DOKUZ EYLÜL UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

MORSE THEORY FOR SINGULAR SPACES

by
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İZMİR

MORSE THEORY FOR SINGULAR SPACES

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M.Sc THESIS EXAMINATION RESULT FORM

We have read the thesis entitled "**MORSE THEORY FOR SINGULAR SPACES**" completed by **SALİHA BAŞTÜRK** under supervision of **ASST. PROF. DR. BEDİA AKYAR MØLLER** and we certify that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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MORSE THEORY FOR SINGULAR SPACES

ABSTRACT

In this thesis, our aim is to understand Morse Theory for singular spaces. To reach this aim, we have studied the Classical Morse Theory which follows from the book of Yukio Matsumoto "An Introduction to Morse Theory". We have considered some theorems of Morse Theory for compact smooth manifolds without boundary and try to understand the proofs of these theorems. Furthermore, we have studied on Whitney stratification of a topological space using the transversality property of the strata.

Afterwards, we have tried to understand how Whitney stratification divides topological spaces and also singular spaces into strata which are smooth submanifolds.

Finally, we have examined the Morse theory for singular spaces using the Whitney stratification which follows from the book of Goresky and MacPherson "Stratified Morse Theory".

Keywords: Non-degenerate critical point, Hessian matrix, index, Morse function, gradient like vector field, handle decomposition, strata, stratification, Whitney stratification, transversality, Morse data, normal and tangential Morse data, singular point, singular space.

TEKİL UZAYLAR İÇİN MORSE TEORİSİ

ÖZ

Bu tezde amacımız tekil uzaylar için, Morse teorisini anlamaktır. Bu amaca ulaşmak için Yukio Matsumoto'nun "An Introduction to Morse Theory" kitabından klasik Morse teorisini inceledik. Morse teorisinde sınırı olmayan, kompakt, pürüzsüz manifoldlar için geçerli olan temel teoremleri ele alıp, ispatlarını anlamaya çalıştık. Bundan başka topolojik uzaylardaki katmanların (enine) diklik özelliğini kullanarak Whitney katmanlamasını inceledik.

Daha sonra Whitney katmanlamasının topolojik uzayları ve hatta tekil uzayları pürüzsüz altmanifoldlar olan katmanlara nasıl ayırdığını anlamaya çalıştık.

Son olarak, tekil uzayların Whitney katmanlamasını kullanarak Morse teorisini Goresky ve MacPherson'ın "Stratified Morse Theory" kitabını kullanarak anlamaya çalıştık.

Anahtar sözcükler : Dejenere olmamış kritik nokta, Hessian matris, indeks, Morse fonksiyonu, gradyant benzeri vektör alanı, kulp ayrıştırması, katmanlar, katmanlama, Whitney katmanlaması, enine diklik, Morse data, normal ve teğetsel Morse data, tekil nokta, tekil uzay.

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CHAPTER ONE
INTRODUCTION

Morse Theory firstly came into mathematics with the paper (Morse, 1925). At nearly same times, Lefschetz studied upon the topology of algebraic varieties, (Lefschetz, 1924). These two studies have been the motivation point of Morse Theory. The philosophy of Morse theory is to determine the relation between the critical points of a differentiable function on a smooth manifold M and topological invariants of M such as Betti numbers, Euler number of M which are given by using homology groups of cell complex of M and Betti numbers, respectively. On the other hand, Euler number of M is also defined by basic elements which is called handle of Morse theory at the critical points of f. Eventually, one can say that Morse theory helps to understand the topology and geometry of M by the handles of singularities of f. Namely, the historical development of Morse Theory is related to the historical progress of algebraic topology. The foundation of Algebraic Topology have been laid by the mathematicians such as Riemann, Betti and Poincaré in the last decade of 1800s. Morse studied on cellular homology and claimed that differentiable manifold with boundary could be cell-decomposed by the book written by Veblen (1922), but he had some difficulties to prove this claim. A year later, he delivered this problem as a thesis subject to his colleague, Cairns. Afterwards, Morse established his first extension concerning Morse Theory by basing it upon his own studies, and using Jacobi vector fields he found out the Morse Index computation method, known Morse Index Theorem. All these studies of Morse guided some mathematicians who were Bott, Thom, Smale and as such and big success in mathematics for a long term. In 1950s, Bott found out some techniques of group theory to calculate the Morse indexes of Lie groups though we are not interested in this subject in this thesis. Thom revealed the existence of a cell complex structure of M by defining a cell for each critical point of f in (Thom, 1949). This study of Thom presents an information about the homotopy type of M. Likewise in (Smale, 1960), developed the handlebody theory and described the handlebodies and handlebody-decomposition of M by defining handles for each critical point of f and using handles. These studies give information about both homotopy types and geometry of M. Smale has also improved new techniques for calculating Morse Indexes corresponding to handlebodies.

On the other hand, in 1974 Mark Goresky and Robert MacPherson started to develop a new homology theory which gave an information about some topological invariants around singular points of singular spaces, but they had some problems to define this homology group and they appealed Morse theory to solve these problems. At that time they needed actually defined Morse theory for singular spaces which had not been developed yet since Morse theory was used only for smooth manifolds. Lazzeri already defined Morse functions on singular space in (Lazzeri, 1973) and Pignoni proved the stability and density of these functions in (Pignoni, 1979). Yet the questions in Goresky and MacPherson minds were "What are the precluding thingsin order to extend Classical Morse Theory to singular spaces and how the critical points of a Morse function could be associated to the topology of singular spaces in their constructed theorem?". Some basic definitions of that subject and the applications of Stratified Morse Theory for singular spaces were given in (Goresky & MacPherson, 1983c) and (Goresky & MacPherson, 1983b). Later on in 1988 they established extensions of fundamental theorems of the Classical Morse Theory for singular spaces in their book (Goresky & MacPherson, 1988). Our aim in this thesis is to understand these theorems (Goresky & MacPherson, 1988) and (Goresky & MacPherson, 1983c).

In last two decades, there have been many studies published on the topology of singular spaces; all these have been based upon Goresky and MacPherson's book "Stratified Morse Theory". In this thesis, our goal is to get these two fundamental theorems that Goresky and MacPherson extended basically for singular spaces, and to achieve this goal we have initially studied on the Classical Morse Theory from (Matsumoto, 2002) developed by Smale and Morse for finite-dimensional compact smooth manifolds. The contents of the chapters of the thesis are given below in details:

The definitions of a compact smooth manifold, critical point, diffeomorphism, Hessian matrix, non-degenerate critical point and Morse function are given in the second chapter with basing upon (Matsumoto, 2002). We have also tried to understand

Morse Lemma and the Existence of Morse Function Theorem using the definitions mentioned above. The definition of Morse Index have been studied by using the standard form of a function defined on Morse Lemma. We have examined the definition of a handle which is given by Smale using Morse Index and the standard form of Morse function around non-degenerate critical point; we have also discussed the relation between the index of handlebodies and the index of Morse function at the critical points. We mentioned the construction of a manifold obtained by attaching handles using their index; then, we have consolidated it with some known examples and figures with details. Furthermore, we have tried to understand that "Handlebody decomposition" informs not only about the topology type of a manifold but also its shape, moreover we have realized that this information constitutes the foundation of Classical Morse Theory.

In the third chapter, we have mainly examined cellular homology to understand the relation between the handle and homology using (Hatcher, 2002) and (Matsumoto, 2002). Thus, we have understood that the handles defined at non-degenerate critical points correspond to the cells. Since these cells are the fundamental elements of cellular homology we have tried to understand this notion with the aid of some examples. We have seen that the Euler number which is a topological invariant can be defined by using the numbers of handles at critical points. On the other hand, we have observed another definition of Euler number with Betti numbers. Then we have examined Morse inequality theorem which is constructed from these two definitions.

As we have mentioned at the beginning, the goal of this thesis is to see how Morse Theory operates on the spaces of singular points. To achieve this goal, we have used theorems and concepts which are given in the second and third chapters. However, all studies done in these two chapters are essentially for the smooth manifolds. To be able to apply these studies on the topological spaces that are of singular points it is necessary to divide singular spaces into smooth submanifolds, (Veblen, 1922). After Veblen, some mathematicians such as Whitney, Thom, Mather, Lojasiewicz and Hardt have developed this idea and constructed "Stratification Theory". On this subject, the most efficient studies have been done by Whitney (1947).

In the fourth chapter, we have especially examined Whitney Stratification which is one of the conditions of theorems of Morse theory for singular spaces, named after and developed by him. We have tried to understand (with examples) how Whitney divides a singular space into strata with some detailed examples. Strata are smooth submanifolds satisfying Whitney's (a) and (b) conditions which are guarantee that singular points vanish in the submanifolds. Moreover, we have examined the "transversality" property in which strata must satisfy so as to apply Morse Theory on singular spaces.

Lastly, in the fifth chapter we have focused on the answers given to the question "How Classical Morse Theory could be applied to the singular spaces?" and we have surveyed the roles of Whitney Stratification on singular spaces in (Gibson et al., 1976) and (Goresky & MacPherson, 1988). As Whitney Stratification ensures that singular points on singular space may disappear, we have tried to see handles which correspond to the Morse data of critical points of Morse function on each stratum with some examples. So, we hope that one can grasp how Morse Theory for singular spaces is obtained.

CHAPTER TWO
MORSE THEORY

Morse theory in a classical sense is the study of relations between functions on a space, the shape of a space and also topological changes of a space. In particular, it gives an information about critical points of a function and the shape of a space with help of the critical points.

Morse theory deals with both finite dimensional and infinite dimensional spaces. In this thesis, we deal with the finite dimensional case and study on Morse theory for general m-dimensional manifolds based on the books (Tu, 2011) and (Matsumoto, 2002).

2.1 Manifolds

Definition 2.1.1. An m-topological manifold M is a topological space with dimension m which satisfies the following properties:

- M is Hausdorff and second countable
- M locally looks like $\mathbb{R}^m$.

Example 2.1.2. Let $\mathbb{R}^m_+$ be the closed upper half-space

$$\mathbb{R}^m_+ = \{x = (x_1, x_2, \ldots, x_m) \in \mathbb{R}^m \mid x_m \geq 0\}$$

with the subspace topology of $\mathbb{R}^m$ and it is a topological m-manifold with boundary since it is a second countable Hausdorff topological space and locally looks like $\mathbb{R}^m$. The boundary of $\mathbb{R}^m_+$ is defined by $x_m = 0$ and we can identify it with $\mathbb{R}^{m-1} = \{(x_1, x_2, \ldots, x_{m-1})\}$

Example 2.1.3. The m-dimensional closed unit disk D^m

$$D^m = \{(x_1, x_2, \ldots, x_m) \in \mathbb{R}^m \mid x_1^2 + x_2^2 + \ldots + x_m^2 \leq 1\}$$

is an m-dimensional topological manifold. It's boundary

$$\partial D^m = S^{m-1} = \{(x_1, x_2, \ldots, x_m) \mid x_1^2 + x_2^2 + \ldots + x_m^2 = 1\}$$

which is the $(m-1)$-dimensional unit sphere, is an $(m-1)$-dimensional topological manifold.

The second property of the Definition 2.1.1 means that each point p in M has a neighborhood U such that there is a homeomorphism $\phi : U \subset M \to U' \subset \mathbb{R}^m$ where U' is an open neighborhood of $\phi(p) = (x_1(p), \ldots, x_m(p))$ in $\mathbb{R}^m$. The pair (U, ϕ) is called a *chart*, U is a coordinate neighborhood and $\phi = (x_1, \ldots, x_m)$ is a coordinate system on U.

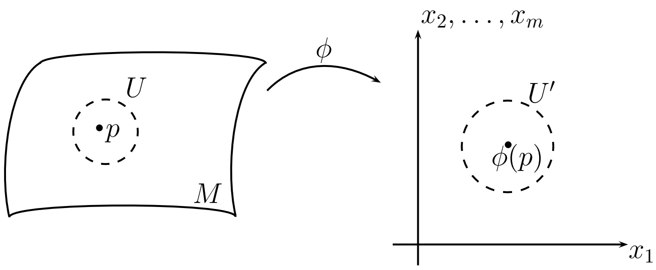

Figure 2.1 A chart (U, ϕ) at $p \in M$.

Definition 2.1.4 (Tu,2011)**.** Two charts (U, ϕ) and (V, ψ) of a topological manifold are C^∞*-compatible* if two maps

$$\phi \circ \psi^{-1} : \psi(U \cap V) \to \phi(U \cap V),$$

$$\psi \circ \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V)$$

are C^∞. These two maps are called *transition functions* between the charts. Moreover if $U \cap V = \emptyset$ then the two charts are also C^∞- compatible.

Definition 2.1.5. A *differentiable (or C^∞) structure* on a topological manifold M is a family $\{(U_i, \phi_i)\}$ of coordinate neighborhoods such that

1. $M = \bigcup_i U_i$, that is, U_i's cover M.
2. (U_i, ϕ_i) and (U_j, ϕ_j) are C^∞-compatible, for any i, j.
3. Any coordinate neighborhood (V, ψ) compatible with every (U_i, ϕ_i) and is itself in $\{(U_i, \phi_i)\}$.

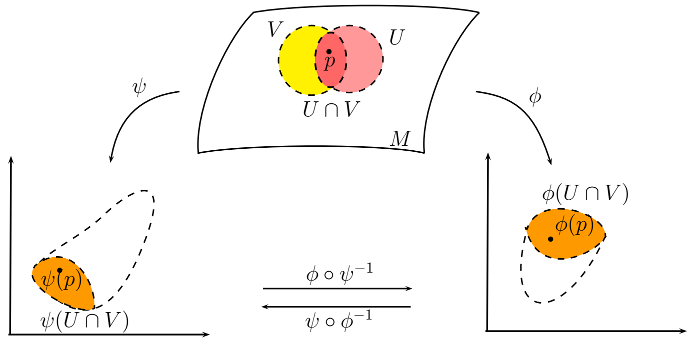

Figure 2.2 Transition functions $\phi \circ \psi^{-1}$ and $\psi \circ \phi^{-1}$.

Definition 2.1.6. A *smooth or C^∞ manifold M* is a topological manifold M together with a differentiable structure on M.

Example 2.1.7. The equation $x^2 + y^2 = 1$ defines the unit circle S^1 in $\mathbb{R}^2$. S^1 can be covered by four semicircles such as U_1, U_2 are lower and upper semicircles and U_3, U_4 are right and left semicircles which are open sets. The coordinate function $\phi_{1,2}(x, y) = x$ can be defined on U_1 and U_2 which are homeomorphisms onto the open interval $(-1, 1)$ in the x-axis. Similarly $\phi_{3,4}(x, y) = y$ are homeomorphisms from U_3 and U_4 onto the open interval $(-1, 1)$ in the y-axis (See in the following figure).

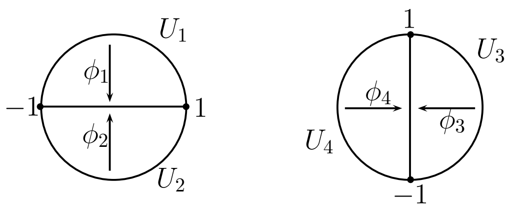

Figure 2.3 Charts on the unit circle.

We can easily check that every non-empty pairwise intersection $(U_i \cap U_j,\ \phi_j^{-1} \circ \phi_i)$ is C^∞.

For example, on $U_1 \cap U_3$

$$\phi_3 \circ \phi_1^{-1}(x) = \phi_3(x,\ \sqrt{1 - x^2}) = \sqrt{1 - x^2}$$

which is C^∞. On $U_1 \cap U_4$ and $U_2 \cap U_3$, respectively

$$\phi_4 \circ \phi_1^{-1}(x) = \phi_4(x, -\sqrt{1 - x^2}) = -\sqrt{1 - x^2}$$

$$\phi_3 \circ \phi_2^{-1}(x) = \phi_3(x, -\sqrt{1 - x^2}) = -\sqrt{1 - x^2}$$

which are C^∞. On $U_2 \cap U_4$

$$\phi_2 \circ \phi_4^{-1}(y) = \phi_4(-\sqrt{1-y^2}, y) = -\sqrt{1-y^2}$$

which are C^∞.

Thus, $\{(U_i, \phi_i)\}_{i=1}^{4}$ is an atlas on S^1 and $S^1 = \bigcup_{i=1}^{4} U_i$. Hence, S^1 is a smooth manifold.

Example 2.1.8. The equation $x^2 + y^2 + z^2 = 1$ defines the unit sphere S^2 in $\mathbb{R}^3$. S^2 can be covered by six open hemispheres U_i with respect to the coordinate functions ϕ_i:

$$U_1 = \{(x,y,z) \in S^2 \ : \ x > 0\},\ \phi_1(x,y,z) = (y,z)$$

$$U_2 = \{(x,y,z) \in S^2 \ : \ x < 0\},\ \phi_2(x,y,z) = (y,z)$$

$$U_3 = \{(x,y,z) \in S^2 \ : \ y > 0\},\ \phi_3(x,y,z) = (x,z)$$

$$U_4 = \{(x,y,z) \in S^2 \ : \ y < 0\},\ \phi_4(x,y,z) = (x,z)$$

$$U_5 = \{(x,y,z) \in S^2 \ : \ z > 0\},\ \phi_5(x,y,z) = (x,y)$$

$$U_6 = \{(x,y,z) \in S^2 \ : \ z < 0\},\ \phi_6(x,y,z) = (x,y)$$

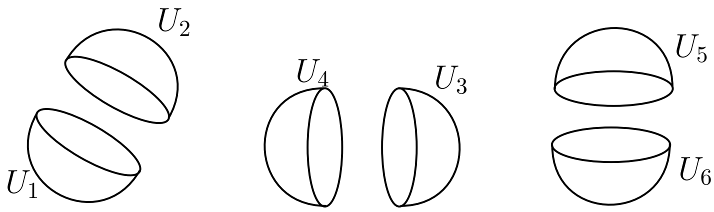

Figure 2.4 Charts on S^2.

One can easily check that every non-empty pairwise intersection $U_i \cap U_j$, $\phi_j^{-1} \circ \phi_i$ is C^∞. For example

$$\phi_3 \circ \phi_1^{-1}(y,z) = \phi_3(\sqrt{1-y^2-z^2}, y, z) = \sqrt{1-y^2-z^2}$$

which is C^∞ on $U_1 \cap U_3$.

$$\phi_6 \circ \phi_4^{-1}(x,z) = \phi_6(x, -\sqrt{1-x^2-z^2}, z) = \sqrt{1-x^2-z^2}$$

which is C^∞ on $U_4 \cap U_6$.

$$\phi_2 \circ \phi_5^{-1}(x,y) = \phi_2(x,y,\sqrt{1-x^2-y^2}) = -\sqrt{1-x^2-y^2}$$

which is C^∞ on $U_5 \cap U_2$. So $\{(U_i,\phi_i)\}_{i=1}^6$ is an atlas on S^2. Hence, $S^2 = \bigcup_{i=1}^6 U_i$ is a smooth manifold.

Definition 2.1.9. A function $f : M \to \mathbb{R}$ is *smooth (or C^∞)* at a point p in M, if there is a chart (U,ϕ) which contains p in the family $\{(U_i,\phi_i)\}$ of coordinate neighborhood of M such that $f \circ \phi^{-1}$, which is defined on the open subset $\phi(U)$ of $\mathbb{R}^m$, is C^∞ at p.

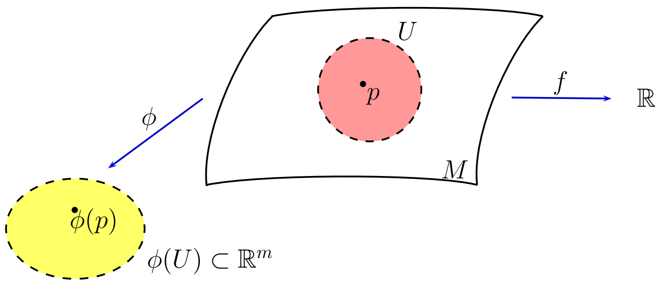

Figure 2.5 A function f is C^∞ if $f \circ \phi^{-1}$ is C^∞ at p.

This definition is independent of the choice of the local coordinate system.

Let N and M be two smooth manifolds with dimensions n and m, respectively and $h : M \to N$ a continuous map. Choose sufficiently small neighborhoods U and V of p and $h(p)$, respectively, such that $h(U) \subset V$ where U and V are in some coordinate neighborhoods $(x_1,x_2,\ldots,x_m)$ and $(y_1,y_2,\ldots,y_n)$. Then we can locally write

$$h(x_1,x_2,\ldots,x_m) = (y_1,y_2,\ldots,y_n), \tag{2.1.1}$$

where each y_i depends on $(x_1,x_2,\ldots,x_m)$. So, we can see that h is a function of m variables $x_1,x_2,\ldots,x_m$ so that

$$y_i = h_i(x_1,x_2,\ldots,x_m) \;\; where \;\; h_i : U \to \mathbb{R}, \; i = 1,2,\ldots,n$$

The map h in the equation (2.1.1) is smooth if and only if h_i is smooth for each i.

Definition 2.1.10. A map $h : M \to N$ is *smooth* on M if the map $h : M \to N$ is smooth at every point $p \in M$.

Definition 2.1.11. Let M be a smooth manifold without boundary and $f : M \to \mathbb{R}$ a smooth function. A point p_0 of M is a *critical point* of f if we have

$$\frac{\partial f}{\partial x_1}(p_0) = 0, \frac{\partial f}{\partial x_2}(p_0) = 0, \ldots, \frac{\partial f}{\partial x_m}(p_0) = 0$$

with respect to a local coordinate system $(x_1, x_2, \ldots, x_m)$ about p_0. A real number $c \in \mathbb{R}$ is a *critical value* of $f : M \to \mathbb{R}$ if $f(p_0) = c$ for some critical point p_0 of f. If the point p_0 is not a critical point then it is called a *regular point*.

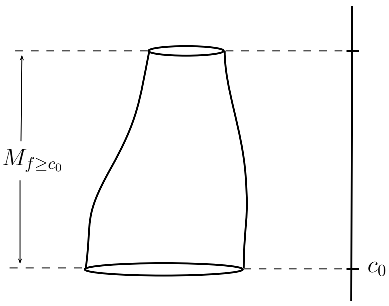

Figure 2.6 A subset $M_{f \geq c_0}$ of M.

For a smooth manifold M without boundary and a real smooth function $f : M \to \mathbb{R}$, the regular values of f help us to investigate M locally. Assume c_0 is a regular value of f, then define a subset $M_{f\geq c_0}$ of M by

$$M_{f\geq c_0} = \{p \in M \mid f(p) \geq c_0\},$$

it is a smooth m-manifold with boundary $\partial M_{f \geq c_0} = M_{f=c_0} = \{p \in M \ : \ f(p) = c_0\}$ (See Figure 2.6).

Example 2.1.12. Let $M = \mathbb{R}^m$ and define $f(x) = 1 - ||x||^2$ for $x = (x_1, \ldots, x_m) \in \mathbb{R}^m$ and consider the regular value $0 \in \mathbb{R}^m$. Then $M_{f \geq c_0} = \{x \in \mathbb{R}^m \ : \ ||x||^2 \leq 1\}$ which is the unit disc D^m, so the boundary $\partial M_{f \geq c_0} = \{x \in \mathbb{R}^m \ : \ ||x||^2 = 1\}$ is the unit sphere S^{m-1}. For another basic example consider the function g on $\mathbb{R}^m$ defined by $g(x) = x_m$. It has 0 as a regular value, since $\frac{\partial g}{\partial x_m} |_{p_0=0}= 1 \neq 0$. So $M_{g\geq c_0} = \mathbb{R}^m_+$, which is the upper half space.

Definition 2.1.13. Let M be an m-manifold and K a subset of M. If for every point p of K, there exists a C^∞-local coordinate system $(x_1, x_2, \ldots, x_m)$ about p of M such that K is described by the equations

$$x_{k+1} = x_{k+2} = \ldots = x_m = 0$$

with this coordinate neighborhood then we say that K is a *k-dimensional submanifold* of M. Every point of a submanifold admits a local coordinate system so K itself is a k-dimensional manifold.

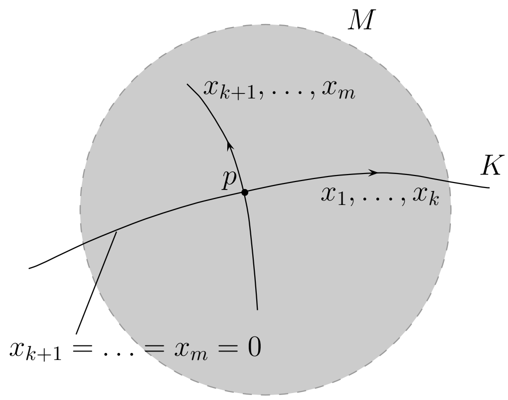

Figure 2.7 Submanifold K of M.

Theorem 2.1.14 (The Implicit function theorem)**.** *Let M be an m-manifold and $f : M \to \mathbb{R}$ a smooth function defined on M. If c_0 is regular value of f, then the subset*

$$f^{-1}(c_0) = \{p \in M \mid f(p) = c_0\}$$

of M is an $(m-1)$-dimensional submanifold of M.

Definition 2.1.15. A function $f : M \to \mathbb{R}$ is *smooth* at a point p in M if the following conditions hold:

- If p is an interior point of M, then f is smooth with respect to local coordinate system $(x_1, x_2, \ldots, x_m)$ in a suitably small neighborhood of p.

- If p is a boundary point of M and we express f with respect to a local coordinate system $(x_1, x_2, \ldots, x_m)$ with $x_m \geq 0$ in a sufficiently small neighborhood of p then $f(x_1, x_2, \ldots, x_m)$ can be extended to a smooth function of m variables

  $$\tilde{f}(x_1, x_2, \ldots, x_m)$$

  defined with respect to the coordinate system $(x_1, x_2, \ldots, x_m)$, $x_i \in \mathbb{R}$, $\forall i$. In other words $\tilde{f}|_{\{x_m \geq 0\}} = f$.

Definition 2.1.16. A homeomorphism $h : M \to N$ is a *diffeomorphism* if both $h : M \to N$ and $h^{-1} : N \to M$ are smooth functions on M and N, respectively.

A diffeomorphism $h : M \to N$ maps the boundary of M onto the boundary of N.

Definition 2.1.17 (W.Boothby)**.** Let $h : M \to N$ be a smooth function between smooth manifolds. If h is a homeomorphism and $h^{-1} : N \to M$ is smooth function then h is a diffeomorphism. The diffeomorphism $h : M \to N$ maps the boundary of M onto the boundary of N.

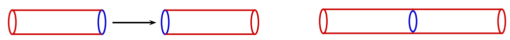

Figure 2.8 Attaching of two pipes.

Let us take two pipes then we can attach them to each other along their boundaries. Now we have a new pipe as in the above figure. This is a very simple case that can be seen in daily life. If we think these pipes are cylinders in $\mathbb{R}^3$ in a mathematical sense, then we see that we get a new cylinder when we attach cylinders to each other along their boundaries. This basic procedure gives a question. In generally, Does a new manifold, which is obtained by attaching two manifolds with boundaries along their boundaries, has also boundary?

The positive extension of this notion is given in (Matsumoto, 2002) with details.

Theorem 2.1.18 (Matsumoto (2002))**.** *Let M_1 and M_2 be manifolds with boundary and $\varphi : \partial M_1 \to \partial M_2$ a diffeomorphism between their boundaries. Then we can construct a new manifold $M = M_1 \cup_\varphi M_2$ by gluing the boundaries of M_1 and M_2 using the diffeomorphism φ, in other words, φ identifies each point p in ∂M_1 with the point $\varphi(p)$ in ∂M_2.*

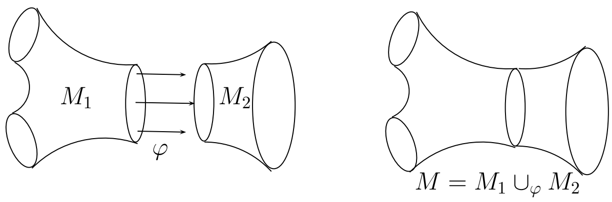

Figure 2.9 Gluing manifolds with boundary.

We assume that the boundaries of M_1 and M_2 have connected components more then one. In this case it is not allowed to attach a part of connected components of the boundary of M_1 to the entire boundary of M_2.

Theorem 2.1.19 (Matsumoto (2002)). *Let $M = M_1 \cup_\varphi M_2$ and $N = N_1 \cup_\psi N_2$ be the manifolds obtained by gluing manifolds along their boundaries where $\varphi : \partial M_1 \to \partial M_2$ and $\psi : \partial N_1 \to \partial N_2$ are diffeomorphism. Suppose that we have diffeomorphisms $h_1 : M_1 \to N_1$ and $h_2 : M_2 \to N_2$ such that*

$$\psi \circ h_1(p) = h_2 \circ \varphi(p)$$

for every point p in ∂M_1. Then there exists a diffeomorphism $H = h_1 \cup h_2 : M \to N$ obtained by gluing h_1 and h_2 along the boundaries.

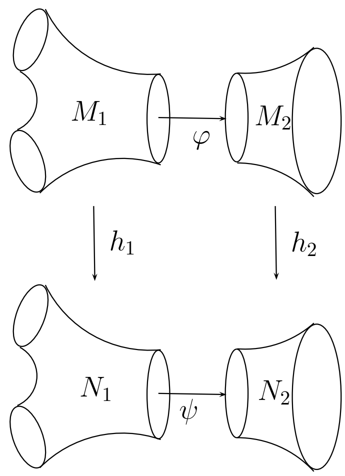

Figure 2.10 Gluing diffeomorphisms.

Now, we can examine an application of the above theorem.

Example 2.1.20. Let $M_1 = N_1 = \mathbb{R}^2_+ = \{(x, y) \mid y \geq 0\}$ be the upper half-plane and $M_2 = N_2 = \mathbb{R}^2_- = \{(x, y) \mid y \leq 0\}$ the lower half-plane. Let φ and ψ be the identity maps so that $M = N = \mathbb{R}^2$.

Define maps h_1 and h_2 by

$$\begin{cases} h_1(x, y) = (x + y, y) & (if\ y \geq 0) \\ h_2(x, y) = (x, y) & (if\ y \leq 0) \end{cases}$$

Then both $h_1 : M_1 \to N_1$ and $h_2 : M_2 \to N_2$ are diffeomorphisms but simply putting these maps together does not yield a diffeomorphism of $\mathbb{R}^2$ onto $\mathbb{R}^2$. Thus we modify h_1 as $\tilde{h}_1(x, y) = (x + \rho(y)y, y)$, where $\rho(y)$ is a smooth function such that $0 \leq \rho(y) \leq 1$,

$\rho(y) = 0$ for $y \leq \varepsilon$ and $\rho(y) = 1$ for $y \geq 2\varepsilon$ where $\varepsilon > 0$ is sufficiently small. So with this modification, the function $H : \mathbb{R}^2 \to \mathbb{R}^2$ such that $H|_{\mathbb{R}^2_+} = \tilde{h}_1$ and $H|_{\mathbb{R}^2_-} = \tilde{h}_2$ is a diffeomorphism which follows from the previous theorem.

2.2 Morse Functions and Gradient-like Vector Fields

In this section, we need a function which is defined on the manifold $M_{\leq c} = \{p \in M | f : M \to \mathbb{R} \ and \ f(p) = c\}$ to understand the topological changes of this manifold where the function f is called "Morse function". We give some definitions to construct Morse function and its existence theorem. Afterwards we give the Morse Lemma using the standard forms of Morse function around the critical points. Finally we investigate the vector fields, gradient and gradient-like vector fields and the vector fields of the quadratic forms from (Matsumoto, 2002).

2.2.1 Morse Functions

Definition 2.2.1. Let p_0 be a critical point of $f : M \to \mathbb{R}$. The *Hessian* of the function f at p_0 is defined to be the square matrix

$$H_f(p_0) = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(p_0) \right)_{m \times m}.$$

Notice that the Hessian matrix is symmetric, since $\frac{\partial^2 f}{\partial x_i \partial x_j}(p_0) = \frac{\partial^2 f}{\partial x_j \partial x_i}(p_0)$.

Remark 2.2.2. If one has a new coordinate system $(y_1, y_2, \ldots, y_m)$, the second order partial derivatives of f with respect to new coordinate system can be computed as

$$\frac{\partial^2 f}{\partial y_k \partial y_l}(p_0) = \sum_{i,j=1}^{m} \frac{\partial x_i}{\partial y_k}(p_0) \frac{\partial x_j}{\partial y_l}(p_0) \frac{\partial^2 f}{\partial x_i \partial x_j}(p_0).$$

Thus one has the following lemma.

Lemma 2.2.3 (Lemma 2.12, Matsumoto, 2002)**.** *Let* $(y_1, y_2, \ldots, y_m)$ *and* $(x_1, x_2, \ldots, x_m)$ *be two coordinate systems at a critical point* p_0 *and let* $\mathcal{H}_f(p_0)$ *and* $H_f(p_0)$ *be the*

Hessian of f with respect to these coordinate systems, respectively. Then $\mathcal{H}_f(p_0)$ and $H_f(p_0)$ are related as

$$\mathcal{H}_f(p_0) = J^t(p_0)H_f(p_0)J(p_0),$$

where $J(p_0) = \left(\frac{\partial x_i}{\partial y_j}(p_0)\right)_{m\times m}$ is a Jacobian (matrix) of the coordinate transformation from $(y_1, y_2, \ldots, y_m)$ to $(x_1, x_2, \ldots, x_m)$ evaluated at p_0.

Definition 2.2.4. A critical point p_0 is said to be *non-degenerate* if $\det H_f(p_0) \neq 0$. Otherwise it is called *degenerate*.

Corollary 2.2.5 (Matsumoto, 2002)**.** *The property of a critical point p_0 of a function $f : M \to \mathbb{R}$ being non-degenerate or degenerate does not depend on the choice of a coordinate system at p_0.*

Definition 2.2.6. A function $f : M \to \mathbb{R}$ is called a *Morse function* if every critical points of f are non-degenerate.

Example 2.2.7. Let us consider the unit sphere $S^2 = \{(x,y,z) \mid x^2 + y^2 + z^2 = 1\}$ in $\mathbb{R}^3$ and $f : S^2 \to \mathbb{R}$is a projection on the last vector such that $f(x,y,z) = z = \pm\sqrt{1-(x^2+y^2)}$ which is called the height function. If we take the first partial derivatives of f with respect to x, y and z, respectively then we get

$$\frac{\partial f}{\partial x} = \frac{\pm x}{\sqrt{1-(x^2+y^2)}}, \quad \frac{\partial f}{\partial y} = \frac{\pm y}{\sqrt{1-(x^2+y^2)}}, \quad \frac{\partial f}{\partial z} = 0.$$

These partial derivatives are equal to 0 when $(x,y,z) = (0,0,\pm 1)$. So $p_0 = (0,0,1)$ and $p_1 = (0,0,-1)$ are the north and south poles, respectively, are the critical points of f. In order to see that f is a Morse function, we must show that p_0 and p_1 are both non-degenerate. To reach this aim we compute the Hessian of f with respect to (x,y).

For the north pole $p_0 = (0,0,1)$

$$\frac{\partial^2 f}{\partial x^2}\,|_{p_0=(0,0,1)} = -1, \quad \frac{\partial^2 f}{\partial y^2}\,|_{p_0=(0,0,1)} = -1, \quad \frac{\partial^2 f}{\partial x \partial y}\,|_{p_0=(0,0,1)} = 0.$$

We get $H_f(p_0) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\det H_f(p_0) = 1 \neq 0$. So the north pole p_0 is a non-degenerate critical point of f. Similarly for the south pole $p_1 = (0,0,-1)$

$$\frac{\partial^2 f}{\partial x^2}\,|_{p_1=(0,0,-1)} = 1, \quad \frac{\partial^2 f}{\partial y^2}\,|_{p_1=(0,0,-1)} = 1, \quad \frac{\partial^2 f}{\partial x \partial y}\,|_{p_1=(0,0,-1)} = 0.$$

Now we get $H_f(p_1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\det H_f(p_1) = 1 \neq 0$. So, the south pole p_1 is also a non-degenerate critical point of f. As a result the height function $f(x,y,z) = z$ on S^2 is a Morse function with exactly two critical points.

Now, we investigate the Morse Lemma and the proof for a function of two variables. Afterwards we give the generalization of the lemma for a function f with m-variables.

Theorem 2.2.8 (Morse Lemma (Matsumoto, 2002))**.** *Let p_0 be a non-degenerate critical point of a function f of two variables. Then we can choose a local coordinate system (X,Y) about p_0 such that the function f which is expressed with respect to (X,Y) takes one of the following three standard forms*
i) $f(X,Y) = X^2 + Y^2 + c$

ii) $f(X,Y) = X^2 - Y^2 + c$

iii) $f(X,Y) = -X^2 - Y^2 + c$
where $c = f(p_0)$ is a constant and $p_0 = (0,0)$.

Proof. Choose any local coordinate system (x,y) near the point p_0 where $p_0(0,0)$ in these coordinates. Since p_0 is a non-degenerate critical point of f, we have

$$\det H_f(p_0) = \begin{vmatrix} \frac{\partial^2 f}{\partial x^2}(p_0) & \frac{\partial^2 f}{\partial x \partial y}(p_0) \\ \frac{\partial^2 f}{\partial y \partial x}(p_0) & \frac{\partial^2 f}{\partial y^2}(p_0) \end{vmatrix} = \frac{\partial^2 f}{\partial x^2}(p_0)\frac{\partial^2 f}{\partial y^2}(p_0) - \left(\frac{\partial^2 f}{\partial x \partial y}(p_0)\right)^2 \neq 0.$$

We claim that $\frac{\partial^2 f}{\partial x^2}(p_0) \neq 0$. Now we must prove this assumption for all cases. If we have $\frac{\partial^2 f}{\partial x^2}(p_0) \neq 0$ then the assumption is true. If $\frac{\partial^2 f}{\partial y^2}(p_0) \neq 0$ then $\frac{\partial^2 f}{\partial x^2}(p_0) \neq 0$, by interchanging the x-axis and the y-axis. So we can say that the assumption is also satisfied. Now suppose that $\frac{\partial^2 f}{\partial x^2}(p_0) \neq 0$, $\frac{\partial^2 f}{\partial y^2}(p_0) \neq 0$ and $\frac{\partial^2 f}{\partial x \partial y}(p_0) \neq 0$ then we get

$$H_f(p_0) = \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix}, \ a \neq 0.$$

Since p_0 is a non-degenerate critical point, we can say that $a \neq 0$. If we introduce a new local coordinate system (X,Y) by

$$x = X - Y, \ y = X + Y$$

then the Jacobian matrix is

$$J = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

for the change of coordinates from (X,Y) to (x,y). Thus the Hessian $\mathcal{H}_f$ with respect to (X,Y) becomes

$$\mathcal{H}_f(p_0) = J^t(p_0)H_f(p_0)J(p_0) = \begin{pmatrix} 2a & 0 \\ 0 & -2a \end{pmatrix}.$$

This equality satisfies that

$$\frac{\partial^2 f}{\partial X^2}(p_0) = 2a \neq 0, \ \frac{\partial^2 f}{\partial Y^2}(p_0) = -2a \neq 0.$$

So, we have shown that our claim is true for all cases. Now we use this assumption in the following part of the proof.

Suppose that we have a function $z = f(x,y)$ defined near the origin with $f(p_0) = 0$ where $p_0 = (0,0)$. From the fundamental fact of calculus there are functions $g(x,y)$ and $h(x,y)$ such that we can write

$$f(x,y) = xg(x,y) + yh(x,y)$$

in some neighborhood of the origin $p_0 = (0,0)$ such that

$$\frac{\partial f}{\partial x}(p_0) = g(p_0), \qquad \frac{\partial f}{\partial y}(p_0) = h(p_0).$$

Firstly we will prove this fact. Suppose that $z = f(x,y)$ is defined for the xy-plane and choose an arbitrary point (x,y) which will stay fixed. Consider a function $f(tx,ty)$ with parameter t. If we differentiate f with respect to t and integrate it, then we obtain the original form of f. In particular, if we look at its definite integral from 0 to 1 where $f(0,0) = 0$, then we have

$$\begin{aligned} f(x,y) &= \int_0^1 \frac{df(tx,ty)}{dt} dt \\ &= \int_0^1 \left\{ x\frac{\partial f}{\partial x}(tx,ty) + y\frac{\partial f}{\partial y}(tx,ty) \right\} dt \\ &= x\int_0^1 \frac{\partial f}{\partial x}(tx,ty)dt + y\int_0^1 \frac{\partial f}{\partial y}(tx,ty)dt \\ &= xg(x,y) + yh(x,y). \end{aligned} \tag{2.2.1}$$

On the other hand if we consider the second equation of the equality (2.2.1) then we define

$$g(x,y) = \int_0^1 \frac{\partial f}{\partial x}(tx,ty)dt \text{ and } h(x,y) = \int_0^1 \frac{\partial f}{\partial y}(tx,ty)dt.$$

Thus we have shown that $f(x,y) = xg(x,y) + h(x,y)$ and $\frac{\partial f}{\partial x}(0,0) = g(0,0)$, $\frac{\partial f}{\partial y}(0,0) = h(0,0)$ by substituting $(x,y) = (0,0)$. Since we assume that the origin $p_0 = (0,0)$ is a critical point of f.

Now, we have

$$\frac{\partial f}{\partial x}(0,0) = g(0,0) \text{ and } \frac{\partial f}{\partial y}(0,0) = h(0,0).$$

If we apply some procedure of calculus to the functions $g(x,y)$ and $h(x,y)$ with suitable functions $h_{11}, h_{12}, h_{21}, h_{22}$, then we can write

$$cg(x,y) = xh_{11}(x,y) + yh_{12}(x,y) \tag{2.2.2}$$

$$h(x,y) = xh_{21}(x,y) + yh_{22}(x,y). \tag{2.2.3}$$

When we put the equalities (2.2.2) and (2.2.3) on the equality of $f(x,y) = xg(x,y) + yh(x,y)$, we obtain

$$\begin{array}{r} f(x,y) = x(xh_{11}(x,y) + yh_{12}(x,y)) + y(xh_{21}(x,y) + yh_{22}(x,y)) \\ h(x,y) = x^2h_{11}(x,y) + xy(h_{12}(x,y) + h_{21}(x,y)) + y^2h_{22}(x,y) \end{array}.$$

If we set $H_{11} = h_{11}$, $H_{12} = \frac{h_{12}+h_{21}}{2}$ and $H_{22} = h_{22}$ then we have

$$f(x,y) = x^2H_{11} + 2xyH_{12} + y^2H_{22}. \tag{2.2.4}$$

From equality (2.2.4) we obtain the second partial derivatives of f with respect to (x,y) at $p_0 = (0,0)$ as follows:

$$\frac{\partial^2 f}{\partial x^2}(p_0) = 2H_{11}(p_0), \quad \frac{\partial^2 f}{\partial x \partial y}(p_0) = \frac{\partial^2 f}{\partial y \partial x}(p_0) = 2H_{12}(p_0), \quad \frac{\partial^2 f}{\partial y^2}(p_0) = 2H_{22}(p_0).$$

At the beginning of this proof we have assumed that $\frac{\partial^2 f}{\partial x^2}(p_0) \neq 0$. So $H_{11}(p_0) \neq 0$ in some neighborhood of p_0.

Now we define a new x-coordinate X near the origin $p_0 = (0,0)$ by

$$X = \sqrt{|H_{11}|}\left(x + \frac{H_{12}}{H_{11}}y\right). \tag{2.2.5}$$

The Jacobian from (x,y) to (X,y) evaluated at the origin is not zero, so (X,y) is also a local coordinate system for some neighborhood of $p_0 = (0,0)$.

Now we take the square of X then we get

$$\begin{aligned} X^2 &= |H_{11}|\left(x^2 + 2\tfrac{H_{12}}{H_{11}}xy + \tfrac{H_{12}^2}{H_{11}^2}y^2\right) \\ &= \begin{cases} H_{11}x^2 + 2H_{12}xy + \frac{H_{12}^2}{H_{11}}y^2 & if \;\; H_{11} > 0 \\ -H_{11}x^2 - 2H_{12}xy - \frac{H_{12}^2}{H_{11}}y^2 & if \;\; H_{11} < 0. \end{cases} \end{aligned}$$

For $H_{11} > 0$ then we substitute $x = \frac{X}{\sqrt{H_{11}}} - \frac{H_{12}}{\sqrt{H_{11}}}y$. If we put this substitution of x in the equality $f(x,y) = x^2H_{11} + 2xyH_{12} + y^2H_{22}$, we see that

$$f = X^2 + \left(H_{22} - \frac{H_{12}^2}{H_{11}}\right)y^2.$$

Similarly, for $H_{11} < 0$ we see that

$$f = -X^2 + \left(H_{22} - \frac{H_{12}^2}{H_{11}}\right)y^2.$$

If we consider

$$\begin{aligned} \det H_f(p_0) &= \tfrac{\partial^2 f}{\partial x^2}(p_0)\tfrac{\partial^2 f}{\partial y^2}(p_0) - \left(\tfrac{\partial^2 f}{\partial x \partial y}(p_0)\right)^2 \\ &= 4(H_{11}(p_0)H_{22}(p_0) - H_{12}(p_0)). \end{aligned}$$

We obtain

$$H_{11}(p_0)H_{22}(p_0) - H_{12}(p_0) = \frac{\det H_f(p_0)}{4} \neq 0$$

since $\det H_f(p_0) \neq 0$ where $p_0 = (0,0)$ is a non-degenerate critical point of f.

Now we choose a new y-coordinate near the origin $p_0 = (0,0)$ which is denoted by Y as follows:

$$Y = \sqrt{\left|\frac{H_{11}H_{22} - H_{12}^2}{H_{11}}\right|}\,y.$$

If we rewrite the equalities of $f = X^2 + \left(H_{22} - \frac{H_{12}^2}{H_{11}}\right)y^2$ for $H_{11} > 0$ and $f = -X^2 + \left(H_{22} - \frac{H_{12}^2}{H_{11}}\right)y^2$ for $H_{11} < 0$, then f has the following expression:

$$f = \begin{cases} X^2 + Y^2 & \text{if } H_{11} > 0 \text{ and } K > 0 \\ X^2 - Y^2 & \text{if } H_{11} > 0 \text{ and } K < 0 \\ -X^2 + Y^2 & \text{if } H_{11} < 0 \text{ and } K < 0 \\ -X^2 - Y^2 & \text{if } H_{11} < 0 \text{ and } K < 0 \end{cases}$$

where $K = H_{11}H_{22} - H_{12}^2$ and the standard form $f = -X^2 + Y^2$ is the "90° rotation" of the standard form $f = X^2 - Y^2$. □

Now we can give the Morse Lemma for f of m-variables.

Theorem 2.2.9 (Morse Lemma)**.** *Let p_0 be a non-degenerate critical point of $f : M \to \mathbb{R}$. Then we can choose a local coordinate system $(x_1, x_2, \ldots, x_m)$ about p_0 such that the coordinate representation of f with respect to these coordinates has the following standard form*

$$f = -x_1^2 - x_2^2 - \ldots - x_\lambda^2 + x_{\lambda+1}^2 + \ldots + x_m^2 + c,$$

where p_0 corresponds to the origin $(0, 0, \ldots, 0)$ and c is a constant which is equal to $f(p_0)$ and $0 \le \lambda \le m$.

Proof. See in Matsumoto (2002),pg(44,46) □

Remark 2.2.10. The number λ is the number of minus signs in the standard form. It is also the number of negative diagonal entries of the Hessian $H_f(p_0)$ after diagonalization.

Definition 2.2.11. The number λ is called the *index* of a non-degenerate critical point p_0, where it is an integer with $0 \le \lambda \le m$.

Example 2.2.12. Consider the function $z = xy$. The origin is the critical point of the function. The Hessian matrix at the origin is

$$H_f(0) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \det H_f(0) = -1 \neq 0.$$

So the origin is non-degenerate and the function is a Morse function. Now one can rewrite the function $z = xy$ using the new coordinates (X, Y) to construct the standard form

$$x = X - Y$$
$$y = X + Y.$$

One gets

$$z = xy = (X - Y)(X + Y) = X^2 - Y^2.$$

Therefore the index of the origin is 1 because the number of minus sign in the standard form at the origin, is 1.

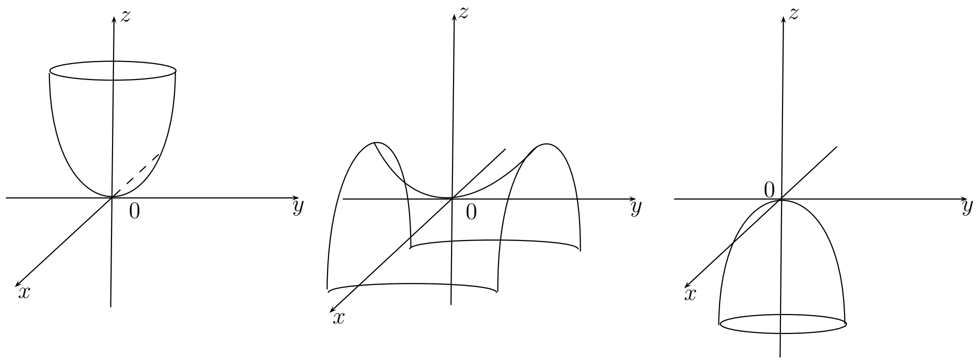

Figure 2.11 The graph of $z = x^2 + y^2$, $z = x^2 - y^2$ and $z = -x^2 - y^2$, respectively from left.

Now we give the "Existence Theorem for Morse Functions" on a closed manifold, that is, we consider the manifold as compact without boundary. Since a topological manifold is a topological space, we start with giving the definition of compactness for a topological space.

Definition 2.2.13. A topological space X is *compact* if among any infinite numbers of open sets $U_{n_1}, \ldots, U_{n_k}, \ldots$ where $k \in \mathbb{Z}$ which cover X:

$$X = \bigcup_{i=1}^{\infty} U_i,$$

there exist finite number of open sets $U_{n_1}, U_{n_2}, \ldots, U_{n_k}$ which still cover X, that is,

$$X = \bigcup_{i=1}^{k} U_i.$$

If the manifold M is compact then $M = \bigcup_{i=1}^{k} U_i$, where $U_1, U_2, \ldots, U_k$ are coordinate neighborhoods.

Definition 2.2.14. Let $f : M \to \mathbb{R}$ be a Morse function and $g : M \to \mathbb{R}$ a smooth function. Then f and g are *C^2-close* on a compact set K which is contained in a coordinate neighborhood of M if the following three inequalities hold at every point p in K:

1. $|f(p)-g(p)| < \varepsilon$, where $\varepsilon > 0$ is a positive real number.
2. $|\frac{\partial f}{\partial x_i}(p) - \frac{\partial g}{\partial x_i}(p)| < \varepsilon$, where $i = 1,2,\ldots,m$.
3. $|\frac{\partial^2 f}{\partial x_i \partial x_j}(p) - \frac{\partial^2 g}{\partial x_i \partial x_j}(p)| < \varepsilon$, where $i,j = 1,2,\ldots,m$.

Theorem 2.2.15 (The Existence of Morse Function)**.** *Let M be a closed m-manifold and $g : M \to \mathbb{R}$ a smooth function defined on M. Then there exists a Morse function $f : M \to \mathbb{R}$ arbitrarily C^2-close to $g : M \to \mathbb{R}$.*

Theorem 2.2.15 implies that there are many Morse functions defined on M. Because one can define many smooth functions on M and also there exists a function which is C^2-close to them. But defining a simple Morse function is not easy. This procedure is very complicated and technical. For example, a constant function $g : M \to \mathbb{R}$ such that $g(p) = c_0$, $\forall p \in M$, is certainly smooth, so there is a Morse function $f : M \to \mathbb{R}$ which is close to g, but f cannot be a constant function. If f is a constant function then $\frac{\partial^2 f}{\partial x_i \partial x_j} = 0$ and $\det H_f(p_0) = 0$. So the critical point p_0 is degenerate. Thus we cannot choose f as a constant function.

2.2.2 Gradient-like Vector Fields

"Gradient-like vector field" plays an important role when we consider how critical points of a given Morse function $f : M \to \mathbb{R}$ are related to each other when we investigate handle decompositions of the manifold M. To understand this relation firstly we give the definition of tangent vectors, vector fields and gradient-like vector fields from (Tu, 2011), (Boothby, 1986) and (Matsumoto, 2002).

Definition 2.2.16. Let M denote a smooth manifold of dimension m. We define the *tangent space* T_pM of M at p is the set of all mappings $X_p : C^\infty(p) \to \mathbb{R}$ satisfy the two conditions, $\forall \alpha,\beta \in \mathbb{R}$ and $f,g \in C^\infty(p)$.

1. $X_p(\alpha f + \beta g) = \alpha(X_p f) + \beta(X_p g)$

2. $X_p(fg) = (X_p(f))g(p) + f(p)(X_p(g))$

with the vector space operations in T_pM defined by

$$(X_p + Y_p)f = X_p f + Y_p f$$

$$(\alpha X_p)f = \alpha(X_p f)$$

where $X_p \in T_pM$ is the *tangent vector* of M at p.

Example 2.2.17. Let $\gamma : (a,b) \to \mathbb{R}^m$ be a smooth curve in $\mathbb{R}^m$ which is defined by the coordinates $(x_1, x_2, \ldots, x_m)$ of $\mathbb{R}^m$ as follows

$$\gamma(t) = (x_1(t), x_2(t), \ldots, x_m(t))$$

where $a < t < b$, $0 \in (a,b)$ and $\gamma(0) = p$. The tangent vector of $\gamma(t)$ is the velocity vector.

For example; the velocity vector v of the curve $\gamma(t)$ at $t = 0$ is given by

$$v = \gamma'(0) = \frac{d\gamma}{dt}(0) = \left(\frac{dx_1}{dt}(0), \frac{dx_2}{dt}(0), \ldots, \frac{dx_m}{dt}(0)\right).$$

If γ lies in a smooth manifold M, then this velocity vector $\gamma'(0) = \dfrac{d\gamma}{dt}(0)$ is also a tangent vector which is in T_pM of M at p.

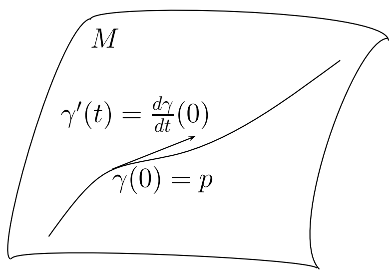

Figure 2.12 The curve in M.

Definition 2.2.18. Let $f : M \to \mathbb{R}$ be a real valued function defined in a neighborhood of $p \in M$. We consider a smooth curve $\gamma : (a,b) \to M$ with local coordinate system $(x_1, \ldots, x_m)$ of M, then γ can be written as

$$\gamma(t) = (x_1(t), \ldots, x_m(t))$$

such that $\gamma(0) = p$ and $\gamma'(t) = \left(\frac{dx_1}{dt}(t), \ldots, \frac{dx_m}{dt}(t)\right) \in T_pM$. If we restrict f to the curve γ and differentiate of f along γ at p then we get

$$
\begin{aligned}
\frac{df}{dt}(\gamma(t))|_{t=0} &= \frac{d}{dt} f(x_1(t), x_2(t), \ldots, x_m(t))|_{t=0} \\
&= \sum_{i=1}^{m} \frac{\partial f}{\partial x_i}(p) \frac{dx_i}{dt}(0) \\
&= \sum_{i=1}^{m} v_i \frac{\partial f}{\partial x_i}(p) = v \cdot f \qquad (2.2.6)
\end{aligned}
$$

where $\gamma'(0) = (v_1, v_2, \ldots, v_m) \in T_pM$. The derivative of f along γ at $t = 0$ is called the *directional derivative* of f in the direction v.

Remark 2.2.19. One can easily see that $v \cdot f > 0$ if and only if the function $f(\gamma(t))$ is an increasing function of t near $t = 0$.

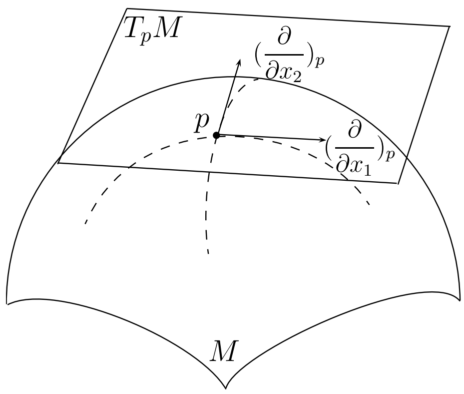

Figure 2.13 Basis Vectors of T_pM.

Definition 2.2.20. If $(U, \phi) = (U, (x_1, x_2, \ldots, x_m))$ is a coordinate neighborhood in M then a *vector field* X on U is given by

$$X = \xi_1 \frac{\partial}{\partial x_1} + \xi_2 \frac{\partial}{\partial x_2} + \ldots + \xi_m \frac{\partial}{\partial x_m}$$

where $\xi_1, \xi_2, \ldots, \xi_m$ are functions defined on U and $\left(\frac{\partial}{\partial x_i}\right)$ are the basis vectors of the tangent space of M, $\forall i = 1, \ldots, m$. This means that X is a function which assigns to each point p in U to the tangent vector

$$\xi_1(p)(\frac{\partial}{\partial x_1})_p + \xi_2(p)(\frac{\partial}{\partial x_2})_p + \ldots + \xi_m(p)(\frac{\partial}{\partial x_m})_p.$$

If the coefficient functions $\xi_1, \xi_2, \ldots, \xi_m$ are smooth then we say that X is a *smooth vector field* on U. So, we say that X is a smooth vector field on M if X is smooth on every coordinate neighborhood (U_i, ϕ_i) on M.

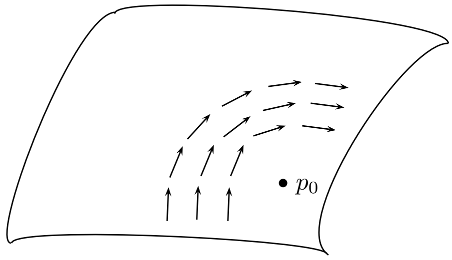

Figure 2.14 A Vector Field Around p_0

Let $f : U \subseteq \mathbb{R}^m \to \mathbb{R}$ be a function defined by coordinate system $(x_1, x_2, \ldots, x_m)$. We know that $grad\ f = \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_m}\right)$ can be defined as a vector field according to fundamental facts in calculus. If we take the directional derivative of f in the direction $grad\ f$, which can be denoted by X_f,then we get

$$X_f = \frac{\partial f}{\partial x_1}\frac{\partial}{\partial x_1} + \frac{\partial f}{\partial x_2}\frac{\partial}{\partial x_2} + \ldots + \frac{\partial f}{\partial x_m}\frac{\partial}{\partial x_m}.$$

So X_f is called the *gradient vector field* of the function f where we choose $\xi_i = \frac{\partial f}{\partial x_i}$, $\forall i$.

If we differentiate f with respect to X_f, we have

$$X_f \cdot f = \left(\sum_{i=1}^{m} \frac{\partial f}{\partial x_i}\frac{\partial}{\partial x_i}\right) \cdot f = \sum_{i=1}^{m} \left(\frac{\partial f}{\partial x_i}\right)^2 = |gradf|^2 \geq 0.$$

The second equality follows from (2.2.6).

If p is not a critical point of f then $(X_f \cdot f)_p > 0$. If p is a critical point of f then

$$\frac{\partial f}{\partial x_1}(p) = \frac{\partial f}{\partial x_2}(p) = \ldots = \frac{\partial f}{\partial x_m}(p) = 0.$$

In the other words, the gradient vector field of f always points in a direction into which f is increasing, outside the critical points of f.

Example 2.2.21. Let $f = -x_1^2 - x_2^2 - \ldots - x_\lambda^2 + x_{\lambda+1}^2 + \ldots + x_m^2$ be a Morse function in a standard form. The gradient vector field of f seems in the following figure, $0 < \lambda < m$ and written as

$$-2x_1\frac{\partial}{\partial x_1} - 2x_2\frac{\partial}{\partial x_2} - \ldots - 2x_\lambda\frac{\partial}{\partial x_\lambda} + 2x_{\lambda+1}\frac{\partial}{\partial x_{\lambda+1}} + \ldots + 2x_m\frac{\partial}{\partial x_m}.$$

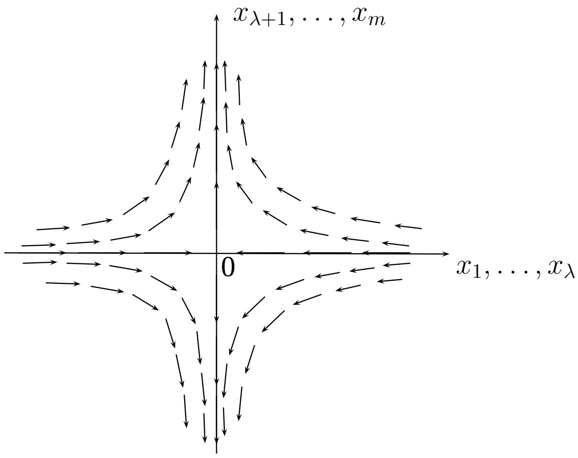

Figure 2.15 Gradient Vector Field of f for $0 < \lambda < m$.

In case of $\lambda = 0$,the standard form of the above Morse function f is $x_1^2 + x_2^2 + \ldots + x_m^2$. So, the gradient vector field of f is written as

$$2x_1 \frac{\partial}{\partial x_1} + 2x_2 \frac{\partial}{\partial x_2} + \ldots + 2x_m \frac{\partial}{\partial x_m}$$

which seems like in the following figure.

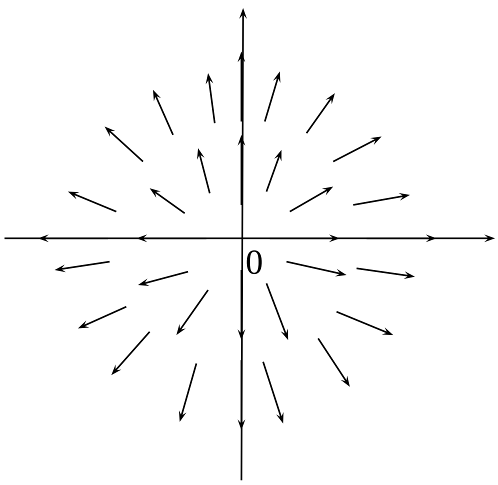

Figure 2.16 Gradient Vector Field of f for $\lambda = 0$.

If $\lambda = m$ then $-x_1^2 - x_2^2 - \ldots - x_m^2$ is the standard form of f. So, the gradient vector field of f is written as

$$-2x_1 \frac{\partial}{\partial x_1} - 2x_2 \frac{\partial}{\partial x_2} - \ldots - 2x_m \frac{\partial}{\partial x_m}$$

which seems like in the following figure.

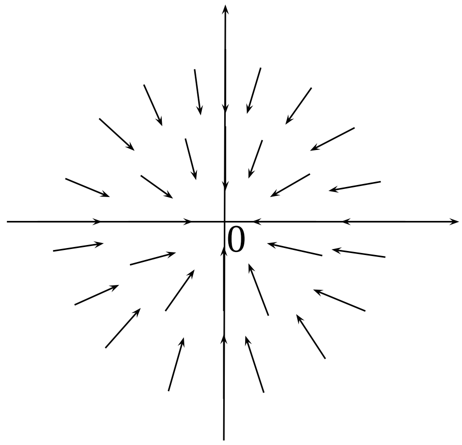

Figure 2.17 Gradient Vector Field of f for $\lambda = m$.

Definition 2.2.22. We say that X is a *gradient-like vector field* for a Morse function $f : M \to \mathbb{R}$ if the following conditions hold:

1. $X_f \cdot f > 0$ away from the critical points of f.
2. If p_0 is a critical point of f of index λ, then p_0 has a sufficiently small neighborhood U with a suitable coordinate system $(x_1, x_2, \ldots, x_m)$ such that f has a standard form

$$f = -x_1^2 - x_2^2 - \ldots - x_\lambda^2 + x_{\lambda+1}^2 + \ldots + x_m^2 + f(p_0)$$

   and X can be written as its gradient vector field:

$$X = -2x_1 \frac{\partial}{\partial x_1} - 2x_2 \frac{\partial}{\partial x_2} - \ldots - 2x_\lambda \frac{\partial}{\partial x_\lambda} + 2x_{\lambda+1} \frac{\partial}{\partial x_{\lambda+1}} + \ldots + 2x_m \frac{\partial}{\partial x_m}$$

   where $\frac{\partial f}{\partial x_i} = -2x_i$ for $0 < i \leq \lambda$ and $\frac{\partial f}{\partial x_i} = 2x_i$ for $\lambda + 1 \leq i \leq m$.

Remark 2.2.23. If we look at the definitions of vector field, gradient vector field and gradient-like vector field we can reach some results as follows:

i) A gradient vector field is also a vector field such that coefficient functions are the partial derivatives of a function which is defined on a chart of a smooth manifold M. A gradient vector field of a function at $p \in M$ gives an information locally, that is; we can survey the magnetic field only on some neighborhood of $p \in M$

ii) If $p \in M$ is a critical point of f then

$$\frac{\partial f}{\partial x_1}(p) = \frac{\partial f}{\partial x_2}(p) = \ldots = \frac{\partial f}{\partial x_m}(p) = 0.$$

Furthermore; $\langle X_f \cdot f \rangle_p = 0$. If $p \in M$ is not a critical point of f then $\langle X_f \cdot f \rangle_p > 0$, that means that if we move in the direction of gradient vector field of f then we see that the points of f which are outside of the critical points always increase.

iii) A gradient-like vector field is a generalization and globalization of a gradient vector field in Morse theory. In other words, one can obtain a vector field which is the gradient-like vector field by a gradient vector field of a Morse function in Morse theory. One can see that f is a Morse Function which follows from the definition of a gradient-like vector field for f (See in Matsumoto (2002)). Moreover a gradient-like vector field is a gradient vector field at any critical point p of f. We know that f has a standard form in a neighborhood of $p \in M$ from the Morse Lemma. So we always calculate the coefficient functions with respect to the standard form of the Morse function of f at any critical point $p \in M$. On the other hand the conditions of the definition of a gradient-like vector field says that the derivative of f is always positive outside the critical points of f and also near the critical point of f.

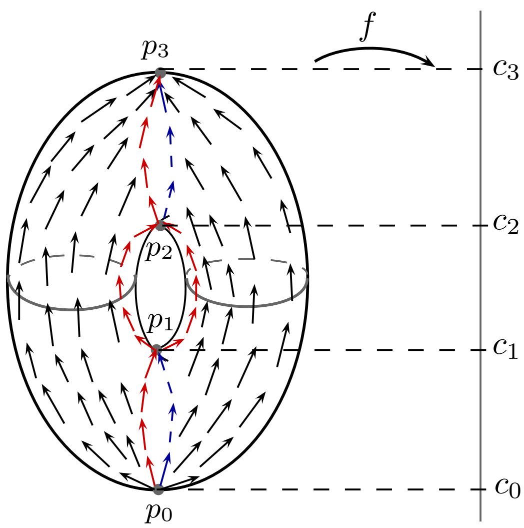

Figure 2.18 Gradient-like vector field of a height function f on the torus $\mathbb{T}$.

For example, let us think that f as a height function on a torus, then the gradient-like vector field of f points "upward". See Figure (2.18).

Theorem 2.2.24 (Matsumoto (2002)). *Suppose that $f : M \to \mathbb{R}$ is a Morse function on a compact manifold M. Then there exists a gradient-like vector field X for f.*

Proof. See Matsumoto (2002). □

Now we give the fundamental theorem of Morse theory which gives an information about the topology and geometry of a part of the manifold using the regularity of Morse function in some interval.

Theorem 2.2.25. *If $f : M \to \mathbb{R}$ has no critical value in the interval $[a,b]$, then $M_{[a,b]}$ is diffeomorphic to the product*

$$f^{-1}(a) \times [0,1]$$

where $M_{[a,b]} = \{p \in M \mid a \leq f(p) \leq b\}$.

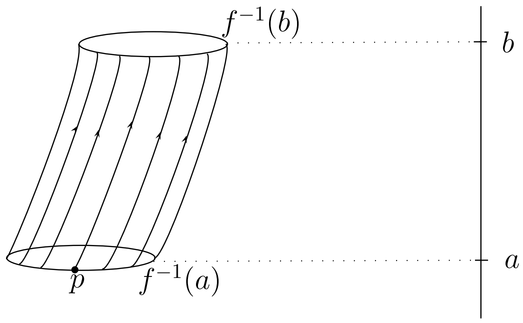

Figure 2.19 If there is a no critical point in $[a,b]$, then $M_{[a,b]} \approx f^{-1}(a) \times [0,1]$.

2.3 Handle Decompositions of Manifolds

In the previous section, we have described the theory of Morse functions for general manifolds. In this section, we use the result from Section 2.2 to search handlebody decompositions of compact manifolds. We give the general theory of handlebodies.

Let M be a closed manifold and $f : M \to \mathbb{R}$ a Morse function on M. The set $M_{\leq c}$ is given by

$$M_{\leq c} = \{p \in M \mid f(p) \leq c\}$$

for a value c of f. We investigate how $M_{\leq c}$ changes when the parameter c changes.

Theorem 2.3.1. *If f has no critical values in the real interval $[a,b]$, then M_a and M_b are diffeomorphic: $M_a \cong M_b$.*

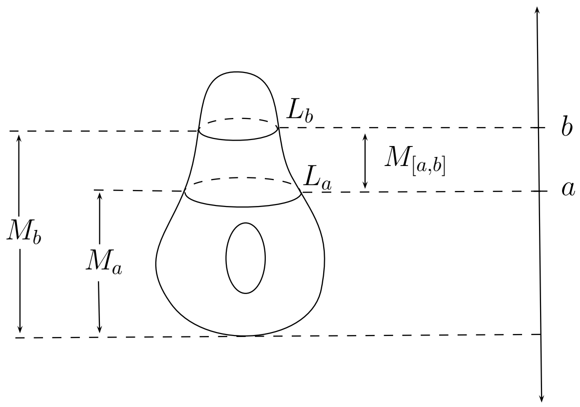

Figure 2.20 There is no critical point between L_a and L_b.

Let $f(p_i) = c_i$ where p_i's are the critical points of f. We have $c_0 < c_1 < \ldots < c_n$, where c_0 is a minimum value and c_n is a maximum value of the Morse function f.

We start with the minimum value. Now, suppose that p_0 is the only point which gives the minimum value then we write f in a standard form

$$f = x_1^2 + x_2^2 + \ldots + x_m^2.$$

So the values of f cannot be less than c_0 and the standard form f has no negative signs. This means that the index of p_0 is necessarily 0.

For sufficiently small positive number $\varepsilon > 0$, we have $M_{c_0-\varepsilon} = 0$ and $M_{c_0+\varepsilon} = \{(x_1, x_2, \ldots, x_m) \mid x_1^2 + x_2^2 + \ldots + x_m^2 \leq \varepsilon\}$, that is, $M_{c_0+\varepsilon}$ is diffeomorphic to the m-dimensional disk D^m.

If c_i is not the minimum value of index 0 then we add an m-dimensional disk facing upward and $M_{c_0+\varepsilon}$ becomes diffeomorphic to $M_{c_0-\varepsilon} \sqcup D^m$ (disjoint union).

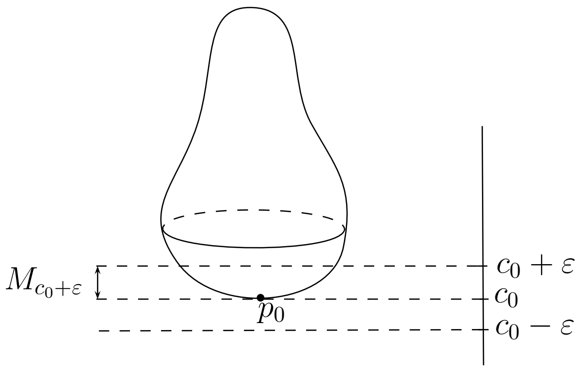

Figure 2.21 $M_{c+\varepsilon}$ diffeomorphic to m-dimensional disc D^m.

Example 2.3.2. Let M be a 3-dimensional closed manifold and c_0 the minimum value of the Morse function of $f : M \to \mathbb{R}$. Then $M_{c_0+\varepsilon} = \{(x_1, x_2, x_3) \mid x_1^2 + x_2^2 + x_3^2 \leq \varepsilon\}$ is diffeomorphic to 3-dimensional disk, which is an ordinary solid ball.

Example 2.3.3. Let $f : M \subseteq \mathbb{R}^3 \to \mathbb{R}$ be a Morse function with a coordinate system (x, y) about $p_0 \in M$ which is a critical value such that $f(p_0) = c_0$. We can write f locally in a standard form

$$f = x^2 + y^2 + c_0,$$

so the index of f is zero at p_0. If c_0 is a minimum value of f, then $M_{c_0-\varepsilon} = \emptyset$. Since $M_{c_0+\varepsilon}$ is defined by

$$\begin{aligned} M_{c_0+\varepsilon} &= \{p \in M | f(p) \leq c_0 + \varepsilon\} \\ &= \{(x, y) | x^2 + y^2 \leq \varepsilon\} \end{aligned}$$

which is a bowl diffeomorphic to the 2-disk D^2 as in the following figure.

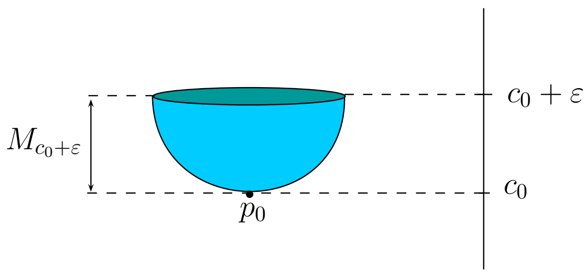

Figure 2.22 The case when the index of p_0 is zero.

Let us assume that p_0 is a critical point of f such that $f(p_0) = c_0$ and the index of f at p_0 is 1. The standard form of f is $f = -x^2 + y^2 + c_0$. So the handle at p_0 is the 1-handle $D^1 \times D^1$.

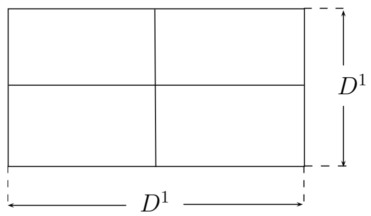

Figure 2.23 A 1-handle $D^1 \times D^1$

The following Figure 2.24 is a graph near a critical point of index one.

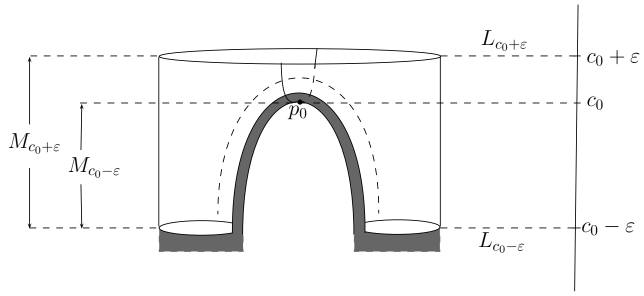

Figure 2.24 $M_{c_0+\varepsilon} \cong M_{c_0-\varepsilon} \cup D^1 \times D^1$.

Let us assume that p_0 is a critical point of f such that $f(p_0) = c_0$ and the index of f at p_0 is 2. The standard form of f is $f = -x^2 - y^2 + c_0$, then the 2-handle is diffeomorphic to D^2.

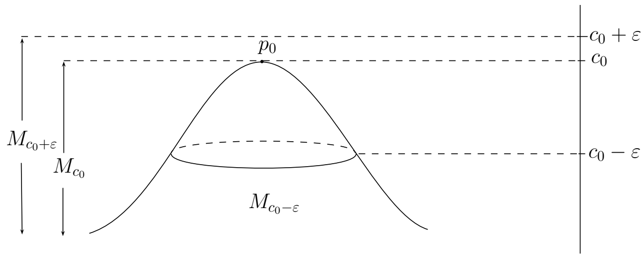

Figure 2.25 $f = -x^2 - y^2 + c_0$.

Definition 2.3.4. The m-dimensional (upward) disk which appears at a critical point of index 0 is called a 0-*handle* or an *m-dimensional* 0-*handle*. (See Figure 2.27 for a 0-handle)

Definition 2.3.5. The product of the λ-disk and $(m-\lambda)$-disk $D^\lambda \times D^{m-\lambda}$ which appears at a critical point of index λ is called a λ-*handle* or *m-dimensional* λ-*handle*.

If c_n is the maximum value of $f : M \to \mathbb{R}$ such that $f(p_n) = c_n$, $p_n \in M$ then f cannot take values larger than c_n, so that the quadratic point of the following standard form for f has no positive signs:

$$f = -x_1^2 - x_2^2 - \ldots - x_m^2 + c_n.$$

Thus, the index of p_n is necessarily m.

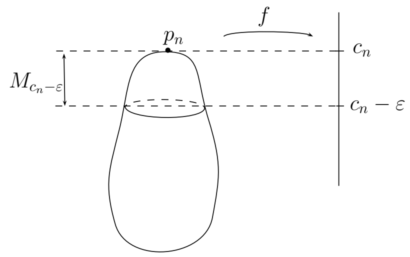

Figure 2.26 $f = -x_1^2 - x_2^2 - \ldots - x_m^2 + c_n$.

Example 2.3.6. Let $S^2 = \{(x,y,z) \mid x^2 + y^2 + z^2 = 1\}$ be a 2-dimensional sphere in $\mathbb{R}^3$ and $f : S^2 \to \mathbb{R}$ the Morse function which is defined by $f(x,y,z) = z$, where $z = \mp\sqrt{1 - x^2 - y^2}$. So $p_0 = (0,0,-1)$ and $p_1 = (0,0,1)$ are the critical points of f and $\{-1,1\}$ are the set of critical values of f. The index of the critical points are 0 and 2, respectively. Thus we have a 0-handle and 2-handle which are glued to each other along their boundaries. Hence, we obtain S^2. Because 0-handle diffeomorphic to (upward) 2-disk and 2-handle diffeomorphic to (downward) 2-disk.

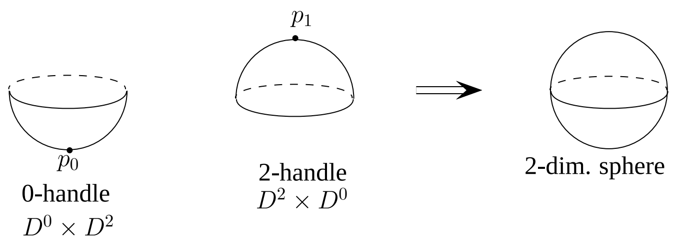

Figure 2.27 The gluing of 0-handle and 2-handle along their boundaries.

If we take a coordinate system about the critical point p_i of index λ and we get f in the standard form,

$$f = -x_1^2 - x_2^2 - \ldots - x_\lambda^2 + x_{\lambda+1}^2 + \ldots + x_m^2 + c_i.$$

The situation around p_i can be given in Figure 2.28 as follows: The darkly shaded area in the Figure 2.28 depicts $M_{c_i-\varepsilon}$, by setting $f(p) \le c_i - \varepsilon$, that is,

$$x_1^2 + x_2^2 + \ldots + x_\lambda^2 - x_{\lambda+1}^2 - \ldots - x_m^2 \ge \varepsilon$$

The light shaded area corresponds to the inequalities

$$x_1^2 + x_2^2 + \ldots + x_\lambda^2 - x_{\lambda+1}^2 - \ldots - x_m^2 \leq \varepsilon$$

$$x_{\lambda+1}^2 + x_{\lambda+2}^2 + \ldots + x_m^2 \leq \delta,$$

where $0 < \delta < \varepsilon$. Thus, this lightly shaded area is called an *m-dimensional handle of index λ* or an *m-dimensional λ-handle* which is constructed by the direct product of the λ-disk and the $(m-\lambda)$-disk

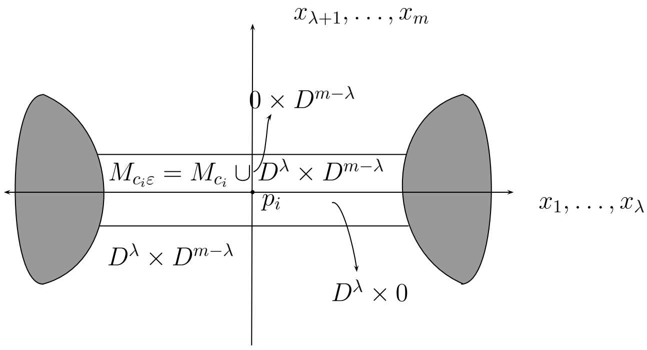

Figure 2.28 A λ-handle

Definition 2.3.7. The λ-disk

$$D^\lambda \times 0 = \{(x_1, x_2, \ldots, x_\lambda, 0, \ldots, 0) \mid x_1^2 + x_2^2 + \ldots + x_\lambda^2 \leq \varepsilon\}$$

is the *core* of the λ-handle $D^\lambda \times D^{m-\lambda}$ and the $m-\lambda$-disk

$$0 \times D^{m-\lambda} = \{(0, \ldots, 0, x_{\lambda+1}, \ldots, x_m) \mid x_{\lambda+1}^2 + \ldots + x_m^2 \leq \delta\}$$

intersecting the core is the *co-core*.

Remark 2.3.8. The core and co-core intersect transversely at the origin, that is, they intersect orthogonally in some coordinate system. The name co-core means that its dual to the core.

The next theorem describes the changes of $M_{\leq c} = \{p \in M \mid f(p) \leq c\}$ as the parameter c passes through the critical value c_i of index λ by attaching a λ-handle around the critical point.

Theorem 2.3.9. *The set $M_{c_i+\varepsilon}$ is diffeomorphic to the manifold obtained by attaching a λ-handle to $M_{c_i-\varepsilon}$:*

$$M_{c_i+\varepsilon} \cong M_{c_i-\varepsilon} \cup D^{\lambda} \times D^{m-\lambda}.$$

If we look at the Figure 2.28 we see that the space $M_{c_i-\varepsilon}$ with a λ-handle $D^{\lambda} \times D^{m-\lambda}$ attached is not "smooth" at the corners of the boundary where the handle meets $M_{c_i-\varepsilon}$. Smoothness of this corners makes $M_{c_i-\varepsilon} \cup D^{\lambda} \times D^{m-\lambda}$ into a C^{∞} manifold M' as follows: (See Figure 2.29.)

We can use the gradient-like vector field X of f for the proof of Theorem 2.3.9 although we are not going to give the proof here. We can see in the Figure 2.29 that the vector field X, after leaving the boundary $\partial M'$ of M', continuous to flow upward till it reaches the boundary $\partial M_{c_i+\varepsilon}$ of $M_{c_i+\varepsilon}$. This shows that M' is diffeomorphic to $M_{c_i+\varepsilon}$.

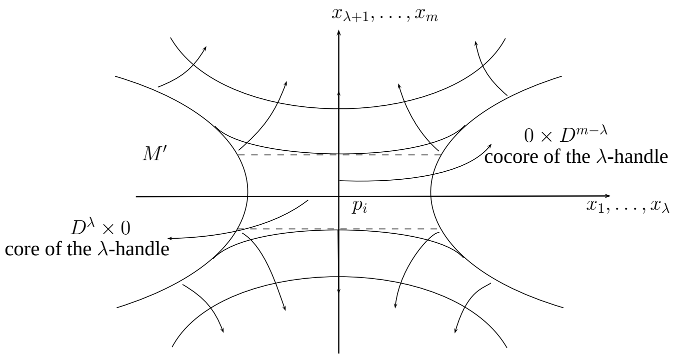

Figure 2.29 The smoothed-out manifold M' after attaching a λ-handle to $M_{c_i-\varepsilon}$.

Now, we can explain the change of the values of f on the core $D^{\lambda} \times 0$ of a λ-handle. Since p_i is the critical points of f such that $f(p_i) = c_i$ and p_i is the origin of the local coordinate system. The value of f is decreasing as it approaches the boundary of the disk and f takes the value $c_i - \varepsilon$. So the core $D^{\lambda} \times 0$ is on "upsidedown" λ-disk. The function f attains the critical value c_i at the center p_i on the co-core $0 \times D^{m-\lambda}$, it takes the value $c_i + \delta$ and its value increases as it approaches to the boundary of the disk. Thus the co-core is an "upright" disk, that is, the core face is down and the co-core faces are up, hence the shape of λ-handle looks like a horse saddle and p_i is a saddle point.

Example 2.3.10. Let M be a torus with one-genus.

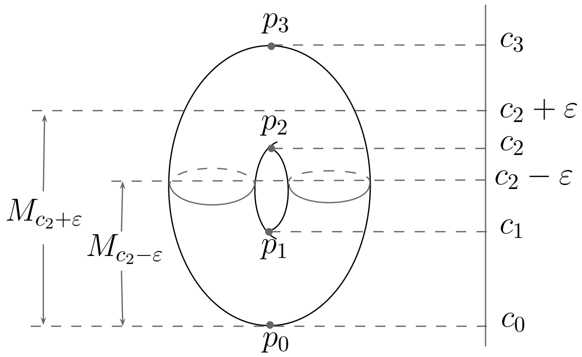

Figure 2.30 There is a critical point between $M_{c_2-\varepsilon}$ and $M_{c_2+\varepsilon}$.

We obtain $M_{c_2+\varepsilon}$ by attaching a 1-handle to $M_{c_2-\varepsilon}$. Thus, $M_{c_2+\varepsilon} \cong M_{c_2-\varepsilon} \cup D^1 \times D^1$.

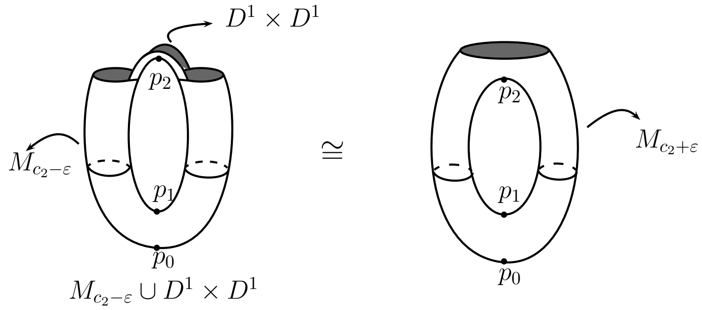

Figure 2.31 $M_{c_2-\varepsilon} \cup D^1 \times D^1 \cong M_{c_2+\varepsilon}$.

We need some preparation for the theorem of handle decomposition of a manifold.

Definition 2.3.11. Let $D^\lambda \times D^{m-\lambda}$ be a λ-handle and c_i a critical value of M. We attach λ-handle $D^\lambda \times D^{m-\lambda}$ to $M_{c_i-\varepsilon}$ by passing $\partial D^\lambda \times D^{m-\lambda}$ along the boundary $\partial M_{c_i-\varepsilon}$ of $M_{c_i-\varepsilon}$. We define a map

$$\varphi : \partial D^\lambda \times D^{m-\lambda} \to \partial M_{c_i-\varepsilon}$$

which is attaching the λ-handle to $M_{c_i-\varepsilon}$ along their boundaries. The map φ is smooth "embedding" which is called the attaching map of the λ-handle. (See Figure2.32.)

Definition 2.3.12. A manifold with boundary obtained from D^m by attaching handles of various indices one after another

$$D^m \cup D^{\lambda_1} \times D^{m-\lambda_1} \cup \ldots \cup D^{\lambda_n} \times D^{m-\lambda_n}$$

is called an *m-dimensional handlebody*. A handlebody is defined in three steps as follows:

1. A disk D^m is an m-dimensional handlebody.

2. The manifold $D^m \cup_{\varphi_1} D^{\lambda_1} \times D^{m-\lambda_1}$ obtained from D^m by attaching a λ_1-handle with an attaching map of class C^∞, $\varphi_1 : \partial D^{\lambda_1} \times D^{m-\lambda_1} \to \partial D^m$ is an m-dimensional handlebody, denoted by $\mathcal{H}(D^m;\varphi_1)$.

3. If $N = \mathcal{H}(D^m;\varphi_1,\varphi_2,\ldots,\varphi_{i-1})$ is an m-dimensional handlebody, then the manifold

$$N \cup_{\varphi_i} D^{\lambda_i} \times D^{m-\lambda_i}$$

is obtained from N by attaching a λ_i-handle $D^{\lambda_i} \times D^{m-\lambda_i}$ with an attaching map of class C^∞,where $\varphi_i : \partial D^{\lambda_i} \times D^{m-\lambda_i} \to \partial N$ and $\mathcal{H}(D^m;\varphi_1,\ldots,\varphi_{i-1},\varphi_i)$ is an m-dimensional handlebody.

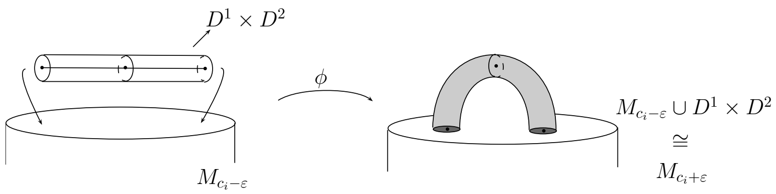

Figure 2.32 A 1-handle.

Theorem 2.3.13 (Handle decomposition of a manifold)**.** *A Morse function $f : M \to \mathbb{R}$ is given on a closed manifold M, a structure of a handlebody on M is determined by f. The handles of this handlebody correspond to the critical points of f and the indices of the handles coincide with the indices of the corresponding critical points.*

This theorem implies that M can be expressed as a handlebody and it is called a handle decomposition of M.

Example 2.3.14. Let $S^m = \{(x_1,\ldots,x_m,x_{m+1}) \mid x_1^2 + \ldots + x_m^2 + x_{m+1}^2 = 1\}$ be the m-dimensional sphere and define a function $f : S^m \to \mathbb{R}$ by

$$f(x_1,\ldots,x_m,x_{m+1}) = x_{m+1},$$

which is a height function with respect to $(m+1)-th$ coordinate. Thus, f is a Morse function and there are only two critical points of f, $(0,0\ldots,-1)$ and $(0,0\ldots,1)$ and their indices are 0 and m, respectively. The handle decomposition of S^m is

$$S^m = D^0 \times D^{m-0} \cup D^m \times D^{m-m} = D^m \cup D^m.$$

Theorem 2.3.15. *(Matsumoto (2002)) If there is a Morse function $f : M \to \mathbb{R}$ on an m-dimensional closed manifold M with only two critical points, then M is homeomorphic to S^m. Furthermore, if $m \leq 6$ then M is diffeomorphic to S^m.*

The following example is a motivating example. Since we can construct a suitable Morse function f then we can find the non-degenerate critical points of f. Finally we construct the handlebody decomposition of $\mathbb{R}P^m$ using the index of non-degenerate critical points of f.

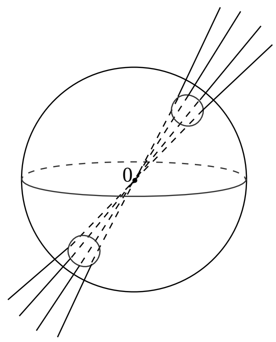

Figure 2.33 Projective space $\mathbb{R}P^m$.

Example 2.3.16 (Projective space $\mathbb{R}P^m$)**.** Let $\mathbb{R}P^m$ be the set of all lines through the origin in the (m+1)-dimensional Euclidean space $\mathbb{R}^{m+1}$. In other words, $\mathbb{R}P^m = S^m \diagup \sim$, $\sim$ identifies the antipodal points. For any point $(x_1,\ldots,x_m,x_{m+1})$ other then the origin, a line that passes through the points $(x_1,\ldots,x_m,x_{m+1})$ and 0 is uniquely determined. Since the line is a "point" of $\mathbb{R}P^m$. The elements of $\mathbb{R}P^m$ are denoted by $[x_1 : \ldots : x_m : x_{m+1}]$. A necessary and sufficient condition for two lines to coincide in $\mathbb{R}^{m+1}$, one of the lines through the point $(y_1,\ldots,y_m,y_{m+1})$ and the origin 0, and the other through the point $(x_1,\ldots,x_m,x_{m+1})$ and the origin 0, is that there exists a non-zero real number α such that

$$(y_1,\ldots,y_m,y_{m+1}) = (\alpha x_1,\ldots,\alpha x_m,\alpha x_{m+1}).$$

Therefore, the above condition is a necessary and sufficient condition for two corresponding points in $\mathbb{R}P^m$ to coincide, that is,

$$[y_1 : \ldots : y_m : y_{m+1}] = [x_1 : \ldots : x_m : x_{m+1}].$$

If we take any point $[x_1 : \ldots : x_m : x_{m+1}]$ in $\mathbb{R}P^m$, we can choose α (in the above condition) such that

$$y_1^2 + \ldots + y_m^2 + y_{m+1}^2 = 1.$$

With this condition, $(y_1, \ldots, y_m, y_{m+1})$ is a point of the unit sphere S^m in $\mathbb{R}^{m+1}$. Furthermore, in $\mathbb{R}P^m$, $[y_1 : \ldots : y_m : y_{m+1}]$ is the same point as $[x_1 : \ldots : x_m : x_{m+1}]$. Therefore the mapping $S^m \to \mathbb{R}P^m$ which assigns $(y_1, \ldots, y_m, y_{m+1})$ to $[y_1 : \ldots : y_m : y_{m+1}]$ is an onto continuous mapping. Any given $[x_1 : \ldots : x_m : x_{m+1}]$ is assigned to the point $(y_1, \ldots, y_m, y_{m+1})$ of the unit sphere. We know that S^m is compact, hence its continuous image $\mathbb{R}P^m$ is also compact.

The map $S^m \to \mathbb{R}P^m$ is called the "projection". The projection is a 2-to-1 mapping which assigns the same point of $\mathbb{R}P^m$ to two points $(y_1, \ldots, y_m, y_{m+1})$ and $(-y_1, \ldots, -y_m, -y_{m+1})$ of S^m.

Define a function $f : \mathbb{R}P^m \to \mathbb{R}$ by

$$f([x_1 : \ldots : x_m : x_{m+1}]) = \frac{a_1 x_1^2 + \ldots + a_m x_m^2 + a_{m+1} x_{m+1}^2}{x_1^2 + \ldots + x_m^2 + x_{m+1}^2}$$

where $a_1, \ldots, a_m, a_{m+1}$ are arbitrarily choosen fixed real constants satisfying $a_1 < \ldots < a_m < a_{m+1}$. If we multiply all the x_i's simultaneously by a, the value of the function is unchanged.

For a fix subscript i, we can consider the set U_i consisting of points $[x_1 : \ldots : x_m : x_{m+1}]$ of $\mathbb{R}P^m$ with $x_i \neq 0$; then U_i is an open set of $\mathbb{R}P^m$. So there is an m-dimensional local coordinate system $(X_1, \ldots, X_m)$ on U_i defined as follows:

$$X_1 = \frac{x_1}{x_i}, \ldots, X_{i-1} = \frac{x_{i-1}}{x_i}, X_i = \frac{x_{i+1}}{x_i}, \ldots, X_m = \frac{x_{m+1}}{x_i}.$$

Now we obtain an expression representing f in terms of local coordinate system $(X_1, X_2, \ldots, X_m)$:

$$f(X_1, X_2, \ldots, X_m) = \frac{a_1 X_1^2 + \ldots + a_{i-1} X_{i-1}^2 + a_i + a_{i+1} X_i^2 + \ldots + a_{m+1} X_m^2}{X_1^2 + \ldots + X_{i-1}^2 + 1 + X_i^2 + \ldots + X_m^2}.$$

To find the critical values, we obtain

$$\frac{\partial f}{\partial X_m} = \frac{2a_{m+1}X_m^2(X_1^2+\ldots+X_{i-1}^2+1+X_i^2+\ldots+X_m^2) - 2X_m(a_1X_1^2+\ldots+a_i+\ldots+a_{m+1}X_m^2)}{(X_1^2+\ldots+X_{i-1}^2+1+X_i^2+\ldots+X_m^2)^2}$$

$$\frac{\partial f}{\partial X_m} = \frac{2X_m[(a_{m+1}-a_1)X_1^2+\ldots+(a_{m+1}-a_m)X_{m-1}^2+(a_{m+1}-a_i)]}{(X_1^2+\ldots+X_m^2+1)^2}$$

by differentiating f with respect to X_m. Since a_{m+1} is the largest real constant, $\frac{\partial f}{\partial X_m} = 0$ if and only if $X_m = 0$.

Now, we consider the restriction $f_{|X_m=0}$ of f on X_m:

$$f|_{X_m=0}(X_1,\ldots,X_{m-1}) = \frac{a_1X_1^2+\ldots+a_{i-1}X_{i-1}^2+a_i+a_{i+1}X_i^2+\ldots+a_mX_{m-1}^2}{X_1^2+\ldots+X_{i-1}^2+1+X_i^2+\ldots+X_{m-1}^2}.$$

If we differentiate $f_{|X_m=0}$ with respect to X_{m-1}, we see that the derivative is 0 if and only if $X_{m-1} = 0$, for the same reason as above. By using the same process, we see that the critical points of f on the coordinate neighborhood U_i must satisfy

$$X_i = \ldots = X_{m-1} = X_m = 0.$$

Next we differentiate f with respect to X_1 and use the fact a_1 is smaller than $a_2, a_3, \ldots, a_{m+1}$ to see that $\frac{\partial f}{\partial X_1} = 0$ if and only if $X_1 = 0$. Furthermore, we differentiate $f_{|X_1=0}$ with respect to X_2 to see that the derivative is 0 if and only if $X_2 = 0$. By repeating this process, then we get that the critical points of f on U_i must satisfy

$$X_1 = X_2 = \ldots = X_{i-1} = 0.$$

Hence, the only critical point of f on U_i is the origin $(0,\ldots,0)$ of the local coordinate system $(X_1, X_2, \ldots, X_m)$, which is the point $[0 : \ldots : 0 : 1 : 0 : \ldots : 0]$ in U_i, where the entry of the i^{th} coordinate is 1.

The Hessian $(\frac{\partial f}{\partial X_i \partial X_j})$ of f at this critical point is

$$\begin{pmatrix} 2(a_1-a_i) & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 2(a_2-a_i) & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 2(a_{i-1}-a_i) & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 2(a_{i+1}-a_i) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 2(a_{m+1}-a_i) \end{pmatrix}$$

where the diagonal entries are not 0 but all the other entries are zero. So the $\det(H_f) \neq 0$. Since the diagonal entries up to and including the $(i-1)^{th}$ entry are negative and the others are positive, as $a_1 < \ldots < a_m < a_{m+1}$. Therefore the critical point at the origin of U_i is non-degenerate and has index $i-1$. Also, the value of the function f at this point is a_i.

Since $\mathbb{R}P^m$ is covered by $(m+1)$ coordinate neighborhoods U_i $(i = 1,2,\ldots,m+1)$, we have shown the following:

The Morse function $f : \mathbb{R}P^m \to \mathbb{R}$ we have constructed here has $(m+1)$ critical points whose indices are $0,1,2,\ldots,m$ in an ascending order. Therefore, the handle decomposition of $\mathbb{R}P^m$ is

$$\mathbb{R}P^m = D^m \cup D^1 \times D^{m-1} \cup \ldots \cup D^{m-1} \times D^1 \cup D^m.$$

In particular the 1-dimensional projective space $\mathbb{R}P^1 = D^1 \times D^1$ is diffeomorphic to the circle S^1.

Furthermore, if $m = 2$ then we have a 2-dimensional projective space $\mathbb{R}P^2$, which is called *projective plane*. Since $\mathbb{R}P^2$ is a 2-dimensional closed manifold and its handle decomposition as follows

$$\mathbb{R}P^2 = D^2 \cup D^1 \times D^1 \cup D^2.$$

This decomposition consists of a 2-dimensional 0-handle, 1-handle and 2-handle attached in this order. At the beginning we have a 2-dimensional(upward) disk then we attach 1-handle $D^1 \times D^1$ to the boundary of 0-handle ∂D^2. There are two ways to attach a 1-handle to a 0-handle.

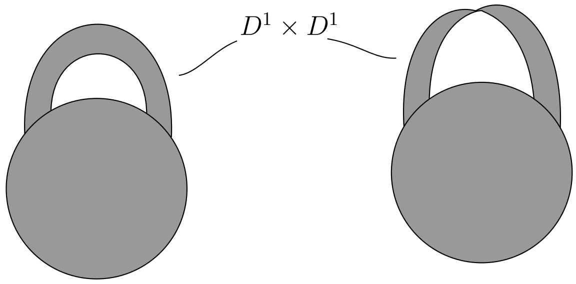

Figure 2.34 Two ways to attach a 2-dimensional 1-handle.

If we attach 2-handle to the left figure then it is impossible to obtain a closed surface (since annulus homeomorphic to $D^2 \cup D^1 \times D^1$) because D^2 is contractible.

So the situation must be as in the right of the Figure 2.34 in case of the projective plane. Here, the union of a 0-handle and 1-handle is homeomorphic to a Möbius band whose boundary is a single circle so we get a closed surface after attaching the 2-handle to the boundary of the Möbius band.

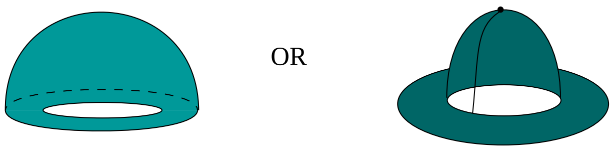

Figure 2.35 If we attach 2-handle to the $D^2 \cup D^1 \times D^1$ in the right of the Figure 2.34, then $D^2 \cup D^1 \times D^1 \cup D^2$ is not a closed surface.

CHAPTER THREE
HOMOLOGY and MORSE INEQUALITY

In this chapter, we study on homology groups of manifolds which tell us their topological invariants such as Betti numbers, Euler number in (Hatcher, 2002) and (Matsumoto, 2002). Although one can consider simplicial homology, singular homology, Čech homology, De Rham homology we prefer using cellular (Morse) homology which is associated to handle decomposition of a manifold and Morse theory. Afterwards, we give the Morse inequality which relates the number of critical points of a Morse function to the Betti number of a manifold and we see that this is the most fundamental result in Morse theory.

3.1 Cellular Homology

In this section we give the definition of an i-cell. An i- cell e^i is the interior of an i-dimensional disk D^i and its closure $\overline{e^i}$ is a closed i-cell. The i-disk D^i is called a *closed i-cell* and denoted by $\overline{e^i}$. For example in the zero-dimensional case e^0 and $\overline{e^0}$ are single points. We also give the definition of cell complex and see relation of cell complexes with handlebodies. See in (Hatcher, 2002) and (Matsumoto, 2002) for further information.

A cell complex X is defined by the following procedure:

1. Start with a discrete set X^0 whose points are 0-cells.

2. Inductively, the q-skeleton X^q obtained from $(q-1)$-skeleton X^{q-1} by attaching q-cells e_i^q via attaching maps $h_i : S^{q-1} \to X^{q-1}$. This means that X^q is the quotient space of the disjoint union $X^{q-1} \sqcup_i D_i^q$ of X^{q-1} with a collection of q-disks D_i^q under the identifications $x \sim h_i(x)$ for $x \in \partial D_i^q$. Thus as a set, $X^q = X^{q-1} \sqcup_i \bar{e}_i^q$ where each e_i^q is an open q-disk.

3. One can either stop this inductive process at a finite stage, and set $X = X^q$ for some $q < \infty$, or continue this process infinitely many times and set $X = \bigcup_q X^q$.

Definition 3.1.1. X^q is a *cell complex.*

We consider a space X obtained from a 0-cell by attaching a closed m-cell $\overline{e^m}$. We know that there is a unique continuous map $h : \partial\overline{e^m} \to \overline{e^0}$ which means that the map collapses $\partial\overline{e^m}$ to a single point $\overline{e^0}$. Then the m-dimensional cell complex $X = X^m = \overline{e^m} \cup_h \overline{e^0}$ is homeomorphic to the m-dimensional sphere S^m. Each m-dimensional unit sphere can be obtained in this way $S^m = e^0 \cup \overline{e^m}$. In Example 2.3.14, S^m has been constructed by attaching a 0-handle to the boundary of m-handle ∂D^m. This is a basic example to see that handlebody decomposition of a manifold has a cell complex structure.

Definition 3.1.2. Let X be a cell complex and e^q a q-dimensional cell in a q-dimensional Euclidean space, so that there is a finite sequence of vector fields $V = \langle v_1, v_2, \ldots, v_q\rangle$ such that they constitute an ordered basis of the tangent space at every point of e^q. Such a finite sequence is considered to determine an orientation of e^q. Assume that there is a similar finite sequence

$$W = \langle w_1, w_2, \ldots, w_q\rangle$$

on e^q, then we say that V and W determine the same orientation of e^q if the transformation matrix A from V to W has a positive determinant at every point of e^q.

If σ is a permutation of $\{1, 2, \ldots, q\}$, then a necessary and sufficient condition for $\langle v_{\sigma(1)}, v_{\sigma(2)}, \ldots, v_{\sigma(q)}\rangle$ and $\langle v_1, v_2, \ldots, v_q\rangle$ to determine the same orientation of e^q is that σ is an even permutation. A cell with an orientation is denoted by $\langle e^q\rangle$.

Suppose that the number of q-cells of X is denoted by k_q and each q-cell has an orientation. We can consider a vector space $C_q(X)$ with integer coefficient which has the set $\{\langle e^q_1\rangle, \ldots, \langle e^q_{k_q}\rangle\}$ as a basis. Each element c of $C_q(X)$ is a formal sum of the q-cells with integer coefficients such that

$$c = a_1\langle e^q_1\rangle + a_2\langle e^q_2\rangle + \cdots + a_{k_q}\langle e^q_{k_q}\rangle,$$

and it is called a *q-chain* of X. Formally $C_q(X)$ is called *q-dimensional chain group* of X. If X does not contain a q-cell clearly $C_q(X) = 0$.

There exist a boundary homomorphism $\partial_q : C_q(X) \to C_{q-1}(X)$ defined by the following equation for each oriented q-cell $\langle e^q \rangle$ in X,

$$\partial_q(\langle e_k^q \rangle) = a_{k1}\langle e_1^{q-1}\rangle + a_{k2}\langle e_2^{q-1}\rangle + \cdots + a_{kk_{q-1}}\langle e_{k_{q-1}}^{q-1}\rangle, \text{ where } k = 1,2,\ldots,k_q.$$

$\partial\langle e_k^q\rangle$ is attached to the $(q-1)$-skeleton by an attaching map $h : \partial\langle e_k^q\rangle \to X^{q-1}$. The orientation of $\partial\langle e_k^q\rangle$ is induced from the orientation of $\langle e_k^q\rangle$. The chain group $C_{q-1}(X)$ is generated by $\langle e_1^{q-1}\rangle, \ldots, \langle e_k^{q-1}\rangle$ and $a_{k1}, \ldots, a_{kk_{q-1}} \in \mathbb{Z}$. The boundary homomorphism ∂_q clearly satisfies the relation $\partial_q \circ \partial_{q+1}$, $\forall q$. Now we want to talk about the coefficients a_{kl} in the above equation, where $l = 1,2,\ldots,k_{q-1}$. Assume that $X^{q-1} \neq \emptyset$ and $\langle e_l^{q-1}\rangle$ be an oriented $(q-1)$-cell. If $\langle e_k^q\rangle$ is attached to $\langle e_l^{q-1}\rangle$ by using the surjective attaching map $h : \partial\langle e_k^q\rangle \to X^{q-1}$, then h winds $\langle e_l^{q-1}\rangle$ some number of times. Because of this reason the number a_{kl} is called the *covering degree* of h.

In detail, since $\partial\overline{e_k^q} \cong S^{q-1}$, the attaching map h of $\overline{e_k^q}$ is thought of as a map $h : S^{q-1} \to X^{q-1}$. Take a point p of e_l^{q-1} and a neighborhood U, if we perturb h continuously then we get a map of class C^∞ on $h^{-1}(U)$. If p is not a critical value of $h|_{h^{-1}(U)} : h^{-1}(U) \to U$, then the inverse image $h^{-1}(p)$ consists of a set of finitely many points $\{q_1, q_2, \ldots, q_r\}$ in S^{q-1}.

The following example makes it clear that the composition of boundary homomorphisms vanishes.

Example 3.1.3. Assume that X is a triangle which is obtained as follows:

$$X = (e_0^0 \sqcup e_1^0 \sqcup e_2^0) \cup_{h_1} (e_0^1 \sqcup e_1^1 \sqcup e_2^1) \cup_{h_2} (e_0^2)$$

where $i = 1,2,3$ and $h_1 : \partial\overline{e_i^1} \to (e_0^0 \sqcup e_1^0 \sqcup e_2^0)$ and $h_2 : \partial\overline{e_0^2} \to Y$, Y is a 1-skeleton of X, are attaching maps, where

$$\partial_3\langle e_0^0, e_1^0, e_2^0\rangle = \langle e_1^0, e_2^0\rangle - \langle e_0^0, e_2^0\rangle + \langle e_0^0, e_1^0\rangle$$

$$\partial_2(\partial_3\langle e_0^0, e_1^0, e_2^0\rangle) = \langle e_2^0\rangle - \langle e_1^0\rangle - \langle e_2^0\rangle + \langle e_0^0\rangle + \langle e_1^0\rangle - \langle e_0^0\rangle = 0.$$

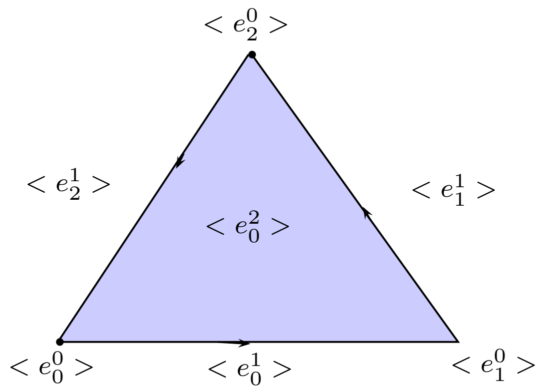

Figure 3.1 Triangle.

Definition 3.1.4. A sequence which consists all chain groups of a cell complex X with boundary homomorphisms

$$\ldots \to C_{q+1}(X) \xrightarrow{\partial_{q+1}} C_q(X) \xrightarrow{\partial_q} C_{q-1}(X) \xrightarrow{\partial_{q-1}} \cdots \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\partial_0} \{0\}$$

is called the *chain complex* of X.

For a fixed q, we set

$$\text{Ker } \partial_q := \{c \in C_q(X) \mid \partial_q(c) = 0\}$$

$$\text{Im } \partial_{q+1} := \{c \in C_q(X) \mid c = \partial_{q+1}(c') \text{ for some } c' \in C_{q+1}(X)\}$$

and call Ker ∂_q the *q-dimensional cyclic group*, Im ∂_{q+1} the *q-dimensional boundary group*. These are subgroups of $C_q(X)$ and we have

$$\text{Im } \partial_{q+1} \subset \text{Ker } \partial_q \subset C_q(X).$$

Definition 3.1.5. We define the q-dimensional homology group $H_q(X)$ by

$$H_q(X) := \frac{\text{Ker } \partial_q}{\text{Im } \partial_{q+1}}$$

The elements of the group $H_q(X)$ are called *homology classes*.

Example 3.1.6. Let us consider $S^m = e^0 \cup e^m$ and see the chain complex of it as follows:

For $m = 1$ we have a circle S^1 which is constructed by attaching the boundary of the closed 1-cell $\partial\overline{e^1}$ to a 0-cell e^0. So we get only a 1-cell and a 0-cell, there is no cell with any dimensional. Thus the chain complex of S^1 is in the following form

$$0 \to C_1(S^1) \to C_0(S^1) \to 0.$$

For $m = 2$, we get 2-dimensional sphere $S^2 = e^0 \cup e^2$. Since we have a 2-cell and a 0-cell then the chain complex of S^2 is of the form

$$0 \to C_2(S^2) \to 0 \to C_0(S^2) \to 0.$$

Generally for $m \geq 1$, m-dimensional sphere $S^m = e^0 \cup e^m$ have two kinds of cells, one of these are m-cells and the other is 0-cell. So the chain complex of the S^m is of the form

$$\cdots \to 0 \to C_m(S^m) \to 0 \to \cdots \to C_0(S^m) \to 0.$$

Here, $C_m(S^m) \cong C_0(S^m) \cong \mathbb{Z}$ and all the other groups $C_q(S^m)$ are 0. If $m \geq 2$, then $C_{m-1}(S^m) = 0$, and if $m = 1$, then it follows from

$$\partial(\langle e^1 \rangle) = \langle e^0 \rangle - \langle e^0 \rangle = 0.$$

Therefore, we obtain

$$\mathrm{Ker}\,\partial := \{c \in C_m(S^m) \mid \partial(c) = 0\} \cong \mathbb{Z}$$

$$\mathrm{Im}\,\partial := \{c \in C_m(S^m) \mid c = \partial(c') \text{ where } c' \in C_{m+1}(S^m)\} \cong 0$$

and the following result :

$$H_q(S^m) \cong \begin{cases} \mathbb{Z} & , \text{ if } q = m \text{ or } q = 0 \\ \{0\} & , \text{ otherwise.} \end{cases}$$

3.2 Morse Inequality

In this section we need some homotopy theory in order to understand the key point of the relation between handles and cells. Then we see the relation between Morse theory and homology theory by Morse inequality theorem. For more details please read (Matsumoto, 2002).

Definition 3.2.1. Let X and Y be topological spaces and $f, g : X \to Y$ continuous maps. We say that f is *homotopic to* g if there exists a continuous map $H : X \times [0,1] \to Y$ such that, $\forall x \in X$,

$$H(x,0) = f(x),\ H(x,1) = g(x).$$

This is a continuous map H is called a *homotopy* from f to g and it is denoted by $f \simeq g$.

Definition 3.2.2. Let X and Y be two cell complexes. If there exist continuous maps $f : X \to Y$ and $g : Y \to X$ such that

$$g \circ f \simeq id_X \text{ and } f \circ g \simeq id_Y$$

the cell complexes X and Y are said to be *homotopy equivalent* and denote by $X \simeq Y$.

Betti number has an important role in Morse theory which is directly related with Morse inequality. We give the Morse inequality and the Euler number formula.

Definition 3.2.3. The rank of the homology group $H_q(X)$ is called the *q-dimensional Betti number* of X and denoted by $b_q(X)$,

$$b_q(X) := \text{rank } H_q(X)\ q = 0, 1, 2, \ldots$$

Theorem 3.2.4. *(Euler number) Let X be an m-dimensional cell complex, and k_q the number of q-cell contained in X. Then we have*

$$\chi(X) = \sum_{q=0}^{m} (-1)^q k_q = \sum_{q=0}^{m} (-1)^q b_q(X) \text{ where } q \in \mathbb{Z}$$

which is called the Euler number *or* Euler-Poincaré characteristic.

For the proof of this theorem please read (Massey, 1991).

Example 3.2.5. The m-sphere S^m is obtained by $e^0 \cup e^m$, then we have

$$\chi(S^m) = (-1)^0 k_0 + (-1)^m k_m = 1 + (-1)^m$$

$\chi(S^m) = 2$ if m is even and $\chi(S^m) = 0$ if m is odd.

The Euler number $\chi(X)$ is a homotopy invariant of X, that is, if two spaces are homotopy equivalent then they have the same Euler number.

Example 3.2.6. The m-disk D^m is homotopy equivalent to a single point $\{x_0\}$. Indeed we have

$$k_q = \begin{cases} 1 & , \quad \text{if } q = m \\ 0 & , \quad q \neq m \end{cases}$$

for D^m and

$$\tilde{k}_q = \begin{cases} 1 & , \quad \text{if } q = 0 \\ 0 & , \quad q \neq 0 \end{cases}$$

for a single point $\{x_0\}$. So we get $\chi(D^m) = 1 = \chi(\{x_0\})$.

Theorem 3.2.7 (Morse Inequality in (Matsumoto, 2002))**.** *Let M be a closed m-manifold, and $f : M \to \mathbb{R}$ a Morse function on M. For the number k_λ of critical points of index λ and the λ-dimensional Betti number $b_\lambda(M)$ of M, the following inequality holds*

$$k_\lambda \geq b_\lambda(M). \tag{3.2.1}$$

Remark 3.2.8. We obtain the Betti numbers $b_\lambda(M)$ by using the shape of M, we can see that the number of critical points of a Morse function on M is restricted by the shape of M from the inequality 3.2.1. In a special case, if $b_\lambda(M) > 0$, then a Morse function on M must have at least one critical point of index λ. Morse theory deals with the shape of a manifold and functions on a manifold, and the Morse inequality gives this relation in a beautiful and understandable way.

Now we give the definition of mapping cylinder to understand the important theorem which gives the relation between cells and handles.

Definition 3.2.9. For a given continuous map $h : K \to X$ between topological spaces, the space obtained from X by attaching $K \times [0,1]$

$$X \cup_h K \times [0,1]$$

by identifying the point "at the bottom" $(x,0)$ of the direct product $K \times [0,1]$ and the point $h(x)$ of X, for each point x of K, is called the *mapping cylinder* of h, and denoted by M_h.

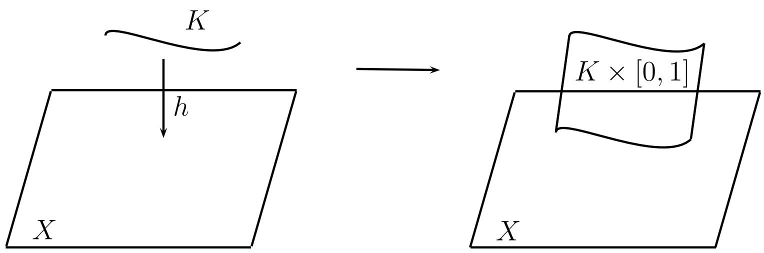

Figure 3.2 The mapping cylinder M_h of $h : K \to X$

Example 3.2.10. The unit m-disk D^m is homeomorphic to the mapping cylinder M_c of the map $c : S^{m-1} \to \{p\}$ which collapses the $(m-1)$-sphere to a single point p. We consider the map

$$D^m \to M_c = \{p\} \cup_c S^{m-1} \times [0,1]$$

sending the center 0 of D^m to the point p in M_c and sending the point x_t to the point (x,t) of M_c, where x_t is on the line segment from the center 0 of D^m to a point x on the boundary S^{m-1}, with the distance t from 0.

Theorem 3.2.11. *(Matsumoto, 2002) Let N be an m-dimensional handlebody. If the largest index of the handles contained in N is n, then N is homotopy equivalent to a certain n-dimensional cell complex X. More precisely:*

i. *There exists a continuous map $h : \partial N \to X$ from the boundary ∂N of N to X such that N is homeomorphic to the mapping cylinder M_h of h.*

ii. *There is a one-to-one correspondence between the i-handles of N and the i-cells of X.*

Corollary 3.2.12. *Let N be an m-dimensional handlebody. If k_q is the number of i-handles contained in N then the Euler number of N is given by*

$$\mathcal{X}(N) = \sum_{q=0}^{m} (-1)^q k_q.$$

Now we give the proof of the Morse inequality (3.2.7).

Proof. (Morse Inequality) Let $f : M \to \mathbb{R}$ be a Morse function on a closed m-manifold M. We denote the number of critical points of index q by k_q. Consider the handle decomposition defined by f. Then by Theorem 3.2.11. M can be identified with a cell complex X and there is a one-to-one correspondence between cells contained in X and the handles of M. The number of q-cells of X equals the number, k_q, of q-handles of M.

Consider the chain complex of X

$$\cdots \to C_q(X) \to C_{q-1}(X) \to \cdots \to C_1(X) \to C_0(X) \to \{0\}.$$

The rank of $C_q(X)$ is equal to k_q of q-cells of X for each $q = 0, 1, 2, \ldots, m$. Since the q-dimensional homology group $H_q(X)$ is obtained from a subgroup

$$\text{Ker } \partial_q = \{c \in C_q(X) \mid \partial_q(c) = 0\}$$

by taking the quotient by a smaller subgroup

$$\text{Im } \partial_{q+1} = \{c \in C_q(X) \mid c = \partial_{q+1}(c') \text{ for some } c' \in C_{q+1}(X)\}$$

of Ker ∂, we have

$$k_q = \text{rank } C_q(X) \geq \text{rankKer}\partial_q \geq \text{rank } H_q(X).$$

By identifying M and X we have

$$b_q(M) = b_q(X) = \text{rank } H_q(X)$$

so that we obtain $k_q \geq b_q(M) = \text{rank } H_q(X)$ where $q = 0, 1, 2, \ldots, m$ from the above inequality. □

Example 3.2.13. We want to determine the homology groups of the complex projective space $\mathbb{CP}^m$ of complex dimension m. We obtain a handle decomposition

$$\mathbb{CP}^m = h^0 \cup h^2 \cup \cdots \cup h^{2m},$$

of $\mathbb{CP}^m$ where h^λ is a λ-handle $D^\lambda \times D^{m-\lambda}$ where $\lambda = 0, 2, \ldots, 2m$. By using Theorem 3.2.11, we can represent $\mathbb{CP}^m$ as a cell complex from this handle decomposition

$$\mathbb{CP}^m = h^0 \cup h^2 \cup \cdots \cup h^{2m}.$$

So the chain groups $C_q(\mathbb{CP}^m)$ of $\mathbb{CP}^m$ are given by

$$H_q(\mathbb{CP}^m) = \begin{cases} \mathbb{Z}(\text{or } \mathbb{C}) & , \text{ if } q \text{ is even } 0 \leq q \leq 2m \\ \{0\} & , \text{ otherwise.} \end{cases}$$

CHAPTER FOUR
STRATIFIED SPACES

This chapter is a motivation for Morse theory for singular spaces and a preparation to understand two fundamental theorems of the generalizations of Morse theory for singular spaces which will be given in the following chapter. Singular spaces can be decomposed into smooth manifolds in order to get rid of singular points. To reach this aim, we study with stratification, Whitney stratification for smooth manifolds and transversality property from (Gibson et al., 1976) and (Chéniot, 2011). Furthermore we see why the stratification is not sufficient for our purpose and how Whitney stratification solves this insufficiency. In this chapter we give the definition of *"stratification"* and *"Whitney stratification"* for smooth manifolds.

4.1 Stratification and Some Basic Definitions

Definition 4.1.1. Let V be a subset of a smooth manifold M. Then a *stratification* of V is a partition $\mathcal{S}$ of V into submanifolds of M which satisfies the *"Locally Finiteness Condition"*, that is, every point in V has a neighborhood in M which meets only finitely many members of $\mathcal{S}$. The members of $\mathcal{S}$ are called the *strata*.

Example 4.1.2. There is a natural stratification on a manifold M with boundary. One of the stratum of M is the boundary ∂M and the other stratum is the complement $M \setminus \partial M$.

Now, we can give an example for stratification by dimension of an algebraic set $V \subseteq \mathbb{R}^m$.

Example 4.1.3. Recall that an algebraic set V in $\mathbb{R}^m$

$$V = \{p \in \mathbb{R}^m \mid f_i(p) = 0 \ where \ f_i \in \mathbb{R}[x_1, \ldots, x_m]\}$$

is the locus of zeros of a collection of polynomials. The set ΣV is the set of all singular points of V which is another algebraic set with strictly lower dimension and $V - \Sigma V$ is a smooth manifold. We can obtain a filtration

$$V = V_k \supseteq V_{k-1} \supseteq \ldots \supseteq V_0 \supseteq V_{-1} = \emptyset$$

by defining $V_k = V$ and

$$V_{i-1} = \begin{cases} \Sigma V_i & if \ \dim V_i = i \\ V_i & if \ \dim V_i < i \end{cases}$$

where $i = 1, 2, \ldots, k$. So we get finitely many differences $V_i - V_{i-1}$, each of which is a smooth manifold of dimension i or the empty set. Thus we construct a finite stratification of V by using a non-empty set $V_i - V_{i-1}$, $\forall i$, as strata.

Now, we give a motivation example by using Whitney umbrella and construct its stratification by defining strata. We use this example to understand differences of stratification and Whitney stratification. Moreover we see that why we need Whitney stratification in Morse theory for singular spaces.

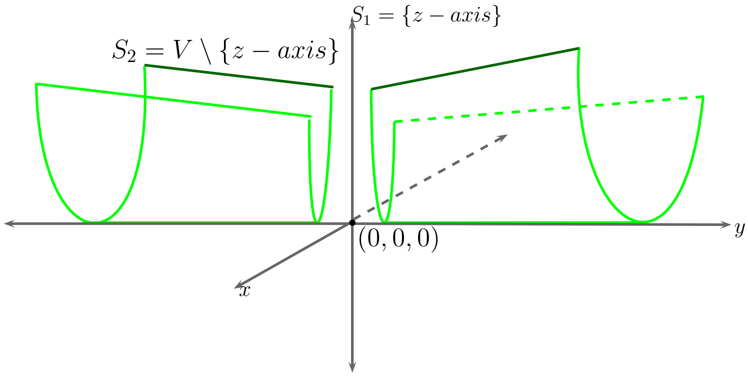

Figure 4.1 Stratification of Whitney Umbrella.

Example 4.1.4. (The Whitney umbrella) Let $V \subseteq \mathbb{R}^3$ be defined by $x^2 = zy^2$.

The filtration of V is given by the following setting

$$V_0 = \emptyset \qquad\qquad V_1 = \{z - axis\} \qquad\qquad V_2 = V.$$

We have stratum which are denoted by S_0, S_1 and S_2, where $S_0 = V_0 \setminus V_{-1} = \emptyset - \emptyset = \emptyset$, $S_1 = V_1 \setminus V_0 = \{z - axis\}$ and $S_2 = V_2 \setminus V_1 = V \setminus \{z - axis\}$. Since $S_0 = \emptyset$ we only have two strata and show them in Figure 4.1. Thus S_1 is a line and S_2 is a 2-dimensional smooth manifold of V with two connected components.

Example 4.1.5. An *affine algebraic variety* over $\mathbb{C}$ is the zero set of a finite family of polynomials $f_1, \dots, f_k \subseteq \mathbb{C}[x_1, \dots, x_n]$. Affine algebraic varieties over $\mathbb{C}$ can be made into stratified spaces. For example, the variety $C = \{x^2 + y^2 - z^2 = 0\} \subseteq \mathbb{C}^3$ is a double cone (It is easiest to picture the real points of such a space). It has a singular point at the origin. If we choose the strata to be $C \setminus \{origin\}$ and $\{origin\}$ then we can stratify the variety C.

Remark 4.1.6. We can stratify the affine varieties as follows: Let $V \subseteq \mathbb{C}^n$ be defined by the vanishing of the polynomials $f_1, \dots, f_k$. We denote $\dim V = n - k$, so the $\operatorname{codim} V = k$ in $\mathbb{C}^n$. We have a $(k \times n)$ Jacobian matrix which is determined by f_i in which the $(i, j) - th$ entry is the polynomial $\partial f_i / \partial x_j$. Let $\Sigma(V) \subseteq V$ be the subset on which this matrix has a rank less than k. Then $\Sigma(V)$ is a subvariety which is called the singular locus and we get $S_{n-k} = V \setminus \Sigma(V)$ is a manifold of complex dimension $(n - k)$. This is the first stratum of our variety. We can construct all strata by induction on complex dimension. Let $\Sigma^{n-j}(V) = \Sigma(\Sigma^{n-j-1}(V))$ be the singular locus of $\Sigma^{n-j-1}(V)$. So $S_{n-j} = \Sigma^{n-j}(V) \setminus \Sigma^{n-j+1}(V)$ are all stratum of the stratification of V, $\forall j$. Thus the Jacobian matrix is just the gradient of affine variety and the singular locus is given by all points on the varieties where the gradient vanishes. In both cases above the vanishing point is just the origin.

Example 4.1.7. Consider *Whitney cusp* which is an algebraic variety defined by the equation $x^3 + z^2 x^2 - y^2 = 0$.

The picture of "The Whitney cusp" in the real case is shown in Figure 4.2 since the complex codimension and dimension of V is 1 and 2, respectively. So we have a Jacobian matrix (gradient of V), where $k = 1$ and $n = 3$, which is

$$\begin{pmatrix} 3x^2 + 2xz^2 & -2y & 2zx^2 \end{pmatrix}$$

We know that $\Sigma V \subseteq V$ is a subvariety and is the singular locus ΣV of the Whitney cusp V is

$$\Sigma V = \{(x, y, z) \in \mathbb{R}^3 \mid x = y = 0 \ and \ z \in \mathbb{R}\}$$

and it is obviously a submanifold. So there are only two strata S_1 and S_2 in the stratification where $V_0 = \Sigma^2(V) = \emptyset,\ V_1 = \Sigma V = \{z-axis\}$ and $V_2 = V$

$$S_1 = V_1 \setminus V_0 = \Sigma V \setminus \Sigma^2(V) = \Sigma V \setminus \Sigma(z-axis) = \Sigma V \setminus \emptyset = \{z-axis\}$$
$$S_2 = V_2 \setminus V_1 = V \setminus \Sigma V = V \setminus \{z-axis\}.$$

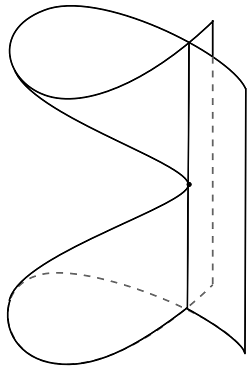

Figure 4.2 Whitney cusp in $\mathbb{R}^3$.

4.2 Whitney Stratification

In this section, we have studied Whitney stratification from (Gibson et al., 1976) and (Chéniot, 2011). Stratification aims to decompose a space (M, V) into simpler pieces, namely manifolds although this is not enough for stratified Morse theory. We require that the geometry of V looks the same at every point in a given stratum or the local topology type of (M, V) is constant on each stratum.

For this purpose the stratification of Figure 4.1 is unsufficient. Because, if we take a point on the z-axis and the figure of the intersection of a neighborhood centered at that point with V then we obtain three different types of the geometry of V (and also three different topology types of V).

We can easily see that something happens very special at the origin where the local topology type changes and geometry of V looks different for $z < 0,\ z = 0,\ z > 0$ in the given stratum S_i in the Whitney umbrella (See in the Example 4.1.4). In order to make the topology stable (fixed) of the smooth pieces of M. We work with three strata instead of two, then we obtain a second stratification of the Whitney umbrella. Namely, we construct our new stratification by choosing the sets $V_{-1} = \emptyset,\ V_0 = origin,\ V_1 = \{z-axis\},\ V_2 = V$. Then we get three strata $S_0 = \{(0,0,0)\},\ S_1 = \{z-axis\} \setminus \{(0,0,0)\},\ S_2 = V \setminus \{z-axis\}$. So the local topology type of the Whitney umbrella is constant on each stratum. There was a problem on the Whitney's mind and he asked that "What happens

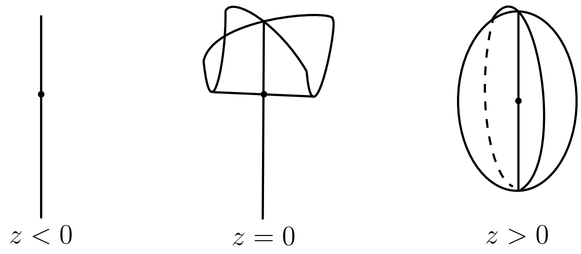

Figure 4.3 The topology type of Whitney umbrella when $z<0$, $z=0$ and $z>0$.

at a point on a stratum where the local topology type changes?". To give an answer to this question Whitney discovered his stratification.

Definition 4.2.1 (Kirwan (1988))**.** Let $V \subseteq M$ be a stratified space with strata S_i. We say that the stratification is a *Whitney stratification* if the Whitney's conditions (a) and (b) are satisfied for any two strata, $S_\alpha, S_\beta \in \mathcal{S}$ such that $S_\beta \subseteq \overline{S_\alpha}$. Whitney's conditions:

a) If a sequence of points $x_i \in S_\alpha$ tends to a point $x \in S_\beta$ then the tangent space $T_x S_\beta$ to x at S_β is contained in the limit of tangent spaces $T_{x_i} S_\alpha$ provided that this limit exists.

b) If a sequence of points $y_i \in S_\beta$ and $x_i \in S_\alpha$ both tend to the same point $x \in S_\beta$ then the limit of the lines joining x_i to y_i is contained in the limit of the tangent spaces to S_α at x_i, provided that both limits exist.

Let us explain these two conditions with aid of the following examples.

Example 4.2.2. We can say that the stratification of the Whitney umbrella in Figure 4.1 is not a Whitney stratification. Let us take a sequence $(x_i) \in S_2$ which tends to the origin $0 \in S_1$ then the tangent space $T_0 S_1$ at the origin is not contained in the limit of tangent spaces $T_{x_i} S_2$ which is denoted by K in the following figure.

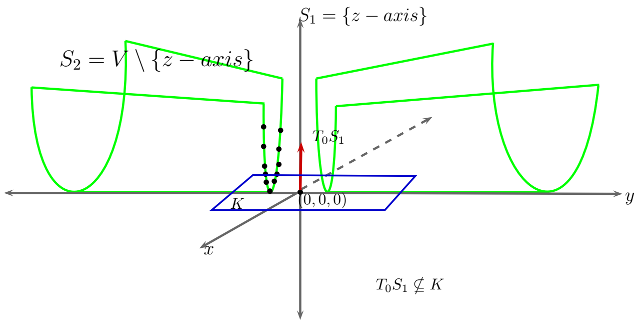

Figure 4.4 Tangent space of S_2 at the origin and the limit of tangent spaces of S_1 at x_i

On the other hand the stratification of Figure 4.4 is a Whitney stratification. Now, we can generalize that on a connected component of a stratum of a Whitney stratification the local topology type is constant (or the geometry of V looks the same at every point in a given stratum).

Example 4.2.3. The stratification of the Whitney cusp which is not a Whitney stratification in Figure 4.2, since S_1 and S_2 do not satisfy the Whitney's condition (b): Let us take a sequence $(x_i) \in S_1$ and $(y_i) \in S_2$ both tend to the origin $0 \in S_1$. If we look at the limit of the lines joining x_i to y_i which is denoted by l is not contained in the limit of the tangent spaces $T_{x_i}S_2$ which is denoted by K in the following figure.

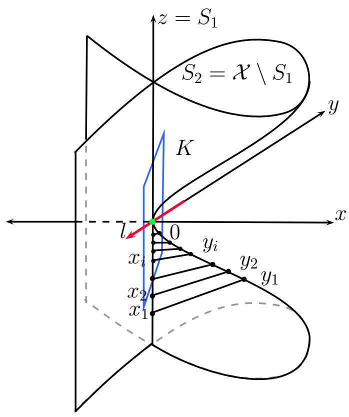

Figure 4.5 $\mathbb{R}^3 \supset \mathcal{M} : y^2 + x^3 - z^2x^2 = 0$

4.3 Transversality

In the previous section we have defined the stratification by dimension and Whitney stratification. In this section we give the transversality to build up new stratification using an old stratification by using (Gibson et al., 1976).

Definition 4.3.1. Let M be a smooth manifold and S_α and S_β two smooth submanifolds of M. We say that S_α and S_β are *transverse* at $x \in S_\alpha \cap S_\beta$ which is denoted by $S_\alpha \pitchfork_{x\in M} S_\beta$ if we have

$$T_x S_\alpha + T_x S_\beta = T_x M.$$

Indeed if $S_\alpha \pitchfork_{x\in M} S_\beta$ then $S_\alpha \cap S_\beta$ is a submanifold and $T_x(S_\alpha \cap S_\beta) = T_x S \cap T_x S_\beta$, i.e.,

$$\mathrm{codim}(S_\alpha \cap S_\beta) = \mathrm{codim}(S_\alpha) + \mathrm{codim}(S_\beta).$$

In the following examples we explain what transversality means:

Example 4.3.2. a) Let us take a plane $\mathcal{P}$ and a line l in $\mathbb{R}^3$ where $\mathcal{P} \cap l = \{p_0\}$. In this example, $\mathcal{P}$ and l intersect transversely at $p_0 \in \mathbb{R}^3$ if they intersect as in the following figure

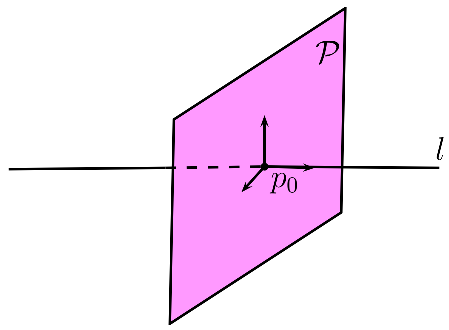

Figure 4.6 The transversality of $\mathcal{P}$ and l at p_0

Now, we can see the transversality of $\mathcal{P}$ and l by using the Definition 4.3.1. If we choose $M = \mathbb{R}^3$, $S_\alpha = \mathcal{P}$ and $S_\beta = l$ then we have $T_{p_0}\mathcal{P} \cong \mathcal{P}$ and $T_{p_0} l \cong l$. So the basis vector of $T_{p_0}\mathcal{P}$ and $T_{p_0} l$ generate $\mathbb{R}^3 \cong T_{p_0}\mathbb{R}^3$. Hence the condition of Definition 4.3.1 is satisfied.

Additionally, $\mathcal{P} \cap l = \{p_0\}$ is a 0-dimensional submanifold,

$$T_{p_0}(\mathcal{P} \cap l) \cong T_{p_0}\mathcal{P} \cap T_{p_0} l \cong \{p_0\}$$

and

$$\mathrm{codim}(\mathcal{P}) = 1, \ \ \mathrm{codim}(l) = 2 \ \text{ and } \ \mathrm{codim}(\{p_0\}) = 3.$$

If $\mathcal{P} \pitchfork l$ at $p_0 \in \mathbb{R}^3$ then we have

$$\mathrm{codim}(\mathcal{P}) + \mathrm{codim}(l) = \mathrm{codim}(\{p_0\}) = 3.$$

b) Let us take a plane $\mathcal{P} \in \mathbb{R}^3$ and a line l in $\mathcal{P}$ where $\mathcal{P} \cap l = \{l\}$. In this case, we want to say that $\mathcal{P}$ and l do not intersect transversely at $p_0 \in \mathbb{R}^3$ if they intersect in Figure 4.7.

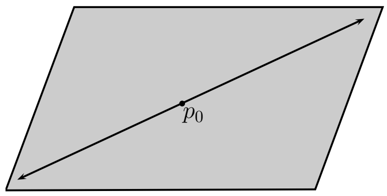

Figure 4.7 The transversality of a plane and a line when the line lies in the plane

Indeed, since $M = \mathbb{R}^3$, $S_\alpha = \mathcal{P}$ and $S_\beta = l$ then we have $T_{p_0}\mathcal{P} \cong \mathcal{P}$ and $T_{p_0} l \cong l$. So in this figure the basis vector of $T_{p_0}\mathcal{P}$ and $T_{p_0} l$ do not generate $\mathbb{R}^3$. On the other hand, $\mathcal{P} \cap l = l$ is a 1-dimensional smooth submanifold and $T_{p_0}(\mathcal{P} \cap l) \cong T_{p_0}\mathcal{P} \cap T_{p_0} l \cong T_{p_0} l$ but

$$\mathrm{codim}(\mathcal{P}) + \mathrm{codim}(l) = \mathrm{codim}(\{p_0\}) = 3$$

$$\mathrm{codim}(\mathcal{P} \cap l) = \mathrm{codim}(l) = 2.$$

Thus, these two equalities are not equal to each other.

Example 4.3.3. Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be two planes which intersect transversely in $\mathbb{R}^3$ along a line l as in the following figure.

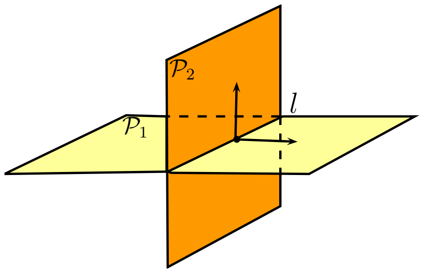

Figure 4.8 The transversality of $\mathcal{P}_1$ and $\mathcal{P}_2$

Indeed, the tangent space of $\mathbb{R}^3$ is generated by the tangent spaces of $\mathcal{P}_1$ and $\mathcal{P}_2$. Hence the transverselity condition is satisfied for $\mathcal{P}_1 \cap \mathcal{P}_2$. Additionally, if $\mathcal{P}_1 \pitchfork \mathcal{P}_2$ then

$$\operatorname{codim}(\mathcal{P}_1) = 1, \quad \operatorname{codim}(\mathcal{P}_2) = 1, \quad \operatorname{codim}(\mathcal{P}_1 \cap \mathcal{P}_2) = 2$$

then we get

$$\operatorname{codim}(\mathcal{P}_1) + \operatorname{codim}(\mathcal{P}_2) = \operatorname{codim}(l) = 2.$$

Example 4.3.4. Let l_1 and l_2 be two lines in $\mathbb{R}^3$ which intersect at point p_0 in $\mathbb{R}^3$, in the following figure.

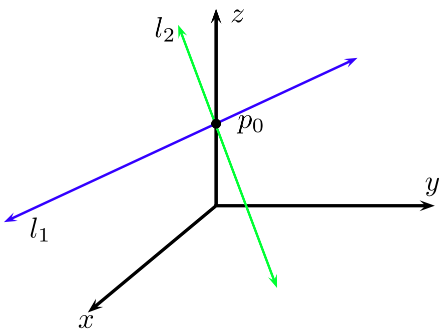

Figure 4.9 The transversality of two lines in $\mathbb{R}^3$

The tangent space of l_1 and l_2 do not generate the tangent space of $\mathbb{R}^3$ since their tangent spaces are homeomorphic to $\mathbb{R}$. Hence we say that l_1 and l_2 do not intersect transversely at p_0 in $\mathbb{R}^3$. We know that

$$\operatorname{codim}(l_1) = 2, \quad \operatorname{codim}(l_2) = 2, \quad \operatorname{codim}(\{p_0\}) = 3$$

then we get

$$\operatorname{codim}(l_1) + \operatorname{codim}(l_2) = 4 \;\text{ and }\; \operatorname{codim}(\{p_0\}) = 3.$$

So

$$\operatorname{codim}(l_1) + \operatorname{codim}(l_2) \neq \operatorname{codim}(\{p_0\}).$$

On the other hand if we take l_1 and l_2 in a plane $\mathcal{P}$ which intersect each other at a point p_0 in $\mathcal{P}$. We can say that l_1 and l_2 intersect transversely at p_0 in $\mathcal{P}$. Since the tangent spaces of l_1 and l_2 generate the tangent space of $\mathcal{P}$. Additionally $l_1 \pitchfork l_2$ at $p_0 \in \mathcal{P}$ and

$$\operatorname{codim}(l_1) = 1, \ \operatorname{codim}(l_2) = 1 \ \text{ then } \ \operatorname{codim}(l_1) + \operatorname{codim}(l_2) = 2 = \operatorname{codim}(l_1 \cap l_2).$$

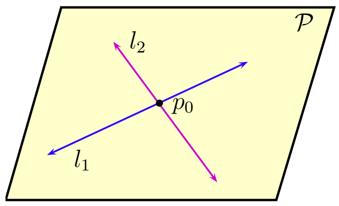

Figure 4.10 The transversality of two lines in a plane

Example 4.3.5. Let M be a smooth manifold and l a line in $\mathbb{R}^3$ such that $M \cap l = \emptyset$. They intersect transversely, since the tangent space of M and the tangent space of l generate $\mathbb{R}^3$. Additionally

$$\operatorname{codim}(M) = 1, \quad \operatorname{codim}(l) = 2 \ \text{ then } \ \operatorname{codim}(M) + \operatorname{codim}(l) = 3.$$

But $M \cap l = \emptyset$ and $\operatorname{codim}(M \cap l) = 4$, so $\operatorname{codim}(M) + \operatorname{codim}(l) \neq \operatorname{codim}(M \cap l)$.

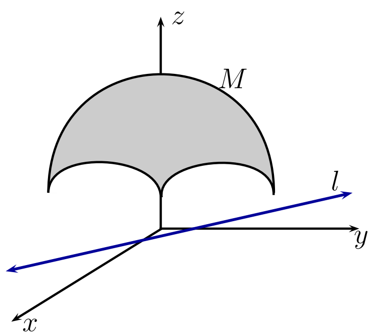

Figure 4.11 The transversality of a surface and a line in $\mathbb{R}^3$

Definition 4.3.6. Let $\mathcal{S} \subset M$ and $\mathcal{S}' \subset M'$ be two stratifications of $V \subset M$ and $V' \subset M'$, respectively and $f : M \to M'$ a smooth map. We say that f is transverse to $\mathcal{S}'$ at x in

M if for each stratum S_i in $\mathcal{S}$ and for each stratum S_i' in $\mathcal{S}'$ the map $f|_{S_i} : S_i \to M'$ is transverse to S_i', $\forall \in S_i$ such that $f(x) \in S_i$, i.e,

$$df_x(T_x S_i) + T_{f(x)} S_i' = T_{f(x)} M'.$$

It follows that the map $f : M \to M'$ takes a neighborhood of x in V transversely to V' and that

$$V \cap f^{-1}(V') = (f|_V)^{-1}(V')$$

is Whitney stratified by strata of the form $S_i \cap f^{-1}(S_i')$ where $S_i \in \mathcal{S}$ is a stratification of V and $S_i' \in \mathcal{S}'$ is a stratification of V', then $\mathcal{S}$ is transverse to $\mathcal{S}'$ which is written as $\mathcal{S} \pitchfork \mathcal{S}'$.

Corollary 4.3.7. *(Gibson et al., 1976) Let $\mathcal{S}'$ be a stratification of a subset V' of a smooth manifold M' and let $f : M \to M'$ be a smooth map which is transverse to $\mathcal{S}'$, i.e. transverse to all the strata of $\mathcal{S}'$. We can obtain a stratification $\mathcal{S}$ of $f^{-1}(V')$ by taking the strata $S' \in \mathcal{S}'$ of the form $f^{-1}(S')$. We call $\mathcal{S}$ the induced stratification on $f^{-1}(V')$.*

Corollary 4.3.8. *(Gibson et al., 1976) If $\mathcal{S}'$ is a Whitney stratification then $\mathcal{S}$ is also a Whitney stratification.*

Proof. We construct f as the composite functions $F : M \to graph\ f$ with $F(x) = (x, f(x))$ and π the restriction to $graph\ f$ of the projection $M \times M' \to M'$. F maps $\mathcal{S}$ diffeomorphically to a stratification $\mathcal{S}''$ of a subset of $graph\ f$. We must show that $\mathcal{S}''$ is a Whitney stratification.

For this purpose we will observe that the product stratification on $M \times V'$ is a Whitney stratification which is transverse to $graph\ f$. Since $\mathcal{S}$ and $\mathcal{S}'$ are stratification of $f^{-1}(V')$ and V', respectively, we can obtain a Whitney stratification of $f^{-1}(V') \times V'$ by taking the strata which are the sets of the form $f^{-1}(S') \times S'$ with $f^{-1}(S') \in \mathcal{S}$ and $S' \in \mathcal{S}'$. So we construct a product stratification $\mathcal{S} \times \mathcal{S}' = \mathcal{S}''$ which is Whitney stratification for a subset $f^{-1}(V') \times V' \subseteq graph\ f$. Now, if we induce the stratification on

$$(M \times V') \cap graph\ f,$$

then we get an induced Whitney stratification which is followed from the transversality of the product stratification to *graph* f. Hence, $\mathcal{S}''$ is Whitney stratification. $\square$

Corollary 4.3.9. *(Gibson et al., 1976) Let $\mathcal{S}$ be a stratification of a subset V of a smooth manifold M and let $U \subseteq M$. The inclusion map $U \hookrightarrow M$ is transverse to $\mathcal{S}$, so there is an induced stratification on $U \cap V$ which is denoted by $\mathcal{S}|_U$ such that*

$$\mathcal{S}|_U = S_i \cap U | \, S_i \in \mathcal{S} \;\; and \;\; S_i \cap U \neq \emptyset$$

On the contrary, if $\mathcal{S}$ is a Whitney stratification then $\mathcal{S}|_U$ is also a Whitney stratification.

CHAPTER FIVE
MORSE THEORY FOR SINGULAR SPACES

Until here we have given Morse theory for compact smooth manifolds and have reminded the handles of a smooth manifold with cell complexes. In Veblen (1922) has been given the natural idea for dividing a singular spaces into smooth submanifolds. At that point the idea of Stratification theory has begun to appear partially in the study of simplicial complexes and regular cell complexes. (Whitney, 1947) Whitney stratified sets have been triangulated up to homeomorphism, the class of Whitney stratified sets coincides with the class of simplicial complexes and the class of regular cell complexes. After all Whitney has given an idea how one can apply Morse theory to singular spaces. Using the Whitney stratification one can divide singular spaces into some smooth submanifolds which are called the strata. After that we can locally apply the classical Morse theory to each stratum.

In this chapter, we give some basic notions for singular spaces defining tangential and normal Morse data of singular spaces. Afterwards we obtain Morse data which is the handlebody of a singular space. Finally, we give two fundamental theorems of Whitney stratified space from (Goresky & MacPherson, 1988) which are also singular spaces and we try to understand the general notion of these theorems.

5.1 Some Basic Definitions

Definition 5.1.1. A point on a topological space is not defined or not "well-behaved", for example not differentiable,it is called *singular point*. A topological space is a *singular space* which has at least one singular point.

Definition 5.1.2. A *Morse data* at a critical point $p \in X$ of a Morse function f is defined to be the pair (A, B) of topological spaces A and B with the properties: $B \subset A$ and the change in $X_{\leq c}$ can be described by gluing in A along B, where $c = f(p)$.

Example 5.1.3. In the second chapter we have given the handle decomposition of a

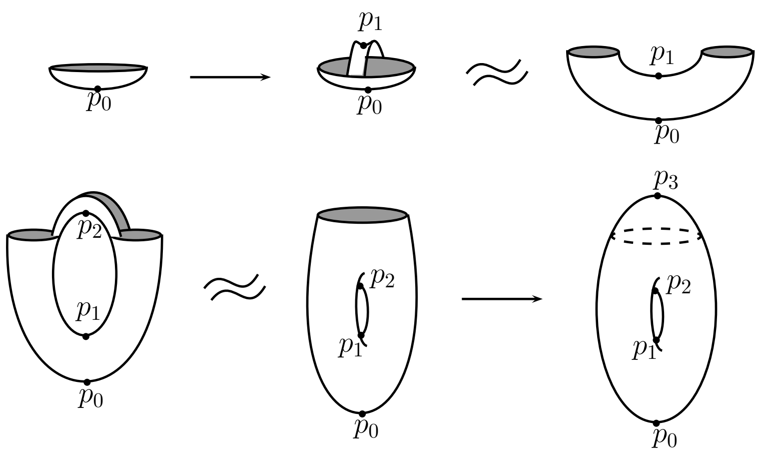

Figure 5.1 The handle decomposition of a torus

torus and we have seen the topological changes while passing through critical values c_i. (See Figure 5.1)

Now we see the Morse data of the torus and relation between the Morse data and handle of the torus as follows:

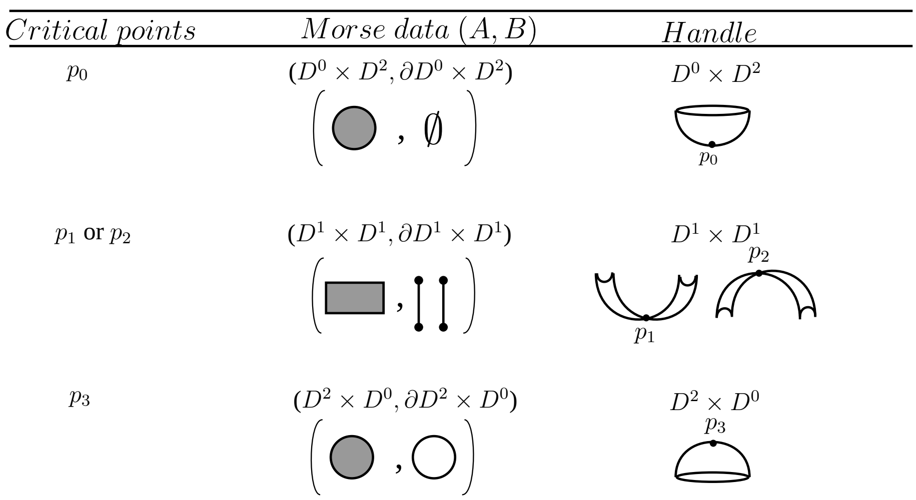

Figure 5.2 Morse data and handle of the torus $\mathcal{T}$

We describe the topological changes of torus $\mathcal{T}$ by Morse data when c crosses the critical values c_0, c_1 and c_3 of the height function on torus $\mathcal{T}$. We will define the Morse data at p as the product of normal Morse data at p and tangential Morse data at p. Here the Morse data and handle have the same topological type.

5.2 Morse Theory For Singular Spaces

In this thesis we aim to give the generalize Morse theory which can be applied to singular spaces. We consider a singular space $\mathcal{B}$ which is embedded in $\mathbb{R}^3$. If we look at the topological type of $\mathcal{B}$ we can say that $\mathcal{B}$ may be obtained from the torus $\mathcal{T}$ by shrinking the circle going around the left side to a point and stretching a taut disc across the circle around the hole. We want to call $\mathcal{B}$ as a *Bally Torus*. Now we need a Morse function f and we can choose it as the height function as before

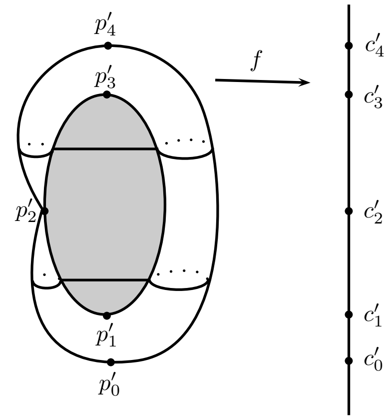

Figure 5.3 f is the height function on $\mathcal{B}$

We can easily say that the topological type of $\mathcal{B}_{\leq c}$ changes only when c passes through one of the (critical) values $c'_0, c'_1, c'_2, c'_3, c'_4$ which are the images of the (critical) points $p'_0, p'_1, p'_2, p'_3, p'_4$. So we can generalize the Morse theory to singular space by using a general definition of critical points.

We need a Whitney stratification of a space X which is a decomposition of X into a submanifolds (*strata*) satisfying the Whitney condition given in Definition 4.2.1. For the space $\mathcal{B}$ we give Whitney stratification by using the Whitney conditions (a) and (b). If we look at the Figure 5.3 we see that the singular set consists of a circle S^1 which bounds the disc. We know that the strata must be the smooth submanifolds and we can choose the largest stratum as the complement of this circle. The circle itself is nonsingular but the point p'_2 has a different kind of singularity there. If we choose the smallest stratum as p'_2 then the rest of the circle is the middle stratum. (See Figure 5.4)

Now we suppose that X is a compact Whitney stratified subspace of a manifold M and f is the restriction to X of a smooth function on M. We define a critical point of f to be a critical point of the restriction of f to any stratum. We get all zero-dimensional strata as the singular points. A critical value is the value of f at a critical point, as before. Now we can give a similar theorem to Theorem 2.3.1 for the stratified Morse theory.

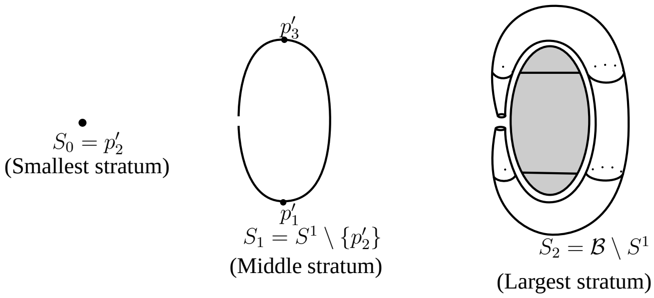

Figure 5.4 The strata of $\mathcal{B}$

Theorem 5.2.1. *(Goresky & MacPherson, 1988) As c varies within the open interval between two adjacent critical values, the topological type of $X_{\leq c}$ remains constant.*

Example 5.2.2. Now we want to investigate the topological type of $\mathcal{B}_{\leq c}$ while c crosses a critical value c'_i, $0 \leq i \leq 4$.

- If $c < c'_0$ then $\mathcal{B}_{\leq c} = \emptyset$
- If c crosses c'_0 then $\mathcal{B}_{\leq c}$ changes by adding an upward two-disc.
- If c crosses the critical values c'_1, c'_2, c'_3, then the change in $\mathcal{B}_{\leq c}$ is described by the Morse data. All can be seen in the Figure 5.5.
- If c crosses c'_4 then $\mathcal{B}_{\leq c}$ changes by gluing in a downward two-disc along its boundary.

Definition 5.2.3. X is a *Whitney stratified subspace* of a manifold M, if $D(p)$ is the small disc in M which is transverse to the stratum S_i containing p such that

$$D(p) \cap S_i = \{p\} \text{ and } \dim(D(p)) = \dim M - \dim S.$$

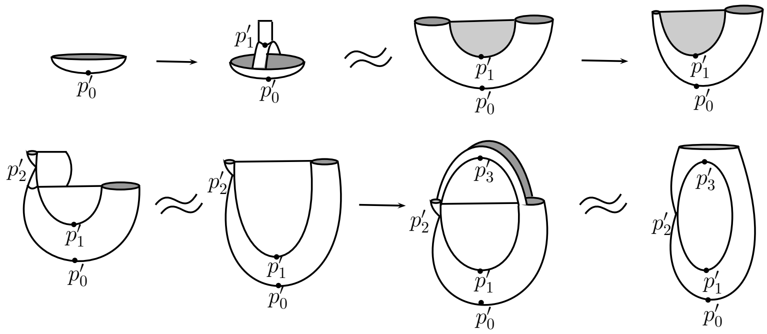

Figure 5.5 The topological type of $\mathcal{B}_{\leq c}$ when c crosses a critical value c_i

Example 5.2.4. Let us consider the Bally Torus $\mathcal{B}$ and the stratum of $\mathcal{B}$ in order to see the small discs $D(p_i')$ which are transverse to S_i containing p_i' where $i = 0, 1, 2, 3, 4$.

1. For the critical points p_0', p_4' which lie in the largest stratum, we will define the small discs $D(p_0')$ and $D(p_4')$ by using the top dimensional stratum $S_2 = \mathcal{B} \setminus S^1$.

   If we choose $D(p_0') = D^1$ and $D(p_4') = D^1$ which are transverse to S_2 then we get the following figure such that $D(p_0') \cap S_2 = \{p_0'\}$, $D(p_4') \cap S_2 = \{p_4'\}$

   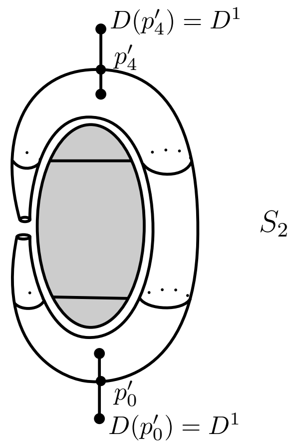

   Figure 5.6 The small discs $D(p_i')$ are transverse to S_i at p_i'

   $$\dim(D(p_0')) = \dim(D(p_4')) = \dim(\mathbb{R}^3) - \dim(S_2) = 3 - 2 = 1 = \dim(D^1).$$

2. For the critical points p_1' and p_3' which lie in the middle stratum S_1 of $\mathcal{B}$, we will define the small discs $D(p_1')$ and $D(p_3')$ using $S_1 = S^1 \setminus \{p_2'\}$. If we choose $D(p_1') = D(p_3') = D^2$ which are transverse to S_1 then we obtain the following figure such

that $D(p'_1) \cap S_1 = \{p'_1\},\ D(p'_3) \cap S_1 = \{p'_3\}$ where $D(p'_1) \cap S_1 = \{p'_1\},\ D(p'_3) \cap S_1 = \{p'_3\}$ and

$$\dim(D(p'_1)) = \dim(D(p'_3)) = \dim(\mathbb{R}^3) - \dim(S_1) = 3 - 1 = 2 = \dim(D^2).$$

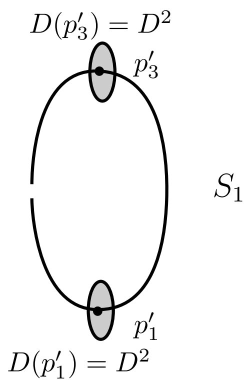

Figure 5.7 The small discs $D(p'_1)$ and $D(p'_3)$ transverse to S_1 at p'_1 and p'_3

3. For the critical point p'_2 which lies in the smallest stratum S_0 of $\mathcal{B}$, we can choose the small disc $D(p'_2)$ using the $S_0 = \{p'_2\}$. If we define $D(p'_2) = D^3$ then the conditions are satisfied for $D(p'_2)$ and S_0 with the same procedure. We get the following figure:

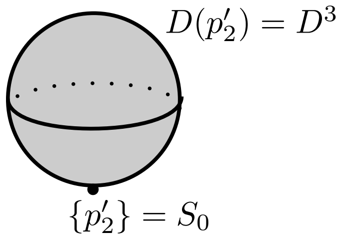

Figure 5.8 The small disc $D(p'_2)$ is a 3-dimensional unit disc at p'_2

Definition 5.2.5. The intersection of $D(p)$ with X is called the *normal slice* at p and denoted by $N(p)$. In other words

$$N(p) = D(p) \cap X.$$

Definition 5.2.6. The boundary of the normal slice is denoted by $L(p) = \partial D(p) \cap X$ which is called the *link* of the stratum S_p.

Remark 5.2.7. (Goresky & MacPherson, 1988) The normal slice $N(p)$ is a key construction for a singular space. Topologically, $N(p)$ is the cone over the link of the stratum S_p. The topological type of a link may be thought of as measuring the singularity type of X along the stratum S. If X is nonsingular along S, then the link $L(p)$ is a sphere. Whitney conditions guarantee that the connected components of S containing p has a neighborhood which is a fibre bundle over S and whose fibre is $N(p)$.

Example 5.2.8. Let us consider the Bally Torus $\mathcal{B}$ and define the normal slices and links of $\mathcal{B}$ for the critical points $p'_0, p'_1, \ldots, p'_4$. Using the example 5.2.4 for the small discs $D(p'_i)$ where $i = 0, \ldots, 4$.

(1) Now we want to obtain $N(p'_2)$ by using $D(p'_2)$ for the critical point p'_2 of $\mathcal{B}$. We know that $D(p'_2) = D^3$ then we get

$$N(p'_2) = D(p'_2) \cap \mathcal{B} = D^3 \cap \mathcal{B}$$

$$L(p'_2) = \partial N(p'_2) = \partial D(p'_2) \cap \mathcal{B} = S^2 \cap \mathcal{B}.$$

We can see clearly $N(p'_2)$ and $L(p'_2)$ in the Figure 5.9 and Figure 5.10 :

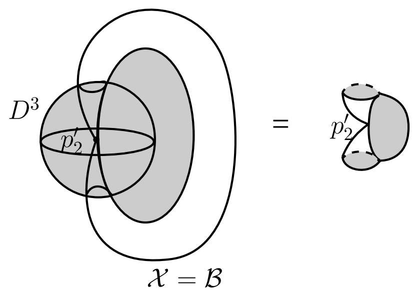

Figure 5.9 Normal slice $N(p'_2)$ at p'_2

(2) Similarly we will obtain $N(p'_0)$ and $N(p'_4)$ using the small discs $D(p'_0)$ and $D(p'_4)$ for the critical points p'_0 and p'_4 of $\mathcal{B}$, respectively.

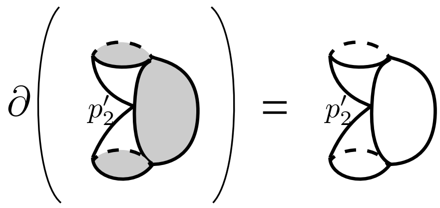

Figure 5.10 The link $L(p_2')$ of S_2 at p_2'

We know that $D(p_0') = D^1$ and $D(p_4') = D^1$ then we get

$$N(p_0') = D(p_0') \cap \mathcal{B} = D^1 \cap \mathcal{B} = \{p_0'\}$$

$$N(p_4') = D(p_4') \cap \mathcal{B} = D^1 \cap \mathcal{B} = \{p_4'\}.$$

We can see clearly $N(p_0')$ and $N(p_4')$ in the Figure 5.11.

$$L(p_0') = \partial N(p_0') = \partial D(p_0') \cap \mathcal{B} = \emptyset$$

$$L(p_4') = \partial N(p_4') = \partial D(p_4') \cap \mathcal{B} = \emptyset$$

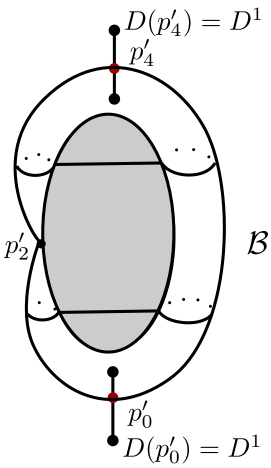

Figure 5.11 Bally torus $\mathcal{B}$ with small discs at p_0' and p_4'

(3) We can use the same procedure as in the (1) and (2) for the critical points p_1', p_3' of $\mathcal{B}$. The two-dimensional two discs $D(p_1') = D(p_3') = D^2$ are transverse to S_1 at the critical points p_1' and p_3', respectively.

Figure 5.12 $N(p_1')$ and $N(p_3')$

So we get the normal slices $N(p_1') = D(p_1') \cap \mathcal{B}$, $N(p_3') = D(p_3') \cap \mathcal{B}$ which can be seen in the following figure: We can place $N(p_1')$ and $N(p_3')$ in the Bally Torus as follows: where the purple colors denote $N(p_1')$ and $N(p_3')$. If we take the

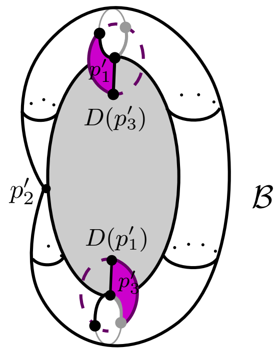

Figure 5.13 The Bally torus $\mathcal{B}$
with the normal slices

boundary of the normal slices then we get the links of the stratum S_1, $L(p_1') = \partial N(p_1') = \partial D(p_1') \cap \mathcal{B}$ and $L(p_3') = \partial N(p_3') = \partial D(p_3') \cap \mathcal{B}$ which can be seen in the following figure:

Figure 5.14 $L(p_1')$ and $L(p_3')$

Definition 5.2.9. (Normal Morse Data) Let $p \in X$ be any critical point. We can define *normal Morse data* at p by using the pair of spaces (A, B), which is called the Morse data, where

$$A = \{x \in N(p) \mid c - \varepsilon \leq f(x) \leq c + \varepsilon,\ f(p) = c\}$$

$$B = \{x \in N(p) \mid f(x) = c - \varepsilon,\ \text{ for very small } \varepsilon\}.$$

Thus we can imagine the normal Morse data at p as Morse data for the restriction of f to the normal slice at p.

Definition 5.2.10. (Tangential Morse Data) We can define the *tangential Morse data* at p to be the Morse data for the restriction of f to the stratum S_p of X containing p.

Now we are ready to state the fundamental theorem of the stratified Morse theory.

Theorem 5.2.11. *(Goresky & MacPherson, 1988) Let f be a Morse function on a compact Whitney stratified space X. Then Morse data measuring the change in the topological type of $X_{\leq c}$ as c crosses the critical value p is the product of the normal Morse data at p and the tangential Morse data at p.*

Theorem 5.2.11 is very naturel and geometrically evident in examples but the proof takes 100 pages and Goresky and McPherson prove rigorously in Goresky & MacPherson (1988).

Remark 5.2.12. The notion of product of pairs used in this theorem is the standard definition in topology, as in the following type:

$$(A,B)\times(A',B') = (A\times A', A\times A' \cup B\times A')$$

Now we give the example by using the Bally Torus $\mathcal{B}$.

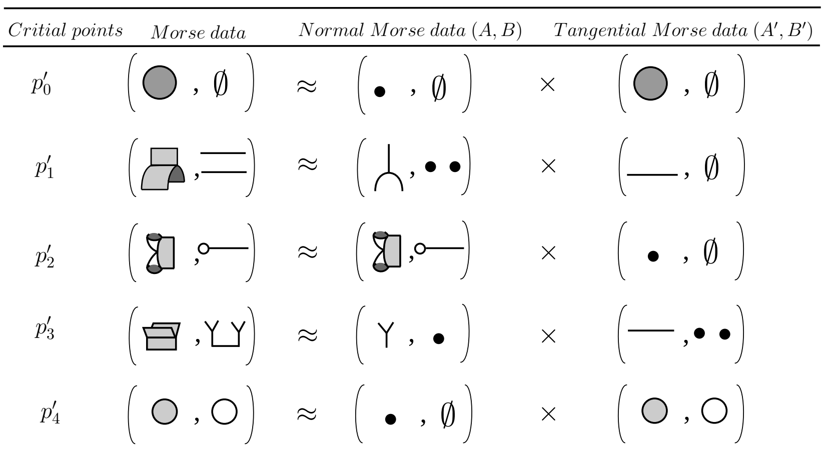

Figure 5.15 Morse data of the Bally torus $\mathcal{B}$

CHAPTER SIX
CONCLUSION

In this thesis, we tried to understand the fundamental theorems of Morse theory for singular spaces. To reach this aim, we studied the classical Morse theory and Whitney stratification of a topological space. In the fourth part, we studied stratification and Whitney stratification of a topological space. Thus, we learnt why we need Whitney stratification in Morse theory for singular spaces. Finally, we gave the fundamental theorems of Morse theory for singular space.

Future Work: Now, we can say that one can try to understand the proofs of these theorems by learning "Moving the Wall" and "fringed set" from (Goresky & MacPherson, 1988) for future plans. Afterwards, intersection homology can be studied. So one can hoped to understand why and where Goresky & MacPherson needed to define a new homology which is called intersection homology and try to learn how and where Goresky & MacPherson get together the Morse theory and intersection homology. Furthermore, one can learn what is the relations between intersection homology and topological invariants of singular spaces.

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