DOKUZ EYLÜL UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

KAPLANSKY'S THEOREM FOR VECTOR BUNDLES

by
Sinem ODABAŞI

July, 2010
İZMİR

KAPLANSKY'S THEOREM FOR VECTOR BUNDLES

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M.Sc THESIS EXAMINATION RESULT FORM

We have read the thesis entitled **"KAPLANSKY'S THEOREM FOR VECTOR BUNDLES"** completed by **SİNEM ODABAŞI** under supervision of **ASSIST. PROF. DR. ENGİN MERMUT** with the contribution of **ASSOC. PROF. DR. SERGIO ESTRADA** as the co-supervisor. We certify that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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KAPLANSKY'S THEOREM FOR VECTOR BUNDLES

ABSTRACT

In this master thesis, we focus on two classes of modules: The projective R-modules and the almost projective R-modules for a commutative ring R with unity. Then we center on the category of quasi-coherent sheaves over some special projective schemes and the several 'new' notions of (infinite dimensional) vector bundles attained to these classes as proposed by Drinfeld. We prove structural results relative to the different generalization of vector bundles in terms of certain filtrations of locally countably generated quasi-coherent sheaves. In the case in which the vector bundles are built from the class of projective R-modules, our structural theorem yields a version of Kaplansky's Theorem for infinite dimensional vector bundles on these special projective schemes.

Keywords: Projective module, countably generated projective module, almost projective module, Kaplansky's theorem for projective modules, quasi-coherent sheaf, projective scheme, filtration, infinite dimensional vector bundle.

VEKTÖR DEMETLERİ İÇİN KAPLANSKY’NİN TEOREMİ

ÖZ

Bu master tezinde birim elemanlı değişmeli bir R halkası için projektif ve hemen hemen projektif R-modülleri olmak üzere iki modül sınıfı üzerinde çalışıldı. Daha sonra özel bazı projektif şemalar üzerindeki yarı tutarlı desteler kategorisine ve Drinfeld’in önerdiği gibi bu sınıflara eşleştirilen sonsuz boyutlu vektör demetlerinin birkaç ‘yeni’ kavramı üzerine yoğunlaşıldı. Son olarak Kaplansky’nin teoremini bu yeni tanımlı vektör demetlerine adapte edildi. Yani, sonsuz boyutlu bir vektör demetinin yerel sayılabilir çoklukta üretilmiş vektör destelerini filtre edilerek elde edilebileceği gösterildi.

Anahtar sözcükler: Projektif modül, sayılabilir çoklukta üreteçli projektif modül, hemen hemen projektif modül, projektif modüller için Kaplansky’nin teoremi, yarı tutarlı deste, projektif şema, filtrasyon, sonsuz boyutlu vektör demeti.

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CHAPTER ONE

INTRODUCTION

Let $X = \operatorname{Spec} R$ be an affine scheme for a commutative ring R with unity. It is known that the category of all quasi-coherent sheaves on X is equivalent to R-Mod (see Hartshorne (1977, Corollary 5.5)). In this equivalence, finite dimensional vector bundles on X correspond to finitely generated projective modules (Serre, 1958). Then a (classical) vector bundle on an arbitrary scheme X corresponds to a quasi–coherent sheaf $\mathcal{F}$ such that, for each affine open subset $U = \operatorname{Spec} R$, the corresponding R-module of sections $\Gamma(U, \mathcal{F})$ is finitely generated and free.

Drinfeld (2006) asks the following key problem: 'Is there a reasonable notion of not necessarily finite dimensional vector bundles on a scheme?'. In the same paper he proposes several possible answers to this question. Each one of these involves different classes of modules. In this thesis, we focus on two classes: the projective R-modules and the almost projective R-modules for a commutative ring R. Then we center on the category of quasi-coherent sheaves over the projective scheme $\mathbb{P}^n_R = (\operatorname{Proj} S, O_{\operatorname{Proj} S})$ where $S = R[x_0, \ldots, x_n]$ for a commutative ring R and the several 'new' notions of (infinite dimensional) vector bundles attained to these classes. We prove structural results relative to the different generalization of vector bundles in terms of filtrations of certain locally countably generated quasi-coherent sheaves.

For the case $n = 1$ and when infinite dimensional vector bundles are locally projective quasi-coherent sheaves on $\mathbb{P}^1_R$, our Theorem 5.2.3 may be seen as the analogous of Grothendieck's theorem on the decomposition of finite dimensional vector bundles on $\mathbb{P}^1_k$, where k is a field, as a direct sum of line bundles (Grothendieck, 1957). Moreover, when X is affine, our theorem coincides with Kaplansky's theorem on the decomposition of a projective module as a direct sum of countably generated projective modules. Therefore, our result can be thought as a 'generalized' version of

Kaplansky's theorem for the category $\mathfrak{Qco}(\mathbb{P}^n_R)$ of quasi-coherent sheaves on $\mathbb{P}^n_R$.

Let us make a brief summary of the contents of this thesis. In the first chapter, we introduce all basic notions and terminology concerning to sheaves, schemes and quasi–coherent sheaves, as well as, projective schemes and twisted sheaves. We have used Hartshorne (1977), Mumford (1999), Eisenbud & Harris (2000), Liu (2002) as main sources for this chapter.

Once we have given all basic definitions, we introduce in Chapter 2 the category of quasi-coherent $\mathbf{R}$-modules associated to a quiver Q. Namely we see in Section 2.1 that given any arbitrary quiver Q, we can associate a representation $\mathbf{R}$ of Q by (commutative) rings and the category of quasi-coherent modules over $\mathbf{R}$ (see Definition 2.1.1). We analyze some of the main properties of this category and conclude that it is a Grothendieck category, whenever the representation of rings $\mathbf{R}$ satisfies the following property: given an edge $v \to w$ in Q, the ring $\mathbf{R}(w)$ is flat as a $\mathbf{R}(v)$-module. This is needed to ensure that the kernel of a morphism between two quasi-coherent $\mathbf{R}$-modules is quasi-coherent (see Lemma 2.1.3). Then, in Section 2.2, we prove that the category $\mathfrak{Qco}(X)$ of quasi-coherent sheaves on a scheme X is isomorphic to the category of quasi-coherent $\mathbf{R}$-modules over a certain quiver. We illustrate this equivalence by constructing the isomorphic category of quasi-coherent $\mathbf{R}$-modules corresponding to the category $\mathfrak{Qco}({\mathbb{P}^n}_R)$ of quasi-coherent sheaves over the projective scheme $\mathbb{P}^n_R$. This construction is crucial for our purpose in Chapter 5. For some basic introduction to the category theory, see the book Adámek, Herrlich & Stecker (1990). This chapter is mainly based on Enochs & Estrada (2005). But we provide detailed proofs that do not appear in Enochs & Estrada (2005) as well as the explicit constructions of Subsection 2.2.1.

We devote Chapter 3 to focus on the category of quasi-coherent sheaves on the projective line $\mathbb{P}^1_R$. In this case we prove that the family of twisted sheaves $\{O(n)$:

$n \in \mathbb{Z}\}$ is a family of generators for $\mathfrak{Qco}(\mathbb{P}^1_R)$ (see Proposition 3.1.4). This is known for the category $\mathfrak{Co}(\mathbb{P}^n_A)$ of coherent sheaves on $\mathbb{P}^n(A)$, where A is a commutative Noetherian ring (see Hartshorne (1977, Corollary 5.18)). Our proof has the advantage that it works for any arbitrary commutative ring and for all quasi-coherent sheaves, so not just for the coherent ones. But it has the disadvantage it only works for $n = 1$. We also prove in Corollary 3.1.5 that $\mathfrak{Qco}(\mathbb{P}^1_R)$ admits no other projective than zero, so many of classical results in module theory involving the existence of a projective generator, can not be extended to this setup. We finish this section stating in Theorem 3.2.2 a well-known theorem concerning to the decomposition of (classical) vector bundles over the projective line. This was originally proved by Grothendieck (1957) in case $k = \mathbb{C}$, but the version we present here works for any arbitrary field. We point out that Grothendieck's theorem constitutes a particular case of the kind of filtrations we study in the last chapter. The contents of this chapter are included in the papers Enochs, Estrada, García Rozas & Oyonarte, (2003, 2004a, 2004b), Enochs, Estrada & Torrecillas, (2006).

In Chapter 4, we present, at the level of modules, our main tools to find structural theorems on infinite dimensional vector bundles on $\mathbb{P}^n_R$. Namely, we give the notion of filtration with respect to a class $\mathcal{C}$ of modules (also called direct transfinite extensions with respect to $\mathcal{C}$) and analyze some closure properties of such filtrations that will be needed in the sequel, like Eklof's lemma (Lemma 4.1.6). In Section 4.2 we state the most important technical tool of this thesis: the Hill Lemma (Lemma 4.2.3). Starting from a given filtration $\mathcal{M} = (M_\alpha \mid \alpha \leq \sigma)$ of a module M, Hill Lemma allows to expand this single filtration into a large family satisfying additional properties, namely those stated in Lemma 4.2.3. For some terms in the set theory, like regular cardinals, see the book Jech (2003). The material of this chapter is contained in Göbel & Trlifaj (2006).

Chapter 5 represents our original contribution to the subject of study in this thesis. We use all the previous tools to find structural theorems for two of the generalizations of vector bundles purposed by Drinfeld (2006). The first of such generalizations involves the class of *almost projective* R-modules (see Definition 5.1.1) and the second is the class of projective R-modules. The main idea of the proof is to use both Proposition 5.1.5 and Hill Lemma to make compatible all the individual filtrations at the level of modules to the level of quasi-coherent sheaves. In the second case we obtain a version of Kaplansky's theorem (see Theorem 5.2.3) for infinite dimensional vector bundles.

Finally we would like to point up that there are some open questions in the line of research initiated in this thesis: In the same paper, Drinfeld (2006) also purposes the class of so called *flat Mittag-Leffler* modules to generalize infinite dimensional vector bundles. We would like to know if our techniques can apply to this setting to obtain new structural theorems. Also we would like to extend the class of schemes in which our structural theorems holds.

In this thesis, *all rings are assumed to be commutative rings with unity. Unless otherwise stated, R always denotes a commutative ring with unity.*

1.1 Basic Notions on Sheaves

Let X be a topological space. Then we can construct a category $\mathfrak{Top}(X)$ from the topological space X by taking objects as open subsets U of X and morphisms as the canonical inclusions when $U \subseteq V$.

A *presheaf* of rings $\mathcal{F}$ is a contravariant functor from $\mathfrak{Top}(X)$ to the category of commutative rings. That is, a presheaf $\mathcal{F}$ consists of two data:

- for every open subset U of X, a commutative ring $\mathcal{F}(U)$ and

- for every inclusion $V \subseteq U$ of open subsets of X, a ring homomorphism

$$\rho_{UV} : \mathcal{F}(U) \longrightarrow \mathcal{F}(V)$$

which is called the restriction map,

satisfying the following properties:

(i) $\mathcal{F}(\emptyset) = 0$;

(ii) $\rho_{UU} = \mathrm{id}_U$;

(iii) If we have $W \subseteq V \subseteq U$, then $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$.

We can define in the same way presheaves of abelian groups, presheaves of algebras over a fixed ring, etc. by changing the terminal category of the presheaf $\mathcal{F}$. But in our study, we focus on the presheaves of rings.

For an element s of $\mathcal{F}(U)$, we shall sometimes denote $\rho_{UV}(s)$ shortly by $s|_V$.

Definition 1.1.1. A presheaf of rings $\mathcal{F}$ on a topological space X is said to be a *sheaf* if for each open subset U of X and for each open covering $\{U_i\}_{i\in I}$ of U, the sequence

$$0 \longrightarrow \mathcal{F}(U) \xrightarrow{f} \prod_{i\in I} \mathcal{F}(U_i) \xrightarrow{p-q} \prod_{i,j\in I} \mathcal{F}(U_i \cap U_j) \tag{1.1.1}$$

is exact, where $f : s \longmapsto \{\rho_{UU_i}(s)\}_{i\in I}$, $p : \{s_i\}_{i\in I} \longmapsto \{\rho_{U_i\cap U_j}(s_i)\}_{i,j\in I}$ and $q : \{s_i\}_{i\in I} \longmapsto \{\rho_{U_i\cap U_j}(s_j)\}_{i,j\in I}$, for all $s \in \mathcal{F}(U)$ and $\{s_i\}_{i\in I} \in \prod_{i\in I} \mathcal{F}(U_i)$.

For any open covering $\{U_i\}_{i\in I}$ of an open subset U of X, the exactness in the sequence (1.1.1) is equivalent to the following two conditions:

(i) If $s \in \mathcal{F}(U)$ such that $s|_{U_i} = 0$ for every i, then $s = 0$.

(ii) If we have elements $s_i \in \mathcal{F}(U_i)$ for each i with the property that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for every $i, j \in I$, then there is an element $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$ for all i.

Condition (ii) implies the existence of such an element and (i) implies the uniqueness of it.

An element of $\Gamma(U, \mathcal{F}) := \mathcal{F}(U)$ is called the *section of the preasheaf $\mathcal{F}$ over the open set U*. In particular, an element of $\Gamma(X, \mathcal{F}) := \mathcal{F}(X)$ is called a *global section*.

Definition 1.1.2. Let $\mathcal{F}$ and $\mathcal{G}$ be preasheaves on a topological space X. A morphism η from $\mathcal{F}$ to $\mathcal{G}$ is a natural transformation between the functors $\mathcal{F}$ and $\mathcal{G}$, that is, to each open subset U of X, there is a ring homomorphism $\eta_U : \mathcal{F}(U) \to \mathcal{G}(U)$ such that the diagram

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[r, "\eta_U"] \arrow[d, "\rho_{UV}"'] & \mathcal{G}(U) \arrow[d, "\rho'_{UV}"] \\
\mathcal{F}(V) \arrow[r, "\eta_V"] & \mathcal{G}(V)
\end{tikzcd}$$

is commutative for all open subsets $V \subseteq U$ in X, where ρ_{UV} and ρ'_{UV} are the restriction maps of $\mathcal{F}$ and $\mathcal{G}$, respectively.

Definition 1.1.3. Let $\mathcal{F}$ be a preasheaf of rings on a topological space X, and $x \in X$. The *stalk* of $\mathcal{F}$ at x is the ring

$$\mathcal{F}_x := \varinjlim_{x \in U} \mathcal{F}(U),$$

where the direct limit is taken over the open neighborhoods U of x.

Basically, the stalk of $\mathcal{F}$ at x consists of the equivalence classes of the disjoint union of $\mathcal{F}(U)$, where U runs through all open neighborhoods of x,

$$\mathcal{F}_x = \big(\bigsqcup_{x \in U} \mathcal{F}(U)\big) / \sim$$

such that $a \sim b$ for $a \in \mathcal{F}(U)$ and $b \in \mathcal{F}(V)$ if and only if there exists an open neighborhood $W \subseteq U \cap V$ such that $a|_W = b|_W$. So, an element of $\mathcal{F}_x$ is represented by $s_x := < U, s >$ for some open neighborhood U of x such that $s \in \mathcal{F}(U)$. Actually, s_x is the image of the section $s \in \mathcal{F}(U)$ in the stalk $\mathcal{F}_x$.

If we have a continuous map $f : X \to Y$ between topological spaces X, Y and a sheaf $\mathcal{F}$ on X, then it is natural to define the functor $f_*\mathcal{F}$ on Y as follows: for an open subset U of Y, $(f_*\mathcal{F})(U) := (\mathcal{F}(f^{-1}(U)))$ and $\rho_{f^{-1}(U)f^{-1}(V)}$ is its restriction map when $V \subseteq U$. Clearly, $f_*\mathcal{F}$ is a sheaf on Y.

Definition 1.1.4. A *ringed (topological) space (locally ringed in local rings)* involves a topological space X endowed with a sheaf of rings O_X on X such that the stalk $O_{X,x}$ is a local ring for every $x \in X$. We denote it by (X, O_X). The sheaf O_X is called *the structure sheaf* of (X, O_X).

Definition 1.1.5. A *morphism of ringed spaces* (X, O_X) and (Y, O_Y) consists of a continuous map $f : X \to Y$ and a morphism $f^\# : O_Y \to f_* O_X$ such that for every $x \in X$, the homomorphism $f^\#_* : O_{Y,x} \to O_{X,x}$ induced by $f^\#$ on the stalks $O_{Y,x}$ and $O_{X,x}$ is a local homomorphism.

1.2 Basic Notions on Schemes

In this section, we define the notion of schemes. In order to define it, we start with the basic notion of an affine scheme.

Let R be a ring. The *prime spectrum* of the ring R is defined to be

$$\operatorname{Spec} R := \{P \subset R \mid P \text{ is a prime ideal of } R\}.$$

Since a prime ideal is a proper ideal, R is not in $\operatorname{Spec} R$.

For each ideal I of R, define the set

$$V(I) := \{P \in \operatorname{Spec} R \mid I \subseteq P\}.$$

We have the following properties.

Proposition 1.2.1. *(by Hartshorne (1977, Lemma 2.2.1)) For any ring R, we have:*

(i) $V(I) \cup V(J) = V(I \cap J)$ *for every pair of ideals I, J of R.*

(ii) $\bigcap_\lambda V(I_\lambda) = V(\sum_\lambda I_\lambda)$ *for every family* $\{I_\lambda\}_\lambda$ *of ideals of R.*

(iii) $V(R) = \emptyset$ *and* $V(\emptyset) = \operatorname{Spec} R$.

In view of Proposition 1.2.1, the set $X := \operatorname{Spec} R$ can be considered as a topological space by defining closed subsets of X to be all sets of the form $V(I)$ where I runs through all ideals of R. So, the open subsets are of the form $D(I) := \operatorname{Spec} R \setminus V(I)$ for some ideal I of R. This topology is called as the *Zariski topology*. The basic open subsets of X are defined to be open subsets of the form

$$X_f := D(f) = D(fR) = \operatorname{Spec} R \setminus V(fR),$$

where f is an element of R. It is easy to see that the family of the basic open subsets $\{X_f\}_{f \in R}$ forms a base for the topological space $X = \operatorname{Spec} R$. Actually, the following proposition shows us that there is a relation between the basic open subsets of $\operatorname{Spec} R$ and R.

Proposition 1.2.2. *(Mumford, 1999, Proposition 2.1.2) For a family* $\{f_a\}_{a \in A}$ *of elements in a ring R, the equality*

$$\operatorname{Spec} R = \bigcup_{a \in A} X_{f_a}$$

holds if and only if the ideal I generated by $\{f_a\}_{a \in A}$ *is equal to R.*

A topological space is called *quasi-compact* if each of its open covers has a finite subcover. The distinction between compact spaces and quasi-compact spaces exists because some people assume that a space that is compact must be a Hausdorff space. From this point of view, we can say that $X = \operatorname{Spec} R$ is a quasicompact topological space. It is not compact because $\operatorname{Spec} R$ is not Hausdorff in general.

Now we will define the structure sheaf O_X of rings on the prime spectrum $X = \operatorname{Spec} R$ with the Zariski topology.

Let $f \in R$. Then, we have a homeomorphism between

$$X_f \longleftrightarrow \operatorname{Spec}(R_f)$$

by $P \mapsto PR_f$, where R_f is localization of R at f and $P \in X_f$.

Lemma 1.2.3. *(Mumford, 1999) Let $X = \operatorname{Spec} R$. Then, for f and g in R we have the following:*

(i) $X_f \cap X_g = X_{fg}$.

(ii) $X_g \subseteq X_f$ *if and only if* $g \in \sqrt{\langle f \rangle}$.

Proof. (i) Let P be a prime ideal of R. Then by property of the prime ideal, f and g are not in P if and only if $f.g$ is not in P. So this proves the equality.

(ii) We know that $\sqrt{\langle f \rangle} = \bigcap_{f \in P} P$, where the intersection is over all prime ideals P of R that contains f. So, g is not in $\sqrt{\langle f \rangle}$ if and only if there is a prime ideal P containing f such that $g \notin P$. That is, there is a prime ideal P containing f such that $g \notin P$ if and only if $X_g \nsubseteq X_f$.

□

For a basic open subset X_f of $X = \operatorname{Spec} R$, define

$$\mathcal{O}_X(X_f) := R_f.$$

By Lemma 1.2.3, in case $X_g \subseteq X_f$, we have $g^m = af$ for some $a \in R$ and $m \in \mathbb{N}$. So, define the restriction map

$$\rho_{X_f X_g} : R_f \longrightarrow R_g$$

by $\rho_{X_f X_g}(\frac{r}{f^n}) := \frac{a^m r}{g^{nm}}$ for $r \in R$ and $n \in \mathbb{N}$.

But, in order to be a sheaf, one also need to define $\mathcal{O}_X(U)$ for open subsets U of X different from the basic open subsets, that is, the previous construction for basic open subsets must be extended to all open subsets. In fact, it is proved the existence of such an extension. In order to do this, we need to give some concepts.

Definition 1.2.4. Let X be a topological space and $\mathcal{B}$ be a base of the topological space X. A presheaf $\mathcal{F}_0$ of rings on $\mathcal{B}$ that is considered as the full subcategory of $\mathfrak{Top}(X)$ is said to be *$\mathcal{B}$-sheaf on X* if for any open subset U of X in $\mathcal{B}$ and any open cover $\{U_i\}_{i \in I}$ of U with $U_i \in \mathcal{B}$ for each $i \in I$, it satisfies the following axioms :

(i) If $s \in \mathcal{F}_0(U)$ such that $\rho_{UU_i}(s) = 0$ for each $i \in I$, then $s = 0$.

(ii) If we have sections $s_i \in \mathcal{F}_0$ for each $i \in I$ such that $\rho_{U_iW}(s_i) = \rho_{U_jW}(s_j)$ for all $i, j \in I$ and all open subsets $W \subseteq U_i \cap U_j$ where $W \in \mathcal{B}$, then there is an element s of $\mathcal{F}_0(U)$ such that $\rho_{UU_i}(s) = s_i$ for each $i \in I$.

From this definition, we obtain the following proposition which will be very useful in the sequel.

Proposition 1.2.5. *(Eisenbud & Harris, 2000, Proposition I.12) Let $\mathcal{B}$ be a base of open subsets on X.*

(i) Every $\mathcal{B}$-sheaf on X extends uniquely to a sheaf on X.

(ii) Given sheaves $\mathcal{F}$ and $\mathcal{G}$ on X and a collection of maps

$$\widetilde{\varphi}(U) : \mathcal{F}(U) \to \mathcal{G}(U) \text{ for each } U \in \mathcal{B}$$

commuting with restrictions, there is a unique morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of sheaves such that $\varphi(U) = \widetilde{\varphi}(U)$ for all $U \in \mathcal{B}$.

The following lemmas help us to use Proposition 1.2.5.

Lemma 1.2.6. *(Mumford, 1999, Lemma 2.1.1) Let R be a ring and let $X = \operatorname{Spec} R$. If $X_f = \bigcup_{\alpha \in A} X_{f_\alpha}$ for some $f \in R$ and a collection $\{f_\alpha\}_{\alpha \in A}$ of elements in R, and if for some $a \in R_f$,*

$$\rho_{X_f X_{f_\alpha}}(a) = 0 \quad \text{for all} \quad \alpha \in A,$$

then $a = 0$.

Lemma 1.2.7. *(Mumford, 1999, Lemma 2.1.2) Let R be a ring and let $X = \operatorname{Spec} R$. Suppose that*

$$X_f = \bigcup_{\alpha \in A} X_{f_\alpha}$$

for some $f \in R$ and a collection $\{f_\alpha\}_{\alpha \in A}$ of elements in R. If we have elements $g_\alpha \in R_{f_\alpha}$ for each $\alpha \in A$ such that for every $\alpha, \beta \in A$,

$$\rho_{X_{f_\alpha} X_{f_\alpha f_\beta}}(g_\alpha) = \rho_{X_{f_\beta} X_{f_\alpha f_\beta}}(g_\beta),$$

then there exists $g \in R_f$ satisfying

$$g_\alpha = \rho_{X_f X_{f_\alpha}}(g) \quad \text{for all } \alpha \in A.$$

We know that the family $\mathcal{B}$ of basic open subsets of $X = \operatorname{Spec} R$ is a base for $X = \operatorname{Spec} R$. And we have just defined above $\mathcal{O}_X$ on all basic open subsets. Thanks to the

Lemmas 1.2.6, 1.2.7, clearly we can say that $\mathcal{O}_X$ is a $\mathcal{B}$-sheaf. Hence, by Proposition 1.2.5 we can extend this $\mathcal{B}$-sheaf to a sheaf of rings on $\operatorname{Spec} R$, denoted by $\mathcal{O}_{\operatorname{Spec} R}$. This sheaf $\mathcal{O}_{\operatorname{Spec} R}$ is called the *structure sheaf* of $X = \operatorname{Spec} R$ and any sheaf isomorphic as a locally ringed space to the structure sheaf $\mathcal{O}_{\operatorname{Spec} R}$ of $\operatorname{Spec} R$ for some ring R is called an *affine scheme*. When we talk about an affine scheme, we always write $(\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$ for some ring R. For any open subset U of a ringed space $(X, \mathcal{O}_X)$, it is easy to see that the structure sheaf on U can be constructed by restricting $\mathcal{O}_X$ to U; denote it $\mathcal{O}_X|_U$. So, $(U, \mathcal{O}_X|_U)$ is a ringed space. An open subset U of any ringed space $(X, \mathcal{O}_X)$ whose restriction $(U, \mathcal{O}_X|_U)$ to U is affine is called an *affine open subset*. Recall that each basic open subset $X_f = D(f)$, $f \in R$, of an affine scheme $\operatorname{Spec} R$ can be written as

$$X_f = \operatorname{Spec} R_f$$

and moreover each one can define a sheaf $\mathcal{O}_{X_f}$ of rings over X_f, by restricting the structure sheaf $\mathcal{O}_{\operatorname{Spec} R}$ of $\operatorname{Spec} R$ to the basic open subset X_f. Therefore, the basic open subset X_f is an affine open subset.

Finally, we have all datas in order to define a scheme.

Definition 1.2.8. A *scheme* X is a topological space together with a sheaf $\mathcal{O}_X$ of rings on X such that $(X, \mathcal{O}_X)$ is locally affine, that is, X is covered by a collection $\{U_i\}_{i \in I}$ of affine open subsets of X.

A scheme is obtained by pasting the affine schemes together and the affine schemes are the generalization of the affine spaces.

Example 1.2.9. Let k be an algebraically closed field. We know that $k[x]$ is principial ideal domain. So, the prime ideals in $k[x]$ are either 0 or $\langle x - \alpha \rangle$, where $\alpha \in k$. Then,

$$\operatorname{Spec} k[x] = \{0\} \cup \{\langle x - \alpha \rangle \mid \alpha \in k\}.$$

Take an ideal I of $k[x]$. Then $I = \langle f(x) \rangle$ for some $f(x) \in k[x]$. Since k is algebraically closed field,

$$f(x) = a_0 \prod_{j=1}^{l} (x - \alpha_j)^{m_j},$$

where $\alpha_j \in k$ and $m_j \in \mathbb{N}$ for each $j = 1, \ldots l$. Hence,

$$V(I) = \{\langle x - \alpha_1 \rangle, \ldots, \langle x - \alpha_l \rangle\}$$

It follows that

$$D(I) = \{0\} \cup \{\langle x - \alpha \rangle |\ \alpha \in k \text{ and } \alpha \neq \alpha_j,\ j = 1, \ldots, l\}$$

Identify the ideal $\langle x - \alpha \rangle$ with $\alpha \in k$. But there is no point in k corresponding to the ideal 0, which is said *generic point*. So, we have

$$\operatorname{Spec} k[x] = \mathbb{A}^1 \cup \{0\},$$

where $\mathbb{A}^1$ is the affine line.

1.2.1 Projective Schemes

In this subsection, we construct and discuss a very important example of schemes on which our problem is focused: projective schemes. In fact, in terms of polynomial equations, it concerns homogeneous equations. This type of scheme is the generalization of the projective space.

Let

$$S = \bigoplus_{d \geq 0} S_d$$

be a graded ring. An element of S in S_d is called a *homogeneous element of degree d*.

An ideal I of S is said to be *homogeneous* if it is generated by homogeneous elements. This is equivalent to $I = \bigoplus_{d \geq 0} I \cap S_d$. The ideal of S

$$S_+ := \bigoplus_{d \geq 1} S_d$$

is called the *irrelevant ideal* of S. Then, take the set

$$\operatorname{Proj} S := \{ \text{ all the homogeneous prime ideals of S not containing } S_+ \}.$$

The reason why we ignore this irrelevant ideal is to generalize the projective space. What we will do in the following is to endow this point set $\operatorname{Proj} S$ with structure of a scheme. As in the case of an affine scheme, the set $V_+(I)$ is define as

$$V_+(I) := \{ P \in \operatorname{Proj} S | \, I \subseteq P \}$$

for a homogeneous ideal I of S.

Proposition 1.2.10. *(by Hartshorne (1977, Lemma 2.2.4)) For a graded ring S, we have:*

(i) $V_+(I) \cup V_+(J) = V_+(I \cap J)$ *for every pair of homogeneous ideals I, J of S.*

(ii) $\bigcap_\lambda V_+(I_\lambda) = V_+(\sum_\lambda I_\lambda)$ *for every family* $\{I_\lambda\}_\lambda$ *of homogeneous ideals of S.*

(iii) $V(S) = \emptyset$ *and* $V(\emptyset) = \operatorname{Proj} S$.

Then, by taking closed subsets of $\operatorname{Proj} S$ as subsets of the form $V_+(I)$ where I is a homogeneous ideal of S, $\operatorname{Proj} S$ can be endowed with a topology, called *Zariski topology* on $\operatorname{Proj} S$. As in the case of affine scheme, we may define the basic open subsets of $\operatorname{Proj} S$ for a given homogeneous element f of S as

$$D_+(f) := \operatorname{Proj} S \setminus V(Sf).$$

It remains to define the structure sheaf on $\operatorname{Proj} S$. Clearly, the basic open subsets cover the topological space $\operatorname{Proj} S$. It can be defined a sheaf on $\operatorname{Proj} S$ by giving its datas on these basic open sets. In fact, we can restrict ourselves to the basic open subsets $D_+(f)$ where $f \in S_+$. Since $V_+(S_+) = \bigcap_i V_+(f_i) = \emptyset$ where f_i's are the homogeneous elements that generate R_+,

$$\operatorname{Proj} S = \bigcup_i D_+(f_i).$$

Then it follows that for every homogeneous $g \in S$, we have

$$D_+(g) = \bigcup_i D_+(gf_i)$$

with $gf_i \in S_+$ for every i.

If $f \in S$ is homogeneous, we denote by $S_{(f)}$ the subring of the localization S_f at f such that it contains the elements of degree zero in the localization S_f. That is, the elements of $S_{(f)}$ are of the form sf^{-n}, $n \geq 0$, $\deg s = n \deg f$ and s is homogeneous.

Lemma 1.2.11. *(Liu, 2002, Lemma 3.36) Let $f \in S_+$ be homogeneous element of degree r and $g \in S$.*

(i) There is a canonical homeomorphism $\theta : D_+(f) \to \operatorname{Spec} R_{(f)}$.

(ii) If $D_+(g) \subseteq D_+(f)$, then $\theta(D_+(g)) = D(\alpha)$, where $\alpha = g^r f^{-\deg g} \in S_{(f)}$ and we have a canonical morphism $S_{(f)} \to S_{(g)}$ which induces an isomorphism $(S_{(f)})_\alpha \cong S_{(g)}$.

(iii) If I be a homogeneous ideal of S, then the image of $V_+(I) \cap D_+(f)$ under θ is the closed set $V_+(I_{(f)})$, where $I_{(f)} := IS_f \cap S_{(F)}$.

(iv) If I is an ideal of S generated by homogeneous elements $\{h_1, \ldots, h_n\}$, then for any $f \in S_1$, $I_{(f)}$ is generated by the h_i/f^{degh_i} where $i = 1, \ldots, n$.

By Lemma 1.2.11, it is natural to define $O_{\operatorname{Proj} S}(D_+(f)) := S_{(f)}$ for a homogeneous element f of S. We have the structure sheaf only on the basic open sets. The following proposition helps us to prove the existence of its extension to $\operatorname{Proj} S$.

Proposition 1.2.12. *(Liu, 2002, Lemma 3.38) Let S be graded ring. There is a unique sheaf of rings on* $\operatorname{Proj} S$ *such that for any homogeneous* $f \in S_+$*, the basic open subset*

$$(D_+(f), \mathcal{O}_{\operatorname{Proj} S}|_{D_+(f)})$$

is isomorphic to the affine scheme $(\operatorname{Spec} S_{(f)}, \mathcal{O}_{\operatorname{Spec} S_{(f)}})$ *as locally ringed spaces.*

Proof. Let $X = \operatorname{Proj} S$ and let $\mathcal{B}$ be the base for X consisting of the basic open subsets $D_+(f)$ with $f \in S_+$. For any $D_+(f)$ in $\mathcal{B}$, define

$$\mathcal{O}_X(D_+(f)) := S_{(f)}.$$

By Lemma 1.2.11, we can easily say that $\mathcal{O}_X$ is a $\mathcal{B}$-sheaf. So we can uniquely extend it to a sheaf $\mathcal{O}_X$ on X. And also the ringed space $(D_+(f), \mathcal{O}_X|_{D_+(f)})$ is isomorphic to the affine scheme $(\operatorname{Spec} S_{(f)}, \mathcal{O}_{\operatorname{Spec} S_{(f)}})$. □

So for a graded ring S, we have endowed $\operatorname{Proj} S$ with sheaf $\mathcal{O}_{\operatorname{Proj} S}$. This notion is similar to the affine prime spectrum, but the projective case differs from it by the homogeneity requirement.

Example 1.2.13. Let $S := R[x_0, x_1, \ldots, x_n]$ be the polynomial ring over R with indeterminates $x_0, x_1, \ldots, x_n$ and let S_d be the set of all the homogeneous polynomials of degree d for each $d \in \mathbb{N}$. By definition, for a homogeneous element f of degree d,

$$S_{(f)} = \left\{ \frac{g}{f^m} \mid g \in S,\ g \text{ is homogeneous with } \deg g = m \deg f, m \geq 0 \right\}.$$

It is easy to see that

$$\mathcal{O}_{\operatorname{Proj} S}(D_+(f)) = \mathcal{O}_{\operatorname{Proj} S}(D_+(f^n))$$

for any homogeneous element f of S and $n \in \mathbb{N}$. For the homogeneous element x_i of degree 1, we have

$$\mathcal{O}_{\operatorname{Proj} S}(D_+(x_i)) = S_{(x_i)} \cong R\left[\frac{x_0}{x_i},\ldots,\frac{x_{i-1}}{x_i},\frac{x_{i+1}}{x_i},\ldots,\frac{x_n}{x_i}\right] \tag{1.2.1}$$

and

$$\begin{aligned}\mathcal{O}_{\operatorname{Proj} S}(D_+(x_i x_j)) = S_{(x_i.x_j)} &= (S_{(x_i)})_{(x_j/x_i)} \\ &\cong R\left[\frac{x_l}{x_m}\right]_{l\in\{0,\ldots,n\},m=i,j}\end{aligned} \tag{1.2.2}$$

If f_i are homogeneous elements of S for each $i \in \{0,\ldots,m\}$, then we have

$$\bigcap_{i=0}^{m} D_+(f_i) = D_+(\prod_{i=0}^{m} f_i).$$

Thus, for a homogeneous element $\prod_{l=1}^{m} x_{i_l}^{j_l}$ where $0 \le i_1 \le \ldots \le i_m \le n$ and $j_l \in \mathbb{N}$ for each $l = 0,\ldots,m$, we have

$$\begin{aligned}\mathcal{O}_{\operatorname{Proj} S}(D_+(\prod_{l=1}^{m} x_{i_l}^{j_l})) = S_{(\prod_{l=1}^{m} x_{i_l}^{j_l})} &= S_{(\prod_{l=1}^{m} x_{i_l})} \\ &\cong R\left[\frac{x_s}{x_t}\right]_{s\in\{0,\ldots,n\},t\in\{i_1,\ldots,l_m\}}.\end{aligned} \tag{1.2.3}$$

Also, for a homogeneous element g which is a factor of some homogeneous element f, we can reach to $S_{(f)}$ by inverting the other factors of f except for g in $S_{(g)}$. For example, $S_{(x_{i_1}\ldots x_{i_{l+1}})}$ where $0 \le i_1 < i_2 < \ldots < i_l \le n$ can be obtained from $S_{(x_{i_1}\ldots x_{i_l})}$ by inverting all the elements $x_{i_{l+1}}/x_{i_j}$ where $1 \le j \le l$.

By Proposition 1.2.12, we have the structure sheaf $\mathcal{O}_{\operatorname{Proj} S}$ on $\operatorname{Proj} S$ where $S = R[x_0,\ldots,x_n]$ by taking the datas obtained above on the basic open subsets. We use the notation $\mathbb{P}_R^n$ for the ringed space $(\operatorname{Proj} S, \mathcal{O}_{\operatorname{Proj} S})$ and the notation $\mathbb{A}_R^{n+1}$ for the ringed space $(\operatorname{Spec} S, O_{\operatorname{Spec} S})$ where $S = R[x_0, x_2, \ldots, x_n]$.

For any open subset U of a scheme (X, O_X), the ringed space $(U, O_X|_U)$ is again a scheme and is called the *open subscheme* of X. But unfortunately, we can not say the same thing for a closed subset of X. Here we will not give the notion of a closed subscheme of a scheme. See Hartshorne (1977, Section 2.3).

Definition 1.2.14. A *projective scheme* over a ring R is an R-scheme that is isomorphic to a closed subscheme of $\mathbb{P}_R^n$ for some $n > 0$.

The following example explains the relation between the notion of a projective scheme and the notion of a projective space.

Example 1.2.15. (Liu, 2002, Lemma 3.43) Let k be a field and V be a finite dimensional vector space over k. We have the following equivalence relation $\sim$ on $V/\{0\}$: $u \sim v$ if there exists $\lambda \in k$ different from zero such that $u = \lambda v$. So $\mathbb{P}(V) := V/\sim$ is a projective space in the sense of the classical projective geometry. A point of projective space represents the line passing through zero and in the direction of this point. Let us take $V = k^{n+1}$. For $\alpha = (\alpha_0, ..., \alpha_n) \in k^{n+1} \setminus 0$, the equivalence class $[\alpha]$ is a point of $\mathbf{P}(k^{n+1})$ which is denoted by $[\alpha] = \alpha = (\alpha_0 : ... : \alpha_n)$. These α_i's are called the homogeneous coordinates of $[\alpha]$ and, as usual, we use the notation $[\alpha] = \alpha = [\alpha_0 : \ldots : \alpha_n]$.

Assume $\alpha_0 \neq 0$. Then the ideal I of $k[x_0, \ldots, x_n]$ generated by $\alpha_j x_i - \alpha_i x_j$, $0 \leq i, j \leq n$ is clearly homogeneous and it is a prime ideal since $k[x_0, \ldots, x_n]/I \cong k[x_0]$. Also I doesn't contain the irrelevant ideal. So I is in $\mathbb{P}_k^n$. In fact I is rational. Since $x_i - a_0^{-1} a_i x_i \in I$ for every i, it follows that $I \in D_+(x_0)$. So I corresponds to the ideal of $k[x_i/x_0]_i$ generated by $\{x_i/x_0 - \alpha_i/\alpha_0\}_i$ by Lemma 1.2.11-(ii). Therefore $k(I) = k$.

Define

$$\tau : \mathbb{P}(k^{n+1}) \longrightarrow \mathbb{P}_k^n$$

such that $\tau([\alpha]) = I$ as defined above. Then τ is a bijection from the projective space $\mathbb{P}(k^{n+1})$ onto the set $\mathbb{P}^n_k(k)$ of rational points of $\mathbb{P}^n_k$ (see Liu (2002, Definition 2.19, Definition 3.30, Example 3.29, Lemma 3.43)).

Let $[\beta] \in \mathbb{P}(k^{n+1})$ be a point with homogeneous coordinates $[\beta_0 : \ldots : \beta_n]$ such that $\tau([\beta]) = \tau([\alpha])$. Then $\beta_0 \neq 0$ because otherwise $x_0 \in \tau([\beta])$. By considering the points $\tau([\beta]), \tau([\alpha])$ in $D_+(x_0) \cong \mathbb{A}^n_k$, we obtain $\alpha_0^{-1} = \beta_0^{-1}\beta_i$ for every i. So $\beta_i = (\alpha_0^{-1}\beta_0)\alpha_i$ for every i. It follows that $[\beta] = [\alpha]$. That is, τ is injective.

Let $p \in \mathbb{P}^n_k(n)$. We may assume, for example, that $p \in D_+(x_0)$. Let α_i be the image of $x_i/x_0 \in O(D_+(x_0))$ in $k = k(p)$. Consider the point $[\alpha] \in \mathbb{P}(k^{n+1})$ with homogeneous coordinates $[\alpha_0 : \ldots : \alpha_n]$. Then we have $\tau([\alpha]) = p$. This implies the surjectivity of τ.

1.3 Quasi-coherent Sheaves

Let (X, O_X) be a scheme. An O_X*-module* is a sheaf $\mathcal{F}$ of abelian groups on X, plus, a $\Gamma(U, O_X)$-module structure on $\Gamma(U, \mathcal{F})$ for each open subset U of X such that if we have open subsets $V \subseteq U$ of X, then the diagram

$$\begin{tikzcd} \Gamma(U, O_X) \times \Gamma(U, \mathcal{F}) \arrow[r] \arrow[d] & \Gamma(U, \mathcal{F}) \arrow[d] \\ \Gamma(V, O_X) \times \Gamma(V, \mathcal{F}) \arrow[r] & \Gamma(V, \mathcal{F}) \end{tikzcd}$$

commutes.

Definition 1.3.1. Let (X, O_X) be a ringed space. An O_X-module $\mathcal{F}$ is said to be *quasi-coherent* if for every $x \in X$, there exists an open neighborhood U of x such that the following sequence of O_X-modules is exact for some indexing sets I and J

$$O_X^{(J)}|U \to O_X^{(I)}|U \to \mathcal{F}|U \to 0.$$

$\mathcal{F}$ is said to be *coherent* if the sets I and J are finite.

1.3.1 Quasi-coherent Sheaves on an Affine Scheme

Now we will classify quasi-coherent sheaves on the affine scheme $X = \operatorname{Spec} R$.

Let M be an R-module. It is well-known that the localization M_f of M at f, where $f \in R$, is an R_f-module and the localization M_P of M at a prime ideal P of R is an R_P-module. For a basic open subset $D(f)$ of $X = \operatorname{Spec} R$, assign M_f. Then similar properties hold as in Lemmas 1.2.6 and 1.2.7. As the structure sheaf $\mathcal{O}_{\operatorname{Spec} R}$ was constructed, it can be proved that there is a unique $\mathcal{O}_X$-module $\widetilde{M}$ on $X = \operatorname{Spec} R$ such that $\Gamma(D(f), \widetilde{M}) = M_f$ for all $f \in R$ and $\Gamma(U, \widetilde{M})$ is a $\Gamma(U, \mathcal{O}_{\operatorname{Spec} R})$-module for each open subset U of X. For open subsets $V \subseteq U$, the restriction map f_{UV} of $\widetilde{M}$ is a $\Gamma(U, \mathcal{O}_{\operatorname{Spec} R})$-module homomorphism by considering $\Gamma(V, \mathcal{O}_{\operatorname{Spec} R})$ as a $\Gamma(U, \mathcal{O}_{\operatorname{Spec} R})$-module with respect to the restriction map ρ_{UV} of the affine scheme $\mathcal{O}_{\operatorname{Spec} R}$.

Proposition 1.3.2. *(Liu, 2002, Proposition 5.1.5) For the affine scheme $X = \operatorname{Spec} R$, we have:*

(i) If $\{M_i\}_i$ is a family of R-modules, then $\widetilde{(\bigoplus_i M_i)} \cong \bigoplus_i (\widetilde{M_i})$.

(ii) A sequence of R-modules $L \to M \to N$ is exact if and only if the associated sequence of $\mathcal{O}_X$-modules $\widetilde{L} \to \widetilde{M} \to \widetilde{N}$ is exact.

(iii) For any R-module M, the sheaf $\widetilde{M}$ is quasi-coherent.

(iv) Let M, N be two R-modules. Then we have a canonical isomorphism

$$\widetilde{M \otimes_R N} \cong \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N}.$$

$\widetilde{M}$ is said to be the *sheaf associated to* M *on* $\operatorname{Spec} R$. The following theorem gives us a different view to the quasi-coherent sheaves by means of the sheaf associated to some modules.

Theorem 1.3.3. *(Liu, 2002, Theorem 5.1.7) Let* X *be a scheme and* $\mathcal{F}$ *be an* $\mathcal{O}_X$*-module. Then* $\mathcal{F}$ *is quasi-coherent if and only if for every affine open subset* U *of* X*, we have*

$$\mathcal{F}|U \cong \widetilde{\mathcal{F}(U)}.$$

The next proposition gives us the reduced version of the previous theorem.

Proposition 1.3.4. *(Hartshorne, 1977, Proposition 2.5.4) Let* X *be a scheme and* $\mathcal{F}$ *be an* $\mathcal{O}_X$*-module. Then* $\mathcal{F}$ *is quasi-coherent if and only if* $\mathcal{F}$ *is locally in the form of the sheaf modules associated to some modules, that is,* X *can be covered by affine open subsets* $\{U_i = \operatorname{Spec} R_i\}_i$*, such that there is a collection of* R_i*-module* M_i *with* $\mathcal{F}|_{U_i} \cong \widetilde{M_i}$*.*

By (Hartshorne, 1977, Proposition (2.5.2-(b,e))), we can say that an $\mathcal{O}_X$-module $\mathcal{F}$ for some scheme X is quasi-coherent if and only if it satisfies the following conditions on the affine open subsets:

(i) Let $V \subseteq U$ be two affine open subsets of the scheme X. Then we have an isomorphism of $\mathcal{O}_X(V)$-modules given by

$$\mathcal{O}_X(V) \otimes_{\mathcal{O}_X(U)} \mathcal{F}(U) \xrightarrow{\mathrm{id} \otimes f_{UV}} \mathcal{O}_X(V) \otimes_{\mathcal{O}_X(V)} \mathcal{F}(V) \cong \mathcal{F}(U)$$

where $f_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$ is the restriction map of the $\mathcal{O}_X$-module $\mathcal{F}$.

(ii) If $W \subseteq V \subseteq U$ for affine open subsets W, V, U, then the composition

$$\mathcal{F}(U) \xrightarrow{f_{UV}} \mathcal{F}(V) \xrightarrow{f_{VW}} \mathcal{F}(W)$$

gives $\mathcal{F}(U) \xrightarrow{f_{UW}} \mathcal{F}(W)$.

1.3.2 Quasi-coherent Sheaves on a Projective Scheme

In this section, we focus on quasi-coherent sheaves on a projective scheme, which is our main concern in this thesis. On such a scheme $X = \operatorname{Proj} S$, there are some special sheaves of the form $O_X(n)$, called as twisted sheaves, that play an essential role in the sheaf theory.

Let S be a graded ring, and let $M = \bigoplus_{n\in\mathbb{N}} M_n$ be a graded S-module, that is, $S_n M_m \subseteq M_{n+m}$ for every $n \geq 0$ and $m \in \mathbb{Z}$. Now we will construct a quasi-coherent sheaf on $X = \operatorname{Proj} S$ in the following way: Let f be a homogeneous element of the irrelevant ideal S_+. By $M_{(f)}$ we denote the submodule of the localization M_f of M at f containing the elements of degree zero, that is,

$$M_{(f)} := \{mf^{-n} \in M_f \mid m \in M_{ndegf} \text{ and } n \in \mathbb{N}\}.$$

Clearly, $M_{(f)}$ is a $B_{(f)}$-module. Then, we have the same result as in the affine case for constructing a quasi-coherent O_X-module on $\operatorname{Proj} S$.

Proposition 1.3.5. *(Liu, 2002, Proposition 5.1.17) With the notation above, there exists a unique quasi-coherent O_X-module $\widetilde{M}$ such that*

(i) For any nonnilpotent homogeneous element $f \in S_+$, $\widetilde{M}|_{D_+(f)}$ is the quasi-coherent sheaf $\widetilde{M_{(f)}}$ on $D_+(f) \cong \operatorname{Spec} S_{(f)}$.

(ii) For any $p \in \operatorname{Proj} S$, $\widetilde{M_{(p)}}$ is isomorphic to $M_{(p)}$.

Remark 1.3.6. Let $M = \bigoplus_{n\geq 0} M_n$ be a graded S-module. Let $N = \bigoplus_{n\geq n_0} M_n$ for some $n_0 > 0$. Then $\widetilde{M} = \widetilde{N}$. Because $M_{(f)} = N_{(f)}$ for every homogeneous element $f \in S$. This implies, in particular, that $\widetilde{M}$ does not determine M, contrary to the affine case.

One of the most important examples of quasi-coherent sheaves on a projective scheme is the twisted sheaf. Firstly we will define a twisting of a graded ring S and, after that, a twisting of a quasi coherent $\mathcal{O}_X$-module.

Definition 1.3.7. Let S be a graded ring. For any $n \in \mathbb{Z}$, let $S(n)$ denote the graded S-module defined by $S(n)_d = S_{n+d}$. $S(n)$ is called as *'twist' of* S. Let $X = \operatorname{Proj} S$. Given $n \in \mathbb{Z}$, the *twisted sheaf* $\mathcal{O}_X(n)$ is the $\mathcal{O}_X$-module $\widetilde{S(n)}$. The twisted sheaf $\mathcal{O}_X(1)$ is known as *the twisting sheaf of Serre*.

Actually, they play an important role in the theory of projective schemes. Note that for any homogeneous element f of degree 1 in S, we have

$$S(n)_{(f)} = \left\{ \frac{s}{f^m} | s \in S_{n+m} \text{ for all } m \geq 0 \right\}.$$

Definition 1.3.8. Let $X = \mathbb{P}^n_R$ and let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. For $n \in \mathbb{Z}$, $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n)$ is denoted by $\mathcal{F}(n)$ and is called the *twist of* $\mathcal{F}$. For the affine open subset U of X, we have

$$\mathcal{F}(U) = \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{O}_X(n)(U).$$

In fact, the global sections of the twists of $\mathcal{F}$ have information about the sheaf. In this way, the direct sum of all the global sections of its twists is defined as the *graded* S*-module associated to* $\mathcal{F}$, that is, it is the group

$$\Gamma_* \mathcal{F} = \bigoplus_{n \in \mathbb{Z}} \Gamma(X, \mathcal{F}(n)).$$

It has the structure of a graded S-module. Because if $s \in S_d$, then s determines a global section $s \in \Gamma(X, \mathcal{O}_X(d))$ in a natural way. Then, for any $t \in \Gamma(X, \mathcal{F}(n))$, we define the product $s \cdot t$ in $\Gamma(X, \mathcal{F}(n+d))$ by taking the tensor product $s \otimes t$ and using the natural map $\mathcal{F}(n) \otimes \mathcal{O}_X(d) \cong \mathcal{F}(n+d)$.

On an affine scheme, a quasi-coherent sheaf $\mathcal{F}$ is determined by its global sections $\mathcal{F}(X)$. The following proposition is an analogue of this result for projective schemes. It shows that the quasi-coherent sheaves on the projective scheme that we are interested in have a special form.

Proposition 1.3.9. *(Hartshorne, 1977, Proposition 5.15) Let $S = R[x_0, \ldots, x_n]$ be the polynomial ring over a ring R and $X = \mathbb{P}^n_R$. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Then $\Gamma_*\mathcal{F}$ is a graded S-module and there is a natural isomorphism*

$$\beta : \widetilde{\Gamma_*(\mathcal{F})} \to \mathcal{F}.$$

The next theorem of Serre is one of the most important results in the category of the quasi-coherent sheaves over the projective scheme;for the proof see Hartshorne (1977, Theorem 5.17).

Theorem 1.3.10. *(Serre, 1955) Let X be a projective scheme over a Noetherian ring R, let $O_X(1)$ be the twisting sheaf of Serre on X, and let $\mathcal{F}$ be a coherent O_X-module. Then there is an integer n_0 such that for all $n \geq n_0$, the sheaf $\mathcal{F}(n)$ can be generated by a finite number of global sections.*

As a corollary of this theorem, we obtain that, over a Noetherian ring, the twisted sheaves $\{O(n) : \ n \in \mathbb{Z}\}$ form a family of generators of the category of the coherent sheaves on a projective scheme; in fact, so is for the category of quasi-coherent sheaves.

Corollary 1.3.11. *(Hartshorne, 1977, Corollary 5.18) Let X be projective over a noetherian ring R. Then any coherent sheaf $\mathcal{F}$ on X can be written as a quotient of a sheaf ε, where ε is a finite direct sum of the twisted sheaves $O(n_i)$ for various integers n_i.*

CHAPTER TWO

$\mathfrak{Q}co(X)$ AS A CATEGORY OF REPRESENTATIONS

Let X be a scheme and $\mathfrak{Q}co(X)$ be the category of quasi-coherent sheaves on X. The aim of this chapter is to give a new and simpler category that is isomorphic to $\mathfrak{Q}co(X)$. So, it allow us to work in $\mathfrak{Q}co(X)$ more easily.

We start by defining the category of quasi-coherent $\mathbf{R}$-modules associated to a quiver.

2.1 The Category of Quasi-coherent R-Modules

A *quiver* Q is a directed graph which is given by the pair (V,E), where E denotes the set of all edges of the quiver Q and V is the set of all vertices. An edge a of the quiver Q from a vertex v_1 to a vertex v_2 is denoted by $a : v_1 \to v_2$.

A *representation* $\mathbf{R}$ of a quiver Q in the category of rings means that for each vertex $v \in V$ we have a ring $R(v)$ and a ring homomorphism

$$\mathbf{R}(a) : R(v) \longrightarrow R(w),$$

for each edge $a : v \to w$.

Now it is reasonable to talk about an $\mathbf{R}$-module. An $\mathbf{R}$-*module* $\mathbf{M}$ is given by an $R(v)$-module $M(v)$, for each vertex $v \in V$, and an $R(v)$-linear morphism

$$\mathbf{M}(a) : M(v) \longrightarrow M(w)$$

for each edge $a : v \to w \in E$. Since $\mathbf{R}(a)$ is a ring homomorphism for an edge $a : v \to w$, the $R(w)$-module $M(w)$ can be thought as a $R(v)$-module.

Definition 2.1.1. Let $Q=(V,E)$ be a quiver and $\mathbf{R}$ be its representation in the category of rings. An $\mathbf{R}$-module $\mathbf{M}$ is *quasi-coherent* if for each edge $a: v \to w$, the morphism

$$\mathrm{id}_{R(w)} \otimes_{R(v)} \mathbf{M}(a) : R(w) \otimes_{R(v)} M(v) \to R(w) \otimes_{R(w)} M(w)$$

is an $R(w)$-module isomorphism.

For a fixed quiver Q and a representation $\mathbf{R}$, the category $\mathbf{R}$-Mod is given by a family of all $\mathbf{R}$-modules. Any morphism $f: \mathbf{M} \to \mathbf{N}$ between $\mathbf{R}$-modules $\mathbf{M}$ and $\mathbf{N}$ consists of $R(v)$-module homomorphisms $f_v : M(v) \to N(v)$ for every vertex v such that the following diagram

$$\begin{tikzcd} M(v) \arrow[r, "\mathbf{M}(a)"] \arrow[d, "f_v"'] & M(w) \arrow[d, "f_w"] \\ N(v) \arrow[r, "\mathbf{N}(a)"'] & N(w) \end{tikzcd}$$

commutes for each edge $a: v \to w$ in Q.

The *tensor product* $\mathbf{M} \otimes_{\mathbf{R}} \mathbf{N}$, where $\mathbf{M}$ and $\mathbf{N}$ are $\mathbf{R}$-modules is an $\mathbf{R}$-module such that for each vertex v

$$(\mathbf{M} \otimes_{\mathbf{R}} \mathbf{N})(v) := M(v) \otimes_{R(v)} N(v)$$

with the canonical map $(\mathbf{M} \otimes_{\mathbf{R}} \mathbf{N})(a) := \mathbf{M}(a) \otimes_{\mathbf{R}(v)} \mathbf{N}(a)$ for an edge $a: v \to w$.

Then we obtain the notion of a flat $\mathbf{R}$-module. An $\mathbf{R}$-module $\mathbf{M}$ is flat if and only if $\mathrm{id} \otimes f$ is a monomorphism for any $\mathbf{R}$-module monomorphism $f: \mathbf{N}_1 \to \mathbf{N}_2$. It can be shown easily that the $\mathbf{R}$-module $\mathbf{M}$ is flat in the category of $\mathbf{R}$-modules if and only if for each vertex v, $M(v)$ is a flat $R(v)$-module.

The category of quasi-coherent $\mathbf{R}$-modules for a fixed quiver Q and a fixed representation $\mathbf{R}$ of the quiver Q is defined as the full subcategory of the category

R-Mod that contains all quasi-coherent **R**-modules. We will denote it by $\mathbf{R}_{Qco}$-Mod. Let us investigate the category $\mathbf{R}_{Qco}$-Mod.

Before giving the following lemma, note that a direct sum of **R**-modules is defined componentwise.

Lemma 2.1.2. *Let* **R** *be a representation of a quiver Q. If we have a family* $\{\mathbf{M}_i\}_{i\in I}$ *of quasi-coherent* **R***-modules, then their direct sum* $\mathbf{M}_i$ *is in* $\mathbf{R}_{Qco}$*-Mod, as well.*

Proof. Let $\{\mathbf{M}_i\}_{i\in I}$ be a family of quasi-coherent **R**-modules. The morphism $\alpha_i(a)$: $M_i(v) \to M_i(w)$ denotes the morphism $\mathbf{M}_i(a)$ for each edge $a : v \to w$. By the definition, for each edge v, we have

$$\left(\bigoplus_{i\in I}\mathbf{M}_i\right)(v) := \bigoplus_{i\in I} M_i(v).$$

We know that the tensor product has the distribution property over direct sums. Therefore, for each edge $v \to w$, we have an isomorphism

$$R(w) \otimes_{R(v)} \left(\bigoplus_{i\in I} M_i(v)\right) \cong \bigoplus_{i\in I}(R(w) \otimes_{R(v)} M_i(v))$$

and this isomorphism is natural. Since each $\mathbf{M}_i$ is quasi-coherent, this implies the isomorphism of $\mathrm{id}\otimes(\oplus_{i\in I}\alpha_i(a))$. $\square$

As in the case of the direct sum, we define kernel (Ker) and cokernel (Coker) of any morphism between quasi-coherent **R**-modules componentwise. That is, if $f : \mathbf{M} \to \mathbf{N}$ is a morphism between quasi-coherent **R**-modules, then $(\mathrm{Ker}\, f)(v) := \mathrm{Ker}(f_v)$ and $(\mathrm{Coker}\, f)(v) := \mathrm{Coker}(f_v)$ for each vertex v and morphisms $(\mathrm{Ker} f)(a)$ and $(\mathrm{Coker} f)(a)$ for an edge $a : v \to w$ are obtained by the properties of kernel and cokernel. Easily, it can be shown that these are well-defined. Since the tensor product preserves epimorphisms, clearly we have that cokernel of any morphism between quasi-coherent

R-modules is again quasi-coherent. But it is not true for kernel. So, we need some additional properties. The following lemma answers our problem.

Lemma 2.1.3. *(by Enochs, Estrada, García Rozas & Oyonarte (2003, Proposition 2.1)) Suppose that the representation* $\mathbf{R}$ *of a quiver has a property that* $\mathbf{R}(w)$ *is a flat* $\mathbf{R}(v)$*-module for each edge* $v \to w$*. Then kernel of any morphism* $f : \mathbf{M} \to \mathbf{N}$ *between two quasi-coherent* $\mathbf{R}$*-modules is again quasi-coherent.*

Proof. Suppose the assumption. Then we have a commutative diagram

$$\begin{tikzcd} \mathrm{Ker}(f_v) \arrow[r, hook] \arrow[d] & M(v) \arrow[r, "f_v"] \arrow[d, "\alpha"] & N(v) \arrow[d, "\beta"] \\ \mathrm{Ker}(f_w) \arrow[r, hook] & M(w) \arrow[r, "f_w"] & N(w) \end{tikzcd}.$$

By tensoring it with $R(w)$, we have

$$\begin{tikzcd} R(w)\otimes \mathrm{Ker}(f_v) \arrow[r] \arrow[d] & R(w)\otimes M(v) \arrow[r, "\mathrm{id}\otimes f_v"] \arrow[d, "\mathrm{id}\otimes\alpha"] & R(w)\otimes N(v) \arrow[d, "\mathrm{id}\otimes\beta"] \\ R(w)\otimes \mathrm{Ker}(f_w) \arrow[r] & R(w)\otimes M(w) \arrow[r, "\mathrm{id}\otimes f_w"] & R(w)\otimes N(w) \end{tikzcd}$$

where $\mathrm{id}\otimes\alpha$ is an isomorphism. Since $\mathbf{R}(w)$ is a flat $\mathbf{R}(v)$-module, the top and bottom rows of the first square are monomorphisms. Also $\mathrm{Ker}(\mathrm{id}\otimes f_v) = R(w)\otimes \mathrm{Ker}(f_v)$ and $\mathrm{Ker}(\mathrm{id}\otimes f_w) = R(w)\otimes \mathrm{Ker}(f_w)$. So, the left column of the diagram is an isomorphism. $\square$

Because of the property of modules, it can be directly said that if $f : \mathbf{M} \to \mathbf{N}$ is a morphism whose kernel is $\mathbf{0}$, which is zero on each vertex, then f is a kernel of its cokernel. And vice versa for cokernel. So, under the condition that $R(w)$ is a flat $R(v)$-module for each edge $v \to w$, it follows that the category $\mathbf{R}_{Qco}$-Mod is an abelian category. We will say that the *representation* $\mathbf{R}$ *is flat* if the ring $R(w)$ is a flat $R(v)$-module for each edge $a : v \to w$.

The category $\mathbf{R}_{Qco}$-Mod is cocomplete. Indeed, direct limits always exist thanks to the existence of cokernel and direct sums. It can be also computed componentwise. Since in the category of modules the direct limit is an exact functor, the direct limit in $\mathbf{R}_{Qco}$-Mod preserves monomorphisms on each vertex. So, the direct limit is exact in the category $\mathbf{R}_{Qco}$-Mod.

Finally, this category has a system of generators as a conclusion of Enochs & Estrada (2005, Corollary 3.5). Hence, if the representation $\mathbf{R}$ is flat, then the category of quasi-coherent $\mathbf{R}$-modules is a Grothendieck category.

2.2 A Category Isomorphic to $\mathfrak{Q}co(X)$

In the previous section, we have defined the category $\mathbf{R}$-Mod and in this manner the category $\mathbf{R}_{Qco}$-Mod. After that, we have explained its structure. As we proved before, the category $\mathbf{R}_{Qco}$-Mod is a Grothendieck category for a fixed quiver and a flat representation $\mathbf{R}$. In this section, we show that for every scheme, there is a quiver and its representation $\mathbf{R}$, which is flat, such that the category of quasi-coherent sheaves and the category of quasi-coherent $\mathbf{R}$-modules are isomorphic categories.

Consider the category of quasi-coherent sheaves on a scheme $(X, \mathcal{O}_X)$, denoted by $\mathfrak{Q}co(X)$. By the definition of a scheme, the scheme X has a family $\mathcal{B}$ of affine open subsets which is a base for X such that this family uniquely determines the scheme $(X, \mathcal{O}_X)$. (for example, it is enough to take the family of the affine open subsets covering X and $U \cap V$ for all U, V in this family). And also this family helps to uniquely determine the quasi-coherent $\mathcal{O}_X$-modules. That is, a quasi-coherent $\mathcal{O}_X$-module is determined by giving an $\mathcal{O}_X(U)$-module M_U for each U and a linear map $f_{UV} : M_U \to M_V$ whenever $V \subseteq U$, $V, U \in \mathcal{B}$, satisfying;

(i) $\mathcal{O}_X(V) \otimes_{\mathcal{O}_X(U)} M_U \to \mathcal{O}_X(V) \otimes_{\mathcal{O}_X(V)} M_V$ is an isomorphism with respect to the morphism $\mathrm{id} \otimes f_{UV}$ for all $V \subseteq U$.

(ii) If $W \subseteq V \subseteq U$, where $W, V, U \in \mathcal{B}$, then the composition $M_U \to M_V \to M_W$ gives $M_U \to M_W$.

By this way, we are able to construct a quiver $Q = (V, E)$ with respect to the scheme $(X, \mathcal{O}_X)$. Let $\mathcal{B}$ be a base of the scheme X containing affine open subsets such that $\mathcal{O}_X$ is $\mathcal{B}$-sheaf. Now, define a quiver Q having the family $\mathcal{B}$ as the set of vertices, and an edge between two affine open subsets $U, V \in \mathcal{B}$ as the only one arrow $U \to V$ provided that $V \subsetneq U$. Fix this quiver. Take the representation $\mathbf{R}$ as $R(U) = \mathcal{O}_X(U)$ for each $U \in \mathcal{B}$ and the restriction map $\rho_{UV} : \mathcal{O}_X(U) \to \mathcal{O}_X(V)$ for the edge $U \to V$. Then the functor

$$\Phi : \mathfrak{Q}co(X) \longmapsto \mathbf{R}_{Qco}\text{-Mod},$$

which was defined by above argument, is well-defined. In fact, it is an isomorphism of categories.

The quiver we have just constructed is not unique for the category of quasi-coherent $\mathcal{O}_X$-modules. There can be another base $\mathcal{B}$ such that $\mathcal{O}_X$ is a $\mathcal{B}$-sheaf. But as it was stated, it is enough to take a family of affine open subsets covering X and all intersections $U \cap V$, where U, V are in this family. Since these categories are isomorphic, to study on the category $\mathfrak{Q}co(X)$ are the same as on the category $\mathbf{R}_{Qco}$-Mod. To see and understand something about quasi-coherent sheaves is easier on the category $\mathbf{R}_{Qco}$-Mod. So, in the rest we will use generally this category instead of the category $\mathfrak{Q}co(X)$.

2.2.1 Examples

Example 2.2.1. Let $X = \mathbb{P}^1_R$. Then the affine open subsets $D_+(x_0), D_+(x_1)$ and their intersection $D_+(x_0) \cap D_+(x_1) = D_+(x_0x_1)$ are the desired affine open subsets in order to obtain the isomorphic category $\mathbf{R}_{Qco}$-Mod. Since we have

$$D_+(x_0) \hookleftarrow D_+(x_0x_1) \hookrightarrow D_+(x_1),$$

our quiver Q is

$$Q \equiv \bullet \longrightarrow \bullet \longleftarrow \bullet.$$

And its representation is

$$\mathbf{R} \equiv R[x] \hookrightarrow R[x, x^{-1}] \hookleftarrow R[x^{-1}],$$

by Example 1.2.13, where $x = \frac{x_1}{x_0}$ and so $x^{-1} = \frac{x_0}{x_1}$.

Therefore, an $\mathbf{R}$-module (equivalently an $O_{\mathbb{P}^1_R}$-module) $\mathbf{M}$ is given by

$$\mathbf{M} \equiv M_1 \xrightarrow{f_1} M \xleftarrow{f_2} M_2,$$

where M_1 is an $R[x]$-module, M_2 is an $R[x^{-1}]$-module and M is an $R[x, x^{-1}]$-module. Then $\mathbf{M}$ is quasi-coherent if and only if $\mathrm{id}_{R[x,x^{-1}]} \otimes f_1$ and $\mathrm{id}_{R[x,x^{-1}]} \otimes f_1$ are isomorphisms of $R[x, x^{-1}]$-modules.

Because of the fact that $R[x, x^{-1}] \cong S^{-1}R[x]$ (the localization of $R[x]$ at S) where $S = \{1, x, x^2, \ldots\}$, we have

$$R[x, x^{-1}] \otimes_{R[x]} M_1 \simeq S^{-1}R[x] \otimes_{R[x]} M_1.$$

It is well known that $S^{-1}R[x]\otimes M_1$ is isomorphic to the localization $S^{-1}M_1$ of M_1 at S naturally. Clearly, $R[x,x^{-1}]\otimes_{R[x,x^{-1}]} M \cong M$ as $R[x,x^{-1}]$-modules. So, the morphism $\mathrm{id}_{R[x,x^{-1}]}\otimes f_1$ is precisely the map $S^{-1}f_1$, which is from $S^{-1}M_1$ to M. By the similar argument for the isomorphism $\mathrm{id}_{R[x,x^{-1}]}\otimes f_2$, with $T=\{1,x^{-1},\ldots\}$ instead of S, we obtain the following equivalent condition for being quasi-coherent $\mathbf{R}$-modules on the scheme $\mathbb{P}^1_R$; an $\mathbf{R}$-module $\mathbf{M}$ is quasi-coherent if and only if $S^{-1}f_1$ and $T^{-1}f_2$ are isomorphisms where $S=\{1,x,x^2,\ldots\}$ and $T=\{1,x^{-1},x^{-2},\ldots\}$. It means that we may obtain the $R[x,x^{-1}]$-module M by inverting x in the $R[x]$-module M_1 and also by inverting x^{-1} in the $R[x^{-1}]$-module M_2.

Example 2.2.2. Let $X=\mathbb{P}^2_R$. Then take the basic affine open subsets $D_+(x_0)$, $D_+(x_1), D_+(x_2)$ and all possible intersections $D_+(x_0)\cap D_+(x_1)=D_+(x_0x_1)$, $D_+(x_0)\cap D_+(x_2)=D_+(x_0x_2), D_+(x_1)\cap D_+(x_2)=D_+(x_1x_2)$,
$D_+(x_0)\cap D_+(x_1)\cap D_+(x_2)=D_+(x_0x_1x_2)$. We have:

$$O_{\mathbb{P}^2_R}(D(x_0)) = R[x_0,x_1,x_2]_{(x_0)} \cong R[x,y],$$

$$O_{\mathbb{P}^2_R}(D(x_1)) = R[x_0,x_1,x_2]_{(x_1)} \cong R[x^{-1},z],$$

$$O_{\mathbb{P}^2_R}(D(x_2)) = R[x_0,x_1,x_2]_{(x_2)} \cong R[y,z^{-1}],$$

$$O_{\mathbb{P}^2_R}(D(x_0x_1)) = R[x_0,x_1,x_2]_{(x_0x_1)} \cong R[x,x^{-1},y,z],$$

$$O_{\mathbb{P}^2_R}(D(x_0x_2)) = R[x_0,x_1,x_2]_{(x_0x_2)} \cong R[x,y,y^{-1},z^{-1}],$$

$$O_{\mathbb{P}^2_R}(D(x_1x_2)) = R[x_0,x_1,x_2]_{(x_1x_2)} \cong R[x^{-1},y^{-1},z,z^{-1}], \text{ and}$$

$$O_{\mathbb{P}^2_R}(D(x_0x_1x_2)) = R[x_0,x_1,x_2]_{(x_1x_2x_3)} \cong R[x,x^{-1},y,y^{-1},z,z^{-1}],$$

where $x = x_1/x_0$, $y = x_2/x_0$, $z = x_2/x_1$. Thus, our quiver is

$$\begin{tikzcd}
 & \bullet & \\
\bullet \arrow[ur] & \bullet \arrow[u] & \bullet \arrow[ul] \\
\bullet \arrow[u] \arrow[ur] & \bullet \arrow[ul] \arrow[ur] & \bullet \arrow[ul] \arrow[u]
\end{tikzcd}$$

and its representation is

$$\begin{tikzcd}
 & R[x,x^{-1},y,y^{-1},z,z^{-1}] & \\
R[x,x^{-1},y,z] \arrow[ur] & R[x,y,y^{-1},z^{-1}] \arrow[u] & R[x^{-1},y^{-1},z,z^{-1}] \arrow[ul] \\
R[x,y] \arrow[u] \arrow[ur] & R[x^{-1},z] \arrow[ul] \arrow[ur] & R[y,z^{-1}] \arrow[ul] \arrow[u]
\end{tikzcd}$$

Example 2.2.3. Let $X = \mathbb{P}^n_R$ where $n \in \mathbb{N}$. Then again take a base containing affine open subsets $D_+(x_i)$ for all $i = 0, \ldots n$, and all possible intersections. In this case, our base contains basic open subsets of this form

$$D_+(\prod_{i \in v} x_i),$$

where $v \subseteq \{0, 1, \ldots, n\}$.

So, the vertices of our quiver are all subsets of $\{0, 1, \ldots, n\}$ and we have only one edge $v \to w$ for each $v \subseteq w \subseteq \{0, 1, \ldots, n\}$ since $D_+(\prod_{i \in w} x_i) \subseteq D_+(\prod_{i \in v} x_i)$. Its representation has

$$O_{\mathbb{R}^n_R}(D(\prod_{i \in v} x_i)) = R[x_0, \ldots, x_n]_{(\prod_{i \in v} x_i)}$$

on each vertex v, and by Example 1.2.13, it is isomorphic to the polynomial ring on the ring R with the variables $\frac{x_j}{x_i}$ where $j = 0, \ldots, n$ and $i \in v$. We will denote this polynomial ring by $R[v]$. Then the representation $\mathbf{R}$ with respect to this quiver has vertex $R(v) = R[v]$ and edges $R[v] \hookrightarrow R[w]$ as long as $v \subseteq w$.

Finally, an **R**-module **M** is quasi-coherent if and only if

$$S_{vw}^{-1} f_{vw} : S_{vw}^{-1} M(v) \longrightarrow S_{vw}^{-1} M(w) = M(w)$$

is an isomorphism as $R[w]$-modules for each $f_{vw} : M(v) \to M(w)$ where S_{vw} is the multiplicative group generated by the set $\{x_j/x_i |\ j \in w \setminus v,\ i \in v\} \cup \{1\}$ and $v \subset w$.

CHAPTER THREE

MORE ON $\mathfrak{Qco}(\mathbb{P}^1_R)$

In this chapter we go on a trip through the category $\mathfrak{Qco}(\mathbb{P}^1_R)$ to understand its structure. We prove that $\mathfrak{Qco}(\mathbb{P}^1_R)$ does not have any projective object. But one of the best part of this category is that it has a nice family of generators although it has no projective objects. Grothendieck (1957) characterized all vector bundles on $\mathbb{P}^1_k$ by using these nice generators. To deal with these, we use the category isomorphic to $\mathfrak{Qco}(\mathbb{P}^1_R)$ introduced in the previous chapter. So, throughout this chapter, we always consider the representation $\mathbf{R}$ of $\mathbb{P}^1_R$ as

$$\mathbf{R} \equiv R[x] \hookrightarrow R[x, x^{-1}] \hookleftarrow R[x^{-1}]$$

where R is commutative ring and a quasi coherent sheaf over $\mathbb{P}^1_R$ is a representation of $\mathbf{R}$ of the form

$$M \xrightarrow{f} P \xleftarrow{g} N$$

with an $R[x]$-module M, an $R[x^{-1}]$-module N and an $R[x, x^{-1}]$-module P and the homomorphisms f, g preserving their module structures such that

$$S^{-1}f : S^{-1}M \longrightarrow S^{-1}P = P$$

and

$$T^{-1}g : T^{-1}N \longrightarrow T^{-1}P = P$$

are $R[x, x^{-1}]$ isomorphisms, where $S = \{1, x, x^2, \ldots\}$ and $T = \{1, x^{-1}, x^{-2}, \ldots\}$. Since $R[x, x^{-1}]$ is the localization of $R[x]$ at S and localizations preserve exactness, $R[x, x^{-1}]$ is a flat $R[x]$-module. And by the same argument, it is also a flat $R[x^{-1}]$-module. So, according to the argument in Chapter 2, the category of quasi-coherent sheaves on $\mathbb{P}^1_R$ is a Grothendieck category.

3.1 Serre's Twisted Sheaves on $\mathbb{P}^1_k$ as Representations of Quivers

Firstly, let us classify all the quasi-coherent sheaves of the form

$$R[x] \xrightarrow{f} R[x,x^{-1}] \xleftarrow{g} R[x^{-1}].$$

Proposition 3.1.1. *(Enochs, Estrada & Torrecillas, Proposition 14.3.1) Any quasi-coherent sheaf of the form*

$$R[x] \xrightarrow{f} R[x,x^{-1}] \xleftarrow{g} R[x^{-1}]$$

is isomorphic to a quasi-coherent sheaf

$$R[x] \hookrightarrow R[x,x^{-1}] \xleftarrow{x^n} R[x^{-1}]$$

for some $n \in \mathbb{Z}$.

Proof. Let

$$d : R[x^{-1}] \longrightarrow S^{-1}R[x] = R[x,x^{-1}]$$

be the homomorphism $(S^{-1}f)^{-1} \circ g$. Then we have a diagram

$$\begin{tikzcd}
R[x] \arrow[r, hook] \arrow[d, "\mathrm{id}"] & R[x,x^{-1}] \arrow[d, "S^{-1}f"] & R[x^{-1}] \arrow[l, "d"'] \arrow[d, "id"] \\
R[x] \arrow[r, "f"] & R[x,x^{-1}] & R[x^{-1}] \arrow[l, "g"']
\end{tikzcd}$$

The representation

$$R[x] \hookrightarrow R[x,x^{-1}] \xleftarrow{d} R[x^{-1}]$$

is quasi-coherent. Indeed, $T^{-1}d = T^{-1}(S^{-1}f)^{-1} \circ T^{-1}g$ and $T^{-1}(S^{-1}f)^{-1} = (S^{-1}f)^{-1}$. Since $T^{-1}g$ is an isomorphism, $T^{-1}d$ is an isomorphism. Therefore

$T^{-1}d(1)$ is a unit of $R[x,x^{-1}]$. Hence, $d(1) = T^{-1}d(-1) = ux^{-n}$ for some unit u and some $n \in \mathbb{Z}$, that is, $d = ux^n$. We can omit u because

$$R[x] \hookrightarrow R[x,x^{-1}] \xleftarrow{ux^n} R[x^{-1}]$$

and

$$R[x] \hookrightarrow R[x,x^{-1}] \xleftarrow{x^n} R[x^{-1}]$$

are isomorphic. Therefore we may say that $d = x^n$ for some $n \in \mathbb{Z}$, as desired. $\square$

In fact, the twisted sheaf referred in Proposition 3.1.1 is unique. It is easy to prove that $O(n)$ and $O(m)$ are isomorphic if and only if $n = m$.

In Subsection 1.3.2, we motivated twisted sheaves. Essentially, in terms of quasi-coherent **R**-modules, a representation

$$R[x] \hookrightarrow R[x,x^{-1}] \xleftarrow{x^n} R[x^{-1}]$$

with $n \in \mathbb{Z}$ corresponds to the unique twisted sheaf (or line bundle) of degree n on $\mathbb{P}^1_R$, which is denoted by $O_{\mathbb{P}^1_R}(n)$.

Definition 3.1.2. Any $R[x] \hookrightarrow R[x,x^{-1}] \xleftarrow{x^n} R[x^{-1}]$ with $n \in \mathbb{Z}$ is denoted by $O_{\mathbb{P}^1_R}(n)$ and called as a *twisted sheaf*. We shall shortly use the notation $O(n)$ for $O_{\mathbb{P}^1_R}(n)$.

Proposition 3.1.3. *(Enochs, Estrada & Torrecillas, Proposition 14.3.3) For any couple of integers n,m,*

$$O(n) \otimes O(m) \cong O(n+m).$$

Proof. If we take the canonical isomorphisms $\alpha_1 : R[x] \otimes R[x] \to R[x]$, $\alpha_2 : R[x,x^{-1}] \otimes$

$R[x,x^{-1}] \to R[x,x^{-1}]$ and $\alpha_3 : R[x^{-1}] \otimes R[x^{-1}] \to R[x^{-1}]$, we have the following diagram

$$\begin{tikzcd}
R[x] \otimes R[x] \arrow[r, hook] \arrow[d, "\alpha_1"] & R[x,x^{-1}] \otimes R[x,x^{-1}] \arrow[d, "\alpha_2"] & R[x^{-1}] \otimes R[x^{-1}] \arrow[l, "x^n \otimes x^m"'] \arrow[d, "\alpha_3"] \\
R[x] \arrow[r, hook] & R[x,x^{-1}] & R[x^{-1}] \arrow[l, "x^{n+m}"']
\end{tikzcd}$$

This proves the isomorphism $O(n) \otimes O(m) \cong O(n+m)$. $\square$

The following proposition shows us that the family $\{O(n) : n \in \mathbb{Z}\}$ constitutes a family of flat generators for the category $\mathfrak{Qco}(\mathbb{P}^1_R)$.

Proposition 3.1.4. *(Enochs, Estrada & Torrecillas, Proposition 3.4) The family $\{O(n) : n \in \mathbb{Z}\}$ generates the category of quasi-coherent sheaves on $\mathbb{P}^1_R$.*

Proof. Let $\mathbf{M}$ be a quasi-coherent sheaf on $\mathbb{P}^1_R$. So, it is of the form

$$M \xrightarrow{f} P \xleftarrow{g} N.$$

Let m be an element of M. Define the morphism $\alpha_m : R[x] \to M$ by $\alpha_m(1) := m$. With respect to m, define $\beta_m : R[x,x^{-1}] \to P$ by $\beta_m(1) := f(m)$. Since $\mathbf{M}$ is quasi-coherent, we have the isomorphism $T^{-1}g : T^{-1}N \to T^{-1}P = P$. So, for $f(m) \in P$, there exists a unique $x^n k \in T^{-1}N$ for some $n \in \mathbb{N}$ such that $T^{-1}g(x^n k) = x^n g(k) = f(m)$. Then define a morphism $\gamma_m : R[x^{-1}] \to N$ by $\gamma_m(1) := k$. And by taking the twisted sheaf $O(-n)$, the following diagram

$$\begin{tikzcd}
R[x] \arrow[r, hook] \arrow[d, "\alpha_m"] & R[x,x^{-1}] \arrow[d, "\beta_m"] & R[x^{-1}] \arrow[l, "x^{-n}"'] \arrow[d, "\gamma_m"] \\
M \arrow[r, "f"] & P & N \arrow[l, "g"']
\end{tikzcd}$$

is commutative.

Let $n \in N$. Likewise, define $\gamma_n : R[x^{-1}] \to N$ by $\gamma_n(1) := n$ and $\beta_n : R[x,x^{-1}] \to P$ by $\beta_n(1) := g(n)$. Again by the condition of quasi-coherence, there is a unique $v/x^{n'} \in S^{-1}M$ such that $f(v)/x^{n'} = g(n)$ for some $n' \in \mathbb{N}$. Accordingly, define the morphism $\alpha_n : R[x] \to M$ by $\alpha_n(1) := v$. Thus, we have a morphism

$$(\alpha_n, \beta_n, \gamma_n) : (R[x] \xrightarrow{x^{n'}} R[x,x^{-1}] \hookleftarrow R[x^{-1}]) \longrightarrow (M \xrightarrow{f} P \xleftarrow{g} N).$$

Since the representation $R[x] \xrightarrow{x^{n'}} R[x,x^{-1}] \hookleftarrow R[x^{-1}]$ is isomorphic to the twisted sheaf $O(-n')$, we may see it as $O(-n')$.

Finally if $p \in P$, then there exists a unique $m/x^i \in S^{-1}M$ and $x^j n \in T^{-1}N$ such that $f(m)/x^i = x^j g(n) = p$ where $i, j \in \mathbb{N}$. So, define the morphisms $\alpha_p : R[x] \to M$ by $\alpha_p(1) = m$, $\beta_p : R[x,x^{-1}] \to P$ by $\beta_p(1) := p$ and $\gamma_p : R[x^{-1}] \to N$ by $\gamma_p(1) := n$. Hence, the diagram

$$\begin{tikzcd}
R[x] \arrow[r, "x^i"] \arrow[d, "\alpha_p"] & R[x,x^{-1}] \arrow[d, "\beta_p"] & R[x^{-1}] \arrow[l, "x^{-j}"'] \arrow[d, "\gamma_p"] \\
M \arrow[r, "f"] & P & N \arrow[l, "g"']
\end{tikzcd}$$

is commutative. And again by Proposition 3.1.1, the representation on the top row of the diagram is isomorphic to a unique twisted sheaf $O(l)$ for some $l \in \mathbb{Z}$.

In the above argument, what we have done is to cover all elements of M,N and P by the twisted sheaves. By considering the direct sum of all $O(n)$ that we found, we obtain an epimorphism $\bigoplus O(n) \to (M \xrightarrow{f} P \xleftarrow{g} N)$. $\square$

Above we proved that the category of quasi-coherent sheaves on $\mathbb{P}^1_R$ is generated by the twisted sheaves. Now we will show that this category has no projective objects.

Corollary 3.1.5. *(Enochs, Estrada, García Rozas & Oyonarte, 2004b, Corollary 2.3) There is no nonzero projective object in the category $\mathfrak{Qco}(\mathbb{P}^1_R)$.*

Proof. Suppose for the contrary that $\mathcal{P}$ is a nonzero projective object in $\mathfrak{Qco}(\mathbb{P}^1_R)$. By Proposition 3.1.4 there is an epimorphism $\bigoplus_{n\in\mathbb{Z}} O(n)^{(x_n)} \to \mathcal{P}$ where $x_n \in \mathbb{N}$ for each $n \in \mathbb{Z}$. Suppose that we have a morphism $O(n) \to \mathcal{P}$ given by $(\alpha,\beta,\gamma)_n$ as in the diagram above. Then for any natural number n_0 we have two morphisms $O(n+n_0) \to \mathcal{P}$ given by $(x^{n_0}\alpha, x^{n_0}\beta, \gamma)_{n+n_0}$ and $(\alpha,\beta,x^{-n_0}\gamma)_{n+n_0}$. The two morphisms give the morphism

$$O(n+n_0) \oplus O(n+n_0) \to \mathcal{P}.$$

Applying this to the sum above, we obtain an epimorphism

$$\bigoplus O(n+m)^{(x_n)} \oplus \bigoplus O(n+n_0)^{(x_n)} \longrightarrow \mathcal{P}$$

for each n_0. Since $\mathcal{P}$ is projective, there will be a section, namely $(\gamma_1,\gamma,\gamma_2)$. We will prove that we can find a certain natural number n_0 such that there does not exist a nonzero morphism between $\mathcal{P}$ and $\bigoplus O(n+n_0)$. If $0 \neq m \in M$, then $f(m) = g(a)/x^{-1}$, that is, $g(a) = x^{-1} f(m)$ for some $a \in N$. Suppose that

$$\gamma_1(m) = (\ldots, \rho_1(x), \ldots, \rho_k(x), \ldots)$$

and

$$\gamma_2(a) = (\ldots, q_1(x^{-1}), \ldots, q_t(x^{-1}), \ldots).$$

Then

$$\gamma \circ g(a) = \gamma(x^{-1} f(m)) = x^{-1}\gamma(f(m)) = \gamma_1(m) = x^{-1}(\rho_1(x), \ldots, \rho_k(x))$$

Thus $\operatorname{ord}(x^{-1}\ \rho_i(x)) \geq -1$ for every $1 \leq i \leq k$. But, by the commutativity of the diagram, we also have

$$\gamma \circ g(a) = \bigoplus x^{-(n+n_0)}(\gamma_2(a)) = \bigoplus x^{-(n+n_0)}(q_1(x^{-1}), \ldots, q_t(x^{-1}))$$
$$= x^{-n_0}(r_1(x^{-1}), \ldots, r_t(x^{-1})).$$

with $\mathrm{ord}(x^{-n_0} r_i(x^{-1})) \leq -n_0$ for all $1 \leq i \leq t$. So, $n_0 = l+1$ gives the desired contradiction. □

3.2 Decomposition of Finite Dimensional Vector Bundles: Grothendieck's Theorem

As we said before, Grothendieck characterized all the vector bundles on the projective line $\mathbb{P}^1(k)$, where k is a field. In this section, we state Grothendieck's Theorem in terms of $\mathbf{k}_{Qco}$-Mod. We also prove the existence of an adjoint pair between the categories $R[x]$-Mod and $\mathfrak{Qco}(\mathbb{P}^1_R)$.

Let H be a functor from $\mathfrak{Qco}(\mathbb{P}^1_R)$ to $R[x]$-Mod defined as follows: For a representation $\mathbf{M} \equiv A \to C \leftarrow B$ in $\mathfrak{Qco}(\mathbb{P}^1_R)$, $H(\mathbf{M})$ is the $R[x]$-module A, and for a morphism $(\alpha, \beta, \gamma) : \mathbf{M} \to \mathbf{N}$ between two representations, $H((\alpha, \beta, \gamma)) = \alpha$.

Proposition 3.2.1. *(Enochs, Estrada, García Rozas & Oyonarte, 2004b, Proposition 3.1) The functor H has a right adjoint.*

Proof. Let D be the functor from $R[x]$-Mod to $\mathfrak{Qco}(\mathbb{P}^1_R)$ defined as follows: for an $R[x]$-module L,

$$D(L) \equiv L \hookrightarrow S^{-1}L \xleftarrow{id} S^{-1}L$$

and for an $R[x]$-morphism $\alpha : L \to N$,

$$D(\alpha) = (\alpha, S^{-1}\alpha, S^{-1}\alpha).$$

To prove that D is the right adjoint of H, for each $\mathbf{M} \in \mathfrak{Q}co(\mathbb{P}^1_n)$ and $R[x]$-module L, we must define an isomorphism τ of Abelian groups

$$\Phi_{\mathbf{M}L} : \mathrm{Hom}_{\mathfrak{Q}co_{\mathbb{P}^1_R}}(\mathbf{M}, D(L)) \to \mathrm{Hom}_{R[x]}(H(\mathbf{M}), L).$$

Define $\Phi_{\mathbf{M}L}((\alpha,\beta,\gamma)) = \alpha$ for every morphism $(\alpha,\beta,\gamma) : \mathbf{M} \to D(L)$. It is easy to see that $\Phi_{\mathbf{M}L}$ is a homomorphism of abelian groups.

Let (α',β',γ') and (α,β,γ) be two morphisms between $A \xrightarrow{f} C \xleftarrow{g} B$ and $L \hookrightarrow S^{-1}L \xleftarrow{id} S^{-1}L$ such that $\alpha = \alpha'$. By commutativity, we have $\beta' o f = \alpha' = \alpha = \beta \circ f$. So $S^{-1}(\beta' \circ f) = S^{-1}(\beta \circ f)$. This implies the equality $\beta' \circ S^{-1}f = \beta \circ S^{-1}f$. Since $S^{-1}f$ is an isomorphism, we have $\beta' = \beta$. That is, $\Phi_{\mathbf{M}L}$ is a monomorphism.

Suppose that we are given a morphism $\alpha : A \to L$. Define the morphisms $\beta := S^{-1}\alpha \circ (S^{-1}f)^{-1} : C \to S^{-1}L$ and $\gamma := S^{-1}\alpha \circ (S^{-1}f)^{-1} \circ g : B \to S^{-1}L$. Then (α,β,γ) satisfies the commutativity between $A \xrightarrow{f} C \xleftarrow{g} B$ and $L \hookrightarrow S^{-1}L \xleftarrow{id} S^{-1}L$ and $\tau((\alpha,\beta,\gamma)) = \alpha$. Hence $\Phi_{\mathbf{M}L}$ is an epimorphism.

It follows that $\Phi_{\mathbf{M}L}$ is a group isomorphism. Thus, (H,D) is an adjoint pair. $\square$

By symmetry, the functor

$$D' : R[x^{-1}]\text{-Mod} \to \mathfrak{Q}co_{\mathbb{P}^1_R},$$

$D'(N) \equiv T^{-1}N \to T^{-1}N \leftarrow N$, is the right adjoint of the functor $H' : \mathfrak{Q}co_{\mathbb{P}^1_R} \to R[x^{-1}]$-Mod, $H'(M \to P \leftarrow N) = N$.

Proposition 3.2.2. *The functor H does not have a left adjoint.*

Proof. Suppose for the contrary that H has a left adjoint $T : R[x]\text{-Mod} \to \mathfrak{Qco}(\mathbb{P}^1_R)$. Then,

$$\operatorname{Hom}_{\mathfrak{Qco}\mathbb{P}^1_R}(T(L), \mathbf{M}) \cong \operatorname{Hom}_{R[x]}(L, H(\mathbf{M}))$$

for any $R[x]$-module L and a quasi-coherent sheaf $\mathbf{M}$. Clearly, the functor H is an exact functor. We know that the left adjoint of an exact functor preserves projective objects. Here, $R[x]$ is a projective object in the category $R[x]$-Mod. So, $T(R[x])$ must be projective. But, in the category $\mathfrak{Qco}(\mathbb{P}^1_R)$ there is no nonzero projective object. This implies $T(R[x]) = \mathbf{0}$ where $\mathbf{0} \equiv 0 \to 0 \leftarrow 0$. Since (T, H) is an adjoint pair, if we apply this to the pair $(R[x], \mathbf{R})$,where $\mathbf{R} \equiv R[x] \hookrightarrow R[x, x^{-1}] \hookleftarrow R[x^{-1}]$, we obtain an isomorphism

$$\operatorname{Hom}_{\mathfrak{Qco}\mathbb{P}^1_R}(\mathbf{0}, \mathbf{R}) \cong \operatorname{Hom}_{R[x]}(R[x], R[x])$$

which is impossible. This contradicts with our assumption. So the functor H has no left adjoint. □

Now we are turning to the case of a field, that is, we take $R = k$ for a field k. It is well known that vector bundles over the projective line, $\mathbb{P}^1(k)$, are the direct sum of the line bundles in a unique way. This is Grothendieck's Theorem (see Grothendieck (1957)). The representation in $\mathfrak{Qco}(\mathbb{P}^1_k)$ which corresponds to a vector bundle is $M \to P \leftarrow N$, where M is a finitely generated and free $k[x]$-module and N is a finitely generated and free $k[x^{-1}]$-module. For an elementary proof see Enochs, Estrada & Torrecillas (2006).

Theorem 3.2.3. *(Grothendieck, 1957) Each representation in $\mathfrak{Qco}(\mathbb{P}^1_k)$ of the form $M \to P \leftarrow N$, with M, N finitely generated and free, is a direct sum of*

$$O(j_i) \equiv k[x] \hookrightarrow k[x, x^{-1}] \xleftarrow{x^{j_i}} k[x^{-1}] \qquad i = 1, \ldots, n$$

where $j_i \in \mathbb{Z}$ *for* $i = 1\ldots,n$ *with* $j_1 \leq j_2 \leq \ldots \leq j_n$*. Moreover the integers* $\{j_1,\ldots,j_n\}$ *are uniquely determined.*

CHAPTER FOUR

FILTRATION IN R-MOD

In this chapter, we introduce the notion of a filtration with respect to a class $\mathcal{C}$ of modules. The closure under this type of $\mathcal{C}$-filtrations is often an important property of the class, and recently the closure of a class under its filtrations has began to play an important role in several areas of mathematics, for instance in Quillen's theory of model categories and in the category of cotorsion pairs.

After that we state Hill's Lemma which allows to expand a single $\mathcal{C}$-filtration to a large family satisfying additional properties. This tool is essential in our main concern in Chapter 5 of this thesis.

4.1 Modules Filtered by a Class and Closure Properties

Definition 4.1.1. (i) Let μ be an ordinal and $\mathcal{A} = (A_\alpha | \ \alpha \leq \mu)$ be a sequence of modules. Let $(f_{\alpha\beta} | \ \alpha \leq \beta \leq \mu)$ be a sequence of monomorphisms with $f_{\alpha\beta} \in \mathrm{Hom}_R(A_\alpha, A_\beta)$ such that $\mathcal{D} = \{A_\alpha, f_{\alpha\beta} | \ \alpha \leq \beta \leq \mu\}$ is a direct system of modules. $\mathcal{D}$ is called *continuous*, provided that $A_0 = 0$ and $A_\alpha = \varinjlim_{\beta<\alpha} A_\beta$ for all limit ordinals $\alpha \leq \mu$.

If all the maps $f_{\alpha\beta}$ are inclusions, then the sequence $\mathcal{A}$ is called a *continuous chain of modules*. Since the category of modules is a Grothendieck category, the direct limit is exact and a continuous chain is just a sequence $(A_\alpha | \ \alpha \leq \mu)$ of modules satisfying $A_0 = 0$, $A_\alpha \subseteq A_{\alpha+1}$ for all $\alpha < \mu$ and $A_\alpha = \bigcup_{\beta<\alpha} A_\beta$ for all limit ordinals $\alpha \leq \mu$.

(ii) Let M be a module and $\mathcal{C}$ be a class of modules.

M is $\mathcal{C}$-filtered, provided that there are an ordinal κ and a continuous chain

$(M_\alpha|\ \alpha \leq \kappa)$ of submodules of M such that $M = M_\kappa$, and each of the modules $M_{\alpha+1}/M_\alpha$ is isomorphic to an element of $\mathcal{C}$, where $\alpha < \kappa$. The chain $(M_\alpha|\ \alpha \leq \kappa)$ is called a *$\mathcal{C}$-filtration* of M.

The following proposition shows that the direct sum decomposition is a particular case of a filtration.

Proposition 4.1.2. *Let R be a ring and $\mathcal{L}$ be a class of R-modules. Suppose an R-module N has a direct sum decomposition $\bigoplus_{i\in I} N_i$ such that $N_i \in \mathcal{L}$ for each $i \in I$. Then N is $\mathcal{L}$-filtered.*

Proof. Suppose we have an R-module satisfying the assumption. We know that every set can be well-ordered. So, we may think of the index set I as well-ordered. Take $M_i := \bigoplus_{j<i} N_j$ for all $i < I$. Then $M_{i+1}/M_i \cong N_i$ is in the class $\mathcal{L}$ for each $i < I$. Clearly, the family $(M_i|\ i \leq I)$ is an $\mathcal{L}$-filtration of N. $\square$

Let M be an R-module and λ be an ordinal. The R-module M is said to be λ-*presented* if there is an exact sequence $R^{(I)} \to R^{(\lambda)} \to M \to 0$ for some $I \leq \lambda$. Let $\mathcal{P}$ be the class of all projective R-modules. By the class $\mathcal{P}^{<\aleph_1}$, we mean the class of all $<\aleph_1$-presented projective R-modules.

Proposition 4.1.3. *Let P be a projective R-module. Then P is $\mathcal{P}^{<\aleph_1}$-filtered if and only if P is a direct sum of countably generated projective modules.*

Proof. Suppose that P is a direct sum of countably generated projective modules. So $P = \bigoplus_{i\in I} P_i$ where P_i is a countably generated projective R-module for each $i \in I$. Since for any cardinal λ, the notions λ-presented and λ-generated are the same for projective modules, we can say that P_i's are $<\aleph_1$-presented projective modules. So, this part of proposition is the result of Proposition 4.1.2.

Suppose P is $\mathcal{P}^{<\aleph_1}$-filtered. Then it has a filtration $(M_\alpha|\ \alpha \le \sigma)$ such that $M_{\alpha+1}/M_\alpha \in \mathcal{P}^{<\aleph_1}$ where $\alpha < \sigma$. Take $P_\alpha := M_{\alpha+1}/M_\alpha$ for $\alpha < \sigma$. By transfinite induction we will prove that $M_\beta \cong \bigoplus_{\alpha<\beta} P_\alpha$ for all $\beta \le \sigma$. Clearly, $M_1 \cong P_0$. If $\beta < \sigma$ and the assumption holds for each ordinal $\alpha < \beta$, then, since $M_{\beta+1}/M_\beta$ is projective, $M_{\beta+1} \cong (M_{\beta+1}/M_\beta) \oplus M_\beta \cong P_\beta \oplus (\bigoplus_{\alpha<\beta} P_\alpha) = \bigoplus_{\alpha\le\beta} P_\alpha$. If β is a limit ordinal where $\beta \le \sigma$ and the assumption holds for all $\alpha < \beta$, then $M_\beta = \varinjlim_{\alpha<\beta} M_\alpha \cong \varinjlim_{\alpha<\beta}(\bigoplus_{\alpha'<\alpha} P_\alpha) = \bigoplus_{\alpha<\beta} P_\alpha$.

By transfinite induction, our claim follows. So we have $P = M_\sigma \cong \oplus_{\alpha<\sigma} P_\alpha$, where P_α is a countably generated projective module for all $\alpha < \sigma$. This proves the statement.

$\square$

Now we can state Kaplansky's Theorem (Anderson, (1992, Corollary 26.2)) in terms of filtrations. And by Proposition 4.1.3, this is equivalent to its original formulation.

Theorem 4.1.4 (Kaplansky's Theorem). *Every projective module is $\mathcal{P}^{<\aleph_1}$-filtered.*

Definition 4.1.5. Let R be a ring. For a class $\mathcal{L}$ of R-modules, the class ${}^\perp\mathcal{L}$ is defined to be the class of all modules M such that $\mathrm{Ext}^1_R(M,L) = 0$ for all objects $L \in \mathcal{L}$, that is, every exact sequence of R-modules

$$0 \longrightarrow L \longrightarrow P \longrightarrow M \longrightarrow 0$$

splits for all $L \in \mathcal{L}$. The class $\mathcal{L}^\perp$ consists of all modules R-modules M such that $\mathrm{Ext}^1_R(L,M) = 0$, that is, every exact sequence of R-modules

$$0 \longrightarrow M \longrightarrow P \longrightarrow L \longrightarrow 0$$

splits for all $L \in \mathcal{L}$.

For a class $\mathcal{L}$ of R-modules, the class of all R-modules having $\mathcal{L}$-filtration is denoted by the notation $\mathrm{Filt}(\mathcal{L})$. Clearly, $\mathcal{L} \subseteq \mathrm{Filt}(\mathcal{L})$ for any class $\mathcal{L}$. The following lemma ,which is known as Eklof Lemma, states that $\mathrm{Filt}({}^{\perp}\mathcal{L}) = {}^{\perp}\mathcal{L}$.

Lemma 4.1.6. *(Eklof, 1977, Theorem 1.2) Let R be a ring and $\mathcal{L}$ be a class of R-modules. If M is a module having an ${}^{\perp}\mathcal{L}$-filtration, then $M \in {}^{\perp}\mathcal{L}$.*

4.2 Expanding a Single Filtration: Hill Lemma

Hill Lemma helps us to expand the filtration of module with some good properties. It is one of the most important tools to prove our main result in this thesis.

Definition 4.2.1. Let σ be an ordinal and let $\mathcal{M} = (M_\alpha \mid \alpha \leq \sigma)$ be a continuous chain of modules. Consider a family of modules $(A_\alpha \mid \alpha < \sigma)$ such that $M_{\alpha+1} = M_\alpha + A_\alpha$ for each $\alpha < \sigma$. A subset S of σ is said to be *closed*, if every $\beta \in S$ satisfies

$$M_\beta \cap A_\beta \subseteq \sum_{\alpha \in S, \alpha < \beta} A_\alpha.$$

If an element $x \in M_\beta$ for some $\beta < \sigma$, then the *height* of x, denoted by $\mathrm{hgt}(x)$, is defined to be the least ordinal α such that $\alpha < \sigma$ and $x \in M_{\alpha+1}$. For any subset S of σ, we define

$$M(S) := \sum_{\alpha \in S} A_\alpha.$$

Example 4.2.2. Following the notation for $\mathcal{M}$ above, since $M_0 = 0$, we have $M_1 = A_0$. So $M_1 = \sum_{\beta<1} A_\beta$. If for $\alpha < \sigma$ the equation $M_\alpha = \sum_{\beta<\alpha} A_\beta$ holds, then $M_{\alpha+1} = M_\alpha + A_\alpha = \sum_{\beta<\alpha+1} A_\beta$. If α is a limit ordinal and $M_\beta = \sum_{\gamma<\beta} A_\gamma$ holds for all $\beta < \alpha$, since $\mathcal{M}$ is a continuous chain,

$$M_\alpha = \bigcup_{\beta<\alpha} M_\beta = \sum_{\beta<\alpha} M_\beta = \sum_{\beta<\alpha} A_\beta.$$

So, by transfinite induction, we have $M_\alpha = \sum_{\beta<\alpha} A_\beta$ for all $\alpha \leq \sigma$. If we intersect the sum $M_\alpha = \sum_{\beta<\alpha} A_\beta$ with A_α for an ordinal $\alpha \leq \sigma$, then we obtain $M_\alpha \cap A_\alpha \subseteq \sum_{\beta<\alpha} A_\beta$. It follows that the ordinal $\alpha (= \{\beta | \beta \in \alpha\})$ is closed where $\alpha \leq \sigma$.

Finally, we shall state Hill Lemma which will assist to solve our problem.

Lemma 4.2.3 (Hill Lemma)**.** *(Göbel & Trlifaj, 2006, Theorem 4.2.6) Let R be a ring, κ an infinite regular cardinal and $\mathcal{C}$ a set of $< \kappa$-presented modules. Let M be a module with a $\mathcal{C}$-filtration $\mathcal{M} = (M_\alpha | \ \alpha \leq \sigma)$ for some ordinal σ. Then there is a family $\mathcal{H}$ consisting of submodules of M such that:*

(i) $\mathcal{M} \subseteq \mathcal{H}$.

(ii) *$\mathcal{H}$ is closed under arbitrary sums and intersections (that is, $\mathcal{H}$ is a complete sublattice of the lattice of submodules of M).*

(iii) *If $N, P \in \mathcal{H}$ such that $N \subseteq P$, then there exists a $\mathcal{C}$-filtration $(\overline{P}_\gamma | \gamma \leq \tau)$ of the module $\overline{P} = P/N$ $\tau \leq \sigma$ such that and for each $\gamma < \tau$, there is a $\beta < \sigma$ with $\overline{P}_{\gamma+1}/\overline{P}_\gamma$ isomorphic to $M_{\beta+1}/M_\beta$.*

(iv) *If $N \in \mathcal{H}$ and X is a subset of M of cardinality $< \kappa$, then there is a $P \in \mathcal{H}$ such that $N \cup X \subseteq P$ and P/N is κ-presented.*

CHAPTER FIVE

FILTRATION IN $\mathfrak{Q}co(\mathbb{P}^n_R)$

In the present chapter, we use all the previous notions and constructions to find a Grothendieck type theorem for infinite-dimensional vector bundles on $\mathbb{P}^n_R$.

In contrast to the finite-dimensional case for $\mathbb{P}^1_k$ when k is a field, it seems unlikely that we can obtain any kind of result like claiming that any infinite-dimensional vector bundle is a direct sum of locally countably generated vector bundles. Our result supports the claim that it is worthwhile studying on filtrations of quasi-coherent sheaves.

Drinfeld (2006) purposes different classes of modules to generalize finite dimensional vector bundles. In the next section, we deal with the first of these generalizations, obtained by considering the class of almost projective modules.

5.1 Filtration of Locally Almost Projective Quasi-Coherent Sheaves

Let us recall from Drinfeld (2006) the definition of an almost projective module. As we will see, this notion generalizes the notion of a projective module.

Definition 5.1.1. Let R be a ring. An *elementary almost projective R-module* is an R-module isomorphic to a direct sum of a projective R-module and a finitely generated one. An *almost projective R-module* is a direct summand of an elementary almost projective module.

That is, an almost projective R-module T is a direct summand of $P \oplus M$, where P is a projective R-module and M is a finitely generated R-module. Clearly, every projective R-module is almost projective. But the converse is not true.

Proposition 5.1.2. *Every almost projective module is a direct sum of countably generated almost projective modules.*

Proof. Let T be an almost projective R-module. Then there exist a projective R-module P and a finitely generated R-module M such that T is a direct summand of $P \oplus M$. By Kaplansky's theorem, we know that P is a direct sum of countably generated projective R-modules, say $P = \bigoplus_{i \in I} P_i$. Then, there exists an R-module K such that

$$\left(\bigoplus_{i \in I} P_i\right) \oplus M = T \oplus K.$$

Since T is a direct summand of a direct sum of countably generated modules, T is again a direct sum of countably generated modules by Anderson (1992, Theorem 26). Say $T = \bigoplus_{j \in J} T_j$ for some index set J where T_j is a countably generated module for every $j \in J$. Clearly, each T_j is a direct summand of $P \oplus M$. This implies that T_j is an almost projective R-module for each $j \in J$ and T is a direct sum of countably generated almost projective modules. $\square$

By using almost projective modules, we define locally almost projective quasi-coherent sheaves. They generalize the classical notion of finite-dimensional vector bundle.

Definition 5.1.3. Let $(X, \mathcal{O}_X)$ be a scheme. We say that a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ is *locally almost projective* if for every basic affine open subset U of X there exists an almost projective $\mathcal{O}(U)$-module T such that $\mathcal{F}|_U \cong \widetilde{T}$.

Firstly, we will study on filtrations of locally almost projective $\mathcal{O}_X$-modules on $X = \mathbb{P}^n_R$ for $n \in \mathbb{N}$. After that, we will present and conclude our problem as a special case.

Let $\mathcal{T}$ be a locally (countably generated) almost projective quasi-coherent $\mathcal{O}_{\mathbb{P}^n_R}$-module. If we let $\mathbf{R}$ be the representation of $\mathbb{P}^n_R$ by rings given in Example

2.2.3, we know that there exists a unique quasi-coherent **R**-module **T** such that $T(v)$ corresponds to the (countably generated) almost projective $R[v]$-module given by $\mathcal{T}(v)$. Conversely, if **T** is an **R**-module such that $T(v)$ is an (countably generated) almost projective $R[v]$-module for each vertex v, then there is a unique locally (countably generated) almost projective quasi-coherent $O_{\mathbb{P}^n_R}$-module $\mathcal{T}$ (recall that in our notation the polynomial ring $R[v]$ refers to the polynomial ring over R with variables x_j/x_i for $j \in \{0,\dots,n\}$ and $i \in v$). From now on, we fix the ring representation **R** of $\mathbb{P}^n_R$ as the representation constructed in Example 2.2.3. The following proposition will allow us to find a family of locally countably generated generators in $\mathfrak{Qco}(\mathbb{P}^n_R)$. Before that, we need the next lemma.

Lemma 5.1.4. *Let* $\mathbf{R}' \equiv R[v] \hookrightarrow R[w]$ *be a part of the ring representation* **R** *of* $\mathbb{P}^n_R$ *where* $v \subseteq w \subseteq \{0,\dots,n\}$. *Suppose that we have a quasi-coherent* $\mathbf{R}'$*-module*

$$M(v) \xrightarrow{f} M(w)$$

and two countable subsets $X(v)$ *and* $X(w)$ *of* $M(v)$ *and* $M(w)$, *respectively. Then there exists a quasi-coherent* $\mathbf{R}'$*-submodule*

$$M'(v) \longrightarrow M'(w)$$

of $M(v) \xrightarrow{f} M(w)$ *such that* $X(v) \subseteq M'(v) \subseteq M(v)$, $X(w) \subseteq M'(w) \subseteq M(w)$ *and* $M'(v), M'(w)$ *are countably generated modules over* $R[v]$ *and* $R[w]$, *respectively.*

Proof. Let $y \in X(w)$. Then, because of the quasi-coherence, there exists $x_y \in M(v)$ such that $y = \frac{f(x_y)}{z}$, $z \in S_{vw}$ where S_{vw} is the multiplicative set generated by the set $\{x_j/x_i \mid j \in w \setminus v, i \in v\} \cup \{1\}$.

Take the submodule $M'(v)$ of $M(v)$ generated by $X(v) \cup Y$ where Y consists of all of x_y which has been found for each $y \in X(w)$ as above . Since $|X(v) \cup Y| \leq \aleph_0 +$

$\aleph_0 = \aleph_0$, $M'(v)$ is countably generated. Let $M'(w)$ be the $R[w]$-submodule of $M(w)$ generated by $f(M'(v))$. Clearly $M'(w)$ is a countably generated submodule of $M(w)$ containing $X(w)$. The submodule

$$M'(v) \xrightarrow{f|_{M'(v)}} M'(w)$$

is quasi-coherent. Since the morphism $S_{vw}f$ is an isomorphism (by condition of quasi-coherence), we need only to show that $S_{vw}f|_{M'(v)}$ is a homomorphism onto the $R[w]$-module $M'(w) = R[w]f(M'(v))$. Indeed, $S_{vw}f(M'(v)) = f(M'(v))_{S_{vw}}$ is equal to the $R[w]$-module generated by $f(M'(v))$, that is, to the $R[w]$-module $M'(w)$. This implies that $M'(v) \to M'(w)$ is a quasi-coherent $\mathbf{R}'$-submodule of $M(v) \to M(w)$. $\square$

Proposition 5.1.5. *Let* $\mathbf{M}$ *be a quasi-coherent sheaf on* $\mathbb{P}^n_R$. *If* $X(v) \subseteq M(v)$ *is a countable subset for each* $v \subseteq \{0,1,\ldots,n\}$, *then there exists a quasi-coherent submodule* $\mathbf{M}'$ *of* $\mathbf{M}$ *such that* $X(v) \subseteq M'(v) \subseteq M(v)$ *and* $M'(v)$ *is a countably generated* $R[v]$*-module for all* $v \subseteq \{0,1,\ldots,n\}$.

Proof. Let $E = \{e_i : 0 \leq l \leq k\}$ be the set of all arrows defining the quiver of $\mathbb{P}^n_R$ for some natural number k. We will construct by induction a family of $\mathbf{R}$-submodules $\mathbf{M}^{(m)}$ of $\mathbf{M}$ satisfying:

(i) $X(v) \subseteq M^{(m)}(v) \subseteq M(v)$ is countably generated for each v and $m, l \in \mathbb{N}$.

(ii) Whenever $m \equiv l \pmod{(k+1)}$ for $m \in \mathbb{N}$ where $l \in E$, $M^{(m)}(v) \to M^{(m)}(w)$ satisfies the quasi-coherent condition on the edge l.

(iii) $\mathbf{M}^{(m)} \subseteq \mathbf{M}^{(m+1)}$ for all $m \in \mathbb{N}$.

When $m \geq n+1$, think e_m as e_l, where $m \equiv l \pmod{(k+1)}$ and $l \in E$. Let us consider the edge $e_0 : v \to w$. By applying Lemma 5.1.4 to this edge, we obtain $T_0^{(0)}(v) \to T_0^{(0)}(w)$ satisfying the quasi-coherent condition. And say $T_0^{(0)}(u) := X(u)$

for all $u \subseteq \{0,\dots,n\}$ different from v and w. Now from $\mathbf{T}_0^{(0)}$, by taking $T_1^{(0)}(u)$ as the $R[u]$-module generated by the sets $\{T_0^{(0)}(u), f_{u',u}(T_0^{(0)}) | f_{u',u} : M(u') \to M(u)\}$ where each morphism $f_{u',u}$ denotes the morphism $\mathbf{M}(a)$ where $a : u' \to u$ $(u,u' \subseteq \{0,\dots,n\})$, we obtain a locally countably generated $\mathbf{R}$-submodule $\mathbf{T}_1^{(0)}$. But it is possible that we may have lost the quasi-coherent condition on $e_0 : v \to w$. So, we again apply Lemma 5.1.4 to obtain $\mathbf{T}_2^{(0)}$ such that $T_2^{(0)}(v) \to T_2^{(0)}(w)$ satisfies the quasi-coherent condition. And by the same argument above, we can construct an $\mathbf{R}$-submodule $\mathbf{T}_3^{(0)}$. Continuing in this way, we obtain the family $\{\mathbf{T}_n^{(0)}\}_{n\in\mathbb{N}}$.

Define the first term $\mathbf{M}^{(0)}$ as the direct union of this family on $n \in \mathbb{N}$. Now assume we have constructed $\mathbf{M}^{(m)}$ for $m \in \mathbb{N}$. Let us define $\mathbf{M}^{(m+1)}$. Take the edge $e_{m+1} : v \to w$. We apply Lemma 5.1.4 to $M^{(m)}(v) \to M^{(m)}(w)$ to obtain $T_0^{(m+1)}(v) \to T_0^{(m+1)}(w)$ which satisfies the quasi-coherent condition. Define $T_0^{(m+1)}(u) := M^{(m)}(u)$ for every $u \neq v,w$. From this, we can construct an $\mathbf{R}$-submodule $\mathbf{T}_1^{(m+1)}$ of $\mathbf{M}$ by the same method we did above. Again applying Lemma 5.1.4 to $T_1^{(m+1)}(v) \to T_1^{(m+1)}(w)$, we find $\mathbf{T}_2^{(m+1)}$ such that $T_2^{(m+1)}(v) \to T_2^{(m+1)}(w)$ is quasi-coherent. By proceeding in the same way, we obtain the family $\{\mathbf{T}_n^{(m+1)}\}_{n\in\mathbb{N}}$. So, define $\mathbf{M}^{(m+1)} := \bigcup_{n\geq 0} \mathbf{T}_n^{(m+1)}$. So we have constructed inductively the desired family $\{\mathbf{M}^{(m)}\}_{m\in\mathbb{N}}$.

Finally, if we let $M'(v) := \bigcup_{m\in\mathbb{N}} M^{(m)}(v)$ for all $v \subseteq \{0,1,2,\dots,n\}$, we see that the properties of being an $\mathbf{R}$-module and the quasi-coherence condition on each edge are cofinal. So, it follows that $\mathbf{M}'$ is a quasi-coherent $\mathbf{R}$-submodule of $\mathbf{M}$ containing $X(v)$ for all $v \subseteq \{0,1,2,\dots,n\}$ (let I be a directed set. A subset J of I is said to be *cofinal* in I if for a given $i \in I$, there is a j in J such that $i \leq j$). Clearly, $\mathbf{M}'$ is locally countably generated, since

$$|M'(v)| = |\bigcup_{n\in\mathbb{N}} M^{(m)}(v)|$$

for each v and countable union of countable sets is again countable. $\square$

Let $\mathcal{S}$ be the class of all countably generated almost projective R-modules. We already know that the direct sum decomposition is a special case of the filtration by Proposition 4.1.2. Since, every almost projective R-module is a direct sum of countably generated almost projective modules by Proposition 5.1.2, we can say that every almost projective R-module has an $\mathcal{S}$-filtration.

Let $\mathcal{S}_v$ be the class of all countably generated almost projective $R[v]$-modules for each $v \subseteq \{0,\ldots,n\}$, $\mathcal{L}$ be the class of all locally countably generated almost projective quasi-coherent $\mathbf{R}$-modules on $\mathbb{P}^n_R$ and $\mathcal{C}$ be the class of all locally almost projective quasi-coherent $\mathbf{R}$-modules. Then the class $\mathcal{L}$ contains quasi-coherent $\mathbf{R}$-modules $\mathbf{M}$ such that $M(v) \in \mathcal{S}_v$ for each edge $v \subseteq \{0,\ldots,n\}$.

Theorem 5.1.6. *(by Estrada, Guil Asensio, Prest & Trlifaj (2009, Theorem 3.8)) Every quasi-coherent* $\mathbf{R}$*-module in the class* $\mathcal{C}$ *has an* $\mathcal{L}$*-filtration.*

Proof. Let $\mathbf{T}$ be a quasi-coherent $\mathbf{R}$-module belonging to the class $\mathcal{C}$. By Proposition 5.1.2, we know that each $T(v)$ has an $\mathcal{S}_v$-filtration $\mathcal{M}_v$ for all $v \subseteq \{0,1,2,\ldots,n\}$. Let $\mathcal{H}_v$ be the family associated to $\mathcal{M}_v$ by Hill Lemma 4.2.3 and $\{m_{v,\alpha} | \alpha < \tau_v\}$ be an $R[v]$-generating set of the $R[v]$-module $M(v)$. Without lost of generality, we can assume that for some ordinal τ, $\tau = \tau_v$ for all v.

We will construct an $\mathcal{L}$-filtration $(\mathbf{M}_\alpha |\ \alpha \leq \tau)$ for $\mathbf{T}$ by induction on α. Let $\mathbf{M}_0 = 0$. Assume that $\mathbf{M}_\alpha$ is defined for some $\alpha < \tau$ such that $M_\alpha(v) \in \mathcal{H}_v$ and $m_{v,\beta} \in M(v)$ for all $\beta < \alpha$ and all $v \subseteq \{0,1,2,\ldots,n\}$. Set $N_{v,0} = M_\alpha(v)$. By Hill Lemma-(iv), there is a module $N_{v,1} \in \mathcal{H}_v$ such that $N_{v,0} \subseteq N_{v,1}$ and $N_{v,1}/N_{v,0}$ is countably generated.

By Proposition 5.1.5 (with $\mathbf{M}$ replaced by $\mathbf{T}/\mathbf{M}_\alpha$, and $X(v) = N_{v,1}/M_\alpha(v)$) there is a quasi-coherent $\mathbf{R}$-submodule $\mathbf{T}_1$ of $\mathbf{T}$ such that $\mathbf{M}_\alpha \subseteq \mathbf{T}_1$ and $\mathbf{T}_1/\mathbf{M}_\alpha$ is locally countably generated. Then $T_1(v) = N_{v,1} + \langle T_v \rangle$ for a countably subset $T_v \subseteq T_1(v)$, for each v. Again by help of Hill Lemma-(iv), there is a module $N_{v,2} \in \mathcal{H}_v$ such that

$T_1(v) = N_{v,1} + \langle T_v \rangle \subseteq N_{v,2}$ and $N_{v,2}/N_{v,1}$ is countably generated.

Proceeding similarly, we obtain a countable chain $(\mathbf{T}_n | \ n < \aleph_0)$ of quasi-coherent $\mathbf{R}$-submodule of $\mathbf{T}$, as well as a countable chain $(N_{v,n} | n < \aleph_0)$ of $R[v]$-submodules of $T(v)$, for each v. Let $\mathbf{M}_{\alpha+1} = \bigcup_{n<\aleph_0} \mathbf{T}_n$. Then $\mathbf{M}_{\alpha+1}$ is a quasi-coherent subsheaf of $\mathbf{T}$ satisfying $M_{\alpha+1}(v) = \bigcup_{n<\aleph_0} T'_n(v)$ for each v. By Hill Lemma-(ii), we deduce that $M_{\alpha+1}(v) \in \mathcal{H}_v$ and $M_{\alpha+1}(v)/M_\alpha(v)$ is a countably generated almost projective $R[v]$-module. Therefore $\mathbf{M}_{\alpha+1}/\mathbf{M}_\alpha \in \mathcal{L}$.

Assume $\mathbf{M}_\beta$ has been defined for all $\beta < \alpha$ where α is a limit ordinal $\leq \tau$. Then we define $\mathbf{M}_\alpha := \bigcup_{\beta<\alpha} \mathbf{M}_\beta$.

Since $m_{v,\alpha} \in M_{\alpha+1}(v)$ for all v and $\alpha < \tau$, we have $M_\tau(v) = M(v)$. So $(\mathbf{M}_\alpha | \ \alpha \leq \tau)$ is an $\mathcal{L}$-filtration of $\mathbf{T}$. □

5.2 A Version of Kaplansky's Theorem for Infinite Dimensional Vector Bundles

Definition 5.2.1. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module where $(X, \mathcal{O}_X)$ is scheme. Then $\mathcal{F}$ is said to be *free* if it is isomorphic to a direct sum of copies of $\mathcal{O}_X$. It is said to be of *finite rank* if this sum is finite. It is said to be *locally free* if X can be covered by open subsets U for which $\mathcal{F}|_U$ is a free $\mathcal{O}_X|_U$-module.

Then a locally free $\mathcal{O}_X$-module $\mathcal{F}$ of finite rank is a coherent $\mathcal{O}_X$-module where $\mathcal{F}(U)$ is a free $\mathcal{O}_X(U)$-module of finite rank for all affine open subsets U of X. Actually, it is known that there is a bijection between the class of the vector bundles in the sense of classical algebraic geometry and the class of the locally free coherent $\mathcal{O}_X$-modules of finite rank. So, in Sheaf Theory, the definition of vector bundle is taken as locally free coherent $\mathcal{O}_X$-module of finite rank. But in our study, we will drop the conditions finiteness and freeness. This leads to Drinfeld's definition of infinite-dimensional vector bundles.

Definition 5.2.2. (Drinfeld, 2006, Section 2) Let $(X, \mathcal{O}_X)$ be a scheme. A quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ is said to be a *vector bundle* (in the sense Drinfeld (2006)) if $\mathcal{F}(U)$ is a projective $\mathcal{O}_X(U)$-module for every affine open subset U of X.

So, if $\mathbf{R}$ is a representation of scheme $(X, \mathcal{O}_X)$, a vector bundle $\mathcal{M}$ corresponds to a unique element $\mathbf{M}$ in $\mathbf{R}_{Qco}$-Mod such that each $M(u)$ is a projective $R(u)$-module for every vertex u. In Drinfeld (2006), it is stated that the notion in 5.2.2 is a local property. That is, conversely, if $\mathbf{M}$ is an $\mathbf{R}$-module such that $M(u)$ is a projective $R(u)$-module for each vertex u, then there exists a unique vector bundle $\mathcal{M}$ on the scheme X.

Now, we restrict ourselves to vector bundles in the sense of Drinfeld on $\mathbb{P}^n_R$ for $n \in \mathbb{N}$. As we did in the Example 2.2.3, we will represent each vector bundle $\mathcal{P}$ on $\mathbb{P}^n_R$ by $\mathbf{P}$ such that $P(v)$ is a projective $R[v]$-module for each $v \subseteq \{0, \ldots, n\}$. Since a projective $R[v]$-module $P(v)$ is also an almost projective $R[v]$-module, a vector bundle $\mathbf{P}$ on $\mathbb{P}^n_R$ is a locally almost projective $\mathbf{R}$-module on $\mathbb{P}^n_R$. Finally, we finish this resarch by giving our major concern in this study as a version of Theorem 5.1.6 which we call Kaplansky's theorem for vector bundles on $\mathbb{P}^n_R$.

Let $\mathcal{S}'_v$ be the class of all countably generated projective $R[v]$-modules for each $v \subseteq \{0, \ldots, n\}$, $\mathcal{L}'$ be the class of all locally countably generated vector bundles on $\mathbb{P}^n_R$ and $\mathcal{C}'$ be the class of all vector bundles on $\mathbb{P}^n_R$. Then the same method in the proof of Theorem 5.1.6 works if we replace the classes $\mathcal{S}_v, \mathcal{L}, \mathcal{C}$ with these classes $\mathcal{S}'_v, \mathcal{L}', \mathcal{C}'$. So, it proves the following theorem.

Theorem 5.2.3. *Every vector bundle on $\mathbb{P}^n_R$ is a filtration of locally countably generated vector bundles.*

CHAPTER SIX

CONCLUSIONS

In this thesis, we focused on two classes: the projective R-modules and the almost projective R-modules for a commutative ring R. Then we centered on the category of quasi-coherent sheaves over the projective scheme $\mathbb{P}^n_R = (\operatorname{Proj} S, O_{\operatorname{Proj} S})$ where $S = [x_0, \ldots, x_n]$ for a commutative ring R and the several 'new' notions of (infinite dimensional) vector bundles attained to these classes. We proved structural results relative to the different generalization of vector bundles in terms of filtrations of certain locally countably generated quasi-coherent sheaves. The first of such generalizations involves the class of *locally almost projective* O_X-modules (see Definition 5.1.1) and the second is the class of infinite dimensional vector bundles.

For the case $n = 1$ and when infinite dimensional vector bundles are locally projective quasi-coherent sheaves on $\mathbb{P}^1_R$, our Theorem 5.2.3 may be seen as the analogous of Grothendieck's theorem on the decomposition of finite dimensional vector bundles on $\mathbb{P}^1_k$, where k is a field, as a direct sum of line bundles (Grothendieck, 1957). Moreover, when X is affine, our theorem coincides with Kaplansky's theorem on the decomposition of a projective module as a direct sum of countably generated projective modules. Therefore, our result can be thought as a 'generalized' version of Kaplansky's theorem for the category $\mathfrak{Qco}(\mathbb{P}^n_R)$ of quasi-coherent sheaves on $\mathbb{P}^n_R$.

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NOTATION

R	a commutative ring with unity
k	a field
$\mathbb{Z}$	the ring of integers
$\mathbb{N}$	the set of natural numbers $\{0, 1, \ldots\}$
id_A	an identity map from a set A to A
$\mathfrak{Top}(X)$	the category consisting of open subsets of a topological space X
$\operatorname{Spec} R$	the spectrum of a commutative ring R
S_+	the irrelevant ideal of a graded ring S
$\operatorname{Proj} S$	the set of homogeneous prime ideals of S not containing S_+ for a graded ring S
$\mathcal{O}_X$	the structure sheaf of a scheme X
$\Gamma(U, \mathcal{F})$	the image $\mathcal{F}(U)$ where $\mathcal{F}$ is a sheaf
$V(I)$	the set of prime ideals of a commutative ring R containing its ideal I
$D(I)$	$\operatorname{Spec} R \setminus V(I)$
X_f	the basic open subset $D(f)$ of $\operatorname{Spec} R$, where $f \in R$
ρ_{UV}	the restriction map of a sheaf X from open subsets U to V of X
$s\|_V$	the image $\rho_{UV}(s)$ of the element $s \in \mathcal{F}(U)$ where $\mathcal{F}$ is a sheaf
$\mathcal{F}_x$	the stalk of the presheaf $\mathcal{F}$ at x
$\mathbb{P}^n_R$	the projective scheme $(\operatorname{Proj} S, \mathcal{O}_{\operatorname{Proj} S})$ where $S = R[x_0, \ldots, x_n]$
$\mathbb{A}^{n+1}_R$	the affine scheme $(\operatorname{Spec} S, \mathcal{O}_{\operatorname{Spec} S})$ where $S = R[x_0, \ldots, n]$
$\mathbb{P}(k^{n+1})$	the projective space over the vector space k^{n+1} for a field k
R_f	the localization of the commutative ring R at f, where $f \in R$
R_P	the localization of the commutative ring R at the prime ideal P
M_f	localization of the R-module M at f, where $f \in R$
M_P	the localization of the R-module M at the prime ideal P
$\mathfrak{Qco}(X)$	the category of quasi-coherent sheaves on the scheme X

$\widetilde{M}$	the sheaf associated to the R-module M on the scheme $\operatorname{Spec} R$
$S(n)$	the twist of the graded ring S for an integer n
$\mathcal{O}_X(n)$	the twisted sheaf $\widetilde{S(n)}$ where $X = \operatorname{Proj} S$ and S is a graded ring for an integer n
$\mathcal{O}_X(1)$	the twisting sheaf of Serre
$\Gamma_* \mathcal{F}$	the graded S-module associated to $\mathcal{O}_X$-module $\mathcal{F}$
Q	a quiver (V, E) with the vertex set V and the edge set E
$\mathbf{R}$	a representation of a quiver Q in the category of rings
$\mathbf{R}_{Qco}$-Mod	the category of quasi-coherent $\mathbf{R}$-modules where $\mathbf{R}$ is a representation of a quiver in the category of rings
$R[v]$	the polynomial ring over the ring R with variables $\frac{x_j}{x_i}$, $j = 0, \ldots, n$ and $i \in v \subseteq \{0, \ldots, n\}$
S_{vw}	the multiplicative group generated by $\{\frac{x_j}{x_i} \mid\ j \in w \setminus v, i \in v\} \cup \{1\}$ where $v \subseteq w \subseteq \{0, \ldots, n\}$
$\mathcal{C}^{\kappa}$	the subclass of the class $\mathcal{C}$ of R-modules containing $< \kappa$-presented objects for a cardinal κ
${}^{\perp}\mathcal{L}$	the class of R-modules M such that $\operatorname{Ext}_R(M, L) = 0$ for all objects $L \in \mathcal{L}$, where $\mathcal{L}$ is a class of R-modules
$\mathcal{L}^{\perp}$	the class of R-modules M such that $\operatorname{Ext}_R(L, M) = 0$ for all objects $L \in \mathcal{L}$, where $\mathcal{L}$ is a class of R-modules
$\operatorname{Filt}(\mathcal{L})$	the class of R-modules having $\mathcal{F}$-filtration

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