DOKUZ EYLÜL UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

SOLUTIONS OF DYNAMICAL SYSTEMS

by
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July, 2009
İZMİR

SOLUTIONS OF
DYNAMICAL SYSTEMS

A Thesis Submitted to the
Graduate School of Natural and Applied Sciences of Dokuz Eylül University
In Partial Fulfilment of the Requirements for the Master of Science in
Mathematics

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July, 2009
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M.Sc THESIS EXAMINATION RESULT FORM

We have read the thesis entitled **”SOLUTIONS OF DYNAMICAL SYSTEMS”** completed by **CEM ÇELİK** under supervision of **YRD. DOÇ. DR. MELDA DUMAN** and we certify that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ACKNOWLEDGEMENTS

I would like to express my sincere gratitude to my supervisor Yrd. Doç. Dr. Melda Duman for her advice, continual presence, guidance, encouragement and endless patience during the course of this research and I would like to thank all staff in Mathematics Department of the Faculty of Arts and Science for their valuable knowledge and time sharing with me during my research. Also I would like to express my thanks to Graduate School of Natural and Applied Sciences of Dokuz Eylül University for its technical support during my research. Moreover I wish to thank to the Faculty of Arts and Science and Dokuz Eylül University for their all support.

Cem ÇELİK

SOLUTIONS OF DYNAMICAL SYSTEMS

ABSTRACT

The main purpose of this work is to apply two different methods for finding the behaviour of the eigenvalues and corresponding eigenfunctions of the Bratu problem which has strongly non-linear term. For this reason, we use two variational methods, such as the variational iteration method and the Rayleigh-Ritz method. The results shows that, we can find the behaviour of the eigenvalues and eigenfunctions of the Bratu problem by using the two methods efficiently.

Keywords: nonlinear eigenvalue problems, Bratu problem, variational iteration method, Rayleigh-Ritz method, two-point boundary value problem.

DİNAMİK SİSTEMLERİN ÇÖZÜMLERİ

ÖZ

Bu çalışmanın temel amacı lineer olmayan Bratu probleminin özdeğer ve karşılık gelen özfonksiyonlarının davranışlarını bulabilmek için iki farklı metodu uygulamaktır. Bu sebeple, varyasyonel iterasyon metodu ve Rayleigh-Ritz metodunu kullandık. Sonuçlar, bu iki metodu kullanarak Bratu probleminin özdeğerleri ve özfonksiyonlarının davranışlarını bulabildiğimizi gösterir.

Anahtar sözcükler: lineer olmayan özdeğer problemi, Bratu Problemi, varyasyonel iterasyon metodu, Rayleigh-Ritz metodu, iki nokta sınır değer problemi.

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CHAPTER ONE
INTRODUCTION

1.1 Introduction

In a dynamical system, bifurcation means sudden change (spliting) in solution that occurs while parameters are being smoothly varied.
Consider the following non-linear boundary value problem

$$u''(t) + \lambda F(t, u(t)) = 0, \quad 0 < t < 1 \tag{1.1.1}$$

$$u(0) = u(1) = 0, \tag{1.1.2}$$

where the parameter $\lambda > 0$, and $F : (0,1] \times [0,\infty) \to [0,\infty)$ is continuous and is not identically zero on any subset of $(0,1] \times [0,\infty)$.
We investigate positive solutions of the Bratu equation with the homogeneous Dirichlet boundary conditions which has strongly non-linearity, where $F(t, u(t)) = e^{u(t)}$, so that it is a special case of (1.1.1) and (1.1.2). The Bratu problem is a non-linear elliptical partial differential equation

$$\Delta u + \lambda e^{u} = 0, \quad \text{in } \Omega \tag{1.1.3}$$

$$u = 0, \quad \text{on } \partial\Omega \tag{1.1.4}$$

where $\lambda > 0$, Ω is bounded domain in $\mathbb{R}^N$, and Δ is the Laplace operator. The problem arises in the fuel ignition model found in thermal combustion theory, the model of thermal reaction process, the Chandrasekhar model of the expansion of the universe, questions in geometry and relativity concerning the Chandrasekhar model, chemical reaction theory, radiative heat transfer and nanotechnology. (Caglar et al., 2008, Frank-Kamenetskii, 1969)

The problem (1.1.3) and (1.1.4) given in one dimensional space Ω, is known as the Liouville-Gelfand problem or Bratu problem, was studied by Liouville (Liouville, 1853) and Bratu (Bratu, 1914).Also, Gelfand published a comprehensive paper (Gelfand, 1959) that included a detailed review of equation (1.1.3), (1.1.4) in $\mathbb{R}^N$, for $N \in \{1,2,3\}$, see Jacobsen (2001).

The existence of positive solutions of (1.1.3), (1.1.4) can be proved by the Choi's theorem (Choi, 1991) which states as follows:

Theorem 1.1.1. *Let $g(t)$ be positive defined on the interval $(0,1)$ and $g(t) \in C^1(0,1)$. Then, there exist a $\lambda^* > 0$ such that the boundary value problem*

$$u''(t) + \lambda g(t) e^{u(t)} = 0, \quad 0 < t < 1$$

$$u(0) = u(1) = 0,$$

has a positive solution in $C^2(0,1] \cap C[0,1]$ for $0 < \lambda < \lambda^*$. Moreover, $g(t)$ can be singular at $t = 0$, but is at most $O(\frac{1}{t^{2-\delta}})$ as $t \to 0^+$ for some $\delta > 0$.

(Agarwal et al., 1999, Choi, 1991)

For solving Bratu problem, there are several methods, such as shooting method (Gelfand, 1959), finite difference method (Buckmire, 2004), collocation method (Boyd, 1986), Adomian decomposition method (Wazwaz, 2005). In bifurcation theory the main question is to find how many solutions exist for a given value λ, known as multiplicity of λ, and to know how the solutions vary as the parameter λ varies. Gelfand

(Gelfand, 1959) shows that the solutions are unique for $\lambda \leq 0$ and for a single positive value λ^*, called critical value, do not exist for $\lambda > \lambda^*$, and two solutions exists for $0 < \lambda < \lambda^*$. In this work, we concern about the case of $\lambda > 0$.

This work has been organized as follows:

In Chapter 2, we give some preliminary definitions about the calculus of variations in order to apply the variational iteration method and the Rayleigh-Ritz method.

The main purpose of Chapter 3 is to describe the construction of the general explicit solutions for all real λ and the explicit solution for $\lambda > 0$ with the homogeneous Dirichlet boundary conditions.

In Chapter 4, we give a brief description of the two variational methods, which are He's variational iteration method (VIM) and the Rayleigh-Ritz method. We apply the variational iteration method to the 1-D Bratu problem with homogeneous Dirichlet boundary conditions and compare to the numerical results of our approximate solutions between variational iteration method and the Rayleigh-Ritz method.

CHAPTER TWO

THE CALCULUS OF VARIATIONS

2.1 Basic concepts of the calculus of variations

The fundamental problem of the calculus of variations is in fact seeking the maximum and the minimum values of functions of curves, expressed by certain definite integrals

$$J[y(x)] = \int_a^b F(x, y(x), y'(x))\mathrm{d}x. \tag{2.1.1}$$

Here $J[y(x)]$ is a functional of $y(x)$ which is from continuously differentiable function to real numbers. For example, if

$$F(x, y(x), y'(x)) = \sqrt{1 + (y'(x))^2}, \tag{2.1.2}$$

then $J[y(x)]$ is the arc length of the curve $y(x)$. In order to find the shortest plane curve y(x) joining points (a, A) and (b, B), we need to determine $y(x)$ for which the integral

$$J[y(x)] = \int_a^b \sqrt{1 + (y'(x))^2}\mathrm{d}x \tag{2.1.3}$$

takes a minimum value, satisfying

$$y(a) = A \quad y(b) = B. \tag{2.1.4}$$

Lemma 2.1.1 (Fundamental Lemma)**.** *If $f(x)$ is a continuous function in the interval $[a, b]$, and if*

$$\int_a^b f(x)\eta(x)\mathrm{d}x = 0 \tag{2.1.5}$$

for every $\eta(x)$ *such that continuously differentiable in the interval* $[a,b]$ *and satisfying*

$$\eta(a) = \eta(b) = 0 \tag{2.1.6}$$

then $f(x)$ *is identically zero in the interval* $[a,b]$.

Let define a new function

$$y(x) + \alpha\eta(x) \tag{2.1.7}$$

where α is small parameter, $\eta(x)$ satisfies (2.1.6) and $y(x)$ yields an extremum of the integral (2.1.1). Substituting (2.1.7) into (2.1.1), we obtain

$$J[\alpha] = \int_a^b F(x, y(x) + \alpha\eta(x), y'(x) + \alpha\eta'(x))\mathrm{d}x. \tag{2.1.8}$$

Since $y(x)$ gives an extramum of $J[y(x)]$, (2.1.8) must have an extramum for the value $\alpha = 0$, so that its derivative must vanish for $\alpha = 0$, that is,

$$0 = \frac{\mathrm{d}J[\alpha]}{\mathrm{d}\alpha}|_{\alpha=0} = \int_a^b \left[F_y(x, y(x), y'(x))\eta(x) + F_{y'}(x, y(x), y'(x))\eta'(x)\right] \mathrm{d}x. \tag{2.1.9}$$

Using integration by parts, we have

$$\begin{aligned} J'[0] =& F_{y'}(x, y(x), y'(x))\eta(x)|_a^b \\ &+ \int_a^b \left[F_y(x, y(x), y'(x)) - \frac{\mathrm{d}}{\mathrm{d}x}F_{y'}(x, y(x), y'(x))\right]\eta(x)\mathrm{d}x. \end{aligned} \tag{2.1.10}$$

From (2.1.6) and Fundamental Lemma 2.1.1 , $y(x)$ satisfy the following differential equation

$$F_y - \frac{\mathrm{d}}{\mathrm{d}x}F_{y'} = 0, \tag{2.1.11}$$

called Euler-Lagrange's equation.

Here the change $\alpha\eta(x)$ in $y(x)$ is called variation of y and is denoted by δy

$$\delta y = \alpha\eta(x). \tag{2.1.12}$$

Corresponding to this change in $y(x)$, for a fixed value of x, the functional F changes by an amount ΔF, where

$$\Delta F = F(x, y + \alpha\eta, y' + \alpha\eta') - F(x, y, y'). \tag{2.1.13}$$

If the right-hand of (2.1.13) is expanded in powers of α, there follows

$$\Delta F = \frac{\partial F}{\partial y}\alpha\eta + \frac{\partial F}{\partial y'}\alpha\eta' + (\text{terms involving higher powers of } \alpha). \tag{2.1.14}$$

Here the first two term of right-hand side is called variation of F,

$$\delta F = \frac{\partial F}{\partial y}\alpha\eta + \frac{\partial F}{\partial y'}\alpha\eta'. \tag{2.1.15}$$

In the case when $F = y'$, (2.1.15) yields

$$\delta y' = \alpha\eta'. \tag{2.1.16}$$

From (2.1.12) and (2.1.16) equation (2.1.15) can be rewritten in the form

$$\delta F = \frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\delta y'. \tag{2.1.17}$$

It is easily verified by the definition of variation that

$$\delta(F_1F_2) = F_1\delta F_2 + F_2\delta F_1, \tag{2.1.18}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\delta y) = \delta\frac{\mathrm{d}y}{\mathrm{d}x}. \tag{2.1.19}$$

(Leis and Hildebrandt, 1988, Smirnov and Sneddon, 1964)

CHAPTER THREE
THE BRATU EQUATION

3.1 General Explicit Solution

The well known one dimensional Bratu Equation is defined by

$$u'' + \lambda e^u = 0. \tag{3.1.1}$$

Here λ can take positive and negative values, but we are interested in the case which λ takes positive values. In this case, λ is known as Frank-Kamenetskii parameter on chemistry. The explicit solution for Bratu Equation can be found by using Liouville's trick (Cohen and Benavides, 2007) which used on a 2-D hyperbolic second order partial differential equation $u_{xy} + \lambda e^u = 0$. Liouville used $v = \frac{u_x}{2}$ transformation for the 2-D hyperbolic equation. The transformation can be used on a 1-D elliptic equation $u'' + \lambda e^u = 0$ as $v = \frac{u'}{2}$. Here we mention about the expressions of solution for 1-D Bratu equation

$$u'' + \lambda e^u = 0. \tag{3.1.2}$$

Let use transformation $v = \frac{u'}{2}$. Then

$$v' = \frac{u''}{2} = \frac{-\lambda e^u}{2} \tag{3.1.3}$$

or

$$v'' = \frac{-\lambda e^u}{2} u'. \tag{3.1.4}$$

As a result, we obtain the equivalent simpler ordinary differential equation

$$v'' = 2v'v, \tag{3.1.5}$$

and integration of (3.1.5) gives

$$v' = v^2 + k, \quad \forall k \in R. \tag{3.1.6}$$

The solution can be found by elementary integration using the method of separation of variables;

$$v(x) = \begin{cases} -1/(x+l) & k = 0 \\ c\tan(c(x+l)) & k > 0 \\ c\coth(c(x+l)) & k < 0 \\ c\tanh(c(x+l)) & k < 0 \\ \mp c & k \leq 0 \end{cases} \tag{3.1.7}$$

where $c > 0$, $l \in R$ and $c^2 = |k|$ (here (c,l) and $(-c,-l)$ define the same solution, hence we can use $c > 0$). Therefore the general solution for the 1-D Bratu equation (3.1.1) can be found by substituting (3.1.7) into

$$u(x) = \ln\left(\frac{-2v'(x)}{\lambda}\right) \tag{3.1.8}$$

as

$$u(x)=\begin{cases}\ln\left(\frac{-2}{\lambda(x+l)^2}\right) & k=0 \text{ and } \lambda<0\\ \ln\left(\frac{-2c^2}{\lambda\cos^2(c(x+l))}\right) & k>0 \text{ and } \lambda<0\\ \ln\left(\frac{-2c^2}{\lambda\sinh^2(c(x+l))}\right) & k<0 \text{ and } \lambda<0\\ \ln\left(\frac{2c^2}{\lambda\cosh^2(c(x+l))}\right) & k<0 \text{ and } \lambda>0\\ \mp cx+l & k\leq 0 \text{ and } \lambda=0\end{cases} \tag{3.1.9}$$

(Cohen and Benavides, 2007)

3.2 Explicit Solution for Dirichlet Boundary Conditions

In Section 2.1, we mention about the general explicit solution of 1-D Bratu equation. Here, more specifically, we are interested in the 1-D Bratu equation (3.1.1) in $(0,1)$ for $\lambda>0$ with the zero Dirichlet boundary conditions

$$u(0)=u(1)=0. \tag{3.2.1}$$

Imposing the boundary conditions (3.2.1) to (3.1.9) for $\lambda>0$, we see that it must be

$$\cosh^2(cl)=\cosh^2(c(1+l)). \tag{3.2.2}$$

Since $c\neq 0$, it follows that $l=-\frac{1}{2}$. Substitution of the value l into $u(x)=\ln\left(\frac{2c^2}{\lambda\cosh^2(c(x+l))}\right)$ and substitution of $c=\frac{\theta}{2}$ give

$$u(x)=-2\ln\left(\frac{\cosh\left(\frac{\theta}{2}\left(x-\frac{1}{2}\right)\right)}{\cosh\left(\frac{\theta}{4}\right)}\right) \tag{3.2.3}$$

with

$$\theta = \sqrt{2\lambda}\cosh\left(\frac{\theta}{4}\right). \tag{3.2.4}$$

Figure 3.1 shows that there exist a unique solution λ_c of (3.2.4), so does $u(x)$. For $0 < \lambda < \lambda_c$ there are two solutions and for the values $\lambda > \lambda_c$ there is no solution of (3.2.4). Here, in the case that we have two solutions, one of the solutions is known as the lower solution and the other is the upper solution.

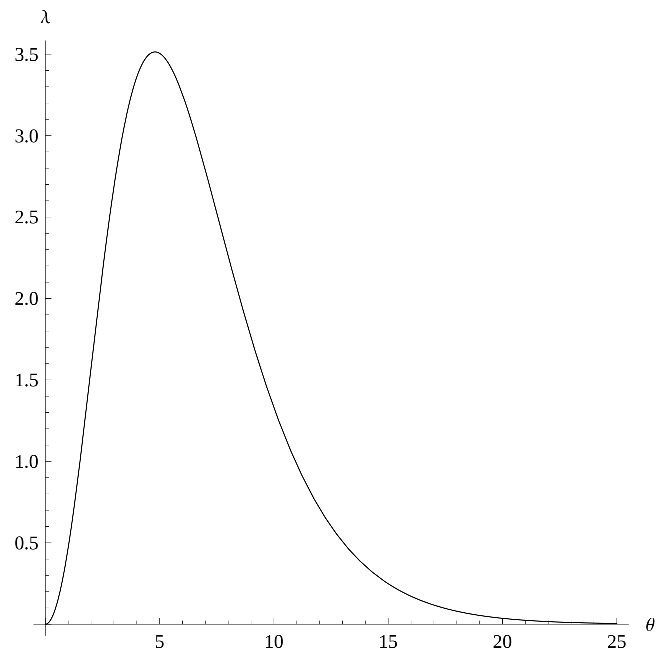

Figure 3.1 Bifurcation diagram of the exact solution of Bratu problem

The maximum value $\lambda = \lambda_c$ can be obtained by solving (3.2.4) and

$$1 = \sqrt{2\lambda_c}\sinh\left(\frac{\theta_c}{4}\right)\frac{1}{4}. \tag{3.2.5}$$

Therefore the division of (3.2.5) by (3.2.4) gives

$$\frac{\theta_c}{4} = \coth\left(\frac{\theta_c}{4}\right). \qquad (3.2.6)$$

Solving (3.2.6) numerically, one can find

$$\theta_c = 4.79871456 \qquad (3.2.7)$$

and

$$\lambda_c = 3.513830719. \qquad (3.2.8)$$

(Buckmire, 2003)

CHAPTER FOUR

TWO VARIATIONAL METHODS

In this chapter we mention about two variational methods as variational iteration method and Rayleigh-Ritz method to obtain approximate solutions of Bratu equation with homogeneous Dirichlet boundary conditions. We will compare the numerical results of the approximate solutions obtained by variational iteration method with Rayleigh-Ritz method.

4.1 Variational Iteration Method

The variational iteration method (VIM) was proposed by He (1997). The method based on the use of restricted variations and correction functionals. Many author shows that the variational iteration method is applicable for many types of problems for solution of non-linear ordinary differential equations (He, 1997, 1999, He and Wu, 2007) and partial differential equations (Hemeda, 2008, Wazwaz, 2007). The convergence of the variational iteration method was investigated in the articles Tatari and Dehghan (2007), Salkuyeh (2008).

Consider the following general differential equation

$$Lu + Nu = g(t) \tag{4.1.1}$$

where L is a linear operator, N is a non-linear operator, and $g(t)$ is an inhomogeneous term.

According to the variational principle, we can construct a correct functional as follows;

$$u_{n+1}(t) = u_n(t) + \int_0^t \mu \{Lu_n(s) + N\tilde{u}_n(s) - g(s)\} \mathrm{d}s \tag{4.1.2}$$

where μ is a general Lagrange multiplier, which can be found optimally by the variational theory, n denotes the nth approximation, and $\tilde{u}_n$ is considered as restricted variation, i.e., $\delta\tilde{u}_n = 0$. The successive approximations u_{n+1}, $(n = 0, 1, \ldots)$ of the solution u can be obtained after finding the Lagrange multiplier and by using the selected initial function u_0.

For construction of Lagrange multiplier, consider the following autonomous equation

$$u'' = f(u). \tag{4.1.3}$$

Its correction functional can be written in the form

$$u_{n+1}(t) = u_n(t) + \int_0^t \mu(t,s) \left\{u_n''(s) - f(\tilde{u}_n(s))\right\} \mathrm{d}s. \tag{4.1.4}$$

Taking the variation both sides of (4.1.4) with respect to u_n, accounting that $\delta\tilde{u}_n = 0$ and using integration by parts, we see that

$$\begin{aligned}
\delta u_{n+1}(t) &= \delta u_n(t) + \delta \int_0^t \mu \left\{u_n''(s) - f(\tilde{u}_n(s))\right\} \mathrm{d}s \\
&= \delta u_n(t) + \int_0^t \left\{(\delta\mu)u_n''(t) + \mu(\delta u_n''(t)) - (\delta\mu) f(\tilde{u}_n) + \mu(\delta f(\tilde{u}_n(s)))\right\} \mathrm{d}s \\
&= \delta u_n(t) + \int_0^t \mu(\delta u_n''(s)) \mathrm{d}s \\
&= \delta u_n(t) + \mu \delta u_n'(s)|_{s=t} - \mu' \delta u_n(s)|_{s=t} + \int_0^t \mu'' \delta u_n(s) \mathrm{d}s \\
&= 0.
\end{aligned}$$

Thus, we obtain the following stationary conditions,

$$\begin{aligned} \frac{\mathrm{d}^2\mu(s)}{\mathrm{d}s^2} &= 0 \\ 1 - \frac{\mathrm{d}\mu(s)}{\mathrm{d}s}\,|_{s=t} &= 0 \\ \mu(s)|_{s=t} &= 0. \end{aligned} \tag{4.1.5}$$

The Lagrange multiplier, therefore, can be easily identified as

$$\mu = s - t. \tag{4.1.6}$$

Hence, we have the following iteration formula,

$$u_{n+1}(t) = u_n(t) + \int_0^t (s-t)\left\{u_n''(s) - f(u_n(s))\right\} \mathrm{d}s. \tag{4.1.7}$$

(He and Wu, 2007)

4.2 Application of the variational iteration method

In this section, we will apply the variational iteration method to (3.1.2) and (3.2.1) using the shooting method.

By the symmetry of the solution in the interval $[0,1]$, it must be

$$u(t) = u(1-t). \tag{4.2.1}$$

Therefore,

$$u'(t) = -u'(1-t) \tag{4.2.2}$$

and for $t = \frac{1}{2}$, we have

$$u'(\frac{1}{2}) = 0. \qquad (4.2.3)$$

Thus, we can use (4.2.3) as an additional condition. Hence it is enough to study on the half interval $[0, \frac{1}{2}]$.

In order to apply the method of shooting, we can use the following initial conditions

$$u(0) = 0, \quad u'(0) = b \qquad (4.2.4)$$

where b is a real constant. To obtain b we will impose the condition $u'(\frac{1}{2}) = 0$ to found variational iteration solution of initial value problem (3.1.1) and (4.2.4). Then b is determined by the equation

$$G_n(\lambda, b) = u'_{n+1}(\frac{1}{2}; \lambda, b) - u'(\frac{1}{2}; \lambda) = 0 \qquad (4.2.5)$$

for a fixed λ. Applying the variational iteration procedure to the Bratu equation (3.1.1), from (4.1.7), we have the following iterative formula,

$$u_{n+1}(t) = u_n(t) + \int_0^t (s-t)\left\{u''_n(s) + \lambda e^{u_n(s)}\right\} \mathrm{d}s. \qquad (4.2.6)$$

Because of the exponential non-linearity, the integral can not be solved directly. For eliminating the exponential non-linearity, we use Taylor decomposition at u_0, that is,

$$e^{u_n} = e^{u_0} + e^{u_0}\frac{(u_k - u_0)}{1!} + e^{u_0}\frac{(u_k - u_0)^2}{2!} + e^{u_0}\frac{(u_k - u_0)^3}{3!} + \cdots. \qquad (4.2.7)$$

Let denote u_n by $u_n(t,\lambda)$. Substitution (4.2.7) into (4.2.6) with N term gives

$$\begin{aligned} u_{n+1}(t,\lambda) =& u_n(t,\lambda) \\ &+ \int_0^t (s-t)\left\{u_n''(s,\lambda)+\lambda\left(1+\sum_{i=1}^{N}\frac{(u_n(s,\lambda)-u_0(s,\lambda))^i}{i!})\right)\right\}\mathrm{d}s. \end{aligned} \tag{4.2.8}$$

We begin with initial function

$$u_0(t) = bt \tag{4.2.9}$$

satisfying initial conditions (4.2.4). We obtain approximate solutions $u_3(t,\lambda,b)$ from (4.2.8). Then, we solve (4.2.5), namely, $u_3'(\frac{1}{2},\lambda,b) = 0$ by numerically in a numerical way. The results obtained by the variational iteration method are shown in Section 4.5.

4.3 The Rayleigh-Ritz Method

The purpose of this section is to apply one of the alternative variational method, the Rayleigh-Ritz method, and compare the numerical results found by the variational iteration method and Rayleigh-Ritz method.

Consider the problem of seeking a function $y(t)$ that minimizes the functional

$$J[y(t)] = \int_a^b F(t,y(t),y'(t))\mathrm{d}t \tag{4.3.1}$$

with the conditions

$$y(a) = y_0 \quad y(b) = y_1. \tag{4.3.2}$$

Assume that we are able to approximate $y(t)$ by a linear combination of linearly independent functions (coordinate functions) of the type

$$y(t) \approx c_0\varphi_0(t) + c_1\varphi_1(t) + c_2\varphi_2(t) + \cdots + c_N\varphi_N(t) \tag{4.3.3}$$

where φ_i satisfies the conditions (4.3.2) and we will need to determine the constant coefficients $c_0, \ldots, c_N$. Substituting (4.3.3) into (4.3.1), we get the function in the form

$$J(c_0, c_1, \ldots, c_N). \tag{4.3.4}$$

We must determine the constants $c_0, \ldots, c_N$, which minimizes the function (4.3.4), therefore

$$\frac{\partial J}{\partial c_i} = 0, \quad i = 0, \ldots, N. \tag{4.3.5}$$

4.4 Application of the Rayleigh-Ritz method

Now we apply the Rayleigh-Ritz method to Bratu equation (3.1.1) with homogeneous Dirichlet boundary conditions (3.2.1), we get the following functional

$$J = \int_0^1 \left(\frac{1}{2}(u'(t))^2 - \lambda e^{u(t)} \right) \mathrm{d}t. \tag{4.4.1}$$

We can choose the following test function satisfying the boundary conditions (3.2.1)

$$u(t) = A\sin(\pi t) \tag{4.4.2}$$

which is proposed by Amore and Fernandez (2009).

The substitution (4.4.2) into (4.4.1) yields

$$J(A) = \int_0^1 \left(\frac{1}{2} A^2 \pi^2 \cos^2(\pi t) - \lambda e^{A \sin(\pi t)} \right) \mathrm{d}t \tag{4.4.3}$$

or

$$J(A) = \frac{A^2 \pi^2}{4} - \lambda \left[I_0(A) + L_0(A) \right] \tag{4.4.4}$$

where $I_v(z) = \frac{1}{\pi} \int_0^{\pi} e^{z \cos\theta} \cos(v\theta) \mathrm{d}\theta$ and $L_v(z) = \frac{2\left(\frac{1}{2}z\right)^v}{\sqrt{\pi}\Gamma\left(v+\frac{1}{2}\right)} \int_0^{\pi/2} \sinh(z \cos\theta) \sin^{2v}\theta \mathrm{d}\theta$ are the modified Bessel function of the first kind and the modified Struve function, respectively. From the minimum condition (4.3.5) , we obtain

$$\lambda = \frac{A\pi^3}{2 + 2\pi I_1(A) + \pi L_{-1}(A) + \pi L_1(A)}. \tag{4.4.5}$$

4.5 Numerical Experiments and comparison to the two variational methods

In this section, we give numerical results of approximate solutions belonging to some chosen λ. All computations are performed using Mathematica package.

Figure 4.2 and 4.3 shows that the behaviour of the all eigenvalues λ corresponding to the approximate solutions found by the variational iteration method and the Rayleigh-Ritz method, respectively. As shown in tables the variational iteration method solution is a good approximation when we search for lower solutions, and also the approximate solution is remarkable when we find upper solutions. In the variational iteration method, the errors arise from truncation of Taylor expansion, iterative procedure and application of shooting method (from numerical roots of (4.2.5)).

In Figure 4.2, we observe that there are noisy solutions of (4.2.5) for b, when b is near 0. Therefore, we cannot use the variational iteration method to obtain solutions for small b.

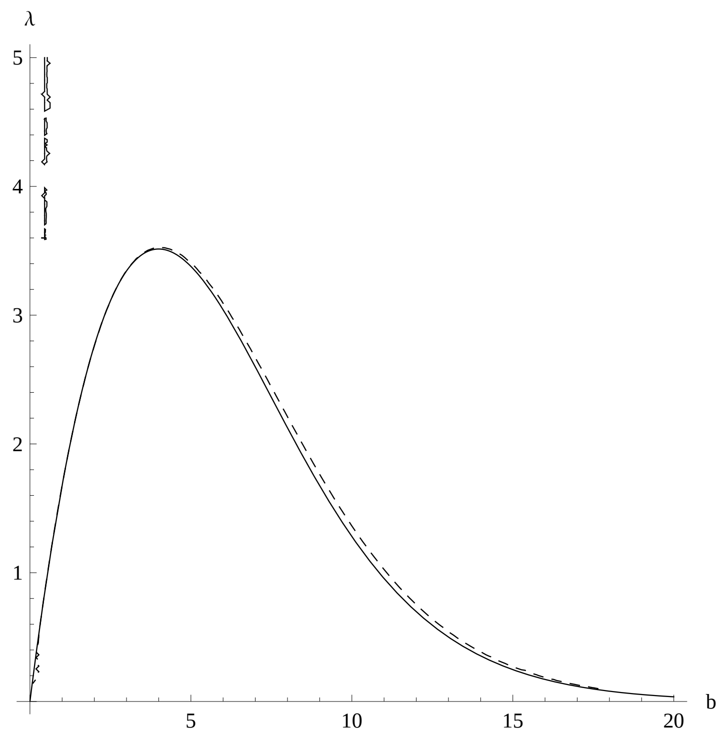

Figure 4.2 Bifurcation diagram (b,λ) obtained by the VIM solution u_2 (illustrated by dashed line) and the exact solution of Bratu problem (continuous line), where $u'(0) = b$ is the shooting parameter.

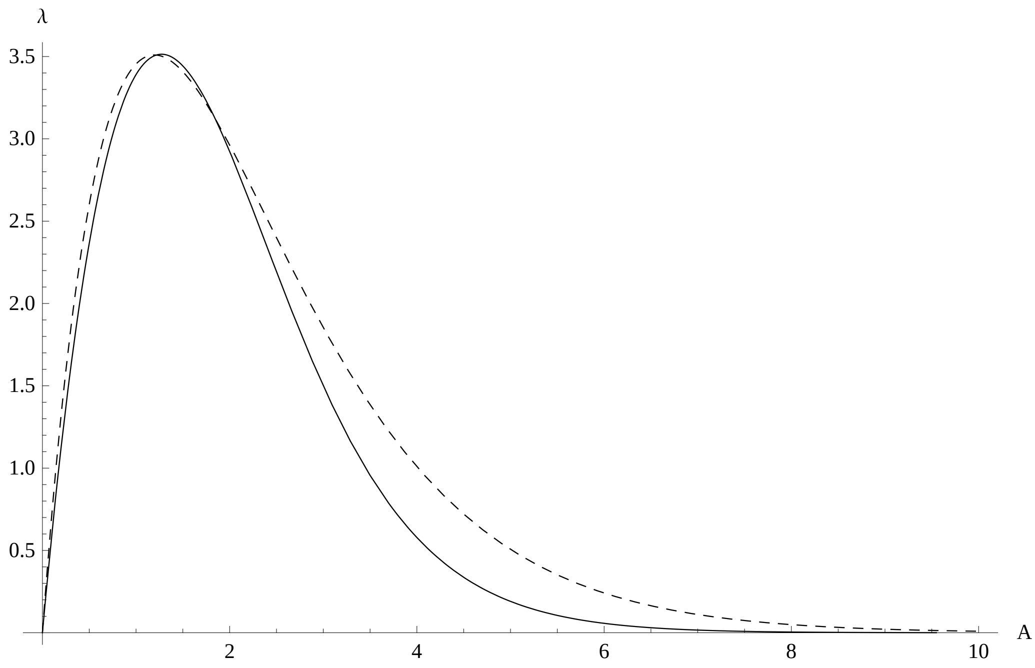

Figure 4.3 Bifurcation diagram (A,λ) of the Ritz solution for $u(t) = A\sin(\pi t)$ (illustrated by dashed line) and the exact solution of Bratu problem (continuous line)

Table 4.1 The numerical results of $u_3(t,\lambda)$ obtained by VIM when $N = 3$ and $u(t,\lambda)$ obtained by Ritz for $\lambda = 1$ (lower solution)

$t \in (0,1)$	VIM	Ritz	Abs Error VIM	Abs Error Ritz
0.1	0.0498469	0.0446824	1.55098×10^{-7}	5.16437×10^{-3}
0.2	0.0891902	0.0849910	3.07120×10^{-7}	4.19891×10^{-3}
0.3	0.1176100	0.1169800	4.55207×10^{-7}	6.28987×10^{-4}
0.4	0.1347910	0.1375180	5.88191×10^{-7}	2.72811×10^{-3}
0.5	0.1405400	0.1445950	6.50597×10^{-7}	4.05615×10^{-3}
0.6	0.1347910	0.1375180	5.88191×10^{-7}	2.72811×10^{-3}
0.7	0.1176100	0.1169800	4.55207×10^{-7}	6.28987×10^{-4}
0.8	0.0891902	0.0849910	3.08047×10^{-7}	4.19891×10^{-3}
0.9	0.0498469	0.0446824	1.55174×10^{-7}	5.16437×10^{-3}

Table 4.2 The numerical results of $u_3(t,\lambda)$ obtained by VIM when $N=3$ and $u(t,\lambda)$ obtained by Ritz for $\lambda=1$ (upper solution)

$t\in(0,1)$	VIM	Ritz	Abs Error VIM	Abs Error Ritz
0.1	1.10052	1.24024	2.32479×10^{-2}	1.62967×10^{-1}
0.2	2.16800	2.35908	4.56091×10^{-2}	2.36684×10^{-1}
0.3	3.14087	3.24699	6.34765×10^{-2}	1.69595×10^{-1}
0.4	3.87256	3.81707	6.64095×10^{-2}	1.09142×10^{-2}
0.5	4.14051	4.01350	4.90410×10^{-2}	7.79664×10^{-2}
0.6	3.87256	3.81707	6.64095×10^{-2}	1.09142×10^{-2}
0.7	3.14087	3.24699	6.34765×10^{-2}	1.69595×10^{-1}
0.8	2.16800	2.35908	4.56091×10^{-2}	2.36684×10^{-1}
0.9	1.10052	1.24024	2.32479×10^{-2}	1.62967×10^{-1}

Table 4.3 The numerical results of $u_3(t,\lambda)$ obtained by VIM when $N=3$ and $u(t,\lambda)$ obtained by Ritz for $\lambda=2$(lower solution)

$t\in(0,1)$	VIM	Ritz	Abs Error VIM	Abs Error Ritz
0.1	0.114411	0.104189	6.33700×10^{-7}	1.02219×10^{-2}
0.2	0.206420	0.198179	1.25163×10^{-6}	8.24012×10^{-3}
0.3	0.273881	0.272770	1.80598×10^{-6}	1.10933×10^{-3}
0.4	0.315091	0.320660	2.08161×10^{-6}	5.57098×10^{-3}
0.5	0.328954	0.337162	1.88510×10^{-6}	8.20982×10^{-3}
0.6	0.315091	0.320660	2.08161×10^{-6}	5.57098×10^{-3}
0.7	0.273881	0.272770	1.80598×10^{-6}	1.10933×10^{-3}
0.8	0.206420	0.198179	1.25163×10^{-6}	8.24012×10^{-3}
0.9	0.114411	0.104189	6.33700×10^{-7}	1.02219×10^{-2}

Table 4.4 The numerical results of $u_3(t,\lambda)$ obtained by VIM when $N=3$ and $u(t,\lambda)$ obtained by Ritz for $\lambda=2$ (upper solution)

$t\in(0,1)$	VIM	Ritz	Abs Error VIM	Abs Error Ritz
0.1	0.832844	0.883853	1.93689×10^{-2}	7.03784×10^{-2}
0.2	1.617310	1.681190	3.76951×10^{-2}	1.01577×10^{-1}
0.3	2.297240	2.313960	5.19853×10^{-2}	6.86990×10^{-2}
0.4	2.775880	2.720220	5.60759×10^{-2}	4.13324×10^{-4}
0.5	2.944280	2.860210	4.87532×10^{-2}	3.53228×10^{-2}
0.6	2.775880	2.720220	5.60759×10^{-2}	4.13324×10^{-4}
0.7	2.297240	2.313960	5.19853×10^{-2}	6.86990×10^{-2}
0.8	1.617310	1.681190	3.76951×10^{-2}	1.01577×10^{-1}
0.9	0.832844	0.883853	1.93689×10^{-2}	7.03784×10^{-2}

Table 4.5 The numerical results of $u_3(t,\lambda)$ obtained by VIM when $N=3$ and $u(t,\lambda)$ obtained by Ritz for $\lambda=3$ (lower solution)

$t \in (0,1)$	VIM	Ritz	Abs Error VIM	Abs Error Ritz
0.1	0.215659	0.201701	1.16028×10^{-4}	1.40739×10^{-2}
0.2	0.394092	0.383658	2.27620×10^{-4}	1.06614×10^{-2}
0.3	0.528107	0.528060	3.29243×10^{-4}	3.76226×10^{-4}
0.4	0.611414	0.620772	4.15044×10^{-4}	8.94280×10^{-3}
0.5	0.639683	0.652718	4.63596×10^{-4}	1.25716×10^{-2}
0.6	0.611414	0.620772	4.15044×10^{-4}	8.94280×10^{-3}
0.7	0.528107	0.528060	3.29243×10^{-4}	3.76226×10^{-4}
0.8	0.394092	0.383658	2.27620×10^{-4}	1.06614×10^{-2}
0.9	0.215659	0.201701	1.16028×10^{-4}	1.40739×10^{-2}

Table 4.6 The numerical results of $u_3(t,\lambda)$ obtained by VIM when $N=3$ and $u(t,\lambda)$ obtained by Ritz for $\lambda=3$ (upper solution)

$t \in (0,1)$	VIM	Ritz	Abs Error VIM	Abs Error Ritz
0.1	0.60525	0.60754	1.34382×10^{-2}	1.57203×10^{-2}
0.2	1.15427	1.15561	2.60603×10^{-2}	2.73960×10^{-2}
0.3	1.60688	1.59056	3.61313×10^{-2}	1.98093×10^{-2}
0.4	1.90994	1.86982	4.09108×10^{-2}	7.82234×10^{-4}
0.5	2.01514	1.96604	3.98712×10^{-2}	9.22686×10^{-3}
0.6	1.90994	1.86982	4.09108×10^{-2}	7.82234×10^{-4}
0.7	1.60688	1.59056	3.61313×10^{-2}	1.98093×10^{-2}
0.8	1.15427	1.15561	2.60603×10^{-2}	2.73960×10^{-2}
0.9	0.60525	0.60754	1.34382×10^{-2}	1.57203×10^{-2}

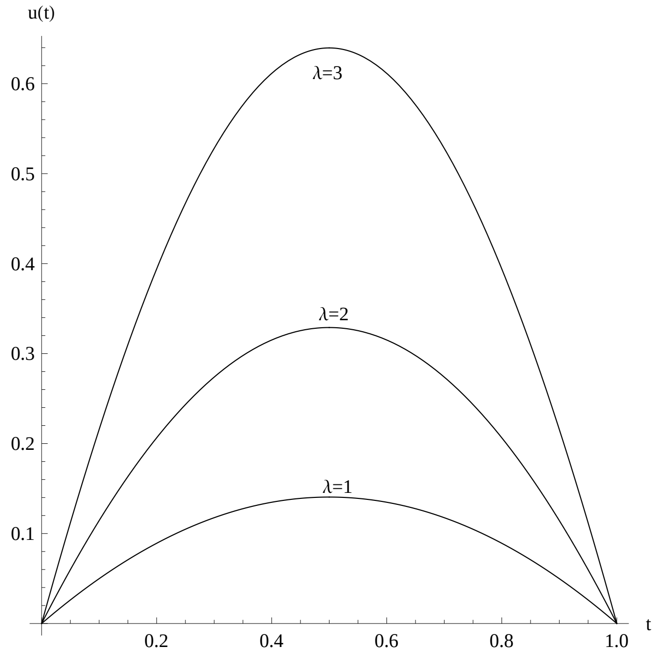

Figure 4.4 The approximate solutions $u_3(t,\lambda)$ obtained by VIM for $\lambda = 1$, $\lambda = 2$ and $\lambda = 3$ when $N = 3$ (lower solutions)

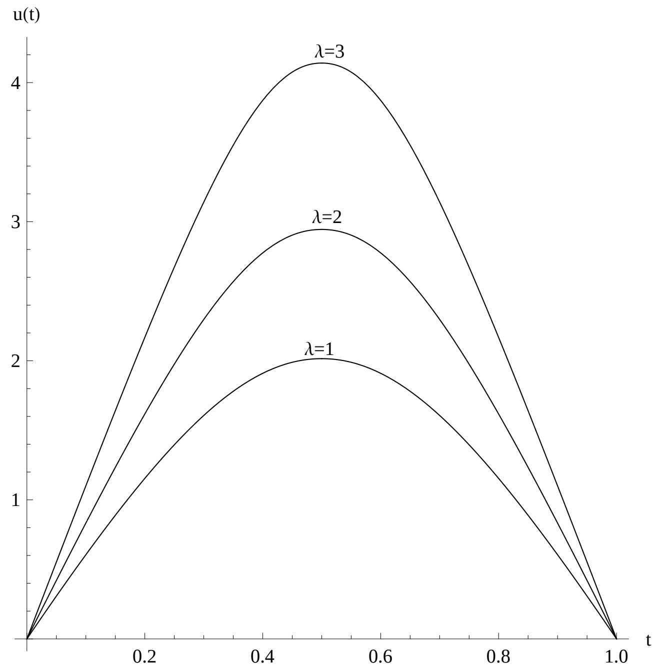

Figure 4.5 The approximate solutions $u_3(t,\lambda)$ obtained by VIM for $\lambda = 1$, $\lambda = 2$ and $\lambda = 3$ when $N = 3$ (upper solutions)

CHAPTER FIVE

CONCLUSION

Two different variational methods, the variational iteration method with shooting method and the Rayleigh-Ritz method were applied to Bratu problem (3.1.1), (3.2.1). Application of the two methods are very easy. The accuracy of two methods depends on the trial function for the Rayleigh-Ritz method and the initial function for the variational iteration method. The Rayleigh-Ritz method is used to compare its solutions with the solutions obtained by the variational iteration method. As a result, we obtain the variational iteration solutions better than the Rayleigh-Ritz solutions when we look for the lower solutions, and the upper solutions obtained by the two methods has almost same accuracy. The iteratively integrals of the non-linear part of the equation is the difficulty of the method of variational iteration. Therefore, in variational iteration method, we use Taylor expansion for the non-linear term near selected initial function. For this reason, we get unexpected shooting parameters. Hence if one can find better approximation to non-linear term, the method will have high accuracy.

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