DOKUZ EYLÜL UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

NEW DIFFERENCE METHOD FOR ORDINARY DIFFERENTIAL EQUATIONS

by
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İZMİR

NEW DIFFERENCE METHOD FOR ORDINARY DIFFERENTIAL EQUATIONS

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M.Sc THESIS EXAMINATION RESULT FORM

We have read the thesis entitled **"NEW DIFFERENCE METHOD FOR ORDINARY DIFFERENTIAL EQUATIONS"** completed by **TUĞÇE UYGURTÜRK** under supervision of **PROF.DR. ŞENNUR SOMALI** and we certify that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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NEW DIFFERENCE METHOD
FOR ORDINARY DIFFERENTIAL EQUATIONS

ABSTRACT

The new high order of accuracy difference scheme for approximating the solution of a linear boundary value problem is constructed. Numerical results demonstrate the efficiency of the method.

Keywords: Second order boundary value problems, difference method, Taylor expression

ADİ DİFERANSİYEL DENKELEMLER İÇİN YENİ FARK YÖNTEMİ

ÖZ

Bu çalışmada lineer sınır değer problemi için sonuca iyi bir yaklaşımla yakınsayan ve hata değeri az olan yeni fark yöntemi ele alındı. Sayısal sonuçlar yöntemin kuvvetli olduğunu gösterdi.

Anahtar sözcükler: Sınır değer problemi, geliştirilen yeni fark yöntemi, Taylor açılımı

CONTENTS

Page

CHAPTER ONE

INTRODUCTION

We consider the boundary value problem of the form:

$$-y''(t)+a(t)y(t)=f(t),\ 0<t\leq T,\quad y(0)=y_0,\ \ y(T)=y_T, \qquad (1)$$

where *a(t)* and *f(t)* are continuous on [0,T]. A standard theoretical result states that if $a(t)>0$ for $t\in[0,T]$, then the boundary value problem (1) has a unique solution. These types of problems are solved by using standard finite difference methods for numerical approximation. To solve a boundary value problem by the method of finite differences, every derivative appearing in the equation is replaced by an appropriate difference approximation. Central differences are usually preferred because they lead to greater accuracy. In each case the finite difference representation is an $O(h^2)$ approximation to the respective derivative.Fox (1990), Epperson (2001) and Ross (1984).

In Chapter 2, we discuss construction of the higher order of accuracy two-step difference schemes generated by an exact difference scheme for the numerical solution of a boundary value problem for the second order differential equation. (Ashyralyev). In Chapter 3, new difference method is investigated by presenting the numerical results.

CHAPTER TWO

NEW DIFFERENCE METHOD FOR ORDINARY DIFFERENTIAL EQUATIONS

We consider the boundary value problem of the form:

$$-y''(t)+a(t)y(t)=f(t),\ 0<t\le T, \tag{1}$$

$$y(0)=y_0, \tag{2a}$$

$$y(T)=y_T \tag{2b}$$

assuming *a(t)* and *f(t)* to be such that problem (1) has a unique smooth solution defined on [*0,T*]. However, for existence of a unique solution of the boundary value problem, we need some estimate from below of *a(t).*

For the construction of difference schemes we consider the uniform grid space:

$$[0,T]_h=\{t_k=kh,k=0,1,...,N,Nh=T\}.$$

with $h>0$ and N is a fixed positive integer.

The construction of the two step difference scheme of an arbitrary high order of accuracy for the approximate solutions of the boundary value problem (1) is based on the following theorem (Allaberen).

THEOREM: Let $y(t_k)$ be a solution of the problem (1) at the grid points $t = t_k$ then $\{y(t_k)\}_0^N$ is the solution of the boundary value problem for the second order difference equations:

$$
\begin{aligned}
&-\frac{1}{h^2}(y(t_{k+1}) - 2y(t_k) + y(t_{k-1})) + \frac{1}{h^2}\Bigg[(1 - \frac{1}{h}\int_{t_k}^{t_{k+1}} J^2(t_{k+1}, s)ds J(t_k, t_{k-1}))y(t_{k-1}) \\
&+(-2+\frac{1}{h}\int_{t_{k-1}}^{t_k} J^2(t_k, s)ds J^2(t_{k+1}, t_k) + \frac{1}{h}\int_{t_k}^{t_{k+1}} J^2(t_{k+1}, s)ds)y(t_k) \\
&+\left(1 - \frac{1}{h}\int_{t_{k-1}}^{t_k} J^2(t_k, s)ds J(t_{k+1}, t_k)\right)y(t_{k+1})\Bigg] \\
&= \frac{1}{h^3}\Bigg[\int_{t_{k-1}}^{t_k} J^2(t_k, s)ds J(t_{k+1}, t_k)\int_{t_k}^{t_{k+1}}\left\{\int_z^{t_{k+1}} J(t_{k+1}, s)J(z, s)ds\right\}f(z)dz \\
&+\int_{t_k}^{t_{k+1}} J^2(t_{k+1}, s)ds\int_{t_{k-1}}^{t_k}\left\{\int_{t_{k-1}}^{z} J(t_k, s)J(z, s)ds\right\}f(z)dz\Bigg],
\end{aligned}
\tag{3}
$$

$$1 \le k \le N - 1,\ y(0) = y_0,\ y(T) = y_T,$$

where

$$J(t, s) = e^{-\int_s^t b(\lambda)d\lambda},$$

$b(t)$ is a solution of the equation

$$b^2(t) - b'(t) = a(t) \tag{4}$$

and the following estimate holds:

$$b(t) \ge b_0 > 0,\ 0 \le t \le T. \tag{5}$$

PROOF: Using relation (4), we can obviously write the equivalent boundary value problem for a system of first order linear differential equations

$$\begin{cases} y'(t) + b(t)y(t) = v(t),\ y(0) = y_0,\ y(T) = y_T, \\ -v'(t) + b(t)v(t) = f(t). \end{cases}$$

Integrating these, we at once obtain

$$\begin{cases} y(t) = e^{-\int_0^t b(\lambda)d\lambda} y_0 + \int_0^t e^{-\int_s^t b(\lambda)d\lambda} v(s)ds, \\ v(t) = e^{-\int_t^T b(\lambda)d\lambda} v(T) + \int_t^T e^{-\int_t^z b(\lambda)d\lambda} f(z)dz. \end{cases}$$

That is,

$$\begin{cases} y(t) = J(t,0)y_0 + \int_0^t J(t,s)v(s)ds, \\ v(t) = J(T,t)v(T) + \int_t^T J(z,t)f(z)dz. \end{cases}$$

From these formulas and the condition $y(T)=y_T$ it follows that

$$y(T) = J(t,0)y_0 + \int_0^T J^2(T,s)dsv(T)$$

$$+ \int_0^T J(T,s)\int_s^T J(z,s)f(z)dzds.$$

Since $\int_0^T e^{-2\int_s^T b(\lambda)d\lambda} ds \neq 0,$

$$v(T) = (\int_0^T J^2(T,s)ds)^{-1}\Big[y_T - J(T,0)y_0$$

$$-\int_0^T\int_s^T J(t,s)J(z,s)f(z)dzds\Big].$$

So we have that

$$v(t) = J(T,t)(\int_0^T J^2(T,s)ds)^{-1}\Big[y(T) - J(T,0)y_0$$

$$-\int_0^T (\int_s^T J(T,s)J(z,s)dz) f(z)ds\Big] + \int_t^T J(z,t) f(z)dz$$

and

$$y(t) = J(t,0)y_0 + \int_0^t J(t,s)(J(T,s)(\int_0^T J^2(T,\tau)d\tau)^{-1})\Big[y(T) - J(T,0)y_0$$

$$-\int_0^T (\int_\tau^T J(T,\tau)J(z,\tau) f(z)dz)d\tau\Big] + \int_s^T J(z,s) f(z)dz)ds.$$

By an interchange of the order of integration

$$y(t) = (\int_0^T J^2(T,s)ds)^{-1}\left[\left\{\int_t^T J^2(T,s)dsJ(t,0)y_0 + \int_0^t J(t,s)J(T,s)dsy(T)\right\}\right.$$

$$-\int_0^t J(t,s)J(T,s)ds \times \int_0^T \left\{\int_0^z J^2(z,s)ds\right\} f(z)dz\Big] + \int_0^t \left\{\int_0^z J(t,s)J(z,s)ds\right\} f(z)dz$$

$$+\int_t^T \left\{\int_0^t J(t,s)J(z,s)ds\right\} f(z)dz = u(t) + w(t) + g(t), \qquad (6)$$

where

$$u(t) = (\int_0^T J^2(T,s)ds)^{-1}\int_t^T J^2(T,s)dsJ(t,0)y_0,$$

$$w(t) = (\int_0^T J^2(T,s)ds)^{-1}\int_0^t J(t,s)J(T,s)dsy(T),$$

$$g(t) = -(\int_0^T J^2(T,s)ds)^{-1}\int_0^t J(t,s)J(T,s)ds$$

$$\times\int_0^T \left\{\int_0^z J^2(z,s)\right\} f(z)dz$$

$$+\int_0^t \left\{\int_0^z J(t,s)J(z,s)ds\right\} f(z)dz + \int_t^T \left\{\int_0^t J(t,s)J(z,s)ds\right\} f(z)dz$$

Putting $t = t_{k+1},\ t = t_k,\ t = t_{k-1}$ into the formula for $u(t)$, we can write

$$u(t_{k-1}) = (\int_0^T J^2(T,s)ds)^{-1} \int_{t_{k-1}}^T J^2(T,s)dsJ(t_{k-1},0)y_0,$$

$$u(t_k) = (\int_0^T J^2(T,s)ds)^{-1} \int_{t_k}^T J^2(T,s)dsJ(t_k,0)y_0$$

$$= (\int_0^T J^2(T,s)ds)^{-1} (\int_{t_{k-1}}^T J^2(T,s)dsJ(t_k,0) - \int_{t_{k-1}}^{t_k} J^2(T,s)dsJ(t_k,0))y_0$$

$$= J(t_k,t_{k-1})u(t_{k-1}) - (\int_0^T J^2(T,s)ds)^{-1} \int_{t_{k-1}}^{t_k} J^2(T,s)dsJ(t_k,0)y_0$$

$$= J(t_k,t_{k-1})u(t_{k-1}) - (\int_0^T J^2(T,s)ds)^{-1} \int_{t_{k-1}}^{t_k} J^2(t_k,s)dsJ^2(T,t_k)J(t_k,0))y_0,$$

$$u(t_{k+1}) = (\int_0^T J^2(T,s)ds)^{-1} \int_{t_{k+1}}^T J^2(T,s)dsJ(t_{k+1},0)y_0$$

$$= (\int_0^T J^2(T,s)ds)^{-1} (\int_{t_k}^T J^2(T,s)dsJ(t_{k+1},0) - \int_{t_k}^{t_{k+1}} J^2(T,s)dsJ(t_{k+1},0))y_0$$

$$= J(t_{k+1},t_k)u(t_k) - (\int_0^T J^2(T,s)ds)^{-1} \int_{t_k}^{t_{k+!}} J^2(T,s)dsJ(t_{k+1},0))y_0$$

$$= J(t_{k+1},t_k)u(t_k).$$

Hence we obtain the following relation between $u(t_{k+1}),\ u(t_k),\ u(t_{k-1})$:

$$(\int_{t_k}^{t_{k+1}} J^2(t_{k+!},s)ds)^{-1}(u(t_{k+1}) - J(t_{k+1},t_k)u(t_k))$$

$$= -(\int_0^T J^2(T,s)ds)^{-1} J^2(T,t_{k+1})J(t_{k+1},0))y_0,$$

$$(\int_{t_{k-1}}^{t_k} J^2(t_k,s)ds)^{-1}(u(t_k) - J(t_k,t_{k-1})u(t_{k-1}))$$

$$= -J(t_{k+1},t_k)(\int_0^T J^2(T,s)ds)^{-1} J^2(T,t_{k+1})J(t_{k+1},0))y_0.$$

Therefore,

$$(\int_{t_{k-1}}^{t_k} J^2(t_k,s)ds)^{-1}(u(t_k)-J(t_k,t_{k-1})u(t_{k-1}))$$

$$=J(t_{k+1},t_k)(\int_{t_k}^{t_{k+1}} J^2(t_{k+!},s)ds)^{-1}(u(t_{k+!})-J(t_{k+1},t_k)u(t_k))$$

or

$$\int_{t_{k-1}}^{t_k} J^2(t_k,s)dsJ(t_{k+1},t_k)u(t_{k+1})-\left[\int_{t_k}^{t_{k+1}} J^2(t_{k+1},s)ds+\int_{t_{k-1}}^{t_k} J^2(t_k,s)dsJ^2(t_{k+1},t_k)\right]u(t_k)$$

$$+\int_{t_k}^{t_{k+1}} J^2(t_{k+1},s)dsJ(t_k,t_{k-1})u(t_{k-1})=0 \qquad (7)$$

In a similar manner, using the definition of $J(t,s)$ and the formula for $w(t)$, we can obtain

$$\int_{t_{k-1}}^{t_k} J^2(t_k,s)dsJ(t_{k+1}t_k)w(t_{k+1})-\left[\int_{t_k}^{t_{k+1}} J^2(t_{k+1},s)ds+\int_{t_{k-1}}^{t_k} J^2(t_k,s)dsJ^2(t_{k+1},t_k)\right]w(t_k)$$

$$+\int_{t_k}^{t_{k+1}} J^2(t_{k+1},s)dsJ(t_k,t_{k-1})w(t_{k-1})=0. \qquad (8)$$

Now putting $t=t_{k+1}$, $t=t_k$, $t=t_{k-1}$ into the formula for $g(t)$ and using the definition of *J(t,s)* we can obtain

$$\int_{t_{k-1}}^{t_k} J^2(t_k,s)dsJ(t_{k+1},t_k)g(t_{k+1})-\left[\int_{t_k}^{t_{k+1}} J^2(t_{k+1},s)ds+\int_{t_{k-1}}^{t_k} J^2(t_k,s)dsJ^2(t_{k+1},t_k)\right]g(t_k)$$

$$+\int_{t_k}^{t_{k+1}} J^2(t_{k+1},s)dsJ(t_k,t_{k-1})g(t_{k-1})$$

$$=-\int_{t_{k-1}}^{t_k} J^2(t_k,s)dsJ(t_{k+1},t_k)((\int_0^T J^2(T,s)ds)^{-1}\int_0^{t_{k+1}} J(t_{k+1},s)J(T,s)ds$$

$$\times\int_0^T\left\{\int_0^z J^2(z,s)ds\right\}f(z)dz)+\left[\int_{t_k}^{t_{k+1}} J^2(t_{k+1},s)ds+\int_{t_{k-1}}^{t_k} J^2(t_k,s)dsJ^2(t_{k+1},t_k)\right]$$

$$\times(\int_0^T J^2(T,s)ds)^{-1}\int_0^{t_k} J(t_k,s)J(T,s)ds\times\int_0^T\left\{\int_0^z J^2(z,s)ds\right\}f(z)dz$$

$$-\int_{t_k}^{t_{k+1}} J^2(t_{k+1},s)dsJ(t_k,t_{k-1})(\int_0^T J^2(T,s)ds)^{-1}\int_0^{t_{k-1}} J(t_{k-1},s)J(T,s)ds$$

$$\times\int_0^T\left\{\int_0^z J^2(z,s)ds\right\}f(z)dz+\int_{t_{k-1}}^{t_k} J^2(t_k,s)dsJ(t_{k+1},t_k)$$

$$(\int_0^{t_{k+1}}\left\{\int_0^z J(t_{k+1},s)J(z,s)ds\right\}f(z)dz+\int_{t_{k+1}}^{T}\left\{\int_0^{t_{k+1}} J(t_{k+1},s)J(z,s)ds\right\}f(z)dz)$$

and it leads,

$$-\left[\int_{t_k}^{t_{k+1}} J^2(t_{k+1},s)ds+\int_{t_{k-1}}^{t_k} J^2(t_k,s)dsJ^2(t_{k+1},t_k)\right]$$

$$\times(\int_0^{t_k}\left\{\int_0^z J(t_k,s)J(z,s)ds\right\}f(z)dz+\int_{t_k}^{T}\left\{\int_0^{t_k} J(t_k,s)J(z,s)ds\right\}f(z)dz)$$

$$+\int_{t_k}^{t_{k41}} J^2(t_{k+1},s)dsJ(t_k,t_{k-1})(\int_0^{t_{k-1}}\left\{\int_0^z J(t_{k-1},s)J(z,s)ds\right\}f(z)dz)$$

$$+\int_{t_{k-1}}^{T}\left\{\int_0^{t_{k-1}} J(t_{k-1},s)J(z,s)ds\right\}f(z)dz)$$

$$=\int_{t_{k-1}}^{t_k} J^2(t_k,s)dsJ(t_{k+1},t_k(\int_{t_k}^{t_{k+1}}\left\{\int_0^z J(t_{k+1},s)J(z,s)ds\right\}f(z)dz-\int_{t_k}^{t_{k+1}}\left\{\int_0^{t_{k+1}} J(t_{k+1},s)J(z,s)ds\right\}f(z)dz$$

$$-\int_{t_k}^{t_{k+1}}\left\{\int_0^{t_{k+1}} J(t_{k+1},s)J(z,s)ds\right\}f(z)dz)$$

$$-\left[\int_{t_k}^{t_{k+1}} J^2(t_{k+1},s)ds+\int_{t_{k-1}}^{t_k} J^2(t_k,s)J^2(t_{k+1},t_k)ds\right]$$

$$+\int_{t_{k-1}}^{t_k} J^2(t_k,s)dsJ(t_{k+1},t_k)(\int_0^{t_k}\left\{\int_0^z J(t_{k+1},s)J(z,s)ds\right\}f(z)dz$$

$$-\int_{t_k}^{T}\left\{\int_0^{t_{k+1}} J(t_{k+1},s)J(z,s)ds\right\}f(z)dz)$$

$$\times(\int_0^{t_k}\left\{\int_0^z J(t_k,s)J(z,s)ds\right\}f(z)dz+\int_{t_k}^{T}\left\{\int_0^{t_k} J(t_k,s)J(z,s)ds\right\}f(z)dz)$$

$$+\int_{t_k}^{t_{k+1}} J^2(t_{k+1},s)dsJ(t_k,t_{k-1})(\int_0^{t_k}\left\{\int_0^z J(t_{k-1},s)J(z,s)ds\right\}f(z)dz$$

$$+\int_{t_k}^{T}\left\{\int_0^{t_{k-1}} J(t_{k-1},s)J(z,s)ds\right\}f(z)dz)$$

$$+\int_{t_k}^{t_{k+1}} J^2(t_{k+1},s)dsJ(t_k,t_{k-1})(-\int_{t_{k-1}}^{t_k}\left\{\int_0^{z} J(t_{k-1},s)J(z,s)ds\right\}f(z)dz$$

$$+\int_{t_{k-1}}^{t_k}\left\{\int_0^{t_{k-1}} J(t_{k-1},s)J(z,s)ds\right\}f(z)dz)$$

$$=-\int_{t_{k-1}}^{t_k} J^2(t_k,s)dsJ(t_{k+1},t_k)(\int_{t_k}^{t_{k+1}}\left\{\int_z^{t_{k+1}} J(t_{k+1},s)J(z,s)ds\right\}f(z)dz$$

$$-\int_{t_k}^{t_{k+1}} J^2(t_{k+1},s)ds\int_{t_{k-1}}^{t_k}\left\{\int_{t_{k-1}}^{z} J(t_k,s)J(z,s)ds\right\}f(z)dz \tag{9}$$

Before the investigating the relation between $y(t_{k-1}),\ y(t_k),\ y(t_{k+1})$ the goal is to find how the below complex formula can be obtained;

$$g(t)=-A^{-1}K\int_0^t J(t,s)J(T,s)ds+\int_0^t\int_0^z J(t,s)J(z,s)dsdz+\int_t^T\int_0^t J(t,s)J(z,s)f(z)dsdz\,,$$

where

$$K=\int_0^T\int_0^z J(T,s)J(z,s)f(z)dsdz\,,$$

$$A=\int_0^T J^2(T,s)ds\,,$$

and

$$a_k=\int_{t_{k-1}}^{t_k} J^2(t_k,s)ds\,,$$

$$a_kJ(t_{k+1},t_k)g(t_{k+1})-(a_{k+1}+a_kJ^2(t_{k+1},t_k))g(t_k)+a_{k+1}J(t_k,t_{k-1})g(t_{k-1})=$$

$$-a_kJ(t_{k+1},t_k)\left[\underbrace{-g(t_{k+1})+J(t_{k+1},t_k)g(t_k)}_{(i)}\right]-a_{k+1}\left[\underbrace{g(t_k)-J(t_k,t_{k-1})}_{(ii)}\right]g(t_{k-1})$$

We separate the parts (*i*), (*ii*) which help us to analyze the formula. Now consider the part (*i*);

$$-g(t_{k+1})+J(t_{k+1},t_k)g(t_k)$$

By expanding the integral's boundary values term by term, some integrals are in the following form:

$$= A^{-1}K\int_{t_k}^{t_{k+1}} J(t_{k+1},s)J(T,s)ds - \int_{t_k}^{t_{k+1}}\int_0^z \ldots - \int_{t_{k+1}}^{T}\int_{t_k}^{t_{k+1}} \ldots + \int_{t_k}^{t_{k+1}}\int_0^{t_k} \ldots dsdz$$

$$I_1 = \int_{t_k}^{t_{k+1}}\int_0^z \ldots dsdz \quad , \; t_k \le z \le t_{k+1}, \; 0 \le s \le z,$$

$$I_2 = \int_{t_k}^{t_{k+1}}\int_0^{t_k} \ldots dsdz \quad , \; t_k \le z \le t_{k+1}, \; 0 \le s \le t_k,$$

$$I_3 = \int_{t_{k+1}}^{T}\int_{t_k}^{t_{k+1}} \ldots dsdz \quad , \; t_{k+1} \le z \le T, \; t_k \le s \le t_{k+1}.$$

By an interchange of the order of integration and using the double integral properties we will get;

$$-a_k J(t_{k+1},t_k)(-g(t_{k+1}) + J(t_{k+1},t_k)g(t_k))$$

$$= -(\int_{t_{k-1}}^{t_k} J^2(t_k,s)ds)J(t_{k+1},t_k) \times \int_{t_k}^{t_{k+1}}\int_z^{t_{k+!}} J(t_{k+1},s)J(z,s)f(z)dsdz$$

Using the same consideration part (*ii*) can be obtained as the part (*i*).

So using the last formula and the identities (7) and (8) we obtain the following relation between $y(t_{k+1}),\; y(t_k),\; y(t_{k-1})$

$$\int_{t_{k-1}}^{t_k} J^2(t_k,s)dsJ(t_{k+1},t_k)y(t_{k+1}) - \left[\int_{t_k}^{t_{k+1}} J^2(t_{k+1},s)ds\right.$$

$$\left. + \int_{t_{k-1}}^{t_k} J^2(t_k,s)dsJ^2(t_{k+1},t_k)\right] y(t_k) + \int_{t_k}^{t_{k+1}} J^2(t_{k+1},s)dsJ(t_k,t_{k-1})y(t_{k-1})$$

$$= -\int_{t_{k-1}}^{t_k} J^2(t_k,s)dsJ(t_{k+1},t_k)\int_{t_k}^{t_{k+1}}\left\{\int_z^{t_{k+1}} J(t_{k+1},s)J(z,s)ds\right\} f(z)dz$$

$$-\int_{t_k}^{t_{k+1}} J^2(t_{k+1},s)ds\int_{t_{k-1}}^{t_k}\left\{\int_{t_{k-1}}^{z} J(t_k,s)J(z,s)ds\right\} f(z)dz.$$

From this it follows (3) for any k, $1 \le k \le N-1$. Theorem is proved.

Let us note that the boundary value problem (3) is called the two step exact difference scheme for solution of the boundary value problem (1). Note that for $b(t)$ we have the Riccati differential equation (4). Therefore for the smooth $a(t)$ there exists a smooth positive solution of this differential equation defined on the segment $[0,T]$.

Now we will consider the applications of this exact difference scheme. From (3), it is clear that for the approximate solutions of the problem is necessary to approximate the expressions

$$\frac{1}{h}\int_{t_k}^{t_{k+1}} J^2(t_{k+1},s)ds,\ \frac{1}{h}\int_{t_{k-1}}^{t_k} J^2(t_k,s)ds,\ J(t_k,t_{k-1}),\ J(t_{k+1},t_k)$$

and

$$\int_{t_k}^{t_{k+1}}\left\{\int_z^{t_{k+1}} J(t_{k+1},s)J(z,s)ds\right\}f(z)dz,\ \int_{t_{k-1}}^{t_k}\left\{\int_{t_{k-1}}^{z} J(t_k,s)J(z,s)ds\right\}f(z)dz \qquad (10)$$

Let us remark that in constructing difference schemes it is important to know how to construct a right hand side $Q_k^{p,q}$ that satisfies

$$\frac{1}{h^3}\left[\int_{t_{k-1}}^{t_k} J^2(t_k,s)dsJ(t_{k+1},t_k)\int_{t_k}^{t_{k+1}}\left\{\int_z^{t_{k+1}} J(t_{k+1},s)J(z,s)ds\right\}f(z)dz\right.$$

$$\left.+\int_{t_k}^{t_{k+1}} J^2(t_{k+1},s)ds\int_{t_{k-1}}^{t_k}\left\{\int_{t_{k-1}}^{z} J(t_k,s)J(z,s)ds\right\}f(z)dz\right]-Q_k^{p,q}=O(h^{p+q}) \qquad (11)$$

and is sufficiently simple. The choice formula $Q_k^{p,q}$ is not unique. Using Taylor's formula with respect to the function

$$\int_z^{t_{k+1}} J(t_{k+1},s)J(z,s)dsf(z)$$

on the variable z we obtain

$$\frac{1}{h^2}\int_{t_k}^{t_{k+1}}\{J(t_{k+1},s)J(z,s)ds\}f(z)dz$$

$$=\sum_{m=1}^{p+q}\sum_{\lambda=0}^{m}\binom{m}{\lambda}\beta_{m-\lambda}(t_{k+1})f^{(\lambda)}(t_{k+1})\frac{(-1)^m h^{m-1}}{(m+1)!}+O(h^{p+q}), \qquad (12)$$

where

$$\left\{\begin{aligned} &\beta_0(t_{k+1})=0, \beta_1(t_{k+1})=-1, \beta_2(t_{k+1})=0, \\ &\beta_m(t_{k+1})=\sum_{\lambda=0}^{m-2}\binom{m-2}{\lambda}a^{(m-2-\lambda)}(t_{k+1})\beta_\lambda(t_{k+1}),\ 3\le m\le p+q. \end{aligned}\right\} \tag{13}$$

Using Taylor's formula with respect to the function

$$\int_{t_{k-1}}^{z} J(t_k,s)J(z,s)ds\} f(z)$$

on the variable z, we obtain

$$\frac{1}{h^2}\int_{t_{k-1}}^{t_k}\left\{\int_{t_{k-1}}^{z} J(t_k,s)J(z,s)ds\right\} f(z)dz$$

$$=\sum_{m=1}^{p+q}\binom{m}{\lambda}\beta_{m-\lambda}(t_{k-1})f^{(\lambda)}(t_{k-1})\frac{h^{m-1}}{(m+1)!}+O(h^{p+q}), \tag{14}$$

where

$$\left\{\begin{aligned} &\beta_0(t_{k-1})=0,\ \beta_1(t_{k-1})=J(t_k,t_{k-1}),\ \beta_2(t_{k-1})=0, \\ &\beta_m(t_{k-1})=\sum_{\lambda=0}^{m-2}\binom{m-2}{\lambda}a^{(m-2-\lambda)}(t_{k-1})\beta_\lambda(t_{k-1}),\ 3\le m\le p+q. \end{aligned}\right. \tag{15}$$

Moreover for the construction of $Q_k^{p,q}$ we will use (12) and (14). However, first of all let us study the approximate formulas for the expressions

$$J(t_k,t_{k-1}),\ J(t_{k+1},t_k),\ \frac{1}{h}\int_{t_k}^{t_{k+1}} J^2(t_{k+1},s)ds,\ \frac{1}{h}\int_{t_{k-1}}^{t_k} J^2(t_k,s)ds,$$

That will be needed further for the construction of difference schemes of a high order of accuracy for the approximate solution of the problem (1).

Let us consider the approximate for the expressions

$$J(t_k,t_{k-1})=e^{-\int_{t_{k-1}}^{t_k} b(s)ds}=e^{-b(t_{k-1})h}-h(b(t_k)-b(t_{k-1}))e^{-b(t_{k-1})h}$$

$$+\sum_{n=1}^{p+q-1}\frac{(-h)^{n+1}}{(n+1)!}\sum_{i=0}^{n}\binom{n}{i}\beta_i(t_k)$$

$$\times\left[(b(t_k)-b(t_{k-1}))(-b(t_{k-1}))^{n-i}+\sum_{j=1}^{n-i}\binom{n-i}{j}b^{(j)}(t_k)(-b(t_{k-1}))^{n-i-j}\right]e^{-b(t_{k-1})h}+O(h^{p+q+1}) \quad (16)$$

where

$$\begin{cases} \beta_i(s)=1,\ i=0 \\ \beta_i(s)=\beta_{i-1}^{'}(s)+\beta_{i-1}(s)b(s),\ 1\le i\le p+q. \end{cases} \quad (17)$$

To obtain the above formula we use such a following method:

$$J(t_k,t_{k-1})=e^{-\int_{t_{k-1}}^{t_k}b(s)ds}=e^{-b(t_k)h}-e^{-b(t_k)h}+e^{-\int_{t_{k-1}}^{t_k}b(s)ds}$$

$$=e^{-b(t_k)h}+\left[e^{-b(t_k)(t_k-z)}e^{-\int_{t_{k-1}}^{z}b(s)ds}\right]_{z=t_{k-1}}^{z=t_k}$$

$$=e^{-b(t_k)h}+\int_{t_{k-1}}^{t_k}\frac{d}{dz}\left[e^{-a(t_k)(t_k-z)}e^{-\int_{t_{k-1}}^{z}b(s)ds}\right]dz$$

$$=e^{-b(t_k)h}+\int_{t_{k-1}}^{t_k}([b(t_k)-b(z)]e^{-b(t_k)(t_k-z)}e^{-\int_{t_{k-1}}^{z}b(s)ds})_{(t_{k-1})}^{(n)}dz$$

$$+\sum_{n=1}^{\infty}([b(t_k)-b(z)]e^{-b(t_k)(t_k-z)}e^{-\int_{t_{k-1}}^{z}b(s)ds})_{(t_{k-1})}^{(n)}\frac{1}{n!}\int_{t_{k-1}}^{t_k}(z-t_{k-1})^{(n)}dz$$

$$=e^{-b(t_k)h}+h(b(t_k)-b(t_{k-1}))e^{-b(t_k)h}$$

$$+\sum_{n=1}^{\infty}\frac{(h)^{n+1}}{(n+1)!}([b(t_k)-b(z)]$$

$$\times e^{-b(t_k)(t_k-z)}e^{-\int_{t_{k-1}}^{z}b(s)ds})_{(t_{k-1})}^{(n)}\sum_{i=0}^{n}\binom{n}{i}([b(t_k)-b(z)]e^{-b(t_k)(t_k-z)})^{i}(e^{-\int_{t_{k-1}}^{z}b(s)ds})^{n-i}$$

Same procedure can be applied for $J(t_{k+1}, t_k)$.

For the other formula

$$\frac{1}{h}\int_{t_{k-1}}^{t_k} J^2(t_k, s)ds$$

$$= \sum_{m=0}^{p+q-1} \binom{m}{\lambda} \beta_{m-\lambda}(t_k) f^{(\lambda)}(t_k) \frac{(-h)^m}{(m+1)!} + O(h^{p+q}), \tag{18}$$

where

$$\begin{cases} \beta_i(s) = 1, \ i = 0, \\ \beta_i(s) = \beta_{i-1}{}^{(')}(s) + \beta_{i-1}(s)2b(s), \ 1 \le i \le p+q. \end{cases} \tag{19}$$

For the formula

$$\frac{1}{h}\int_{t_k}^{t_{k+1}} J^2(t_{k+1}, s)ds$$

$$= \sum_{m=0}^{p+q-1} \binom{m}{\lambda} \beta_{m-\lambda}(t_{k+1}) f^{(\lambda)}(t_{k+1}) \frac{(-h)^m}{(m+1)!} + O(h^{p+q}), \tag{20}$$

where

$$\begin{cases} \beta_i(s) = 1, \ i = 0, \\ \beta_i(s) = \beta_{i-1}{}^{(')}(s) + \beta_{i-1}(s)2b(s), \ 1 \le i \le p+q. \end{cases} \tag{21}$$

CHAPTER THREE

NUMERICAL RESULTS

To illustrate the high accuracy of the new difference method for second order linear boundary value problem, the following test problem is considered. We compared the errors which are obtained by standard finite difference method and new difference method.

Example 3.1:

$$-y''(t)+y(t)=1$$
$$y(0)=0,$$
$$y(0.5)=1,$$

where $a(t)=f(t)=1$.

The corresponding first order system is:

$$\begin{cases} y'(t)+b(t)y(t)=v(t) \\ -v'(t)+b(t)v(t)=1, \end{cases}$$

where

$$b^2(t)-b'(t)=1$$

with the solution

$$b(t)=\frac{c+e^{2t}}{c-e^{2t}},$$

for any c, such that $c\neq e^{2t}$ and $b(t)>0$, $\forall t\in[0,1]$.

Let us choose $c=8$ for the test problem.

It is observed that the errors in the new difference method less than the error in standard finite difference method, although same step sizes are used. All numerical computations are performed by Mathematica.

Table 3.1: Errors between approximate and exact values

	New Difference Method (p+q=4)	**Finite difference method**
h=1/3	8.96271×10^{-6}	0.0000459231
	0.0000158814	0.0000420156
h=1/6	1.22941×10^{-8}	7.52659×10^{-6}
	2.416×10^{-8}	0.0000115121
	3.4944×10^{-8}	0.0000123857
	4.2592×10^{-8}	0.0000105326
	4.01216×10^{-8}	6.29958×10^{-6}
h=1/12	1.9329×10^{-11}	1.05938×10^{-6}
	3.84228×10^{-11}	1.88294×10^{-6}
	5.71811×10^{-11}	2.48522×10^{-6}
	7.54253×10^{-11}	2.88×10^{-6}
	9.28432×10^{-11}	3.08032×10^{-6}
	1.08892×10^{-10}	3.09853×10^{-6}
	1.22617×10^{-10}	2.94637×10^{-6}
	1.32299×10^{-10}	2.63494×10^{-6}
	1.34762×10^{-10}	2.17479×10^{-6}
	1.23911×10^{-10}	1.57597×10^{-6}
	8.75457×10^{-11}	8.4801×10^{-7}

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