DOKUZ EYLÜL UNIVERSITY

GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

ACTUARIAL VALUATION OF PENSION PLANS BY STOCHASTIC INTEREST RATES APPROACH

by

Dilek KESGİN

November, 2012

İZMİR

ACTUARIAL VALUATION OF PENSION PLANS BY STOCHASTIC INTEREST RATES APPROACH

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İZMİR

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We have read the thesis entitled **"ACTUARIAL VALUATION OF PENSION PLANS BY STOCHASTIC INTEREST RATES APPROACH"** completed by **DİLEK KESGİN** under supervision of **ASSOC. PROF. DR. GÜÇKAN YAPAR** and we certify that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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Dilek KESGİN

ACTUARIAL VALUATION OF PENSION PLANS
BY STOCHASTIC INTEREST RATES APPROACH

ABSTRACT

Interest rates which have been deterministic are used in calculations of actuarial present values, reserve, mortality, premium concerning pension plans. Interest rates had been preferred a constant value while life contingencies were determined to be random during pension time of insured. These cases landed risk measures all establishments that constituted the pension system.

In this study, interest rates which are the most uncertain risks at issue are considered stochastic to decrease the effect of inflation in the actuarial valuations. Also, applications were made based on the procedures and principles in the draft resolution of ministerial cabinet relevant to Banks, Insurance Companies, Reinsurance Undertakings, Chambers of Commerce, Chambers of Industry, Bourses and the special retirement fund where consist all of these establishment personals. As a result of the applications, some results were obtained with reference to how will happen calculations of the pension system both pluses and minuses after cession term.

Keywords: Stochastic interest rates, pension system, risk measurement, actuarial valuations.

EMEKLİLİK PLANLARININ STOKASTİK FAİZ ORANLARI YAKLAŞIMIYLA AKTÜERYAL OLARAK DEĞERLENDİRİLMESİ

ÖZ

Emeklilik planlamalarına dair aktüeryal peşin değer, rezerv, sağ kalım süresi ve prim hesaplamalarında genellikle rastgele olmayan faiz oranları kullanılmıştır. Sigortalı kişinin hayatta kalma olasılığı emeklilik süresi boyunca rastgele olarak belirlenirken, faiz oranları sabit olarak tercih edilmiştir. Bu durumda emeklilik sistemini oluşturan birçok kuruluşa çeşitli risk unsurları yüklemiştir.

Bu çalışmada söz konusu risklerin en belirsizi olan faiz unsuru stokastik düşünülerek hesaplamalarda daha net sonuçlar elde edilmeye çalışılmıştır. Ayrıca; Bankalar, Sigorta ve Reasürans Şirketleri, Ticaret Odaları, Sanayi Odaları, Borsalar ve bunların teşkil ettikleri birlikler personeli için kurulmuş bulunan sandıkların iştirakçilerinin Sosyal Güvenlik Kurumu'na devrine ilişkin esas ve usuller hakkındaki bakanlar kurulu karar taslağına yönelik hesaplamalar stokastik faiz oranlarıyla yapılmıştır. Uygulamaların sonucu olarak, devir işleminden sonra Emeklilik Sistemi'nin artı ve eksileriyle nasıl olacağıyla ilgili olarak bazı sonuçlar elde edilmiştir.

Anahtar sözcükler: Stokastik faiz oranları, emeklilik sistemi, risk unsurları, aktüeryal değerler.

CONTENTS

CHAPTER ONE
INTRODUCTION

1.1 Introduction

An establishment which provides the insurance services must have taken the decisions in the light of actuarial equivalence principles to fulfill all of its liabilities; on the contrary, it can be faced with elements of risk. One of the most important problems in actuarial equivalence calculations is interest rates because of indeterminacy and variability; therefore, interest rates must be accepted the stochastic into long-term financial transactions.

The applications of this study have been performed using the stochastic interest rates according to a draft resolution that is published about foundation funds by the ministerial cabinet. There isn't a new attempt to transfer from the foundation funds to Social Security System; on the other hand, ongoing efforts in this direction have been continuing for a long time. Consequently, Social Security Institution has been taken necessary step to gather under a single roof all of foundation funds with temporary twentieth article of the Social Security and General Health Insurance law.

1.1.1 Foundation Funds

Foundation is called the administrative control system of the funds. Foundation Funds have been undertaken the function of the Social Security Institution, are the Social Insurance Institutions where have the qualifications of the Social Security Institution which is established by the laws, have been containing state assistances which are presented by public social security as a minimum with regards to the social security rights. Seventeen piece foundation funds which are established as for that temporary twentieth article of the law no 506 have been consisting of Banks, Insurance Companies, Reinsurance Undertakings, Chambers of Commerce, Chambers of Industry, Bourses and their subsidiaries. Table 1.1 is given to show these foundation funds's name. Also, in next Tables and Figures, the numbers corresponding to the names of the foundation funds in Table 1.1 will be used instead of the foundation funds names.

Table 1.1 Classification of names of the foundation funds

Number	Names of Foundation Funds
1	Türkiye İş Bankası A.Ş. Mensupları Emekli Sandığı Vakfı
2	Yapı ve Kredi Bankası A.Ş. Emekli Sandığı Vakfı
3	Akbank T.A.Ş. Mensupları Tekaüt Sandığı Vakfı
4	Türkiye Vakıflar Bankası T.A.O. Memur ve Hizmetlileri Emekli ve Sağlık Yardım Sandığı Vakfı
5	Türkiye Garanti Bankası A.Ş. Emekli ve Yardım Sandığı Vakfı
6	T.C. Ziraat Bankası A.Ş. ve Türkiye Halk Bankası A.Ş. Mensupları Emekli ve Yardım Sandığı Vakfı
7	Türkiye Halk Bankası A.Ş. Mensupları Emekli ve Yardım Sandığı Vakfı (Pamukbank T.A.Ş.)
8	Türkiye Odalar Borsalar ve Birlik Personeli Sigorta ve Emekli Sandığı Vakfı
9	Şekerbank T.A.Ş. Emeklileri Sandığı
10	Fortis Bank A.Ş. Mensupları Emekli Sandığı ve Dış Bank Personeli Güvenlik Vakfı
11	Anadolu Anonim Türk Sigorta Şirketi Memurları Emekli Sandığı Vakfı (Anadolu Sigorta)
12	Türkiye Sinai Kalkınma Bankası Mensupları Munzam Sosyal Güvenlik ve Yardımlaşma Vakfı
13	Esbank Eskişehir Bankası T.A.Ş. Mensupları Emekli Sandığı Vakfı
14	Mapfre Genel Sigorta
15	Milli Reasürans T.A.Ş. Mensupları Emekli ve Sağlık Sandığı Vakfı
16	Liberty Sigorta
17	İmar Bankası T.A.Ş. Memur ve Müstahdemleri Yardım ve Emekli Sandığı Vakfı

The relevant legislations which will be used during the cession process are the temporary twentieth article and the additional thirty sixth article of the law no 506, the temporary twenty third article of the law no 5411 (canceled) and the temporary twentieth article of the law no 5510. The relationships of the Ministry of Labor and Social Security with foundation funds are as below:

- The approval authority on the subject of the status change
- The financial audit authority
- The surveillance authority arising from establishment under the state guarantee of the social security according to sixtieth article of the constitution

İstanbul Bankası, Türkiye Öğretmenler Bankası, Tam Sigorta, Ankara Anonim Türk Sigorta Şirketi, Türkiye Kredi Bankası, Türk Ticaret Bankası, Tütün Bank Foundation Funds have been transferred to the Social Security Institution with regard to the additional thirty sixth article of the law no 506 up till now. The current cession is different from the previous cession owing to the following reasons:

- Only, the participations of the foundation funds, and individuals who are granted with pensions or incomes, and their survivors are included in the scope of this act will take place transferring them to the Social Security Institution
- The takeover with actives and passives of the foundation funds isn’t in question

Regulations which are made in respect of the temporary twentieth article of the law no 5510 are envisaged as below:

- Protection of existing rights of the foundation fund participations
- Technical interest rate is taken as 9.8 percent
- Determined cash value is received, maximal fifteen years, in equal annual installments, for each year separately
- The cash value is accepted by a commission
- Processes of increase, decrease, discontinuation and reassignment due to state changes in pensions and income are restricted according to the law no 5510

1.1.2 Statistics of Foundation Funds

As from 2011, insured situation of the foundation funds which are established according to the temporary twentieth article of the law no 506 is given Table 1.2.

Table 1.2 Insured situation of the foundation funds as from 2011

Number	**Insured**					
	Active	**Passive**	**Beneficiary**	**Total**	**General Total Ratio (%)**	**Active/Passive Ratio**
1	24.839	26.716	39.190	90.745	26,00	0,93
2	14.796	12.762	22.631	50.189	14,38	1,16
3	16.175	11.581	18.161	45.917	13,15	1,40
4	12.276	8.109	16.339	36.724	10,52	1,51
5	16.623	7.742	11.818	36.183	10,37	2,15
6	11.126	3.378	7.529	22.033	6,31	3,29
7	9.883	2.716	8.000	20.599	5,90	3,64
8	5.194	4.522	8.028	17.744	5,08	1,15
9	3.529	3.798	6.223	13.550	3,88	0,93
10	3.295	824	3.590	7.709	2,21	4,00
11	902	502	846	2.250	0,64	1,80
12	346	519	601	1.466	0,42	0,67
13	8	736	571	1.315	0,38	0,01
14	449	126	291	866	0,25	3,56
15	158	330	276	764	0,22	0,48
16	191	217	233	641	0,18	0,88
17	6	233	140	379	0,11	0,03
General Total	119.796	84.811	144.467	349.074	100,00	1,41

Distribution of the foundation funds according to total insured number and general total ratio is obtained as shown in the Figure 1.1; similarly, Change of active/passive ratio is attained using the values of the active and passive depend on each foundation fund, based on the data given in the Table 1.2.

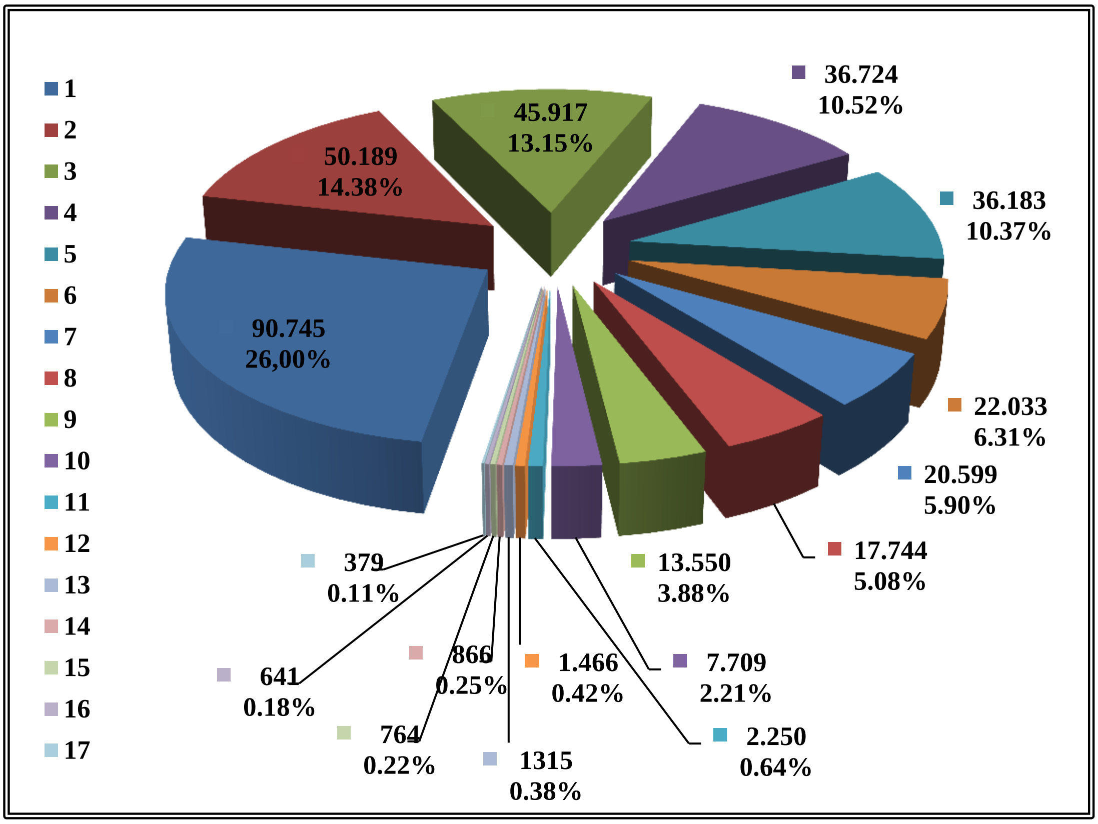

Figure 1.1 Distribution of the foundation funds according to total insured and general total ratio

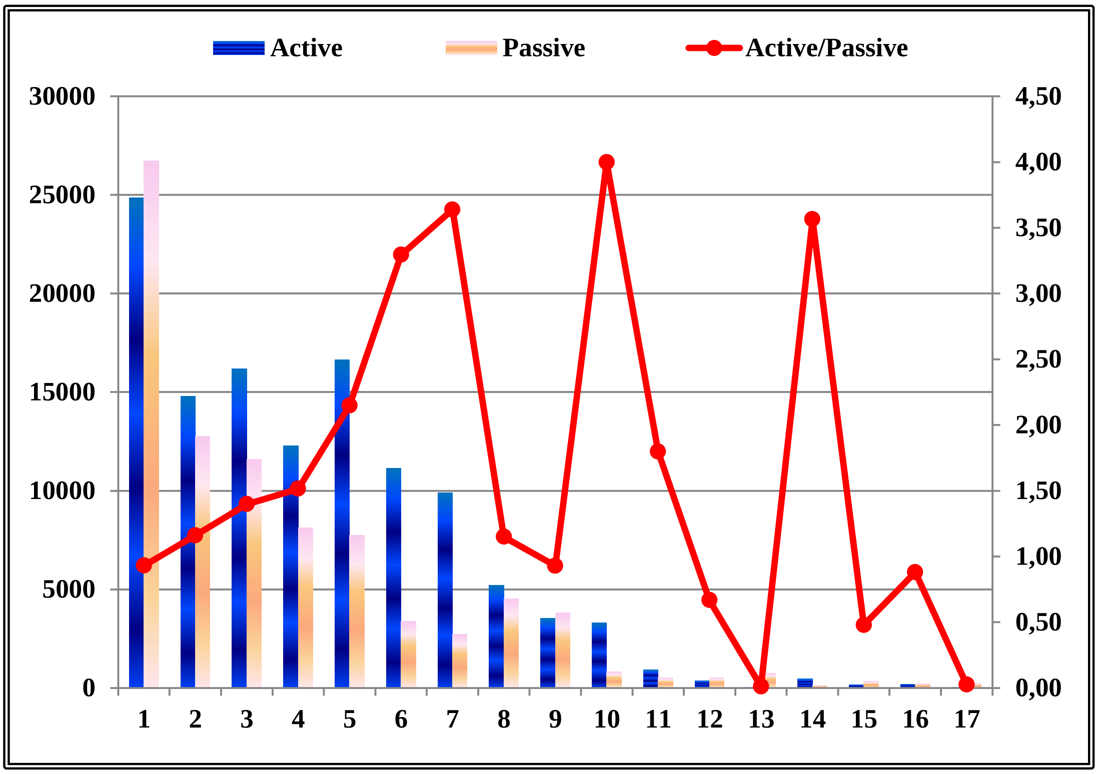

Figure 1.2 Change of active/passive ratios of the foundation funds according to the active and passive numbers in 2011 year

Insured numbers of foundation funds which are established as for that temporary twentieth article of the law no 506 is given as such in Table 1.3, based on the data given between 1994 and 2011 years.

Table 1.3 Insured numbers of foundation funs between 1994 and 2011 years

Years	Insured				
	Active	**Passive**	**Beneficiary**	**Total**	**Active/Passive Ratio**
1994	71.073	47.114	139.838	258.025	1,51
1995	70.854	51.948	168.445	291.247	1,36
1996	71.465	58.744	177.814	308.023	1,22
1997	74.494	63.116	177.442	315.052	1,18
1998	77.526	65.757	174.802	318.085	1,18
1999	78.861	69.428	184.581	332.870	1,14
2000	78.495	71.266	173.808	323.569	1,10
2001	73.090	75.162	174.436	322.688	0,97
2002	71.641	77.738	174.923	324.302	0,92
2003	70.925	71.595	153.021	295.541	0,99
2004	73.412	74.367	153.662	301.441	0,99
2005	75.685	76.027	155.449	307.161	1,00
2006	85.358	78.082	134.829	298.269	1,09
2007	95.341	79.388	136.121	310.850	1,20
2008	105.707	81.042	136.469	323.218	1,30
2009	109.668	82.459	139.078	331.205	1,33
2010	114.534	83.599	143.388	341.521	1,37
2011	119.796	84.811	144.467	349.074	1,41

Change of Active/Passive ratio of the foundation funds is obtained as shown in the Figure 1.3, using the values of the active and passive depend on each year; similarly, change of active/passive ratio of the foundation funds according to the total insured numbers (total of active, passive and beneficiary numbers) between 1994 and 2011 years is showed as such in Figure 1.4, based on the data given in the Table 1.3.

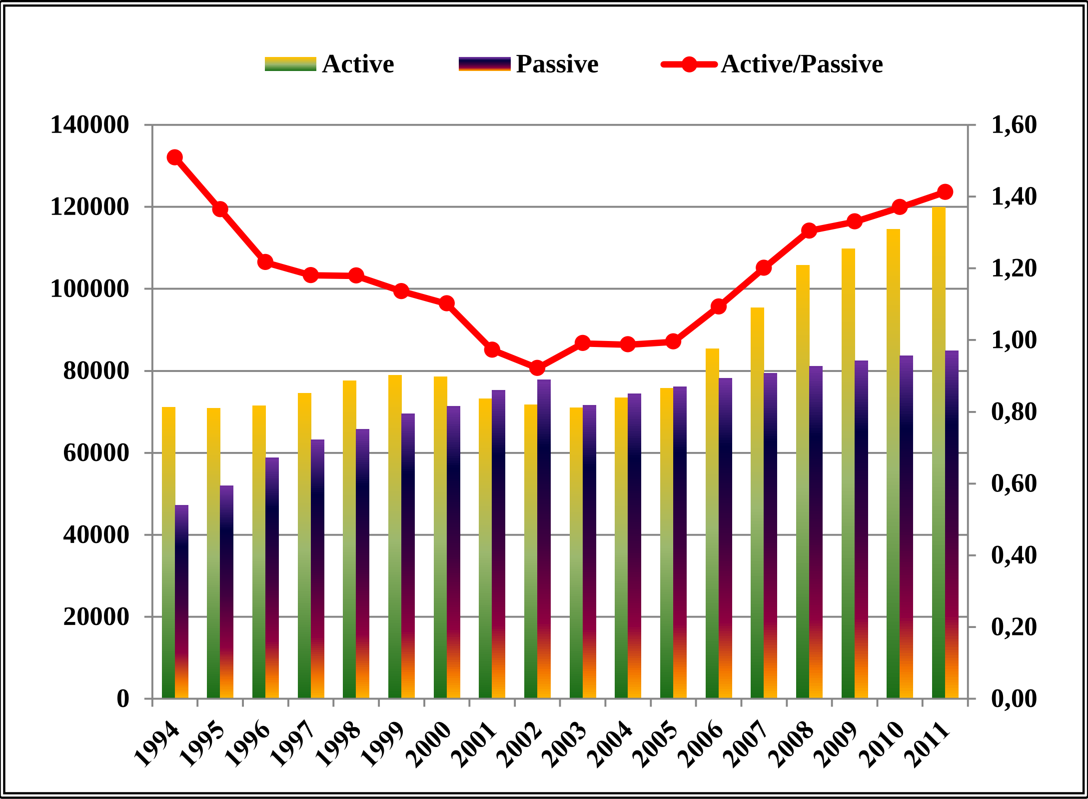

Figure 1.3 Change of active/passive ratio of the foundation funds according to the active and passive numbers between 1994 and 2011 years

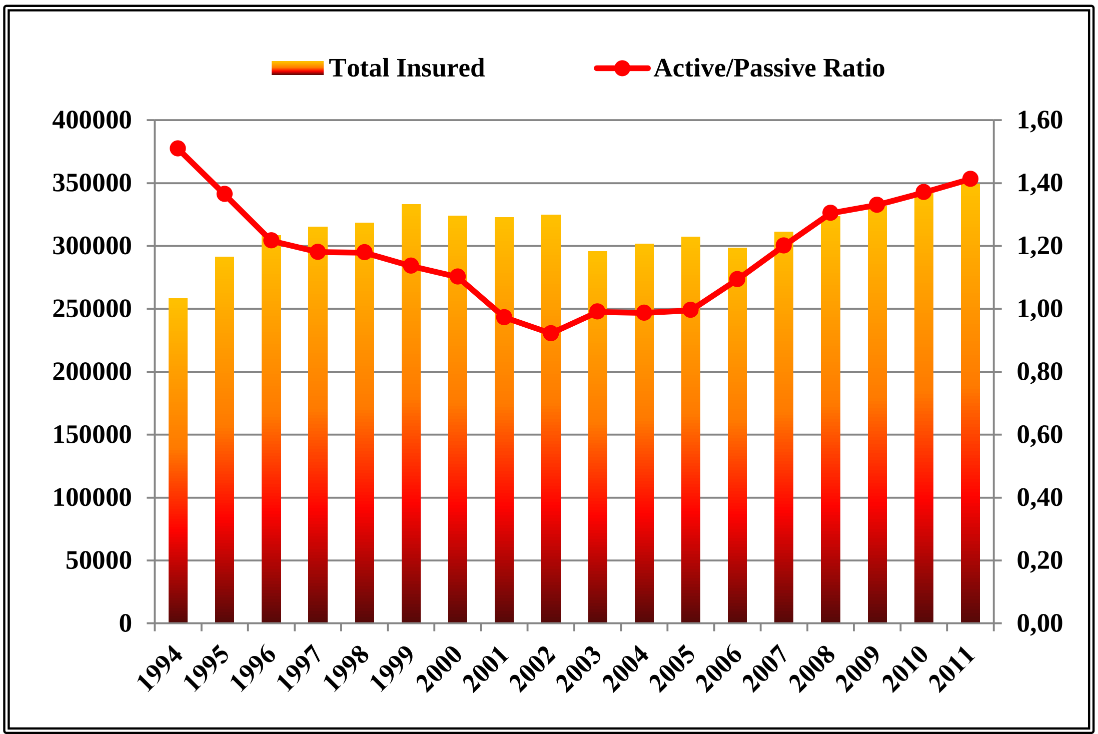

Figure 1.4 Change of active/passive ratio of the foundation funds according to the total insured numbers between 1994 and 2011 years

1.2 Literature Overview

Interest rates which have been deterministic are used in calculations of actuarial present values, reserve, mortality, premium concerning pension plans. Interest rates had been preferred a constant value while life contingencies were determined to be random during pension time of insured. These cases landed risk measures; as a result of these reasons, stochastic interest rates were started to use for actuarial models. Lots of papers interested in stochastic interest rates have been published for different insurance models since long years. Some of them were presented chronologically in the following paragraph.

(A. H. Pollard & J. H. Pollard, 1969) presented a study to compute the moments of actuarial random variables. They discussed some calculations, defined certain specialties of the random variables involved and obtained some numerical examples. Consequently, they argued the problem of retention limits and reassurance arrangements.

(Boyle, 1976) carried out a study to present a theory by using varying rates of interest. Stochastic process was constituted with the help of an investment model which is the one year returns and the returns are independent from year to year. Several special results were determined using properties of the lognormal distribution.

(Bellhouse & Panjer, 1980, 1981) made a statement about characteristics of stochastic interest for continuous and discrete models. They presented some experientially supported models of interest rate and researched about how to change the structures of life contingencies functions, premiums, reserves in depth when life time and interest rate had a random fluctuation. A general theory was developed to make an evaluation associated with the risk measures of interest. Furthermore, they forwarded their study with conditional autoregressive interest rate models and obtained numerical results for interest, insurance and annuity functions.

(Giaccotto, 1986) used the stochastic interest rates to compute insurance functions. A general method was developed for both the actuarial case and the equilibrium approach. In calculations of the actuarial case, Interest rates were accepted deterministic and random. In calculations of the equilibrium approach, the present value of two life insurance functions was derived using the Vasicek model for pricing zero coupon bonds.

(Dhaene, 1989) created a method to calculate moments of insurance functions using the force of interest which is supposed to follow an autoregressive integrated moving average process.

(Vanneste, Goovaerts, De Schepper & Dhaene, 1997) obtained the moment generating function of the annuity certain by using the stochastic interest rates which were written in the way that a time discretization of the Wiener process as an n-fold integral and created a simple assessment of the corresponding distribution function. The present method is easier than others to calculations and can be applied to IBNR results, as well as to pension funds calculations.etc.

(Marcea & Gaillardetz, 1999) studied on life insurance reserves in a stochastic mortality and interest rates environment for the general portfolio. In this study, Monte Carlo simulation and the assumption of large portfolio methods were used to find the first two moments of the prospective loss random variable. In the calculations, they benefited from the discrete model.

(Zaks, 2001, 2009) analyzed the accumulated value of some annuities-certain over a period of years where the interest rate is a stochastic under some limitation. He presented two methods to derive moments of the expected value and the variance of the accumulated value. One of the methods is more suitable with regards to the simplicity of calculation than the other. His study presented some novelty and showed recursive relationships for the variance of the accumulated values and obtained these relationships. Besides, in his recently study, the future value of the expected value and the variance for various cash flows were evaluated.

(Beekman & Fuelling, 1990, 1992) studied extra randomness in certain annuity models, interest and mortality randomness in some annuities. For certain annuities, they presented a model which can be used when interest rates and future life times are stochastic. For the mean values and standard deviations of the present values of future cash flows, they found some expressions which can be used in determining contingency reserves for possible adverse interest and mortality experience for collections of life annuity contracts. Also, they determined certain boundary crossing probabilities for the stochastic process component of the model. In their recently study, they utilized the Wiener stochastic process for an alternative model which has extensive boundary crossing probabilities. Additionally, the last model is much more randomness than an earlier model.

(Wilkie, 1987) argued the stochastic investment models which involved four series as the Retail Prices Index, an index of share dividend yields, an index of share yields, and the yield on ‘consols’. Relating to the expense charges of unit trusts and to guarantees incorporated in index linked life annuities were defined in detail.

(Burnecki, Marciniuk & Weron, 2003) built accumulated values of annuities certain with payments varying in arithmetic and geometric procession by using the stochastic interest rates. First and second moment aside from variance of the accumulated values, which leads to a correction of main results from (Zaks, 2001), was calculated using recursive relations.

(Perry, Stadje & Yosef, 2003) obtained the expected values of annuities when interest rates had a stochastic nature that reflected Brownian motion with a switchover at some positive level at which the drift and variance parameters change. The lifetime of annuity was determined under the exponential distribution. Their study can be extended to the case of some switchover levels and other related models.

(Huang & Cairns, 2006) aimed to obtain a proper contribution rate for described a benefit pension plans under the stochastic interest rates and random rates of return.

They offered two methods; one of them is short-term interest rates to control contribution rate fluctuation, other of them is three assets (cash, bonds and equities) to permit comparison of various asset strategies. Applications were made for unconditional means and variances.

(Hoedemakers, Darkiewicz & Goovaerts, 2005) performed a study on the distribution of life annuities with stochastic interest rates. In their paper, they purposed to use the theory of comonotonic risks developed by Dhaene et al. and also, they obtained some conservative estimates both for high quantiles and stop-loss premiums for an individual policy and for a whole portfolio. Nevertheless, they explained that the method has very high accuracy with some numerical examples.

(Satıcı & Erdemir, 2009) analyzed term insurance and whole life insurance under the stochastic interest rate approach. They chose a proper distribution through real interest rates taking into account both annual interest rates on deposits and consumer price index rates. The goals of this study, applications were actualized for deterministic and random interest rates by using the actuarial present value of whole life insurance. Then, comparisons were made about obtained results.

1.3 Thesis Outline

This thesis is constituted in four chapters. In Chapter 1, both definitional and numerical information are given relevant to the draft resolution of ministerial cabinet published about temporary twentieth article of the Social Security and General Health Insurance law which will use applications and related to the structure of the foundation funds, as an introduction. In Chapter 2, theoretical knowledge is described concerning interest, mortality, life insurance models, life annuity models, premiums and reserves. In Chapter 3, applications are made depending upon the defined subjects in chapter 1 and chapter 2. Finally, In Chapter 4, Conclusions are told about what expects to the Social Security System with pluses and minuses in the future.

CHAPTER TWO
ACTUARIAL FORMULAS

2.1 Interest

Under this section will be focused on the basic interest concepts which will assist calculation of the life insurance premiums. The most important variable is the interest variable to determine life insurance premiums and the amount of deposit.

Interest may be described by (Ruckman & Francis, 2005, s.1) as "The payment by one party (the borrower) for the use of an asset that belongs to another party (the lender) over a period of time".

2.1.1 Interest Rate

Money is used as a medium of exchange in the purchase of goods and services in our daily lives such that the interest rate is explained as a tool used in this change. The interest rate is usually expressed as a percentage or a decimal and is symbolized by "i". At any time t, the amount of money is represented by $A(t)$, in this case the money that directed to investment at $t=0$ is called as the principal or the capital and is indicated with $A(0)=b$. The amount of interest obtained for the any period t is expressed with $I(t)$. The amount of interest obtained from any time t up to time $(t+s)$ is given with following formula.

$$I(s)=A(t+s)-A(t) \qquad (2.1.1)$$

The annual interest rate from any time t up to time $(t+1)$ is given:

$$i=\frac{A(t+1)-A(t)}{A(t)} \qquad (2.1.2)$$

2.1.2 Accumulated Value and Accumulation Function

At the time $t > 0$, when the a certain amount of money reaches a value, this value is called as accumulated value and is symbolized by $A(t)$. The accumulated value at any time $t \geq 0$ is given:

$$A(t)=A(0).a(t)=b.a(t) \tag{2.1.3}$$

In equation (2.1.3), $a(t)$ is expressed as accumulation function which gives the accumulated value at time $t \geq 0$ of a deposit of 1 unit. At a given time t, the difference between the accumulated value and the principal is defined as the amount of interest. Hence, the time t when may be measured in many different units (days, months, decades, etc.) is determined time from the date of investment. The difference between the accumulated value of the money earned during the nth period and the accumulated value of the money earned during the $(n-1)$th period is described the amount of interest earned during the nth period from the date of investment by $I(n)$.

$$I(n) = A(n) - A(n-1) \quad ; \quad n \geq 1 \tag{2.1.4}$$

2.1.3 The Effective Rate of Interest

(Kellison, 1991) says that about the effective rate of interest "The effective rate of interest i is the amount of money that one unit invested at the beginning of a period will earn during the period, where interest is paid at the end of the period" (s.4). The following equations are obtained according to this description.

$$i = a(1) - a(0) \quad \Rightarrow \quad a(1) = 1 + i \tag{2.1.5}$$

$$i = \frac{a(1) - a(0)}{a(0)} = \frac{(1+i)-1}{1} \tag{2.1.6}$$

If we want to express the effective rate of interest concerning the any n th period using the accumulated value, it is defined as following.

$$i(n) = \frac{A(n) - A(n-1)}{A(n-1)} = \frac{I(n)}{A(n-1)} \quad ; \quad n \geq 1 \tag{2.1.7}$$

2.1.4 Simple and Compound Interest

If one unit is invested in a savings account with simple interest, the amount of interest earned during each period is constant. The accumulated value of one unit at the end of the first period is $1+i$, at the end of the second period it is $1+2i$, etc. Hence, the accumulation function may be obtained as:

$$a(t) = (1+ti) \quad ; \quad t \geq 0 \tag{2.1.8}$$

At the time t, the accumulated value of the principal which is invested in a savings account that pays simple interest at a rate of i per year is:

$$A(t) = A(0)(1+ti) \tag{2.1.9}$$

If one unit is invested in a savings account with compound interest, the total investment of principal and interest earned to date is kept invested at all times. When you invests one unit in a savings account, $1+i$ accumulates at the end of the first period. Thus, the principal happens $1+i$ at the beginning of the second period and this amount earns interest of $i(1+i)$ during the second period; after this, $(1+i)+i(1+i)=(1+i)^2$ accumulates at the end of the second period. Then, the principal happens $(1+i)^2$ at the beginning of the third period and this amount earns interest of $i(1+i)^2$ during the third period; after this, $(1+i)^2+i(1+i)^2=(1+i)^3$ accumulates at the end of the third period. Continuing these calculations indefinitely, in conclusion, the accumulation function may be obtained as:

$$a(t)=(1+i)^t \quad ; \quad t\geq 0 \tag{2.1.10}$$

At the time t, the accumulated value of the principal which is invested in a savings account that pays compound interest at a rate of i per year is:

$$A(t)=A(0)(1+i)^t \tag{2.1.11}$$

For the simple interest operation, the effective rate of interest concerning the any nth period is:

$$i(n)=\frac{a(n)-a(n-1)}{a(n-1)}=\frac{(1+ni)-(1+(n-1)i)}{(1+(n-1)i)}=\frac{i}{1+(n-1)i} \tag{2.1.12}$$

For the compound interest operation, the effective rate of interest concerning the any nth period is:

$$i(n)=\frac{a(n)-a(n-1)}{a(n-1)}=\frac{(1+i)^n-(1+i)^{n-1}}{(1+i)^{n-1}}=\frac{(1+i)-1}{1}=i \tag{2.1.13}$$

2.1.5 Present Value

The value at time 0 (the value of an investment at the beginning of a period) of the accumulated value at the time $t\geq 0$ (the value at the end of the period) is known as present value. This value is symbolized by v and is defined as:

$$v=\frac{1}{1+i}=(1+i)^{-1}=a^{-1}(1) \tag{2.1.14}$$

We understand from the formula (2.1.14) that the reciprocal of the accumulation function $a^{-1}(t)$ is called discount function (the present value function).

For simple interest, the present value of a deposit of one unit and $A(t)$ to be made in t years is:

$$a^{-1}(t)=\frac{1}{1+ti}=(1+ti)^{-1} \tag{2.1.15}$$

$$A(0)=\frac{A(t)}{(1+ti)} \tag{2.1.16}$$

For compound interest, the present value of a deposit of one unit and $A(t)$ to be made in t years is:

$$a^{-1}(t)=\frac{1}{(1+i)^t}=(1+i)^{-t}=v^t \tag{2.1.17}$$

$$A(0)=\frac{A(t)}{(1+i)^t}=A(t)v^t \tag{2.1.18}$$

2.1.6 The Effective Rate of Discount

(Kellison, 1991) says that about the effective rate of discount "The effective rate of interest was defined as a measure of interest paid at the end of the period. The effective rate of discount, denoted by d, as a measure of interest paid at the beginning of the period" (s.12). Relationships between the variables i, d and v may be defined as following:

$$d=iv=(1-v)=\frac{i}{1+i} \tag{2.1.19}$$

$$i=\frac{d}{1-d} \tag{2.1.20}$$

If we want to express the effective rate of discount concerning the any n th period using the accumulated value, it is defined as following.

$$d(n)=\frac{A(n)-A(n-1)}{A(n)}=\frac{I(n)}{A(n)} \quad ; \quad n\geq 1 \tag{2.1.21}$$

For annually simple rate and compound rate of discount of d, the present values of a payments of one unit to be made in t years are:

$$A(0)=A(t)(1-td) \quad ; \quad (Simple \;\; Rate \;\; of \;\; Discount) \tag{2.1.22}$$

$$A(0)=A(t)(1-d)^{t} \quad ; \quad (Compound \;\; Rate \;\; of \;\; Discount) \tag{2.1.23}$$

For annually simple rate and compound rate of discount of d, the accumulated values after t years of a deposit of one unit are:

$$A(t)=A(0)(1-td)^{-1} \quad ; \quad (Simple \;\; Rate \;\; of \;\; Discount) \tag{2.1.24}$$

$$A(t)=A(0)(1-d)^{-t} \quad ; \quad (Compound \;\; Rate \;\; of \;\; Discount) \tag{2.1.25}$$

2.1.7 Constant Force of Interest

Constant force of interest may be described by (Ruckman & Francis, 2005, s.17) as "The case of interest is considered that is compounded continuously. A continuously compounded interest rate is called the force of interest, at time t is denoted δ_t, is the instantaneous change in the account value, expressed as an annualized percentage of the current value ". The constant force of interest rate can be obtained with regards to the annual effective interest rate i as following:

$$\delta = \frac{A'(t)}{A(t)} = \frac{A(0)(1+i)^t \, ln(1+i)}{A(0)(1+i)^t} = ln(1+i) \qquad (2.1.26)$$

Relationships between the variables i, d and v may be defined rearranging $\delta = ln(1+i)$ as following:

$$\delta = ln(1-d)^{-1} \qquad (2.1.27)$$

$$\delta = ln(1+i) \quad \Rightarrow \quad e^{\delta} = 1+i \quad \Rightarrow \quad i = 1-e^{\delta} \qquad (2.1.28)$$

$$1+i = e^{\delta} \quad \Rightarrow \quad v = (1+i)^{-1} = e^{-\delta} \qquad (2.1.29)$$

For a constant force of interest of δ, the accumulated value after t years of a payment of one unit is:

$$A(t) = A(0)e^{\delta t} \qquad (2.1.30)$$

For a constant force of interest of δ, the present value of a payment of one unit to be made in t years is:

$$A(0) = A(t)e^{-\delta t} \qquad (2.1.31)$$

2.1.8 Varying Force of Interest

Differently from section (2.1.7), now, the force of interest will be varying over time. From the time t_1 up to the time t_2 of a payment of one unit, where $t_1 < t_2$, the accumulated value function of the varying force of interest is defined as:

$$a(t) = exp\left[\int_{t_1}^{t_2} \delta_t \, dt\right] \tag{2.1.32}$$

For the varying force of interest, the present value function at time t_1 of a payment of one unit at time t_2 is defined as:

$$a^{-1}(t) = exp\left[-\int_{t_1}^{t_2} \delta_t \, dt\right] \tag{2.1.33}$$

For the varying force of interest, the accumulated value at time t_2 of an amount of money at time t_1 is defined as:

$$A(t_2) = A(t_1) exp\left[\int_{t_1}^{t_2} \delta_t \, dt\right] \tag{2.1.34}$$

For the varying force of interest, the present value at time t_1 of an amount of accumulation at time t_2 is defined as:

$$A(t_1) = A(t_2) exp\left[-\int_{t_1}^{t_2} \delta_t \, dt\right] \tag{2.1.35}$$

2.1.9 Discrete Changes in Interest Rates

In this section, the effective rate of interest will change in the given period of time, but it won't be continuous in this situation. If i_t is the effective interest rate in relation to the any tth $(t \geq 1)$ period of time, the accumulated value function is defined for the discrete changes in interest rates as:

$$a(t) = (1+i_1)(1+i_2)...(1+i_t) = \prod_{k=1}^{t}(1+i_k) \tag{2.1.36}$$

The present value function is defined for the discrete changes in interest rates as:

$$a^{-1}(t) = (1+i_1)^{-1}(1+i_2)^{-1}...(1+i_t)^{-1} = \prod_{k=1}^{t}(1+i_k)^{-1} = \prod_{k=1}^{t}v_k \tag{2.1.37}$$

The accumulated value is defined for the discrete changes in interest rates as:

$$A(t) = A(0)\prod_{k=1}^{t}(1+i_k) \tag{2.1.38}$$

The present value is defined for the discrete changes in interest rates as:

$$A(0) = A(t)\prod_{k=1}^{t}(1+i_k)^{-1} \tag{2.1.39}$$

2.2 Main Annuities

An annuity can be explained as a regular series of payments made at uniform periodic intervals (such as annually or monthly) and all the same amount. There are two types' annuities according to the payments period in the field of banking and insurance business. The first of these, a certain annuity is annuity where the payments continue for a certain period. The second of these, a contingent annuity is an annuity where the payments continue for an uncertain period. Usually, Payments in the banking system enters a certain annuity type, because the time and amount of payment are previously determined. But, Payments in the insurance system are connected to the condition whether or not an event occurs. The possibility of an event is one of the basic principles of insurance.

2.2.1 Annuity-Immediate

An annuity is described an annuity-immediate when the payments of one unit are occurred at the end of each period (at annual intervals) for a series of n payments. For this series, the rate of interest is accepted i from year to year.

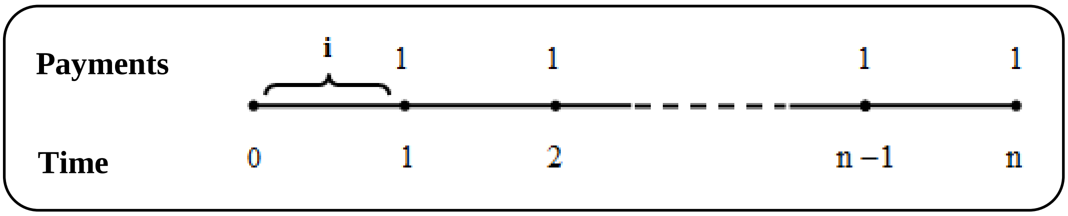

Figure 2.1 Time and payments diagram for the annuity-immediate

The present value of the annuity-immediate is denoted by $a_{\overline{n}|}$ and can be formulated using the generic geometric progression formula as following:

$$a_{\overline{n}|} = v + v^2 + v^3 + \ldots + v^{n-1} + v^n = v\left(1 + v + v^2 + v^3 + \ldots + v^{n-1}\right)$$

$$= v\left(\frac{1-v^n}{1-v}\right) = v\left(\frac{1-v^n}{iv}\right) = \frac{1-v^n}{i} \qquad (2.2.1)$$

The accumulated value of the annuity-immediate is denoted by $s_{\overline{n}|}$, can be formulated multiplying the annuity-immediate present value by the n year accumulated value function.

$$s_{\overline{n}|} = a_{\overline{n}|}(1+i)^n = \left(\frac{1-v^n}{i}\right)(1+i)^n = \frac{(1+i)^n - (1+i)^n v^n}{i} = \frac{(1+i)^n - 1}{i} \qquad (2.2.2)$$

2.2.2 Annuity-Due

An annuity is described an annuity-due when the payments of one unit are occurred at the start of each period (at annual intervals) for a series of n payments.

The only difference from the annuity-immediate is that each payment has been shifted one year earlier. The rate of interest is accepted i from year to year.

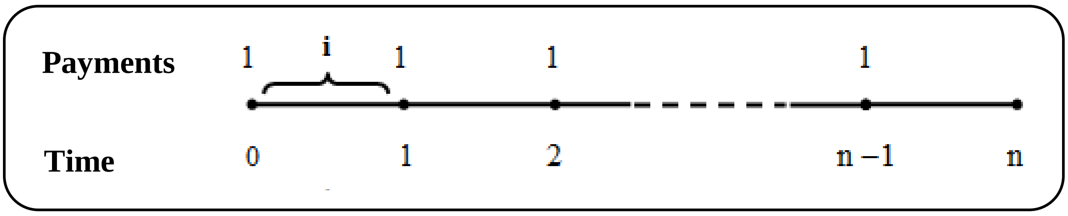

Figure 2.2 Time and payments diagram for the annuity-due

The present value of the annuity-due is denoted by $\ddot{a}_{\overline{n}|}$ and can be formulated using the generic geometric progression formula as following:

$$\ddot{a}_{\overline{n}|} = 1 + v + v^2 + v^3 + \ldots + v^{n-1} = \frac{1-v^n}{1-v} = \frac{1-v^n}{iv} = \frac{1-v^n}{d} \tag{2.2.3}$$

The accumulated value of the annuity-due is denoted by $\ddot{s}_{\overline{n}|}$, can be formulated multiplying the annuity-due present value by the n year accumulated value function.

$$\ddot{s}_{\overline{n}|} = \ddot{a}_{\overline{n}|}(1+i)^n = \left(\frac{1-v^n}{d}\right)(1+i)^n = \frac{(1+i)^n - (1+i)^n v^n}{d} = \frac{(1+i)^n - 1}{d} \tag{2.2.4}$$

2.2.3 Continuously Payable Annuities

An annuity is described a continuously paid annuity when the payments of one unit are occurred at the start or end of each annual time period and continuously.

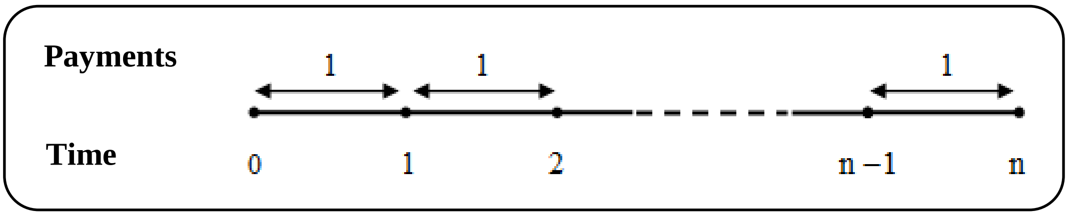

Figure 2.3 Time and payments diagram for the continuously paid annuity

The present value of the continuous annuity is denoted by $\bar{a}_{\overline{n}|}$ and can be formulated for n interest conversion periods using the constant force of interest $\delta = ln(1+i)$, such that all such payments are integrated since the differential expression $v^t\, dt$ is the present value of the payment dt made at exact moment t.

$$\bar{a}_{\overline{n}|} = \int_0^n 1 v^t\, dt = \left(\frac{v^t}{ln\, v} \bigg|_0^n \right) = \frac{v^n - 1}{ln\, v} = \frac{1 - v^n}{ln(1+i)} = \frac{1 - v^n}{\delta} \tag{2.2.5}$$

The accumulated value of the continuous annuity is denoted by $\bar{s}_{\overline{n}|}$, can be formulated multiplying the continuously payable annuity present value by the n year accumulated value function.

$$\bar{s}_{\overline{n}|} = \bar{a}_{\overline{n}|}(1+i)^n = \left(\frac{1 - v^n}{\delta} \right)(1+i)^n = \frac{(1+i)^n - (1+i)^n v^n}{\delta} = \frac{(1+i)^n - 1}{\delta} \tag{2.2.6}$$

2.2.4 Deferred Annuities

An annuity is described a deferred annuity when the payments of one unit are occurred at some point after the first time period. A deferred annuity can be defined for both an annuity-immediate and an annuity-due.

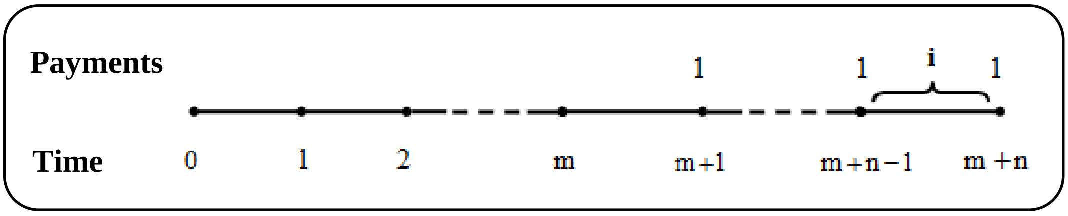

Figure 2.4 Time and payments diagram for the deferred annuity-immediate

(Ruckman & Francis, 2005, s.36) formulates the deferred annuity-immediate present value for the annual effective interest rate i that “The present value at time 0 of an n year annuity immediate that starts in m years where the first payment of one unit occurs at time $m+1$ years and the last payment occurs at time $m+n$ years is”:

$$ {}_{m|}a_{\overline{n}|} = v^m a_{\overline{n}|} \qquad (2.2.7) $$

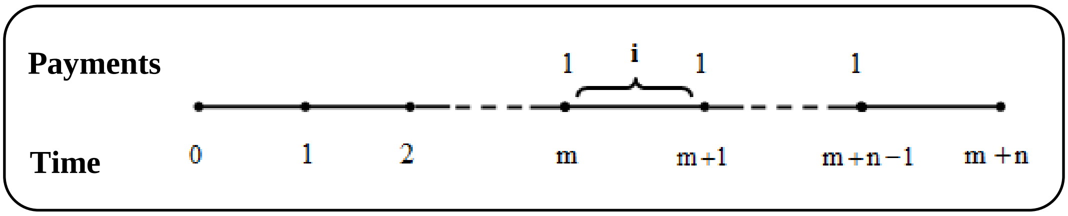

Figure 2.5 Time and payments diagram for the deferred annuity-due

(Ruckman & Francis, 2005, s.36) formulates the deferred annuity-due present value for the annual effective interest rate i that "The present value at time 0 of an n year annuity due that starts in m years where the first payment of one unit occurs at time m years and the last payment occurs at time $m+n-1$ years is":

$$ {}_{m|}\ddot{a}_{\overline{n}|} = v^m \ddot{a}_{\overline{n}|} \qquad (2.2.8) $$

Accumulated values of deferred annuities may be obtained by combining the accumulated value functions from section 2.1.

2.2.5 Perpetuities

An annuity is described a perpetuity when the payments of one unit are continue forever at annual intervals for an infinite series of $n=\infty$ payments. For this series, the rate of interest is accepted i from year to year. Three types of perpetuities are considered. The first type of these, the present value of the perpetuity-immediate is denoted by $a_{\overline{\infty}|}$ and can be formulated using the generic geometric progression formula as following:

$$ a_{\overline{\infty}|} = v + v^2 + v^3 + \ldots = v\left(1 + v + v^2 + v^3 + \ldots\right) = \frac{v}{1-v} = \frac{v}{iv} = \frac{1}{i} \qquad (2.2.9) $$

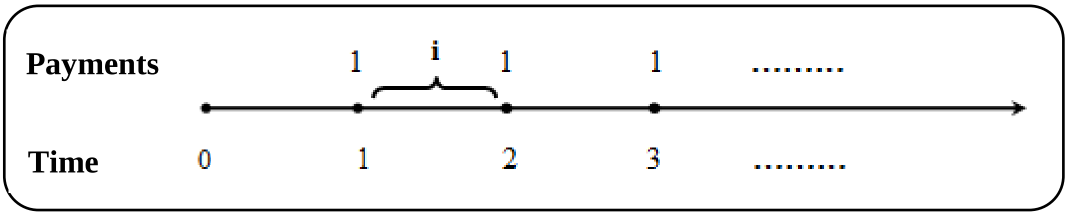

Figure 2.6 Time and payments diagram for the perpetuity-immediate

The second type of these, the present value of the perpetuity-due is denoted by $\ddot{a}_{\overline{\infty|}}$ and can be formulated using the generic geometric progression formula as following:

$$\ddot{a}_{\overline{\infty|}} = 1 + a_{\overline{\infty|}} = 1 + v + v^2 + v^3 + \ldots = \frac{1}{1-v} = \frac{1}{d} \qquad (2.2.10)$$

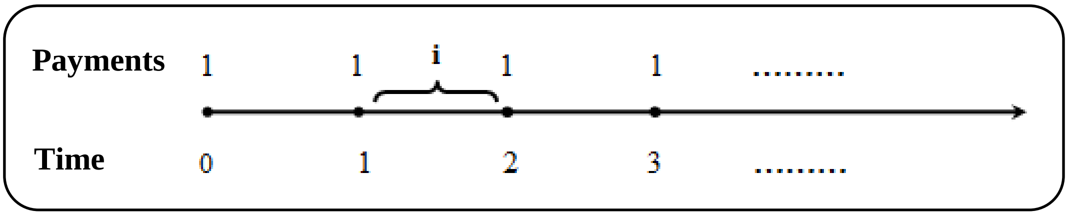

Figure 2.7 Time and payments diagram for the perpetuity-due

The third type of these, the present value of the continuously payable perpetuity is denoted by $\bar{a}_{\overline{\infty|}}$ and can be formulated as following:

$$\bar{a}_{\overline{\infty|}} = \int_0^{\infty} 1 v^t \, dt = \left(\frac{v^t}{\ln v} \bigg|_0^{\infty} \right) = \frac{v^{\infty} - 1}{\ln v} = \frac{1}{\ln(1+i)} = \frac{1}{\delta} \qquad (2.2.11)$$

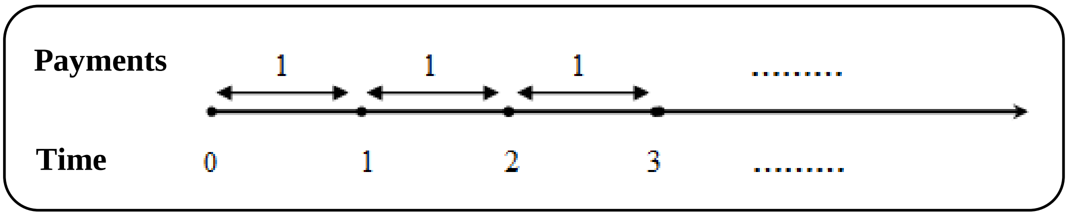

Figure 2.8 Time and payments diagram for the continuously payable perpetuity

The accumulated values of the perpetuities don't obtain, since the payments continue forever.

2.3 Survival Models & Life Tables

A survival model is a probabilistic model of a random variable that deals with death in biological organisms and failure in mechanical systems. Assume that B is a benefit function, v^n is the n year's present value discount factor, i is an effective annual rate of interest; if a random event occurs, the random present value of the payment, Z will be Bv^n. Otherwise, if a random event doesn't occur, Z will be $0(zero)$. Z can describe both discrete and continuous random variable as follows:

$$Z=\begin{cases} Bv^n & ; \quad a\ random\ event\ occurs \\ 0 & ; \quad a\ random\ event\ doesn't\ occur \end{cases} \tag{2.3.1}$$

The expected value of the random present value of payment $E[Z]$ is called the actuarial present value of the insurance. X represents the time until death of a newborn life.

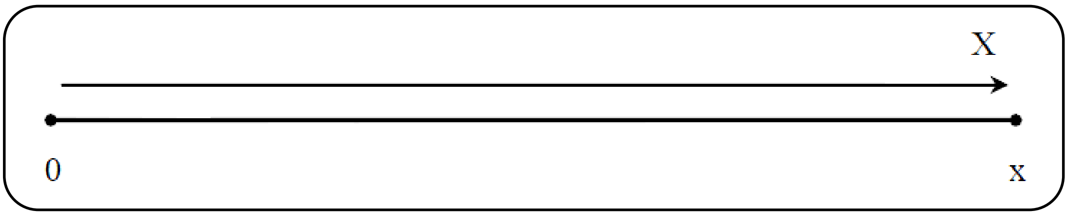

Figure 2.9 The random lifetime

2.3.1 Discrete Survival Models and Mortality Table

Mortality Tables *(life tables)* can be defined as a table of death rates and survival rates for a population. Obtained numerical values for all certain values of x can set a precedent for discrete survival models used in insurance applications. In the mortality table, the radix that is symbolized by l_0 is called the number of newborn lives. This constant describes with numbers such as $1.000, 10.000, 100.000, \ldots$ so that it usually can be increased as the multiples of 10. The ages that are symbolized by x are indicated by the first column in the table and takes integer values in the range of

$[0,w]$. w is the first integer age at which there are no remaining lives in the mortality table. The survivors of that group to age x are represented by the second column in which are symbolized by l_x. The numbers of death in the age range $[x,x+1]$ are presented by the third column in which are symbolized by d_x. It is computed is:

$$d_x = l_x - l_{x+1} \qquad (2.3.2)$$

In the mortality table, the probability of death is usually symbolized by q and so the probability that a life currently age x will die within 1 year is defined in the fourth column in which is denoted by q_x and we have:

$$q_x = \frac{l_x - l_{x+1}}{l_x} = \frac{d_x}{l_x} \qquad (2.3.3)$$

In the mortality table, the possibilities of life is usually symbolized by p and so the probability that a life currently age x will survive 1 year is defined in the fifth column in which is denoted by p_x and we have:

$$p_x = \frac{l_{x+1}}{l_x} \qquad (2.3.4)$$

From equations (2.3.3) and (2.3.4) can be obtained the following results as:

$$p_x + q_x = 1 \qquad (2.3.5)$$

There are lots of special symbols for the more general events that x will survive the different periods of time. Some of them; the conditional probability of surviving to age $x+n$, given alive at age x is had as follows:

$$ {}_{n}p_{x} = \frac{l_{x+n}}{l_{x}} \tag{2.3.6} $$

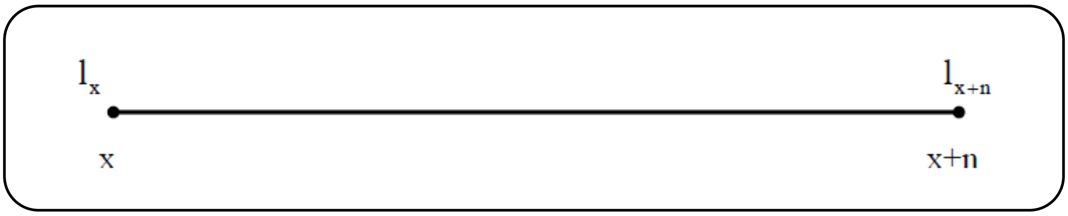

Figure 2.10 A life currently age x will survive n years

The probability that a life currently age x will die within n year is denoted by ${}_{n}q_{x}$ and we have:

$$ {}_{n}q_{x} = \frac{d_{x} + d_{x+1} + ... + d_{x+n-1}}{l_{x}} = \frac{l_{x} - l_{x+n}}{l_{x}} \tag{2.3.7} $$

The probability that a life currently age x will survive for m years and then die within 1 year is denoted by ${}_{m|}q_{x}$ and we have:

$$ {}_{m|}q_{x} = \frac{l_{x+m} - l_{x+m+1}}{l_{x}} = \frac{d_{x+m}}{l_{x}} \tag{2.3.8} $$

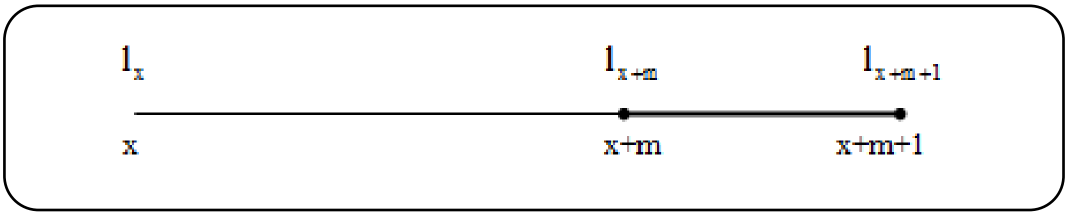

Figure 2.11 A life currently age x will survive for m years and then die within 1 year

The probability that an entity known to be alive at age x will fail between ages $x+m$ and $x+m+n$ is represented by ${}_{m|n}q_{x}$ and we have:

$$ {}_{m|n}q_{x} = \frac{d_{x+m} + d_{x+m+1} + ... + d_{x+m+n-1}}{l_{x}} = \frac{l_{x+m} - l_{x+m+n}}{l_{x}} \tag{2.3.9} $$

The point to consider in equations (2.3.8) and (2.3.9) is that the notation " $|$ " between m and n is called deferment.

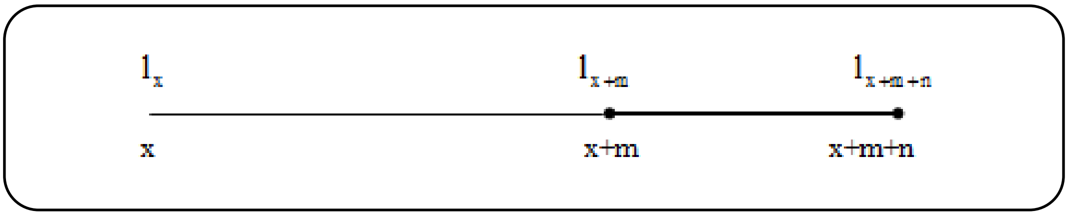

Figure 2.12 A life currently age x will survive for m years and then die within n years

2.3.2 Continuous Survival Models

In this section, four different mathematical functions will be formulated the distribution of X, the random lifetime of a newborn life.

2.3.2.1 Cumulative Distribution Function of X

The cumulative distribution function of the random lifetime of a newborn life X is denoted by $F_X(x)$, is a continuous type random variable and a non-decreasing function with $F_X(0)=0$ and $F_X(w)=1$. We have:

$$F_X(x) = Pr(X \le x) = \int_0^x f_X(u)\,du = {}_xq_0 \quad ; \quad x \ge 0 \tag{2.3.10}$$

2.3.2.2 Probability Density Function of X

The probability density function of the random lifetime of a newborn life X is denoted by $f_X(x)$, is a continuous type random variable and a non-negative function on the interval $[0,w)$ with $\int_0^w f_X(x)\,dx = 1$ and we have:

$$f_X(x) = F_X'(x) = \frac{d}{dx}F_X(x) \quad ; \quad (\textit{wherever the derivative exists}) \qquad (2.3.11)$$

The probability that a newborn life dies between ages x and z $(x < z)$ is:

$$Pr(x < X \leq z) = \int_x^z f_X(u)\,du = F_X(z) - F_X(x) \qquad (2.3.12)$$

2.3.2.3 Survival Function of X

The survival function of the random lifetime of a newborn life X is denoted by $s_X(x)$, represents the probability that a newborn life dies after age x, is a continuous type random variable and a non-increasing function with $s_X(0) = 1$ and $s_X(w) = s_X(\infty) = 0$. We have:

$$s_X(x) = Pr(X > x) = 1 - Pr(X \leq x) = 1 - F_X(x) = {}_x p_0 = \frac{l_x}{l_0} \qquad (2.3.13)$$

The probability that a newborn life dies between ages x and z $(x < z)$ is:

$$Pr(x < X \leq z) = \int_x^z f_X(u)\,du = s_X(x) - s_X(z) \qquad (2.3.14)$$

The relationship of the probability density function of X with the survival function of X is defined as below:

$$f_X(x) = -s_X'(x) = -\frac{d}{dx}s_X(x) \qquad (2.3.15)$$

2.3.2.4 The Force of Mortality

The force of mortality is denoted by $\mu_X(x)$, for each age x , represents the value of the conditional probability density function of X at exact age x, is a piece-wise continuous and a non-negative function with $\int_0^w \mu_X(t)\,dt = \infty$. We have:

$$\mu_X(x)\Delta x \cong Pr\left(x < X \le x+\Delta x \mid X > x\right) = \frac{F_X(x+\Delta x)-F_X(x)}{1-F_X(x)} \cong \frac{f_X(x)\Delta x}{1-F_X(x)} \quad (2.3.16)$$

$$\mu_X(x) = \frac{f_X(x)}{1-F_X(x)} = \frac{f_X(x)}{s_X(x)} = \frac{-\frac{d}{dx}s_X(x)}{s_X(x)} = -\frac{d}{dx}\ln s_X(x) = \frac{l_x'}{l_x} \quad (2.3.17)$$

If we firstly integrate both sides of equation $\mu_X(x) = -\frac{d}{dx}\ln s_X(x)$ from 0 to x and secondly on taking exponentials, the survival function of X is obtained as following:

$$\mu_X(x) = -\frac{d}{dx}\ln s_X(x) \;\Rightarrow\; \int_0^x \mu_X(t)\,dt = -\ln s_X(x)$$

$$\Rightarrow exp\left[\int_0^x \mu_X(t)\,dt\right] = exp\left[-\ln s_X(x)\right]$$

$$\Rightarrow s_X(x) = exp\left[-\int_0^x \mu_X(t)\,dt\right] \quad (2.3.18)$$

2.3.3 Complete – Future – Lifetime

$T(x)$ is called the complete future lifetime at age x, is defined on the interval $[0, w-x]$. Numerically, $T(x) = X - x \mid X > x$ is the value of the complete future lifetime of a person that has survived until age $x(X > x)$. The future time lived after age x is $X - x$.

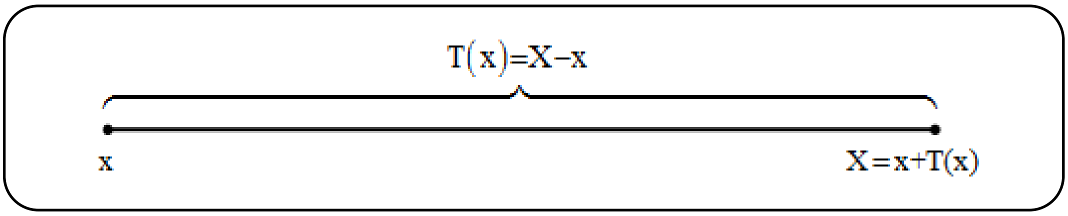

Figure 2.13 The complete future lifetime

2.3.3.1 Survival Function of $T(x)$

The survival function of the continuous random variable $T(x)$ is denoted by $s_{T(x)}(t)$, represents the probability that x is alive at age $x+t$. We have:

$$
\begin{aligned}
s_{T(x)}(t) &= {}_t p_x = Pr\left(T(x) > t\right) = Pr\left(X > x+t \mid X > x\right) \\
&= \frac{Pr\left(X > x+t \cap X > x\right)}{Pr\left(X > x\right)} = \frac{Pr\left(X > x+t\right)}{Pr\left(X > x\right)} = \frac{s_X(x+t)}{s_X(x)} \qquad (2.3.19)
\end{aligned}
$$

2.3.3.2 Cumulative Distribution Function of $T(x)$

The cumulative distribution function of $T(x)$ is denoted by $F_{T(x)}(t)$. We have:

$$
\begin{aligned}
F_{T(x)}(t) &= {}_t q_x = Pr\left(T(x) < t\right) = Pr\left(X \le x+t \mid X > x\right) \\
&= 1 - Pr\left(X > x+t \mid X > x\right) = 1 - \frac{s_X(x+t)}{s_X(x)} \qquad (2.3.20)
\end{aligned}
$$

2.3.3.3 Probability Density Function of $T(x)$

The probability density function of $T(x)$ is denoted by $f_{T(x)}(t)$. We have:

$$f_{T(x)}\left(t\right)=\frac{d}{dt}F_{T(x)}\left(t\right)=-\frac{d}{dt}\frac{s_X\left(x+t\right)}{s_X\left(x\right)}=\frac{f_X\left(x+t\right)}{s_X\left(x\right)} \quad ; \quad 0\le t\le w-x \qquad (2.3.21)$$

$$f_{T(x)}\left(t\right)=\frac{f_X\left(x+t\right)}{s_X\left(x\right)}=\frac{s_X\left(x+t\right)\mu_X\left(x+t\right)}{s_X\left(x\right)} \quad \Rightarrow \quad \mu_X\left(x+t\right)=\frac{f_X\left(x+t\right)}{s_X\left(x+t\right)} \qquad (2.3.22)$$

2.3.4 Curtate – Future – Lifetime

$K\left(x\right)$ is called the curtate future lifetime at age x and the possible values of $K\left(x\right)$ are the numbers $K\left(x\right)=0,1,2,3,...,w-x-1$. Numerically, $K\left(x\right)=\left[T\left(x\right)\right]$ is the value of curtate future lifetime of a person that has survived until at age x, is the greatest integer in $T\left(x\right)$. As a result of these, we have $k\le T\left(x\right)<k+1$.

2.3.4.1 Probability Density Function of $K\left(x\right)$

The probability density function of $K\left(x\right)$ is denoted by $f_{K(x)}\left(k\right)$. We have:

$$\begin{aligned} f_{K(x)}\left(k\right) &= {}_{k|}q_x = Pr\left(K\left(x\right)=k\right)=Pr\left(k\le T\left(x\right)<k+1\right) \\ &= Pr\left(x+k\le X<x+k+1 \middle| X>x\right) \\ &= \frac{d_{x+k}}{l_x}=\frac{l_{x+k}-l_{x+k+1}}{l_x} \quad ; \quad k=0,1,2,...,w-x-1 \end{aligned} \qquad (2.3.23)$$

2.3.4.2 Cumulative Distribution Function of $K\left(x\right)$

The cumulative distribution function of $K\left(x\right)$ is denoted by $F_{K(x)}\left(k\right)$. We have:

$$F_{K(x)}\left(k\right)=Pr\left(K\left(x\right)\le k\right)=Pr\left(K\left(x\right)=0\right)+...+Pr\left(K\left(x\right)=k\right)={}_{k+1}q_x \qquad (2.3.24)$$

2.3.4.3 *Survival Function of* $K(x)$

The survival function of the discrete random variable $K(x)$ is denoted by $s_{K(x)}(k)$, represents the curtate future lifetime after age x. We have:

$$s_{K(x)}(k) = Pr(K(x) > k) = 1 - F_{K(x)}(k) = 1 - {}_{k+1}q_x = {}_{k+1}p_x \qquad (2.3.25)$$

2.3.5 *The Life Table Functions* L_x *and* T_x

"The functions L_x and T_x are useful devices in the calculation of life expectancy. They are defined in terms of the life table function, l_x" (Gauger, 2006, s.21).

The total number of people-years lived after age x by the survivors to age x is denoted by T_x and is described as follows:

$$T_x = \int_x^w l_y dy = L_x + L_{x+1} + \ldots + L_{w-1} \qquad (2.3.26)$$

The number of people-years lived by the survivors to age x during the next year is denoted by L_x and is described as follows:

$$L_x = \int_x^{x+1} l_y dy \qquad (2.3.27)$$

2.3.6 *The Expected Value of* X, $T(x)$ *and* $K(x)$

In this section, we will obtain equations for some commonly used characteristics of the distributions of X, $T(x)$ and $K(x)$, such as the life expectancy of these distributions.

2.3.6.1 *Life Expectancy*

The expected value of time until death of a newborn life is called as life expectancy and is denoted by $\overset{0}{e}_0$ for a continuous and positive-valued random variable X . Thus,

$$\overset{0}{e}_0 = E\left[X\right] = \int_0^\infty x\, f_X\left(x\right)dx \tag{2.3.28}$$

The equation (2.3.28) is developed using integrasyon by parts $\left(u = x \;\; \Rightarrow \;\; du = dx \quad and \quad f_X\left(x\right)dx = dv \;\; \Rightarrow \;\; -s_X\left(x\right) = v\right)$, as follow:

$$\overset{0}{e}_0 = E\left[X\right] = \int_0^\infty x\, f_X\left(x\right)dx = \underbrace{\left(-x\, s_X\left(x\right)\Big|_0^\infty\right)}_{0} + \int_0^\infty s_X\left(x\right)dx = \int_0^\infty s_X\left(x\right)dx \tag{2.3.29}$$

2.3.6.2 *Complete Life Expectancy*

The expected value of $T\left(x\right)$ (the complete future life time at age x) is called as the complete life expectancy and is denoted by $\overset{0}{e}_x$. Thus,

$$\overset{0}{e}_x = E\left[T\left(x\right)\right] = \int_0^{w-x} t\, f_{T(x)}\left(t\right)dt = \underbrace{\int_0^{w-x} s_{T(x)}\left(t\right)dt}_{integrasyon\;\; by\;\; parts}$$

$$= \int_0^{w-x} \frac{l_{x+t}}{l_x}dt = \frac{\int_0^{w-x} l_{x+t}\, dt}{l_x} = \underbrace{\frac{\int_x^{w} l_y dy}{l_x}}_{substitute\;\; y=x+t} = \frac{T_x}{l_x} \tag{2.3.30}$$

The expected value of $T(x)\wedge n$ (the random number of years lived by x in the next n years) is called as the temporary complete life expectancy and is denoted by $\overset{0}{e}_{x:\overline{n}|}$. Thus, $T(x)\wedge n$ defined as:

$$T(x)\wedge n = \begin{cases} T(X) & ; \quad T(x)\le n \\ n & ; \quad T(x) > n \end{cases} \tag{2.3.31}$$

As a result, we have:

$$\overset{0}{e}_{x:\overline{n}|} = E\left[T(x)\wedge n\right] = \underbrace{\int_0^n s_{T(x)}(t)\,dt}_{integrasyon\ by\ parts} = \frac{\int_0^n l_{x+t}\,dt}{l_x} = \underbrace{\frac{\int_x^{x+n} l_y\,dy}{l_x}}_{substitute\ y=x+t} = \frac{T_x - T_{x+n}}{l_x} \tag{2.3.32}$$

2.3.6.3 *Curtate Life Expectancy*

The expected value of $K(x)$ (the curtate future life time at age x) is called as the curtate life expectancy and is denoted by e_x. Thus,

$$\begin{aligned} e_x = E\left[K(x)\right] &= \sum_{k=0}^{w-x-1} k\,Pr\left(K(x)=k\right) = \sum_{k=0}^{w-x-1} k\ {}_{k|}q_x = \sum_{k=0}^{w-x-1} k\,\frac{d_{x+k}}{l_x} \\ &= \frac{d_{x+1} + 2d_{x+2} + \ldots + (w-x-1)\,d_{w-1}}{l_x} \\ &= \frac{l_{x+1} - l_{x+2} + 2\left(l_{x+2} - l_{x+3}\right) + \ldots + (w-x-1)\left(l_{w-1} - l_w\right)}{l_x} \\ &= \frac{l_{x+1} + l_{x+2} + l_{x+3} + \ldots + l_{w-1}}{l_x} = p_x + {}_2p_x + {}_3p_x + \ldots + {}_{w-x-1}p_x \end{aligned} \tag{2.3.33}$$

The expected value of $K(x)\wedge n$ (the random number of full years lived by the life x in the next n years) is called as the temporary curtate life expectancy and is denoted by $e_{x:\overline{n}|}$. Thus, we have:

$$e_{x:\overline{n}|} = E\left[K(x)\wedge n\right] = p_x + {}_2p_x + \ldots + {}_np_x = \frac{l_{x+1} + l_{x+2} + \ldots + l_{x+n}}{l_x} \qquad (2.3.34)$$

2.3.6.4 Central Mortality Rate

"The n year central mortality rate denoted by ${}_nm_x$ computes a weighted average of the force of mortality over the range from age x to age $x+n$" (Gauger, 2006, s.27). Thus, we have:

$${}_nm_x = \frac{\int_x^{x+n} s_X(y)\mu_X(y)dy}{\int_x^{x+n} s_X(y)dy} = \underbrace{\frac{\int_0^n s_X(x+t)\mu_X(x+t)dt}{\int_0^n s_X(x+t)dt}}_{substitute\ \ y=x+t} = \frac{\int_0^n {}_tp_x\mu_X(x+t)dt}{\int_0^n {}_tp_x dt}$$

$$= \frac{{}_nq_x}{\overset{0}{e}_{x:\overline{n}|}} = \frac{{}_nd_x / l_x}{T_x - T_{x+n} / l_x} = \frac{{}_nd_x}{T_x - T_{x+n}} = \frac{{}_nd_x}{L_x + L_{x+1} + \ldots + L_{x+n-1}} \qquad (2.3.35)$$

2.3.6.5 The Function $a(x)$

The average number of years lived between ages x and $x+1$ by those of the survivorship group who die between those ages is represented by $a(x)$ and is the conditional expected value $E\left[T(x)\middle|T(x)\le 1\right]$ since the event $T(x)\le 1$ indicates that the life x dies within a year. Thus, we have:

$$a(x) = E\left[T(x)\middle|T(x)\le 1\right] = \frac{\overset{0}{e}_{x:\overline{1}|} - p_x}{q_x} = \frac{L_x - l_{x+1}}{d_x} \qquad (2.3.36)$$

2.4 Life Insurance

The spouse, child, mother and father who are left behind can live much important negativity to continue their life, if a person dies unexpectedly. On the other hand, not only the death but also the period of retirement can be negative for people, if they don't have enough incomes to continue their life. For these reasons, in order to minimize these negatives, some life insurance products were developed. By the end of this section, we will be able to describe calculates related to the moments and probabilities of several standard life insurance policies.

2.4.1 Discrete Whole Life Insurance

Whole life insurance is called a life insurance contract that pays a death benefit when the policyholder dies, no matter when this may occur. A discrete whole life insurance is supposed that any death benefit is paid on the policy anniversary following death. A payment of one unit is made at a time $K(x)+1$ years after the contract is issued at age x. (Gauger, 2006)

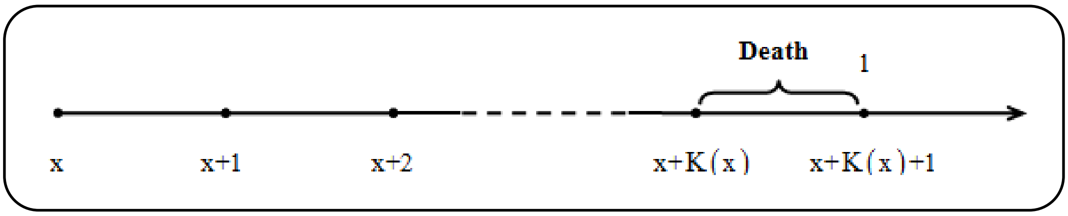

Figure 2.14 Death age and payment diagram for discrete whole life insurance

For this insurance model, the random present value of benefit is defined as:

$$Z = v^{K(x)+1} \quad ; \quad K(x) = 0,1,\ldots,w-x-1 \tag{2.4.1}$$

The actuarial present value of the one unit benefit is defined as the expected value of Z and symbolized by A_x. Then we have:

$$A_x = E[Z] = E\left[v^{K(x)+1}\right] = \sum_{k=0}^{w-x-1} v^{k+1}\ {}_{k|}q_x \qquad (2.4.2)$$

In equation (2.4.2), ${}_{k|}q_x$ is known as a probability function of the curtate lifetime variable $K = K(x)$. If a payment of b_{K+1} is made at a time $K(x)+1$ years after the contract is issued at age x, the actuarial present value of the benefit is defined as:

$$E\left[b_{K+1}v^{K+1}\right] = b_{K+1}E\left[v^{K+1}\right] = b_{K+1}A_x \qquad (2.4.3)$$

2.4.2 Continuous Whole Life Insurance

(Gauger, 2006) A continuous whole life insurance is supposed that the death benefit is paid at the time of death. A payment of one unit is made at a time $T(x)$ years after the contract is issued at age x. For this insurance model, the random present value of benefit is defined as:

$$Z = v^{T(x)} \quad ; \quad 0 < T(x) < w - x \qquad (2.4.4)$$

The actuarial present value of the one unit benefit is defined as the expected value of Z and symbolized by $\bar{A}_x$. Then we have:

$$\bar{A}_x = E[Z] = E\left[v^{T(x)}\right] = \int_0^{w-x} v^t\ f_{T(x)}(t)\,dt = \int_0^{w-x} v^t\ {}_t p_x\ \mu(x+t)\,dt \qquad (2.4.5)$$

In equation (2.4.5), $f_{T(x)}(t)$ is known as a probability function of the complete lifetime variable $T = T(x)$. If a payment of b_T is made at a time $T(x)$ years after the contract is issued at age x, the actuarial present value of the benefit is defined as:

$$E\left[b_T v^T\right] = b_T E\left[v^T\right] = b_T \bar{A}_x \qquad (2.4.6)$$

The most important condition for an insurance company is the probability that Z exceeds $E[Z]$. About this subject, (Gauger, 2006) says that:

- If $Z > E[Z]$ then the insurance company makes a loss on the policy
- If $Z < E[Z]$ then the insurance company makes a profit on the policy
- If $Z = E[Z]$ then the insurance company breaks even on the policy, with zero profit.

2.4.3 Other Types of Life Insurance Policies

Life insurance benefit payments may be made not only at the time of death but also according to a certain condition. In this section, we will analyze several types of life insurance policies. Time of benefit payments will vary from person to person according to either on death or survival to a certain age.

2.4.3.1 Term Life Insurance

"n-year term life insurance provides for a payment only if the insured dies within the n-year term of an insurance commencing at issue" (Bowers, Gerber, Hickman, Jones, Nesbitt, 1997, s.94). For discrete model, the random present value of benefit Z and the amount of the benefit paid b_{K+1} are defined as:

$$b_{K+1} = \begin{cases} 1 & ; \quad K = 0,1,...,n-1 \\ 0 & ; \quad K \geq n \end{cases} \tag{2.4.7}$$

$$Z = \begin{cases} v^{K+1} & ; \quad K \leq n-1 \\ 0 & ; \quad K \geq n \end{cases} \tag{2.4.8}$$

The actuarial present value of the one unit benefit is defined as the expected value of Z and symbolized by $A^{1}_{x:\overline{n|}}$. Then we have:

$$A^{1}_{x:\overline{n}|} = E[Z] = \sum_{k=0}^{n-1} v^{k+1}\, Pr(K = k) = \sum_{k=0}^{n-1} v^{k+1}\ {}_{k|}q_x \tag{2.4.9}$$

For continuous model, the random present value of benefit Z and the amount of the benefit paid b_T is defined as:

$$b_T = \begin{cases} 1 & ;\ T \le n \\ 0 & ;\ T > n \end{cases} \tag{2.4.10}$$

$$Z = \begin{cases} v^T & ;\ T \le n \\ 0 & ;\ T > n \end{cases} \tag{2.4.11}$$

The actuarial present value of the one unit benefit is defined as the expected value of Z and symbolized by $\bar{A}^{1}_{x:\overline{n}|}$. Then we have:

$$\bar{A}^{1}_{x:\overline{n}|} = E[Z] = \int_0^n v^t\ f_{T(x)}(t)\,dt \tag{2.4.12}$$

2.4.3.2 Deferred Life Insurance

"An n-year deferred life insurance provides for a benefit following the death of the insured only if the insured dies at least n years following policy issue" (Bowers, Gerber, Hickman, Jones, Nesbitt, 1997, s.103). For discrete model, the random present value of benefit Z and the amount of the benefit paid b_{K+1} are defined as:

$$b_{K+1} = \begin{cases} 0 & ;\ K \le n-1 \\ 1 & ;\ K \ge n \end{cases} \tag{2.4.13}$$

$$Z = \begin{cases} 0 & ;\ K \le n-1 \\ v^{K+1} & ;\ K \ge n \end{cases} \tag{2.4.14}$$

The actuarial present value of the one unit benefit is defined as the expected value of Z and symbolized by ${}_{n|}A_x$. Then we have:

$$ {}_{n|}A_x = E[Z] = \sum_{k=n}^{w-x-1} v^{k+1}\, Pr(K=k) = \sum_{k=n}^{w-x-1} v^{k+1}\, {}_{k|}q_x \tag{2.4.15} $$

For continuous model, the random present value of benefit Z and the amount of the benefit paid b_T is defined as:

$$ b_T = \begin{cases} 0 & ; \ T \le n \\ 1 & ; \ T > n \end{cases} \tag{2.4.16} $$

$$ Z = \begin{cases} 0 & ; \ T \le n \\ v^T & ; \ T > n \end{cases} \tag{2.4.17} $$

The actuarial present value of the one unit benefit is defined as the expected value of Z and symbolized by ${}_{n|}\bar{A}_x$. Then we have:

$$ {}_{n|}\bar{A}_x = E[Z] = \int_{n}^{w-x} v^t\, f_{T(x)}(t)\, dt \tag{2.4.18} $$

2.4.3.3 Pure Endowment Life Insurance

“An n-year pure endowment life insurance provides for a payment at the end of the n years if and only if the insured survives at the least n years from the time of policy issue” (Bowers, Gerber, Hickman, Jones, Nesbitt, 1997, s.101). The timing of the benefit payment is the same for both the discrete and continuous models. As a result of this, the random present value of benefit Z and the amount of the benefit paid b_T are defined as:

$$b_T = \begin{cases} 0 & ; \quad K \le n \quad ; \quad T \le n \\ 1 & ; \quad K > n \quad ; \quad T > n \end{cases} \tag{2.4.19}$$

$$Z = \begin{cases} 0 & ; \quad K \le n \quad ; \quad T \le n \\ v^n & ; \quad K > n \quad ; \quad T > n \end{cases} \tag{2.4.20}$$

The actuarial present value of the one unit benefit is defined as the expected value of Z and symbolized by $A_{x:\overset{1}{\overline{n|}}}$ or ${}_nE_x$. Then we have:

$$A_{x:\overset{1}{\overline{n|}}} = {}_nE_x = E\left[Z\right] = v^n \ {}_np_x \tag{2.4.21}$$

2.4.3.4 Endowment Life Insurance

"An n-year endowment life insurance provides for an amount to be payable either following the death of the insured or upon the survival of the insured to the end of the n-year term, whichever occurs first" (Bowers, Gerber, Hickman, Jones, Nesbitt, 1997, s.101). For discrete model, the random present value of benefit Z and the amount of the benefit paid b_{K+1} are defined as:

$$b_{K+1} = 1 \quad ; \quad K \ge 0 \tag{2.4.22}$$

$$Z = \begin{cases} v^{K+1} & ; \quad K \le n-1 \\ v^n & ; \quad K \ge n \end{cases} \tag{2.4.23}$$

The actuarial present value of the one unit benefit is defined as the expected value of Z and symbolized by $A_{x:\overline{n|}}$. Then we have:

$$A_{x:\overline{n|}} = E\left[Z\right] = \sum_{k=0}^{n-1} v^{k+1} \ {}_{k|}q_x + v^n \ {}_np_x = A^1_{x:\overline{n|}} + A_{x:\overset{1}{\overline{n|}}} \tag{2.4.24}$$

For continuous model, the random present value of benefit Z and the amount of the benefit paid b_T is defined as:

$$b_T = 1 \quad ; \quad T > 0 \tag{2.4.25}$$

$$Z = \begin{cases} v^T & ; \ T \le n \\ v^n & ; \ T > n \end{cases} \tag{2.4.26}$$

The actuarial present value of the one unit benefit is defined as the expected value of Z and symbolized by $\bar{A}_{x:\overline{n}|}$. Then we have:

$$\bar{A}_{x:\overline{n}|} = E[Z] = \int_0^n v^t \ f_{T(x)}(t)\,dt + v^n \ {}_n p_x = \bar{A}^{1}_{x:\overline{n}|} + A_{x:\overline{n}|}^{\ \ 1} \tag{2.4.27}$$

As a result of all of these, symbolic representations between the actuarial present values of the some insurance policies are given as such in Table 2.1.

Table 2.1 Relationships between the actuarial present values of the insurance policies (Gauger, 2006)

Insurance Type	Discrete	Continuous
Whole Life Insurance	$A_x = A^1_{x:\overline{n}|} + {}_{n|}A_x$	$\bar{A}_x = \bar{A}^1_{x:\overline{n}|} + {}_{n|}\bar{A}_x$
Deferred Life Insurance	${}_{n|}A_x = v^n \ {}_n p_x \ A_{x+n}$	${}_{n|}\bar{A}_x = v^n \ {}_n p_x \ \bar{A}_{x+n}$
Endowment Life Insurance	$A_{x:\overline{n}|} = A^1_{x:\overline{n}|} + A_{x:\overline{n}|}^{\ \ 1}$	$\bar{A}_{x:\overline{n}|} = \bar{A}^1_{x:\overline{n}|} + A_{x:\overline{n}|}^{\ \ 1}$

2.4.4 The Variance for Life Insurance Models

How to calculate the actuarial present value $E[Z]$ is learned in the previous sections, according to both the discrete and continuous models of several life insurance models. In this section, we will define the variance of Z using standard variance formula, in order to “analyze aggregate risk for a group of independent lives of the same age that all purchase the same insurance contract and calculate the probability that the insurance company makes an aggregate profit or a loss across all of these policies (Gauger, 2006, s.61)". By this way, we have:

$$Var(Z) = E\left[Z^2\right] - \left(E[Z]\right)^2 \tag{2.4.28}$$

Making some arrangements will be enough to find the expected value of Z^2. First arrangement will be $E\left[Z^2\right] = E\left[b_{K+1}^2 v^{2(K+1)}\right]$ for discrete model and second arrangement will be $E\left[Z^2\right] = E\left[b_T^2 v^{2T}\right]$ for continuous model. If the force of interest is δ in the continuous model, then; $E\left[Z^2\right] = E\left[b_T^2 v^{2T}\right] E\left[b_T^2 e^{-2\delta T}\right]$, because of relationship $v^T = e^{-\delta T}$ between the present value v^T with the force of interest δ. As a result, $E\left[Z^2\right]$ is equal to $E\left[Z\right]$ calculated using double the original force of interest. Symbolic representations in concern with the second moments of the actuarial present values are given as such in Table 2.2.

Table 2.2 Symbolic representations in concern with the second moment of the actuarial present values

Insurance Type	**Discrete**	**Continuous**
Whole Life Insurance	${}^2A_x$	${}^2\bar{A}_x$
Term Life Insurance	${}^2A^1_{x:\overline{n}\rceil}$	${}^2\bar{A}^1_{x:\overline{n}\rceil}$
Deferred Life Insurance	${}_{n\vert}{}^2A_x$	${}_{n\vert}{}^2\bar{A}_x$
Pure Endowment Life Insurance	${}^2A_{x:\overline{n}\rceil}^{\;\;1}$	${}^2A_{x:\overline{n}\rceil}^{\;\;1}$
Endowment Life Insurance	${}^2A_{x:\overline{n}\rceil}$	${}^2\bar{A}_{x:\overline{n}\rceil}$

Another important rule might have been anticipated under the assumption of a uniform distribution of deaths between fractional ages, then; we have:

$$E\left[\bar{Z}\right] = \frac{i}{\delta} E\left[Z\right] \tag{2.4.29}$$

$$E\left[\bar{Z}^2\right] = \frac{2i + i^2}{2\delta} E\left[Z^2\right] \tag{2.4.30}$$

2.4.5 Aggregate Life Insurance Models

If any insurance company sells the same life insurance policy (n people living independently, ages of all of them x), the random present value of each life insurance policy for $i = 1,2,...,n$ that pays a benefit one unit is denoted Z_i. In this case, the aggregate random present value symbolized by S and defined for the group of insurance policies as:

$$S = Z_1 + Z_2 + ... + Z_n \tag{2.4.31}$$

We know that a sum of many independent and identically distributed random variables (if each variable has finite mean and variance) can approximate the normal distribution according to the central limit theorem. So, we have:

$$E[S] = nE[Z] \tag{2.4.32}$$

$$Var(S) = nVar(Z) \tag{2.4.33}$$

The fund that created by the group of the insurance policies is symbolized F and for meeting all liabilities, we want to calculate whether F exceeds S. Hence, the probability of F must be approximately greater than or equal to S.

$$Pr(S \le F) = Pr\left(\frac{S - E[S]}{\sqrt{Var(S)}} \le \frac{F - E[S]}{\sqrt{Var(S)}} \right) \cong \underbrace{\Phi\left(\frac{F - nE[Z]}{\sqrt{nVar(Z)}} \right)}_{CDF\ of\ the\ standard\ normal\ distribution} \tag{2.4.34}$$

The insurance company will want to guarantee itself against the risk measurements and will make the risk charge to meet all benefit liabilities. Hence, fund F is the $100(1-\alpha)\%$ percentile of the distribution of S:

$$F = E[S] + z_\alpha \sqrt{Var(S)} = nE[Z] + z_\alpha \sqrt{nVar(Z)} \quad ; \quad \left(\alpha = Pr\left(N(0,1) > z_\alpha\right)\right) \quad (2.4.35)$$

In equation (2.4.35), z_α is the standard normal distribution random variable and $Pr(S \leq F) = (1-\alpha)$ is the cumulative area under the standard normal distribution for a given z_α. As a result of these, a single contract premium amount that charged each of the n policyholders is:

$$Premium = \frac{F}{n} = \underbrace{E[Z]}_{single\ benefit\ premium} + \underbrace{\frac{z_\alpha}{\sqrt{n}} \overbrace{\sqrt{Var(Z)}}^{coefficient\ of\ variation}}_{risk\ charge\ per\ policy} \quad (2.4.36)$$

2.5 Life Annuity

In this section, payments will be conditioned on survival differently from section 2.4. “A life annuity is a series of payments made continuously or at equal intervals (such as months, quarters, years) while a given life survives” (Bowers, Gerber, Hickman, Jones, Nesbitt, 1997, s.133). By the end of this section, we will be able to describe calculates related to the moments and probabilities of several standard life annuity policies.

2.5.1 Discrete Whole Life Annuity

A discrete whole life annuity is analyzed two main types; firstly of these is annuity-due (payments are made at the beginnings of the payment intervals), secondly of these is annuity-immediate (payments are made at the ends of such intervals).

For life annuity-due, payments of one unit are made at the start of each year, for as long as x is alive and there are $K(x)+1$ payments in this annuity model.

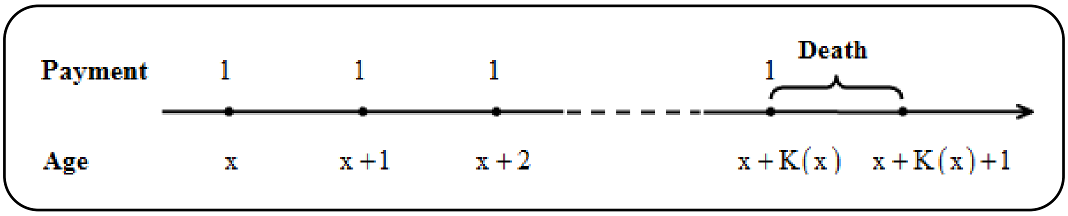

Figure 2.15 The series of payments associated with life annuity-due

For the life annuity-due model, the random present value of payments is defined:

$$Y = \ddot{a}_{\overline{K(x)+1|}} = 1 + v + v^2 + ... + v^{K(x)} = \frac{1 - v^{K(x)+1}}{d} \quad ; \quad K(x) = 0,1,2,...,w - x - 1 \qquad (2.5.1)$$

The actuarial present value of the one unit per year for x is defined as the expected value of Y and symbolized by $\ddot{a}_x$. Then we have:

$$\ddot{a}_x = E[Y] = E\left[\ddot{a}_{\overline{K(x)+1|}}\right] = \sum_{k=0}^{w-x-1} \ddot{a}_{\overline{k+1|}} Pr(K(x) = k) = \sum_{k=0}^{w-x-1} \ddot{a}_{\overline{k+1|}} \, {}_{k|}q_x \qquad (2.5.2)$$

For the life annuity-due model, If a payment of b is made per year for x, the actuarial present value is defined as:

$$E\left[b\ddot{a}_{\overline{K(x)+1|}}\right] = bE\left[\ddot{a}_{\overline{K(x)+1|}}\right] = b\ddot{a}_x \qquad (2.5.3)$$

The equation of (2.5.2) may be reorganized by making some adjustments concerning the probability of ${}_{k|}q_x$. Such that:

$${}_{k|}q_x = \frac{d_{x+k}}{l_x} = \frac{l_{x+k} - l_{x+k+1}}{l_x} = \frac{l_{x+k}}{l_x} - \frac{l_{x+k+1}}{l_x} = {}_k p_x - {}_{k+1} p_x \qquad (2.5.4)$$

An alternative formula can be written utilizing from the relation in the equation of (2.5.4). Then we have (Gauger,2006):

$$\ddot{a}_x = E[Y] = \sum_{k=0}^{w-x-1} \ddot{a}_{\overline{k+1|}}\ {}_{k|}q_x = \sum_{k=0}^{w-x-1} \left(1+v+...+v^k\right)\left({}_k p_x - {}_{k+1} p_x\right)$$

$$= 1\left(1-p_x\right)+\left(1+v\right)\left(p_x - {}_2 p_x\right)+...+\left(1+v+...+v^{w-x-1}\right)\left({}_{w-x-1} p_x - {}_{w-x} p_x\right)$$

$$= 1 + p_x\left(1+v-1\right)+...+ {}_{w-x-1} p_x \left(1+v+...+v^{w-x-1} - 1 - v - ... - v^{w-x-2}\right)$$

$$= 1 + vp_x + v^2\ {}_2 p_x + ... + v^{w-x-1}\ {}_{w-x-1} p_x$$

$$= \sum_{k=0}^{w-x-1} v^k\ {}_k p_x \qquad (2.5.5)$$

The random present value of the life annuity-due can be rewritten depending upon a connection between the life annuity-due and a discrete whole life insurance as:

$$Y = \ddot{a}_{\overline{K(x)+1|}} = \frac{1-v^{K(x)+1}}{d} = \frac{1-Z}{d} \quad ; \quad Z = v^{K(x)+1} \quad ; \quad K(x) = 0,1,2,...,w-x-1 \qquad (2.5.6)$$

The expected value of the equation of (2.5.6) is obtained as:

$$E[Y] = E\left[\ddot{a}_{\overline{K(x)+1|}}\right] = \ddot{a}_x = \frac{1-E\left[v^{K(x)+1}\right]}{d} = \frac{1-E[Z]}{d} = \frac{1-A_x}{d} \qquad (2.5.7)$$

As a result of linear relation of (2.5.6), we have a variance relation for the random present value of the life annuity-due as:

$$Var(Y) = Var\left(\frac{1-Z}{d}\right) = \left(-\frac{1}{d}\right)^2 Var(Z) \quad ; \quad \left(Where \ \underbrace{Var(aZ+b)}_{a \ and \ b \ constant} = a^2 Var(Z)\right)$$

$$= \frac{1}{d^2}\left(E\left[Z^2\right] - \left(E[Z]\right)^2\right) = \frac{{}^2A_x - \left(A_x\right)^2}{d^2} \qquad (2.5.8)$$

For life annuity-immediate, payments of one unit are made at the end of each year, for as long as x is alive and there are $K(x)$ payments in this annuity model.

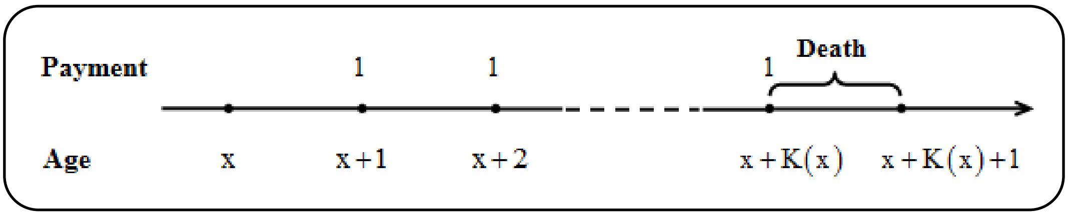

Figure 2.16 The series of payments associated with life annuity-immediate

For the life annuity-immediate model, the random present value of payments is defined as:

$$Y_1 = a_{\overline{K(x)}|} = v + v^2 + ... + v^{K(x)} = \frac{1 - v^{K(x)}}{i} \quad ; \quad K(x) = 1,2,...,w-x-1 \qquad (2.5.9)$$

The actuarial present value of the one unit per year for x is defined as the expected value of Y_1 and symbolized by a_x. Then we have:

$$a_x = E[Y_1] = E\left[a_{\overline{K(x)}|}\right] = \sum_{k=1}^{w-x-1} a_{\overline{k}|} Pr(K(x) = k) = \sum_{k=1}^{w-x-1} a_{\overline{k}|\ k|}q_x \qquad (2.5.10)$$

“The only difference between these payments and those under a life annuity-due is that no payment is made at issue” (Gauger, 2006).

$$a_x = E[Y_1] = E[Y-1] = E[Y] - 1 = \ddot{a}_x - 1 \qquad (2.5.11)$$

For the life annuity-immediate model, If a payment of b is made per year for x, the actuarial present value is defined as:

$$E\left[ba_{\overline{K(x)}|}\right] = bE\left[a_{\overline{K(x)}|}\right] = ba_x \qquad (2.5.12)$$

2.5.2 Continuous Whole Life Annuity

For continuous model of a life annuity, payments of one unit are made continuously each year, while the life x is surviving. The continuous payment stream provides for payments until death. Since one unit is paid for $T(x)$ years, the random present value of payments is defined as:

$$Y = \bar{a}_{\overline{T(x)}|} = \frac{1-v^{T(x)}}{\delta} = \frac{1-e^{-\delta T(x)}}{\delta} \quad ; \; 0 < T(x) < w-x \tag{2.5.13}$$

The actuarial present value of the one unit per year for x is defined as the expected value of Y and symbolized by $\bar{a}_x$. Then we have:

$$\bar{a}_x = E[Y] = E\left[\bar{a}_{\overline{T(x)}|}\right] = \int_0^{w-x} \bar{a}_{\overline{t}|}\, f_{T(x)}(t)\,dt = \int_0^{w-x} \frac{1-e^{-\delta t}}{\delta}\, f_{T(x)}(t)\,dt \tag{2.5.14}$$

The equation (2.5.14) is developed using integrasyon by parts $\left(u = \frac{1-e^{-\delta t}}{\delta} \Rightarrow du = e^{-\delta t}dt \quad and \quad dv = f_{T(x)}(t)\,dt \quad \Rightarrow \quad v = -s_{T(x)}(t) = -{}_t p_x \right)$ as follow:

$$\bar{a}_x = \int_0^{w-x} \frac{1-e^{-\delta t}}{\delta}\, f_{T(x)}(t)\,dt = \underbrace{\left(\frac{1-e^{-\delta t}}{\delta}\left(-{}_t p_x\right)\Bigg|_0^{w-x} \right)}_{0} - \int_0^{w-x} \left(-{}_t p_x\right) e^{-\delta t}dt$$

$$= \int_0^{w-x} v^t\, {}_t p_x\, dt \tag{2.5.15}$$

For the continuous life annuity model, If a payment of b is made continuously for x, the actuarial present value is defined as:

$$E\left[b\bar{a}_{\overline{T(x)}|}\right] = bE\left[\bar{a}_{\overline{T(x)}|}\right] = b\bar{a}_x \tag{2.5.16}$$

The random present value of the continuous life annuity can be rewritten depending upon a connection between the continuous life annuity and a continuous whole life insurance as:

$$Y=\bar{a}_{\overline{T(x)}|}=\frac{1-v^{T(x)}}{\delta}=\frac{1-e^{-\delta T(x)}}{\delta}=\frac{1-\bar{Z}}{\delta}\quad ;\quad \bar{Z}=v^{T(x)}\quad ;\quad 0<T(x)<w-x \qquad (2.5.17)$$

The expected value of the equation of (2.5.17) is obtained as:

$$E[Y]=E\left[\bar{a}_{\overline{T(x)}|}\right]=\bar{a}_x=\frac{1-E\left[v^{T(x)}\right]}{\delta}=\frac{1-E\left[\bar{Z}\right]}{\delta}=\frac{1-\bar{A}_x}{\delta} \qquad (2.5.18)$$

We can derive a continuous relation from the equation of (2.5.18) as following:

$$\bar{a}_x=\frac{1-\bar{A}_x}{\delta}\ \Rightarrow\ 1=\bar{A}_x+\delta\,\bar{a}_x \qquad (2.5.19)$$

As a result of linear relation of (2.5.17), we have a variance relation for the random present value of the continuous life annuity as:

$$Var(Y)=Var\left(\frac{1-\bar{Z}}{\delta}\right)=\left(-\frac{1}{\delta}\right)^2 Var\left(\bar{Z}\right)\quad ;\quad \left(Where\ \ \underbrace{Var\left(aZ+b\right)}_{a\ and\ b\ constant}=a^2Var\left(Z\right)\right)$$

$$=\frac{1}{\delta^2}\left(E\left[\bar{Z}^2\right]-\left(E\left[\bar{Z}\right]\right)^2\right)=\frac{{}^2\bar{A}_x-\left(\bar{A}_x\right)^2}{\delta^2} \qquad (2.5.20)$$

2.5.3 Other Types of Life Annuity Models

Life annuity payments may be made not only at the time of survival but also according to a certain condition. In this section, we will analyze several types of life annuity models. Time of payments will vary from person to person according to survival up to a certain age.

2.5.3.1 Temporary Life Annuities

"For an n-year temporary life annuity, payments are made while x survives only during the next n years. Put it differently, Payments cease on the earlier of the death of the policyholder or the expiration of n years after the date of issue" (Gauger, 2006, s.89). For discrete model, the random present value of an n-year temporary life annuity-due is defined as:

$$Y = \frac{1-Z}{d} = \begin{cases} \ddot{a}_{\overline{K(x)+1}|} = \dfrac{1-v^{K(x)+1}}{d} & ; \quad K(x) \le n-1 \\ \ddot{a}_{\overline{n}|} = \dfrac{1-v^{n}}{d} & ; \quad K(x) \ge n \end{cases} \tag{2.5.21}$$

Where

$$Z = \begin{cases} v^{K(x)+1} & ; \quad K(x) \le n-1 \\ v^{n} & ; \quad K(x) \ge n \end{cases} \qquad (endovment \quad insurance) \tag{2.5.22}$$

The actuarial present value of the one unit per year is defined as the expected value of Y and symbolized by $\ddot{a}_{x:\overline{n}|}$. Then we have:

$$\ddot{a}_{x:\overline{n}|} = E[Y] = \sum_{k=0}^{n-1} \ddot{a}_{\overline{k+1}|}\ {}_{k|}q_x + \sum_{k=n}^{w-x-1} \ddot{a}_{\overline{n}|}\ {}_{k|}q_x = \sum_{k=0}^{n-1} v^k\ {}_k p_x \tag{2.5.23}$$

The equation (2.5.23) can be rewritten depending upon a connection between the n-year temporary life annuity-due and a discrete term life insurance as:

$$\ddot{a}_{x:\overline{n}|} = E[Y] = \frac{1-E[Z]}{d} = \frac{1-A_{x:\overline{n}|}}{d} \quad \Rightarrow \quad 1 = A_{x:\overline{n}|} + d\ddot{a}_{x:\overline{n}|} \tag{2.5.24}$$

As a result of linear relation of (2.5.21), we have a variance relation for the random present value of the n-year temporary life annuity-due as:

$$Var(Y) = Var\left(\frac{1-Z}{d}\right) = \left(-\frac{1}{d}\right)^2 Var(Z) \quad ; \quad \left(Where \ \underbrace{Var(aZ+b)}_{a \ and \ b \ constant} = a^2 Var(Z) \right)$$

$$= \frac{1}{d^2}\left(E\left[Z^2\right] - \left(E[Z]\right)^2\right) = \frac{{}^2A_{x:\overline{n}|} - \left(A_{x:\overline{n}|}\right)^2}{d^2} \tag{2.5.25}$$

For continuous model, the random present value of an n-year temporary life annuity is defined as:

$$Y = \frac{1-\bar{Z}}{\delta} = \begin{cases} \bar{a}_{\overline{T(x)}|} = \dfrac{1-v^{T(x)}}{\delta} & ; \ T(x) \le n \\ \bar{a}_{\overline{n}|} = \dfrac{1-v^n}{\delta} & ; \ T(x) > n \end{cases} \tag{2.5.26}$$

Where

$$\bar{Z} = \begin{cases} v^{T(x)} & ; \ T(x) \le n \\ v^n & ; \ T(x) > n \end{cases} \qquad (endowment \ \ insurance) \tag{2.5.27}$$

The actuarial present value of the one unit per year is defined as the expected value of Y and symbolized by $\bar{a}_{x:\overline{n}|}$. Then we have:

$$\bar{a}_{x:\overline{n}|} = E[Y] = \int_0^n \bar{a}_{\overline{t}|} \ f_{T(x)}(t)\,dt + \int_n^{w-x} \bar{a}_{\overline{n}|} \ f_{T(x)}(t)\,dt = \int_0^n v^t \ {}_t p_x \, dt \tag{2.5.28}$$

The equation (2.5.28) can be rewritten depending upon a connection between the n-year continuous temporary life annuity and a continuous term life insurance as:

$$\bar{a}_{x:\overline{n}|} = \frac{1-E\left[\bar{Z}\right]}{\delta} = \frac{1-\bar{A}_{x:\overline{n}|}}{\delta} \quad \Rightarrow \quad 1 = \bar{A}_{x:\overline{n}|} + \delta \bar{a}_{x:\overline{n}|} \tag{2.5.29}$$

As a result of linear relation of (2.5.26), we have a variance relation for the random present value of the n-year continuous temporary life annuity as:

$$Var(Y) = Var\left(\frac{1-\bar{Z}}{\delta}\right) = \left(-\frac{1}{\delta}\right)^2 Var(\bar{Z}) \quad ; \quad \left(Where \ \underbrace{Var(aZ+b)}_{a \ and \ b \ constant} = a^2 Var(Z)\right)$$

$$= \frac{1}{\delta^2}\left(E\left[\bar{Z}^2\right] - \left(E\left[\bar{Z}\right]\right)^2\right) = \frac{{}^2\bar{A}_{x:\overline{n}|} - \left(\bar{A}_{x:\overline{n}|}\right)^2}{\delta^2} \tag{2.5.30}$$

2.5.3.2 Deferred Life Annuities

"For an n-year deferred life annuity, payments are made while x survives only after an n-year period (known as a waiting period). Put it differently, Payments begin n years after issue and continue until the policyholder's death" (Gauger, 2006, s.90). For discrete model, the random present value of an n-year deferred life annuity-due is defined as:

$$Y = \begin{cases} 0 & ; \ K(x) \le n-1 \\ \ddot{a}_{\overline{K(x)+1}|} - \ddot{a}_{\overline{n}|} = \dfrac{\left(v^n - v^{K(x)+1}\right)}{d} & ; \ K(x) \ge n \end{cases} \tag{2.5.31}$$

The actuarial present value is defined as the expected value of Y and symbolized by ${}_{n|}\ddot{a}_x$. The prefix "${}_{n}|$" explains that payments won't start until n years after issue. Then we have:

$${}_{n|}\ddot{a}_x = E[Y] = \sum_{k=n}^{w-x-1}\left(\ddot{a}_{\overline{k+1}|} - \ddot{a}_{\overline{n}|}\right){}_{k|}q_x = \sum_{k=n}^{w-x-1} v^k \, {}_k p_x \tag{2.5.32}$$

The equation (2.5.32) can be rewritten depending upon the actuarial present value of the life annuity-due for as long as $x+n$ is alive as following:

$$ {}_{n|}\ddot{a}_x = v^n \, {}_n p_x \sum_{m=0}^{w-x-n-1} v^m \, {}_m p_{x+n} = v^n \, {}_n p_x \, \ddot{a}_{x+n} \quad ; \quad (m = k - n) \tag{2.5.33} $$

Y, Y_1 and Y_2 respectively denote the random present value random variables for discrete whole life, n-year temporary and n-year deferred annuity-due of one unit per year on a life x (Gauger, 2006, s.97). As previously observed, we have:

$$ Y = Y_1 + Y_2 \tag{2.5.34} $$

And

$$ Y_1 Y_2 = \ddot{a}_{\overline{n}|} Y_2 \tag{2.5.35} $$

So, we have:

$$ Var(Y) = Var(Y_1 + Y_2) = Var(Y_1) + Var(Y_2) + 2Cov(Y_1, Y_2) \tag{2.5.36} $$

Where

$$ Cov(Y_1, Y_2) = E[Y_1 Y_2] - E[Y_1] E[Y_2] \tag{2.5.37} $$

By using the equations (2.5.35) and (2.5.36), we have a variance relation the present value variable associated with an n-year deferred life annuity-due of one unit per year on x:

$$ \begin{aligned} Var(Y_2) &= Var(Y) - Var(Y_1) - 2\left(E[Y_1 Y_2] - E[Y_1] E[Y_2]\right) \\ &= \frac{{}^2A_x - (A_x)^2}{d^2} - \frac{{}^2A_{x:\overline{n}|} - \left(A_{x:\overline{n}|}\right)^2}{d^2} - 2\left(E\left[\ddot{a}_{\overline{n}|} Y_2\right] - \ddot{a}_{x:\overline{n}|} \, {}_{n|}\ddot{a}_x\right) \\ &= \frac{{}^2A_x - (A_x)^2 - {}^2A_{x:\overline{n}|} + \left(A_{x:\overline{n}|}\right)^2}{d^2} - 2\left[{}_{n|}\ddot{a}_x \left(\ddot{a}_{\overline{n}|} - \ddot{a}_{x:\overline{n}|}\right)\right] \end{aligned} \tag{2.5.38} $$

For continuous model, the random present value of an n-year temporary life annuity is defined as:

$$Y = \begin{cases} 0 & ; \; T(x) \le n \\ \bar{a}_{\overline{T(x)}|} - \bar{a}_{\overline{n}|} = \dfrac{\left(v^n - v^{T(x)}\right)}{\delta} & ; \; T(x) > n \end{cases} \tag{2.5.39}$$

The actuarial present value is defined as the expected value of Y and symbolized by ${}_{n|}\bar{a}_x$. Then we have:

$${}_{n|}\bar{a}_x = E[Y] = \int_n^{w-x} \left(\bar{a}_{\overline{t}|} - \bar{a}_{\overline{n}|}\right) f_{T(x)}(t)\, dt = \int_n^{w-x} v^t \, {}_t p_x \, dt \tag{2.5.40}$$

The equation (2.5.40) can be rewritten depending upon the actuarial present value of the continuous life annuity for as long as $x+n$ is alive as following:

$${}_{n|}\bar{a}_x = v^n \, {}_n p_x \int_0^{w-x-n} v^t \, {}_t p_{x+n} \, dt = v^n \, {}_n p_x \, \bar{a}_{x+n} \tag{2.5.41}$$

Y, Y_1 and Y_2 respectively denote the random present value random variables for continuous whole life, n-year temporary and n-year deferred annuity of one unit per year on a life x (Gauger, 2006, s.97). As previously observed, we have:

$$\bar{Y} = \bar{Y}_1 + \bar{Y}_2 \tag{2.5.42}$$

And

$$\bar{Y}_1 \bar{Y}_2 = \bar{a}_{\overline{n}|} \bar{Y}_2 \tag{2.5.43}$$

So, we have:

$$Var\left(\bar{Y}\right)=Var\left(\bar{Y}_1+\bar{Y}_2\right)=Var\left(\bar{Y}_1\right)+Var\left(\bar{Y}_2\right)+2Cov\left(\bar{Y}_1,\bar{Y}_2\right) \tag{2.5.44}$$

Where

$$Cov\left(\bar{Y}_1,\bar{Y}_2\right)=E\left[\bar{Y}_1\bar{Y}_2\right]-E\left[\bar{Y}_1\right]E\left[\bar{Y}_2\right] \tag{2.5.45}$$

By using the equations (2.5.43) and (2.5.44), we have a variance relation the present value variable associated with an n-year continuous deferred life annuity of one unit per year on x:

$$\begin{aligned}
Var\left(\bar{Y}_2\right)&=Var\left(\bar{Y}\right)-Var\left(\bar{Y}_1\right)-2\left(E\left[\bar{Y}_1\bar{Y}_2\right]-E\left[\bar{Y}_1\right]E\left[\bar{Y}_2\right]\right)\\
&=\frac{{}^2\bar{A}_x-\left(\bar{A}_x\right)^2}{\delta^2}-\frac{{}^2\bar{A}_{x:\overline{n}|}-\left(\bar{A}_{x:\overline{n}|}\right)^2}{\delta^2}-2\left(E\left[\bar{a}_{\overline{n}|}\bar{Y}_2\right]-\bar{a}_{x:\overline{n}|}\,{}_{n|}\bar{a}_x\right)\\
&=\frac{{}^2\bar{A}_x-\left(\bar{A}_x\right)^2-{}^2\bar{A}_{x:\overline{n}|}+\left(\bar{A}_{x:\overline{n}|}\right)^2}{\delta^2}-2\left[{}_{n|}\bar{a}_x\left(\bar{a}_{\overline{n}|}-\bar{a}_{x:\overline{n}|}\right)\right]
\end{aligned} \tag{2.5.46}$$

2.5.3.3 Life Annuity Certain

For an n-year life annuity certain, payments are guaranteed for first n year whether or not the annuitant is alive. Put it differently, if the annuitant is living after the guaranteed number of payments have been made, the income continuous for life. If the annuitant dies within the guarantee period, the balance is paid to a beneficiary (http://www.allbusiness.com). For discrete model, the random present value of an n-year life annuity certain is defined as:

$$Y=\left\{\begin{array}{lll}\ddot{a}_{\overline{n}|}=\dfrac{1-v^n}{d} & ; & K\left(x\right)\le n-1\\ \ddot{a}_{\overline{K(x)+1}|}=\dfrac{1-v^{K(x)+1}}{d} & ; & K\left(x\right)\ge n\end{array}\right\}=\ddot{a}_{\overline{n}|}+Y_1 \tag{2.5.47}$$

In the above equation (2.5.47), Y_1 is defined as the random present value of n-year deferred life annuity-due. The actuarial present value is defined as the expected value of Y and symbolized by $\ddot{a}_{\overline{x:n}|}$. Then we have:

$$\ddot{a}_{\overline{x:n}|} = E\left[Y\right] = \sum_{k=0}^{n-1} \ddot{a}_{\overline{n}|}\ {}_{k|}q_x + \sum_{k=n}^{w-x-1} \ddot{a}_{\overline{k+1}|}\ {}_{k|}q_x = \ddot{a}_{\overline{n}|} + \sum_{k=n}^{w-x-1} v^k\ {}_k p_x = \ddot{a}_{\overline{n}|} + {}_{n|}\ddot{a}_x \qquad (2.5.48)$$

As a result of linear relation of (2.5.47), we have a variance relation for the random present value of the n-year certain life annuity-due as:

$$Var\left(Y\right) = Var\left(\ddot{a}_{\overline{n}|} + Y_1\right) = Var\left(Y_1\right) \ ; \ \left(Where \ \ Var\left(Y_1 + \underset{constant}{b} \right) = Var\left(Y_1\right) \right) \qquad (2.5.49)$$

We understand from equation (2.5.49) that the variance relation of the n-year certain life annuity-due is equal the variance relation of the n-year deferred life annuity-due. This relation was given in the equation (2.5.38).

For continuous model, the random present value of an n-year life annuity certain is defined as:

$$Y = \begin{Bmatrix} \bar{a}_{\overline{n}|} = \dfrac{1-v^n}{\delta} & ; \ T\left(x\right) \le n \\ \bar{a}_{\overline{T(x)}|} = \dfrac{1-v^{T(x)}}{\delta} & ; \ T\left(x\right) > n \end{Bmatrix} = \bar{a}_{\overline{n}|} + Y_1 \qquad (2.5.50)$$

In the above equation (2.5.50), Y_1 is defined as the random present value of n-year continuous deferred life annuity. The actuarial present value is defined as the expected value of Y and symbolized by $\bar{a}_{\overline{x:n}|}$. Then we have:

$$\bar{a}_{\overline{x:n}|} = E\left[Y\right] = \int_0^n \bar{a}_{\overline{n}|}\ f_{T(x)}\left(t\right)dt + \int_n^{w-x} \bar{a}_{\overline{t}|}\ f_{T(x)}\left(t\right)dt = \bar{a}_{\overline{n}|} + \int_0^n v^t\ {}_t p_x\, dt = \bar{a}_{\overline{n}|} + {}_{n|}\bar{a}_x \qquad (2.5.51)$$

As a result of linear relation of (2.5.50), we have a variance relation for the random present values of an n-year continuous certain life annuity as:

$$Var(Y) = Var\left(\bar{a}_{\overline{n}|} + Y_1\right) = Var(Y_1) \quad ; \quad \left(Where \quad Var\left(Y_1 + \underset{constant}{b}\right) = Var(Y_1)\right) \qquad (2.5.52)$$

We understand from equation (2.5.52) that the variance relation of the n-year continuous certain life annuity is equal the variance relation of the n-year continuous deferred life annuity. This relation was given in the equation (2.5.46).

2.5.4 Aggregate Life Annuity Models

If any insurance company sells the same life annuity policy (n people living independently, ages of all of them x), the random present value of each life annuity policy for $i = 1,2,...,n$ that a payment of one unit is denoted Y_i. In this case, the aggregate random present value symbolized by S and defined for the group of annuity policies as:

$$S = Y_1 + Y_2 + ... + Y_n \qquad (2.5.53)$$

We know that a sum of many independent and identically distributed random variables (if each variable has finite mean and variance) can approximate the normal distribution according to the central limit theorem. So, we have:

$$E[S] = n\,E[Y] \qquad (2.5.54)$$

$$Var(S) = nVar(Y) \qquad (2.5.55)$$

The fund that created by the group of the annuity policies is symbolized F and is adequate to provide the annuity payments for all n lives. We want to calculate whether F exceeds S. Hence, the probability of F must be approximately greater than or equal to S.

$$Pr\left(S \le F\right) = Pr\left(\frac{S - E\left[S\right]}{\sqrt{Var\left(S\right)}} \le \frac{F - E\left[S\right]}{\sqrt{Var\left(S\right)}}\right) \cong \underbrace{\Phi\left(\frac{F - nE\left[Y\right]}{\sqrt{nVar\left(Y\right)}}\right)}_{CDF\ of\ the\ standard\ normal\ distribution} \tag{2.5.56}$$

The insurance company will want to guarantee itself against the risk measurements and will make the risk charge to provide the annuity payments for all n lives. Hence, fund F is the $100\left(1-\alpha\right)\%$ percentile of the distribution of S:

$$F = E\left[S\right] + z_\alpha \sqrt{Var\left(S\right)} = nE\left[Y\right] + z_\alpha \sqrt{nVar\left(Y\right)} \ ; \ \left(\alpha = Pr\left(N\left(0,1\right) > z_\alpha\right)\right) \tag{2.5.57}$$

In equation (2.5.57), z_α is the standard normal distribution random variable and $Pr\left(S \le F\right) = \left(1-\alpha\right)$ is the cumulative area under the standard normal distribution for a given z_α. As a result of these, a single contract premium amount that needed per life is a premium of:

$$Premium = \frac{F}{n} = \underbrace{E\left[Y\right]}_{single\ benefit\ premium} + \underbrace{\frac{z_\alpha}{\sqrt{n}} \overbrace{\sqrt{Var\left(Y\right)}}^{coefficient\ of\ variation}}_{risk\ charge\ per\ policy} \tag{2.5.58}$$

2.6 Commutation Functions

The Commutation Functions may be described by (Slud, 2001, s.147) as “A computational device to ensure that net single premiums for life annuities, endowments, and insurances from the same life table and figured at the same interest rate, for lives of differing ages and for policies of differing durations, can all be obtained from a single table look-up”. In this section, we will derive new formulas using discrete survival models and mortality table functions.

2.6.1 Commutation Functions for Whole Life Annuity

In the previous section 2.5, the actuarial present value of the whole life annuity-due was described as the formula (2.5.5). When this formula was written as expansion in a series, the following equation is obtained.

$$\ddot{a}_x = \sum_{k=0}^{w-x-1} v^k \; {}_k p_x = \frac{l_x + v\,l_{x+1} + v^2\,l_{x+2} + v^3\,l_{x+3} + ... + v^{w-x-1}\,l_{w-1}}{l_x} \qquad (2.6.1)$$

The numerator and denominator of the equation (2.6.1) are multiplied by v^x, the following equation is obtained.

$$\ddot{a}_x = \frac{v^x\,l_x + v^{x+1}\,l_{x+1} + v^{x+2}\,l_{x+2} + v^{x+3}\,l_{x+3} + ... + v^{w-1}\,l_{w-1}}{v^x\,l_x} \qquad (2.6.2)$$

Where

$$D_x = v^x\,l_x$$

$$D_{x+1} = v^{x+1}\,l_{x+1}$$

....

$$D_{w-1} = v^{w-1}\,l_{w-1}$$

And

$$N_x = D_x + D_{x+1} + D_{x+2} + ... + D_{w-1}$$

$$N_{x+1} = D_{x+1} + D_{x+2} + ... + D_{w-1}$$

....

$$N_{w-1} = D_{w-1}$$

Are described, a new formula is defined to express the actuarial present value of the whole life annuity-due with D_x and N_x commutation functions as following:

$$\ddot{a}_x = \frac{D_x + D_{x+1} + D_{x+2} + ... + D_{w-1}}{D_x} = \frac{N_x}{D_x} \tag{2.6.3}$$

The other new formula is defined to express the actuarial present value of the whole life annuity-immediate with D_x and N_x commutation functions, using the equation (2.5.11), as following:

$$\ddot{a}_x = 1 + a_x = 1 + \frac{D_{x+1} + D_{x+2} + ... + D_{w-1}}{D_x} = 1 + \frac{N_{x+1}}{D_x} \quad \Rightarrow \quad a_x = \frac{N_{x+1}}{D_x} \tag{2.6.4}$$

2.6.2 Commutation Functions for Temporary Life Annuity

In the previous section 2.5, the actuarial present value of the temporary life annuity-due was described as the formula (2.5.23). When this formula was written as expansion in a series, the following equation is obtained.

$$\ddot{a}_{x:\overline{n}|} = \sum_{k=0}^{n-1} v^k \, {}_k p_x = \frac{l_x + v\, l_{x+1} + v^2\, l_{x+2} + v^3\, l_{x+3} + ... + v^{n-1}\, l_{x+n-1}}{l_x} \tag{2.6.5}$$

The numerator and denominator of the equation (2.6.5) are multiplied by v^x, the following equation is obtained.

$$\ddot{a}_{x:\overline{n}|} = \frac{v^x l_x + v^{x+1}\, l_{x+1} + v^{x+2}\, l_{x+2} + v^{x+3}\, l_{x+3} + ... + v^{x+n-1}\, l_{x+n-1}}{v^x l_x} \tag{2.6.6}$$

With some rearrangements, Where

$$N_x = D_x + D_{x+1} + \ldots + D_{x+n-1} + D_{x+n} + \ldots + D_{w-1}$$

$$N_{x+n} = D_{x+n} + D_{x+n+1} + \ldots + D_{w-1}$$

$$-$$

$$N_x - N_{x+n} = D_x + D_{x+1} + \ldots + D_{x+n-1}$$

Is described, a new formula is defined to express the actuarial present value of the temporary life annuity-due with D_x and N_x commutation functions as following:

$$\ddot{a}_{x:\overline{n}|} = \frac{D_x + D_{x+1} + \ldots + D_{x+n-1}}{D_x} = \frac{N_x - N_{x+n}}{D_x} \tag{2.6.7}$$

The other new formula is defined to express the actuarial present value of the temporary life annuity-immediate with D_x and N_x commutation functions as following:

$$a_{x:\overline{n}|} = \sum_{k=1}^{n} v^k \, {}_k p_x = \frac{D_{x+1} + \ldots + D_{x+n-1}}{D_x} = \frac{N_{x+1} - N_{x+n+1}}{D_x} \tag{2.6.8}$$

2.6.3 Commutation Functions for Deferred Life Annuity

In the previous section 2.5, the actuarial present value of the deferred life annuity-due was described as the formula (2.5.32). When this formula was written as expansion in a series, the following equation is obtained.

$${}_{n|}\ddot{a}_x = \sum_{k=n}^{w-x-1} v^k \, {}_k p_x = \frac{v^n l_{x+n} + v^{n+1} l_{x+n+1} + v^{n+2} l_{x+n+2} + \ldots + v^{w-x-1} l_{w-1}}{l_x} \tag{2.6.9}$$

The numerator and denominator of the equation (2.6.9) are multiplied by v^x, the following equation is obtained.

$$ {}_{n|}\ddot{a}_x = \frac{v^{x+n} l_{x+n} + v^{x+n+1} l_{x+n+1} + v^{x+n+2} l_{x+n+2} + \ldots + v^{w-1} l_{w-1}}{v^x l_x} \tag{2.6.10} $$

A new formula is defined to express the actuarial present value of the deferred life annuity-due with D_x and N_x commutation functions as following:

$$ {}_{n|}\ddot{a}_x = \frac{D_{x+n} + D_{x+n+1} + \ldots + D_{w-1}}{D_x} = \frac{N_{x+n}}{D_x} \tag{2.6.11} $$

The other new formula is defined to express the actuarial present value of the deferred life annuity-immediate with D_x and N_x commutation functions as following:

$$ {}_{n|}a_x = \sum_{k=n+1}^{w-x-1} v^k \, {}_k p_x = \frac{D_{x+n+1} + D_{x+n+2} + \ldots + D_{w-1}}{D_x} = \frac{N_{x+n+1}}{D_x} \tag{2.6.12} $$

2.6.4 Commutation Functions for Whole Life Insurance

In the previous section 2.4, the actuarial present value of the whole life insurance was described as the formula (2.4.2). When this formula was written as expansion in a series, the following equation is obtained.

$$ A_x = \sum_{k=0}^{w-x-1} v^{k+1} \, {}_{k|}q_x = \frac{v\, d_x + v^2 d_{x+1} + \ldots + v^{w-x} d_{w-1}}{l_x} \tag{2.6.13} $$

The numerator and denominator of the equation (2.6.13) are multiplied by v^x, the following equation is obtained.

$$ A_x = \frac{v^{x+1} d_x + v^{x+2} d_{x+1} + \ldots + v^w d_{w-1}}{v^x l_x} \tag{2.6.14} $$

Where

$C_x = v^{x+1} d_x$

$C_{x+1} = v^{x+2} d_{x+1}$

....

$C_{w-1} = v^w d_{w-1}$

And

$M_x = C_x + C_{x+1} + ... + C_{w-1}$

$M_{x+1} = C_{x+1} + C_{x+2} + ... + C_{w-1}$

....

$M_{w-1} = C_{w-1}$

Are described, a new formula is defined to express the actuarial present value of the whole life insurance with C_x, D_x and M_x commutation functions as following:

$$A_x = \frac{C_x + C_{x+1} + ... + C_{w-1}}{D_x} = \frac{M_x}{D_x} \tag{2.6.15}$$

2.6.5 Commutation Functions for Term Life Insurance

In the previous section 2.4, the actuarial present value of the term life insurance was described as the formula (2.4.9). When this formula was written as expansion in a series, the following equation is obtained.

$$A^1_{x:\overline{n}|} = \sum_{k=0}^{n-1} v^{k+1}\ {}_{k|}q_x = \frac{v\,d_x + v^2\,d_{x+1} + ... + v^n\,d_{x+n-1}}{l_x} \tag{2.6.16}$$

The numerator and denominator of the equation (2.6.16) are multiplied by v^x, the following equation is obtained.

$$A^{1}_{x:\overline{n}|} = \frac{v^{x+1}\, d_x + v^{x+2}\, d_{x+1} + \ldots + v^{x+n}\, d_{x+n-1}}{v^x\, l_x} \tag{2.6.17}$$

With some rearrangements, Where

$$M_x = C_x + C_{x+1} + \ldots + C_{x+n-1} + C_{x+n} + \ldots + C_{w-1}$$

$$M_{x+n} = C_{x+n} + C_{x+n+1} + \ldots + C_{w-1}$$

—

$$M_x - M_{x+n} = C_x + C_{x+1} + \ldots + C_{x+n-1}$$

Is described, a new formula is defined to express the actuarial present value of the term life insurance with C_x, D_x and M_x commutation functions as following:

$$A^{1}_{x:\overline{n}|} = \frac{C_x + C_{x+1} + \ldots + C_{x+n-1}}{D_x} = \frac{M_x - M_{x+n}}{D_x} \tag{2.6.18}$$

2.6.6 Commutation Functions for Deferred Life Insurance

In the previous section 2.4, the actuarial present value of the deferred life insurance was described as the formula (2.4.15). When this formula was written as expansion in a series, the following equation is obtained.

$${}_{n|}A_x = \sum_{k=n}^{w-x-1} v^{k+1}\; {}_{k|}q_x = \frac{v^{n+1}\, d_{x+n} + v^{n+2}\, d_{n+n+1} + \ldots + v^{w-x}\, d_{w-1}}{l_x} \tag{2.6.19}$$

The numerator and denominator of the equation (2.6.19) are multiplied by v^x, the following equation is obtained.

$${}_{n|}A_x = \frac{v^{x+n+1}\, d_{x+n} + v^{x+n+2}\, d_{n+n+1} + \ldots + v^{w}\, d_{w-1}}{v^x\, l_x} \tag{2.6.20}$$

A new formula is defined to express the actuarial present value of the deferred life insurance with C_x, D_x and M_x commutation functions as following:

$$ {}_{n|}A_x = \frac{C_{x+n} + C_{x+n+1} + ... + C_{w-1}}{D_x} = \frac{M_{x+n}}{D_x} \tag{2.6.21} $$

2.6.7 Commutation Functions for Pure Endowment Life Insurance

In the previous section 2.4, the actuarial present value of the pure endowment life insurance was described in the formula (2.4.21) as following:

$$ A_{x:\overline{n|}}^{\;\;1} = {}_nE_x = \frac{v^n\, l_{x+n}}{l_x} \tag{2.6.22} $$

The numerator and denominator of the equation (2.6.22) are multiplied by v^x, the following equation is obtained.

$$ A_{x:\overline{n|}}^{\;\;1} = {}_nE_x = \frac{v^{x+n}\, l_{x+n}}{v^x\, l_x} \tag{2.6.23} $$

A new formula is defined to express the actuarial present value of the pure endowment life insurance with D_x commutation function as following:

$$ A_{x:\overline{n|}}^{\;\;1} = {}_nE_x = \frac{D_{x+n}}{D_x} \tag{2.6.24} $$

2.6.8 Commutation Functions for Endowment Life Insurance

In the previous section 2.4, the actuarial present value of the endowment life insurance was described in the formula (2.4.24). The formula (2.4.24) is used to express again the actuarial present value of the endowment life insurance with D_x and M_x commutation functions as following:

$$A_{x:\overline{n}|} = \frac{M_x - M_{x+n} + D_{x+n}}{D_x} \tag{2.6.25}$$

2.7 Premiums

We have obtained some actuarial present value formulas for life insurance and life annuity models up till now. In this section, we will derive new formulas using previously learned sections and learn how to find the appropriate premium to be paid each year. “We will study three types of insurance models, which are differential by the assumed time of the death benefit payment, and the manner of premium payment” (Gauger, 2006, s.123). Related to each insurance model is a random variable defined as the insurer’s loss function at issue. It is described as the random variable of the present value of benefits to be paid by the insurer less the annuity of premiums to be paid by the insured.

$$L = L(P) = Z - PY \tag{2.7.1}$$

In equation (2.7.1), Z is defined as the random present value variable for insurance benefit, Y is defines as the random present value variable for a life annuity of one unit per year, P is defined as the annual premium amount. We know that:

- If $L(P) > 0$, the insurer will have lost money
- If $L(P) < 0$, the insurer will have made money
- If $L(P) = 0$, the insurer will have broken even

“The insurance company determines the premiums based on the equivalence principle which says that the expected loss of an insurance contract must be zero.

This occurs when the actuarial present value of charges to the insured is equal to the actuarial present value of the benefit payments" (B. Finan, 2012, s.409).

$$E\left[L(P)\right]=0 \tag{2.7.2}$$

By the end of this section, we will be able to calculate level premiums for a diversity of insurance models based on the actuarial equivalence principle between the actuarial present values of the premiums with benefits.

$$E\left[present \quad value \quad of \quad benefits\right]=E\left[present \quad value \quad of \quad premiums\right] \tag{2.7.3}$$

2.7.1 Fully Discrete Premiums

In a fully discrete model, "the sum insured is payable at the end of the policy year in which death occurs, and the first premium is payable when the insurance is issued. Subsequent premiums are payable on anniversaries of the policy issue date while the insured survives during the contractual premium payment period" (Bowers, Gerber, Hickman, Jones, Nesbitt, 1997, s.180).

The loss random variable for fully discrete whole life insurance is given by:

$$L_x=v^{K(x)+1}-P\ddot{a}_{\overline{K(x)+1}|} \quad ; \quad \left(Where \quad K(x)=0,1,2,...,w-x-1\right) \tag{2.7.4}$$

The benefit premium for this insurance is denoted by P_x, and the actuarial present value of the L_x is:

$$E\left[L_x\right]=E\left[v^{K(x)+1}\right]-E\left[P_x\ddot{a}_{\overline{K(x)+1}|}\right]=0 \;\Rightarrow\; A_x-P_x\ddot{a}_x=0 \;\Rightarrow\; P_x=\frac{A_x}{\ddot{a}_x} \tag{2.7.5}$$

For fully discrete whole life insurance model, the aggregate premiums and benefits are shown in the diagram below:

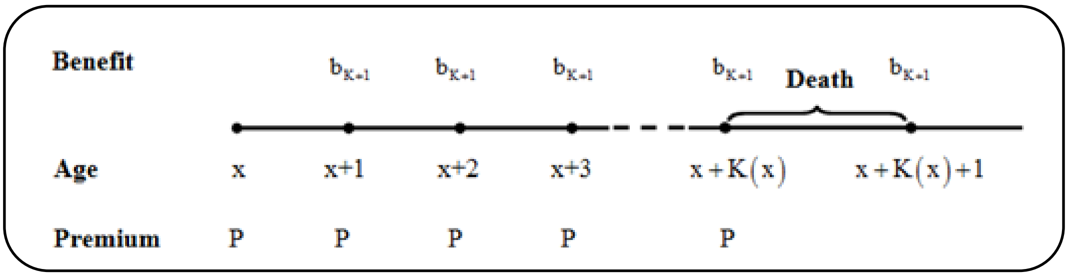

Figure 2.17 The aggregate premiums and benefits associated with fully discrete model

On the other hand, according to the figure (2.17), Equation (2.7.5) may be obtained from following formula:

$$l_x P + l_{x+1} P v + \ldots + l_{x+K(x)} P v^{K(x)} = b_{K+1} d_x v + b_{K+1} d_{x+1} v^2 + \ldots + b_{K+1} d_{x+K(x)} v^{K(x)+1} \qquad (2.7.6)$$

According to these ideas, we will determine a table to display other types of fully discrete annual benefit premiums.

Table 2.3 Formulas for other types of fully discrete annual benefit premiums

Fully Discrete Annual Benefit Premiums		
Insurance Model	**Premium Formula**	**Loss**
n-year term	$P^1_{x:\overline{n}\|} = \dfrac{A^1_{x:\overline{n}\|}}{\ddot{a}_{x:\overline{n}\|}}$	$L^1_{x:\overline{n}\|} = \underbrace{v^{K+1}}_{I(K\le n-1)} - P\ddot{a}_{\overline{\min(K+1,n)}\|}$
n-year endowment	$P_{x:\overline{n}\|} = \dfrac{A_{x:\overline{n}\|}}{\ddot{a}_{x:\overline{n}\|}}$	$L_{x:\overline{n}\|} = v^{\min(n,K+1)} - P\ddot{a}_{\overline{\min(K+1,n)}\|}$
n-year pure endowment	$P_{x:\overline{n}\|}^{\ \ 1} = \dfrac{A_{x:\overline{n}\|}^{\ \ 1}}{\ddot{a}_{x:\overline{n}\|}}$	$L_{x:\overline{n}\|}^{\ \ 1} = \underbrace{v^{n}}_{I(K>n-1)} - P\ddot{a}_{\overline{\min(K+1,n)}\|}$
n-year deferred	${}_{n\|}P_x = \dfrac{{}_{n\|}A_x}{\ddot{a}_x}$	${}_{n\|}L_x = \underbrace{v^{K+1}}_{I(K\ge n)} - P\ddot{a}_{\overline{K+1}\|}$
n-year deferred annuity-due	$P\left({}_{n\|}\ddot{a}_x\right) = \dfrac{A_{x:\overline{n}\|}^{\ \ 1}\,\ddot{a}_{x+n}}{\ddot{a}_{x:\overline{n}\|}}$	$L\left({}_{n\|}\ddot{a}_x\right) = \underbrace{\ddot{a}_{\overline{K+1-n}\|}v^n}_{I(K\ge n)} - P\ddot{a}_{\overline{\min(K+1,n)}\|}$

2.7.2 Fully Continuous Premiums

In a fully continuous model, sum insured is payable at the time of death occurs, and the first premium is payable when the insurance is issued. Subsequent premiums are payable as a continuous level life annuity per year until death occurs at age $x+T(x)\ \left(K(x)\le T(x)<K(x)+1\right)$.

The loss random variable for fully continuous whole life insurance is given by:

$$\bar{L}_x = v^{T(x)} - \bar{P}\bar{a}_{\overline{T(x)}|} \quad ; \quad \left(Where\ \ 0<T(x)<w-x\right) \tag{2.7.7}$$

The benefit premium for this insurance is denoted by $\bar{P}_x$, and the actuarial present value of the $\bar{L}_x$ is:

$$E\left[\bar{L}_x\right] = E\left[v^{T(x)}\right] - E\left[\bar{P}_x\bar{a}_{\overline{T(x)}|}\right] = 0 \ \Rightarrow\ \bar{A}_x - \bar{P}_x\bar{a}_x = 0 \ \Rightarrow\ \bar{P}_x = \frac{\bar{A}_x}{\bar{a}_x} \tag{2.7.8}$$

For fully continuous whole life insurance model, the aggregate premiums and benefits are shown in the diagram below:

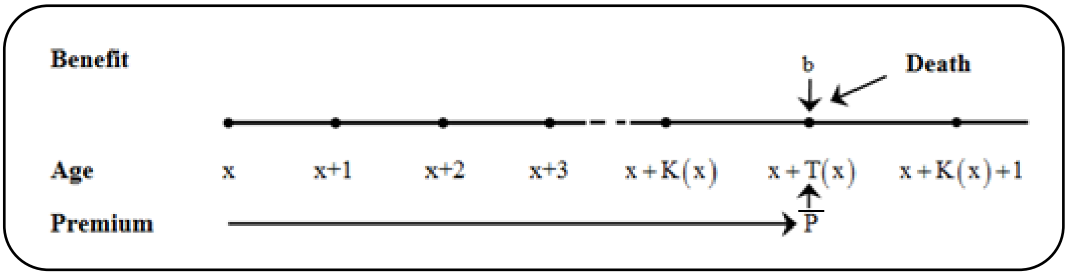

Figure 2.18 The aggregate premiums and benefits associated with fully continuous model

On the other hand, according to the figure (2.18), Equation (2.7.8) may be obtained from following formula:

$$\bar{P}_x\bar{a}_{\overline{T(x)}|} = bv^{T(x)} \quad ; \quad \left(0<T(x)<w-x\right) \tag{2.7.9}$$

According to these ideas, we will determine a table to display other types of fully continuous benefit premiums.

Table 2.4 Formulas for other types of fully continuous benefit premiums

Fully Continuous Benefit Premiums		
Insurance Model	**Premium Formula**	**Loss**
n-year term	$\bar{P}^{1}_{x:\overline{n}\vert} = \dfrac{\bar{A}^{1}_{x:\overline{n}\vert}}{\bar{a}_{x:\overline{n}\vert}}$	$\bar{L}^{1}_{x:\overline{n}\vert} = v^{T}_{I(T\le n)} - \bar{P}\bar{a}_{\overline{\min(T,n)}\vert}$
n-year endowment	$\bar{P}_{x:\overline{n}\vert} = \dfrac{\bar{A}_{x:\overline{n}\vert}}{\bar{a}_{x:\overline{n}\vert}}$	$\bar{L}_{x:\overline{n}\vert} = v^{\min(T,n)} - \bar{P}\bar{a}_{\overline{\min(T,n)}\vert}$
n-year pure endowment	$\bar{P}_{x:\overline{n}\vert}^{\ \ 1} = \dfrac{A_{x:\overline{n}\vert}^{\ \ 1}}{\bar{a}_{x:\overline{n}\vert}}$	$\bar{L}_{x:\overline{n}\vert}^{\ \ 1} = v^{T}_{I(T>n)} - \bar{P}\bar{a}_{\overline{\min(T,n)}\vert}$
n-year deferred	${}_{n\vert}\bar{P}_{x} = \dfrac{{}_{n\vert}\bar{A}_{x}}{\bar{a}_{x}}$	${}_{n\vert}\bar{L}_{x} = v^{T}_{I(T>n)} - \bar{P}\bar{a}_{\overline{T}\vert}$
n-year deferred annuity	$\bar{P}\left({}_{n\vert}\bar{a}_{x}\right) = \dfrac{A_{x:\overline{n}\vert}^{\ \ 1}\,\bar{a}_{x+n}}{\bar{a}_{x:\overline{n}\vert}}$	$\bar{L}\left({}_{n\vert}\bar{a}_{x}\right) = \underbrace{\bar{a}_{\overline{T-n}\vert}v^{n}}_{I(T(x)>n)} - \bar{P}\bar{a}_{\overline{\min(T,n)}\vert}$

2.7.3 Semi-Continuous Premiums

In a semi-continuous model, “is a policy with a continuous benefit and payments made with a discrete annuity-due. That is, the benefit is paid at the moment of death and the premiums are paid at the beginning of the year while insured is a live” (B. Finan, 2012, s.456).

The loss random variable for semi-continuous whole life insurance is given by:

$$\overline{SL}_{x} = v^{T(x)} - P\ddot{a}_{\overline{K(x)+1}\vert} \quad ; \quad \left(0 < T\left(x\right) < w - x \, ; K\left(x\right) = 0,1,\ldots,w - x - 1\right) \qquad (2.7.10)$$

The benefit premium for this insurance is denoted by $P\left(\bar{A}_x\right)$, and the actuarial present value of the $\overline{SL}_x$ is:

$$E\left[\overline{SL}_x\right]=E\left[v^{T(x)}\right]-E\left[P\ddot{a}_{\overline{K(x)+1|}}\right]=0 \;\Rightarrow\; \bar{A}_x-P\ddot{a}_x=0 \;\Rightarrow\; P\left(\bar{A}_x\right)=\frac{\bar{A}_x}{\ddot{a}_x} \qquad (2.7.11)$$

For semi-continuous whole life insurance model, the aggregate premiums and benefits are shown in the diagram below:

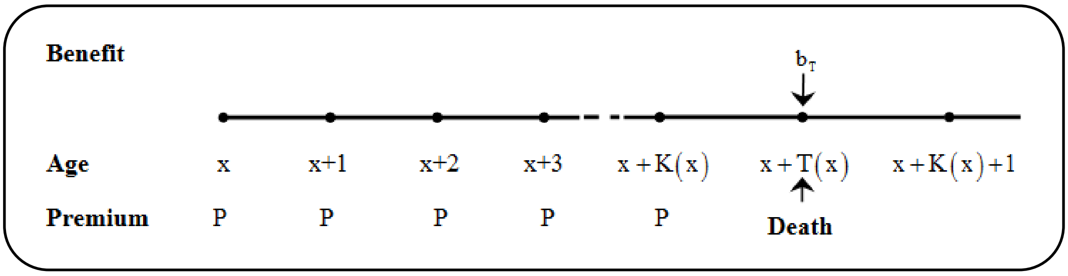

Figure 2.19 The aggregate premiums and benefits associated with semi-continuous model

On the other hand, according to the figure (2.19), Equation (2.7.11) may be obtained from following formula:

$$l_x P+l_{x+1}P\,v+\ldots+l_{x+K(x)}P\,v^{K(x)}=b_T v^{T(x)} \qquad (2.7.12)$$

According to these ideas, we will determine a table to display other types of semi-continuous benefit premiums.

Table 2.5 Formulas for other types of semi-continuous benefit premiums

Semi-Continuous Benefit Premiums		
Insurance Model	**Premium Formula**	**Loss**
n-year term	$P\left(\bar{A}^{1}_{x:\overline{n}\rceil}\right)=\dfrac{\bar{A}^{1}_{x:\overline{n}\rceil}}{\ddot{a}_{x:\overline{n}\rceil}}$	$\bar{S}\bar{L}^{1}_{x:\overline{n}\rceil} = \underset{I(T\le n)}{v^{T}} - P\ddot{a}_{\overline{\min(K+1,n)}\rceil}$
n-year endowment	$P\left(\bar{A}_{x:\overline{n}\rceil}\right)=\dfrac{\bar{A}_{x:\overline{n}\rceil}}{\ddot{a}_{x:\overline{n}\rceil}}$	$\bar{S}\bar{L}_{x:\overline{n}\rceil} = v^{\min(n,T)} - P\ddot{a}_{\overline{\min(K+1,n)}\rceil}$
n-year deferred	$P\left({}_{n\mid}\bar{A}_{x}\right)=\dfrac{{}_{n\mid}\bar{A}_{x}}{\ddot{a}_{x}}$	${}_{n\mid}\bar{S}\bar{L}_{x} = \underset{I(T>n)}{v^{T}} - P\ddot{a}_{\overline{K+1}\rceil}$

2.8 Reserves

"The benefit reserve at time t is the conditional expectation of the difference between the present value of future benefits and future benefit premiums, the conditioning event being survivorship of the insured to time t" (Bowers, Gerber, Hickman, Jones, Nesbitt, 1997, s.205).

There are two methods to calculate reserves. One of those is called as the insurer's prospective loss function at duration t and is defined as the difference between the present value of future benefits and the present value of future benefit premiums.

$$ {}_{t}L = PV\left(\textit{future benefit at } t\right) - PV\left(\textit{future premiums at } t\right) \qquad (2.8.1)$$

The reserve ${}_{t}V$ is called the conditional expected value of the prospective loss random variable, the conditional event being survivorship of the insured to time t.

$$ {}_{t}V = E\left[{}_{t}L \mid T > t\right] = APV\ of\ \textit{future benefits} - APV\ of\ \textit{future premiums} \qquad (2.8.2)$$

The second of these is called as the insurer's retrospective loss function at time t, its conditional expectation is defined as the difference between the accumulated value of past benefits paid and the accumulated value of past premiums received, the conditional event being survivorship of the insured to time t.

$$ {}_tV = AV\ of\ \ premiums\ received - AV\ of\ benefits\ paid \tag{2.8.3}$$

By the end of this section, we will be able to know all benefit reserve symbols for a plan of insurances.

2.8.1 Reserves for Fully Discrete General Insurances

${}_kV$ is expressed as the terminal benefit reserve for year k and is defined as the reserve at the end of the k years. π_k is denoted as the non-level benefit premium payable at the beginning of policy year $k+1$. As a result of these, ${}_kV+\pi_k$ is called as the initial benefit reserve for year $k+1(k=0,1,2,...)$. We assume that the death benefit is b_{k+1} and death occurs in the $(k+1)$th policy year, the initial benefit reserve for $k+1$ is defined as (B. Finan, 2012, s.550):

$$ {}_kV+\pi_k = vq_{x+k}b_{k+1} + vp_{x+k}\ {}_{k+1}V \tag{2.8.4}$$

"If we move the term ${}_kV$ to the right side, replace p_{x+k} by $1-q_{x+k}$ and then group the two reserve terms on the right side we obtain" (Gauger, 2006, s.174):

$$ \pi_k = vq_{x+k}\left(b_{k+1} - {}_{k+1}V\right) + \left(v\ {}_{k+1}V - {}_kV\right) \tag{2.8.5}$$

In the equation (2.8.5), "The expression $b_{k+1} - {}_{k+1}V$ is known as the net amount at risk and is the amount of money the insurer will have to produce from sources other than the insured's benefit reserve if the insured dies in policy year $k+1$" (B. Finan, 2012, s.560). This is illustrated in the following figure 2.20.

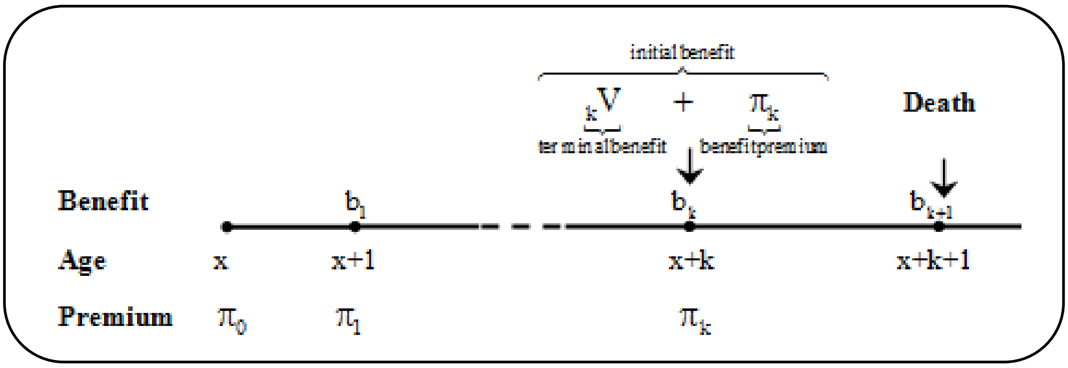

Figure 2.20 The representation of the initial benefit reserve and terminal benefit reserve

According to the figure 2.20, the actuarial present value of the benefits $(APVB)$ is symbolized as following:

$$APVB = \sum_{k=0}^{\infty} b_{k+1} v^{k+1} \, {}_k p_x q_{x+k} \tag{2.8.6}$$

Also, the actuarial present value of the benefit premium stream $(APVP)$ is symbolized as following:

$$APVP = \sum_{k=0}^{\infty} \pi_k v^k \, {}_k p_x \tag{2.8.7}$$

Based on the actuarial equivalence principle between the actuarial present values of the premiums with benefits at time 0, we write:

$$APVB = APVP \quad \Rightarrow \quad \sum_{k=0}^{\infty} b_{k+1} v^{k+1} \, {}_k p_x q_{x+k} = \sum_{k=0}^{\infty} \pi_k v^k \, {}_k p_x \tag{2.8.8}$$

2.8.2 Fully Discrete Benefit Reserves

"A fully discrete whole life insurance of one unit benefit issued to x. This policy is still in force k years later, when the life x has survived to age $x+k$" (Gauger, 2006, s.153).

The prospective loss random variable for this insurance is given by:

$$ {}_{k}L_{x} = v^{K(x)-k+1} - P_{x}\ddot{a}_{\overline{K(x)-k+1|}} \quad ; \quad K(x) = k, k+1, \ldots \quad ; \quad k = 0, 1, 2, \ldots \tag{2.8.9} $$

The benefit reserve for this insurance is denoted by ${}_{k}V_{x}$, and is the conditional expectation of the loss function, is known as the prospective k th terminal reserve of the policy, is symbolized as:

$$ {}_{k}V_{x} = E\left[{}_{k}L_{x} \middle| K(x) \ge k \right] = A_{x+k} - P_{x}\ddot{a}_{x+k} \tag{2.8.10} $$

An alternative to finding the reserve is the retrospective method. The fully discrete whole life insurance is symbolized by this method as following:

$$ {}_{h}V_{x} = E\left[{}_{h}L_{x} \middle| K(x) \ge h = k \right] = P_{x}\ddot{s}_{x:\overline{h|}} - {}_{h}k_{x} \tag{2.8.11} $$

Where $\ddot{s}_{x:\overline{h|}}$ (the actuarial accumulated value of the premiums paid during the first h years) is described as:

$$ \ddot{s}_{x:\overline{h|}} = \frac{\ddot{a}_{x:\overline{h|}}}{{}_{h}E_{x}} \tag{2.8.12} $$

And ${}_{h}k_{x}$ (the accumulated cost of insurance) is described as:

$$ {}_{h}k_{x} = \frac{A^{1}_{x:\overline{h|}}}{{}_{h}E_{x}} \tag{2.8.13} $$

According to these ideas, we will determine a table to display other types of fully discrete benefit reserves. Benefit reserves symbols of other insurances, the prospective loss random variable and formulas for other insurance are defined in Table 2.6.

Table 2.6 Formulas for other types of fully discrete benefit reserves, age at issue *x*, duration *k*, one unit benefit

FULLY DISCRETE BENEFIT RESERVES			
Insurance Model	**Loss Random Variable**	**Symbol**	**Prospective Formulas**
n-year term	${}_kL^1_{x:\overline{n}\mid} = v^{K-k+1}_{I(K<n)} - P^1_{x:\overline{n}\mid}\ddot{a}_{\overline{\min\{(K-k+1,n-k)\}}\mid}$	${}_kV^1_{x:\overline{n}\mid} = {}_kV\left(A^1_{x:\overline{n}\mid}\right)$	$\begin{cases} A^1_{x+k:\overline{n-k}\mid} - P^1_{x:\overline{n}\mid}\ddot{a}_{x+k:\overline{n-k}\mid} & ;\ k<n \\ 0 & ;\ k=n \end{cases}$
n-year endowment	${}_kL_{x:\overline{n}\mid} = v^{\min\{(K-k+1,n-k)\}} - P_{x:\overline{n}\mid}\ddot{a}_{\overline{\min\{(K-k+1,n-k)\}}\mid}$	${}_kV_{x:\overline{n}\mid} = {}_kV\left(A_{x:\overline{n}\mid}\right)$	$\begin{cases} A_{x+k:\overline{n-k}\mid} - P_{x:\overline{n}\mid}\ddot{a}_{x+k:\overline{n-k}\mid} & ;\ k<n \\ 1 & ;\ k=n \end{cases}$
n-year deferred annuity	${}_kL\left({}_{n\mid}\ddot{a}_x\right) = \begin{cases} \underbrace{\ddot{a}_{\overline{K-n+1}\mid}v^{n-k}}_{I(K<n)} - P\left({}_{n\mid}\ddot{a}_x\right)\ddot{a}_{\overline{\min\{(K-k+1,n-k)\}}\mid} & ;\ k<n \\ \ddot{a}_{\overline{K-n+1}\mid} & ;\ k\geq n \end{cases}$	${}_kV\left({}_{n\mid}\ddot{a}_x\right)$	$\begin{cases} {}_{n-k\mid}\ddot{a}_{x+k} - P\left({}_{n\mid}\ddot{a}_x\right)\ddot{a}_{x+k:\overline{n-k}\mid} & ;\ k<n \\ \ddot{a}_{x+k} & ;\ k\geq n \end{cases}$
n-year pure endowment	${}_kL_{x:\overline{n}\mid}{}^{1} = v^{n-k}_{I(K\geq n)} - P_{x:\overline{n}\mid}{}^{1}\ddot{a}_{\overline{\min\{(K-k+1,n-k)\}}\mid}$	${}_kV_{x:\overline{n}\mid}{}^{1} = {}_kV\left(A_{x:\overline{n}\mid}{}^{1}\right)$	$\begin{cases} A_{x+k:\overline{n-k}\mid}{}^{1} - P_{x:\overline{n}\mid}{}^{1}\ddot{a}_{x+k:\overline{n-k}\mid} & ;\ k<n \\ 1 & ;\ k=n \end{cases}$

We can derive a lot of equations of the prospective reserve formula for a discrete whole life contract, but we will concentrate on the derivations of four in the context in this insurance model. The first of these is called as the annuity ratio formula and expresses the reserve as a ratio of two annuities due:

$$ {}_kV_x = A_{x+k} - P_x\ddot{a}_{x+k} = \left(1 - d\ddot{a}_{x+k}\right) - P_x\ddot{a}_{x+k} = 1 - \left(d + P_x\right)\ddot{a}_{x+k} = 1 - \frac{\ddot{a}_{x+k}}{\ddot{a}_x} \qquad (2.8.14) $$

The second of these is known as the benefit formula or the insurance ratio formula, and is obtained from the annuity ratio formula, and is defined as:

$$ {}_kV_x = 1 - \frac{\ddot{a}_{x+k}}{\ddot{a}_x} = \frac{\ddot{a}_x - \ddot{a}_{x+k}}{\ddot{a}_x} = \frac{\left[\left(1 - A_x\right) - \left(1 - A_{x+k}\right)\right]/d}{\left(1 - A_x\right)/d} = \frac{A_{x+k} - A_x}{1 - A_x} \qquad (2.8.15) $$

The third of these is known as the paid-up insurance formula and is the derivation from the prospective formula as following:

$$ {}_kV_x = A_{x+k} - P_x\ddot{a}_{x+k} = A_{x+k} - P_x\frac{A_{x+k}}{P_{x+k}} = A_{x+k}\left(1 - \frac{P_x}{P_{x+k}}\right) \qquad (2.8.16) $$

The fourth of these is known as the premium difference formula and it is:

$$ {}_kV_x = A_{x+k} - P_x\ddot{a}_{x+k} = P_{x+k}\ddot{a}_{x+k} - P_x\ddot{a}_{x+k} = \ddot{a}_{x+k}\left(P_{x+k} - P_x\right) \qquad (2.8.17) $$

2.8.3 Reserves for Fully Continuous General Insurances

For a general fully continuous insurance issued to x, ${}_t\bar{V}$ is expressed as the tth terminal benefit reserve for year t and is defined as the reserve at the time of the t years. π_t is denoted as the annual rate of benefit premiums at time $t\left(0 \le t < \infty\right)$ and payable continuously. We assume that the death benefit is b_t and payable at the moment of death t (B. Finan, 2012, s.557).

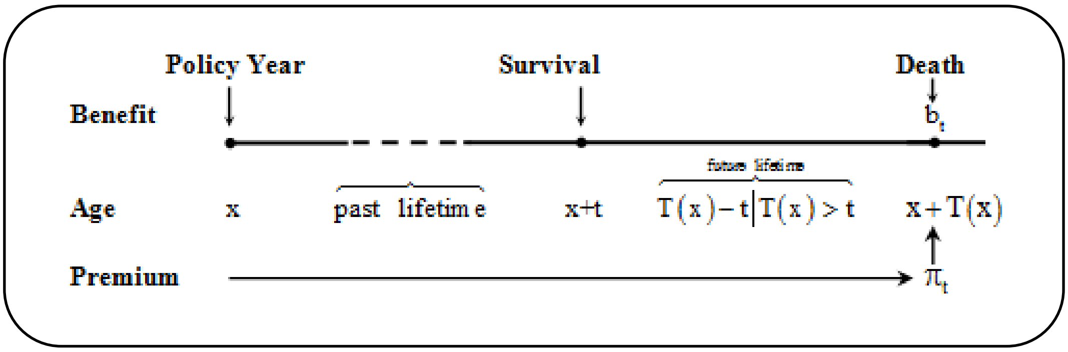

Figure 2.21 The representation of the general fully continuous insurance

According to the figure 2.21, the actuarial present value of the benefits $(APVB)$ is symbolized as following:

$$APVB = \int_{0}^{\infty} b_t v^t \, {}_t p_x \mu(x+t)\, dt \tag{2.8.18}$$

Also, the actuarial present value of the benefit premiums $(APVP)$ is symbolized:

$$APVP = \int_{0}^{\infty} \pi_t v^t \, {}_t p_x dt \tag{2.8.19}$$

Based on the actuarial equivalence principle between the actuarial present values of the premiums with benefits at time 0, we write:

$$APVB = APVP \quad \Rightarrow \quad \int_{0}^{\infty} b_t v^t \, {}_t p_x \mu(x+t)\, dt = \int_{0}^{\infty} \pi_t v^t \, {}_t p_x dt \tag{2.8.20}$$

2.8.4 Fully Continuous Benefit Reserves

"A fully continuous whole life insurance of one unit benefit is issued on a life age x. Assume that this policy is still in force t years later, that is, the life x has survived to age $x+t$" (Gauger, 2006, s.156).

The prospective loss random variable for this insurance at age $x+t$ is given by:

$$ {}_t\bar{L}_x = v^{T-t} - \bar{P}_x \bar{a}_{\overline{T-t}|} \quad ; \quad T(x) > t \tag{2.8.21}$$

The benefit reserve for this insurance is denoted by ${}_t\bar{V}_x$, and is the conditional expectation of the loss function, is known as the prospective formula at time t, is symbolized as:

$$ {}_t\bar{V}_x = E\left[{}_t\bar{L}_x \middle| T(x) > t \right] = \bar{A}_{x+t} - \bar{P}_x \bar{a}_{x+t} \tag{2.8.22}$$

An alternative to finding the reserve is the retrospective method. The fully continuous whole life insurance is symbolized by this method as following:

$$ {}_t\bar{V}_x = E\left[{}_t\bar{L}_x \middle| T(x) > t \right] = \bar{P}_x \bar{s}_{x:\overline{t}|} - {}_t\bar{k}_x \tag{2.8.23}$$

Where $\bar{s}_{x:\overline{t}|}$ (the actuarial accumulated value of the premiums paid during the first t years) is described as:

$$ \bar{s}_{x:\overline{t}|} = \frac{\bar{a}_{x:\overline{t}|}}{{}_tE_x} \tag{2.8.24}$$

And ${}_t\bar{k}_x$ (the actuarial accumulated value of past benefits) is described as:

$$ {}_t\bar{k}_x = \frac{\bar{A}^{1}_{x:\overline{t}|}}{{}_tE_x} \tag{2.8.25}$$

According to these ideas, we will determine a table to display other types of fully continuous benefit reserves. Benefit reserves symbols of other insurances, the prospective loss random variable and formulas for other insurance are defined in Table 2.7.

Table 2.7 Formulas for other types of fully continuous benefit reserves, age at issue *x*, duration *t*, one unit benefit

FULLY CONTINUOUS BENEFIT RESERVES			
Insurance Model	**Loss Random Variable**	**Symbol**	**Prospective Formulas**
n-year term	${}_t\bar{L}^1_{x:\overline{n}\rvert} = \underset{I(T\le n)}{v^{T-t}} - \bar{P}^1_{x:\overline{n}\rvert}\bar{a}_{\overline{\min\{(T-t,n-t)\}}\rvert}$	${}_t\bar{V}^1_{x:\overline{n}\rvert} = {}_t\bar{V}\left(\bar{A}^1_{x:\overline{n}\rvert}\right)$	$\begin{cases} \bar{A}^1_{x+t:\overline{n-t}\rvert} - \bar{P}^1_{x:\overline{n}\rvert}\bar{a}_{x+t:\overline{n-t}\rvert} & ;\ t<n \\ 0 & ;\ t=n \end{cases}$
n-year endowment	${}_t\bar{L}_{x:\overline{n}\rvert} = v^{\min\{(T-t,n-t)\}} - \bar{P}_{x:\overline{n}\rvert}\bar{a}_{\overline{\min\{(T-t,n-t)\}}\rvert}$	${}_t\bar{V}_{x:\overline{n}\rvert} = {}_t\bar{V}\left(\bar{A}_{x:\overline{n}\rvert}\right)$	$\begin{cases} \bar{A}_{x+t:\overline{n-t}\rvert} - \bar{P}_{x:\overline{n}\rvert}\bar{a}_{x+t:\overline{n-t}\rvert} & ;\ t<n \\ 1 & ;\ t=n \end{cases}$
n-year deferred annuity	${}_t\bar{L}\left({}_{n\rvert}\bar{a}_x\right) = \begin{cases} \underbrace{\bar{a}_{\overline{T-n}\rvert}v^{n-t}}_{I(T>n)} - \bar{P}\left({}_{n\rvert}\bar{a}_x\right)\bar{a}_{\overline{\min\{(T-t,n-t)\}}\rvert} & ;\ t\le n \\ \bar{a}_{\overline{T-n}\rvert} & ;\ t>n \end{cases}$	${}_t\bar{V}\left({}_{n\rvert}\bar{a}_x\right)$	$\begin{cases} {}_{n-t\rvert}\bar{a}_{x+t} - \bar{P}\left({}_{n\rvert}\bar{a}_x\right)\bar{a}_{x+t:\overline{n-t}\rvert} & ;\ t\le n \\ \bar{a}_{x+t} & ;\ t>n \end{cases}$
n-year pure endowment	${}_t\bar{L}_{x:\overline{n}\rvert}^{\;\;1} = \underset{I(T>n)}{v^{n-t}} - \bar{P}_{x:\overline{n}\rvert}^{\;\;1}\bar{a}_{\overline{\min\{(T-t,n-t)\}}\rvert}$	${}_t\bar{V}_{x:\overline{n}\rvert}^{\;\;1} = {}_t\bar{V}\left(A_{x:\overline{n}\rvert}^{\;\;1}\right)$	$\begin{cases} \bar{A}_{x+t:\overline{n-t}\rvert}^{\;\;\;1} - \bar{P}_{x:\overline{n}\rvert}^{\;\;1}\bar{a}_{x+t:\overline{n-t}\rvert} & ;\ t<n \\ 1 & ;\ t=n \end{cases}$

We can derive a lot of equations of the prospective reserve formula for a continuous whole life contract, but we will concentrate on the derivations of four in the context in this insurance model. The first of these is called as the annuity ratio formula and expresses the reserve as a ratio of two continuous annuities:

$$ {}_t\bar{V}_x = \bar{A}_{x+t} - \bar{P}_x \bar{a}_{x+t} = \left(1-\delta\bar{a}_{x+t}\right) - \left(\frac{1-\delta\bar{a}_x}{\bar{a}_x}\right)\bar{a}_{x+t} = 1 - \frac{\bar{a}_{x+t}}{\bar{a}_x} \tag{2.8.26} $$

The second of these is known as the benefit formula or the insurance ratio formula, and is obtained from the annuity ratio formula, and is defined as:

$$ {}_t\bar{V}_x = 1 - \frac{\bar{a}_{x+t}}{\bar{a}_x} = 1 - \frac{\delta^{-1}\left(1-\bar{A}_{x+t}\right)}{\delta^{-1}\left(1-\bar{A}_x\right)} = \frac{\bar{A}_{x+t} - \bar{A}_x}{1-\bar{A}_x} \quad ; \quad \bar{A}_x + \delta\bar{a}_x = 1 \tag{2.8.27} $$

The third of these is known as the paid-up insurance formula and is the derivation from the prospective formula as following:

$$ {}_t\bar{V}_x = \bar{A}_{x+t} - \bar{P}_x \bar{a}_{x+t} = \bar{A}_{x+t}\left(1 - \bar{P}_x \frac{\bar{a}_{x+t}}{\bar{A}_{x+t}}\right) = \bar{A}_{x+t}\left(1 - \frac{\bar{P}_x}{\bar{P}_{x+t}}\right) \tag{2.8.28} $$

The fourth of these is known as the premium difference formula and is the derivation from the prospective formula as following:

$$ {}_t\bar{V}_x = \bar{A}_{x+t} - \bar{P}_x \bar{a}_{x+t} = \bar{P}_{x+t}\bar{a}_{x+t} - \bar{P}_x \bar{a}_{x+t} = \bar{a}_{x+t}\left(\bar{P}_{x+t} - \bar{P}_x\right) \tag{2.8.29} $$

2.8.5 Semi-Continuous Benefit Reserves

We have had encounter with semi-continuous contracts in section (2.7.3). Accordingly, “A semi-continuous whole life insurance of one unit benefit is issued on a life age x. Assume that this policy is still in force k years later, the life x has survived to age $x+k$” (Gauger, 2006, s.157).

The prospective loss random variable at age $x+k$ at time k is given by:

$$ {}_{k}L\left(\bar{A}_x\right)=v^{T-t}-P\left(\bar{A}_x\right)\ddot{a}_{\overline{K(x)-k+1}|}=\bar{Z}_{x+t}-P\left(\bar{A}_x\right)\ddot{Y}_{x+k} \quad ; \; T(x)>t \quad ; \; K(x)\geq k \quad (2.8.30)$$

The benefit reserve for this insurance is denoted by ${}_{k}V\left(\bar{A}_x\right)$, and is the conditional expectation of the loss function, is known as the prospective k th terminal reserve of the policy, is symbolized as:

$$ {}_{k}V\left(\bar{A}_x\right)=E\left[{}_{t}\bar{L}_x \middle| T(x)>t;K(x)\geq k\right]=\bar{A}_{x+t}-P\left(\bar{A}_x\right)\ddot{a}_{x+k} \qquad (2.8.31)$$

An alternative to finding the reserve is the retrospective method. The semi-continuous whole life insurance is symbolized by this method as following:

$$ {}_{k}V\left(\bar{A}_x\right)=P\left(\bar{A}_x\right)\ddot{s}_{x:\overline{k}|}-{}_{t}\bar{k}_x \qquad (2.8.32)$$

Where $\ddot{s}_{x:\overline{k}|}\left(k=h\right)$ (the actuarial accumulated value of the premiums paid during the first t years) is described in formula (2.8.12) and ${}_{t}\bar{k}_x$ (the actuarial accumulated value of past benefits) is described in formula (2.8.25).

Other terminal reserve expressions for contracts with immediate payment of claims and premium payments made at the start of the year are developed taking account of sections (2.8.2) and (2.8.4).

2.9 Stochastic (Random) Interest Rate Approaches

In this section, “we consider interest rates to be stochastic, which mean that the future interest rates are random variables. Assumptions about statistical distribution of the future interest rates and some conclusion about the financial cash flows associated with them can be obtained” (Ruckman & Francis, 2005, s.279).

The random variable i_t is defined to be the interest rate applicable from time $(t-1)$ to time t. Accumulated value at time n years an investment of one unit is symbolized as:

$$AV_n = (1+i_1)(1+i_2)...(1+i_n) \tag{2.9.1}$$

The expected value of the equation (2.9.1) is symbolized as:

$$E[AV_n] = E[(1+i_1)(1+i_2)...(1+i_n)] \tag{2.9.2}$$

The variance of the equation (2.9.1) is symbolized as:

$$Var[AV_n] = E[AV_n^2] - (E[AV_n])^2 \tag{2.9.3}$$

For using the equations (2.9.2) and (2.9.3), we must know the probability distribution of the values of i_t. There are two simple interest rate models to find the future interest rates. Those are called fixed interest rate model and varying interest rate model. For fixed interest rate model, "the initial interest rate is determined in the first year and the subsequent interest rates are then fixed at that initial interest rate. Therefore, the future interest rates in this model are perfectly correlated" (Ruckman & Francis, 2005, s.280). For varying interest rate model, "the interest rate in each year is independent of the interest rates in the other years" (Ruckman & Francis, 2005, s.280).

The present value at time 0 of one unit payable at time n years is symbolized as:

$$PV_n = \left(\frac{1}{1+i_1}\right)\left(\frac{1}{1+i_2}\right)...\left(\frac{1}{1+i_n}\right) \tag{2.9.4}$$

The expected value of the equation (2.9.4) is symbolized as:

$$E\left[PV_n\right]=E\left[\left(\frac{1}{1+i_1}\right)\left(\frac{1}{1+i_2}\right)...\left(\frac{1}{1+i_n}\right)\right] \tag{2.9.5}$$

The variance of the equation (2.9.4) is symbolized as:

$$Var\left[PV_n\right]=E\left[PV_n^{\,2}\right]-\left(E\left[PV_n\right]\right)^2 \tag{2.9.6}$$

"The actual present value at time 0 isn't known until all n of the interest rates are revealed, but the expected present value can be calculated at time 0 if the distribution of the interest rates is known" (Ruckman & Francis, 2005, s.282).

"If the future interest rates are independent and identically distributed, then the expected accumulated value at time n years of one unit invested now is" (Ruckman & Francis, 2005, s.284):

$$\begin{aligned}
E\left[AV_n\right]&=E\left[(1+i_1)(1+i_2)...(1+i_n)\right] \ ; \ \left(from\,the\,equation\,(2.8.2)\right)\\
&=E\left[(1+i_1)\right]E\left[(1+i_2)\right]...E\left[(1+i_n)\right] \ ; \ \left(Where\,i_1,i_2,...,i_n\,are\,independent\right)\\
&=\left(1+E\left[i_1\right]\right)\left(1+E\left[i_2\right]\right)...\left(1+E\left[i_n\right]\right)\\
&=\left(1+E\left[i_t\right]\right)^n \ ; \ \left(int\,erest\,rates\,are\,identically\,distributed\right)\\
&=\left(1+\bar{i}\right)^n \ ; \ \left(where\,\bar{i}=E\left[i_t\right]\,for\,t=1,2,...,n\right)
\end{aligned} \tag{2.9.7}$$

"If the future interest rates are independent and identically distributed, then the variance of the accumulated value at time n years of one unit invested now is" (Ruckman & Francis, 2005, s.285):

$$Var\left[AV_n\right]=E\left[AV_n^{\,2}\right]-\left(E\left[AV_n\right]\right)^2=\left[\left(1+\bar{i}\right)^2+s^2\right]^n-\left(1+\bar{i}\right)^{2n} \tag{2.9.8}$$

Where

$$Var[i_t] = s^2 \;\Rightarrow\; E\left[i_t^2\right] - \left(E[i_t]\right)^2 = E\left[i_t^2\right] - \bar{i}^{\,2} = s^2 \;\Rightarrow\; E\left[i_t^2\right] = s^2 + \bar{i}^{\,2} \qquad (2.9.9)$$

"If the future interest rates are independent and identically distributed, then the expected value of the present value of one unit payable at time n years is" (Ruckman & Francis, 2005, s.286):

$$E[PV_n] = E\left[\frac{1}{(1+i_1)(1+i_2)...(1+i_n)}\right] \;;\; \left(\mathit{from\,the\,equation}(2.8.5)\right)$$

$$= E\left[\frac{1}{(1+i_1)}\right] E\left[\frac{1}{(1+i_2)}\right] ... E\left[\frac{1}{(1+i_n)}\right] \;;\; \left(\mathit{Where}\; i_1, i_2, ..., i_n \;\mathit{are\;independent}\right)$$

$$= \left(E\left[\frac{1}{1+i_t}\right]\right)^n \;;\; \left(\mathit{interest\;rates\;are\;identically\;distributed}\right)$$

$$= \bar{v}^{\,n} \;;\; \left(\mathit{where}\;\bar{v} = E\left[\frac{1}{1+i_t}\right] \mathit{for}\; t = 1,2,...,n\right) \qquad (2.9.10)$$

"Knowledge of the variance s^2 isn't sufficient to allow us to find a convenient expression for the variance of PV_n. If it is necessary to calculate it, the calculation can often be performed based on first principles" (Ruckman & Francis, 2005, s.287).

CHAPTER THREE
APPLICATIONS

3.1 Introduction

In this study, we aimed to explain difference between the stochastic interest rates and the deterministic interest rates. We made applications using the assumptions which located in the draft resolution of ministerial cabinet. Some controversial issues related to calculation of liabilities in opposition to the “Social Security Institution” of foundation funds which are established according to temporary twentieth article of the law no 506 have been existed. Some of them are as below:

- The most controversial issue in terms of the non-state actors; inflation rate assumption which located in the draft resolution of ministerial cabinet and in parallel with the technical interest rate. When as, technical interest rate that will use for the calculation of the present value both temporary twentieth article of law no 5510 and the draft resolution of ministerial cabinet is determined 9,80%.

- Funds and employers’ representatives have been demanded to be removed from the text of the calculation of the liability which will be made according to this phrase and the phrase of the “inflation rate” in located the exposure draft. Because, they are claimed that the technical interest rate (9,80%) is adjusted for inflation (real interest rate) and it must not take into consideration. The main reason underlying of the objection is that incumbent liability (the amount of the liability which is necessary for the cession) of the each foundation fund will increase when the inflation rate is used in calculations.

- Undersecretariat of Treasury and Ministry of Development presented an opinion in the direction of taking into account of the inflation rate in the calculations. Because, in today's conditions, the real interest rate in the market is much lower than mentioned the inflation-adjusted real interest rate (9,80%). Therefore, “Inflation rate” was added to the text taking into consideration.

In this study, we will obtain some results about how will change incumbent liability of the each foundation fund for both technical interest rate (9,80%) and real interest rate (stochastic). By this means, we can comment about controversial issues of the draft resolution of ministerial cabinet.

3.2 Basic Concepts for Calculations

During the implementation of legal decision interested in the temporary twentieth article of the law no 5510:

a) **Institution:** Refers to “Social Security Institution (SSI)”.

b) **Fund:** Refers to the foundation funds which are subject to the temporary twentieth article of the law no 506.

c) **Salary and Income:** Refer to the disablement, old age and survivor’s pensions and the permanent incapacity income and survivor’s income which are assigned in case of an occupational accident or professional disease, all of which are defined in law no 5510.

d) **Dependents:** Refer to the spouse, children and parents which a person is liable to look after as per law no 5510. In the Table 1.2, general total numbers of beneficiaries are given as 144.467. Nine percent (9,00%) of the beneficiaries are accepted as dependents. Under these assumptions, other properties are given in the Table 3.1.

Table 3.1 Distribution in terms of type, number, age, salary of dependents

DEPENDENTS			
Type	Mean Number	Mean Age	Mean Salary
Spouse - Mother and Father - Children (>=25 age)	10.502	45	800 ₺
Children (<25 age)	2.500	15	300 ₺

e) Earning as Basis to Premium: Refers to the earning taken as basis to premium, defined in the foundation voucher of the relative fund. This value is assumed as 3.500₺ in the part of application.

f) Contributors: Refer to individuals who work, have worked and quitted, voluntarily pay premiums, receive salary and/or income and who received full settlement under funds which are subject to the temporary twentieth article of the law no 506. These persons are defined as actives and passives in the Table 1.2. Other assumptions about these persons are given in the Table 3.2.

Table 3.2 Distribution in terms of sex, number, age, salary of contributors

CONTRIBUTORS				
Properties	Active		Passive	
	Male	Female	Male	Female
Number	58.734	61.062	44.132	40.679
Mean Age	31	29	60	58
Mean Salary	1.800₺	1.800₺	1.800₺	1.800₺

g) Inflation Rate: Refers to the rate of change in the general consumer price index in the medium term program for the two years following the endorsement of the fund to the institution (given in the Table 3.3), and the amount that corresponds to each year which will be gradually equalized to the average inflation rate in the 2003-2008 EU Euro Zone after 25 years (given in the Table 3.4).

Table 3.3 The general consumer price index in the medium term program (Türkiye Cumhuriyeti Kalkınma Bakanlığı, 2012)

THE GENERAL CONSUMER PRICE INDEX			
Appearance	**Years**		
	2013	**2014**	**Other Years**
Medium Term	5,30%	5,00%	5,00%

Table 3.4 The average inflation rate in the 2003-2008 EU Euro Zone (Eurostat, 2012)

EURO ZONE ANNUAL AVERAGE INFLATION RATE IN 2003-2008						
Years						**Average in 2003-2008**
2003	**2004**	**2005**	**2006**	**2007**	**2008**	
2,10%	2,20%	2,20%	2,20%	2,10%	3,30%	2,35%

As a result of Tables 3.3 and 3.4; inflation rate is obtained between 2013 – 2039 years as the following Table 3.5. For other years, inflation rate continues with constant value identified at 2039 year.

Table 3.5 Annual inflation rates according to the draft resolution of ministerial cabinet

AVERAGE ANNUAL RATE OF INFLATION						
Years	**2013**	**2014**	**2015**	**2016**	**2017**	**2018**
Inflation Rate (%)	5,30	5,00	4,89	4,79	4,68	4,58
Years	**2019**	**2020**	**2021**	**2022**	**2023**	**2024**
Inflation Rate (%)	4,47	4,36	4,26	4,15	4,05	3,94
Years	**2025**	**2026**	**2027**	**2028**	**2029**	**2030**
Inflation Rate (%)	3,83	3,73	3,62	3,52	3,41	3,30
Years	**2031**	**2032**	**2033**	**2034**	**2035**	**2036**
Inflation Rate (%)	3,20	3,09	2,99	2,88	2,77	2,67
Years	**2037**	**2038**	**2039**	**2040**	**2041**	**2042**
Inflation Rate (%)	2,56	2,46	2,35	2,35	2,35	2,35

h) Development Rate: Refers to the percentage the change in the Fixed Rate Gross National Product in the medium term program for the years following the endorsement of the fund to the institution, and the change rate anticipated in the last year of the program (shown in the Table 3.6). For other years, development rate continues with constant value identified at 2014 year.

Table 3.6 The fixed rate gross national product in the medium term program (Türkiye Cumhuriyeti Kalkınma Bakanlığı, 2012)

THE FIXED RATE GROOS NATIONAL PRODUCT			
Appearance	**Years**		
	2013	**2014**	**Other Years**
Medium Term	4,00%	5,00%	5,00%

i) **Updating Coefficient:** Shall mean the value found by adding the whole number (1) to the total of 100% of the rate of change in the general index of consumer prices of the final basis year declared by Turkish Statistics Institution according to December of each year and 30% of the development rate of gross domestic product with fixed prices. The assigned salaries and income are considered for each year after endorsement; the possible salaries and income are considered as being increased by the updating coefficient for each year after endorsement. In these applications, updating coefficient is obtained as following using Table 3.3 and 3.6 for 2013, 2014 and other years.

Table 3.7 The updating coefficient according to the medium term program

THE UPDATING COEFFICIENT			
Appearance	**Years**		
	2013	**2014**	**Other Years**
Medium Term	7,50%	7,50%	7,50%

j) **The Rate for The Increase in Income Taken Basis to Premium:** The division of geometric mean of the annual change which would be calculated with the earnings taken as basis to premiums for December for the last five years prior (Table 3.9) to endorsement to the geometric mean of the annual change in the consumer price index published by the Turkey Statistics Institution for the same period (Table 3.10) is the coefficient for the increase in income taken basis to premium. As a result of Tables 3.9 and 3.10, this value is obtained as:

$$The\,coefficient\;\;for\,the\,increase\,in\,earning\,basis\,to\;premium = \frac{9,89}{7,99} = 1,24 \qquad (3.2.1)$$

The rate for increase in income taken basis to premium for each year is the product of for increase in income taken basis to premium and the inflation rate mentioned in heading 3.2 paragraphs (g). Rate of increase is obtained between 2013 – 2039 years as the following Table 3.8. For other years, rate of increase continues with constant value identified at 2039 year.

Table 3.8 The rate for increase in income taken basis to premium for each year

THE RATE OF INCREASE IN INCOME TAKEN BASIS TO PREMIUM						
Years	**2013**	**2014**	**2015**	**2016**	**2017**	**2018**
Rate of Increase (%)	6,59	6,19	6,06	5,93	5,80	5,67
Years	**2019**	**2020**	**2021**	**2022**	**2023**	**2024**
Rate of Increase (%)	5,53	5,40	5,27	5,14	5,01	4,88
Years	**2025**	**2026**	**2027**	**2028**	**2029**	**2030**
Rate of Increase (%)	4,75	4,62	4,48	4,35	4,22	4,09
Years	**2031**	**2032**	**2033**	**2034**	**2035**	**2036**
Rate of Increase (%)	3,96	3,83	3,70	3,57	3,43	3,30
Years	**2037**	**2038**	**2039**	**2040**	**2041**	**2042**
Rate of Increase (%)	3,17	3,04	2,91	2,91	2,91	2,91

Table 3.9 The geometric mean of the annual change of the earnings taken as basis to premiums for 2008-2012 December (http://www.alomaliye.com/)

THE EARNINGS TAKEN AS BASİS TO PREMIUM FOR 2007-2012 (₺)						
MONTHS	**YEARS**					
	2007	**2008**	**2009**	**2010**	**2011**	**2012**
January	3.656,40	3.954,60	4.329,00	4.738,50	5.177,40	5.762,40
February	3.656,40	3.954,60	4.329,00	4.738,50	5.177,40	5.762,40
March	3.656,40	3.954,60	4.329,00	4.738,50	5.177,40	5.762,40
April	3.656,40	3.954,60	4.329,00	4.738,50	5.177,40	5.762,40
May	3.656,40	3.954,60	4.329,00	4.738,50	5.177,40	5.762,40
June	3.656,40	3.954,60	4.329,00	4.738,50	5.177,40	5.762,40
July	3.802,50	4.151,70	4.504,50	4.943,40	5.440,50	6.113,40
August	3.802,50	4.151,70	4.504,50	4.943,40	5.440,50	6.113,40
September	3.802,50	4.151,70	4.504,50	4.943,40	5.440,50	6.113,40
October	3.802,50	4.151,70	4.504,50	4.943,40	5.440,50	6.113,40
November	3.802,50	4.151,70	4.504,50	4.943,40	5.440,50	6.113,40
December	3.802,50	4.151,70	4.504,50	4.943,40	5.440,50	6.113,40
TOTAL	44.753,40	48.637,80	53.001,00	58.091,40	63.707,40	71.254,80
Annual Change of the Earnings Taken as Basis to Premiums of the Month of 2008-2012 December (%)		9,18	8,50	9,74	10,06	12,37
Geometric Mean of the Annual Change of the Earnings Taken as Basis to Premiums of the Month of 2008-2012 December (%)						9,89

Table 3.10 The geometric mean of the annual change in the consumer price index for 2008-2012 December (Türkiye Istatistik Kurumu [TUIK], 2012)

2007-2012 CONSUMER PRICE INDEX NUMBERS						
MONTHS	**YEARS**					
	2007	**2008**	**2009**	**2010**	**2011**	**2012**
January	135,84	146,94	160,90	174,07	182,60	201,98
February	136,42	148,84	160,35	176,59	183,93	203,12
March	137,67	150,27	162,12	177,62	184,70	203,96
April	139,33	152,79	162,15	178,68	186,30	207,05
May	140,03	155,07	163,19	178,04	190,81	206,61
June	139,69	154,51	163,37	177,04	188,08	204,76
July	138,67	155,40	163,78	176,19	187,31	204,29
August	138,70	155,02	163,29	176,90	188,67	205,43
September	140,13	155,72	163,93	179,07	190,09	207,55
October	142,67	159,77	167,88	182,35	196,31	**≈**
November	145,45	161,10	170,01	182,40	199,70	**≈**
December	145,77	160,44	170,91	181,85	200,85	**≈**
Annual Change of Consumer Price Index of the Month of 2008-2012 December (%)		10,06	6,53	6,40	10,45	7,40
Geometric Mean of Annual Change of Consumer Price Index of the Month of 2008-2012 December (%)						7,99

3.3 Present Value Calculations

The present value for liabilities of each fund, for each contributor, including the ones resigned from the fund by the date of endorsement, is calculated with regard to the provisions below, considering the income and expenditures of the fund as per law no 5510.

- The real interest rate (depending on the annual inflation rate) is designated according to 9,80% the technical interest rate.

- "CSO 1980 Female and Male Life Tables" are used for the morbidity probabilities with regard to age.

- For both the premiums and liabilities, "Female Income and Expense Tables" using the "CSO 1980 Female Life Tables" and "Male Income and Expense Tables" using the "CSO 1980 Male Life Tables" are created in "EXCEL 2007". Rows are enumerated until 99 age from 0 age, symbolized with "x" and represents the current age of the person. Columns are enumerated until 99 from 0, symbolized with "n" and represents that payments for no more than n years for temporary life annuities or represents that payments will not start until n years after issue for deferred life annuities.

- In the present value calculation, $\ddot{a}_x$ is called as "pension coefficient". For deterministic model, pension coefficient may be defined using the commutation functions; isn't possible to stochastic model. This value is found using EXCEL 2007 from 0 age to 99 age depending upon both stochastic model and deterministic model in "Female Income and Expense Tables For Pension Coefficient" and "Male Income and Expense Tables For Pension Coefficient". "Income Tables" are changed according to both stochastic interest rates and the rates of increase to premium, and "Expense Tables" are changed according to both stochastic interest rates and the rates of increase to salary. Under these circumstances; different pension coefficient is obtained for all of “Female Income and Expense Tables” and “Male Income and Expense Tables”.

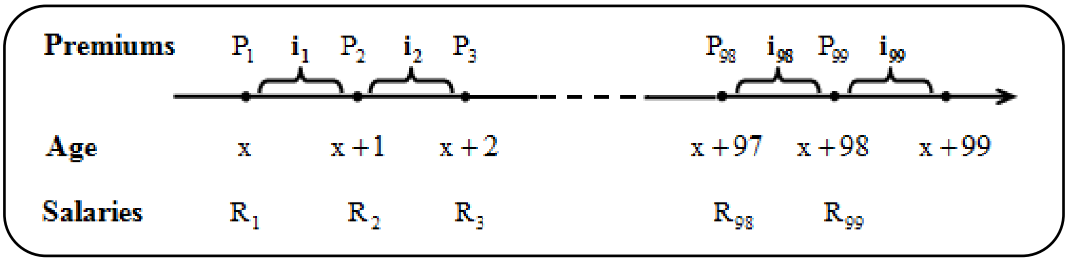

Figure 3.1 The series of payments associated with life annuity-due for stochastic rates

According to Figure 3.1; the rate of increase to premiums are symbolized with P and the rate of increase to salaries are symbolized with R. Assume that annually interest rates are changed randomly. The rate of increase to premium and salaries is accepted 0,00% at the start of each year. For these conditions; the present value of the amount collected from individuals who survived x years is denoted by l_x ; the present value of the amount collected from individuals who survived $x+1$ years is denoted by $l_{x+1}V_1$; the present value of the amount collected from individuals who survived $x+2$ years is denoted by $l_{x+2}V_1V_2$; these calculations are continued until 99 age…in conclusion, these values are added for each x age and divided by the number of people at age x. Then, the present value of this system or pension coefficient may be obtained as:

$$\ddot{a}_x = \frac{l_x + l_{x+1}\overbrace{\frac{1}{(1+i_1)}}^{V_1} + l_{x+2}\overbrace{\frac{1}{(1+i_1)}}^{V_1}\overbrace{\frac{1}{(1+i_2)}}^{V_2} + \ldots + l_{x+99}\overbrace{\frac{1}{(1+i_1)}}^{V_1}\cdots\overbrace{\frac{1}{(1+i_{98})}}^{V_{98}}\overbrace{\frac{1}{(1+i_{99})}}^{V_{99}}}{l_x} \qquad (3.3.1)$$

Assume that annually interest rates and the rate of increase to premiums are changed randomly. In this case, “Female and Male Income Tables for Pension Coefficient” is obtained. For these conditions; the present value of the amount collected from individuals who survived x years is denoted by l_x ; the present value of the amount collected from individuals who survived $x+1$ years is denoted by $l_{x+1}V_1\left(1+P_1\right)$; the present value of the amount collected from individuals who survived $x+2$ years is denoted by $l_{x+2}V_1V_2\left(1+P_1\right)\left(1+P_2\right)$; these calculations are continued until 99 age…in conclusion, these values are added for each x age and divided by the number of people at age x. Then, the present value of this system or pension coefficient may be obtained as:

$$\ddot{a}_x=\frac{l_x+l_{x+1}\overbrace{\frac{1}{\left(1+i_1\right)}}^{V_1}\left(1+P_1\right)+l_{x+2}\left[\overbrace{\frac{1}{\left(1+i_1\right)}}^{V_1}\overbrace{\frac{1}{\left(1+i_2\right)}}^{V_2}\right]\left[\left(1+P_1\right)\left(1+P_2\right)\right]+\ldots}{l_x}$$

$$\frac{\ldots+l_{x+99}\left[\overbrace{\frac{1}{\left(1+i_1\right)}}^{V_1}\cdots\overbrace{\frac{1}{\left(1+i_{98}\right)}}^{V_{98}}\overbrace{\frac{1}{\left(1+i_{99}\right)}}^{V_{99}}\right]\left[\left(1+P_1\right)\ldots\left(1+P_{98}\right)\left(1+P_{99}\right)\right]}{l_x} \qquad (3.3.2)$$

Assume that annually interest rates, the rate of increase to salaries are changed randomly. In this case, “Female and Male Expense Tables for Pension Coefficient” is obtained. For these conditions; the present value of the amount collected from individuals who survived x years is denoted by l_x; the present value of the amount collected from individuals who survived $x+1$ years is denoted by $l_{x+1}V_1\left(1+R_1\right)$; the present value of the amount collected from individuals who survived $x+2$ years is denoted by $l_{x+2}V_1V_2\left(1+R_1\right)\left(1+R_2\right)$; these calculations are continued until 99 age…in conclusion, these values are added for each x age and divided by the number of people at age x. Then, the present value of this system or pension coefficient may be obtained as:

$$\ddot{a}_x = \frac{l_x + l_{x+1}\overbrace{\frac{1}{(1+i_1)}}^{v_1}(1+R_1) + l_{x+2}\left[\overbrace{\frac{1}{(1+i_1)}}^{v_1}\overbrace{\frac{1}{(1+i_2)}}^{v_2}\right]\left[(1+R_1)(1+R_2)\right] + \ldots}{l_x}$$

$$\frac{\ldots + l_{x+99}\left[\overbrace{\frac{1}{(1+i_1)}}^{v_1}\ldots\overbrace{\frac{1}{(1+i_{98})}}^{v_{98}}\overbrace{\frac{1}{(1+i_{99})}}^{v_{99}}\right]\left[(1+R_1)\ldots(1+R_{98})(1+R_{99})\right]}{l_x} \quad (3.3.3)$$

Assume that the rate of increase to premium is changed according to the coefficient for the increase in income taken basis to premium from year to year; the real interest rate (depending on the annual inflation rate) is taken as stochastic according to 9,80% technical interest rate; the rate of increase to salary is designated in accordance with 30% of the development rate of gross domestic product with fixed prices in the medium term program from year to year. Under these circumstances, “Female Income for Pension Coefficient” is shown as Table 3.11; “Male Income for Pension Coefficient” is shown as Table 3.12; “Female Expense for Pension Coefficient” is shown as Table 3.13 and “Male Expense for Pension Coefficient” is shown as Table 3.14.

Table 3.11 Female income for pension coefficient (the rate of increase to premium according to the coefficient for the increase in income taken basis to premium; the real interest rate as stochastic according to 9,80% technical interest rate; the rate of increase to salary in accordance with 30% of the development rate of gross domestic product with fixed prices in the medium term program)

x \ n	**0**	**1**	**2**	**3**	**...**	**96**	**97**	**98**	**99**	**Total**	$\ddot{a}_x$
0	1.000.000,00	963.238,71	928.733,64	894.665,58	**...**	123,94	72,90	36,06	11,69	20.645.794,94	20,646
1	997.110,00	962.400,69	927.981,36	893.958,80	**...**	77,37	38,28	12,41	0,00	20.608.579,61	20,668
2	996.242,51	961.621,14	927.248,26	893.270,45	**...**	40,62	13,17	0,00	0,00	20.572.339,55	20,650
3	995.435,56	960.861,46	926.534,27	892.591,56	**...**	13,98	0,00	0,00	0,00	20.534.986,53	20,629
4	994.649,16	960.121,60	925.830,11	891.939,97	**...**	0,00	0,00	0,00	0,00	20.496.388,27	20,607
5	993.883,28	959.391,91	925.154,25	891.297,78	**...**	0,00	0,00	0,00	0,00	20.456.447,56	20,582
6	993.127,93	958.691,55	924.488,14	890.673,87	**...**	0,00	0,00	0,00	0,00	20.415.062,13	20,556
7	992.402,95	958.001,29	923.841,00	890.059,30	**...**	0,00	0,00	0,00	0,00	20.372.134,71	20,528
8	991.688,42	957.330,69	923.203,55	889.454,06	**...**	0,00	0,00	0,00	0,00	20.327.542,25	20,498
9	990.994,24	956.670,13	922.575,77	888.840,34	**...**	0,00	0,00	0,00	0,00	20.281.176,14	20,465
10	990.310,45	956.019,60	921.939,19	888.200,37	**...**	0,00	0,00	0,00	0,00	20.232.911,79	20,431
...	**...**	**...**	**...**	**...**	**...**	**...**	**...**	**...**	**...**	**...**	**...**
89	181.200,58	144.597,92	112.922,37	86.128,98	**...**	0,00	0,00	0,00	0,00	708.626,49	3,911
90	149.682,55	117.015,86	89.336,28	66.421,81	**...**	0,00	0,00	0,00	0,00	547.130,23	3,655
91	121.130,61	92.574,76	68.895,24	49.716,06	**...**	0,00	0,00	0,00	0,00	412.210,32	3,403
92	95.830,06	71.392,73	51.567,40	35.829,87	**...**	0,00	0,00	0,00	0,00	301.827,40	3,150
93	73.903,18	53.436,74	37.164,11	24.460,33	**...**	0,00	0,00	0,00	0,00	213.556,84	2,890
94	55.315,79	38.511,33	25.371,19	15.269,61	**...**	0,00	0,00	0,00	0,00	144.745,72	2,617
95	39.865,54	26.290,91	15.838,22	8.017,00	**...**	0,00	0,00	0,00	0,00	92.668,89	2,325
96	27.215,41	16.412,37	8.315,54	2.759,05	**...**	0,00	0,00	0,00	0,00	54.702,36	2,010
97	16.989,49	8.616,98	2.861,79	0,00	**...**	0,00	0,00	0,00	0,00	28.468,27	1,676
98	8.919,99	2.965,54	0,00	0,00	**...**	0,00	0,00	0,00	0,00	11.885,53	1,332
99	3.069,82	0,00	0,00	0,00	**...**	0,00	0,00	0,00	0,00	3.069,82	1,000

Table 3.12 Male income for pension coefficient (the rate of increase to premium according to the coefficient for the increase in income taken basis to premium; the real interest rate as stochastic according to 9,80% technical interest rate; the rate of increase to salary in accordance with 30% of the development rate of gross domestic product with fixed prices in the medium term program)

x \ n	**0**	**1**	**2**	**3**	**...**	**96**	**97**	**98**	**99**	**Total**	$\ddot{a}_x$
0	1.000.000,00	961.992,53	927.346,43	893.168,33	**...**	44,77	25,96	12,72	4,10	20.454.784,14	20,455
1	995.820,00	960.963,19	926.428,36	892.293,03	**...**	27,55	13,50	4,35	0,00	20.406.449,72	20,492
2	994.754,47	960.011,84	925.520,46	891.445,35	**...**	14,32	4,62	0,00	0,00	20.359.829,76	20,467
3	993.769,67	959.071,03	924.641,21	890.643,05	**...**	4,90	0,00	0,00	0,00	20.311.724,73	20,439
4	992.795,77	958.159,91	923.809,03	889.877,10	**...**	0,00	0,00	0,00	0,00	20.261.961,32	20,409
5	991.852,62	957.297,57	923.014,56	889.165,19	**...**	0,00	0,00	0,00	0,00	20.210.430,56	20,376
6	990.959,95	956.474,29	922.276,15	888.489,43	**...**	0,00	0,00	0,00	0,00	20.156.997,27	20,341
7	990.107,72	955.709,11	921.575,22	887.831,95	**...**	0,00	0,00	0,00	0,00	20.101.499,15	20,302
8	989.315,64	954.982,77	920.893,25	887.183,83	**...**	0,00	0,00	0,00	0,00	20.043.776,69	20,260
9	988.563,76	954.276,09	920.221,00	886.500,70	**...**	0,00	0,00	0,00	0,00	19.983.642,44	20,215
10	987.832,22	953.579,47	919.512,43	885.747,17	**...**	0,00	0,00	0,00	0,00	19.920.921,01	20,166
...	**...**	**...**	**...**	**...**	**...**	**...**	**...**	**...**	**...**	**...**	**...**
89	181.200,58	144.597,92	112.922,37	86.128,98	**...**	0,00	0,00	0,00	0,00	294.015,47	3,609
90	149.682,55	117.015,86	89.336,28	66.421,81	**...**	0,00	0,00	0,00	0,00	220.463,73	3,414
91	121.130,61	92.574,76	68.895,24	49.716,06	**...**	0,00	0,00	0,00	0,00	161.660,29	3,217
92	95.830,06	71.392,73	51.567,40	35.829,87	**...**	0,00	0,00	0,00	0,00	115.508,88	3,012
93	73.903,18	53.436,74	37.164,11	24.460,33	**...**	0,00	0,00	0,00	0,00	79.989,62	2,794
94	55.315,79	38.511,33	25.371,19	15.269,61	**...**	0,00	0,00	0,00	0,00	53.232,69	2,555
95	39.865,54	26.290,91	15.838,22	8.017,00	**...**	0,00	0,00	0,00	0,00	33.567,24	2,288
96	27.215,41	16.412,37	8.315,54	2.759,05	**...**	0,00	0,00	0,00	0,00	19.574,48	1,991
97	16.989,49	8.616,98	2.861,79	0,00	**...**	0,00	0,00	0,00	0,00	10.091,38	1,668
98	8.919,99	2.965,54	0,00	0,00	**...**	0,00	0,00	0,00	0,00	4.184,13	1,330
99	3.069,82	0,00	0,00	0,00	**...**	0,00	0,00	0,00	0,00	1.075,66	1,000

Table 3.13 Female expense for pension coefficient (the rate of increase to premium according to the coefficient for the increase in income taken basis to premium; the real interest rate as stochastic according to 9,80% technical interest rate; the rate of increase to salary in accordance with 30% of the development rate of gross domestic product with fixed prices in the medium term program)

x \ n	**0**	**1**	**2**	**3**	**...**	**96**	**97**	**98**	**99**	**Total**	$\ddot{a}_x$
0	1.000.000,00	965.712,45	933.510,02	901.576,22	**...**	158,55	93,49	46,37	15,07	21.558.452,79	21,558
1	997.110,00	964.872,28	932.753,88	900.863,98	**...**	98,98	49,09	15,96	0,00	21.517.215,83	21,580
2	996.242,51	964.090,74	932.017,01	900.170,31	**...**	51,97	16,89	0,00	0,00	21.476.815,07	21,558
3	995.435,56	963.329,10	931.299,35	899.486,18	**...**	17,88	0,00	0,00	0,00	21.435.157,03	21,533
4	994.649,16	962.587,34	930.591,57	898.829,56	**...**	0,00	0,00	0,00	0,00	21.392.104,22	21,507
5	993.883,28	961.855,77	929.912,23	898.182,40	**...**	0,00	0,00	0,00	0,00	21.347.554,21	21,479
6	993.127,93	961.153,62	929.242,70	897.553,67	**...**	0,00	0,00	0,00	0,00	21.301.399,47	21,449
7	992.402,95	960.461,59	928.592,23	896.934,36	**...**	0,00	0,00	0,00	0,00	21.253.537,40	21,416
8	991.688,42	959.789,27	927.951,50	896.324,44	**...**	0,00	0,00	0,00	0,00	21.203.839,66	21,382
9	990.994,24	959.127,01	927.320,49	895.705,98	**...**	0,00	0,00	0,00	0,00	21.152.192,33	21,344
10	990.310,45	958.474,81	926.680,64	895.061,07	**...**	0,00	0,00	0,00	0,00	21.098.465,51	21,305
...	**...**	**...**	**...**	**...**	**...**	**...**	**...**	**...**	**...**	**...**	**...**
89	181.200,58	144.969,27	113.503,12	86.794,27	**...**	0,00	0,00	0,00	0,00	712.832,85	3,934
90	149.682,55	117.316,38	89.795,73	66.934,87	**...**	0,00	0,00	0,00	0,00	550.086,98	3,675
91	121.130,61	92.812,51	69.249,57	50.100,08	**...**	0,00	0,00	0,00	0,00	414.216,86	3,420
92	95.830,06	71.576,08	51.832,61	36.106,63	**...**	0,00	0,00	0,00	0,00	303.131,76	3,163
93	73.903,18	53.573,98	37.355,24	24.649,27	**...**	0,00	0,00	0,00	0,00	214.359,79	2,901
94	55.315,79	38.610,23	25.501,68	15.387,56	**...**	0,00	0,00	0,00	0,00	145.205,67	2,625
95	39.865,54	26.358,43	15.919,68	8.078,93	**...**	0,00	0,00	0,00	0,00	92.907,19	2,331
96	27.215,41	16.454,51	8.358,31	2.780,36	**...**	0,00	0,00	0,00	0,00	54.808,59	2,014
97	16.989,49	8.639,11	2.876,51	0,00	**...**	0,00	0,00	0,00	0,00	28.505,11	1,678
98	8.919,99	2.973,15	0,00	0,00	**...**	0,00	0,00	0,00	0,00	11.893,14	1,333
99	3.069,82	0,00	0,00	0,00	**...**	0,00	0,00	0,00	0,00	3.069,82	1,000

Table 3.14 Male expense for pension coefficient (the rate of increase to premium according to the coefficient for the increase in income taken basis to premium; the real interest rate as stochastic according to 9,80% technical interest rate; the rate of increase to salary in accordance with 30% of the development rate of gross domestic product with fixed prices in the medium term program)

x \ n	0	1	2	3	...	96	97	98	99	Total	$\ddot{a}_x$
0	1.000.000,00	964.463,07	932.115,68	900.067,41	...	57,27	33,30	16,35	5,28	21.343.871,00	21,344
1	995.820,00	963.431,10	931.192,89	899.185,34	...	35,25	17,31	5,59	0,00	21.290.666,84	21,380
2	994.754,47	962.477,30	930.280,32	898.331,11	...	18,32	5,92	0,00	0,00	21.239.016,37	21,351
3	993.769,67	961.534,07	929.396,55	897.522,62	...	6,27	0,00	0,00	0,00	21.185.714,67	21,319
4	992.795,77	960.620,62	928.560,10	896.750,75	...	0,00	0,00	0,00	0,00	21.130.583,15	21,284
5	991.852,62	959.756,06	927.761,53	896.033,35	...	0,00	0,00	0,00	0,00	21.073.507,57	21,247
6	990.959,95	958.930,67	927.019,33	895.352,36	...	0,00	0,00	0,00	0,00	21.014.347,49	21,206
7	990.107,72	958.163,52	926.314,79	894.689,80	...	0,00	0,00	0,00	0,00	20.952.935,54	21,162
8	989.315,64	957.435,32	925.629,32	894.036,68	...	0,00	0,00	0,00	0,00	20.889.107,19	21,115
9	988.563,76	956.726,82	924.953,61	893.348,27	...	0,00	0,00	0,00	0,00	20.822.670,19	21,064
10	987.832,22	956.028,40	924.241,39	892.588,92	...	0,00	0,00	0,00	0,00	20.753.444,50	21,009
...	...	...	...	...	...	...	...	...	...	...	...
89	81.465,89	62.545,33	47.092,53	34.731,48	...	0,00	0,00	0,00	0,00	295.644,71	3,629
90	64.578,82	48.674,65	35.932,54	25.928,79	...	0,00	0,00	0,00	0,00	221.586,85	3,431
91	50.257,18	37.139,73	26.825,44	18.873,31	...	0,00	0,00	0,00	0,00	162.409,34	3,232
92	38.347,23	27.726,67	19.525,97	13.288,69	...	0,00	0,00	0,00	0,00	115.988,48	3,025
93	28.628,13	20.181,96	13.748,23	8.903,96	...	0,00	0,00	0,00	0,00	80.281,08	2,804
94	20.838,13	14.210,12	9.211,87	5.479,94	...	0,00	0,00	0,00	0,00	53.397,86	2,563
95	14.672,12	9.521,35	5.669,44	2.848,47	...	0,00	0,00	0,00	0,00	33.652,08	2,294
96	9.830,91	5.859,91	2.946,98	974,23	...	0,00	0,00	0,00	0,00	19.612,04	1,995
97	6.050,43	3.045,98	1.007,92	0,00	...	0,00	0,00	0,00	0,00	10.104,34	1,670
98	3.145,02	1.041,79	0,00	0,00	...	0,00	0,00	0,00	0,00	4.186,80	1,331
99	1.075,66	0,00	0,00	0,00	...	0,00	0,00	0,00	0,00	1.075,66	1,000

- In the present value calculation, $\ddot{a}_{x:\overline{n}|}$ is called as “premium coefficient”. For deterministic model, premium coefficient may be defined using the commutation functions; isn’t possible to stochastic model. This value is found using EXCEL 2007 tables depending upon the pension coefficient for both stochastic model and deterministic model. “Female Income and Expense Tables for Premium Coefficient” and “Male Income and Expense Tables For Premium Coefficient” are made up using “Female Income and Expense Tables For Pension Coefficient” and “Male Income and Expense Tables For Pension Coefficient”, respectively. “Income Tables” are changed according to both stochastic interest rates and the rates of increase to premium, and “Expense Tables” are changed according to both stochastic interest rates and the rates of increase to salary. Under these circumstances; different premium coefficient is obtained for all of “Female Income and Expense Tables” and “Male Income and Expense Tables”.

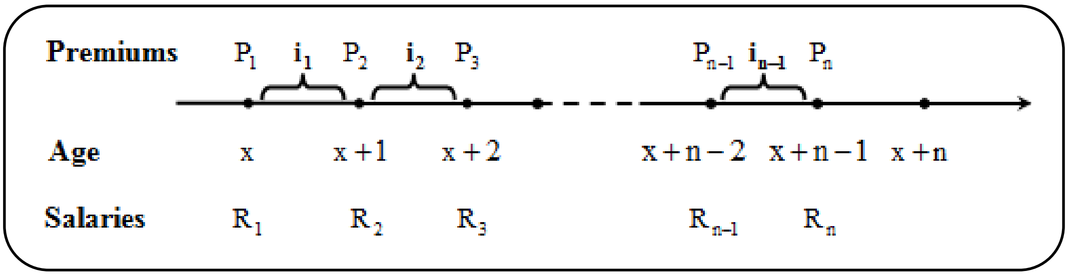

Figure 3.2 The series of payments associated with temporary life annuity-due for stochastic rates

According to Figure 3.2; the rates of increase to premiums are symbolized with P and the rates of increase to salaries are symbolized with R. Assume that annually interest rates are changed randomly. The rates of increase to premiums and salaries is accepted 0,00% at the start of each year. For these conditions; the present value of the amount collected from individuals who survived x years is denoted by l_x ; the present value of the amount collected from individuals who survived $x+1$ years is denoted by $l_{x+1}V_1$; the present value of the amount collected from individuals who survived $x+2$ years is denoted by $l_{x+2}V_1V_2$; these calculations are continued until $(n-1)$ age (payments for no more than n years)…in conclusion, the division of sums obtained consecutively until $(n-1)$

age for each x age using “Female Income and Expense Tables for Pension Coefficient” and “Male Income and Expense Tables for Pension Coefficient” to the number of people at age x is “Female Income and Expense Tables for Premium Coefficient” and “Male Income and Expense Tables for Premium Coefficient”, respectively. As a result, the present value of this system or premium coefficient may be obtained as:

$$\ddot{a}_{x:\overline{n}|} = \frac{l_x + l_{x+1} \overbrace{\frac{1}{(1+i_1)}}^{V_1} + l_{x+2} \overbrace{\frac{1}{(1+i_1)}}^{V_1} \overbrace{\frac{1}{(1+i_2)}}^{V_2} + \ldots + l_{x+n-1} \overbrace{\frac{1}{(1+i_1)}}^{V_1} \cdots \overbrace{\frac{1}{(1+i_{n-1})}}^{V_{n-1}}}{l_x} \qquad (3.3.4)$$

Assume that annually interest rates and the rates of increase to premiums are changed randomly. In this case, “Female and Male Income Tables for Premium Coefficient” is obtained. For these conditions; the present value of the amount collected from individuals who survived x years is denoted by l_x ; the present value of the amount collected from individuals who survived $x+1$ years is denoted by $l_{x+1}V_1(1+P_1)$; the present value of the amount collected from individuals who survived $x+2$ years is denoted by $l_{x+2}V_1V_2(1+P_1)(1+P_2)$; these calculations are continued until $(n-1)$ age (payments for no more than n years)…in conclusion, the division of sums obtained consecutively until $(n-1)$ age for each x age using “Female Income Tables for Pension Coefficient” and “Male Income Tables for Pension Coefficient” to the number of people at age x is “Female Income Tables for Premium Coefficient” and “Male Income Tables for Premium Coefficient”, respectively. As a result, the present value of this system or premium coefficient may be obtained as:

$$\ddot{a}_{x:\overline{n}|} = \frac{l_x + l_{x+1} \overbrace{\frac{1}{(1+i_1)}}^{V_1}(1+P_1) + l_{x+2}\left[\overbrace{\frac{1}{(1+i_1)}}^{V_1}\overbrace{\frac{1}{(1+i_2)}}^{V_2}\right]\left[(1+P_1)(1+P_2)\right] + \ldots}{l_x}$$

$$\frac{\ldots + l_{x+n-1}\left[\overbrace{\frac{1}{(1+i_1)}}^{V_1}\cdots\overbrace{\frac{1}{(1+i_{n-2})}}^{V_{n-2}}\overbrace{\frac{1}{(1+i_{n-1})}}^{V_{n-1}}\right]\left[(1+P_1)\ldots(1+P_{n-2})(1+P_{n-1})\right]}{l_x} \quad (3.3.5)$$

Assume that annually interest rates, the rates of increase to salaries are changed randomly. In this case, “Female and Male Expense Tables for Premium Coefficient” is obtained. For these conditions; the present value of the amount collected from individuals who survived x years is denoted by l_x ; the present value of the amount collected from individuals who survived $x+1$ years is denoted by $l_{x+1}V_1(1+R_1)$; the present value of the amount collected from individuals who survived $x+2$ years is denoted by $l_{x+2}V_1V_2(1+R_1)(1+R_2)$; these calculations are continued until $(n-1)$ age (payments for no more than n years)…in conclusion, the division of sums obtained consecutively until $(n-1)$ age for each x age using “Female Expense Tables for Pension Coefficient” and “Male Expense Tables for Pension Coefficient” to the number of people at age x is “Female Expense Tables for Premium Coefficient” and “Male Expense Tables for Premium Coefficient”, respectively. As a result, the present value of this system or premium coefficient may be obtained as:

$$\ddot{a}_{x:\overline{n}|} = \frac{l_x + l_{x+1}\overbrace{\frac{1}{(1+i_1)}}^{V_1}(1+R_1) + l_{x+2}\left[\overbrace{\frac{1}{(1+i_1)}}^{V_1}\overbrace{\frac{1}{(1+i_2)}}^{V_2}\right]\left[(1+R_1)(1+R_2)\right] + \dots}{l_x}$$

$$\frac{\dots + l_{x+n-1}\left[\overbrace{\frac{1}{(1+i_1)}}^{V_1}\dots\overbrace{\frac{1}{(1+i_{n-2})}}^{V_{n-2}}\overbrace{\frac{1}{(1+i_{n-1})}}^{V_{n-1}}\right]\left[(1+R_1)\dots(1+R_{n-2})(1+R_{n-1})\right]}{l_x} \quad (3.3.6)$$

Assume that the rate of increase to premium is changed according to the coefficient for the increase in income taken basis to premium from year to year; the real interest rate (depending on the annual inflation rate) is taken as stochastic according to 9,80% technical interest rate; the rate of increase to salary is designated in accordance with 30% of the development rate of gross domestic product with fixed prices in the medium term program from year to year. Under these circumstances, sums obtained consecutively until $(n-1)$ age for each x age using “Female Income Tables for Pension Coefficient” is shown in the Table 3.15, using “Male Income Tables for Pension Coefficient” is shown in the Table 3.16, using “Female Expense Tables for Pension Coefficient” is shown in the Table 3.17, using “Male Expense Table for Pension Coefficient” is shown in the Table 3.18. Then, these totals are divided by the number of people at age x and the premium coefficient is obtained for all “Female Income and Expense Tables” and “Male Income and Expense Tables”.

Table 3.15 Sums obtained consecutively until $(n-1)$ age for each x age using the female income table for pension coefficient

x \ n	**0**	**1**	**2**	**3**	**...**	**95**	**96**	**97**	**98**	**99**
0	1.000.000,00	1.963.238,71	2.891.972,34	3.786.637,93	**...**	20.645.550,34	20.645.674,29	20.645.747,19	20.645.783,25	20.645.794,94
1	997.110,00	1.959.510,69	2.887.492,05	3.781.450,85	**...**	20.608.451,55	20.608.528,92	20.608.567,20	20.608.579,61	20.608.579,61
2	996.242,51	1.957.863,66	2.885.111,91	3.778.382,36	**...**	20.572.285,75	20.572.326,38	20.572.339,55	20.572.339,55	20.572.339,55
3	995.435,56	1.956.297,02	2.882.831,30	3.775.422,86	**...**	20.534.972,55	20.534.986,53	20.534.986,53	20.534.986,53	20.534.986,53
4	994.649,16	1.954.770,76	2.880.600,87	3.772.540,85	**...**	20.496.388,27	20.496.388,27	20.496.388,27	20.496.388,27	20.496.388,27
5	993.883,28	1.953.275,19	2.878.429,44	3.769.727,22	**...**	20.456.447,56	20.456.447,56	20.456.447,56	20.456.447,56	20.456.447,56
6	993.127,93	1.951.819,48	2.876.307,63	3.766.981,49	**...**	20.415.062,13	20.415.062,13	20.415.062,13	20.415.062,13	20.415.062,13
7	992.402,95	1.950.404,24	2.874.245,24	3.764.304,55	**...**	20.372.134,71	20.372.134,71	20.372.134,71	20.372.134,71	20.372.134,71
8	991.688,42	1.949.019,11	2.872.222,66	3.761.676,72	**...**	20.327.542,25	20.327.542,25	20.327.542,25	20.327.542,25	20.327.542,25
9	990.994,24	1.947.664,37	2.870.240,14	3.759.080,48	**...**	20.281.176,14	20.281.176,14	20.281.176,14	20.281.176,14	20.281.176,14
10	990.310,45	1.946.330,05	2.868.269,24	3.756.469,62	**...**	20.232.911,79	20.232.911,79	20.232.911,79	20.232.911,79	20.232.911,79
...	**...**	**...**	**...**	**...**	**...**	**...**	**...**	**...**	**...**	**...**
89	181.200,58	325.798,50	438.720,87	524.849,86	**...**	708.626,49	708.626,49	708.626,49	708.626,49	708.626,49
90	149.682,55	266.698,42	356.034,70	422.456,51	**...**	547.130,23	547.130,23	547.130,23	547.130,23	547.130,23
91	121.130,61	213.705,37	282.600,61	332.316,67	**...**	412.210,32	412.210,32	412.210,32	412.210,32	412.210,32
92	95.830,06	167.222,79	218.790,19	254.620,06	**...**	301.827,40	301.827,40	301.827,40	301.827,40	301.827,40
93	73.903,18	127.339,93	164.504,04	188.964,37	**...**	213.556,84	213.556,84	213.556,84	213.556,84	213.556,84
94	55.315,79	93.827,12	119.198,31	134.467,92	**...**	144.745,72	144.745,72	144.745,72	144.745,72	144.745,72
95	39.865,54	66.156,45	81.994,67	90.011,68	**...**	92.668,89	92.668,89	92.668,89	92.668,89	92.668,89
96	27.215,41	43.627,77	51.943,31	54.702,36	**...**	54.702,36	54.702,36	54.702,36	54.702,36	54.702,36
97	16.989,49	25.606,47	28.468,27	28.468,27	**...**	28.468,27	28.468,27	28.468,27	28.468,27	28.468,27
98	8.919,99	11.885,53	11.885,53	11.885,53	**...**	11.885,53	11.885,53	11.885,53	11.885,53	11.885,53
99	3.069,82	3.069,82	3.069,82	3.069,82	**...**	3.069,82	3.069,82	3.069,82	3.069,82	3.069,82

Table 3.16 Sums obtained consecutively until $(n-1)$ age for each x age using the male income table for pension coefficient

x \ n	0	1	2	3	...	95	96	97	98	99
0	1.000.000,00	1.961.992,53	2.889.338,96	3.782.507,29	...	20.454.696,59	20.454.741,36	20.454.767,32	20.454.780,04	20.454.784,14
1	995.820,00	1.956.783,19	2.883.211,55	3.775.504,58	...	20.406.404,33	20.406.431,88	20.406.445,38	20.406.449,72	20.406.449,72
2	994.754,47	1.954.766,31	2.880.286,77	3.771.732,12	...	20.359.810,82	20.359.825,14	20.359.829,76	20.359.829,76	20.359.829,76
3	993.769,67	1.952.840,70	2.877.481,91	3.768.124,96	...	20.311.719,83	20.311.724,73	20.311.724,73	20.311.724,73	20.311.724,73
4	992.795,77	1.950.955,68	2.874.764,72	3.764.641,81	...	20.261.961,32	20.261.961,32	20.261.961,32	20.261.961,32	20.261.961,32
5	991.852,62	1.949.150,18	2.872.164,74	3.761.329,94	...	20.210.430,56	20.210.430,56	20.210.430,56	20.210.430,56	20.210.430,56
6	990.959,95	1.947.434,24	2.869.710,39	3.758.199,82	...	20.156.997,27	20.156.997,27	20.156.997,27	20.156.997,27	20.156.997,27
7	990.107,72	1.945.816,84	2.867.392,05	3.755.224,00	...	20.101.499,15	20.101.499,15	20.101.499,15	20.101.499,15	20.101.499,15
8	989.315,64	1.944.298,41	2.865.191,66	3.752.375,49	...	20.043.776,69	20.043.776,69	20.043.776,69	20.043.776,69	20.043.776,69
9	988.563,76	1.942.839,84	2.863.060,84	3.749.561,54	...	19.983.642,44	19.983.642,44	19.983.642,44	19.983.642,44	19.983.642,44
10	987.832,22	1.941.411,68	2.860.924,11	3.746.671,29	...	19.920.921,01	19.920.921,01	19.920.921,01	19.920.921,01	19.920.921,01
...	...	...	...	...	...	...	...	...	...	...
89	81.465,89	143.851,00	190.702,58	225.167,84	...	294.015,47	294.015,47	294.015,47	294.015,47	294.015,47
90	64.578,82	113.128,79	148.877,48	174.607,52	...	220.463,73	220.463,73	220.463,73	220.463,73	220.463,73
91	50.257,18	87.301,77	113.989,96	132.718,60	...	161.660,29	161.660,29	161.660,29	161.660,29	161.660,29
92	38.347,23	66.002,88	85.428,94	98.615,77	...	115.508,88	115.508,88	115.508,88	115.508,88	115.508,88
93	28.628,13	48.758,39	62.436,28	71.271,99	...	79.989,62	79.989,62	79.989,62	79.989,62	79.989,62
94	20.838,13	35.011,85	44.176,58	49.614,52	...	53.232,69	53.232,69	53.232,69	53.232,69	53.232,69
95	14.672,12	24.169,08	29.809,52	32.636,16	...	33.567,24	33.567,24	33.567,24	33.567,24	33.567,24
96	9.830,91	15.675,81	18.607,71	19.574,48	...	19.574,48	19.574,48	19.574,48	19.574,48	19.574,48
97	6.050,43	9.088,61	10.091,38	10.091,38	...	10.091,38	10.091,38	10.091,38	10.091,38	10.091,38
98	3.145,02	4.184,13	4.184,13	4.184,13	...	4.184,13	4.184,13	4.184,13	4.184,13	4.184,13
99	1.075,66	1.075,66	1.075,66	1.075,66	...	1.075,66	1.075,66	1.075,66	1.075,66	1.075,66

Table 3.17 Sums obtained consecutively until $(n-1)$ age for each x age using the female expense table for pension coefficient

x \ n	0	1	2	3	...	95	96	97	98	99
0	1.000.000,00	1.965.712,45	2.899.222,48	3.800.798,70	...	21.558.139,30	21.558.297,85	21.558.391,34	21.558.437,71	21.558.452,79
1	997.110,00	1.961.982,28	2.894.736,16	3.795.600,14	...	21.517.051,81	21.517.150,79	21.517.199,88	21.517.215,83	21.517.215,83
2	996.242,51	1.960.333,25	2.892.350,26	3.792.520,57	...	21.476.746,21	21.476.798,18	21.476.815,07	21.476.815,07	21.476.815,07
3	995.435,56	1.958.764,66	2.890.064,01	3.789.550,20	...	21.435.139,14	21.435.157,03	21.435.157,03	21.435.157,03	21.435.157,03
4	994.649,16	1.957.236,50	2.887.828,07	3.786.657,63	...	21.392.104,22	21.392.104,22	21.392.104,22	21.392.104,22	21.392.104,22
5	993.883,28	1.955.739,06	2.885.651,29	3.783.833,69	...	21.347.554,21	21.347.554,21	21.347.554,21	21.347.554,21	21.347.554,21
6	993.127,93	1.954.281,55	2.883.524,25	3.781.077,92	...	21.301.399,47	21.301.399,47	21.301.399,47	21.301.399,47	21.301.399,47
7	992.402,95	1.952.864,54	2.881.456,77	3.778.391,12	...	21.253.537,40	21.253.537,40	21.253.537,40	21.253.537,40	21.253.537,40
8	991.688,42	1.951.477,69	2.879.429,18	3.775.753,63	...	21.203.839,66	21.203.839,66	21.203.839,66	21.203.839,66	21.203.839,66
9	990.994,24	1.950.121,25	2.877.441,74	3.773.147,72	...	21.152.192,33	21.152.192,33	21.152.192,33	21.152.192,33	21.152.192,33
10	990.310,45	1.948.785,26	2.875.465,90	3.770.526,97	...	21.098.465,51	21.098.465,51	21.098.465,51	21.098.465,51	21.098.465,51
...	...	...	...	...	...	...	...	...	...	...
89	181.200,58	326.169,85	439.672,97	526.467,24	...	712.832,85	712.832,85	712.832,85	712.832,85	712.832,85
90	149.682,55	266.998,93	356.794,66	423.729,53	...	550.086,98	550.086,98	550.086,98	550.086,98	550.086,98
91	121.130,61	213.943,11	283.192,68	333.292,76	...	414.216,86	414.216,86	414.216,86	414.216,86	414.216,86
92	95.830,06	167.406,13	219.238,74	255.345,37	...	303.131,76	303.131,76	303.131,76	303.131,76	303.131,76
93	73.903,18	127.477,16	164.832,40	189.481,67	...	214.359,79	214.359,79	214.359,79	214.359,79	214.359,79
94	55.315,79	93.926,02	119.427,70	134.815,25	...	145.205,67	145.205,67	145.205,67	145.205,67	145.205,67
95	39.865,54	66.223,97	82.143,65	90.222,58	...	92.907,19	92.907,19	92.907,19	92.907,19	92.907,19
96	27.215,41	43.669,92	52.028,23	54.808,59	...	54.808,59	54.808,59	54.808,59	54.808,59	54.808,59
97	16.989,49	25.628,60	28.505,11	28.505,11	...	28.505,11	28.505,11	28.505,11	28.505,11	28.505,11
98	8.919,99	11.893,14	11.893,14	11.893,14	...	11.893,14	11.893,14	11.893,14	11.893,14	11.893,14
99	3.069,82	3.069,82	3.069,82	3.069,82	...	3.069,82	3.069,82	3.069,82	3.069,82	3.069,82

Table 3.18 Sums obtained consecutively until $(n-1)$ age for each x age using the male expense table for pension coefficient

x \ n	0	1	2	3	...	95	96	97	98	99
0	1.000.000,00	1.964.463,07	2.896.578,76	3.796.646,16	...	21.343.758,80	21.343.816,08	21.343.849,37	21.343.865,72	21.343.871,00
1	995.820,00	1.959.251,10	2.890.443,99	3.789.629,33	...	21.290.608,69	21.290.643,94	21.290.661,24	21.290.666,84	21.290.666,84
2	994.754,47	1.957.231,77	2.887.512,09	3.785.843,21	...	21.238.992,12	21.239.010,45	21.239.016,37	21.239.016,37	21.239.016,37
3	993.769,67	1.955.303,74	2.884.700,29	3.782.222,91	...	21.185.708,40	21.185.714,67	21.185.714,67	21.185.714,67	21.185.714,67
4	992.795,77	1.953.416,39	2.881.976,48	3.778.727,23	...	21.130.583,15	21.130.583,15	21.130.583,15	21.130.583,15	21.130.583,15
5	991.852,62	1.951.608,67	2.879.370,21	3.775.403,55	...	21.073.507,57	21.073.507,57	21.073.507,57	21.073.507,57	21.073.507,57
6	990.959,95	1.949.890,61	2.876.909,94	3.772.262,30	...	21.014.347,49	21.014.347,49	21.014.347,49	21.014.347,49	21.014.347,49
7	990.107,72	1.948.271,24	2.874.586,04	3.769.275,83	...	20.952.935,54	20.952.935,54	20.952.935,54	20.952.935,54	20.952.935,54
8	989.315,64	1.946.750,95	2.872.380,27	3.766.416,95	...	20.889.107,19	20.889.107,19	20.889.107,19	20.889.107,19	20.889.107,19
9	988.563,76	1.945.290,57	2.870.244,18	3.763.592,45	...	20.822.670,19	20.822.670,19	20.822.670,19	20.822.670,19	20.822.670,19
10	987.832,22	1.943.860,62	2.868.102,02	3.760.690,94	...	20.753.444,50	20.753.444,50	20.753.444,50	20.753.444,50	20.753.444,50
...	...	...	...	...	...	...	...	...	...	...
89	81.465,89	144.011,22	191.103,74	225.835,23	...	295.644,71	295.644,71	295.644,71	295.644,71	295.644,71
90	64.578,82	113.253,47	149.186,02	175.114,80	...	221.586,85	221.586,85	221.586,85	221.586,85	221.586,85
91	50.257,18	87.396,91	114.222,35	133.095,65	...	162.409,34	162.409,34	162.409,34	162.409,34	162.409,34
92	38.347,23	66.073,90	85.599,87	98.888,56	...	115.988,48	115.988,48	115.988,48	115.988,48	115.988,48
93	28.628,13	48.810,09	62.558,32	71.462,28	...	80.281,08	80.281,08	80.281,08	80.281,08	80.281,08
94	20.838,13	35.048,25	44.260,11	49.740,05	...	53.397,86	53.397,86	53.397,86	53.397,86	53.397,86
95	14.672,12	24.193,47	29.862,92	32.711,39	...	33.652,08	33.652,08	33.652,08	33.652,08	33.652,08
96	9.830,91	15.690,82	18.637,80	19.612,04	...	19.612,04	19.612,04	19.612,04	19.612,04	19.612,04
97	6.050,43	9.096,42	10.104,34	10.104,34	...	10.104,34	10.104,34	10.104,34	10.104,34	10.104,34
98	3.145,02	4.186,80	4.186,80	4.186,80	...	4.186,80	4.186,80	4.186,80	4.186,80	4.186,80
99	1.075,66	1.075,66	1.075,66	1.075,66	...	1.075,66	1.075,66	1.075,66	1.075,66	1.075,66

- In the present value calculation, ${}_{n|}\ddot{a}_x$ is called as "deferred coefficient". This value is found from the formula ${}_{n|}\ddot{a}_x = \ddot{a}_x - \ddot{a}_{x:\overline{n}|}$ for every age from 0 to 99 using "Female Income and Expense Tables for Pension and Premium Coefficients" and "Male Income and Expense Tables for Pension and Premium Coefficients".

The present value calculations for the liability of salaries, incomes and /or expenses, what would be taken into consideration are explained under the following headings.

3.3.1 Premiums Incoming From Actives

The possible premiums that the individuals who work or voluntarily pay premiums under the fund would pay until they are assigned salaries and/or income as per law no 5510 are considered as income in the present value calculation.

The part to 13,5% of total premium incomes is reserved for professional disease in accordance with active male and female numbers. The part to 20% of total premium incomes is reserved for the disablement, old age and survivor's pensions, the permanent incapacity income and survivor's income, occupational accident in accordance with active male and female numbers. As a result, The part to 33,5% of total premium incomes is accepted the general total of the incomes.

The present value of premiums incoming from actives is formulated for both active male and female members according to mean age which gives in the Table 3.2, as following:

$$\underbrace{58.734}_{active\ males} \times \underset{month}{12} \times \underset{premium}{3.500} \times \underbrace{33,5\%}_{disease\ and\ others} \times \underset{premium\ coefficient\ (male\ income)}{\ddot{a}_{31:\overline{29}|}} \quad ; \quad (Male) \qquad (3.3.7)$$

$$\underbrace{61.062}_{active\ females} \times \underset{month}{12} \times \underset{premium}{3.500} \times \underbrace{33,5\%}_{disease\ and\ others} \times \underset{premium\ coefficient\ (female\ income)}{\ddot{a}_{29:\overline{29}|}} \quad ; \quad (Female) \qquad (3.3.8)$$

3.3.2 Active Liabilities

The present value for the liability of the salary and/or income to be assigned to the individuals who are working or voluntarily paying premiums by the date of endorsement, which equals to the possible premium payment days between the first day of the premium payment to the day the salary and /or income is formulated for both active male and female members according to mean age and mean salary which gives in the Table 3.2, as following:

$$\underbrace{58.734}_{active\ males} \times \underbrace{12}_{month} \times \underbrace{1.800}_{salary} \times \underbrace{{}_{29|}\ddot{a}_{31}}_{deferred\ coefficient\ (male\ expense)} \quad ; \quad (Male) \tag{3.3.9}$$

$$\underbrace{61.062}_{active\ females} \times \underbrace{12}_{month} \times \underbrace{1.800}_{salary} \times \underbrace{{}_{29|}\ddot{a}_{29}}_{deferred\ coefficient\ (female\ expense)} \quad ; \quad (Female) \tag{3.3.10}$$

3.3.3 Passive Liabilities

The present value for the liability of the salary and income to be paid to individuals receiving salaries and/or income by the date of endorsement is formulated for both passive male and female members according to mean age and mean salary which gives in the Table 3.2, as following:

$$\underbrace{44.132}_{passive\ males} \times \underbrace{12}_{month} \times \underbrace{1.800}_{salary} \times \underbrace{\ddot{a}_{60}}_{pension\ coefficient\ (male\ expense)} \quad ; \quad (Male) \tag{3.3.11}$$

$$\underbrace{40.679}_{passive\ females} \times \underbrace{12}_{month} \times \underbrace{1.800}_{salary} \times \underbrace{\ddot{a}_{58}}_{pension\ coefficient\ (female\ expense)} \quad ; \quad (Female) \tag{3.3.12}$$

3.3.4 Dependents

The present value for the liability of dependents of active and passive members is represented under the headings of widow and orphan. Widows are composed of spouse, mother and father, children who is older than or equal to twenty five age (>=25 age); orphans are composed of children who is smaller than twenty five age (<25 age). As a result, the present value for the liability of dependents is formulated for both passive and active members according to mean age, mean salary and mean numbers which give in the Table 3.1 of dependents, as following:

$$\underbrace{10.502}_{\text{widow numbers}} \times \underbrace{12}_{\text{month}} \times \underbrace{800}_{\text{salary}} \times \underbrace{\ddot{a}_{45}}_{\substack{\text{pension coefficient} \\ (\text{male}-\text{female expense})}} \quad ; \quad (Widow) \tag{3.3.13}$$

$$\underbrace{2.500}_{\text{orphan numbers}} \times \underbrace{12}_{\text{month}} \times \underbrace{300}_{\text{premium}} \times \underbrace{\ddot{a}_{15:\overline{10}|}}_{\substack{\text{premium coefficient} \\ (\text{male}-\text{female expense})}} \quad ; \quad (Orphan) \tag{3.3.14}$$

3.3.5 Health Liabilities

In the calculation of the present value for health benefits after the endorsement of the fund, for the individuals receiving salaries and/or income and their beneficiaries, the present value of the amount corresponding to the share proportion of health benefits of the individuals receiving salaries and/or income as a result of service merging, who have worked under more than one fund or social security laws, the cash value of the amount of the health benefits to the individuals who worked and quitted prior the endorsement and their beneficiaries, as the ratio of the salary and/or income to be assigned after endorsement to the amount of premium to be paid since the first premium payment to the last premium payment before assignment of salary and/or income, the cost per capita obtained from the institution database for every age group with regard to sex. The health benefit expenditure for the following years, with regard to age and sex is calculated by increasing the cost per capita obtained from the institution database by the inflation rate.

For the calculation of the health expenses, all of active and passive members are incorporated into the present value. The annual mean expense of per member is assumed 2.000₺. The weighted mean age of all active and passive members is found

42. As a result, the present value of health expenses is formulated as following according to the information;

$$\underbrace{204.607}_{members}\times \underbrace{2.000}_{annual\ health\ expense} \times \underbrace{\ddot{a}_{42}}_{\substack{pension\ coefficient\\(male-female\ expense)}} \quad ; \quad (All\ \ Members) \tag{3.3.15}$$

3.4 Scenarios and Actuarial Valuations

For actuarial valuations, four different scenarios will be proposed and used different scales for each scenario in this section. Different actuarial valuations will be obtained for every scale. Actuarial valuations will be allocated into two parts in the form of premiums and liabilities. The present value calculations in which were given under the previous heading 3.3 will change depending on the scales that will explain in every scenario.

3.4.1 Scenario I

- Six different scales are used,
- Scales don’t have the rate of increase to premium and salary,
- Only, interest rates are accepted deterministic and stochastic.

3.4.1.1 First Scale of the Scenario I

The technical interest rate is accepted deterministic and taken as 9,80%. For first scale of the Scenario I (shown in the Table B.1), valuations of the total premium and liability are obtained according to present value calculations given in the Table C.1. In the end, incumbent liabilities of the foundation funds are determined by subtracting from the sum of premiums to the sum of liabilities in the Table 3.19.

Table 3.19 Actuarial valuations for first scale of the scenario I (9,80% technical interest rate)

PREMIUMS		**LIABILITIES**	
Premiums Incoming from Actives		**Passives**	**16.122.611.353**
Disease	**6.964.794.686**	*Male*	7.946.617.247
Male	3.401.904.975	*Female*	8.175.994.107
Female	3.562.889.711	**Actives**	**1.328.776.773**
Others	**10.318.214.350**	*Male*	594.124.745
Male	5.039.859.222	*Female*	734.652.028
Female	5.278.355.127	**Dependents**	**1.081.931.802**
		Widow	1.020.981.415
		Orphan	60.950.387
		Health Liabilities	**4.215.852.861**
General Total	**17.283.009.036**	**General Total**	**22.749.172.788**
Liabilities of the Foundation Funds		**-5.466.163.753 ₺**	

Obtained results are divided into according to the insured ratio given for each foundation funds as Table 3.20.

Table 3.20 Incumbent liability of the each foundation fund for first scale of the scenario I

Names of Foundation Funds	**Liability (₺)**
Türkiye İş Bankası A.Ş.	1.421.202.576
Yapı ve Kredi Bankası A.Ş.	786.034.348
Akbank	718.800.534
Türkiye Vakıflar Bankası	575.040.427
Türkiye Garanti Bankası A.Ş.	566.841.181
T.C. Ziraat Bankası A.Ş. ve Türkiye Halk Bankası A.Ş.	344.914.933
Türkiye Halk Bankası A.Ş. (Pamukbank T.A.Ş.)	322.503.661
Türkiye Odalar Borsalar Birliği	277.681.119
Şekerbank	212.087.154
Fortis Bank A.Ş. ve Dış Bank	120.802.219
Anadolu Anonim Türk Sigorta Şirketi (Anadolu Sigorta)	34.983.448
Türkiye Sinai Kalkınma Bankası	22.957.888
Esbank Eskişehir Bankası	20.771.422
Mapfre Genel Sigorta	13.665.409
Milli Reasürans	12.025.560
Liberty Sigorta	9.839.095
İmar Bankası	6.012.780

3.4.1.2 Second Scale of the Scenario I

The technical interest rate is accepted deterministic and taken as 7,35%. For second scale of the Scenario I (shown in the Table B.2), valuations of the total premium and liability are obtained according to present value calculations given in the Table C.1. In the end, incumbent liabilities of the foundation funds are determined by subtracting from sum of premiums to sum of liabilities in Table 3.21.

Table 3.21 Actuarial valuations for second scale of the scenario I (7,35% technical interest rate)

PREMIUMS		LIABILITIES	
Premiums Incoming from Actives		**Passives**	**18.976.523.279**
Disease	**8.447.160.148**	*Male*	9.237.757.264
Male	4.122.443.837	*Female*	9.738.766.015
Female	4.324.716.311	**Actives**	**3.012.489.308**
Others	**12.514.311.331**	*Male*	1.328.832.803
Male	6.107.324.203	*Female*	1.683.656.505
Female	6.406.987.128	**Dependents**	**1.319.586.501**
		Widow	1.253.151.158
		Orphan	66.435.343
		Health Liabilities	**5.210.737.853**
General Total	**20.961.471.479**	**General Total**	**28.519.336.941**
Liabilities of the Foundation Funds		**-7.557.865.462 ₺**	

Obtained results are divided into according to the insured ratio given for each foundation funds as Table 3.22.

Table 3.22 Incumbent liability of the each foundation fund for second scale of the scenario I

Names of Foundation Funds	Liability (₺)
Türkiye İş Bankası A.Ş.	1.965.045.020
Yapı ve Kredi Bankası A.Ş.	1.086.821.053
Akbank	993.859.308
Türkiye Vakıflar Bankası	795.087.447
Türkiye Garanti Bankası A.Ş.	783.750.648
T.C. Ziraat Bankası A.Ş. ve Türkiye Halk Bankası A.Ş.	476.901.311
Türkiye Halk Bankası A.Ş. (Pamukbank T.A.Ş.)	445.914.062
Türkiye Odalar Borsalar Birliği	383.939.565
Şekerbank	293.245.180
Fortis Bank A.Ş. ve Dış Bank	167.028.827
Anadolu Anonim Türk Sigorta Şirketi (Anadolu Sigorta)	48.370.339
Türkiye Sinai Kalkınma Bankası	31.743.035
Esbank Eskişehir Bankası	28.719.889
Mapfre Genel Sigorta	18.894.664
Milli Reasürans	16.627.304
Liberty Sigorta	13.604.158
İmar Bankası	8.313.652

3.4.1.3 Third Scale of the Scenario I

The technical interest rate is accepted deterministic and taken as 5,85%. For third scale of the Scenario I (shown in the Table B.3), valuations of the total premium and liability are obtained according to present value calculations given in the Table C.1. In the end, incumbent liabilities of the foundation funds are determined by subtracting from the sum of premiums to the sum of liabilities in the Table 3.23.

Table 3.23 Actuarial valuations for third scale of the scenario I (5,85% technical interest rate)

PREMIUMS		**LIABILITIES**	
Premiums Incoming from Actives		**Passives**	**21.254.144.310**
Disease	**9.663.107.903**	*Male*	10.248.538.891
Male	4.712.940.273	*Female*	11.005.605.419
Female	4.950.167.630	**Actives**	**5.078.586.360**
Others	**14.315.715.412**	*Male*	2.217.124.870
Male	6.982.133.738	*Female*	2.861.461.489
Female	7.333.581.674	**Dependents**	**1.522.623.532**
		Widow	1.452.375.787
		Orphan	70.247.746
		Health Liabilities	**6.076.411.939**
General Total	**23.978.823.316**	**General Total**	**33.931.766.140**
Liabilities of the Foundation Funds		**-9.952.942.824 ₺**	

Obtained results are divided into according to the insured ratio given for each foundation funds as Table 3.24.

Table 3.24 Incumbent liability of the each foundation fund for third scale of the scenario I

Names of Foundation Funds	**Liability (₺)**
Türkiye İş Bankası A.Ş.	2.587.765.134
Yapı ve Kredi Bankası A.Ş.	1.431.233.178
Akbank	1.308.811.981
Türkiye Vakıflar Bankası	1.047.049.585
Türkiye Garanti Bankası A.Ş.	1.032.120.171
T.C. Ziraat Bankası A.Ş. ve Türkiye Halk Bankası A.Ş.	628.030.692
Türkiye Halk Bankası A.Ş. (Pamukbank T.A.Ş.)	587.223.627
Türkiye Odalar Borsalar Birliği	505.609.495
Şekerbank	386.174.182
Fortis Bank A.Ş. ve Dış Bank	219.960.036
Anadolu Anonim Türk Sigorta Şirketi (Anadolu Sigorta)	63.698.834
Türkiye Sinai Kalkınma Bankası	41.802.360
Esbank Eskişehir Bankası	37.821.183
Mapfre Genel Sigorta	24.882.357
Milli Reasürans	21.896.474
Liberty Sigorta	17.915.297
İmar Bankası	10.948.237

3.4.1.4 Fourth Scale of the Scenario I

The technical interest rate is accepted deterministic and taken as 4,40%. For fourth scale of the Scenario I (shown in the Table B.4), valuations of the total premium and liability are obtained according to present value calculations given in Table C.1. In the end, incumbent liabilities of the foundation funds are determined by subtracting from the sum of premiums to the sum of liabilities in the Table 3.25.

Table 3.25 Actuarial valuations for fourth scale of the scenario I (4,40% technical interest rate)

PREMIUMS		LIABILITIES	
Premiums Incoming from Actives		**Passives**	**23.987.879.080**
Disease	**11.152.308.832**	*Male*	11.441.494.303
Male	5.435.600.787	*Female*	12.546.384.777
Female	5.716.708.044	**Actives**	**8.559.052.994**
Others	**16.521.939.010**	*Male*	3.692.593.457
Male	8.052.741.907	*Female*	4.866.459.538
Female	8.469.197.103	**Dependents**	**1.782.237.792**
		Widow	1.707.925.072
		Orphan	74.312.720
		Health Liabilities	**7.201.865.603**
General Total	**27.674.247.842**	**General Total**	**41.531.035.469**
Liabilities of the Foundation Funds		**-13.856.787.627 ₺**	

Obtained results are divided into according to the insured ratio given for each foundation funds as Table 3.26.

Table 3.26 Incumbent liability of the each foundation fund for fourth scale of the scenario I

Names of Foundation Funds	Liability (₺)
Türkiye İş Bankası A.Ş.	3.602.764.783
Yapı ve Kredi Bankası A.Ş.	1.992.606.061
Akbank	1.822.167.573
Türkiye Vakıflar Bankası	1.457.734.058
Türkiye Garanti Bankası A.Ş.	1.436.948.877
T.C. Ziraat Bankası A.Ş. ve Türkiye Halk Bankası A.Ş.	874.363.299
Türkiye Halk Bankası A.Ş. (Pamukbank T.A.Ş.)	817.550.470
Türkiye Odalar Borsalar Birliği	703.924.811
Şekerbank	537.643.360
Fortis Bank A.Ş. ve Dış Bank	306.235.007
Anadolu Anonim Türk Sigorta Şirketi (Anadolu Sigorta)	88.683.441
Türkiye Sinai Kalkınma Bankası	58.198.508
Esbank Eskişehir Bankası	52.655.793
Mapfre Genel Sigorta	34.641.969
Milli Reasürans	30.484.933
Liberty Sigorta	24.942.218
İmar Bankası	15.242.466

3.4.1.5 Fifth Scale of the Scenario I

The technical interest rate is accepted deterministic and taken as 3,00%. For fifth scale of the Scenario I (shown in the Table B.5), valuations of the total premium and liability are obtained according to present value calculations given in the Table C.1. In the end, incumbent liabilities of the foundation funds are determined by subtracting from the sum of premiums to the sum of liabilities in the Table 3.27.

Table 3.27 Actuarial valuations for fifth scale of the scenario I (3,00% technical interest rate)

PREMIUMS		LIABILITIES	
Premiums Incoming from Actives		**Passives**	**27.287.437.944**
Disease	**12.982.705.332**	*Male*	12.855.488.784
Male	6.323.177.770	*Female*	14.431.949.160
Female	6.659.527.563	**Actives**	**14.417.688.126**
Others	**19.233.637.529**	*Male*	6.137.228.396
Male	9.367.670.770	*Female*	8.280.459.730
Female	9.865.966.760	**Dependents**	**2.118.711.640**
		Widow	2.040.074.271
		Orphan	78.637.369
		Health Liabilities	**8.687.609.827**
General Total	**32.216.342.862**	**General Total**	**52.511.447.536**
Liabilities of the Foundation Funds		**-20.295.104.675 ₺**	

Obtained results are divided into according to the insured ratio given for each foundation funds as Table 3.28.

Table 3.28 Incumbent liability of the each foundation fund for fifth scale of the scenario I

Names of Foundation Funds	Liability (₺)
Türkiye İş Bankası A.Ş.	5.276.727.216
Yapı ve Kredi Bankası A.Ş.	2.918.436.052
Akbank	2.668.806.265
Türkiye Vakıflar Bankası	2.135.045.012
Türkiye Garanti Bankası A.Ş.	2.104.602.355
T.C. Ziraat Bankası A.Ş. ve Türkiye Halk Bankası A.Ş.	1.280.621.105
Türkiye Halk Bankası A.Ş. (Pamukbank T.A.Ş.)	1.197.411.176
Türkiye Odalar Borsalar Birliği	1.030.991.317
Şekerbank	787.450.061
Fortis Bank A.Ş. ve Dış Bank	448.521.813
Anadolu Anonim Türk Sigorta Şirketi (Anadolu Sigorta)	129.888.670
Türkiye Sinai Kalkınma Bankası	85.239.440
Esbank Eskişehir Bankası	77.121.398
Mapfre Genel Sigorta	50.737.762
Milli Reasürans	44.649.230
Liberty Sigorta	36.531.188
İmar Bankası	22.324.615

3.4.1.6 Sixth Scale of the Scenario I

The real interest rate (depending on the annual inflation rate) is taken as stochastic according to 9,80%. For sixth scale of the Scenario I (shown in the Table B.6), valuations of the total premium and liability are obtained according to present value calculations given in the Table C.1. In the end, incumbent liabilities of the foundation funds are determined by subtracting from the sum of premiums to the sum of liabilities in the Table 3.29.

Table 3.29 Actuarial valuations for sixth scale of the scenario I (real interest rate)

PREMIUMS		LIABILITIES	
Premiums Incoming from Actives		**Passives**	**21.844.508.917**
Disease	**9.906.525.638**	*Male*	10.554.400.462
Male	4.832.354.118	*Female*	11.290.108.455
Female	5.074.171.521	**Actives**	**4.022.423.548**
Others	**14.676.334.279**	*Male*	1.775.440.021
Male	7.159.043.137	*Female*	2.246.983.527
Female	7.517.291.142	**Dependents**	**1.544.867.159**
		Widow	1.472.479.117
		Orphan	72.388.042
		Health Liabilities	**6.139.244.136**
General Total	**24.582.859.917**	**General Total**	**33.551.043.760**
Liabilities of the Foundation Funds		**-8.968.183.843 ₺**	

Obtained results are divided into according to the insured ratio given for each foundation funds as Table 3.30.

Table 3.30 Incumbent liability of the each foundation fund for sixth scale of the scenario I

Names of Foundation Funds	Liability (₺)
Türkiye İş Bankası A.Ş.	2.331.727.799
Yapı ve Kredi Bankası A.Ş.	1.289.624.837
Akbank	1.179.316.175
Türkiye Vakıflar Bankası	943.452.940
Türkiye Garanti Bankası A.Ş.	930.000.665
T.C. Ziraat Bankası A.Ş. ve Türkiye Halk Bankası A.Ş.	565.892.400
Türkiye Halk Bankası A.Ş. (Pamukbank T.A.Ş.)	529.122.847
Türkiye Odalar Borsalar Birliği	455.583.739
Şekerbank	347.965.533
Fortis Bank A.Ş. ve Dış Bank	198.196.863
Anadolu Anonim Türk Sigorta Şirketi (Anadolu Sigorta)	57.396.377
Türkiye Sinai Kalkınma Bankası	37.666.372
Esbank Eskişehir Bankası	34.079.099
Mapfre Genel Sigorta	22.420.460
Milli Reasürans	19.730.004
Liberty Sigorta	16.142.731
İmar Bankası	9.865.002

3.4.2 Scenario II

- Two different scales are used,
- The rate of increase to salary is determined according to the updating coefficient for both first and second scale,
- The technical interest rate is taken as 9,80% for both first and second scale,
- The only difference between the scales is the rate of increase to premium.

3.4.2.1 First Scale of the Scenario II

The rate of increase to premium is designated in accordance with inflation rate. For first scale of the Scenario II (shown in the Table B.7), valuations of the total premium and liability are obtained according to present value calculations given in the Table C.1. In the end, incumbent liabilities of the foundation funds are determined by subtracting from sum of premiums to sum of liabilities in Table 3.31.

Table 3.31 Actuarial valuations for first scale of the scenario II

PREMIUMS		**LIABILITIES**	
Premiums Incoming from Actives		**Passives**	**29.729.657.308**
Disease	**10.115.524.802**	*Male*	13.885.968.147
Male	4.933.865.751	*Female*	15.843.689.161
Female	5.181.659.051	**Actives**	**20.049.321.533**
Others	**14.985.962.670**	*Male*	8.455.065.437
Male	7.309.430.742	*Female*	11.594.256.096
Female	7.676.531.928	**Dependents**	**2.383.890.507**
		Widow	2.302.378.624
		Orphan	81.511.883
		Health Liabilities	**9.877.415.104**
General Total	**25.101.487.472**	**General Total**	**62.040.284.453**
Liabilities of the Foundation Funds		**-36.938.796.981 ₺**	

Obtained results are divided into according to the insured ratio given for each foundation funds as Table 3.32.

Table 3.32 Incumbent liability of the each foundation fund for first scale of the scenario II

Names of Foundation Funds	**Liability (₺)**
Türkiye İş Bankası A.Ş.	9.604.087.215
Yapı ve Kredi Bankası A.Ş.	5.311.799.006
Akbank	4.857.451.803
Türkiye Vakıflar Bankası	3.885.961.442
Türkiye Garanti Bankası A.Ş.	3.830.553.247
T.C. Ziraat Bankası A.Ş. ve Türkiye Halk Bankası A.Ş.	2.330.838.090
Türkiye Halk Bankası A.Ş. (Pamukbank T.A.Ş.)	2.179.389.022
Türkiye Odalar Borsalar Birliği	1.876.490.887
Şekerbank	1.433.225.323
Fortis Bank A.Ş. ve Dış Bank	816.347.413
Anadolu Anonim Türk Sigorta Şirketi (Anadolu Sigorta)	236.408.301
Türkiye Sinai Kalkınma Bankası	155.142.947
Esbank Eskişehir Bankası	140.367.429
Mapfre Genel Sigorta	92.346.992
Milli Reasürans	81.265.353
Liberty Sigorta	66.489.835
İmar Bankası	40.632.677

3.4.2.2 Second Scale of the Scenario II

The rate of increase to premium is designated using the rate for the increase in income taken basis to premium. For second scale of the Scenario II (shown in the Table B.8), valuations of the total premium and liability are obtained according to present value calculations given in the Table C.1. In the end, incumbent liabilities of the foundation funds are determined by subtracting from the sum of premiums to the sum of liabilities in the Table 3.33.

Table 3.33 Actuarial valuations for second scale of the scenario II

PREMIUMS		**LIABILITIES**	
Premiums Incoming from Actives		**Passives**	**29.729.657.308**
Disease	**11.154.737.866**	*Male*	13.885.968.147
Male	5.438.685.738	*Female*	15.843.689.161
Female	5.716.052.128	**Actives**	**20.049.321.533**
Others	**16.525.537.580**	*Male*	8.455.065.437
Male	8.057.312.205	*Female*	11.594.256.096
Female	8.468.225.375	**Dependents**	**2.383.890.507**
		Widow	2.302.378.624
		Orphan	81.511.883
		Health Liabilities	**9.877.415.104**
General Total	**27.680.275.446**	**General Total**	**62.040.284.453**
Liabilities of the Foundation Funds		**-34.360.009.007 ₺**	

Obtained results are divided into according to the insured ratio given for each foundation funds as Table 3.34.

Table 3.34 Incumbent liability of the each foundation fund for second scale of the scenario II

Names of Foundation Funds	Liability (₺)
Türkiye İş Bankası A.Ş.	8.933.602.342
Yapı ve Kredi Bankası A.Ş.	4.940.969.295
Akbank	4.518.341.184
Türkiye Vakıflar Bankası	3.614.672.948
Türkiye Garanti Bankası A.Ş.	3.563.132.934
T.C. Ziraat Bankası A.Ş. ve Türkiye Halk Bankası A.Ş.	2.168.116.568
Türkiye Halk Bankası A.Ş. (Pamukbank T.A.Ş.)	2.027.240.531
Türkiye Odalar Borsalar Birliği	1.745.488.458
Şekerbank	1.333.168.349
Fortis Bank A.Ş. ve Dış Bank	759.356.199
Anadolu Anonim Türk Sigorta Şirketi (Anadolu Sigorta)	219.904.058
Türkiye Sinai Kalkınma Bankası	144.312.038
Esbank Eskişehir Bankası	130.568.034
Mapfre Genel Sigorta	85.900.023
Milli Reasürans	75.592.020
Liberty Sigorta	61.848.016
İmar Bankası	37.796.010

3.4.3 Scenario III

- Two different scales are used,
- The rate of increase to premium is determined according to the coefficient for the increase in income taken basis to premium for both first and second scale,
- The real interest rate (depending on the annual inflation rate) is taken as stochastic according to 9,80% technical interest rate,
- The only difference between the scales is the rate of increase to salary.

3.4.3.1 First Scale of the Scenario III

The rate of increase to salary is designated in accordance with 30% of the development rate of gross domestic product with fixed prices in the medium term program. For first scale of the Scenario III (shown in the Table B.9), valuations of the total premium and liability are obtained according to present value calculations given in the Table C.1. In the end, incumbent liabilities of the foundation funds are determined by subtracting from sum of premiums to sum of liabilities in Table 3.35.

Table 3.35 Actuarial valuations for first scale of the scenario III

PREMIUMS		**LIABILITIES**	
Premiums Incoming from Actives		**Passives**	**24.871.059.347**
Disease	**11.237.200.915**	*Male*	11.885.047.359
Male	5.478.332.407	*Female*	12.986.011.988
Female	5.758.868.507	**Actives**	**6.987.325.097**
Others	**16.647.705.059**	*Male*	3.050.686.653
Male	8.116.048.011	*Female*	3.936.638.444
Female	8.531.657.048	**Dependents**	**1.820.701.308**
		Widow	1.743.719.734
		Orphan	76.981.574
		Health Liabilities	**7.321.489.231**
General Total	**27.884.905.973**	**General Total**	**41.000.574.984**
Liabilities of the Foundation Funds		**-13.115.669.011 ₺**	

Obtained results are divided into according to the insured ratio given for each foundation funds as Table 3.36.

Table 3.36 Incumbent liability of the each foundation fund for first scale of the scenario III

Names of Foundation Funds	**Liability (₺)**
Türkiye İş Bankası A.Ş.	3.410.073.943
Yapı ve Kredi Bankası A.Ş.	1.886.033.204
Akbank	1.724.710.475
Türkiye Vakıflar Bankası	1.379.768.380
Türkiye Garanti Bankası A.Ş.	1.360.094.876
T.C. Ziraat Bankası A.Ş. ve Türkiye Halk Bankası A.Ş.	827.598.715
Türkiye Halk Bankası A.Ş. (Pamukbank T.A.Ş.)	773.824.472
Türkiye Odalar Borsalar Birliği	666.275.986
Şekerbank	508.887.958
Fortis Bank A.Ş. ve Dış Bank	289.856.285
Anadolu Anonim Türk Sigorta Şirketi (Anadolu Sigorta)	83.940.282
Türkiye Sinai Kalkınma Bankası	55.085.810
Esbank Eskişehir Bankası	49.839.542
Mapfre Genel Sigorta	32.789.173
Milli Reasürans	28.854.472
Liberty Sigorta	23.608.204
İmar Bankası	14.427.236

3.4.3.2 Second Scale of the Scenario III

The rate of increase to salary is found by adding the whole number (1) to 30% of the development rate of gross domestic product with fixed prices in the medium term program. For second scale of the Scenario III (shown in the Table B.10), valuations of the total premium and liability are obtained according to present value calculations given in the Table C.1. In the end, incumbent liabilities of the foundation funds are determined by subtracting from sum of premiums to sum of liabilities in Table 3.37.

Table 3.37 Actuarial valuations for second scale of the scenario III

PREMIUMS		**LIABILITIES**	
Premiums Incoming from Actives		**Passives**	**27.266.204.762**
Disease	**11.237.200.915**	*Male*	12.922.512.430
Male	5.478.332.407	*Female*	14.343.692.332
Female	5.758.868.507	**Actives**	**10.114.344.138**
Others	**16.647.705.059**	*Male*	4.379.803.905
Male	8.116.048.011	*Female*	5.734.540.233
Female	8.531.657.048	**Dependents**	**2.051.429.881**
		Widow	1.971.186.267
		Orphan	80.243.614
		Health Liabilities	**8.324.502.453**
General Total	**27.884.905.973**	**General Total**	**47.756.481.234**
Liabilities of the Foundation Funds		**-19.871.575.261 ₺**	

Obtained results are divided into according to the insured ratio given for each foundation funds as Table 3.38.

Table 3.38 Incumbent liability of the each foundation fund for second scale of the scenario III

Names of Foundation Funds	**Liability (₺)**
Türkiye İş Bankası A.Ş.	5.166.609.568
Yapı ve Kredi Bankası A.Ş.	2.857.532.523
Akbank	2.613.112.147
Türkiye Vakıflar Bankası	2.090.489.717
Türkiye Garanti Bankası A.Ş.	2.060.682.355
T.C. Ziraat Bankası A.Ş. ve Türkiye Halk Bankası A.Ş.	1.253.896.399
Türkiye Halk Bankası A.Ş. (Pamukbank T.A.Ş.)	1.172.422.940
Türkiye Odalar Borsalar Birliği	1.009.476.023
Şekerbank	771.017.120
Fortis Bank A.Ş. ve Dış Bank	439.161.813
Anadolu Anonim Türk Sigorta Şirketi (Anadolu Sigorta)	127.178.082
Türkiye Sinai Kalkınma Bankası	83.460.616
Esbank Eskişehir Bankası	75.511.986
Mapfre Genel Sigorta	49.678.938
Milli Reasürans	43.717.466
Liberty Sigorta	35.768.835
İmar Bankası	21.858.733

3.4.4 Scenario IV

- Two different scales are used,
- The rate of increase to premium is determined according to the inflation rate for both first and second scale,
- The technical interest rate is taken as 9,80%,
- The only difference between the scales is the rate of increase to salary.

3.4.4.1 First Scale of the Scenario IV

The rate of increase to salary is designated in accordance with the general consumer price index in the medium term program. For first scale of the Scenario IV (shown in the Table B.11), valuations of the total premium and liability are obtained according to present value calculations given in the Table C.1. In the end, incumbent liabilities of the foundation funds are determined by subtracting from the sum of premiums to the sum of liabilities in the Table 3.39.

Table 3.39 Actuarial valuations for first scale of the scenario IV

PREMIUMS		**LIABILITIES**	
Premiums Incoming from Actives		**Passives**	**23.632.208.578**
Disease	**10.115.524.802**	*Male*	11.287.446.131
Male	4.933.865.751	*Female*	12.344.762.447
Female	5.181.659.051	**Actives**	**8.039.371.882**
Others	**14.985.962.670**	*Male*	3.473.607.220
Male	7.309.430.742	*Female*	4.565.764.662
Female	7.676.531.928	**Dependents**	**1.747.476.113**
		Widow	1.673.664.977
		Orphan	73.811.136
		Health Liabilities	**7.050.050.122**
General Total	**25.101.487.472**	**General Total**	**40.469.106.695**
Liabilities of the Foundation Funds		**-15.367.619.223 ₺**	

Obtained results are divided into according to the insured ratio given for each foundation funds as Table 3.40.

Table 3.40 Incumbent liability of the each foundation fund for first scale of the scenario IV

Names of Foundation Funds	**Liability (₺)**
Türkiye İş Bankası A.Ş.	3.995.580.998
Yapı ve Kredi Bankası A.Ş.	2.209.863.644
Akbank	2.020.841.928
Türkiye Vakıflar Bankası	1.616.673.542
Türkiye Garanti Bankası A.Ş.	1.593.622.113
T.C. Ziraat Bankası A.Ş. ve Türkiye Halk Bankası A.Ş.	969.696.773
Türkiye Halk Bankası A.Ş. (Pamukbank T.A.Ş.)	906.689.534
Türkiye Odalar Borsalar Birliği	780.675.057
Şekerbank	596.263.626
Fortis Bank A.Ş. ve Dış Bank	339.624.385
Anadolu Anonim Türk Sigorta Şirketi (Anadolu Sigorta)	98.352.763
Türkiye Sinai Kalkınma Bankası	64.544.001
Esbank Eskişehir Bankası	58.396.953
Mapfre Genel Sigorta	38.419.048
Milli Reasürans	33.808.762
Liberty Sigorta	27.661.715
İmar Bankası	16.904.381

3.4.4.2 Second Scale of the Scenario IV

The rate of increase to salary is found by adding the whole number (1) to the general consumer price index in the medium term program. For second scale of the Scenario IV (shown in the Table B.12), valuations of the total premium and liability are obtained according to present value calculations given in the Table C.1. In the end, incumbent liabilities of the foundation funds are determined by subtracting from the sum of premiums to the sum of liabilities in the Table 3.41.

Table 3.41 Actuarial valuations for second scale of the scenario IV

PREMIUMS		**LIABILITIES**	
Premiums Incoming from Actives		**Passives**	**25.817.430.515**
Disease	**10.115.524.802**	*Male*	12.228.793.986
Male	4.933.865.751	*Female*	13.588.636.529
Female	5.181.659.051	**Actives**	**11.569.792.777**
Others	**14.985.962.670**	*Male*	4.953.920.884
Male	7.309.430.742	*Female*	6.615.871.893
Female	7.676.531.928	**Dependents**	**1.965.694.756**
		Widow	1.888.915.072
		Orphan	76.779.684
		Health Liabilities	**8.008.414.344**
General Total	**25.101.487.472**	**General Total**	**47.361.332.392**
Liabilities of the Foundation Funds		**-22.259.844.920 ₺**	

Obtained results are divided into according to the insured ratio given for each foundation funds as Table 3.42.

Table 3.42 Incumbent liability of the each foundation fund for second scale of the scenario IV

Names of Foundation Funds	**Liability (₺)**
Türkiye İş Bankası A.Ş.	5.787.559.679
Yapı ve Kredi Bankası A.Ş.	3.200.965.699
Akbank	2.927.169.607
Türkiye Vakıflar Bankası	2.341.735.686
Türkiye Garanti Bankası A.Ş.	2.308.345.918
T.C. Ziraat Bankası A.Ş. ve Türkiye Halk Bankası A.Ş.	1.404.596.214
Türkiye Halk Bankası A.Ş. (Pamukbank T.A.Ş.)	1.313.330.850
Türkiye Odalar Borsalar Birliği	1.130.800.122
Şekerbank	863.681.983
Fortis Bank A.Ş. ve Dış Bank	491.942.573
Anadolu Anonim Türk Sigorta Şirketi (Anadolu Sigorta)	142.463.007
Türkiye Sinai Kalkınma Bankası	93.491.349
Esbank Eskişehir Bankası	84.587.411
Mapfre Genel Sigorta	55.649.612
Milli Reasürans	48.971.659
Liberty Sigorta	40.067.721
İmar Bankası	24.485.829

CHAPTER FOUR
CONCLUSIONS

Obtained results depending upon the scenarios may be organized as following:

- For Scenario I, we saw that when not have the rates of increase to premium and salary, the highest value of the liability will be for 3,00% technical interest rate. We understand that the liability of the foundation funds will increase when the technical interest rate decreases. Also, the liability of the foundation funds will increase for real (stochastic) interest rates according to the 9,80% technical interest rate.

- For Scenario II, we saw that when the rate of increase to salary was determined according to the updating coefficient and the technical interest rate was taken as 9,80%, the highest value of the liability will be for the rate of increase to premium that designated in accordance with inflation rate. Because, the rates for the increase in income taken basis to premium are bigger than inflation rates from year to year.

- For Scenario III, we saw that when the rate of increase to premium was determined according to the coefficient for the increase in income taken basis to premium and the real interest rate (depending on the annual inflation rate) was taken as stochastic according to 9,80% technical interest rate, the highest value of the liability will be for the rate of increase to salary that found by adding the whole number (1) to 30% of the development rate of gross domestic product with fixed prices in the medium term program.

- For Scenario IV, we saw that when the rate of increase to premium was determined according to the inflation rate and the technical interest rate is taken as 9,80%, the highest value of the liability will be for the rate of increase to salary is found by adding the whole number (1) to the general consumer price index in the medium term program.

In this study, both salary-premium rates and interest rates were thought stochastic. As a result of these, different results were obtained for all stations. Some recommendations will suggest for obtained results in Table 4.1.

Table 4.1 Liabilities of foundation funds according to scenarios

SCENARIOS	SCALES	LIABILITIES (₺)
Scenario I *P = 0,00%* *R = 0,00 %*	***Scale I*** *i = 9,80%*	-5.466.163.753
	Scale II *i = 7,35%*	-7.557.865.462
	Scale III *i = 5,85%*	-9.952.942.824
	Scale IV *i = 4,40%*	-13.856.787.627
	Scale V *i = 3,00%*	-20.295.104.675
	Scale VI *i = reel*	-8.968.183.843
Scenario II *R = updating coefficient* *i = 9,80%*	***Scale I*** *P = inflation rate*	-36.938.796.981
	Scale II *P = basis premium*	-34.360.009.007
Scenario III *P = basis premium* *i = reel*	***Scale I*** *R = 30% development*	-13.115.669.011
	Scale II *R = 30%+1 development*	-19.871.575.261
Scenario IV *P = inflation rate* *i = 9,80%*	***Scale I*** *R = consumer index*	-15.367.619.223
	Scale II *R = consumer index+1*	-22.259.844.920

For Scenario I, the technical interest rate was decreased with rates respectively 2,45%, 1,50%, 1,45%, 1,40% beginning from 9,80%. As a result, liabilities were increased while interest rates were decreasing. The liability of Scale VI associated with the real interest rate was remained in between Scale II and Scale III liabilities.

For Scenario II, premiums were increased with rate of approximately 1,15% in the Scale II. As result, liabilities were increased while premium rates were decreasing.

For Scenario III, salaries were increased with rate of 1,00% according to the development rate of gross domestic product with fixed prices in the medium term program for Scale II. Liabilities were increased while salaries rates were increasing.

For Scenario IV, salaries were increased with rate of 1,00% according to the general consumer price index in the medium term program in the Scale II. As a result, liabilities were increased while salaries rates were increasing.

REFERENCES

Allbusiness (life annuity certain). (n.d.). Retrieved June 18, 2012, from http://www.allbusiness.com/glossaries/life-annuity-certain/4948429-1.html#ixzz22wC2Se2M.

Alomaliye.com (SSI Notices (the earnings taken as basis to premiums for 2008-2012)). (n.d.). Retrieved August 3, 2012, from http://www.alomaliye.com/sskg_ana_sayfa.htm.

Beekman, J. A., & Fuelling, C. P. (1990). Interest and mortality randomness in some annuities. *Insurance: Mathematics and Economics, 9* (2-3), 185-196.

Beekman, J. A., & Fuelling, C. P. (1992). Extra randomness in certain annuity models. *Insurance: Mathematics and Economics, 10* (4), 275-287.

Bellhouse D. R., & Panjer H. H. (1981). Stochastic modeling of interest rates with applications to life contingencies part II. *Journal of Risk and Insurance, 48* (4), 628-637.

Bowers, N. L., Gerber, H. U., Hickman, J. C., Jones, D. A., & Nesbitt, C. J. (1997). *Actuarial mathematics* (2nd ed.). Schaumburg, Illinois: The Society of Actuaries.

Boyle, P. P. (1976). Rates of return as random variables. *American Risk and Insurance Association, 43* (4), 693-713.

Burnecki K., Marciniuk A., & Weron A. (2003). Annuities under random rates of interest—revisited. *Insurance: Mathematics and Economics, 32* (3), 457-460.

Cunningham, R. J., Herzog, T. N., & London, R. L. (2006). *Model for quantifying risk* (4th ed.). USA: Actex Publications Inc.

Dhaene, J. (1989). Stochastic interest rates and autoregressive integrated moving average processes. *Astin Bulletin*, *19* (2), 131-138.

Eurostat (HICP – inflation rate (the average inflation rate in the 2003-2008 EU Euro Zone)). (n.d.). Retrieved June 29, 2012, from http://epp.eurostat.ec.europa.eu/tgm/table.do?tab=table&language=en&pcode=tec00118&tableSelection=1&footnotes=yes&labeling=labels&plugin=1.

Finan, M. B. (2012). *A reading of the theory of life contingency models: A preparation for exam MLC/3L* (1th ed.). USA: Arkansas Tech Universtiy Press.

Gauger, M. A. (2006). *Actuarial models* (2nd ed.). USA: BPP Professional Education.

Giaccotto, C. (1986). Stochastic modelling of interest rates: actuarial vs. equilibrium. *The Journal of Risk and Insurance*, *53* (3), 435-453.

Hoedemakers, T., Darkiewicz, G., & Goovaerts, M. (2005). On the distribution of life annuities with stochastic interest rates. *Insurance: Mathematics and Economics* 1-35.

Huang, H.C., & Cairns, A. J. G. (2006). On the control of defined-benefit pension plans. *Insurance: Mathematics and Economics*, *38* (1), 113-131.

Kellison, S. G. (1991). *The theory of interest* (2nd ed.). USA: Irwin/McGraw-Hill Inc.

Marcea, E., & Gaillardetz, P. (1999). On life insurance reserves in a stochastic mortality and interest rates environment. *Insurance: Mathematics and Economics*, *25* (3), 261-280.

Panjer, H. H., & Bellhouse, D. R. (1980). Stochastic modelling of interest rates with applications to life contingencies. *The Journal of Risk and Insurance*, *47* (1), 91-110.

Perry, D., Stadje, W., & Yosef R. (2003) . Annuities with controlled random interest rates. *Insurance: Mathematics and Economics*, *32* (2), 245-253.

Pollard, A. H., & Pollard, J. H. (1969). A stochastic approach to actuarial functions. *Journal of the Institute of Actuaries*, *95*, 79-113.

Ruckman, C., & Francis, J. (2005). *Financial mathematics* (2nd ed.). USA: BPP Professional Education.

Satıcı, E. (2009). Faiz oranının rastlantı değişkeni olması durumunda tam hayat ve dönem sigortaları. *İstatistikçiler Dergisi*, *2*, 1-8.

Slud, E. V. (2001). *Actuarial mathematics and life-table statistics* (1th ed.) . USA: University of Maryland Press.

TUIK (The consumer price index (TUFE)). (n.d.). Retrieved October 10, 2012, from http://tuikapp.tuik.gov.tr/tufedagitimapp/.

Türkiye Cumhuriyeti Kalkınma Bakanlığı (The Medium Term Program 2013 – 2015; the fixed rate gross national product). (n.d.). Retrieved October 9, 2012, from http://www.dpt.gov.tr/PortalDesign/PortalControls/WebIcerikGosterim.aspx?Enc=83D5A6FF03C7B4FCFFBD2CFE9591A91E.

Türkiye Cumhuriyeti Kalkınma Bakanlığı (The Medium Term Program 2013 – 2015; the general consumer price index). (n.d.). Retrieved October 9, 2012, from http://www.dpt.gov.tr/PortalDesign/PortalControls/WebIcerikGosterim.aspx?Enc=83D5A6FF03C7B4FCFFBD2CFE9591A91E.

Vanneste, M., Goovaerts, M. J., De Schepper, A., & Dhaene, J. (1997). A straightforward analytical calculation of the distribution of an annuity certain with stochastic interest rate. *Insurance: Mathematics and Economics*, *20* (1), 35-41.

Wilkie, A. D. (1987). Stochastic investment models-theory and applications. *Insurance: Mathematics and Economics*, *6* (1), 65-83.

Zaks, A. (2001). Annuities under random rates of interest. *Insurance: Mathematics and Economics*, *28* (1), 1-11.

APPENDICES

APPENDIX A

CSO 1980 refers to "Commissioners Standard Ordinary", the life table published by the "Reinsurance Association of America" between 1938 and 1941 and updated in 1980. Both Male Mortality Table and Female Mortality Table are defined with commutation functions in which the compound interest rate is accepted by 9,80% as following. Calculations were made using EXCEL 2007 from 0 age to 99 age.

Table A.1 CSO 1980 mortality table for male with commutation functions

x	q_x	p_x	l_x	d_x	D_x	N_x	C_x	M_x
0	0,00418	0,99582	1.000.000	4.180	1.000.000	11.028.944	3.807	15.632
1	0,00107	0,99893	995.820	1.066	906.940	10.028.944	884	11.825
2	0,00099	0,99901	994.754	985	825.109	9.122.004	744	10.941
3	0,00098	0,99902	993.770	974	750.721	8.296.895	670	10.197
4	0,00095	0,99905	992.796	943	683.047	7.546.174	591	9.527
5	0,00090	0,99910	991.853	893	621.492	6.863.127	509	8.936
6	0,00086	0,99914	990.960	852	565.512	6.241.635	443	8.426
7	0,00080	0,99920	990.108	792	514.596	5.676.123	375	7.984
8	0,00076	0,99924	989.316	752	468.291	5.161.527	324	7.609
9	0,00074	0,99926	988.564	732	426.171	4.693.236	287	7.284
10	0,00073	0,99927	987.832	721	387.846	4.267.065	258	6.997
11	0,00077	0,99923	987.111	760	352.972	3.879.219	248	6.739
12	0,00085	0,99915	986.351	838	321.221	3.526.247	249	6.492
13	0,00099	0,99901	985.513	976	292.302	3.205.026	264	6.243
14	0,00115	0,99885	984.537	1.132	265.950	2.912.724	279	5.980
15	0,00133	0,99867	983.405	1.308	241.934	2.646.775	293	5.701
16	0,00151	0,99849	982.097	1.483	220.048	2.404.840	303	5.408
17	0,00167	0,99833	980.614	1.638	200.105	2.184.793	304	5.105
18	0,00178	0,99822	978.976	1.743	181.941	1.984.688	295	4.801
19	0,00186	0,99814	977.234	1.818	165.407	1.802.747	280	4.506
20	0,00190	0,99810	975.416	1.853	150.364	1.637.340	260	4.226
21	0,00191	0,99809	973.563	1.860	136.683	1.486.976	238	3.966
22	0,00189	0,99811	971.703	1.837	124.246	1.350.293	214	3.728
23	0,00186	0,99814	969.867	1.804	112.943	1.226.047	191	3.514
24	0,00182	0,99818	968.063	1.762	102.671	1.113.104	170	3.323

Table A.1 (Continued)

x	q_x	p_x	l_x	d_x	D_x	N_x	C_x	M_x
25	0,00177	0,99823	966.301	1.710	93.337	1.010.433	150	3.153
26	0,00173	0,99827	964.591	1.669	84.856	917.096	134	3.002
27	0,00171	0,99829	962.922	1.647	77.149	832.240	120	2.868
28	0,00170	0,99830	961.275	1.634	70.143	755.092	109	2.748
29	0,00171	0,99829	959.641	1.641	63.774	684.949	99	2.640
30	0,00173	0,99827	958.000	1.657	57.982	621.176	91	2.540
31	0,00178	0,99822	956.343	1.702	52.716	563.193	85	2.449
32	0,00183	0,99817	954.640	1.747	47.925	510.478	80	2.364
33	0,00191	0,99809	952.893	1.820	43.568	462.552	76	2.284
34	0,00200	0,99800	951.073	1.902	39.604	418.984	72	2.208
35	0,00211	0,99789	949.171	2.003	35.997	379.381	69	2.136
36	0,00224	0,99776	947.168	2.122	32.715	343.384	67	2.067
37	0,00240	0,99760	945.047	2.268	29.728	310.669	65	2.000
38	0,00258	0,99742	942.779	2.432	27.010	280.941	63	1.935
39	0,00279	0,99721	940.346	2.624	24.536	253.931	62	1.871
40	0,00302	0,99698	937.723	2.832	22.283	229.396	61	1.809
41	0,00329	0,99671	934.891	3.076	20.233	207.113	61	1.748
42	0,00356	0,99644	931.815	3.317	18.367	186.879	60	1.687
43	0,00387	0,99613	928.498	3.593	16.668	168.513	59	1.628
44	0,00419	0,99581	924.905	3.875	15.121	151.845	58	1.569
45	0,00455	0,99545	921.029	4.191	13.714	136.723	57	1.511
46	0,00492	0,99508	916.838	4.511	12.433	123.009	56	1.454
47	0,00532	0,99468	912.328	4.854	11.268	110.576	55	1.399
48	0,00574	0,99426	907.474	5.209	10.208	99.308	53	1.344
49	0,00621	0,99379	902.265	5.603	9.243	89.100	52	1.291
50	0,00671	0,99329	896.662	6.017	8.366	79.857	51	1.238
51	0,00730	0,99270	890.645	6.502	7.568	71.491	50	1.187
52	0,00796	0,99204	884.144	7.038	6.842	63.923	50	1.137
53	0,00871	0,99129	877.106	7.640	6.182	57.081	49	1.087
54	0,00956	0,99044	869.466	8.312	5.581	50.899	49	1.038
55	0,01047	0,98953	861.154	9.016	5.034	45.318	48	990
56	0,01146	0,98854	852.138	9.766	4.537	40.283	47	942
57	0,01249	0,98751	842.372	10.521	4.085	35.746	46	894
58	0,01359	0,98641	831.851	11.305	3.674	31.662	45	848
59	0,01477	0,98523	820.546	12.119	3.300	27.988	44	802
60	0,01608	0,98392	808.427	13.000	2.961	24.687	43	758
61	0,01754	0,98246	795.427	13.952	2.654	21.726	42	715
62	0,01919	0,98081	781.476	14.997	2.374	19.072	41	672

Table A.1 (Continued)

x	q_x	p_x	l_x	d_x	D_x	N_x	C_x	M_x
63	0,02106	0,97894	766.479	16.142	2.121	16.698	41	631
64	0,02314	0,97686	750.337	17.363	1.891	14.577	40	590
65	0,02542	0,97458	732.974	18.632	1.682	12.686	39	550
66	0,02785	0,97215	714.342	19.894	1.493	11.003	38	511
67	0,03044	0,96956	694.448	21.139	1.322	9.510	37	473
68	0,03319	0,96681	673.309	22.347	1.167	8.188	35	437
69	0,03617	0,96383	650.962	23.545	1.028	7.020	34	401
70	0,03951	0,96049	627.416	24.789	902	5.992	32	368
71	0,04330	0,95670	602.627	26.094	789	5.090	31	335
72	0,04765	0,95235	576.533	27.472	688	4.300	30	304
73	0,05264	0,94736	549.061	28.903	597	3.613	29	274
74	0,05819	0,94181	520.159	30.268	515	3.016	27	246
75	0,06419	0,93581	489.891	31.446	441	2.501	26	218
76	0,07053	0,92947	458.445	32.334	376	2.060	24	192
77	0,07712	0,92288	426.111	32.862	319	1.684	22	168
78	0,08390	0,91610	393.249	32.994	268	1.365	20	146
79	0,09105	0,90895	360.255	32.801	223	1.097	19	125
80	0,09884	0,90116	327.454	32.366	185	874	17	107
81	0,10748	0,89252	295.089	31.716	152	689	15	90
82	0,11725	0,88275	263.372	30.880	123	537	13	75
83	0,12826	0,87174	232.492	29.819	99	414	12	62
84	0,14025	0,85975	202.673	28.425	79	315	10	51
85	0,15295	0,84705	174.248	26.651	62	236	9	41
86	0,16609	0,83391	147.597	24.514	48	174	7	32
87	0,17955	0,82045	123.082	22.099	36	127	6	25
88	0,19327	0,80673	100.983	19.517	27	91	5	19
89	0,20729	0,79271	81.466	16.887	20	64	4	14
90	0,22177	0,77823	64.579	14.322	14	44	3	10
91	0,23698	0,76302	50.257	11.910	10	30	2	8
92	0,25345	0,74655	38.347	9.719	7	19	2	5
93	0,27211	0,72789	28.628	7.790	5	12	1	4
94	0,29590	0,70410	20.838	6.166	3	8	1	3
95	0,32996	0,67004	14.672	4.841	2	4	1	2
96	0,38455	0,61545	9.831	3.780	1	2	0	1
97	0,48020	0,51980	6.050	2.905	1	1	0	1
98	0,65798	0,34202	3.145	2.069	0	0	0	0
99	1,00000	0,00000	1.076	1.076	0	0	0	0

Table A.2 CSO 1980 mortality table for female with commutation functions

x	q_x	p_x	l_x	d_x	D_x	N_x	C_x	M_x
0	0,00289	0,99711	1.000.000	2.890	1.000.000	11.073.363	2.632	11.667
1	0,00087	0,99913	997.110	867	908.115	10.073.363	720	9.035
2	0,00081	0,99919	996.243	807	826.343	9.165.248	610	8.315
3	0,00079	0,99921	995.436	786	751.980	8.338.905	541	7.706
4	0,00077	0,99923	994.649	766	684.322	7.586.926	480	7.165
5	0,00076	0,99924	993.883	755	622.764	6.902.604	431	6.685
6	0,00073	0,99927	993.128	725	566.750	6.279.839	377	6.254
7	0,00072	0,99928	992.403	715	515.789	5.713.090	338	5.877
8	0,00070	0,99930	991.688	694	469.415	5.197.301	299	5.539
9	0,00069	0,99931	990.994	684	427.219	4.727.887	268	5.240
10	0,00068	0,99932	990.310	673	388.819	4.300.668	241	4.971
11	0,00069	0,99931	989.637	683	353.875	3.911.849	222	4.730
12	0,00072	0,99928	988.954	712	322.068	3.557.974	211	4.508
13	0,00075	0,99925	988.242	741	293.112	3.235.905	200	4.297
14	0,00080	0,99920	987.501	790	266.750	2.942.794	194	4.097
15	0,00085	0,99915	986.711	839	242.748	2.676.043	188	3.902
16	0,00090	0,99910	985.872	887	220.894	2.433.296	181	3.714
17	0,00095	0,99905	984.985	936	200.997	2.212.402	174	3.533
18	0,00098	0,99902	984.049	964	182.884	2.011.405	163	3.359
19	0,00102	0,99898	983.085	1.003	166.397	1.828.521	155	3.196
20	0,00105	0,99895	982.082	1.031	151.391	1.662.124	145	3.041
21	0,00107	0,99893	981.051	1.050	137.734	1.510.733	134	2.897
22	0,00109	0,99891	980.001	1.068	125.307	1.372.998	124	2.762
23	0,00111	0,99889	978.933	1.087	113.999	1.247.691	115	2.638
24	0,00114	0,99886	977.846	1.115	103.709	1.133.693	108	2.523
25	0,00116	0,99884	976.732	1.133	94.345	1.029.984	100	2.415
26	0,00119	0,99881	975.599	1.161	85.824	935.640	93	2.315
27	0,00122	0,99878	974.438	1.189	78.071	849.816	87	2.222
28	0,00126	0,99874	973.249	1.226	71.016	771.744	81	2.136
29	0,00130	0,99870	972.023	1.264	64.596	700.728	76	2.054
30	0,00135	0,99865	970.759	1.311	58.755	636.132	72	1.978
31	0,00140	0,99860	969.448	1.357	53.438	577.377	68	1.906
32	0,00145	0,99855	968.091	1.404	48.601	523.939	64	1.837
33	0,00150	0,99850	966.687	1.450	44.199	475.338	60	1.773
34	0,00158	0,99842	965.237	1.525	40.193	431.140	58	1.713
35	0,00165	0,99835	963.712	1.590	36.548	390.946	55	1.655
36	0,00176	0,99824	962.122	1.693	33.231	354.398	53	1.600
37	0,00189	0,99811	960.429	1.815	30.212	321.167	52	1.547

Table A.2 (Continued)

x	q_x	p_x	l_x	d_x	D_x	N_x	C_x	M_x
38	0,00204	0,99796	958.614	1.956	27.463	290.955	51	1.495
39	0,00222	0,99778	956.658	2.124	24.961	263.491	50	1.444
40	0,00242	0,99758	954.534	2.310	22.683	238.530	50	1.393
41	0,00264	0,99736	952.224	2.514	20.608	215.847	50	1.343
42	0,00287	0,99713	949.710	2.726	18.719	195.239	49	1.294
43	0,00309	0,99691	946.985	2.926	17.000	176.520	48	1.245
44	0,00332	0,99668	944.059	3.134	15.435	159.520	47	1.197
45	0,00356	0,99644	940.924	3.350	14.010	144.085	45	1.150
46	0,00380	0,99620	937.575	3.563	12.714	130.075	44	1.105
47	0,00405	0,99595	934.012	3.783	11.536	117.360	43	1.061
48	0,00433	0,99567	930.229	4.028	10.464	105.825	41	1.018
49	0,00463	0,99537	926.201	4.288	9.488	95.361	40	977
50	0,00496	0,99504	921.913	4.573	8.601	85.873	39	937
51	0,00531	0,99469	917.340	4.871	7.795	77.271	38	898
52	0,00570	0,99430	912.469	5.201	7.061	69.476	37	860
53	0,00615	0,99385	907.268	5.580	6.395	62.415	36	824
54	0,00661	0,99339	901.688	5.960	5.788	56.020	35	788
55	0,00709	0,99291	895.728	6.351	5.237	50.232	34	753
56	0,00757	0,99243	889.378	6.733	4.735	44.996	33	719
57	0,00803	0,99197	882.645	7.088	4.280	40.260	31	687
58	0,00847	0,99153	875.557	7.416	3.867	35.980	30	655
59	0,00894	0,99106	868.141	7.761	3.492	32.114	28	626
60	0,00947	0,99053	860.380	8.148	3.152	28.622	27	597
61	0,01013	0,98987	852.232	8.633	2.843	25.470	26	570
62	0,01096	0,98904	843.599	9.246	2.563	22.627	26	544
63	0,01202	0,98798	834.353	10.029	2.309	20.064	25	518
64	0,01325	0,98675	824.324	10.922	2.078	17.755	25	493
65	0,01459	0,98541	813.402	11.868	1.867	15.677	25	468
66	0,01600	0,98400	801.535	12.825	1.676	13.810	24	443
67	0,01743	0,98257	788.710	13.747	1.502	12.134	24	419
68	0,01884	0,98116	774.963	14.600	1.344	10.633	23	395
69	0,02036	0,97964	760.363	15.481	1.201	9.289	22	372
70	0,02211	0,97789	744.882	16.469	1.071	8.088	22	349
71	0,02423	0,97577	728.412	17.649	954	7.017	21	328
72	0,02687	0,97313	710.763	19.098	848	6.063	21	307
73	0,03011	0,96989	691.665	20.826	751	5.215	21	286
74	0,03393	0,96607	670.839	22.762	664	4.463	21	265
75	0,03824	0,96176	648.077	24.782	584	3.800	20	245

Table A.2 (Continued)

x	q_x	p_x	l_x	d_x	D_x	N_x	C_x	M_x
76	0,04297	0,95703	623.295	26.783	512	3.216	20	225
77	0,04804	0,95196	596.512	28.656	446	2.704	20	205
78	0,05345	0,94655	567.855	30.352	387	2.258	19	185
79	0,05935	0,94065	537.503	31.901	333	1.871	18	166
80	0,06599	0,93401	505.602	33.365	286	1.538	17	148
81	0,07360	0,92640	472.238	34.757	243	1.253	16	131
82	0,08240	0,91760	437.481	36.048	205	1.010	15	115
83	0,09253	0,90747	401.433	37.145	171	805	14	99
84	0,10381	0,89619	364.288	37.817	142	634	13	85
85	0,11610	0,88390	326.471	37.903	116	492	12	72
86	0,12929	0,87071	288.568	37.309	93	377	11	59
87	0,14332	0,85668	251.259	36.010	74	284	10	48
88	0,15818	0,84182	215.249	34.048	58	210	8	39
89	0,17394	0,82606	181.201	31.518	44	152	7	31
90	0,19075	0,80925	149.683	28.552	33	108	6	24
91	0,20887	0,79113	121.131	25.301	24	75	5	18
92	0,22881	0,77119	95.830	21.927	18	51	4	13
93	0,25151	0,74849	73.903	18.587	12	33	3	9
94	0,27931	0,72069	55.316	15.450	8	21	2	7
95	0,31732	0,68268	39.866	12.650	6	12	2	4
96	0,37574	0,62426	27.215	10.226	3	7	1	3
97	0,47497	0,52503	16.989	8.069	2	3	1	2
98	0,65585	0,34415	8.920	5.850	1	1	1	1
99	1,00000	0,00000	3.070	3.070	0	0	0	0

APPENDIX B

Four different scenarios will be proposed and used different scales for each scenario. Data will be given until 2013 year from 2112 year, because stochastic premium, salary and interest rates changes until 99 age from 0 ages. As a result, scales of each scenario are defined as following tables.

Table B.1 First scale of the Scenario I

Years	Premium Rate	Salary Rate	Interest Rate	Compound Premium	Compound Salary	Compound Interest
2013	0,00%	0,00%	9,80%	1,000	1,000	1,000
2014	0,00%	0,00%	9,80%	1,000	1,000	0,911
2015	0,00%	0,00%	9,80%	1,000	1,000	0,829
2016	0,00%	0,00%	9,80%	1,000	1,000	0,755
2017	0,00%	0,00%	9,80%	1,000	1,000	0,688
2018	0,00%	0,00%	9,80%	1,000	1,000	0,627
2019	0,00%	0,00%	9,80%	1,000	1,000	0,571
2020	0,00%	0,00%	9,80%	1,000	1,000	0,520
2021	0,00%	0,00%	9,80%	1,000	1,000	0,473
2022	0,00%	0,00%	9,80%	1,000	1,000	0,431
2023	0,00%	0,00%	9,80%	1,000	1,000	0,393
2024	0,00%	0,00%	9,80%	1,000	1,000	0,358
2025	0,00%	0,00%	9,80%	1,000	1,000	0,326
2026	0,00%	0,00%	9,80%	1,000	1,000	0,297
2027	0,00%	0,00%	9,80%	1,000	1,000	0,270
2028	0,00%	0,00%	9,80%	1,000	1,000	0,246
2029	0,00%	0,00%	9,80%	1,000	1,000	0,224
2030	0,00%	0,00%	9,80%	1,000	1,000	0,204
2031	0,00%	0,00%	9,80%	1,000	1,000	0,186
2032	0,00%	0,00%	9,80%	1,000	1,000	0,169
2033	0,00%	0,00%	9,80%	1,000	1,000	0,154
2034	0,00%	0,00%	9,80%	1,000	1,000	0,140
2035	0,00%	0,00%	9,80%	1,000	1,000	0,128
2036	0,00%	0,00%	9,80%	1,000	1,000	0,116
2037	0,00%	0,00%	9,80%	1,000	1,000	0,106
2038	0,00%	0,00%	9,80%	1,000	1,000	0,097
2039	0,00%	0,00%	9,80%	1,000	1,000	0,088
2040	0,00%	0,00%	9,80%	1,000	1,000	0,080
...	**...**	**...**	**...**	**...**	**...**	**...**
2110	0,00%	0,00%	9,80%	1,000	1,000	0,000
2111	0,00%	0,00%	9,80%	1,000	1,000	0,000
2112	0,00%	0,00%	9,80%	1,000	1,000	0,000

Table B.2 Second scale of the Scenario I

Years	Premium Rate	Salary Rate	Interest Rate	Compound Premium	Compound Salary	Compound Interest
2013	0,00%	0,00%	7,35%	1,000	1,000	1,000
2014	0,00%	0,00%	7,35%	1,000	1,000	0,932
2015	0,00%	0,00%	7,35%	1,000	1,000	0,868
2016	0,00%	0,00%	7,35%	1,000	1,000	0,808
2017	0,00%	0,00%	7,35%	1,000	1,000	0,753
2018	0,00%	0,00%	7,35%	1,000	1,000	0,701
2019	0,00%	0,00%	7,35%	1,000	1,000	0,653
2020	0,00%	0,00%	7,35%	1,000	1,000	0,609
2021	0,00%	0,00%	7,35%	1,000	1,000	0,567
2022	0,00%	0,00%	7,35%	1,000	1,000	0,528
2023	0,00%	0,00%	7,35%	1,000	1,000	0,492
2024	0,00%	0,00%	7,35%	1,000	1,000	0,458
2025	0,00%	0,00%	7,35%	1,000	1,000	0,427
2026	0,00%	0,00%	7,35%	1,000	1,000	0,398
2027	0,00%	0,00%	7,35%	1,000	1,000	0,370
2028	0,00%	0,00%	7,35%	1,000	1,000	0,345
2029	0,00%	0,00%	7,35%	1,000	1,000	0,321
2030	0,00%	0,00%	7,35%	1,000	1,000	0,299
2031	0,00%	0,00%	7,35%	1,000	1,000	0,279
2032	0,00%	0,00%	7,35%	1,000	1,000	0,260
2033	0,00%	0,00%	7,35%	1,000	1,000	0,242
2034	0,00%	0,00%	7,35%	1,000	1,000	0,226
2035	0,00%	0,00%	7,35%	1,000	1,000	0,210
2036	0,00%	0,00%	7,35%	1,000	1,000	0,196
2037	0,00%	0,00%	7,35%	1,000	1,000	0,182
2038	0,00%	0,00%	7,35%	1,000	1,000	0,170
2039	0,00%	0,00%	7,35%	1,000	1,000	0,158
2040	0,00%	0,00%	7,35%	1,000	1,000	0,147
...	...	...	...	...	...	...
2110	0,00%	0,00%	7,35%	1,000	1,000	0,001
2111	0,00%	0,00%	7,35%	1,000	1,000	0,001
2112	0,00%	0,00%	7,35%	1,000	1,000	0,001

Table B.3 Third scale of the Scenario I

Years	Premium Rate	Salary Rate	Interest Rate	Compound Premium	Compound Salary	Compound Interest
2013	0,00%	0,00%	5,85%	1,000	1,000	1,000
2014	0,00%	0,00%	5,85%	1,000	1,000	0,945
2015	0,00%	0,00%	5,85%	1,000	1,000	0,893
2016	0,00%	0,00%	5,85%	1,000	1,000	0,843
2017	0,00%	0,00%	5,85%	1,000	1,000	0,797
2018	0,00%	0,00%	5,85%	1,000	1,000	0,753
2019	0,00%	0,00%	5,85%	1,000	1,000	0,711
2020	0,00%	0,00%	5,85%	1,000	1,000	0,672
2021	0,00%	0,00%	5,85%	1,000	1,000	0,635
2022	0,00%	0,00%	5,85%	1,000	1,000	0,599
2023	0,00%	0,00%	5,85%	1,000	1,000	0,566
2024	0,00%	0,00%	5,85%	1,000	1,000	0,535
2025	0,00%	0,00%	5,85%	1,000	1,000	0,505
2026	0,00%	0,00%	5,85%	1,000	1,000	0,478
2027	0,00%	0,00%	5,85%	1,000	1,000	0,451
2028	0,00%	0,00%	5,85%	1,000	1,000	0,426
2029	0,00%	0,00%	5,85%	1,000	1,000	0,403
2030	0,00%	0,00%	5,85%	1,000	1,000	0,380
2031	0,00%	0,00%	5,85%	1,000	1,000	0,359
2032	0,00%	0,00%	5,85%	1,000	1,000	0,340
2033	0,00%	0,00%	5,85%	1,000	1,000	0,321
2034	0,00%	0,00%	5,85%	1,000	1,000	0,303
2035	0,00%	0,00%	5,85%	1,000	1,000	0,286
2036	0,00%	0,00%	5,85%	1,000	1,000	0,270
2037	0,00%	0,00%	5,85%	1,000	1,000	0,256
2038	0,00%	0,00%	5,85%	1,000	1,000	0,241
2039	0,00%	0,00%	5,85%	1,000	1,000	0,228
2040	0,00%	0,00%	5,85%	1,000	1,000	0,215
...	...	...	...	...	...	...
2110	0,00%	0,00%	5,85%	1,000	1,000	0,004
2111	0,00%	0,00%	5,85%	1,000	1,000	0,004
2112	0,00%	0,00%	5,85%	1,000	1,000	0,004

Table B.4 Fourth scale of the Scenario I

Years	Premium Rate	Salary Rate	Interest Rate	Compound Premium	Compound Salary	Compound Interest
2013	0,00%	0,00%	4,40%	1,000	1,000	1,000
2014	0,00%	0,00%	4,40%	1,000	1,000	0,958
2015	0,00%	0,00%	4,40%	1,000	1,000	0,917
2016	0,00%	0,00%	4,40%	1,000	1,000	0,879
2017	0,00%	0,00%	4,40%	1,000	1,000	0,842
2018	0,00%	0,00%	4,40%	1,000	1,000	0,806
2019	0,00%	0,00%	4,40%	1,000	1,000	0,772
2020	0,00%	0,00%	4,40%	1,000	1,000	0,740
2021	0,00%	0,00%	4,40%	1,000	1,000	0,709
2022	0,00%	0,00%	4,40%	1,000	1,000	0,679
2023	0,00%	0,00%	4,40%	1,000	1,000	0,650
2024	0,00%	0,00%	4,40%	1,000	1,000	0,623
2025	0,00%	0,00%	4,40%	1,000	1,000	0,596
2026	0,00%	0,00%	4,40%	1,000	1,000	0,571
2027	0,00%	0,00%	4,40%	1,000	1,000	0,547
2028	0,00%	0,00%	4,40%	1,000	1,000	0,524
2029	0,00%	0,00%	4,40%	1,000	1,000	0,502
2030	0,00%	0,00%	4,40%	1,000	1,000	0,481
2031	0,00%	0,00%	4,40%	1,000	1,000	0,461
2032	0,00%	0,00%	4,40%	1,000	1,000	0,441
2033	0,00%	0,00%	4,40%	1,000	1,000	0,423
2034	0,00%	0,00%	4,40%	1,000	1,000	0,405
2035	0,00%	0,00%	4,40%	1,000	1,000	0,388
2036	0,00%	0,00%	4,40%	1,000	1,000	0,371
2037	0,00%	0,00%	4,40%	1,000	1,000	0,356
2038	0,00%	0,00%	4,40%	1,000	1,000	0,341
2039	0,00%	0,00%	4,40%	1,000	1,000	0,326
2040	0,00%	0,00%	4,40%	1,000	1,000	0,313
...	...	...	...	...	...	...
2110	0,00%	0,00%	4,40%	1,000	1,000	0,015
2111	0,00%	0,00%	4,40%	1,000	1,000	0,015
2112	0,00%	0,00%	4,40%	1,000	1,000	0,014

Table B.5 Fifth scale of the Scenario I

Years	Premium Rate	Salary Rate	Interest Rate	Compound Premium	Compound Salary	Compound Interest
2013	0,00%	0,00%	3,00%	1,000	1,000	1,000
2014	0,00%	0,00%	3,00%	1,000	1,000	0,971
2015	0,00%	0,00%	3,00%	1,000	1,000	0,943
2016	0,00%	0,00%	3,00%	1,000	1,000	0,915
2017	0,00%	0,00%	3,00%	1,000	1,000	0,888
2018	0,00%	0,00%	3,00%	1,000	1,000	0,863
2019	0,00%	0,00%	3,00%	1,000	1,000	0,837
2020	0,00%	0,00%	3,00%	1,000	1,000	0,813
2021	0,00%	0,00%	3,00%	1,000	1,000	0,789
2022	0,00%	0,00%	3,00%	1,000	1,000	0,766
2023	0,00%	0,00%	3,00%	1,000	1,000	0,744
2024	0,00%	0,00%	3,00%	1,000	1,000	0,722
2025	0,00%	0,00%	3,00%	1,000	1,000	0,701
2026	0,00%	0,00%	3,00%	1,000	1,000	0,681
2027	0,00%	0,00%	3,00%	1,000	1,000	0,661
2028	0,00%	0,00%	3,00%	1,000	1,000	0,642
2029	0,00%	0,00%	3,00%	1,000	1,000	0,623
2030	0,00%	0,00%	3,00%	1,000	1,000	0,605
2031	0,00%	0,00%	3,00%	1,000	1,000	0,587
2032	0,00%	0,00%	3,00%	1,000	1,000	0,570
2033	0,00%	0,00%	3,00%	1,000	1,000	0,554
2034	0,00%	0,00%	3,00%	1,000	1,000	0,538
2035	0,00%	0,00%	3,00%	1,000	1,000	0,522
2036	0,00%	0,00%	3,00%	1,000	1,000	0,507
2037	0,00%	0,00%	3,00%	1,000	1,000	0,492
2038	0,00%	0,00%	3,00%	1,000	1,000	0,478
2039	0,00%	0,00%	3,00%	1,000	1,000	0,464
2040	0,00%	0,00%	3,00%	1,000	1,000	0,450
...	...	...	...	...	...	...
2110	0,00%	0,00%	3,00%	1,000	1,000	0,057
2111	0,00%	0,00%	3,00%	1,000	1,000	0,055
2112	0,00%	0,00%	3,00%	1,000	1,000	0,054

Table B.6 Sixth scale of the Scenario I

Years	Premium Rate	Salary Rate	Interest Rate	Compound Premium	Compound Salary	Compound Interest
2013	0,00%	0,00%	4,50%	1,000	1,000	1,000
2014	0,00%	0,00%	4,80%	1,000	1,000	0,954
2015	0,00%	0,00%	4,91%	1,000	1,000	0,910
2016	0,00%	0,00%	5,01%	1,000	1,000	0,866
2017	0,00%	0,00%	5,12%	1,000	1,000	0,824
2018	0,00%	0,00%	5,22%	1,000	1,000	0,783
2019	0,00%	0,00%	5,33%	1,000	1,000	0,743
2020	0,00%	0,00%	5,44%	1,000	1,000	0,705
2021	0,00%	0,00%	5,54%	1,000	1,000	0,668
2022	0,00%	0,00%	5,65%	1,000	1,000	0,632
2023	0,00%	0,00%	5,75%	1,000	1,000	0,598
2024	0,00%	0,00%	5,86%	1,000	1,000	0,565
2025	0,00%	0,00%	5,97%	1,000	1,000	0,533
2026	0,00%	0,00%	6,07%	1,000	1,000	0,503
2027	0,00%	0,00%	6,18%	1,000	1,000	0,473
2028	0,00%	0,00%	6,28%	1,000	1,000	0,445
2029	0,00%	0,00%	6,39%	1,000	1,000	0,419
2030	0,00%	0,00%	6,50%	1,000	1,000	0,393
2031	0,00%	0,00%	6,60%	1,000	1,000	0,369
2032	0,00%	0,00%	6,71%	1,000	1,000	0,346
2033	0,00%	0,00%	6,81%	1,000	1,000	0,323
2034	0,00%	0,00%	6,92%	1,000	1,000	0,303
2035	0,00%	0,00%	7,03%	1,000	1,000	0,283
2036	0,00%	0,00%	7,13%	1,000	1,000	0,264
2037	0,00%	0,00%	7,24%	1,000	1,000	0,246
2038	0,00%	0,00%	7,34%	1,000	1,000	0,229
2039	0,00%	0,00%	7,45%	1,000	1,000	0,213
2040	0,00%	0,00%	7,45%	1,000	1,000	0,199
...	...	...	...	...	...	...
2110	0,00%	0,00%	7,45%	1,000	1,000	0,001
2111	0,00%	0,00%	7,45%	1,000	1,000	0,001
2112	0,00%	0,00%	7,45%	1,000	1,000	0,001

Table B.7 First scale of the Scenario II

Years	Premium Rate	Salary Rate	Interest Rate	Compound Premium	Compound Salary	Compound Interest
2013	5,30%	7,50%	9,80%	1,000	1,000	1,000
2014	5,00%	7,50%	9,80%	1,050	1,075	0,911
2015	4,89%	7,50%	9,80%	1,101	1,156	0,829
2016	4,79%	7,50%	9,80%	1,154	1,242	0,755
2017	4,68%	7,50%	9,80%	1,208	1,335	0,688
2018	4,58%	7,50%	9,80%	1,263	1,436	0,627
2019	4,47%	7,50%	9,80%	1,320	1,543	0,571
2020	4,36%	7,50%	9,80%	1,377	1,659	0,520
2021	4,26%	7,50%	9,80%	1,436	1,783	0,473
2022	4,15%	7,50%	9,80%	1,496	1,917	0,431
2023	4,05%	7,50%	9,80%	1,556	2,061	0,393
2024	3,94%	7,50%	9,80%	1,618	2,216	0,358
2025	3,83%	7,50%	9,80%	1,680	2,382	0,326
2026	3,73%	7,50%	9,80%	1,742	2,560	0,297
2027	3,62%	7,50%	9,80%	1,805	2,752	0,270
2028	3,52%	7,50%	9,80%	1,869	2,959	0,246
2029	3,41%	7,50%	9,80%	1,933	3,181	0,224
2030	3,30%	7,50%	9,80%	1,996	3,419	0,204
2031	3,20%	7,50%	9,80%	2,060	3,676	0,186
2032	3,09%	7,50%	9,80%	2,124	3,951	0,169
2033	2,99%	7,50%	9,80%	2,187	4,248	0,154
2034	2,88%	7,50%	9,80%	2,250	4,566	0,140
2035	2,77%	7,50%	9,80%	2,313	4,909	0,128
2036	2,67%	7,50%	9,80%	2,375	5,277	0,116
2037	2,56%	7,50%	9,80%	2,435	5,673	0,106
2038	2,46%	7,50%	9,80%	2,495	6,098	0,097
2039	2,35%	7,50%	9,80%	2,554	6,556	0,088
2040	2,35%	7,50%	9,80%	2,614	7,047	0,080
...	**...**	**...**	**...**	**...**	**...**	**...**
2110	2,35%	7,50%	9,80%	13,287	1.113,323	0,000
2111	2,35%	7,50%	9,80%	13,599	1.196,822	0,000
2112	2,35%	7,50%	9,80%	13,919	1.286,583	0,000

Table B.8 Second scale of the Scenario II

Years	Premium Rate	Salary Rate	Interest Rate	Compound Premium	Compound Salary	Compound Interest
2013	6,56%	7,50%	9,80%	1,000	1,000	1,000
2014	6,19%	7,50%	9,80%	1,062	1,075	0,911
2015	6,06%	7,50%	9,80%	1,126	1,156	0,829
2016	5,93%	7,50%	9,80%	1,193	1,242	0,755
2017	5,80%	7,50%	9,80%	1,262	1,335	0,688
2018	5,67%	7,50%	9,80%	1,334	1,436	0,627
2019	5,53%	7,50%	9,80%	1,408	1,543	0,571
2020	5,40%	7,50%	9,80%	1,484	1,659	0,520
2021	5,27%	7,50%	9,80%	1,562	1,783	0,473
2022	5,14%	7,50%	9,80%	1,642	1,917	0,431
2023	5,01%	7,50%	9,80%	1,724	2,061	0,393
2024	4,88%	7,50%	9,80%	1,808	2,216	0,358
2025	4,75%	7,50%	9,80%	1,894	2,382	0,326
2026	4,62%	7,50%	9,80%	1,982	2,560	0,297
2027	4,48%	7,50%	9,80%	2,071	2,752	0,270
2028	4,35%	7,50%	9,80%	2,161	2,959	0,246
2029	4,22%	7,50%	9,80%	2,252	3,181	0,224
2030	4,09%	7,50%	9,80%	2,344	3,419	0,204
2031	3,96%	7,50%	9,80%	2,437	3,676	0,186
2032	3,83%	7,50%	9,80%	2,530	3,951	0,169
2033	3,70%	7,50%	9,80%	2,624	4,248	0,154
2034	3,57%	7,50%	9,80%	2,717	4,566	0,140
2035	3,43%	7,50%	9,80%	2,811	4,909	0,128
2036	3,30%	7,50%	9,80%	2,903	5,277	0,116
2037	3,17%	7,50%	9,80%	2,995	5,673	0,106
2038	3,04%	7,50%	9,80%	3,087	6,098	0,097
2039	2,91%	7,50%	9,80%	3,176	6,556	0,088
2040	2,91%	7,50%	9,80%	3,269	7,047	0,080
...	...	...	...	...	...	...
2110	2,91%	7,50%	9,80%	24,345	1.113,323	0,000
2111	2,91%	7,50%	9,80%	25,054	1.196,822	0,000
2112	2,91%	7,50%	9,80%	25,783	1.286,583	0,000

Table B.9 First scale of the Scenario III

Years	Premium Rate	Salary Rate	Interest Rate	Compound Premium	Compound Salary	Compound Interest
2013	1,24%	1,20%	4,50%	1,000	1,000	1,000
2014	1,24%	1,50%	4,80%	1,012	1,015	0,954
2015	1,24%	1,50%	4,91%	1,025	1,030	0,910
2016	1,24%	1,50%	5,01%	1,038	1,046	0,866
2017	1,24%	1,50%	5,12%	1,051	1,061	0,824
2018	1,24%	1,50%	5,22%	1,064	1,077	0,783
2019	1,24%	1,50%	5,33%	1,077	1,093	0,743
2020	1,24%	1,50%	5,44%	1,090	1,110	0,705
2021	1,24%	1,50%	5,54%	1,104	1,126	0,668
2022	1,24%	1,50%	5,65%	1,117	1,143	0,632
2023	1,24%	1,50%	5,75%	1,131	1,161	0,598
2024	1,24%	1,50%	5,86%	1,145	1,178	0,565
2025	1,24%	1,50%	5,97%	1,159	1,196	0,533
2026	1,24%	1,50%	6,07%	1,174	1,214	0,503
2027	1,24%	1,50%	6,18%	1,188	1,232	0,473
2028	1,24%	1,50%	6,28%	1,203	1,250	0,445
2029	1,24%	1,50%	6,39%	1,218	1,269	0,419
2030	1,24%	1,50%	6,50%	1,233	1,288	0,393
2031	1,24%	1,50%	6,60%	1,248	1,307	0,369
2032	1,24%	1,50%	6,71%	1,264	1,327	0,346
2033	1,24%	1,50%	6,81%	1,280	1,347	0,323
2034	1,24%	1,50%	6,92%	1,295	1,367	0,303
2035	1,24%	1,50%	7,03%	1,311	1,388	0,283
2036	1,24%	1,50%	7,13%	1,328	1,408	0,264
2037	1,24%	1,50%	7,24%	1,344	1,430	0,246
2038	1,24%	1,50%	7,34%	1,361	1,451	0,229
2039	1,24%	1,50%	7,45%	1,378	1,473	0,213
2040	1,24%	1,50%	7,45%	1,395	1,495	0,199
...	**...**	**...**	**...**	**...**	**...**	**...**
2110	1,24%	1,50%	7,45%	3,305	4,238	0,001
2111	1,24%	1,50%	7,45%	3,346	4,302	0,001
2112	1,24%	1,50%	7,45%	3,387	4,367	0,001

Table B.10 Second scale of the Scenario III

Years	Premium Rate	Salary Rate	Interest Rate	Compound Premium	Compound Salary	Compound Interest
2013	1,24%	2,20%	4,50%	1,000	1,000	1,000
2014	1,24%	2,50%	4,80%	1,012	1,025	0,954
2015	1,24%	2,50%	4,91%	1,025	1,051	0,910
2016	1,24%	2,50%	5,01%	1,038	1,077	0,866
2017	1,24%	2,50%	5,12%	1,051	1,104	0,824
2018	1,24%	2,50%	5,22%	1,064	1,131	0,783
2019	1,24%	2,50%	5,33%	1,077	1,160	0,743
2020	1,24%	2,50%	5,44%	1,090	1,189	0,705
2021	1,24%	2,50%	5,54%	1,104	1,218	0,668
2022	1,24%	2,50%	5,65%	1,117	1,249	0,632
2023	1,24%	2,50%	5,75%	1,131	1,280	0,598
2024	1,24%	2,50%	5,86%	1,145	1,312	0,565
2025	1,24%	2,50%	5,97%	1,159	1,345	0,533
2026	1,24%	2,50%	6,07%	1,174	1,379	0,503
2027	1,24%	2,50%	6,18%	1,188	1,413	0,473
2028	1,24%	2,50%	6,28%	1,203	1,448	0,445
2029	1,24%	2,50%	6,39%	1,218	1,485	0,419
2030	1,24%	2,50%	6,50%	1,233	1,522	0,393
2031	1,24%	2,50%	6,60%	1,248	1,560	0,369
2032	1,24%	2,50%	6,71%	1,264	1,599	0,346
2033	1,24%	2,50%	6,81%	1,280	1,639	0,323
2034	1,24%	2,50%	6,92%	1,295	1,680	0,303
2035	1,24%	2,50%	7,03%	1,311	1,722	0,283
2036	1,24%	2,50%	7,13%	1,328	1,765	0,264
2037	1,24%	2,50%	7,24%	1,344	1,809	0,246
2038	1,24%	2,50%	7,34%	1,361	1,854	0,229
2039	1,24%	2,50%	7,45%	1,378	1,900	0,213
2040	1,24%	2,50%	7,45%	1,395	1,948	0,199
...	**...**	**...**	**...**	**...**	**...**	**...**
2110	1,24%	2,50%	7,45%	3,305	10,970	0,001
2111	1,24%	2,50%	7,45%	3,346	11,244	0,001
2112	1,24%	2,50%	7,45%	3,387	11,526	0,001

Table B.11 First scale of the Scenario IV

Years	Premium Rate	Salary Rate	Interest Rate	Compound Premium	Compound Salary	Compound Interest
2013	5,30%	5,30%	9,80%	1,000	1,000	1,000
2014	5,00%	5,00%	9,80%	1,050	1,050	0,911
2015	4,89%	5,00%	9,80%	1,101	1,103	0,829
2016	4,79%	5,00%	9,80%	1,154	1,158	0,755
2017	4,68%	5,00%	9,80%	1,208	1,216	0,688
2018	4,58%	5,00%	9,80%	1,263	1,276	0,627
2019	4,47%	5,00%	9,80%	1,320	1,340	0,571
2020	4,36%	5,00%	9,80%	1,377	1,407	0,520
2021	4,26%	5,00%	9,80%	1,436	1,477	0,473
2022	4,15%	5,00%	9,80%	1,496	1,551	0,431
2023	4,05%	5,00%	9,80%	1,556	1,629	0,393
2024	3,94%	5,00%	9,80%	1,618	1,710	0,358
2025	3,83%	5,00%	9,80%	1,680	1,796	0,326
2026	3,73%	5,00%	9,80%	1,742	1,886	0,297
2027	3,62%	5,00%	9,80%	1,805	1,980	0,270
2028	3,52%	5,00%	9,80%	1,869	2,079	0,246
2029	3,41%	5,00%	9,80%	1,933	2,183	0,224
2030	3,30%	5,00%	9,80%	1,996	2,292	0,204
2031	3,20%	5,00%	9,80%	2,060	2,407	0,186
2032	3,09%	5,00%	9,80%	2,124	2,527	0,169
2033	2,99%	5,00%	9,80%	2,187	2,653	0,154
2034	2,88%	5,00%	9,80%	2,250	2,786	0,140
2035	2,77%	5,00%	9,80%	2,313	2,925	0,128
2036	2,67%	5,00%	9,80%	2,375	3,072	0,116
2037	2,56%	5,00%	9,80%	2,435	3,225	0,106
2038	2,46%	5,00%	9,80%	2,495	3,386	0,097
2039	2,35%	5,00%	9,80%	2,554	3,556	0,088
2040	2,35%	5,00%	9,80%	2,614	3,733	0,080
...	...	...	...	...	...	...
2110	2,35%	5,00%	9,80%	13,287	113,596	0,000
2111	2,35%	5,00%	9,80%	13,599	119,276	0,000
2112	2,35%	5,00%	9,80%	13,919	125,239	0,000

Table B.12 Second scale of the Scenario IV

Years	Premium Rate	Salary Rate	Interest Rate	Compound Premium	Compound Salary	Compound Interest
2013	5,30%	6,30%	9,80%	1,000	1,000	1,000
2014	5,00%	6,00%	9,80%	1,050	1,060	0,911
2015	4,89%	6,00%	9,80%	1,101	1,124	0,829
2016	4,79%	6,00%	9,80%	1,154	1,191	0,755
2017	4,68%	6,00%	9,80%	1,208	1,262	0,688
2018	4,58%	6,00%	9,80%	1,263	1,338	0,627
2019	4,47%	6,00%	9,80%	1,320	1,419	0,571
2020	4,36%	6,00%	9,80%	1,377	1,504	0,520
2021	4,26%	6,00%	9,80%	1,436	1,594	0,473
2022	4,15%	6,00%	9,80%	1,496	1,689	0,431
2023	4,05%	6,00%	9,80%	1,556	1,791	0,393
2024	3,94%	6,00%	9,80%	1,618	1,898	0,358
2025	3,83%	6,00%	9,80%	1,680	2,012	0,326
2026	3,73%	6,00%	9,80%	1,742	2,133	0,297
2027	3,62%	6,00%	9,80%	1,805	2,261	0,270
2028	3,52%	6,00%	9,80%	1,869	2,397	0,246
2029	3,41%	6,00%	9,80%	1,933	2,540	0,224
2030	3,30%	6,00%	9,80%	1,996	2,693	0,204
2031	3,20%	6,00%	9,80%	2,060	2,854	0,186
2032	3,09%	6,00%	9,80%	2,124	3,026	0,169
2033	2,99%	6,00%	9,80%	2,187	3,207	0,154
2034	2,88%	6,00%	9,80%	2,250	3,400	0,140
2035	2,77%	6,00%	9,80%	2,313	3,604	0,128
2036	2,67%	6,00%	9,80%	2,375	3,820	0,116
2037	2,56%	6,00%	9,80%	2,435	4,049	0,106
2038	2,46%	6,00%	9,80%	2,495	4,292	0,097
2039	2,35%	6,00%	9,80%	2,554	4,549	0,088
2040	2,35%	6,00%	9,80%	2,614	4,822	0,080
...	...	...	...	...	...	...
2110	2,35%	6,00%	9,80%	13,287	284,885	0,000
2111	2,35%	6,00%	9,80%	13,599	301,978	0,000
2112	2,35%	6,00%	9,80%	13,919	320,096	0,000

APPENDIX C

Results related to mean ages to “pension, deferred and premium coefficient” used in the present value calculations are given in the following Table C.1.

Table C.1 Results related to mean ages to “pension coefficient”, “deferred coefficient” and “premium coefficient” used in the present value calculations

PRESENT VALUE COEFFICIENT FOR CALCULATIONS										
Scenarios	**Scales**	**Pension Coefficient**				**Premium Coefficient**			**Deferred Coefficient**	
		Passive Expenses		**Dependents (Widow)**	**Health Expenses**	**Premiums Incoming From Actives**		**Dependents (Orphan)**	**Active Expenses**	
		Male Expense	***Female Expense***	***Mean Male-Female Expense***	***Mean Male-Female Expense***	***Male Income***	***Female Income***	***Mean Male-Female Expense***	***Male Expense***	***Female Expense***
		$\ddot{a}_{60}$	$\ddot{a}_{58}$	$\ddot{a}_{45}$	$\ddot{a}_{42}$	$\ddot{a}_{31:\overline{29}\rceil}$	$\ddot{a}_{29:\overline{29}\rceil}$	$\ddot{a}_{15:\overline{10}\rceil}$	$_{29\rvert}\ddot{a}_{31}$	$_{29\rvert}\ddot{a}_{29}$
Scenario I	**Scale I**	8,336	9,305	10,127	10,302	10,215	10,291	6,772	0,468	0,557
	Scale II	9,691	11,084	12,430	12,734	12,379	12,491	7,382	1,047	1,277
	Scale III	10,751	12,525	14,406	14,849	14,152	14,298	7,805	1,748	2,170
	Scale IV	12,003	14,279	16,940	17,599	16,322	16,512	8,257	2,911	3,690
	Scale V	13,486	16,425	20,235	21,230	18,987	19,235	8,737	4,838	6,278
	Scale VI	11,072	12,849	14,605	15,003	14,511	14,656	8,043	1,399	1,704
Scenario II	**Scale I**	14,567	18,032	22,837	24,138	14,815	14,966	9,057	6,665	8,791
	Scale II	14,567	18,032	22,837	24,138	16,331	16,510	9,057	6,665	8,791
Scenario III	**Scale I**	12,468	14,779	17,296	17,892	16,450	16,633	8,554	2,405	2,985
	Scale II	13,556	16,324	19,552	20,343	16,450	16,633	8,916	3,452	4,348
Scenario IV	**Scale I**	11,841	14,049	16,601	17,228	14,815	14,966	8,201	2,738	3,462
	Scale II	12,829	15,465	18,736	19,570	14,815	14,966	8,531	3,905	5,016