = e^{(k)}. \qquad (1.2.8.b)$$

(Here $e^{(k)} \equiv (0, \dots, 0,1,0, \dots, 0)^T$ *is the kth unit vector in n-space.)* (Keller,H.B.,1968) $\square$

Example: $y'' = tanhy + cos(y')$

$$y(0) - y'(0) = 1 \qquad |1| + |-1| \neq 0$$

$$y(1) + y'(1) = 0 \qquad |1| + |1| \neq 0$$

$$f\big(x, u_1, u_2\big) = tanhu_1 + cosu_2$$

$$(i)\ \frac{\partial f}{\partial u_1} = sech^2 u_1 > 0$$

$$(ii)\ \frac{\partial f}{\partial u_2} = -\sin(u_2)$$

$$\left|\frac{\partial f}{\partial u_2}\right| = |-\sin(u_2)| \leq 1$$

$M = 1$ is a constant.

CHAPTER TWO

DESCRIPTION OF DIFFERENTIAL TRANSFORMATION METHOD

2.1. Description of Differential Transformation Method

The basic definitions and fundamental Theorems 2.1.1 – 2.1.14 of one dimensional differential transform are defined and will be stated in brief in this section. (Chang, S-H. and Chang, I-L., 2008).

Differential transform of function y(x) is defined as follows:

$$Y(k) = \frac{1}{k!}\left[\frac{d^k y(x)}{dx^k}\right]_{x=0} \quad , \tag{2.1.1}$$

where y(x) is the original function and Y(k) is the transformed function, which is also called the T-function. Differential inverse transform of Y(k) is defined as:

$$y(x) = \sum_{k=0}^{\infty} x^k Y(k) \tag{2.1.2}$$

Combining equations (2.1.1) and (2.1.2), we obtain:

$$y(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}\left[\frac{d^k y(x)}{dx^k}\right]_{x=0} \tag{2.1.3}$$

Equation (2.1.3) implies that the concept of differential transform is derived from Taylor series expansion, but the method does not evaluate the derivatives symbolically.

However, relative derivatives are calculated by an iterative ways which are described by the transformed equations of the original functions. In this study, we use the lower case letter to represent the original function and upper case letter represent the transformed function.

From the definitions of equations (2.1.1) and (2.1.2), it is easily proven that the transformed functions comply with the basic mathematics operations shown in theorems.

In actual applications, the function y(x) is expressed by a finite series and equation (2.1.2) can be written as

$$y(x)=\sum_{k=0}^{\infty} x^k Y(k) \tag{2.1.4}$$

Equation (2.1.3) implies that $\sum_{k=m+1}^{\infty} x^k Y(k)$ is negligibly small. In fact, m is decided by the convergence of natural frequency in this study.

The fundamental theorems of the one- dimensional differential transform are:

Theorem 2.1.1. *If* $w(x)=y(x)\mp z(x),\ \textit{then}\ W(k)=Y(k)\mp Z(k)$

Proof: By using the definition of the transform:

$$Y(k)=\frac{1}{k!}\frac{d^k y(x)}{dx^k}\ , \qquad y(x)=\sum_{k=0}^{\infty} x^k Y(k)$$

$$Z(k)=\frac{1}{k!}\frac{d^k z(x)}{dx^k}\ , \qquad z(x)=\sum_{k=0}^{\infty} x^k Z(k)$$

$$Y(k)\mp Z(k)=\frac{1}{k!}\frac{d^k y(x)}{dx^k}\mp\frac{1}{k!}\frac{d^k z(x)}{dx^k}$$

$$Y(k)\mp Z(k)=\frac{1}{k!}\frac{d^k}{dx^k}\left[y(x)\mp z(x)\right] \text{ by the hypothesis.}$$

$$y(x)\mp z(x)=w(x)$$

So,

$$Y(k) \mp Z(k) = \frac{1}{k!}\frac{d^k w(x)}{dx^k} = W(k) \qquad (2.1.5)$$

$$W(x) = \sum_{k=0}^{\infty} x^k Y(k) \mp \sum_{k=0}^{\infty} x^k Z(k)$$

$$= \sum_{k=0}^{\infty} x^k \left(Y(k) \mp Z(k)\right)$$

by using the definition (2.1.5)

$$W(x) = \sum_{k=0}^{\infty} x^k W(k) \quad \square$$

Theorem 2.1.2. *If w(x) = cy(x), then W(k) = cY(k)*

Proof: By using definition of the differential transform

$$W(k) = \frac{1}{k!}\frac{d^k}{dx^k}\left[cy(x)\right],$$

where c is constant. Thus, we have

$$W(k) = c\left[\frac{1}{k!}\frac{d^k y(x)}{dx^k}\right] = cY(k)$$

$$W(k) = cY(k) \quad \square$$

Theorem 2.1.3. *If* $w(x)=\dfrac{dy(x)}{dx}$, *then* $W(k)=(k+1)Y(k+1)$

Proof: By utilizing the definition of transform:

$$y(x)=\sum_{k=0}^{\infty}x^{k}Y(k)$$

$$y(x)=Y(0)+xY(1)+x^{2}Y(2)+x^{3}Y(3)+\ldots \qquad (2.1.6)$$

by taking the derivative of (2.1.6)

$$\frac{dy(x)}{dx}=Y(1)+2xY(2)+3x^{2}Y(3)+\ldots$$

$$\frac{dy}{dx}=\sum_{k=0}^{\infty}x^{k-1}Y(k)$$

by starting the index from k = 0 instead of k = 1 we can obtain $\frac{\mathrm{dy}}{\mathrm{dx}}$ as follows:

$$w(x)=\frac{dy(x)}{dx}=\sum_{k=0}^{\infty}x^{k}(k+1)Y(k+1)$$

Consequently we obtain

$$W(k)=(k+1)Y(k+1) \quad \square$$

Theorem 2.1.4. *If* $w(x)=\dfrac{d^{n}y(x)}{dx^{n}}$, *then* $W(k)=\dfrac{(k+n)!}{k!}Y(k+n)$

Proof: By using the definition of the transform

$$y(x)=\sum_{k=0}^{\infty}x^{k}Y(k)$$

$$y(x)=Y(0)+xY(1)+x^{2}Y(2)+x^{3}Y(3)+\ldots \qquad (2.1.7)$$

step by step if we take the derivative of (2.1.7)

$$\frac{dy(x)}{dx} = Y(1) + 2xY(2) + 3x^2Y(3) + ...$$

$$\frac{dy(x)}{dx} = \sum_{k=0}^{\infty} x^k (k+1)Y(k+1)$$

$$\frac{d^2y(x)}{dx^2} = 2Y(2) + 6xY(3) + ...$$

$$\frac{d^2y(x)}{dx^2} = \sum_{k=0}^{\infty} x^k (k+1)(k+2)Y(k+2)$$

.

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.

$$w(x) = \frac{d^n y(x)}{dx^n} = \sum_{k=0}^{\infty} x^k (k+1)(k+2)....(k+n)Y(k+n).$$

We have

$$W(k) = \frac{(k+n)!}{k!} Y(k+n)$$

This can be proved by mathematical induction. □

Theorem 2.1.5. *If* $w(x) = y(x)z(x)$, then $W(k) = \sum_{m=0}^{k} Y(m)Z(k-m)$

Proof: By the definition of transform,

$$w(x) = \sum_{m=0}^{\infty} x^m Y(m) \ \sum_{j=0}^{\infty} x^j Z(j)$$

$$w(x) = \sum_{k=0}^{\infty} x^k \sum_{m=0}^{k} Y(m)Z(k-m).$$

We get

$$W(k) = \sum_{m=0}^{k} Y(m)Z(k-m)$$

□

Theorem 2.1.6. *If* $w(x) = y_1(x)y_2(x)........y_{n-1}(x)y_n(x)$, *then*

$$W(k) = \sum_{k_{n-1}=0}^{k} \sum_{k_{n-2}=0}^{k_{n-1}} .....\sum_{k_2=0}^{k_3} \sum_{k_1=0}^{k_2} Y_1(k_1)Y_2(k_2-k_1).....Y_{n-1}(k_{n-1}-k_{n-2})xY_n(k_n-k_{n-1})$$

(A. Arikoglu, and I. Ozkol, 2005)

Proof: By using the definition of the transform

$$W(0) = \frac{1}{0!}\left[y_1(x)y_2(x).....y_{n-1}(x)y_n(x)\right]_{x=x_0} = Y_1(0)Y_2(0)....Y_{n-1}(0)Y_n(0) ,$$

$$W(1) = \frac{1}{1!}\frac{d}{dx}\underbrace{\left[y_1(x)y_2(x)......y_{n-1}(x)y_n(x)\right]_{x=x_0}}$$

$$\begin{bmatrix} y_1^{|}(x)y_2(x)....y_{n-1}(x)y_n(x) + y_1(x)y_2^{|}(x).....y_{n-1}(x)y_n(x) \\ + y_1(x)y_2(x).....y_{n-1}{}^{|}(x)y_n(x) + y_1(x)y_2(x)....y_{n-1}(x)y_n^{|}(x) \end{bmatrix}_{x=x_0}$$

$$\begin{aligned} W(1) = {} & Y_1(1)Y_2(0).......Y_{n-1}(0)Y_n(0) + Y_1(0)Y_2(1)......Y_{n-1}(0)Y_n\ (0) \\ & + Y_1(0)Y_2(0).....Y_{n-1}(1)Y_n(0) + Y_1(0)Y_2(0)......Y_{n-1}(0)Y_n(1) \end{aligned}$$

$$\begin{aligned} W(2) = {} & Y_1(1)Y_2(1)Y_3(0).....Y_n(0) + Y_1(1)Y_2(0)Y_3(1)....Y_n(0) + ... \\ & + Y_1(1)Y_2(0)Y_3(0)....Y_n(1) + Y_1(0)Y_2(1)Y_3\ (1).....Y_n(0) + ... \\ & + Y_1(0)Y_2(0).....Y_{n-1}(1)Y_n(1) + ...+ Y_1(2)Y_2(0)Y_3(0)....Y_n(0) \\ & + ...+ Y_1(0)Y_2(0)Y_3(0)......Y_n(2) \end{aligned}$$

In general we have

$$W(k) = \sum_{k_{n-1}=0}^{k} \sum_{k_{n-2}=0}^{k_{n-1}} .....\sum_{k_2=0}^{k_3} \sum_{k_1=0}^{k_2} Y_1(k_1)Y_2(k_2-k_1)....Y_{n-1}(k_{n-1}-k_{n-2})Y_n(k_n-k_{n-1})$$

□

Theorem 2.1.7. *If* w(x) = c, then $W(k) = c\delta(k)$

Proof: By using the definition of the differential transform

$$w(x) = \sum_{K=0}^{\infty} x^k W(k),$$

where w(x) = c. Thus, we get

$$c = W(0) + xW(1) + x^2 W(2) + \ldots$$

From the definition of the polynomials

$$\begin{aligned} &c = W(0) \\ &W(1) = W(2) = \ldots = 0 \end{aligned}$$

So, we have

$$\begin{aligned} W(k) &= \begin{cases} c\,, & k = 0 \\ 0\,, & k \neq 0 \end{cases} \qquad , k = 0,1,2,\ldots \\ &= c\begin{cases} 1\,, & k = 0 \\ 0\,, & k \neq 0 \end{cases} \end{aligned}$$

$$W(k) = c\delta(k) \quad \square$$

Theorem 2.1.8. *If* w(x) = x, *then* $W(k) = \delta(k-1)$

Proof: By using the definition of the transform

$$w(x) = \sum_{k=0}^{\infty} x^k W(k)$$

where w(x) = x

$$x = W(0) + xW(1) + x^2 W(2) + x^3 W(3) + \ldots$$

From the definition of the polynomials

$$W(0) = 0$$
$$W(1) = 1$$
$$W(2) = W(3) = ... = 0$$

$$W(k) = \begin{cases} 1, & k = 1 \\ 0, & k \neq 1 \end{cases}$$
$$W(k) = \begin{cases} 1, & k-1 = 0 \\ 0, & k-1 \neq 0 \end{cases}$$

Finally, we obtain

$$W(k) = \delta(k-1) \quad \square$$

Theorem 2.1.9. *If* $w(x) = x^m$, *then* $W(k) = \delta(k-m)$

Proof: By using the definition of the differential transform

$$w(x) = \sum_{k=0}^{\infty} x^k W(k)$$
$$where \quad w(x) = x^m$$
$$x^m = W(0) + xW(1) + ... + x^m W(m) + ...$$
$$W(0) = W(1) = ... = W(m-1) = ... = 0$$
$$W(m) = 1$$

$$W(k) = \begin{cases} 1, & k = m \\ 0, & k \neq m \end{cases} \qquad k = 0,1,2,...$$
$$W(k) = \begin{cases} 1, & k-m = 0 \\ 0, & k-m \neq 0 \end{cases}$$

Thus we get

$$W(k) = \delta(k-m) \quad \square$$

Theorem 2.1.10. *If* $w(x) = e^{\lambda x}$, *then* $W(k) = \dfrac{\lambda^k}{k!}$

Proof: By using the definition of the transform

$$w(x) = \sum_{k=0}^{\infty} x^k W(k)$$

We use Taylor Series expansion of $e^{\lambda x}$

$$1 + \lambda x + \frac{\lambda^2}{2!}x^2 + \frac{\lambda^3}{3!}x^3 + \ldots = W(0) + xW(1) + x^2W(2) + x^3W(3) + \ldots$$
$$W(0) = 1$$
$$W(1) = \lambda$$
$$W(2) = \frac{\lambda^2}{2}$$
$$W(3) = \frac{\lambda^3}{3!}$$
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$$W(k) = \frac{\lambda^k}{k!} \square$$

Theorem 2.1.11. *If* $w(x) = (1 + x)^m$, *then* $W(k) = \dfrac{m(m-1)\ldots(m-k+1)}{k!}$

Proof: By using the definition of the differential transform

$$w(x) = \sum_{k=0}^{\infty} x^k W(k)$$

We use Binomial theorem of $(1 + x)^m$

$$1+mx+\frac{m(m-1)}{2!}x^2+\frac{m(m-1)(m-2)}{3!}x^3+...=W(0)+xW(1)+x^2W(2)+...$$

$$W(0)=1$$
$$W(1)=m$$
$$W(2)=\frac{m(m-1)}{2!}$$
$$W(3)=\frac{m(m-1)(m-2)}{3!}$$

Since, we obtain

$$W(k)=\frac{m(m-1)....(m-(k-1))}{k!} \quad \square$$

Theorem 2.1.12. *If* $w(x)=\sin(zx)$, *then* $W(k)=\dfrac{z^k}{k!}\sin\left(\dfrac{\pi k}{2!}\right)$

Proof: By using the definition of the transform

$$w(x)=\sum_{k=0}^{\infty}x^kW(k)$$

We use

$$zx-\frac{(zx)^3}{3!}+\frac{(zx)^5}{5!}-\frac{(zx)^7}{7!}+...=W(0)+xW(1)+x^2W(2)+x^3W(3)+...$$

$$W(0)=0 \quad W(2)=0 \quad W(4)=0 \ \ ... \ \ W(2k)=0$$

$$W(1)=z \ \ W(3)=-\frac{z^3}{3!} \quad W(5)=\frac{z^5}{5!} \ \ ... \ \ W(2k+1)=(-1)^k\,\frac{z^{2k+1}}{(2k+1)!}$$

Thus we get

$$W(k)=\frac{z^k}{k!}\sin\left(\frac{\pi k}{2}\right) \quad \square$$

Theorem 2.1.13. *If* $w(x) = \cos(zx)$, *then* $W(k) = \frac{z^k}{k!}\cos\left(\frac{k\pi}{2}\right)$

Proof: By using the definition of the transform

$$w(x) = \sum_{k=0}^{\infty} x^k W(k)$$

We use

$$1 - \frac{z^2}{2!}x^2 + \frac{z^4}{4!}x^4 - \frac{z^6}{6!}x^6 + \ldots = W(0) + xW(1) + x^2W(2) + x^3W(3) + x^4W(4) + \ldots$$

$$W(1) = W(3) = W(5) = \ldots = W(2k+1) = 0$$

$$W(0) = 1$$

$$W(2) = -\frac{z^2}{2!}$$

$$W(4) = \frac{z^4}{4!}$$

$$W(6) = -\frac{z^6}{6!}$$

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$$W(2k) = (-1)^k \frac{z^{2k}}{(2k)!}$$

Finally, we have

$$W(k) = \frac{z^k}{k!}\cos\left(\frac{\pi k}{2}\right) \quad \square$$

Theorem 2.1.14. *If* $w(x)=\int_0^x y(t)dt$, *then* $W(k)=\frac{Y(k-1)}{k}$, *where* $k\geq 1$

(A. Arikoglu, and I. Ozkol, 2005)

Proof: By using equation (2.1.2) the transform of an integral can be found as follows:

$$w(x)=\int_0^x y(t)dt \quad then \quad w(x)=\int_0^x \sum_{k=0}^{\infty} Y(k)t^k dt$$

$$=\sum_{k=0}^{\infty}\int_{x_0}^{x} Y(k)t^k dt$$

$$=\sum_{k=0}^{\infty}\left[Y(k)\frac{t^{k+1}}{k+1}\bigg|_0^x\right]$$

$$=\sum_{k=0}^{\infty}\frac{Y(k)}{(k+1)}x^{k+1}$$

by starting the index from k = 1 instead of k = 0 we can obtain w(x) as follows:

$$w(x)=\sum_{k=1}^{\infty}\frac{Y(k-1)}{k}x^k$$

by using the equations (2.1.1) and (2.1.2) we get:

$$W(k)=\frac{Y(k-1)}{k}, \quad where \quad k\geq 1 \quad and \quad W(0)=0$$

□

CHAPTER THREE

APPLICATIONS

3.1 Applications

In this chapter we solve some second order two points Boundary Value Problems by the differential transformation method and compare the results with either exact or other numerical solutions.

Example 3.1.1: $-y'' = 1 + \alpha^2(\dot{y})^2 \qquad \alpha^2 = 0.49$ (3.1.1)

$y(0) = 0$ (3.1.2)

$y(1) = 0$

Applying the differential transformation of Eq. (3.1.1), it can be obtained that

$$-(k+1)(k+2)Y(k+2) = 1 + \alpha^2\left[\sum_{l=0}^{k}(l+1)Y(l+1)(k-l+1)Y(k-l+1)\right]$$

$$Y(k+2) = \frac{\delta(k) + \alpha^2\left[\sum_{l=0}^{k}(l+1)Y(l+1)(k-l+1)Y(k-l+1)\right]}{-(k+1)(k+2)} \qquad (3.1.3)$$

Use the initial condition

$$y(0) = 0 \qquad y'(0) = A$$

to obtain:

$$y(x) = \sum_{k=0}^{\infty} x^k\, Y(k)$$

$$= Y(0) + xY(1) + x^2Y(2) + x^3Y(3) + \cdots$$

$$y(0) = 0 \qquad\qquad Y(0) = 0 \tag{3.1.4}$$

and

$$\sum_{k=0}^{n} Y(k) = 0 \tag{3.1.5}$$

Let

$$y'(x) = Y(1) + 2xY(2) + 3x^2Y(3) + \cdots$$

$$y'(0) = A = Y(1) + 2Y(2)0 + 3Y(3)0 + \cdots$$

$$Y(1) = A \tag{3.1.6}$$

For each k, substituting Eq. (3.1.4) and (3.1.6) into (3.1.3), and by recursive method,

$$For \quad k = 0 \qquad\qquad Y(2) = -\frac{1 + \alpha^2[Y(1)]^2}{2} \qquad\qquad \alpha^2 = 0.49$$

$$Y(2) = -\frac{1 + \alpha^2 A^2}{2}$$

$For \quad k = 1$

$$Y(3) = \frac{\alpha^2 A[1 + \alpha^2 A^3]}{3}$$

$For \quad k = 2$

$$Y(4) = \frac{\alpha^2(2A^2 + 4A^2\alpha^2 + \alpha^2 A^4 + 4A^5\alpha^4)}{-24}$$

$$\vdots$$

$$y(x) = \sum_{k=0}^{\infty} x^k \, Y(k)$$

$$y(x) = Y(0) + xY(1) + x^2Y(2) + \cdots$$

$$= Y(1) + Y(2) + \cdots + Y(N) + \cdots = 0$$

We use first three result ($Y(1), Y(2), Y(3)$) for approximate solution.

$$Y(1) + Y(2) + Y(3) = 0$$
$$A = 0.4736$$

Then we obtain:

$$Y(1) = A \qquad\qquad Y(1) = 0.4736$$

$$Y(2) = -\frac{1 + \alpha^2 A^2}{2} \qquad\qquad Y(2) = -0.5550$$

$$Y(3) = \frac{\alpha^2 A[1 + \alpha^2 A^3]}{3} \qquad\qquad Y(3) = 0.0814$$

$$Y(4) = \frac{\alpha^2(2A^2 + 4A^2\alpha^2 + \alpha^2 A^4 + 4A^5\alpha^4)}{-24} \qquad\qquad Y(4) = -0.0191$$

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$$y(x) = Y(0) + xY(1) + x^2Y(2) + x^3Y(3) + x^4Y(4) + O(x^5)$$

$$y(0.3) = Y(0) + 0.3Y(1) + 0.3^2Y(2) + 0.3^3Y(3) + 0.3^4Y(4)$$

$$y(0.3) = 0.0940$$

$$y(0.8) = Y(0) + 0.8Y(1) + 0.8^2Y(2) + 0.8^3Y(3) + 0.8^4Y(4)$$

$$y(0.8) = 0.0471$$

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Table 3.1 Numerical Results

x_i	y(DTM)	y(Belmann and Kalaba approach)	y(Quasilinearization, Arsoy, A.)
0.0	0.0	0.0	0.0
0.1	0.0419	0.046571	0.018533
0.2	0.0731	0.082304	0.032400
0.3	0.0940	0.107573	0.0411824
0.4	0.1048	0.122635	0.047016
0.5	0.1055	0.127639	0.048180
0.6	0.0962	0.122635	0.045508
0.7	0.0768	0.107573	0.039184
0.8	0.0471	0.082304	0.029381
0.9	0.0067	0.046571	0.016267
1.0	-0.0447	0.0	0.0

Example 3.1.2:

$$y''(x) = \frac{1}{2}y^3(x) \qquad 1 \le x \le 3 \tag{3.1.7}$$

$$y(1) = \frac{2}{3} \qquad y(3) = \frac{2}{5} \tag{3.1.8}$$

Exact solution is given as

$$y(x) = \frac{2}{x+2}$$

Applying the differential transformation of Eq. (3.1.7), it can be obtained that

$$(k+1)(k+2)Y(k+2) = \frac{1}{2}\sum_{l=0}^{k}\sum_{i=0}^{l} Y(i)Y(l-i)Y(k-l)$$

$$Y(k+2) = \frac{\sum_{l=0}^{k}\sum_{i=0}^{l} Y(i)Y(l-i)Y(k-l)}{2(k+1)(k+2)} \tag{3.1.9}$$

where $Y(0) = \frac{2}{3}$ (Taylor series is opened at x=1) (3.1.10)

$$Y(1) = A \tag{3.1.11}$$

For each k, substituting Eqn. (3.1.10) and (3.1.11) into (3.1.9), and by recursive method,

For $k = 0$

$$Y(2) = \frac{Y(0)Y(0)Y(0)}{4}$$

$$Y(2) = \frac{2}{27}$$

For $k = 1$

$$Y(3) = \frac{Y(0)Y(0)Y(1) + Y(0)Y(1)Y(0) + Y(1)Y(0)Y(0)}{2.6}$$

$$Y(3) = \frac{A}{3^2}$$

For $k = 2$

$$Y(4) = \frac{Y(0)Y(0)Y(2) + Y(0)Y(1)Y(1) + Y(1)Y(0)Y(1) + Y(0)Y(2)Y(0) + Y(1)Y(1)Y(0) + Y(2)Y(0)Y(0)}{2.12}$$

$$Y(4) = \frac{1}{2}\left(\frac{4}{9}\frac{2}{27} + \frac{2}{3}A^2 + \frac{2}{3}A^2 + \frac{4}{9}\frac{2}{27} + \cdots\right)$$

We can write our function as:

$$y(x) = \sum_{k=0}^{\infty} Y(k)(x-1)^k = \sum_{k=0}^{\infty} (\frac{1}{2(k+1)(k+2)} \sum_{l=0}^{k}\sum_{i=0}^{l} Y(i)Y(l-i)\,Y(k-l)(x-1)^k$$

Let y_n be the sum of first n-term.Then $y_m(3) = \frac{2}{5}$.That is

$$\sum_{k=0}^{n} (\frac{1}{2(k+1)(k+2)} \sum_{l=0}^{k}\sum_{i=0}^{l} Y(i)Y(l-i)Y(k-l))2^k \approx \frac{2}{5}$$

$$y(x) = \sum_{k=0}^{\infty} Y(k)(x-1)^k = Y(0) + (x-1)Y(1) + (x-1)^2Y(2) + (x-1)^3Y(3) + \cdots$$

We use four results to find A:

$$\frac{2}{3} + 2A + 4\frac{2}{27} + 8\frac{A}{9} = \frac{2}{5}$$

$$\underline{A = -0.1949}$$

$$Y(0) = \frac{2}{3}$$

$$Y(1) = -0.1949$$

$$Y(2) = \frac{2}{27}$$

$$Y(3) = -\frac{0.1949}{9}$$

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$$y(x) = Y(0) + (x-1)Y(1) + (X-1)^2Y(2) + (x-1)^3Y(3) + O((x-1)^4)$$

$$y(2) = Y(0) + (2-1)Y(1) + (2-1)^2Y(2) + (2-1)^3Y(3)$$

$$y(2) = 0.5242$$

$$y(2.7) = Y(0) + (2.7-1)Y(1) + (2.7-1)^2Y(2) + (2.7-1)^3Y(3)$$

$$y(2.7) = 0.4430$$

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Table 3.2 Numerical Results

x_i	y(DTM)	y(exact solution)	Error
1.0	0.6667	0.6667	0
1.1	0.6479	0.6452	-0.0027
1.2	0.6305	0.6250	-0.0055
1.3	0.6143	0.6061	-0.0082
1.4	0.5992	0.5882	-0.0109
1.5	0.5850	0.5714	-0.0136
1.6	0.5717	0.5556	-0.0162
1.7	0.5591	0.5405	-0.0186
1.8	0.5471	0.5263	-0.0208
1.9	0.5355	0.5128	-0.0226
2.0	0.5242	0.5000	-0.0242
2.1	0.5131	0.4878	-0.0253
2.2	0.5020	0.4762	-0.0258
2.3	0.4909	0.4651	-0.0258
2.4	0.4796	0.4545	-0.0250
2.5	0.4679	0.4444	-0.0235
2.6	0.4558	0.4348	-0.0210
2.7	0.4430	0.4255	-0.0175
2.8	0.4296	0.4167	-0.0129
2.9	0.4152	0.4082	-0.0071
3.0	0.3999	0.4000	0.0001

CHAPTER FOUR

CONCLUSION

This thesis shows that the differential transformation technique can be applied to solve the two point non-linear boundary value problems , which gives the solution in the form of a finite - term Taylor series . We first gave their proofs and then applied to TPBVPs. The method is a powerful tool which enables to find analytical solution in case of non-linear differential equations . This method is better than numerical methods , since it is free from rounding off error , yields a series solution which converges faster than the series obtained by another methods . The numerical results obtained by present method are compared with the analytical solutions . It is shown that the differential transform method can achieve good results in predicting the solution of such problems.

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