DOKUZ EYLÜL UNIVERSITY

GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

ON THE SOLUTIONS OF NON-LINEAR INITIAL VALUE PROBLEMS

by

CANAN KURTARAN

July, 2008

İZMİR

ON THE SOLUTIONS OF NON-LINEAR INITIAL VALUE PROBLEMS

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We have read the thesis entitled **"ON THE SOLUTIONS OF NON-LINEAR INITIAL VALUE PROBLEMS"** completed by **CANAN KURTARAN** under supervision of **PROF. DR. GÜZİN GÖKMEN** and we certify that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ON THE SOLUTIONS OF NON-LINEAR INITIAL VALUE PROBLEMS

ABSTRACT

In this thesis, numerical solution of the second order non-linear initial value problems is considered by differential transform method. This method can easily be applied to non-linear initial value problems and series solutions are obtained. After the transformation, we have formulated series coefficients very simply for the considered problems.

Keywords: Non-Linear Initial Value Problem, Differential Transform Method.

DOĞRUSAL OLMAYAN BAŞLANGIÇ DEĞER PROBLEMLERİNİN ÇÖZÜMLERİ ÜZERİNE

ÖZ

Bu tezde, ikinci dereceden doğrusal olmayan başlangıç değer problemlerinin yaklaşık çözümleri diferensiyel dönüşüm yöntemi ile incelenmiştir. Bu yöntem doğrusal olmayan başlangıç değer problemlerine kolayca uygulanabilmiş ve seri çözümleri oluşturulabilmiştir. Dönüşümden sonra, incelenen problemlerin seri katsayıları elde edilmiştir.

Anahtar sözcükler: Doğrusal Olmayan Başlangıç Değer Problemi, Diferensiyel Dönüşüm Yöntemi.

CONTENTS

CHAPTER ONE

INTRODUCTION

1.1 Previous Studies

In this study, we consider the second order nonlinear differential equation

$$\frac{d^2y}{dt^2} = f\left(t, y, y'\right), \tag{1.1.1}$$

where f is continuous over some subset of the plane. The initial value problem associated with (1.1.1) is to find a function $\varnothing$ satisfying (1.1.1), defined in an interval I containing x_0, x_1 and satisfying the initial conditions $\varnothing(t_0) = x_0$, $\varnothing'(t_0) = x_1$.

These problems may be too complicated to solve analytically. Alternatively, the numerical methods can provide approximate solutions of the problems. The Euler Method, the Taylor Method and the Runge – Kutta methods serve as an introduction to numerical methods for solving systems of differential equations. The differential transformation technique is one of the numerical methods for ordinary(partial) differential equations which uses the form of polynomials as the approximation to the exact solution. However, the Taylor Method requires the calculation of high-order derivatives, a difficult symbolic and complex problem. The concept of differential transformation was first proposed by Zhou in 1986 and it was applied to solve linear and nonlinear initial value problems in electric circuit analysis (Zhou, 1986). This method has been applied to solve a second – order nonlinear differential equation that describes the under damped and over damped motion of a system subject to external excitations (Jang and Chen, 1997). In a recent work, Jang, Chen and Liy introduced the application of the concept of the differential transformation of fixed grid size to approximate solutions of linear and nonlinear initial value problems (Jang, Chen and Liy, 2000). Jang states that “the differential transform is an iterative procedure for obtaining Taylor series solutions of differential equations” (Jang, Chen and Liy, 2001). Although the Taylor series method requires more computational work for large orders, the present method reduces the size of computational domain and is applicable to many problems easily. This method has been applied to eigenvalue problems and Sturm – Lioville eigenvalue problem by Hassan (Hassan, 2002). This technique of fixed grid size is applied to solve higher – order initial value problems by I. H. Abdel – Halim Hassan (Abdel and Hassan, 2004). Ayaz has obtained numerical solution of linear differential – algebraic equations by using this method (Ayaz, 2004). Abbasov used this method to obtain approximate solutions of some linear and nonlinear equations related to engineering problems and observed that the numerical results are in good agreement with the analytical solutions (Abbasov and Bahadir, 2005).

In this thesis, the differential transform method is applied to the nonlinear initial value problems. This method does not evaluate the derivatives symbolically; instead, it calculates the relative derivatives by an iteration procedure described by the transformed equations obtained from the original equations using differential transformation.

1.2 Existence And Uniqueness Theorems

Definition 1: Consider the first order differential equation

$$\frac{dx}{dt} = f(t,x) \quad , \tag{1.2.1}$$

where f is continuous over some subset of the plane. The initial value problem associated with (1.2.1) is to find a function $\varnothing$ satisfying (1.2.1), defined in an interval I containing x_0 and satisfying the initial condition $\varnothing(t_0) = x_0$.

We will give abstract existence and uniqueness theorem first and then apply it to initial value problems.

Let T be a mapping on continuous functions. $T : E \rightarrow E$

$E = \{u(x) : u$ is continuous on the closed interval I and $A \leq u \leq B\}$

Definition 2:A mapping $T : E \to E$ is called a CONTRACTION MAPPING if there is a constant $\alpha, 0 < \alpha < 1$, such that for any $u, v \in E$

$$\max_{x \in I} \left| T(u(x)) - T(v(x)) \right| \le \alpha \max_{x \in I} \left| u(x) - v(x) \right|$$

Theorem 1.2.1. *(Contraction Mapping Theorem – Fixed Point Theorem) Let* $T : E \to E$ *be a contraction mapping. There exists a unique function* $y = \varnothing(x)$ *in E such that*

$$T(\varnothing(x)) = \varnothing(x) \quad for\ all\ x \in I$$
$$(Ty = y)$$

(Erwing Kreyszig, 1978)

Proof. The uniqueness follows from the definition. Suppose $\varnothing, \Psi$ are two functions in E such that

$$T(\varnothing(x)) = \varnothing(x), \quad T(\Psi(x)) = \Psi(x) \quad for\ all \quad x \in I$$

Then

$$\max_{x \in I} \left| \varnothing(x) - \Psi(x) \right| = \max_{x \in I} \left| T(\varnothing(x)) - T(\Psi(x)) \right| \le \alpha. \max_{x \in I} \left| \varnothing(x) - \Psi(x) \right|$$

But $\alpha < 1$ so we must have

$$\max_{x \in I} \left| \varnothing(x) - \Psi(x) \right| = 0 \quad i.e., \ \varnothing(x) = \Psi(x)$$

For existence, we apply an iteration scheme called Picard’s Method. Let $u_0(x)$ be any function in E. Let

$$u_1(x) = T(u_0(x)) \quad then \;\; u_1(x) \in E$$
$$u_2(x) = T(u_1(x)) \quad then \;\; u_2(x) \in E$$
$$.$$
$$.$$
$$.$$
$$u_n(x) = T(u_{n-1}(x)) \quad then \;\; u_n(x) \in E$$

Since T is contraction u_n's get closer each other. For $m \geq k$

$$\begin{aligned}
\max_{x\in I}\left|u_m(x) - u_k(x)\right| &= \max_{x\in I}\left|T(u_{m-1}(x)) - T(u_{k-1}(x))\right| \\
&\leq \alpha. \max_{x\in I}\left|u_{m-1}(x) - u_{k-1}(x)\right| = \alpha. \max_{x\in I}\left|T(u_{m-2}(x)) - T(u_{k-2}(x))\right| \\
&\leq \alpha^2 \max_{x\in I}\left|u_{m-2}(x) - u_{k-2}(x)\right| = \alpha^2 \max_{x\in I}\left|T(u_{m-3}(x)) - T(u_{k-3}(x))\right| \\
&. \\
&. \\
&. \\
&\leq \alpha^k \max_{x\in I}\left|u_{m-k}(x) - u_0(x)\right|
\end{aligned}$$

But $u_n(x) \in I$ for all n, hence $A \leq u_n(x) \leq B \;\; or \;\; |u_n(x)| < C = \max\{|A|, |B|\}$. So for all $m \geq k$ we have

$$\begin{aligned}
\left|u_m(x) - u_k(x)\right| \leq \alpha^k \max_{x\in I}\left|u_{m-k}(x) - u_0(x)\right| &\leq \alpha^k \left\{\left|u_{m-k}(x)\right| + \left|u_0(x)\right|\right\} \\
&\leq \alpha^k (C + C) = 2C\alpha^k \quad for \;\; all \;\; x \in I \qquad (1.2.2)
\end{aligned}$$

Since $0 < \alpha < 1$, $\alpha^k \to 0 \;\; as \;\; k \to \infty$ so

$$\lim_{k\to\infty}\left|u_m(x) - u_k(x)\right| = 0$$

So $\{u_k(x)\}$ form a Cauchy sequence therefore

$$\lim_{k\to\infty} u_k(x) = \varnothing(x) \text{ exists for all } x \in I$$

By taking the limit in (1.2.2) we get

$$\left|\varnothing(x) - u_k(x)\right| \leq 2C\alpha^k \text{ for all } x \in I \qquad (1.2.3)$$

We will show that $\varnothing$ is continuous on I. Let $\varepsilon > 0$ be given and $x_0 \in I$. Since $\alpha^k \to 0$ as $k \to \infty$ there is a number k_0 such that $2C\alpha^{k_0} < \frac{\varepsilon}{3}$.

Hence by (1.2.3)

$$\left|\varnothing(x) - u_{k_0}(x)\right| < \frac{\varepsilon}{3} \quad for \;\; all \;\; x \in I$$

Since u_{k_0} is continuous on I, there exists a $\delta > 0$ such that if $\left|x - x_0\right| < \delta$ then $\left|u_{k_0}(x) - u_{k_0}(x_0)\right| < \frac{\varepsilon}{3}$, Now suppose $\left|x - x_0\right| < \delta$. Then

$$\left|\varnothing(x) - \varnothing(x_0)\right| \le \left|\varnothing(x) - u_{k_0}(x)\right| + \left|u_{k_0}(x) - u_k(x_0)\right| + \left|u_k(x_0) - \varnothing(x_0)\right| < \varepsilon$$

So, $\varnothing(x)$ is continuous on I.

Also since $A \le u_k(x) \le B$ by taking the limit we get

$$A \le \varnothing(x) \le B \quad so \quad \varnothing(x) \in \varepsilon$$

We now show that $T(\varnothing(x)) = \varnothing(x) \quad for \;\; all \;\; x \in I$.

$$\left|T(u_k(x)) - T(\varnothing(x))\right| \le \alpha \;\; \max_{x \in I}\left|u_k(x) - \varnothing(x)\right| \;\; \le 2C\alpha^{k+1}$$

So,

$$\lim_{k \to \infty} T(u_k(x)) = T(\varnothing(x)) \quad for \;\; all \;\; x \in I$$

If we let m = k + 1 in (1.2.2)

$$2C\alpha^k \ge \left|u_{k+1}(x) - u_k(x)\right| = \left|T(u_k(x)) - u_k(x)\right|$$

$$\lim_{k \to \infty}\left(T(u_k(x)) - u_k(x)\right) = 0$$

$$\lim_{k \to \infty} T(u_k(x)) - \lim_{k \to \infty} u_k(x) = 0 = T(\varnothing(x)) - \varnothing(x) = 0$$

Therefore,

$$T(\varnothing(x)) = \varnothing(x) \quad for \;\; all \;\; x \in I$$

We can apply these theorems to the initial value problem.

$$\frac{dx}{dt} = f(t,x)$$
$$x(t_0) = x_0$$

Lemma 1. *A function* $x = \varnothing(t)$ *is a solution to the initial value problem if and only if it is satisfies the integral equation.*

$$\varnothing(t) = x_0 + \int_{t_0}^{t} f(\xi, \varnothing(\xi))\, d\xi \quad for\ all\ t \in I$$

Proof.

$$\frac{dx}{dt} = f(t,x), \quad x(t_0) = x_0$$

By integrating

$$\int_{t_0}^{t} \frac{dx}{dt} dt = \int_{t_0}^{t} f(\xi, x(\xi)) dt \quad for\ all\ t \in I$$

$$\underbrace{x(t)}_{\varnothing(t)} - \underbrace{x(t_0)}_{x_0} = \int_{t_0}^{t} f(\xi, x(\xi)) d\xi$$

$$\varnothing(t) = x_0 + \int_{t_0}^{t} f(\xi, \varnothing(\xi))\, d\xi$$

Lemma 2. *Let f(t,x) be continuous on a rectangle R.*
$R = \{(t,x) : a \le t \le b,\ A \le x \le B\}$ *then there is a constant M such that*

$$|f(t,x)| \le M \quad for\ all\ (t,x) \in R$$

Proof. M is just the maximum of the function $|f(t,x)|$ which is continuous on the closed set R.

So $|f(t,x)| \le M$

Lemma 3. *Let* $\dfrac{\partial f}{\partial x}$ *be continuous on a rectangle R. Then there is a constant K such that*

$$\left| f(t, x_1) - f(t - x_2) \right| \le K \left| x_1 - x_2 \right| \quad \textit{for all } (t, x_1) \in R,\ (t, x_2) \in R$$

Proof. If we take $K = \max\limits_{R} \left| \dfrac{\partial f}{\partial x} \right|$ it follows from the Mean-Value Theorem. In this case we say that f(x,y) satisfies Lipschitz continuity. $\square$

Theorem 1.2.2. *Suppose f(t,x) and* $\dfrac{\partial f}{\partial x}$ *are continuous on R then there is a unique solution to the initial value problem in some interval* $\left| x - x_0 \right| < h$. *(D. H. Griffel, 1993)*

Proof. $\dfrac{dx}{dt} = f(t, x)$

$x(t_0) = x_0$ is to equivalent to find $\varnothing(t)$

$$\varnothing(t) = x_0 + \int_{t_0}^{t} f(\xi, \varnothing(\xi)) d\xi \quad \text{for } \left| t - t_0 \right| < h$$

$$\textit{if } \ T : x_0 + \int_{t_0}^{t} f(\xi, \ ) d\xi$$

$$\varnothing(t) = T(\varnothing(t))$$

So, we can apply Fixed Point Theorem for the existence-uniqueness solution. $\square$

CHAPTER TWO

DIFFERENTIAL TRANSFORM METHOD

2.1 Description of Differential Transform Method

Differential transform of a smooth function y(x) is defined as follows:

$$Y(k) = \frac{1}{k!}\left[\frac{d^k y(x)}{dx^k}\right]_{x=0} \quad , \tag{2.1.1}$$

where y(x) is the original function and Y(k) is the transformed function, which is also called the T-function. Differential inverse transform of Y(k) is defined as:

$$y(x) = \sum_{k=0}^{\infty} x^k Y(k) \tag{2.1.2}$$

Combining (2.1.1) and (2.1.2), we obtain:

$$y(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}\left[\frac{d^k y(x)}{dx^k}\right]_{x=0} \tag{2.1.3}$$

Series (2.1.3) implies that the concept of differential transform is derived from Taylor series expansion, but the method does not evaluate the derivatives symbolically.

However, relative derivatives are calculated by an iterative way which are described by the transformed equations of the original functions. In this thesis, we use the lower case letter to represent the original function and upper case letter represent the transformed function.

From (2.1.1) and (2.1.2), it is easily proven that the transformed functions comply with the basic mathematical operations shown in the theorems.

In actual applications, the function y(x) is expressed by a finite series and (2.1.2) can be written as

$$y(x) = \sum_{k=0}^{m} x^k Y(k) \tag{2.1.4}$$

Series (2.1.3) implies that $\sum_{k=m+1}^{\infty} x^k Y(k)$ is negligibly small. In fact, m is decided by the convergence of natural frequency in this thesis.

The fundamental theorems of the one- dimensional differential transform are given below.

Theorem 2.1.1. *If* $w(x) = y(x) \mp z(x)$, *then* $W(k) = Y(k) \mp Z(k)$

Proof. By using the definition of the transform:

$$Y(k) = \frac{1}{k!}\frac{d^k y(x)}{dx^k} \quad , \qquad y(x) = \sum_{k=0}^{\infty} x^k Y(k)$$

$$Z(k) = \frac{1}{k!}\frac{d^k z(x)}{dx^k} \quad , \qquad z(x) = \sum_{k=0}^{\infty} x^k Z(k)$$

$$Y(k) \mp Z(k) = \frac{1}{k!}\frac{d^k y(x)}{dx^k} \mp \frac{1}{k!}\frac{d^k z(x)}{dx^k}$$

$$Y(k) \mp Z(k) = \frac{1}{k!}\frac{d^k}{dx^k}\left[y(x) \mp z(x)\right] \text{ by the hypothesis.}$$

$$y(x) \mp z(x) = w(x)$$

So,

$$Y(k) \mp Z(k) = \frac{1}{k!}\frac{d^k w(x)}{dx^k} = W(k) \tag{2.1.5}$$

$$w(x)=\sum_{k=0}^{\infty}x^kY(k)\mp\sum_{k=0}^{\infty}x^kZ(k)$$
$$=\sum_{k=0}^{\infty}x^k\left(Y(k)\mp Z(k)\right)$$

By using the definition (2.1.5)

$$w(x)=\sum_{k=0}^{\infty}x^kW(k)$$

□

Theorem 2.1.2. *If w(x) = cy(x), then W(k) = cY(k)*

Proof. By using definition of the differential transform

$$W(k)=\frac{1}{k!}\frac{d^k}{dx^k}\left[cy(x)\right]$$

where c is a constant. Thus, we have

$$W(k)=c\left[\frac{1}{k!}\frac{d^k y(x)}{dx^k}\right]=cY(k)$$
$$W(k)=cY(k)$$

□

Theorem 2.1.3. *If* $w(x)=\frac{dy(x)}{dx}$, *then* $W(k)=(k+1)Y(k+1)$

Proof. By utilizing the definition of transform:

$$y(x)=\sum_{k=0}^{\infty}x^kY(k)$$

$$y(x)=Y(0)+xY(1)+x^2Y(2)+x^3Y(3)+\ldots \qquad (2.1.6)$$

By taking the derivative of (2.1.6)

$$\frac{dy(x)}{dx} = Y(1) + 2xY(2) + 3x^2Y(3) + ...$$

$$\frac{dy}{dx} = \sum_{k=1}^{\infty} x^{k-1}\ k\ Y(k)$$

By starting the index from k = 0 instead of k = 1 we can obtain $\frac{dy}{dx}$ as follows:

$$w(x) = \frac{dy(x)}{dx} = \sum_{k=0}^{\infty} x^k (k+1)Y(k+1)$$

Consequently, we obtain

$$W(k) = (k+1)Y(k+1)$$

□

Theorem 2.1.4. *If* $w(x) = \frac{d^n y(x)}{dx^n}$, *then* $W(k) = \frac{(k+n)!}{k!}Y(k+n)$

Proof. By using the definition of the transform

$$y(x) = \sum_{k=0}^{\infty} x^k Y(k)$$

$$y(x) = Y(0) + xY(1) + x^2Y(2) + x^3Y(3) + ... \qquad (2.1.7)$$

Step by step if we take the derivative of (2.1.7)

$$\frac{dy(x)}{dx} = Y(1) + 2xY(2) + 3x^2Y(3) + ...$$
$$\frac{dy(x)}{dx} = \sum_{k=0}^{\infty} x^k (k+1)Y(k+1)$$
$$\frac{d^2y(x)}{dx^2} = 2Y(2) + 6xY(3) + ...$$
$$\frac{d^2y(x)}{dx^2} = \sum_{k=0}^{\infty} x^k (k+1)(k+2)Y(k+2)$$
.
.
.
$$w(x) = \frac{d^n y(x)}{dx^n} = \sum_{k=0}^{\infty} x^k (k+1)(k+2)...(k+n)Y(k+n).$$

We have

$$W(k) = \frac{(k+n)!}{k!}Y(k+n)$$

This can be proved by mathematical induction. □

Theorem 2.1.5. *If w(x) = y(x) z(x), then* $W(k) = \sum_{m=0}^{k} Y(m)Z(k-m)$

Proof. By the definition of transform,

$$w(x) = \sum_{m=0}^{\infty} x^m Y(m) \sum_{j=0}^{\infty} x^j Z(j)$$
$$w(x) = \sum_{k=0}^{\infty} x^k \sum_{m=0}^{k} Y(m)Z(k-m)$$

We get

$$W(k) = \sum_{m=0}^{k} Y(m)Z(k-m)$$

□

Theorem 2.1.6. *If* $w(x) = y_1(x).y_2(x)...y_{n-1}(x).y_n(x)$, *then*

$$W(k) = \sum_{k_{n-1}=0}^{k} \sum_{k_{n-2}=0}^{k_{n-1}} ... \sum_{k_2=0}^{k_3} \sum_{k_1=0}^{k_2} Y_1(k_1)Y_2(k_2-k_1)...Y_{n-1}(k_{n-1}-k_{n-2})xY_n(k-k_{n-1})$$

(A. Arikoğlu and I. Özkol, 2005)

Proof. By using the definition of the transform

$$W(0)=\frac{1}{0!}\left[y_1(x)y_2(x)...y_{n-1}(x)y_n(x)\right]_{x=x_0}=Y_1(0)Y_2(0)...Y_{n-1}(0)Y_n(0)\,,$$

$$W(1)=\frac{1}{1!}\frac{d}{dx}\underbrace{\left[y_1(x)y_2(x)...y_{n-1}(x)y_n(x)\right]_{x=x_0}}$$

$$=\begin{bmatrix} y_1^{\prime}(x)y_2(x)...y_{n-1}(x)y_n(x)+y_1(x)y_2^{\prime}(x)...y_{n-1}(x)y_n(x)+... \\ +y_1(x)y_2(x)...y_{n-1}{}^{\prime}(x)y_n(x)+y_1(x)y_2(x)...y_{n-1}(x)y_n^{\prime}(x) \end{bmatrix}_{x=x_0}$$

$$\begin{aligned} W(1)=&\,Y_1(1)Y_2(0)...Y_{n-1}(0)Y_n(0)+Y_1(0)Y_2(1)....Y_{n-1}(0)Y_n\ (0)+... \\ &+Y_1(0)Y_2(0)...Y_{n-1}(1)Y_n(0)+Y_1(0)Y_2(0)...Y_{n-1}(0)Y_n(1)\,, \end{aligned}$$

$$\begin{aligned} W(2)=&\,Y_1(1)Y_2(1)Y_3(0)...Y_n(0)+Y_1(1)Y_2(0)Y_3(1)...Y_n(0)+... \\ &+Y_1(1)Y_2(0)Y_3(0)...Y_n(1)+Y_1(0)Y_2(1)Y_3\ (1)....Y_n(0)+... \\ &+Y_1(0)Y_2(0)...Y_{n-1}(1)Y_n(1)+...+Y_1(2)Y_2(0)Y_3(0)...Y_n(0) \\ &+...+Y_1(0)Y_2(0)Y_3(0)...Y_n(2) \end{aligned}$$

In general, we have

$$W(k)=\sum_{k_{n-1}=0}^{k}\ \sum_{k_{n-2}=0}^{k_{n-1}}...\sum_{k_2=0}^{k_3}\ \sum_{k_1=0}^{k_2}Y_1(k_1)Y_2(k_2-k_1)...Y_{n-1}(k_{n-1}-k_{n-2})Y_n(k-k_{n-1})$$

$\square$

Theorem 2.1.7. *If w(x) = c, then* $W(k)=c\delta(k)$

Proof. By using the definition of the differential transform

$$w(x)=\sum_{k=0}^{\infty}x^kW(k)$$

where w(x) = c. Thus, we get

$$c=W(0)+xW(1)+x^2W(2)+...$$

From the definition of the polynomials

$$c=W(0)$$
$$W(1)=W(2)=...=0$$

So, we have

$$W(k)=\begin{cases} c\,, & k=0 \\ 0\,, & k\neq 0 \end{cases} \quad , k=0,1,2,...$$

$$=c\begin{cases} 1\,, & k=0 \\ 0\,, & k\neq 0 \end{cases}$$

$$W(k)=c\delta(k)$$

□

Theorem 2.1.8. *If w(x) = x, then* $W(k)=\delta(k-1)$

Proof. By using the definition of the transform

$$w(x)=\sum_{k=0}^{\infty} x^k W(k)$$

where w(x) = x

$$x=W(0)+xW(1)+x^2W(2)+x^3W(3)+...$$

From the definition of the polynomials

$$W(0)=0$$

$$W(1)=1$$

$$W(2)=W(3)=...=0$$

$$W(k)=\begin{cases} 1\,, & k=1 \\ 0\,, & k\neq 1 \end{cases}$$

$$W(k)=\begin{cases} 1\,, & k-1=0 \\ 0\,, & k-1\neq 0 \end{cases}$$

Finally, we obtain

$$W(k)=\delta(k-1)$$

□

Theorem 2.1.9. *If* $w(x)=x^m$, *then* $W(k)=\delta(k-m)$

Proof. By using the definition of the differential transform

$$
\begin{aligned}
&w(x)=\sum_{k=0}^{\infty}x^kW(k)\\
&where\ \ w(x)=x^m\\
&x^m=W(0)+xW(1)+...+x^mW(m)+...\\
&W(0)=W(1)=...=W(m-1)=...=0\\
&W(m)=1\\
&W(k)=\begin{cases}1\,,\ \ k=m\\ 0\,,\ \ k\neq m\end{cases}\qquad k=0,1,2,...\\
&W(k)=\begin{cases}1\,,\ \ k-m=0\\ 0\,,\ \ k-m\neq 0\end{cases}
\end{aligned}
$$

Thus , we get

$$W(k)=\delta(k-m)$$

□

Theorem 2.1.10. *If* $w(x)=e^{\lambda x}$, *then* $W(k)=\dfrac{\lambda^k}{k!}$

Proof. By using the definition of the transform

$$w(x)=\sum_{k=0}^{\infty}x^kW(k)$$

We use Taylor Series expansion of $e^{\lambda x}$

$$
\begin{aligned}
&1+\lambda x+\frac{\lambda^2}{2!}x^2+\frac{\lambda^3}{3!}x^3+...=W(0)+xW(1)+x^2W(2)+x^3W(3)+...\\
&W(0)=1\\
&W(1)=\lambda\\
&W(2)=\frac{\lambda^2}{2}\\
&W(3)=\frac{\lambda^3}{3!}\\
&.\\
&.\\
&.\\
&W(k)=\frac{\lambda^k}{k!}
\end{aligned}
$$

□

Theorem 2.1.11. *If* $w(x) = (1+x)^m$, *then* $W(k) = \dfrac{m(m-1)...(m-k+1)}{k!}$

Proof. By using the definition of the differential transform

$$w(x) = \sum_{k=0}^{\infty} x^k W(k)$$

We use Binomial theorem of $(1+x)^m$

$$1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + ... = W(0) + xW(1) + x^2W(2) + ...$$
$$W(0) = 1$$
$$W(1) = m$$
$$W(2) = \frac{m(m-1)}{2!}$$
$$W(3) = \frac{m(m-1)(m-2)}{3!}$$

Since, we obtain

$$W(k) = \frac{m(m-1)...(m-(k-1))}{k!}$$

□

Theorem 2.1.12. *If* $w(x) = \sin(zx)$, *then* $W(k) = \dfrac{z^k}{k!}\sin\left(\dfrac{\pi k}{2!}\right)$

Proof. By using the definition of the transform

$$w(x) = \sum_{k=0}^{\infty} x^k W(k)$$

We use

$$zx - \frac{(zx)^3}{3!} + \frac{(zx)^5}{5!} - \frac{(zx)^7}{7!} + ... = W(0) + xW(1) + x^2W(2) + x^3W(3) + ...$$
$$W(0) = 0, \; W(2) = 0, \; W(4) = 0, \; ..., \; W(2k) = 0$$
$$W(1) = z, W(3) = -\frac{z^3}{3!}, \; W(5) = \frac{z^5}{5!}, \; ..., \; W(2k+1) = (-1)^k \frac{z^{2k+1}}{(2k+1)!}$$

Thus, we get

$$W(k)=\frac{z^k}{k!}\sin\left(\frac{\pi k}{2}\right) \qquad \square$$

Theorem 2.1.13. *If* $w(x)=\cos(zx)$, *then* $W(k)=\frac{z^k}{k!}\cos\left(\frac{k\pi}{2}\right)$

Proof. By using the definition of the transform

$$w(x)=\sum_{k=0}^{\infty}x^k W(k)$$

$$1-\frac{z^2}{2!}x^2+\frac{z^4}{4!}x^4-\frac{z^6}{6!}x^6+\ldots=W(0)+xW(1)+x^2W(2)+x^3W(3)+x^4W(4)+\ldots$$

$$W(1)=W(3)=W(5)=\ldots=W(2k+1)=0$$

$$W(0)=1$$

$$W(2)=-\frac{z^2}{2!}$$

$$W(4)=\frac{z^4}{4!}$$

$$W(6)=-\frac{z^6}{6!}$$

.

.

.

$$W(2k)=(-1)^k\frac{z^{2k}}{(2k)!}$$

We use

Finally, we have

$$W(k)=\frac{z^k}{k!}\cos\left(\frac{\pi k}{2}\right)$$

$\square$

Theorem 2.1.14. *If* $w(x)=\int_0^x y(t)dt$ *, then* $W(k)=\dfrac{Y(k-1)}{k}$, *where* $k\geq 1$

(A. Arikoğlu and I. Özkol, 2005)

Proof. By using equation (2.1.2) the transform of an integral can be found as follows:

$$w(x)=\int_0^x y(t)dt \quad then \quad w(x)=\int_0^x \sum_{k=0}^{\infty} Y(k)t^k dt$$
$$=\sum_{k=0}^{\infty}\int_{x_0}^{x} Y(k)t^k dt$$
$$=\sum_{k=0}^{\infty}\left[Y(k)\frac{t^{k+1}}{k+1}\Bigg|_0^x \right]$$
$$=\sum_{k=0}^{\infty}\frac{Y(k)}{(k+1)}x^{k+1}$$

By starting the index from k = 1 instead of k = 0 we can obtain w(x) as follows:

$$w(x)=\sum_{k=1}^{\infty}\frac{Y(k-1)}{k}x^k$$

By using (2.1.1) and (2.1.2) we get:

$$W(k)=\frac{Y(k-1)}{k}, \quad where \quad k\geq 1 \quad and \quad W(0)=0$$

$\square$

CHAPTER THREE

APPLICATIONS

3.1 Applications

In this chapter, we solve four nonlinear initial value problems by the differential transform method compare the results with exact solutions

Example 3.1.1. Consider the second order nonlinear initial value problem

$$-2y'' + y'^2 = -1 \qquad (3.1.1)$$

with initial conditions

$$y(\pi) = 0 \qquad (3.1.2)$$

$$y'(\pi) = 0 \qquad (3.1.3)$$

The exact solution of the problem is $y(x) = -2\ell n\left(\sin\frac{x}{2}\right)$. One can see that the differential transform of equation (3.1.1) can be evaluated by using Theorems (2.1.4) – (2.1.5) and (2.1.7) as follows:

$$-2(k+1).(k+2)Y(k+2) + \sum_{\ell=0}^{k}(\ell+1)(k+1-\ell)Y(\ell+1)Y(k+1-\ell) = -\delta(k) \quad (3.1.4)$$

where Y(k) is the differential transformation of the corresponding function y(x). The initial conditions are transformed as follows:

$$y(x) = \sum_{k=0}^{\infty} Y(k)(x-x_0)^k \quad at \quad x_0 = \pi$$

$$y(x) = Y(0) + Y(1)(x-\pi) + Y(2)(x-\pi)^2 + Y(3)(x-\pi)^3 + ... \qquad (3.1.5)$$

$$y(\pi) = Y(0) + Y(1)(\pi-\pi) + Y(2)(\pi-\pi)^2 + Y(3)(\pi-\pi)^3 + ...$$

$$0 = Y(0) + Y(1)\,0 + Y(2)\,0 + Y(3)\,0 + ...$$

From the definition of polynomials equality

$$Y(0) = 0 \qquad (3.1.6)$$

Differentiating (3.1.5) we find

$$y^{'}(x) = Y(1) + 2Y(2)(x-\pi) + 3Y(3)(x-\pi)^2 + ...$$
$$y^{'}(\pi) = Y(1) + 2Y(2)(\pi-\pi) + 3Y(3)(\pi-\pi)^2 + ...$$
$$0 = Y(1) + 2Y(2)\,0 + 3Y(3)\,0 + ...$$
$$Y(1) = 0 \qquad (3.1.7)$$

By using (3.1.6) and (3.1.7) in (3.1.4), we obtain the following simplified equations, for k = 0,1,2,... and we get

For k = 0

$$(-2).1.2.Y(2) + 1.1.Y(1).Y(1) = -\delta(0)$$
$$Y(2) = \frac{1}{4} \qquad (3.1.8)$$

For k = 1

$$(-2).2.3.Y(3) + \sum_{\ell=0}^{1}(\ell+1)(2-\ell)Y(\ell+1)Y(2-\ell) = \delta(1)$$

From the theorem (2.1.6) $\delta(k) = 0, \quad for\ k \geq 1$

Consequently, we obtain

$$Y(3) = 0 \qquad (3.1.9)$$

$Y(k)\ \ for\ \ k \geq 2$ are easily obtained as follows

$$Y(4) = \frac{1}{96}, \quad Y(5) = 0, \quad Y(6) = \frac{1}{1440}, \quad Y(7) = 0, \quad Y(8) = \frac{17}{322560}$$

and so on. In general, we find

$$Y(2k+1) = 0, \ \ k = 0,1,2,...$$

Substitution of all Y(k) into Eq. (2.1.2) give the solution in a series form:

$$y(x)=\frac{1}{4}(x-\pi)^2+\frac{1}{96}(x-\pi)^4+\frac{1}{1140}(x-\pi)^6+\frac{17}{322560}(x-\pi)^8+O((x-\pi)^9)$$

Table 3.1 Numerical Results

x	y (DTM)	y (Exact)	y (Error)
π	0.0	0.0	0.0
3.22743	0.00184272	0.00184272	$1.55571x10^{-13}$
3.31327	0.0073777	0.0073777	$3.98716x10^{-11}$
3.39911	0.0166254	0.0166254	$1.02494x10^{-9}$
3.48496	0.0296205	0.0296205	$1.0281x10^{-8}$
3.5708	0.0464118	0.0464118	$6.16145x10^{-8}$
3.65664	0.0670637	0.0670639	$2.66712x10^{-7}$
3.74248	0.0916564	0.916573	$9.22752x10^{-7}$
3.82832	0.120288	0.12029	$2.71056x10^{-6}$
3.91416	0.153073	0.15308	$7.02919x10^{-6}$
4.0	0.19015	0.190166	0.0000165273

Example 3.1.2. Consider the second order nonlinear initial value problem

$$y' = xy'' + y''^2 \tag{3.1.10}$$

with initial conditions

$$y(-1)=0 \tag{3.1.11}$$

$$y'(-1)=2 \tag{3.1.12}$$

To solve this problem, we write p instead of y$'$ and obtain Clairaut's equation

$$p = x\frac{dp}{dx}+\left(\frac{dp}{dx}\right)^2 \tag{3.1.13}$$

The equation (3.1.13) has the general solution as follows:

$$p = cx + c^2 \tag{3.1.14}$$

Using the initial conditions (3.1.11) – (3.1.12), the following straight lines obtain for Clairaut's equation

$$p_1(x) = -x + 1 \qquad (3.1.15)$$

$$p_2(x) = 2x + 4 \qquad (3.1.16)$$

Consequently, we have two exact solutions from (3.1.15) and (3.1.16)

$$y_1(x) = -\frac{x^2}{2} + x + \frac{3}{2} \qquad (3.1.17)$$

$$y_2(x) = x^2 + 4x + 3 \qquad (3.1.18)$$

Now, we take the differential transform of ODE and use the initial condition y(-1) = 0, $y^{'}(-1) = 2$ to obtain:

$$(k+1)Y(k+1) = \sum_{\ell=0}^{k} \delta(\ell-1)(k-\ell+1)(k-\ell+2)Y(k-\ell+2)$$

$$+\sum_{\ell=0}^{k} (\ell+1)(\ell+2)Y(\ell+2)(k-\ell+1)(k-\ell+2)Y(k-\ell+2) - (k+1)(k+2)Y(k+2) \qquad (3.1.19)$$

$$Y(0) = 0\ , \quad Y(1) = 2 \qquad (3.1.20)$$

For each k, substituting (3.1.20) into (3.1.19), and by recursive method,

$Y(2) = -\dfrac{1}{2}$ or $Y(2) = 1$, $Y(3) = 0$, $Y(4) = 0$, $Y(5) = 0$, $Y(6) = 0$, ... and so on. In general, we find

Y(k) = 0, k = 3,4,5,6,…

Substitution of Y(0), Y(1), Y(2) into equation (2.1.2) we obtain two polynomial solutions

$$y_1(x) = 2(x+1) - \frac{1}{2}(x+1)^2$$

$$y_2(x) = 2(x+1) + (x+1)^2$$

Table 3.2 Numerical Results

x	y_1 (DTM)	y_1 (Exact)	y_1 (Error)
-1.0	0.0	0.0	0.0
-0.9	0.195	0.195	0.0
-0.8	0.38	0.38	0.0
-0.7	0.555	0.555	0.0
-0.6	0.72	0.72	0.0
-0.5	0.875	0.875	0.0
-0.4	1.02	1.02	0.0
-0.3	1.155	1.155	$2.22045x10^{-16}$
-0.2	1.28	1.28	0.0
-0.1	1.395	1.395	0.0
0.0	1.5	1.5	0.0

x	y_2 (DTM)	y_2 (Exact)	y_2 (Error)
-1.0	0.0	0.0	0.0
-0.9	0.21	0.21	0.0
-0.8	0.44	0.44	$5.55112x10^{-17}$
-0.7	0.69	0.69	0.0
-0.6	0.96	0.96	0.0
-0.5	1.25	1.25	0.0
-0.4	1.56	1.56	0.0
-0.3	1.89	1.89	0.0
-0.2	2.24	2.24	0.0
-0.1	2.61	2.61	$4.44089x10^{-16}$
0.0	3.0	3.0	0.0

Example 3.1.3. Consider the second order nonlinear initial value problem

$$y'' = 1 + y'^2 \qquad (3.1.21)$$

with initial conditions

$$y(0) = 0 \qquad (3.1.22)$$

$$y'(0) = 0 \qquad (3.1.23)$$

The exact solution of the problem is $y(x) = -\ell n(\cos x)$. One can see that differential transform of equation (3.1.21) can be evaluated by using Theorems (2.1.4) – (2.1.5) and (2.1.7) as follows:

$$(k+1)(k+2)Y(k+2)=\delta(k)+\sum_{\ell=0}^{k}(\ell+1)(k-\ell+1)Y(\ell+1)Y(k-\ell+1) \qquad (3.1.24)$$

where Y(k) is the differential transformation of the corresponding function y(x).

The initial conditions are transformed as follows:

$$y(x)=\sum_{k=0}^{\infty}Y(k)(x-x_0)^k \quad \text{at} \quad x_0=0$$

$$y(x)=Y(0)+Y(1)x+Y(2)x^2+Y(3)x^3+\ldots \qquad (3.1.25)$$

We use initial condition (3.1.22)

$$y(0)=Y(0)+Y(1).0+Y(2).0^2+Y(3).0^3+\ldots$$

So, we find

$$Y(0)=0 \qquad (3.1.26)$$

Differentiating (3.1.25) we have

$$y^{'}(x)=Y(1)+2Y(2)x+3Y(3)x^2+\ldots$$

From the initial condition (3.1.23)

$$y^{'}(0)=Y(1)+2Y(2).0+3Y(3).0^2+\ldots$$

$$Y(1)=0 \qquad (3.1.27)$$

By using (3.1.26) and (3.1.27) in (3.1.24), we obtain the following simplified equations, for k=0,1,2,… and we get

For k = 0

$$1.2.Y(2)=\delta(0)+1.1Y(1).Y(1)$$

$$Y(2)=\frac{1}{2} \qquad (3.1.28)$$

For k = 1

$$2.3.Y(3)=\delta(1)+\sum_{\ell=0}^{1}(\ell+1)(2-\ell)Y(\ell+1)Y(2-\ell)$$

By using theorem (2.1.6) $\delta(k)=0$, for $k\geq 1$

Consequently, we obtain

$$Y(3)=0 \tag{3.1.29}$$

$Y(k)$ for $k\geq 2$ are easily obtained as follows:

$$Y(4)=\frac{1}{12},\quad Y(5)=0,\quad Y(6)=\frac{1}{45},\quad Y(7)=0,\quad Y(8)=\frac{17}{2520}$$

and so on. In general, we have

$$Y(2k+1)=0,\quad k=0,1,2,...$$

Substitution of all Y(k) into equation (2.1.2) give the solution in a series form:

$$y(x)=\frac{1}{2}x^2+\frac{1}{12}x^4+\frac{1}{45}x^6+\frac{17}{2520}x^8+O(x^9)$$

Table 3.3 Numerical Results

x	y (DTM)	y (Exact)	y (Error)
0.0	0.0	0.0	0.0
0.1	0.00500836	0.00500836	$6.85216x10^{-16}$
0.2	0.0201348	0.0201348	$3.06797x10^{-12}$
0.3	0.0456917	0.0456917	$4.05201x10^{-10}$
0.4	0.082229	0.082229	$1.31219x10^{-8}$
0.5	0.130584	0.130584	$1.97509x10^{-7}$
0.6	0.191963	0.191965	$1.83835x10^{-6}$
0.7	0.268073	0.268086	0.0000123308
0.8	0.361325	0.361391	0.0000653731
0.9	0.475151	0.475442	0.000291156
1.0	0.614489	0.61526	0.00113793

3.1.4 Simple Pendulum

Consider the second order nonlinear initial value problem

$$\frac{d^2\theta}{dt^2} + \sin\theta = 0 \qquad (3.1.30)$$

with the initial conditions

$$\theta(0) = \frac{\pi}{3} \qquad (3.1.31)$$

$$\theta'(0) = 0 \qquad (3.1.32)$$

Exact solution is obtained as follows:

Integrating the given equation with respect to θ we have

$$\frac{1}{2}.\left(\frac{d\theta}{dt}\right)^2 - \cos\theta = \text{constant} = -\cos(\pi/3)$$

$$\frac{1}{2}.\left(\frac{d\theta}{dt}\right)^2 = \cos\theta - \cos(\pi/3) = \left(1 - 2\sin^2\left(\theta/2\right) - 1 + 2\sin^2\left(\pi/6\right)\right)$$

$$\left(\frac{d\theta}{dt}\right)^2 = 4\left(\sin^2\left(\pi/6\right) - \sin^2\left(\theta/2\right)\right)$$

Thus

$$\frac{d\theta}{dt} = -2\sqrt{\sin^2\left(\pi/6\right) - \sin^2\left(\theta/2\right)}$$

$$-\frac{d\theta}{\sqrt{\sin^2\left(\pi/6\right) - \sin^2\left(\theta/2\right)}} = 2dt$$

The negative sign being taken since $\dot{\theta}$ is initially negative. Hence

$$2t = -\int_{\pi/3}^{\theta} \frac{du}{\sqrt{\sin^2\left(\pi/6\right) - \sin^2\left(u/2\right)}}$$

writing $k=\sin\left(\pi/6\right)$ and $\sin\left(u/2\right)=kv$, we have

$$\frac{1}{2}\cos\left(u/2\right)du = kdv$$

$$du = \frac{2k}{\sqrt{1-k^2v^2}}\,dv$$

$$t = -\int_1^x \frac{k}{\sqrt{\left(1-k^2v^2\right)\left(k^2-k^2v^2\right)}}\,dv$$

$$t = -\int_1^x \frac{dv}{\sqrt{\left(1-v^2\right)\left(1-k^2v^2\right)}}$$

where $kx=\sin\left(\theta/2\right)$. Thus

$$t = \int_0^1 \frac{dv}{\sqrt{\left(1-v^2\right)\left(1-k^2v^2\right)}} - \int_0^x \frac{dv}{\sqrt{\left(1-v^2\right)\left(1-k^2v^2\right)}}$$

$$t = K - sn^{-1}(x)$$

$$sn^{-1}(x) = K - t$$

The modulus of the Jacobian function being k and its period 4K. Then

$$x = sn(K-t)$$

$$x = -sn(t-K)$$

$$x = sn(t+K)$$

that is,

$$\sin\left(\theta/2\right) = \sin\left(\pi/6\right).sn\left(t+K\right)$$

So, the exact solution is obtained as

$$\theta(t) = 2\sin^{-1}\left[\sin\left(\pi/6\right).sn\left(t+K\right)\right]$$

The period is $4K = 2\pi._2F_1\left(\frac{1}{2},\frac{1}{2};1;\sin^2\left(\pi/6\right)\right)$ (C.G.Lambe and C.J. Tranter, 1961)

Approximate solution is obtained as follows:

Now, we apply the differential transform method to ODE and the initial conditions $\theta(0) = \pi/3$, $\theta'(0) = 0$;

$$(k+1)(k+2)Y(k+2) = -F(k) \tag{3.1.33}$$

$$Y(0) = \pi/3, \quad Y(1) = 0 \tag{3.1.34}$$

Trigonometric nonlinearity $f(\theta) = \sin\theta$ and $g(\theta) = \cos\theta$

$$\begin{aligned} F(0) &= [\sin(\theta(t))]_{t=0} = \sin(\theta(0)) = \sin(Y(0)), \\ G(0) &= [\cos(\theta(t))]_{t=0} = \cos(\theta(0)) = \cos(Y(0)) \end{aligned} \tag{3.1.35}$$

To find other transformed functions, we differentiate f(θ) = $\sin\theta$ and g(θ)=$\cos\theta$, obtaining

$$\begin{aligned} \frac{df(\theta)}{dt} &= \cos\theta\frac{d\theta}{dt} = g(\theta)\frac{d\theta}{dt}, \\ \frac{dg(\theta)}{dt} &= -\sin\theta\frac{d\theta}{dt} = -f(\theta)\frac{d\theta}{dt}. \end{aligned} \tag{3.1.36}$$

Applying the differential transform to Eq. (3.1.35), we obtain;

$$\begin{aligned} (k+1)F(k+1) &= \sum_{m=0}^{k}(k+1-m)G(m)Y(k+1-m), \\ (k+1)G(k+1) &= -\sum_{m=0}^{k}(k+1-m)F(m)Y(k+1-m). \end{aligned} \tag{3.1.37}$$

Similarly, replacing k+1 by k gives

$$\begin{aligned} F(k) &= \sum_{m=0}^{k-1}\frac{k-m}{k}G(m)Y(k-m), \quad k \geq 1, \\ G(k) &= -\sum_{m=0}^{k-1}\frac{k-m}{k}F(m)Y(k-m), \; k \geq 1 \end{aligned} \tag{3.1.38}$$

Combine equations (3.1.34) and (3.1.37) to give the recursive relation

$$F(k)=\begin{cases}\sin\big(Y(0)\big), & k=0,\\ \sum_{m=0}^{k-1}\dfrac{k-m}{k}G(m)Y(k-m), & k\geq 1\end{cases} \tag{3.1.39}$$

(S-H Chang and I-L Chang, 2008)

Substituting equation (3.1.31) and k = 0 into equations (3.1.30) and (3.1.39), we get

$$F(0)=\sin\big(Y(0)\big)=\sin\left(\pi/3\right)\ ,\quad Y(2)=-\frac{\sin\left(\pi/3\right)}{2},$$
$$F(0)=0.866025403\ ,\quad Y(2)=-0.433012701$$

Following the same recursive procedure, we find

$$Y(3)=0\ ,\quad Y(4)=0.018042195,\ \ Y(5)=0,$$
$$Y(6)=0.002405626109\ ,\ \ Y(7)=0,\quad Y(8)=0.008398759293$$

In general,

$$Y(2k+1)=0,\quad k=0,1,2,...$$

Substitution of all Y(k) into Eq. (2.1.2) we obtain the solution in a series form:

$$\theta(t)=\pi/3+(-0.433012701)t^2+(0.018042195)t^4+(0.002405626109)t^6$$
$$+(0.008398759293)t^8+O(t^9)$$

Table 3.4 Numerical Results

t	θ(DTM)	θ(Exact)	θ (Error)
0.0	1.0472	1.0472	$4.44089x10^{-16}$
0.1	1.04287	1.04287	$9.54505x10^{-11}$
0.2	1.02991	1.02991	$2.22098x10^{-8}$
0.3	1.00837	1.00837	$5.68446x10^{-7}$
0.4	0.978393	0.978387	$5.66793x10^{-6}$
0.5	0.940142	0.940109	0.0000338461
0.6	0.893905	0.893759	0.000145549
0.7	0.84012	0.839621	0.000499619
0.8	0.779499	0.778045	0.00145422
0.9	0.713189	0.709457	0.00373169
1.0	0.643031	0.634362	0.00866992

CHAPTER FOUR

CONCLUSION

Differential transformation method has been applied to second order non-linear initial value problems. The results for four numerical examples showed that the present method is quite reliable. The method has been successfully applied to non-linear second order initial value problems. The numerical results obtained by present method are also compared with the exact solutions. All computations are made by Mathematica. It is shown that the results are found to be in good agreement with each other.

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