DOKUZ EYLÜL UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

ON BERNSTEIN-SCHOENBERG OPERATOR

by
Gülter BUDAKÇI

July, 2010
İZMİR

ON BERNSTEIN-SCHOENBERG OPERATOR

A Thesis Submitted to the
Graduate School of Natural and Applied Sciences of
Dokuz Eylül University
In Partial Fulfilment of the Requirements for
the Degree of Master of Science in Mathematics

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July, 2010
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M.Sc THESIS EXAMINATION RESULT FORM

We have read the thesis entitled **"ON BERNSTEIN-SCHOENBERG OPERATOR"** completed by **GÜLTER BUDAKÇI** under supervision of **ASSOC. PROF. HALİL ORUÇ** and we certify that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

. . . .
Assoc. Prof. Halil ORUÇ
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Supervisor

. . . .
____

Jury Member

. . . .
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Jury Member

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Prof. Dr. Mustafa SABUNCU
Director
Graduate School of Natural and Applied Sciences

ACKNOWLEDGMENTS

I would like to express my sincere gratitude to my supervisor Associate Prof. Halil Oruç for his continual presence, encouragement, invaluable guidance, and endless patience during the course of this research.

I would also like to thank TÜBİTAK (The Scientific and Technical Research Council of Turkey) for its generous financial support during my M.Sc. research.

Finally, I am also grateful to my family for their endless patience, understanding and confidence to me throughout my life.

Gülter BUDAKÇI

ON BERNSTEIN-SCHOENBERG OPERATOR

ABSTRACT

In this thesis, we investigate the properties of Bernstein-Schoenberg operator on general knot sequences and on the q-integers. Also we use this operator to give another proof of some theorems of Bernstein operator. We give the transformation matrix between spline basis and Bernstein basis.

Keywords: B-splines, Marsden's Identity, Bernstein-Schoenberg Operator, q-integers

BERNSTEIN-SCHOENBERG OPERATÖRÜ ÜZERİNE

ÖZ

Bernstein-Schoenberg Operatörünün özellikleri genel nokta dizilerinde ve q-tamsayı noktalarında incelendi. Bernstein operatörüyle ilgili bazı teoremlerin ispatları bu operatörü kullanılarak yapıldı. Spline bazları ve Bernstein bazları arasındaki geçiş matrisi verildi.

Anahtar sözcükler: B-spline, Marsden Özdeşliği, Bernstein-Schoenberg Operatörü, q-tamsayıları

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CHAPTER ONE

INTRODUCTION

A spline function consists of polynomial pieces on subintervals joined together with certain continuity conditions. Formally, suppose that $k+1$ points $u_0, \ldots, u_k$ have been specified and satisfy $u_0 < \cdots < u_k$. These points are called knots. Suppose also that an integer $m \geq 0$ has been prescribed. A spline function of degree $m-1$ having knots $u_0, \ldots, u_k$ is a function S such that

i. On each interval $[u_{j-1}, u_j)$ is a polynomial of order $\leq m$.

ii. S has a continuous $(m-2)$st derivative on $[u_0, u_k]$.

The term *spline* comes from the flexible spline devices used by shipbuilders and drafters to draw smooth shapes. The theory splines is a good example of an area in mathematics which was developed in response to practical needs. Spline curves were first used as a drafting tool for aircraft and ship building industries. A loft man's spline is a flexible strip of material, which can be clamped or weighted so it will pass through any number of points with smooth deformation.

Lobachevsky investigated B-splines as early as the nineteenth century, they were constructed as convolutions of certain probability distributions. Spline functions are currently used in diverse domains of numerical analysis (interpolation, computer aided geometric design, data smoothing, numerical solution of differential and integral equations, etc.). In 1946, Schoenberg used B-splines for statistical data smoothing, and his paper started the modern theory of spline approximation. Gordon and Reisenfield formally introduced B-splines into computer aided design.

We first give some basics of B-splines which may be found in (Phillips, 2003). For splines of fixed order on a fixed partition, this is a question of choice of basis, since such splines form a linear space. Only three kinds of bases for spline spaces have actually been given serious attention; those consisting of truncated power functions, of cardinal splines, and of B-splines.

B-splines form a basis for spline spaces, see (Phillips, 2003). B-splines are splines which have smallest possible support, in other words, they are zero on a large set. For the evaluation of splines, it is desirable to have basis functions with this property. Moreover, a stable evaluation of B-splines with the aid of a recurrence relation is possible. It is shown that B-splines form a partition of unity.

1.1 B-Splines

Let $\cdots < u_{-2} < u_{-1} < u_0 < u_1 < u_2 < \cdots$ be the knot sequence where $u_{-i} \to -\infty$ as $i \to \infty$ and $u_i \to \infty$ as $i \to \infty$ with $i > 0$.

Definition 1.1.1. The B-splines of order *one* are piecewise constants defined by

$$N_i^1(x) = \begin{cases} 1, & u_i < x \leq u_{i+1}, \\ 0, & \text{otherwise.} \end{cases}$$

and $N_i^2(x)$ are piecewise linear functions on $[u_i, u_{i+2}]$, and zero elsewhere.

(a) $N_i^1(x)$ (b) $N_i^2(x)$

Figure 1.1 Graphs of $N_i^1(x)$ and $N_i^2(x)$

The original definition of the B-spline basis functions uses the idea of divided differences. Hence equivalently we can define B-splines as a multiple of a divided difference of a truncated power where truncated power is defined as

$$(t-x)_+^{m-1} = \begin{cases} (t-x)^{m-1}, & x \le t, \\ 0, & \text{otherwise.} \end{cases} \tag{1.1.1}$$

Theorem 1.1.1. *For any $n \geq 0$ and all i,*

$$N_i^m(x) = (u_{i+m} - u_i).[u_i, \dots, u_{i+m}](t-x)_+^{m-1}, \tag{1.1.2}$$

where $[u_i, \dots, u_{i+m}]$ denotes a divided difference operator of order m that is applied to the truncated power $(t-x)_+^{m-1}$, regarded as a function of the variable t.

See de Boor (1972), Carl de Boor established in the early 1970's a recursive relationship for the B-spline basis. By applying Leibniz' theorem, de Boor was able to derive the following formula for B-spline basis functions

$$N_i^m(x) = \left(\frac{x - u_i}{u_{i+m-1} - u_i} \right) N_i^{m-1}(x) + \left(\frac{u_{i+m} - x}{u_{i+m} - u_{i+1}} \right) N_{i+1}^{m-1}(x),$$

starting with $N_i^1(x)$.

1.1.1 Properties of B-splines

Definition 1.1.2. Let S denote a spline defined on the whole real line. The *interval of support* of the spline S is the smallest closed interval outside which S is zero.

Theorem 1.1.2. *The interval of support of the B-spline N_i^m is $[u_i, u_{i+m}]$, and N_i^m is positive in the interior of this interval.*

The next result shows that the derivative of a spline function is also a spline function.

Theorem 1.1.3. *For $m-1 \geq 0$, we have*

$$\frac{d}{dx}N_i^m(x) = \left(\frac{m-1}{u_{i+m-1}-u_i}\right)N_i^{m-1}(x) - \left(\frac{m-1}{u_{i+m}-u_{i+1}}\right)N_{i+1}^{m-1}(x) \tag{1.1.3}$$

for all real x. For $m=2$, equation (1.1.3) holds for all x except at the three knots u_i, u_{i+1}, and u_{i+2}, where the derivative of N_i^2 is not defined.

In the remainder of the thesis B-splines are computed with a knot sequence $u_0, \ldots, u_k$ and defined over all $\mathbb{R}$, and the algorithms described are independent of the chosen interval $[a,b]$ (with the condition that $u_{m-1} \leq a$ and $u_{k-m+1} \geq b$); in the algorithms described below, we have set $l = k-m$. Notice that dimension of the space is $l+1$. We will see that a spline approximation is

$$S(x) = \sum_{i=0}^{l} a_i N_i^m(x), \tag{1.1.4}$$

a sum of multiples of all B-splines of order m whose interval of support contains one of the subintervals $[u_j, u_{j+1}]$ where $j = m-1, \ldots, k-m$.

1.1.2 Marsden's identity and its consequences

It is obtained in (Marsden, 1970) that

$$(z-x)^{m-1} = \sum_{j=0}^{l} (z-u_{j+1})(z-u_{j+2})\ldots(z-u_{j+m-1})N_j^m(x) \tag{1.1.5}$$

for all real or complex z and all real x restricted to the interval

$$I = \{x : \quad u_{m-1} \leq x \leq u_{k-m+1}\} \tag{1.1.6}$$

It is useful to give the definition of elementary symmetric functions which can be found in (Phillips, 2003) since we use these functions in the next theorem.

Definition 1.1.3. The elementary symmetric function $\sigma_r(x_0, x_1, \ldots, x_n)$, for, $r \geq 1$, is the sum of all products of r distinct variables chosen from the set $\{x_0, x_1, \ldots, x_n\}$, and we define $\sigma_0(x_0, x_1, \ldots, x_n) = 1$. Namely we have

$$\sigma_r(x_0, \ldots, x_n) = \sum_{0 \leq i_1 < i_2 < \cdots < i_r \leq n} x_{i_1} \cdots x_{i_r} \tag{1.1.7}$$

As a consequence of Definition 1.1.3 we have

$$\sigma_r(x_0, x_1, \ldots, x_n) = 0 \quad \text{if} \quad r > n+1. \tag{1.1.8}$$

Since

$$(1 + x_0 x)(1 + x_1 x) \cdots (1 + x_n x) = \sum_{r=0}^{n+1} \sigma_r(x_0, x_1, \ldots, x_n) x^r \tag{1.1.9}$$

the polynomial $(1 + x_0 x)(1 + x_1 x) \cdots (1 + x_n x)$ is the generating function for elementary symmetric functions.

The following theorem illustrates the relationship between monomials and B-splines. It can be proved easily using Marsden's Identity.

Theorem 1.1.4. *For any given integer $r \geq 0$ we can express any monomial x^r as a linear combination of B-splines $N_i^m(x)$, for any fixed $m - 1 \geq r$, in the form*

$$\binom{m-1}{r} x^r = \sum_{i=0}^{l} \sigma_r(u_{i+1}, \ldots, u_{i+m-1}) N_i^m(x) \tag{1.1.10}$$

where $\sigma_r(u_{i+1}, \ldots, u_{i+m-1})$ is the elementary symmetric function of order r in the variables $u_{i+1}, \ldots, u_{i+m-1}$. Furthermore, if $r = 0$ in (1.1.10) we have

$$\sum_{i=0}^{l} N_i^m(x) = 1 \tag{1.1.11}$$

and thus the B-spline of order m form a partition of unity.

Proof. It follows from (1.1.9) that

$$(1+u_{i+1}x)\dots(1+u_{i+m-1}x) = \sum_{r=0}^{m-1} \sigma_r(u_{i+1},\dots,u_{i+m-1})x^r. \qquad (1.1.12)$$

By replacing x by $-1/z$ and multiplying through z^{m-1}, we find that

$$(z-u_{i+1})\dots(z-u_{i+m-1}) = z^{m-1}\sum_{r=0}^{m-1} \sigma_r(u_{i+1},\dots,u_{i+m-1})(-z)^{(-r)}. \qquad (1.1.13)$$

Combining (1.1.5) and (1.1.13) gives

$$(z-x)^{m-1} = \sum_{i=0}^{l}\left(\sum_{r=0}^{m-1}(-1)^r\sigma_r(u_{i+1},\dots,u_{i+m-1})z^{m-1-r}\right)N_i^m(x) \qquad (1.1.14)$$

Equating the coefficients of z^{m-1-r} on both sides gives

$$\binom{m-1}{r}x^r = \sum_{i=0}^{l}\sigma_r(u_{i+1},\dots,u_{i+m-1})N_i^m(x).$$

□

Note that comparing the coefficient of z^{m-1} in (1.1.14) yields

$$\sum_{j=0}^{l} N_j^m(x) = 1 \quad \text{if} \quad x\in I, \qquad (1.1.15)$$

and that of z^{m-2} gives

$$\sum_{j=0}^{l}\xi_j N_j(x) = x \quad \text{if} \quad x\in I \qquad (1.1.16)$$

where

$$\xi_j = \frac{1}{m-1}(u_{j+1}+u_{j+2}+\dots+u_{j+m-1}) \quad j=0,1,\dots,l. \qquad (1.1.17)$$

This is called Greville Abscissae. We see from (1.1.17) and $u_0 < \cdots < u_k$ that Greville Abscissaes are ordered

$$u_0 < \xi_0 < \xi_1 < \ldots < \xi_l < u_k. \tag{1.1.18}$$

1.1.3 Further Results of Marsden's Identity

We have seen in the last section that one can express the monomials as a linear combination of B-splines. So we have a transformation matrix A of size $m \times (l+1)$ between the monomials and the spline basis. That is,

$$\begin{bmatrix} 1 \\ x \\ \vdots \\ x^{m-1} \end{bmatrix} = A \begin{bmatrix} N_0^m(x) \\ N_1^m(x) \\ \vdots \\ N_l^m(x) \end{bmatrix}$$

Let N be the vector containing B-splines and A be the transformation matrix between the monomials and B-splines. It can be seen from the equation (1.1.10) that the entries of A are

$$A_{i,j} = \frac{1}{\binom{m-1}{i}} \sigma_i(u_{j+1}, \ldots, u_{j+m-1}) \tag{1.1.19}$$

for $i = 0, \ldots, m-1$ and $j = 0, \ldots, m+n-2$. Let B be the vector containing Bernstein polynomials and M be the matrix between the monomials and Bernstein basis. Then from (Oruç & Phillips, 2003) we have

$$\begin{bmatrix} 1 \\ x \\ \vdots \\ x^{m-1} \end{bmatrix} = M \begin{bmatrix} B_0^{m-1}(x) \\ B_1^{m-1}(x) \\ \vdots \\ B_{m-1}^{m-1}(x) \end{bmatrix}$$

where $B_i^{m-1}(x)$ dentoes the ith Bernstein basis of degree $m-1$ such that

$$B_i^{m-1}(x) = \binom{m-1}{i} x^i (1-x)^{m-1-i} \tag{1.1.20}$$

and M is an upper triangular matrix such that

$$M_{ij} = \frac{\binom{j}{i}}{\binom{m-1}{i}} \quad \text{for} \quad i = 0,\ldots,m-1, \quad j = 0,\ldots,m-1 \tag{1.1.21}$$

Then it follows that

$$AN = MB \tag{1.1.22}$$

Since M is an invertible matrix we have

$$B = M^{-1}AN \tag{1.1.23}$$

where

$$(M^{-1})_{ij} = (-1)^{j-i}\binom{m-1}{j}\binom{j}{i} \quad \text{for} \quad i = 0,\ldots,m-1, \quad j = 0,\ldots,m-1 \tag{1.1.24}$$

Notice that we generate a transformation matrix between Bernstein basis and spline basis.

CHAPTER TWO

BERNSTEIN-SCHOENBERG OPERATOR

In this chapter we shall discuss the properties of Bernstein-Schoenberg Operator for general knot sequences. In (Schoenberg, 1967) Schoenberg introduced a spline approximation operator which generalised the Bernstein polynomial and we shall refer to as the Bernstein-Schoenberg operator.

We call $S_{m,n}$ the *Bernstein-Schoenberg Operator*; it maps a function f, defined on $[a,b]$, to $S_{m,n}f$, where the function $S_{m,n}f$ evaluated at x is denoted by $S_{m,n}(f;x)$.

In approximation theory it is often useful to have an approximation $S_{m,n}f$ to a function f which is not only close to f but whose graph has a similar shape to that of the graph of f. Goodman discussed in (Goodman, 1994) the advantages of variation diminishing property when designing the curves or constructing approximation operators. Like the Bernstein polynomials Bernstein-Schoenberg operator has variation diminishing and therefore has certain shape preserving properties. Goodman and Sharma discussed in (Goodman & Sharma, 1985) the convexity properties for of Bernstein-Schoenberg operator for special knot sequence.

In the remainder of this note we investigate the operator for the functions f which are defined on the interval $[0,1]$.

2.1 Preliminaries

If $f(x)$ is defined in the interval $[u_0,u_k]$ we construct the spline function

$$S_{m,n}(f;x) = \sum_{j=0}^{l} f(\xi_j)N_j^m(x) \tag{2.1.1}$$

where

- m is the order of B-splines, that is each piecewise polynomial is of degree $m-1$
- n is the number of intervals in $[0,1]$
- $l = m+n-2$
- $\xi_i = \frac{1}{m-1}(u_{i+1}+\ldots+u_{i+m-1})$, the Greville abscissae.

and we have the knot sequence;

$$\begin{gathered} u_0 = u_1 = \cdots = u_{m-1} = 0 \\ u_m \\ \vdots \\ u_{m+n-2} \\ u_{m+n-1} = \cdots = u_{l+m} = 1 \end{gathered} \tag{2.1.2}$$

The importance of taking the first m knots 0 and the last m knots 1 is the fact

$$S_{m,n}(f;0) = f(0) \quad , \quad S_{m,n}(f;1) = f(1)$$

which is known as end-point interpolation. The following figure shows Bernstein-Schoenberg approximation to $f(x) = x^2$.

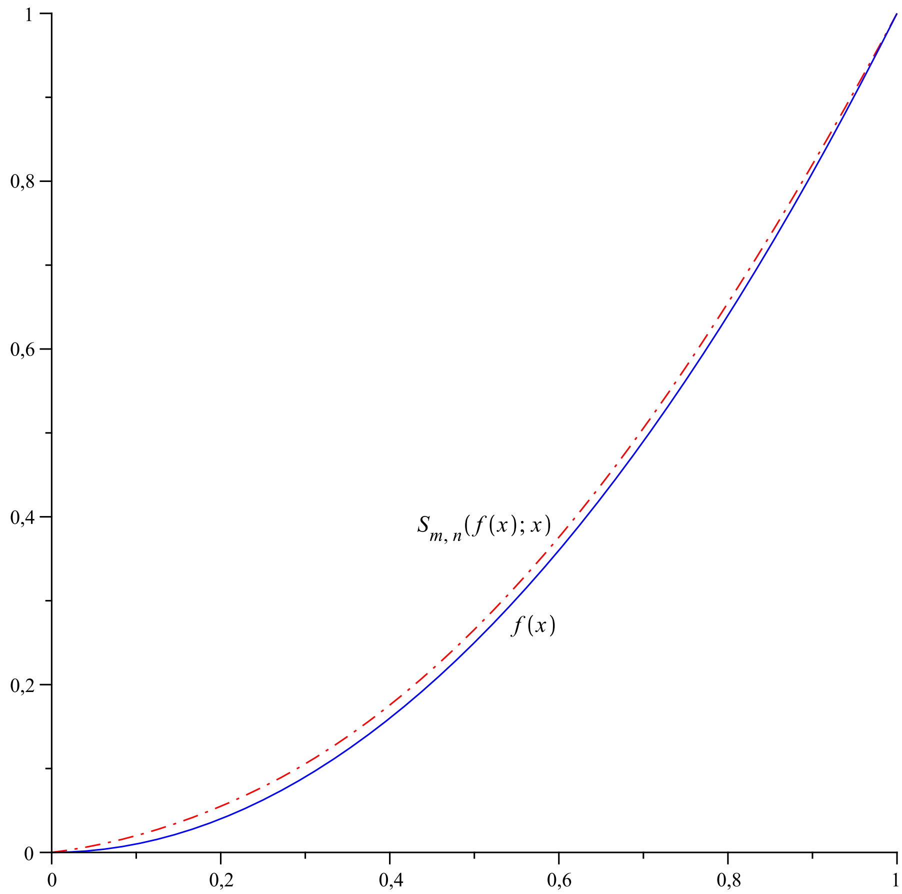

Figure 2.1 The graph of $S_{3,4}(x^2;x)$ and $f(x) = x^2$

Notice that when we choose the knot sequence as above we have $I = [u_0, u_k]$ in (1.1.6). So we can use Marsden's Identity in the whole interval $[0,1]$. It follows from

$$\sum_{i=0}^{l} N_i^m(x) = 1 \quad \text{and} \quad \sum_{i=0}^{l} \xi_i N_i^m(x) = x$$

that $S_{m,n}f(x) = f(x)$ for any linear function $f(x) = ax + b$

2.2 The Relationship Between Bernstein-Schoenberg Operator and Bernstein Operator

The Bernstein polynomials is first introduced by S. Bernstein in 1912. Then it is investigately vastly see (Phillips, 2003) for further information.

Definition 2.2.1. For a given function f on $[0,1]$, we define the Bernstein Polynomial

$$B_n(f;x) = \sum_{i=0}^{n} f\left(\frac{i}{n}\right)\binom{n}{i}x^i(1-x)^{n-i} \tag{2.2.1}$$

for each positive integer n which denotes the degree of the polynomial. We call B_n the *Bernstein Operator*.

One of the most important properties of the Bernstein-Schoenberg operator is that if we select the knot sequence in a special case we obtain Bernstein polynomials. That is, if we choose $n = 1$ in equation (2.1.2) the knot sequence becomes;

$$u_0 = u_1 = \cdots = u_{m-1} = 0$$

$$u_m = u_1 = \cdots = u_{l+m} = 1$$

we obtain

$$S_{m,n}(f;x) = B_{m-1}(f;x) \tag{2.2.2}$$

Therefore Bernstein-Schoenberg operator may be viewed as a generalization to the Bernstein operator.

2.3 Convexity of Bernstein-Schoenberg Operator

In this section we look into the splines for a convex function f. We first give the definition of a convex function.

Definition 2.3.1. A function f is said to be convex on $[a,b]$ if for any $x_1, x_2 \in [a,b]$,

$$\lambda f(x_1) + (1-\lambda)f(x_2) \geq f(\lambda x_1 + (1-\lambda)x_2) \tag{2.3.1}$$

for any $\lambda \in [0,1]$. Geometrically, this is saying that a chord connecting any two points on the convex curve $y = f(x)$ is never below the curve.

Alternatively, if $f : I \to R$ is a twice differentiable function then f is convex if and only if $f''(x) \geq 0$ for all $x \in I$.

From (Goodman, 1994) we state the following important facts.

i. If the function $f \in C[0,1]$ is increasing (respectively decreasing), then $S_{m,n}f$ is increasing (respectively decreasing).

ii. If f is convex on $[0,1]$, then $S_{m,n}f$ is also convex.

However, we propose an alternative proof for the latter property. Firstly we need the *Jensen's Inequality*, see (Webster, 1994).

Jensen's Inequality: Let f be continuous and convex on an interval I. If $x_1, x_2, \ldots, x_n$ are in I and $0 < \lambda_1, \lambda_2, \ldots, \lambda_n < 1$ with $\lambda_1 + \cdots + \lambda_n = 1$, then

$$\lambda_1 f(x_1) + \cdots + \lambda_n f(x_n) \geq f(\lambda_1 x_1 + \cdots + \lambda_n x_n)$$

Theorem 2.3.1. *If $f(x)$ is convex on $[0,1]$ then*

$$S_{m,n}(f;x) \geq f(x) \qquad 0 \leq x \leq 1 \tag{2.3.2}$$

Proof. Let $\xi_i = \frac{1}{m-1}(u_{i+1} + \ldots + u_{i+m-1})$ and $\lambda_i = N_i^m(x)$, we see that $\lambda_i \geq 0$ for all $x \in [0,1]$ and as in (1.1.11)

$$\lambda_0 + \lambda_1 + \ldots + \lambda_l = 1, \tag{2.3.3}$$

and

$$\lambda_0 \xi_0 + \lambda_1 \xi_1 + \ldots + \lambda_l \xi_l = x. \tag{2.3.4}$$

Then we obtain from *Jensen's Inequality* that

$$S_{m,n}(f;x) = \sum_{i=0}^{l} \lambda_i f(\xi_i) \geq f\left(\sum_{i=0}^{l} \lambda_i \xi_i\right) = f(x) \tag{2.3.5}$$

and this completes the proof. $\square$

Theorem 2.3.2. *If f is a convex function defined on $[0,1]$ then $B_{m-1}(f;x)$ is also convex.*

Note that as a special case, for $n = 1$ we have

$$S_{m,1}(f;x) = B_{m-1}(f;x) \geq f(x). \tag{2.3.6}$$

Proof. Our aim is to show that

$$\frac{d^2}{dx^2} S_{m,1}(f;x) \geq 0.$$

Using (1.1.3) we have

$$\begin{aligned}
\frac{d}{dx} S_{m,1}(f;x) &= \frac{d}{dx} \sum_{i=0}^{l} f(\xi_i) N_i^m(x) \\
&= \sum_{i=0}^{l} f(\xi_i) \left[\frac{m-1}{u_{i+m-1} - u_i} N_i^{m-1}(x) - \frac{m-1}{u_{i+m} - u_{i+1}} N_{i+1}^{m-1}(x) \right] \\
&= (m-1) \left\{ \sum_{i=1}^{m-1} f(\xi_i) N_i^{m-1}(x) - \sum_{i=1}^{m-1} f(\xi_{i-1}) N_i^{m-1}(x) \right\} \\
&= (m-1) \sum_{i=1}^{m-1} [f(\xi_i) - f(\xi_{i-1})] N_i^{m-1}(x). \qquad (2.3.7)
\end{aligned}$$

For simplicity set $b_i = f(\xi_i) - f(\xi_{i-1})$. Differentiating the equation (2.3.7) one more time gives

$$\begin{aligned}
\frac{d^2}{dx^2}S_{m,1}(f;x) &= \frac{d}{dx}(m-1)\sum_{i=1}^{m-1} b_i N_i^{m-1}(x) \\
&= (m-1)\sum_{i=1}^{m-1} b_i \left[\frac{m-2}{u_{i+m-2}-u_i}N_i^{m-2}(x) - \frac{m-2}{u_{i+m+1}-u_{i+1}}N_{i+1}^{m-2}(x)\right] \\
&= (m-1)(m-2)\sum_{i=2}^{m-1} b_i N_i^{m-2}(x) - \sum_{i=2}^{m-1} b_{i-1} N_i^{m-2}(x) \\
&= (m-1)(m-2)\sum_{i=2}^{m-1} [b_i - b_{i-1}] N_i^{m-2}(x).
\end{aligned}$$

It follows that

$$b_i - b_{i-1} = \frac{1}{2}\left(\frac{1}{2}f(\xi_i) - f(\xi_{i-1}) + \frac{1}{2}f(\xi_{i-2})\right).$$

Substituting $\xi_i = \frac{i}{m-1}$ gives

$$\frac{d^2}{dx^2}S_{m,1}(f;x) = \frac{1}{2}(m-1)(m-2)\sum_{i=2}^{m-1}\left(\frac{1}{2}f(\frac{i}{m-1}) - f(\frac{i-1}{m-1}) + \frac{1}{2}f(\frac{i-2}{m-1})\right)N_i^{m-2}(x)$$

Notice that $N_i^{m-2}(x) \geq 0$ for all i. So it is enough to show that

$$\frac{1}{2}f\left(\frac{i}{m-1}\right) - f\left(\frac{i-1}{m-1}\right) + \frac{1}{2}f\left(\frac{i-2}{m-1}\right) \geq 0.$$

Since f is convex we have, with

$$\lambda = \frac{1}{2}, \quad x_1 = \frac{i}{m-1} \quad \text{and} \quad x_2 = \frac{i-2}{m-1} \quad \text{in} \quad (2.3.1)$$

$$\frac{1}{2}f\left(\frac{i}{m-1}\right) + \left(1-\frac{1}{2}\right)f\left(\frac{i-2}{m-1}\right) \geq f\left(\frac{1}{2}\frac{i}{m-1} + \left(1-\frac{1}{2}\right)\frac{i-2}{m-1}\right) = f\left(\frac{i-1}{m-1}\right)$$

This completes the proof. $\square$

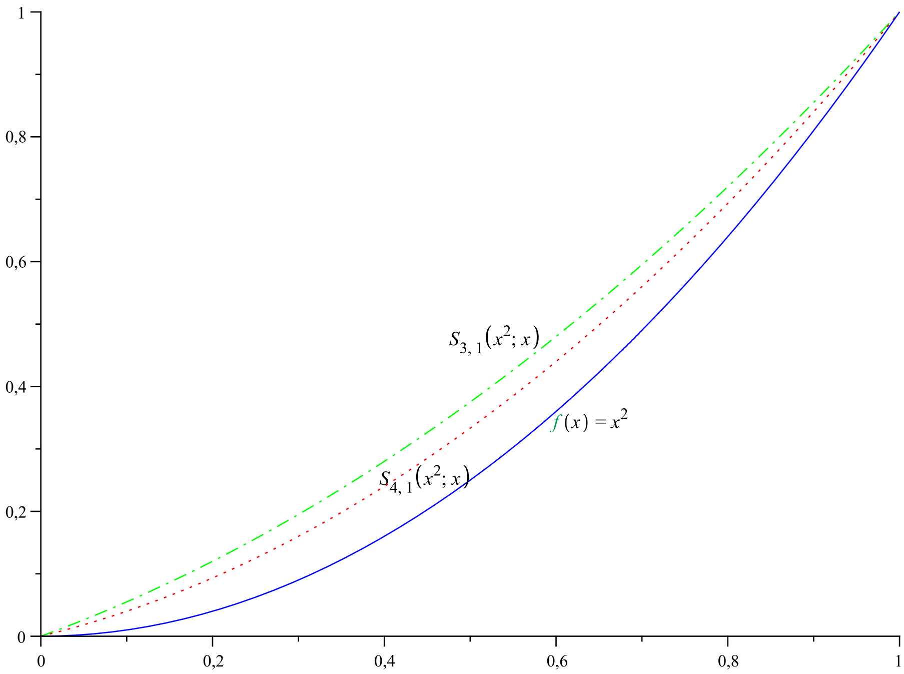

Figure 2.2 The graph of $S_{3,1}(x^2;x)$, $S_{4,1}(x^2;x)$ and $f(x)=x^2$

2.4 Monotonicity of Bernstein-Schoenberg Operator

It can be easily seen that $S_{m,n}$ is a monotone operator. That is, suppose that $f(x) \geq g(x)$, for all $x \in [0,1]$. So,

$$S_{m,n}(f;x) = \sum_{i=0}^{l} f(\xi_i)N_i^m(x) \geq \sum_{i=0}^{l} g(\xi_i)N_i^m(x) = S_{m,n}(g;x)$$

giving $S_{m,n}(f;x) \geq S_{m,n}(g;x)$

As a consequence of monotonicity of $S_{m,n}$ and the fact that $S_{m,n}(1;x)=1$, if $m \leq f(x) \leq M$, $x \in [0,1]$ then $m \leq S_{m,n}(f;x) \leq M$ for all $x \in [0,1]$. (Marsden & Schoenberg, 1966) shows that for the knot sequence

$$u_0 = u_1 = \cdots = u_{m-1} = 0$$

$$u_m = \frac{1}{n}, \ldots, u_l = \frac{n-1}{n}$$

$$u_{l+1} = \cdots = u_{l+m} = 1$$

$S_{m,n}(x^2;x)$ converges to x^2 uniformly as $m \to \infty$. Note that this is also true as $n \to \infty$. It follows from Bohman-Korovkin Theorem that $S_{m,n}$ converges uniformly to the function f where $f \in C[0,1]$ since $S_{m,n}f$ converges uniformly to $f(x) = 1, x, x^2$.

Let us recall Bohman-Korovkin Theorem, see (Kincaid & Cheney, 1996)

Theorem 2.4.1. *(Bohman-Korovkin Theorem) Let L_n ($n \geq 1$) be a sequence of positive linear operators defined on $C[a,b]$ and taking values in the same space. If $\|L_n f - f\|_\infty \to 0$ for the three functions $f(x) = 1, x$, and x^2, then the same is true for all $f \in C[a,b]$.*

2.5 Modulus of Continuity

It is not important that f is continuous or not, we define the modulus of continuity by the equation

$$\omega(f;\delta) = \sup_{|s-t| \leq \delta} |f(s) - f(t)|$$

If f is a continuous function defined on an interval $[a,b]$, then it is uniformly continuous. This means for any $\varepsilon > 0$, there is a $\delta > 0$ such that for all s and t in $[a,b]$,

$$|s-t| < \delta \quad \text{implies} \quad |f(s) - f(t)| < \varepsilon$$

Hence, $\omega(f;\delta) \leq \varepsilon$. In other words, for a continuous function f on a closed and bounded interval, the modulus of continuity $\omega(f;\delta)$ converges to 0 as δ converges to 0.

By the mean value theorem, if f' exists, continuous and $|f'(x)| \leq M$, we have

$$|f(s) - f(t)| = |f(\xi)||s - t| \leq M|s - t|$$

Thus, $\omega(f;\delta) \leq M\delta$.

Theorem 2.5.1. *If f is a function on $[u_0, u_k]$, then the spline function g where $g = \sum_{i=0}^{l} f(u_{i+2}) N_i^m$ satisfies*

$$\sup_{u_0 \leq x \leq u_k} |f(x) - g(x)| \leq (m-1)\omega(f;\delta)$$

where $\delta = \sup_{m-1 \leq i \leq k-m} |u_i - u_{i-1}|$, see (Kincaid & Cheney, 1996)

Let S_k^m denotes the family of all splines which are piecewise polynomials of order $\leq m$ on the intervals $[u_0, u_1], \ldots, [u_{k-1}, u_k]$.

Denote the function *dist*; distance from a function f to a subspace G in a normed space is defined by

$$dist(f, G) = \inf_{g \in G} \|f - g\|$$

From above theorem, we have

$$dist(f, S_k^m) \leq (m-1)\omega(f;\delta). \tag{2.5.1}$$

If f is continuous, then

$$\lim_{\delta \to 0} \quad \omega(f;\delta) = 0.$$

Hence, as the density of the knots is increased, the upper bound in equation (2.5.1) will approach zero, showing that the distance between a continuous function and its spline approximant can be made as close as we wish.

CHAPTER THREE

BERNSTEIN-SCHOENBERG OPERATOR on q-INTEGERS

In this chapter, we investigate the properties of Bernstein-Schoenberg operator that is defined on q-integers, geometrically spaced knot sequence. We denote this operator on q-integers by $S_{m,n}(f;x,q)$.

In the remainder of this chapter we use the knot sequence

$$u_0 = u_1 = \cdots = u_{m-1} = 0$$

$$u_m = \frac{1}{[n]}, \ldots, u_l = \frac{[n-1]}{[n]}$$

$$u_{l+1} = \cdots = u_{l+m} = 1$$

Here $[i]$ denotes a q-integer, defined by

$$[i] = \begin{cases} (1-q^i)/(1-q), & q \neq 1, \\ i, & q = 1. \end{cases} \tag{3.0.2}$$

3.1 B-splines based on q-integers

Koçak and Phillips, (Koçak & Phillips, 1994) studied B-splines based on q-integers, which is a generalization of the similarly particularly simple properties of the uniform B-splines. Notice that in this section we have a fixed real parameter $q > 0$. B-splines on the q-integers are defined by

$$N_i^1(x) = \begin{cases} 1, & [i] < x \leq [i+1], \\ 0, & \text{otherwise,} \end{cases}$$

and recursively ,

$$N_i^m(x) = \left(\frac{x-[i]}{q^i[m-1]}\right) N_i^{m-1}(x) + \left(\frac{[i+m]-x}{q^{i+1}[m-1]}\right) N_{i+1}^{m-1}(x).$$

The B-splines with knots at the q-integers satisfy the relation

$$N_i^m(x) = N_{i+1}^m(qx+1).$$

More generally

$$N_i^m(x) = N_{i+k}^m(q^k x + [k]).$$

Although the uniform B-splines are symmetric about the midpoint of the interval of support, the B-splines with knots at the q-integers are not.

3.2 Properties of Generalized Operator

Since we choose a special knot sequence, the properties for general knot sequence also satisfy. This means;

1. Generalized Bernstein-Schoenberg operator is also linear, i.e,

$$S_{m,n}(\lambda f + g; x, q) = \lambda S_{m,n}(f; x, q) + S_{m,n}(g; x, q)$$

2. If $|f(x))| \leq M$ then $|S_{m,n}(f;x,q)| \leq M$ for all for any $q > 0$.

3. It also has the variation diminishing property.

4. Suppose that f is convex on $[0,1]$ then $S_{m,n}(f;x,q)$ is convex for any $q>0$.

5. If $f(x)$ is convex on $[0,1]$ then

$$S_{m,n}(f;x,q) \geq f(x), \quad \text{for} \quad 0 \leq x \leq 1 \quad \text{and} \quad q > 0. \tag{3.2.1}$$

6. $S^q_{m,n}$ is also a monotone operator.

Remark. A great deal of research papers have appeared on $q-$Bernstein Bézier polynomials which is first introduced by G.M. Phillips in (Phillips, 1997) as a generalization of Bernstein polynomials. See full details in a recent survey paper by G. M. Phillips (Phillips, 2008). He defines q-Bernstein polynomials as;

$$B^q_n(f;x) = \sum_{r=0}^{n} f_r \begin{bmatrix} n \\ r \end{bmatrix} x^r \prod_{s=0}^{n-r-1} (1-q^s x)$$

where $f_r = f\left(\frac{[r]}{[n]}\right)$. The $q-$binomial coefficient $\begin{bmatrix} n \\ i \end{bmatrix}$, which is also called a Gaussian polynomial, in (Andrews, 1998), is defined as

$$\begin{bmatrix} n \\ i \end{bmatrix} = \frac{[n][n-1]\cdots[n-i+1]}{[i][i-1]\cdots[1]} \tag{3.2.2}$$

for $0 \leqslant i \leqslant n$, and has the value 0 otherwise. The generalized Bernstein polynomials $B^q_n f$, holds an interesting relation when we vary the parameter. That is; for $0<q\leq r<1$ and a convex function f convex on $[0,1]$, we have

$$B^r_n(f,x) \leq B^q_n(f,x), \quad 0 \leq x \leq 1$$

However, for $0<q<r<1$ then there is no relation between $S_{m,n}(f;x,q)$ and $S_{m,n}(f;x,r)$ for $n>1$, i.e, we do not have

$$S_{m,n}(f;x,q) \leq S_{m,n}(f;x,r) \quad \text{or} \quad S_{m,n}(f;x,r) \leq S_{m,n}(f;x,q). \tag{3.2.3}$$

Example 3.2.1.

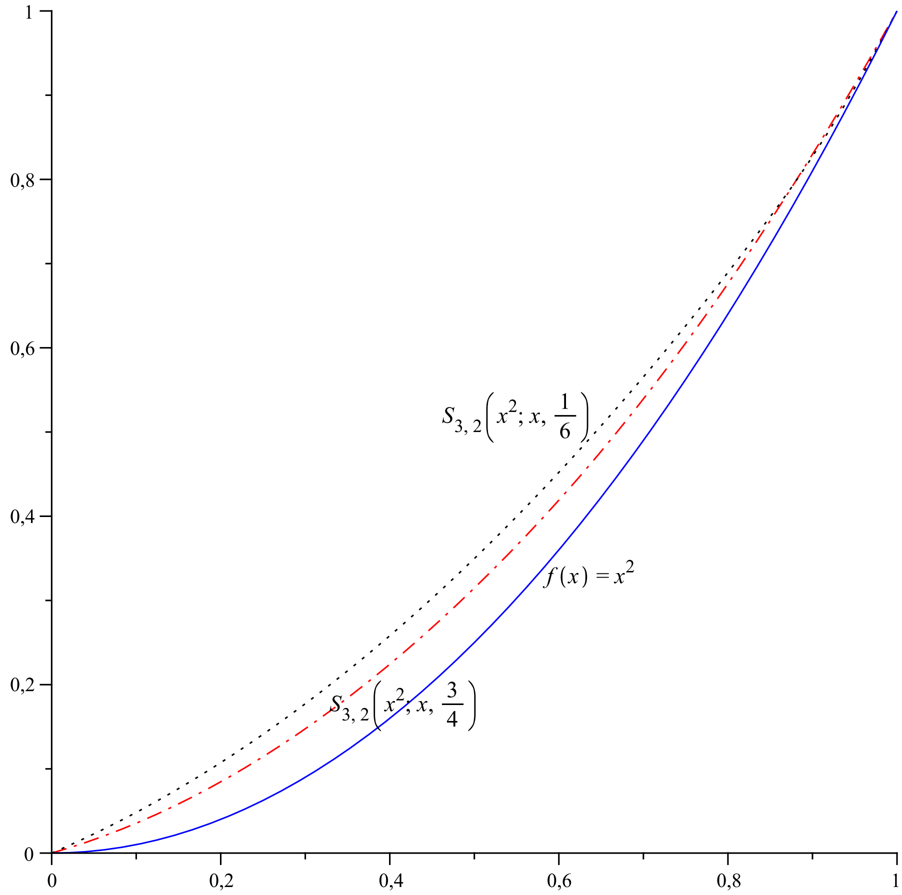

Figure 3.1 The graph of $S_{3,2}(x^2;x,3/4)$, $S_{3,2}(x^2;x,1/6)$ and $f(x)=x^2$

where

$$S_{3,2}(x^2;x,3/4) = \begin{cases} \frac{11}{16}x^2 + \frac{2}{7}x, & 0 < x < \frac{4}{7}, \\ \frac{5}{6}x^2 + \frac{5}{42}x + \frac{1}{21}, & \frac{4}{7} < x < 1, \\ 0, & \text{otherwise} \end{cases} \tag{3.2.4}$$

$$S_{3,2}(x^2;x,1/6) = \begin{cases} \frac{13}{24}x^2 + \frac{3}{7}x, & 0 < x < \frac{6}{7}, \\ 2x^2 - \frac{29}{14}x + \frac{15}{14}, & \frac{6}{7} < x < 1, \\ 0, & \text{otherwise} \end{cases} \tag{3.2.5}$$

Notice that for $x = 0.2$

$$S_{3,2}(x^2; 0.2, 3/4) = 0.0846$$
$$S_{3,2}(x^2; 0.2, 1/6) = 0.1073,$$

we have

$$S_{3,2}(x^2; 0.2, 3/4) - S_{3,2}(x^2; 0.2, 1/6)) = -0.0227 < 0$$

and for $x = 0.9$

$$S_{3,2}(x^2; 0.9, 3/4) = 0.8297$$
$$S_{3,2}(x^2; 0.9, 1/6) = 0.8271.$$

$$S_{3,2}(x^2; 0.9, 3/4) - S_{3,2}(x^2; 0.9, 1/6) = 0.0026 > 0$$

It is easily seen in the next figure that error function changes sign in the interval.

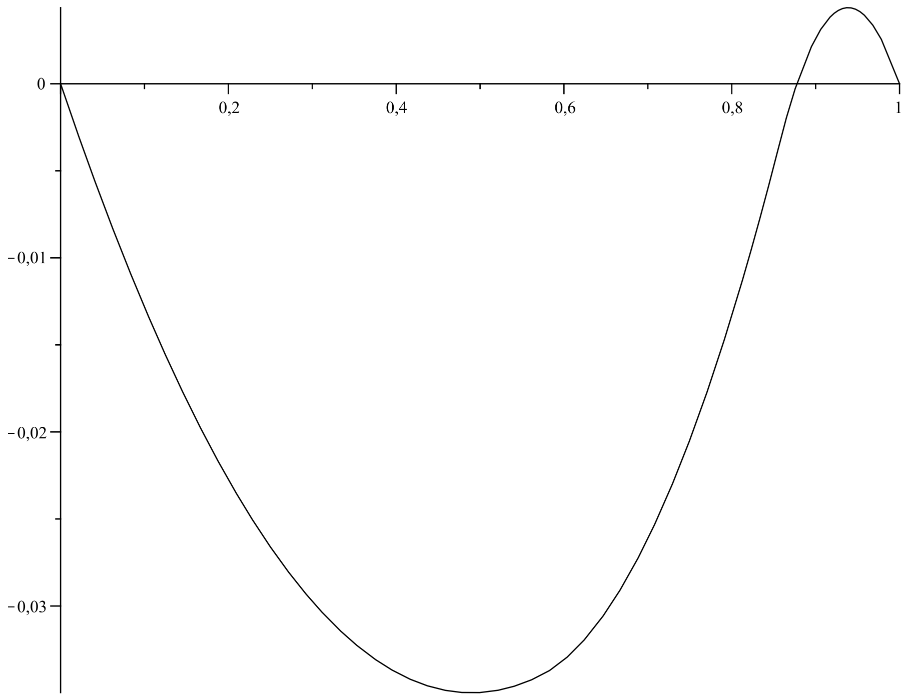

Figure 3.2 The graph of $S_{3,2}(x^2;x,3/4)-S_{3,2}(x^2;x,1/6)$

3.3 Error Analysis for $f(x)=x^2$

Due to the Bohman-Korovskin's theorem, analysing the error between $f(x)=x^2$ and $S_{m,n}(x^2;x,q)$ is vital. The approximating spline function for x^2 is

$$S_{m,n}(x^2;x,q)=\sum_{j=0}^{l}(\xi_j)^2N_j^m(x;q).$$

We define the error function by

$$E_{m,n}(x;q)=\sum_{j=0}^{l}(\xi_j)^2N_j^m(x;q)-x^2. \tag{3.3.1}$$

Since

$$x^2=\sum_{j=0}^{l}\xi_j^{(2)}N_j^m(x;q),$$

then

$$\xi_j^{(2)}=\frac{1}{\binom{m-1}{2}}\sum_{j+1\le r<s\le j+m-1}u_ru_s.$$

Thus we have

$$E_{m,n}(x;q) = \sum_{j=0}^{l} \lambda_j N_j^m(x;q). \tag{3.3.2}$$

Here we set

$$\lambda_j = (\xi_j)^2 - \xi_j^{(2)}$$

After some computations, we find that

$$\lambda_j = \frac{1}{(m-1)^2(m-2)} \sum_{j+1 \le r < s \le j+m-1} (u_s - u_r)^2. \tag{3.3.3}$$

We claim that if $m \geq 3$, we have

i. $E_{m,n}(0;q) = E_{m,n}(1;q) = 0$, $E_{m,n}(x;q) > 0$, if $0 < x < 1$

ii. $E_{m,n}(x;q) = E_{m,n}((1-x);1/q)$ for $0 \leq x \leq 1$

Proof. (i.) Since the Bernstein-Schoenberg operator interpolates the end points, we have

$$E_{m,n}(0;q) = E_{m,n}(1;q) = 0.$$

By equation (3.3.3)

$$E_{m,n}(x;q) = \sum_{j=0}^{l} \lambda_j N_j^m(x;q)$$

$\lambda_j > 0$ for $j = 1, \ldots, l-1$. It can be seen that

$$E_{m,n}(x;q) > 0, \quad \text{if} \quad 0 < x < 1.$$

(ii.) Let u be the knot sequence that we use for $N_j^m(x;q)$ and t be the knot sequence for $N_j^m(x;1/q)$ where

$$u_0 = \cdots = u_{m-1} = 0$$

$$u_m = \frac{1}{[n]_q}, \cdots, u_l = \frac{[n-1]_q}{[n]_q}$$

$$u_{l+1} = \cdots = u_{l+m} = 1$$

and

$$t_0 = \cdots = t_{m-1} = 0$$

$$t_m = \frac{1}{[n]_{1/q}} = 1 - u_l, \cdots, t_l = \frac{[n-1]_{1/q}}{[n]_{1/q}} = 1 - u_m$$

$$t_{l+1} = \cdots = t_{l+m} = 1$$

One may see that $u_i = 1 - t_{l+m-i}$ for $i = 0, \ldots, l$. We have

$$E_{m,n}(x;q) = \sum_{j=0}^{l} \lambda_j N_j^m(x,q) \quad \text{and} \quad E_{m,n}(x;1/q) = \sum_{j=0}^{l} \beta_j N_j^m(x,1/q)$$

where

$$\beta_j = \frac{1}{(m-1)^2(m-2)} \sum_{j+1 \le r < s \le j+m-1} (t_s - t_r)^2. \tag{3.3.4}$$

Our aim is to show that

$$\sum_{j=0}^{l} \lambda_j N_j^m(x;q) = \sum_{j=0}^{l} \beta_{l-j} N_{l-j}^m(1-x;1/q).$$

If we can show that $\lambda_j = \beta_{l-j}$ and $N_j^m(x;q) = N_{l-j}^m(1-x;1/q)$ for $j = 0, \ldots, l$, then the proof will be completed. Using divided differences of truncated powers we have

$$
\begin{aligned}
N_j^m(x;q) &= (u_{j+m}-u_j)[u_j,\ldots,u_{j+m}](u-x)_+^{m-1} \\
&= ((1-t_j)-(1-t_{l+m-j}))[1-t_{l+m-j},\ldots,1-t_{l-j}](u-x)_+^{m-1} \\
&= (t_{l+m-j}-t_{l-j})[t_{l-j},\ldots,t_{l+m-j}](t-(1-x))_+^{m-1} \\
&= N_{l-j}^m(1-x;1/q).
\end{aligned}
$$

To show $\lambda_j = \beta_{l-j}$, an induction argument is imposed.

For $j = 0$;

$$\lambda_0 = A \sum_{1\le r<s\le m-1} (u_s-u_r)^2 = 0,$$

$$\beta_l = A \sum_{l+1\le r<s\le l+m-1} (t_s-t_r)^2 = 0.$$

So, it is true for $j = 0$. Suppose it is true for any arbitrary integer $j = k$ for $0 < k < l$. Our aim is to get $\lambda_{k+1} = \beta_{l-k-1}$. By inductive hypothesis we have

$$A \sum_{k+1\le r<s\le k+m-1} (u_s-u_r)^2 = A \sum_{l-k+1\le r<s\le l-k+m-1} (t_s-t_r)^2. \tag{3.3.5}$$

Since,

$$\lambda_{k+1} = A \sum_{k+2\le r<s\le k+m} (u_s-u_r)^2 = \lambda_k - A\sum_{s=k+2}^{k+m-1} (u_s-u_{k+1})^2 + A\sum_{r=k+2}^{k+m-1} (u_{k+m}-u_r)^2 \tag{3.3.6}$$

we write

$$\sum_{r=k+2}^{k+m-1} (u_{k+m}-u_r)^2 = \sum_{r=k+2}^{k+m-1} ((1-u_r)-(1-u_{k+m}))^2 = \sum_{r=k+2}^{k+m-1} (t_{l+m-r}-t_{l-k})^2 \tag{3.3.7}$$

and

$$\begin{aligned}\sum_{s=k+2}^{k+m-1}(u_s-u_{k+1})^2 &= \sum_{s=k+2}^{k+m-1}((1-u_{k+1})-(1-u_s))^2\\ &= \sum_{s=k+2}^{k+m-1}(t_{l+m-s}-t_{l+m-k-1})^2. \end{aligned} \tag{3.3.8}$$

Substituting (3.3.5), (3.3.7) and (3.3.8) in (3.3.6) gives

$$\begin{aligned}\lambda_{k+1} &= A\sum_{l-k+1\le r<s\le l-k+m-1}(t_s-t_r)^2 - A\sum_{s=k+2}^{k+m-1}(t_{l+m-s}-t_{l+m-k-1})^2\\ &\qquad +A\sum_{r=k+2}^{k+m-1}(t_{l+m-r}-t_{l-k})^2\\ &= A\left(\sum_{l-k+1\le r<s\le l-k+m-1}(t_s-t_r)^2-\sum_{r=l-k+1}^{l-k+m-2}(t_{l+m-k-1}-t_r)^2\right.\\ &\qquad \left.+\sum_{s=l-k+1}^{l-k+m-2}(t_s-t_{l-k})^2\right)\\ &= A\sum_{l-k\le r<s\le l-k+m-2}(t_s-t_r)^2\\ &= \beta_{l-k-1}.\end{aligned}$$

Hence we have $\lambda_j=\beta_{l-j}$ for $j=0,\ldots,l$. This completes the proof. $\square$

CHAPTER FOUR

CONCLUSION

The properties of Bernstein-Schoenberg operator on general knot sequences and on the q-integers are studied. Some special results for Bernstein-Schoenberg operator based on q-integers are obtained. We use this operator to give an alternative proof for a theorem on the convexity of Bernstein Operator. Analytical and geometric properties of Bernstein operator and Bernstein-Schoenberg operator are compared. We show in the second chapter that some properties of them coincide and in the last chapter we give a remark that they also have different properties. We give the transformation matrix between spline basis and Bernstein basis. This gives us a chance to obtain Bèzier curves by using B-splines instead of Bernstein polynomials.

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