

DOKUZ EYLÜL UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

PROPER CLASSES GENERATED BY
SIMPLE MODULES

by
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August, 2013
İZMİR

**PROPER CLASSES GENERATED BY
SIMPLE MODULES**

**A Thesis Submitted to the
Graduate School of Natural and Applied Sciences of Dokuz Eylül University
In Partial Fulfillment of the Requirements for the Degree of Master of Science
in Mathematics**

**by
Zübeyir TÜRKOĞLU**

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M.Sc THESIS EXAMINATION RESULT FORM

We have read the thesis entitled “**PROPER CLASSES GENERATED BY SIMPLE MODULES**” completed by **ZÜBEYİR TÜRKOĞLU** under supervision of **ENGİN MERMUT** and we certify that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.



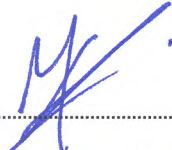
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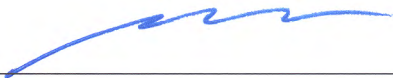


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PROPER CLASSES GENERATED BY SIMPLE MODULES

ABSTRACT

Let R be a ring with unity. A short exact sequence \mathbb{E} of left R -modules is said to be neat-exact if every simple left R -module is projective with respect to it. We call it \mathcal{P} -pure-exact if for every left primitive ideal P of R , the sequence obtained by taking the tensor product of \mathbb{E} from the left by R/P is exact. These give proper classes of short exact sequences of left R -modules. The characterization of N -domains, that is, the commutative domains such that neatness and \mathcal{P} -purity coincide, has been given recently by László Fuchs: they are the commutative domains where every maximal ideal is projective (and so necessarily finitely generated in the commutative domain case). We extend this sufficient condition to commutative rings using the Auslander-Bridger transpose of simple R -modules, that is, we prove that if R is a commutative ring where every maximal ideal is projective and finitely generated, then neatness and \mathcal{P} -purity coincide. Conversely, we show that the necessary condition holds for commutative rings with zero socle, that is, we show that if R is a commutative ring where neatness and \mathcal{P} -purity coincide and if R has zero socle, then every maximal ideal of the ring R is projective and finitely generated.

Keywords: Neat short exact sequence, \mathcal{P} -pure short exact sequence, simple R -module, the Auslander-Bridger transpose, left primitive ideal, proper class, N -domain, commutative rings with zero socle, maximal ideal, projective R -module, injective R -module, flat R -module.

BASİT MODÜLLER TARAFINDAN ÜRETİLEN ÖZSINIFLAR

ÖZ

R birimli bir halka olsun ve \mathbb{E} de sol R -modüllerin bir kısa tam dizisi olsun. Eğer her basit sol R -modül bu kısa tam diziyeye göre projektif ise \mathbb{E} 'ye düzenli-tam dizi denir. Eğer her sol primitif P ideali için \mathbb{E} kısa tam dizisinin solundan R/P ile tensör çarpımı alınarak elde edilen dizi bir kısa tam dizi oluyorsa, \mathbb{E} 'ye \mathcal{P} -saf-tam dizi diyoruz. Bunlar sol R -modüllerin kısa tam dizilerinin öz sınıflarını verir. N -tamlık bölgelerinin karakterizasyonu, yani, \mathcal{P} -saflık ve düzenliliğin denk olduğu değişmeli tamlık bölgelerinin karakterizasyonu László Fuchs tarafından yakın zamanda verilmiştir: Bunlar her maksimal ideali projektif olan (ve değişmeli tamlık bölgesinde olması nedeniyle zorunlu olarak sonlu üretilmiş olan) değişmeli tamlık bölgeleridir. Biz bu yeter koşulu basit R -modüllerin Auslander-Bridger transpozunu kullanarak değişmeli halkalara genelledik, yani, eğer R değişmeli halkası her maksimal ideali projektif olan bir halka ise, \mathcal{P} -saflık ve düzenliliğin denk olduğunu gösterdik. Tersine gerek koşulun kaidesi sıfır olan değişmeli halkalar için sağlandığını gösterdik, yani, eğer \mathcal{P} -saflık ve düzenliliğin denk olduğu değişmeli bir R halkasının kaidesi sıfır ise R halkasının her maksimal ideali projektif ve sonlu üretilendir.

Anahtar Sözcükler : Düzenli kısa tam dizi, \mathcal{P} -saf kısa tam dizi, basit R -modül, Auslander-Bridger transpozu, sol primitif ideal, öz sınıf, N -tamlık bölgesi, kaidesi sıfır olan değişmeli halka, maksimal ideal, projektif R -modül, injektif R -modül, düz R -modül.

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CHAPTER ONE

INTRODUCTION

Throughout this thesis, R denotes an arbitrary ring with unity and an R -module or module means a unital *left* R -module. For the undefined terms in module and ring theory or abelian group theory, see for example Bland (2011) and Fuchs (1970).

A subgroup A of an abelian group B is said to be a **neat subgroup** if $A \cap pB = pA$ for all prime numbers p ; see (Honda (1956) and Fuchs (1970, p.131)). This is a weakening of the condition for being a pure subgroup. For a subgroup A of an abelian group B , the following are equivalent:

(1) A is neat subgroup of B , that is, $A \cap pB = pA$ for all prime numbers p .

(2) The sequence

$$0 \longrightarrow (\mathbb{Z}/p\mathbb{Z}) \otimes A \xrightarrow{1_{\mathbb{Z}/p\mathbb{Z}} \otimes i_A} (\mathbb{Z}/p\mathbb{Z}) \otimes B$$

obtained by applying the functor $(\mathbb{Z}/p\mathbb{Z}) \otimes -$ to the inclusion monomorphism $i_A : A \longrightarrow B$ is exact for all prime numbers p .

(3) The sequence

$$\text{Hom}_{\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z}, B) \longrightarrow \text{Hom}_{\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z}, B/A) \longrightarrow 0$$

obtained by applying the functor $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z}, -)$ to the canonical epimorphism $B \longrightarrow B/A$ is exact for all prime numbers p .

(4) The sequence

$$\text{Hom}_{\mathbb{Z}}(B, \mathbb{Z}/p\mathbb{Z}) \longrightarrow \text{Hom}_{\mathbb{Z}}(A, \mathbb{Z}/p\mathbb{Z}) \longrightarrow 0$$

obtained by applying the functor $\text{Hom}_{\mathbb{Z}}(-, \mathbb{Z}/p\mathbb{Z})$ to the inclusion monomorphism $i_A : A \longrightarrow B$ is exact for all prime numbers p .

(5) A is a complement of a subgroup K of B , that is, $A \cap K = 0$ and A is maximal with respect to this property (equivalently, A is a closed subgroup of B , that is, A has no proper essential extension in B).

There are several reasonable ways to generalize this concept to modules and a natural question is when these are equivalent.

Following Stenström, we say that a submodule A of an R -module B is **neat** in B if for every simple R -module S , the sequence $\text{Hom}_R(S, B) \longrightarrow \text{Hom}_R(S, B/A) \longrightarrow 0$ obtained by applying the functor $\text{Hom}_R(S, -)$ to the canonical epimorphism $B \longrightarrow B/A$ is exact; see (Stenström (1967b, 9.6) and Stenström (1967a, §3)).

Another natural generalization of neat subgroups to modules is what is called ${}_R\mathcal{P}$ -**purity**, see for example Mermut et al. (2009). Denote by \mathcal{P} the collection of all *left primitive ideals* of the ring R ; recall that a (two-sided) ideal P of R is said to be a **left primitive ideal** if it is the annihilator of a simple R -module. We say that a submodule A of an R -module B is ${}_R\mathcal{P}$ -**pure** in B if $A \cap PB = PA$ for all $P \in \mathcal{P}$.

A natural question to ask is when neatness and ${}_R\mathcal{P}$ -purity coincide. Suppose that the ring R is *commutative*. Then \mathcal{P} is the collection of all *maximal ideals* of R . Recently László Fuchs has characterized the commutative domains for which these two notions coincide; see Fuchs (2012). Fuchs calls a ring R to be an **N-domain** if R is a commutative domain such that neatness and ${}_R\mathcal{P}$ -purity coincide. Unlike expected, Fuchs shows that N -domains are not just Dedekind domains; they are exactly the commutative domains whose all *maximal ideals* are projective (and so all maximal ideals are invertible ideals and finitely generated).

Motivated by Fuchs' result for commutative domains, we wish to understand first whether for some class of commutative rings larger than commutative domains, neatness and ${}_R\mathcal{P}$ -purity coincide if and only if all the maximal ideals of the ring are projective and finitely generated. We have first found the answer to be yes if every maximal ideal of the commutative ring R contains a regular element (that is an element that is not a zero-divisor) so that the maximal ideals of R that are invertible in the total quotient ring of R will be just projective ones as in the case of commutative domains (see for example Lam (1999, §2C)). We shall give some examples for these rings that are not domains. Indeed, we can even weaken this condition and just require that the

socle of the commutative ring R is zero, that is, R contains no simple submodules. A bit less to assume is that the commutative ring R contains no simple submodules that are not direct summands of R . See Sections 4.5 and 4.6.

It is known that a proper class of short exact sequences of modules that is projectively generated by a set of finitely presented modules is flatly generated by ‘the’ Auslander-Bridger transpose of these finitely presented modules. So to generalize the sufficiency of the Fuchs’ characterization of N -domains to all commutative rings, we shall show in Section 4.4 that for a commutative ring R , an Auslander-Bridger transpose of a finitely presented simple R -module S of projective dimension 1 is isomorphic to S . This enables us to prove that if R is a commutative ring such that every maximal ideal of R is finitely generated and projective, then neatness and ${}_R\mathcal{P}$ -purity coincide. For the definition of an Auslander-Bridger transpose of a finitely presented R -module, see Section 4.2; for the definition of finitely presented R -modules, see Section 4.1.

We use the language of proper classes of short exact sequences of R -modules to investigate the relations among these concepts by considering the corresponding class of short exact sequences. For the definition, equivalent conditions, terminology, and some properties of proper classes, see Chapter Two, and for furthermore information about the proper classes, see Stenström (1967b), Sklyarenko (1978), MacLane (1963, Ch. 12, §4), Mishina & Skornyakov (1976), Mermut (2004), Alizade & Mermut (2004, §3), Clark et al. (2006, §10) and Al-Takhman et al. (2006). We shall follow the terminology and notation for proper classes given as in Stenström (1967b) and Sklyarenko (1978). The reason for using proper classes is to formulate easily and explicitly some problems of interest for relative injectivity, projectivity, flatness and to use the present technique for them for further investigations of the relations between them along these lines.

Let’s explain the motivating observation in abelian groups in terms of proper classes of short exact sequences of abelian groups. For abelian groups (= \mathbb{Z} -modules), the simple \mathbb{Z} -modules, up to isomorphism, are just $\mathbb{Z}/p\mathbb{Z}$ where p runs through all prime

numbers.

The following are equivalent for a short exact sequence

$$\mathbb{E} : 0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

of abelian groups:

(1) $\text{Im}(f)$ is a neat subgroup of B , that is, $(\text{Im}(f)) \cap pB = p\text{Im}(f)$ for all prime numbers p .

(2) For all prime numbers p , the sequence $\mathbb{Z}/p\mathbb{Z} \otimes \mathbb{E}$, that is

$$0 \longrightarrow (\mathbb{Z}/p\mathbb{Z}) \otimes A \xrightarrow{1_{\mathbb{Z}/p\mathbb{Z}} \otimes f} (\mathbb{Z}/p\mathbb{Z}) \otimes B \xrightarrow{1_{\mathbb{Z}/p\mathbb{Z}} \otimes g} (\mathbb{Z}/p\mathbb{Z}) \otimes C \longrightarrow 0$$

is exact.

(3) For all prime numbers p , the sequence $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z}, \mathbb{E})$, that is, the sequence

$$0 \longrightarrow \text{Hom}_{\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z}, A) \longrightarrow \text{Hom}_{\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z}, B) \longrightarrow \text{Hom}_{\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z}, C) \longrightarrow 0$$

is exact; equivalently, the simple \mathbb{Z} -module $\mathbb{Z}/p\mathbb{Z}$ is projective with respect to \mathbb{E} , that is, for every \mathbb{Z} -module homomorphism $h : \mathbb{Z}/p\mathbb{Z} \longrightarrow C$ there exists a \mathbb{Z} -module homomorphism $\tilde{h} : \mathbb{Z}/p\mathbb{Z} \longrightarrow B$ such that $g \circ \tilde{h} = h$:

$$\begin{array}{ccccccc} \mathbb{E} : 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C \longrightarrow 0 \\ & & & & & \nearrow h & \uparrow \\ & & & & & \tilde{h} & \mathbb{Z}/p\mathbb{Z} \end{array}$$

(4) For all prime numbers p , the sequence $\text{Hom}_{\mathbb{Z}}(\mathbb{E}, \mathbb{Z}/p\mathbb{Z})$, that is,

$$0 \longrightarrow \text{Hom}_{\mathbb{Z}}(C, \mathbb{Z}/p\mathbb{Z}) \longrightarrow \text{Hom}_{\mathbb{Z}}(B, \mathbb{Z}/p\mathbb{Z}) \longrightarrow \text{Hom}_{\mathbb{Z}}(A, \mathbb{Z}/p\mathbb{Z}) \longrightarrow 0$$

is exact; equivalently, the simple \mathbb{Z} -module $\mathbb{Z}/p\mathbb{Z}$ is injective with respect to \mathbb{E} , that is, for every \mathbb{Z} -module homomorphism $h : A \longrightarrow \mathbb{Z}/p\mathbb{Z}$ there exists a \mathbb{Z} -module homomorphism $\tilde{h} : B \longrightarrow \mathbb{Z}/p\mathbb{Z}$ such that $\tilde{h} \circ f = h$:

$$\begin{array}{ccccccc} \mathbb{E} : 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C \longrightarrow 0 \\ & & \downarrow h & \nearrow \tilde{h} & & & \\ & & \mathbb{Z}/p\mathbb{Z} & & & & \end{array}$$

(5) $\text{Im}(f)$ is a complement of a subgroup K of B , that is, $\text{Im}(f) \cap K = 0$ and $\text{Im}(f)$ is maximal with respect to this property (equivalently $\text{Im}(f)$ is a closed subgroup of B which means that $\text{Im}(f)$ has no proper essential extension in B).

\mathbb{E} is said to be a neat exact-sequence if (1) holds. Denote the class of all neat-exact sequences of abelian groups by ${}_{\mathbb{Z}}\mathcal{Neat}$. Denote by ${}_{\mathbb{Z}}\mathcal{Compl}$ the class of all short exact sequences \mathbb{E} of abelian groups such that (5) holds. Denote by $\tau^{-1}(\{\mathbb{Z}/p\mathbb{Z} \mid p \text{ prime}\})$ the class of all short exact sequences \mathbb{E} of abelian groups such that (2) holds. Denote by $\pi^{-1}(\{\mathbb{Z}/p\mathbb{Z} \mid p \text{ prime}\})$ the class of all short exact sequences \mathbb{E} of abelian groups such that (3) holds. Denote by $\iota^{-1}(\{\mathbb{Z}/p\mathbb{Z} \mid p \text{ prime}\})$ the class of all short exact sequences \mathbb{E} of abelian groups such that (4) holds. The equivalence of (1), (2), (3), (4), (5) then means that for abelian groups, these five proper classes of short exact sequences of abelian groups are equal:

$$\begin{aligned} {}_{\mathbb{Z}}\mathcal{Compl} &= {}_{\mathbb{Z}}\mathcal{Neat} \\ &= \pi^{-1}(\{\mathbb{Z}/p\mathbb{Z} \mid p \text{ prime}\}) \\ &= \tau^{-1}(\{\mathbb{Z}/p\mathbb{Z} \mid p \text{ prime}\}) \\ &= \iota^{-1}(\{\mathbb{Z}/p\mathbb{Z} \mid p \text{ prime}\}) \end{aligned}$$

These results have motivated Rafail Alizade (the Ph.D. advisor of my advisor) to ask investigating similar results for modules over some classes of rings with its relations with complements and supplements in modules. My advisor Engin Mermut, following Stenström (1967a,b) and Generalov (1972), has dealt, in his Ph.D. Thesis, with proper classes related with complements (closed submodules) and supplements in R -modules using relative homological algebra via the known two dual proper classes ${}_{R}\mathcal{Compl}$ and ${}_{R}\mathcal{Suppl}$ of short exact sequences in $R\text{-Mod}$, and related other proper classes like ${}_{R}\mathcal{Neat}$ and ${}_{R}\mathcal{Co-Neat}$. The main related proper classes of short exact sequences of R -modules are the proper classes generated projectively, injectively or flatly by simple modules.

In this thesis, we mainly obtain results over commutative rings. Over a commutative ring R , we shall see in Chapter 3 some properties of these proper classes generated by simple modules. We deal with the proper classes ${}_{R}\mathcal{P-Pure} = \tau^{-1}(\{R/P \mid$

P is left primitive ideal of R) and ${}_R\mathcal{Neat} = \pi^{-1}(\{\text{all simple } R\text{-modules}\})$. Over a commutative ring R ,

$${}_R\mathcal{P}\text{-Pure} = \tau^{-1}(\{\text{all simple } R\text{-modules}\}) = \iota^{-1}(\{\text{all simple } R\text{-modules}\})$$

. The natural question is when these proper classes ${}_R\mathcal{P}\text{-Pure}$ and ${}_R\mathcal{Neat}$ are equal over a commutative ring, and this is the main problem for our thesis.

CHAPTER TWO

PROPER CLASSES

In the first section of this chapter, we will see the definitions of a pull back and a push out of a short exact sequence of R -modules. In the second section, we will see the definition, some equivalent conditions and some properties of proper classes of short exact sequences of R -modules. See Stenström (1967b) and Sklyarenko (1978). For the definition of proper classes of short exact sequences of objects in an abelian category, see MacLane (1963, §4 of Ch. 12). In the third section, we will give the definitions of relative projective, relative injective and relative flat R -modules with respect to a proper class. In the fourth section, we will give the definitions of classes of short exact sequences of R -modules that are projectively, injectively or flatly generated by a class of R -modules. For completeness, we shall also give detailed proofs to show that these classes are proper classes. In the last section, we will give the definitions of direct limit of a direct system and the proper class ${}_R\mathcal{P}ure$ (the smallest inductively closed proper class).

2.1 Pull Back and Push Out of a Short Exact Sequence

Let us start with definitions of pull back and push out.

Definition 2.1.1. Given a pair of R -module homomorphisms $\alpha : C' \rightarrow C$ and $\beta : B \rightarrow C$, an R -module P together with R -module homomorphisms $f : P \rightarrow C'$ and $g : P \rightarrow B$ is called a **pull back** of the pair α and β of R -module homomorphisms if the following conditions hold:

- (1) the diagram

$$\begin{array}{ccc} P & \xrightarrow{f} & C' \\ \downarrow g & & \downarrow \alpha \\ B & \xrightarrow{\beta} & C \end{array}$$

commutes.

- (2) If X is another R -module with a pair of R -module homomorphisms $h : X \rightarrow C'$,

$j : X \longrightarrow B$ such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{h} & C' \\ \downarrow j & & \downarrow \alpha \\ B & \xrightarrow{\beta} & C \end{array}$$

commutes, then there exists a unique R -module homomorphism $\theta : X \longrightarrow P$ such that $f \circ \theta = h$ and $g \circ \theta = j$, that is, the following diagram

$$\begin{array}{ccccc} X & & & & \\ & \searrow \theta & & & \\ & & P & \xrightarrow{g} & B \\ & & \downarrow f & & \downarrow \beta \\ & & C' & \xrightarrow{\alpha} & C \end{array}$$

commutes.

Shortly we say that (P, f, g) is a pull back of the pair α and β .

Definition 2.1.2. Given a pair of R -module homomorphisms $\alpha : A \longrightarrow B$, $\beta : A \longrightarrow A'$, an R -module P together with the R -module homomorphisms $f : B \longrightarrow P$ and $g : A' \longrightarrow P$ is called a push out of the pair of α and β if $f \circ \alpha = g \circ \beta$ and if X is another R -module with a pair of R -module homomorphisms $f' : B \longrightarrow X$, $g' : A' \longrightarrow X$ such that $f' \circ \alpha = g' \circ \beta$, then there exists a unique R -module homomorphism $\phi : P \longrightarrow X$ such that $\phi \circ f = f'$ and $\phi \circ g = g'$. In terms of diagrams, we say that P together with f, g is a **push out** if the diagram

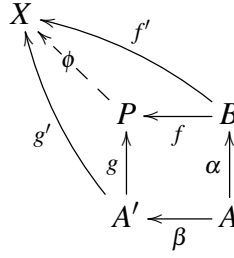
$$\begin{array}{ccc} A & \xrightarrow{\alpha} & B \\ \downarrow \beta & & \downarrow f \\ A' & \xrightarrow{g} & P \end{array}$$

commutes and if X is another R -module with R -module homomorphisms $f' : B \longrightarrow X$, $g' : A' \longrightarrow X$ such that the diagram

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & B \\ \downarrow \beta & & \downarrow f' \\ A' & \xrightarrow{g'} & X \end{array}$$

commutes, then there exists a unique R -module homomorphism $\phi : P \longrightarrow X$ such that

the diagram



commutes.

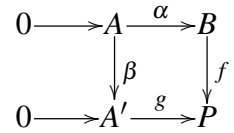
Shortly we say that (P, f, g) is a push out of the pair α and β .

We will give some properties of pull back and push out; for these properties, see for example Vermani (2003) and Maclane (1963, Ch. 3).

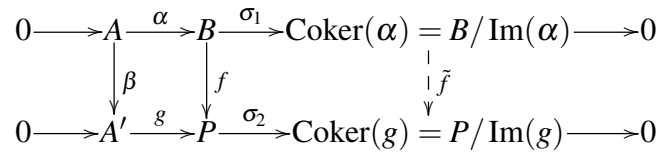
Every pair of R -module homomorphisms $\alpha : C' \rightarrow C$ and $\beta : B \rightarrow C$ has a pull back.

If (P, f, g) and (P', f', g') are two pull backs of the R -module homomorphisms $\alpha : C' \rightarrow C$ and $\beta : B \rightarrow C$, then there exist a unique R -module isomorphism $\theta : P \rightarrow P'$ such that $f' \circ \theta = f$ and $g' \circ \theta = g$. Dually these properties also holds for a push out of a pair of R -module homomorphisms.

We also know that if (P, f, g) is a push out of the pair β and α where α is a monomorphism, then g is a monomorphism, that is we can construct the following commutative diagram:



We can complete this diagram with the cokernels to the following commutative diagram, that is,



where σ_1 and σ_2 are canonical epimorphisms, by properties of push out, $\text{Coker}(\alpha)$ and $\text{Coker}(g)$ are isomorphic via $\tilde{f} : \text{Coker}(\alpha) = B/\text{Im}(\alpha) \rightarrow \text{Coker}(g) = P/\text{Im}(g)$

defined by $\tilde{f}(b + \text{Im}(\alpha)) = f(b) + \text{Im}(g)$ for all $b \in B$. So we can modify the above diagram to the following commutatively

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{\alpha} & B & \xrightarrow{\sigma'_1} & C & \longrightarrow & 0 \\ & & \downarrow \beta & & \downarrow f & & \parallel 1_C & & \\ 0 & \longrightarrow & A' & \xrightarrow{g} & P & \xrightarrow{\sigma'_2} & C & \longrightarrow & 0 \end{array}$$

where $C = \text{Coker}(\alpha)$, $\sigma'_1 = \sigma_1$, $\sigma'_2 = \tilde{f}^{-1} \circ \sigma_2$.

We know that if (P, f, g) is a pull back of the pair β and α where β is an epimorphism, then f is an epimorphis. So we can construct the following diagram:

$$\begin{array}{ccccc} P & \xrightarrow{f} & C' & \longrightarrow & 0 \\ \downarrow g & & \downarrow \alpha & & \\ B & \xrightarrow{\beta} & C & \longrightarrow & 0 \end{array}$$

we can complete this diagram with the kernels to the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \text{Ker}(f) & \xrightarrow{i_1} & P & \xrightarrow{f} & C' & \longrightarrow & 0 \\ & & \downarrow \tilde{g} & & \downarrow g & & \downarrow \alpha & & \\ 0 & \longrightarrow & \text{Ker}(\beta) & \xrightarrow{i_2} & B & \xrightarrow{\beta} & C & \longrightarrow & 0 \end{array}$$

where i_1 and i_2 are inclusion monomorphisms. By the properties of pull back, we know that $\text{Ker}(\beta)$ and $\text{Ker}(f)$ are isomorphic via $\tilde{g} : \text{Ker}(f) \longrightarrow \text{Ker}(\beta)$ defined by $\tilde{g}(x) = g(x)$ for all $x \in \text{Ker}(f)$. So we can modify the above diagram to the following commutatively:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{i'_1} & P & \xrightarrow{f} & C' & \longrightarrow & 0 \\ & & \parallel 1_A & & \downarrow g & & \downarrow \alpha & & \\ 0 & \longrightarrow & A & \xrightarrow{i'_2} & B & \xrightarrow{\beta} & C & \longrightarrow & 0 \end{array}$$

where $A = \text{Ker}(\beta)$, $i'_2 = i_2$ and $i'_1 = i_1 \circ \tilde{g}^{-1}$.

If we have a short exact sequence $\mathbb{E} : 0 \longrightarrow A \xrightarrow{\chi} B \xrightarrow{\sigma} C \longrightarrow 0$ of R -modules and R -module homomorphisms with a given R -module homomorphism $\alpha : A \longrightarrow A'$, then we have the following diagram

$$\mathbb{E} : \quad \begin{array}{ccccccc} 0 & \longrightarrow & A & \xrightarrow{\chi} & B & \xrightarrow{\sigma} & C & \longrightarrow & 0 \\ & & \downarrow \alpha & & & & & & \\ & & A' & & & & & & \end{array}$$

by the construction of push out we can obtain the following commutative diagram of R -module homomorphisms with exact rows

$$\mathbb{E} : \begin{array}{ccccccc} 0 & \longrightarrow & A & \xrightarrow{\chi} & B & \xrightarrow{\sigma} & C \longrightarrow 0 \\ & & \downarrow \alpha & & \downarrow \beta & & \parallel 1_C \\ 0 & \longrightarrow & A' & \xrightarrow{\chi'} & (A' \oplus B)/K & \xrightarrow{\sigma'} & C \longrightarrow 0 \end{array}$$

where $K = \{(\alpha(a), -\chi(a)) \mid a \in A\}$, $\beta(b) = (0, b) + K$ for every $b \in B$, $\chi'(a') = (a', 0) + K$ for all $a' \in A'$ and $\sigma'((a', b) + K) = \sigma(b)$ for all $(a', b) \in A' \oplus B$. We denote by $\alpha\mathbb{E}$ the short exact sequence in the second row of the above diagram, and we call $\alpha\mathbb{E}$ **the push out of the short exact sequence \mathbb{E} with the R -module homomorphism α** . If the following diagram

$$\begin{array}{ccccccc} \mathbb{E} : & 0 & \longrightarrow & A & \xrightarrow{\chi} & B & \xrightarrow{\sigma} & C \longrightarrow 0 \\ & & & \downarrow \alpha & & \downarrow & & \parallel 1_C \\ \mathbb{E}' : & 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C \longrightarrow 0 \end{array}$$

is commutative with exact rows, then $\mathbb{E}' \cong \alpha\mathbb{E}$. \mathbb{E}' is also called a **push out of the short exact sequence \mathbb{E}** .

If we have a short exact sequence $\mathbb{E} : 0 \longrightarrow A \xrightarrow{\chi} B \xrightarrow{\sigma} C \longrightarrow 0$ of R -modules and R -module homomorphisms with a given R -module homomorphism $\gamma : C' \longrightarrow C$, then we have the following diagram

$$\mathbb{E} : \begin{array}{ccccccc} & & & & & C' & \\ & & & & & \downarrow \gamma & \\ 0 & \longrightarrow & A & \xrightarrow{\chi} & B & \xrightarrow{\sigma} & C \longrightarrow 0 \end{array}$$

by the construction of pull back, we can obtain the following commutative diagram of R -module homomorphisms with exact rows

$$\mathbb{E} : \begin{array}{ccccccc} 0 & \longrightarrow & A & \xrightarrow{\chi'} & B' & \xrightarrow{\sigma'} & C' \longrightarrow 0 \\ & & \parallel 1_A & & \downarrow \beta & & \downarrow \gamma \\ 0 & \longrightarrow & A & \xrightarrow{\chi} & B & \xrightarrow{\sigma} & C \longrightarrow 0 \end{array}$$

where $B' = \{(b, c') \in B \oplus C' \mid \sigma(b) = \gamma(c')\}$, $\sigma'(b, c') = c'$ for every $(b, c') \in B'$, $\chi'(a) = (\chi(a), 0)$ for all $a \in A$ and $\beta(b, c') = b$ for all $(b, c') \in B'$. We denote by $\mathbb{E}\gamma$ the short exact sequence in the first row of the above diagram, and we call $\mathbb{E}\gamma$ **the pull back of**

the short exact sequence \mathbb{E} with the R -module homomorphism γ . If the following diagram

$$\begin{array}{ccccccccc} \mathbb{E}' : & 0 & \longrightarrow & A & \longrightarrow & B'' & \longrightarrow & C' & \longrightarrow & 0 \\ & & & \parallel & & \downarrow & & \downarrow & & \\ & & & I_A & & & & \gamma & & \\ \mathbb{E} : & 0 & \longrightarrow & A & \xrightarrow{\chi} & B & \xrightarrow{\sigma} & C & \longrightarrow & 0 \end{array}$$

is commutative with exact rows, then $\mathbb{E}' \cong \mathbb{E}\gamma$. \mathbb{E}' is also called a **pull back of the short exact sequence \mathbb{E}** .

2.2 Proper Classes

Let \mathcal{A} be a class of short exact sequences of R -modules. If

$$\mathbb{E} : 0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

belongs to \mathcal{A} , then we say f is an \mathcal{A} -**monomorphism**, g is an \mathcal{A} -**epimorphism**, both are called \mathcal{A} -**proper**, and \mathbb{E} is called an \mathcal{A} -**proper short exact sequence**. The class \mathcal{A} is said to be a **proper class of short exact sequences** if the following six conditions hold:

- (P1) If \mathbb{E} is in \mathcal{A} , then \mathcal{A} contains every short exact sequence isomorphic to \mathbb{E} .
- (P2) The class \mathcal{A} contains all splitting short exact sequences.
- (P3) The composite of two \mathcal{A} -monomorphisms is an \mathcal{A} -monomorphism, if this composition is defined.
- (P4) The composite of two \mathcal{A} -epimorphisms is an \mathcal{A} -epimorphism, if this composition is defined.
- (P5) If g and f are monomorphism and $g \circ f$ is an \mathcal{A} -monomorphism, then f is an \mathcal{A} -monomorphism.
- (P6) If g and f are epimorphism and $g \circ f$ is an \mathcal{A} -epimorphism, then g is an \mathcal{A} -epimorphism.

By (P2), $0 \longrightarrow 0 \longrightarrow A \xrightarrow{1_A} A \longrightarrow 0$ and $0 \longrightarrow A \xrightarrow{1_A} A \longrightarrow 0 \longrightarrow 0$ are proper short exact sequences, and hence $1_A : A \longrightarrow A$ and $0 : 0 \longrightarrow A$ are \mathcal{A} -monomorphisms and $1_A : A \longrightarrow A$ and $0 : A \longrightarrow 0$ are \mathcal{A} -epimorphisms.

Lemma 2.2.1. (*Montaño (2010, Ch.2) and Maclane (1963, §4 Ch. 12)*)

Proper classes of short exact sequences of R -modules are closed under pull backs and push outs.

Proof. Let \mathcal{A} be a proper class of short exact sequences of R -modules. Let \mathbb{E} be a short exact sequence in \mathcal{A} and $\mathbb{E}\gamma$ a pull back of \mathbb{E} (that is (D, σ', β) a pull back of σ, γ), that is, we have the following commutative diagram of R -module homomorphisms with exact rows:

$$\begin{array}{ccccccc} \mathbb{E} : & 0 & \longrightarrow & A & \xrightarrow{\chi} & B & \xrightarrow{\sigma} & C & \longrightarrow & 0 \\ \mathbb{E}\gamma : & 0 & \longrightarrow & A & \xrightarrow{\chi'} & D & \xrightarrow{\sigma'} & C' & \longrightarrow & 0 \\ & & & \parallel & & \downarrow & & \downarrow & & \\ & & & 1_A & & \beta & & \gamma & & \\ \mathbb{E} : & 0 & \longrightarrow & A & \xrightarrow{\chi} & B & \xrightarrow{\sigma} & C & \longrightarrow & 0 \end{array}$$

Here β need not be a monomorphism. If β is a monomorphism, then by (P5) we can obtain the proof easily: $\beta \circ \chi' = \chi$ and χ is an \mathcal{A} -monomorphism implies that χ' is an \mathcal{A} -monomorphism by (P5). But β need not be a monomorphism. By property of pull backs, $P = \{(b, c') \in B \oplus C' \mid \gamma(c') = \sigma(b)\}$ with the projection R -module homomorphisms onto B and C' is a pull back of the pair σ and γ , and by uniqueness of pull back up to isomorphism, there exists a unique isomorphism $\theta : D \longrightarrow P$. We can embed P into $B \oplus C'$ by the inclusion homomorphism $P \longrightarrow B \oplus C'$. Then for $v = i \circ \theta$ we obtain the following exact sequence

$$0 \longrightarrow D \xrightarrow{v} B \oplus C' \xrightarrow{\sigma \circ \pi_1 - \gamma \circ \pi_2} C$$

where $\pi_1 : B \oplus C' \longrightarrow B$ and $\pi_2 : B \oplus C' \longrightarrow C'$ are the projection epimorphisms. By the the pull back property we have $\pi_1 \circ v = \beta$ and $\pi_2 \circ v = \sigma'$, that is, we have the following commutative diagram:

$$\begin{array}{ccccccc} \mathbb{E}\gamma : & 0 & \longrightarrow & A & \xrightarrow{\chi'} & D & \xrightarrow{\sigma'} & C' & \longrightarrow & 0 \\ & & & \parallel & & \downarrow & & \downarrow & & \\ & & & 1_A & & \beta & & \gamma & & \\ & & & & & \searrow & & \nearrow & & \\ & & & & & B \oplus C' & & & & \\ & & & & & \swarrow & & \searrow & & \\ \mathbb{E} : & 0 & \longrightarrow & A & \xrightarrow{\chi} & B & \xrightarrow{\sigma} & C & \longrightarrow & 0 \end{array}$$

The R -module homomorphisms $v = i \circ \theta$ is clearly a monomorphism but need not be an \mathcal{A} -monomorphism. But $v \circ \chi' = 1_{B \oplus C'} \circ (v \circ \chi') = (i_1 \circ \pi_1 + i_2 \circ \pi_2) \circ v \circ \chi' = i_1 \circ \pi_1 \circ v \circ \chi' + i_2 \circ \pi_2 \circ v \circ \chi' = i_1 \circ \beta \circ \chi' + i_2 \circ \sigma' \circ \chi' = i_1 \circ \beta \circ \chi' + i_2 \circ 0 = i_1 \circ \chi$ where $i_1 : B \rightarrow B \oplus C'$ and $i_2 : C' \rightarrow B \oplus C'$ are inclusion monomorphisms. By (P2), i_1 is an \mathcal{A} -monomorphism and by (P3), $i_1 \circ \chi$ is also an \mathcal{A} -monomorphism. Hence $v \circ \chi' = i_1 \circ \chi$ is an \mathcal{A} -monomorphism and since v, χ' are monomorphisms we obtain by (P5) that χ' is an \mathcal{A} -monomorphism. Thus $\mathbb{E}\gamma$ is a proper short exact sequence. This shows proper classes are closed under pull backs.

Now let us show that proper classes are closed under push outs. By the construction of push out, we can construct the following commutative diagram of R -module homomorphisms with exact rows for a given short exact sequence $\mathbb{E} \in \mathcal{A}$ and for a R -module homomorphism $\alpha : A \rightarrow A'$:

$$\begin{array}{ccccccccc}
\mathbb{E} : 0 & \longrightarrow & A & \xrightarrow{\chi} & B & \xrightarrow{\sigma} & C & \longrightarrow & 0 \\
& & \downarrow \alpha & & \nearrow \pi_2 & & \downarrow \beta & & \parallel 1_C \\
& & A' \oplus B & & & & & & \\
& & \swarrow \pi_1 & & \searrow \pi & & & & \\
\alpha \mathbb{E} : 0 & \longrightarrow & A' & \xrightarrow{\chi'} & (A' \oplus B)/K & \xrightarrow{\sigma'} & C & \longrightarrow & 0
\end{array}$$

where $K = \{(\alpha(a), -\chi(a)) \mid a \in A\}$, $\beta(b) = (0, b) + K$ for every $b \in B$, $\chi'(a') = (a', 0) + K$ for all $a' \in A'$ and $\sigma'((a', b) + K) = \sigma(b)$ for all $(a', b) \in A' \oplus B$ and π is the canonical epimorphism. We have $\sigma' \circ \pi = \sigma \circ \pi_2$ by commutativity of the diagram. Since σ and π_2 are \mathcal{A} -epimorphisms, $\sigma' \circ \pi = \sigma \circ \pi_2$ is an \mathcal{A} -epimorphism by (P4). Then by (P6), σ' is also an \mathcal{A} -epimorphism. Hence we $\alpha \mathbb{E}$ is in the class \mathcal{A} . \square

For a class \mathcal{A} , the properties (PB), (PO), (P5') and (P6') are defined as follows:

(PB) \mathcal{A} is closed under pull backs.

(PO) \mathcal{A} is closed under push outs.

(P5') If $g \circ f$ is an \mathcal{A} -monomorphism, then f is an \mathcal{A} -monomorphism.

(P6') If $g \circ f$ is an \mathcal{A} -epimorphism, then g is an \mathcal{A} -epimorphism.

Theorem 2.2.2. (Montaño (2010, Ch.2), (Maclane, 1963, §4 Ch. 12) and Stenström (1967a)) Let \mathcal{A} is a class of short exact sequences of R -modules. We have then the following equivalences for the definition of proper classes of short exact sequences:

(1) \mathcal{A} is a proper class of short exact sequences, that is, \mathcal{A} satisfies (P1), (P2), (P3), (P4), (P5) and (P6) in the definition of proper class.

(2) \mathcal{A} satisfies properties (P1), (P2), (P3), (P4), (PB) and (PO).

(3) \mathcal{A} satisfies (P1), (P2), (P3), (P4), (P5') and (P6').

Proof. (1) \Rightarrow (2) Follows from Lemma 2.2.1.

(2) \Rightarrow (3) : We want to show that if $g \circ f$ is an \mathcal{A} -monomorphism, then f is also an \mathcal{A} -monomorphism. Suppose $g : D \rightarrow B$ and $f : A \rightarrow D$ and $g \circ f : A \rightarrow B$ are R -module homomorphisms. Construct the following commutative diagram:

$$\begin{array}{ccccccccc} \mathbb{E}' : 0 & \longrightarrow & A & \xrightarrow{f} & D & \xrightarrow{\sigma'} & D/\text{Im}(f) & \longrightarrow & 0 \\ & & \parallel & & \downarrow g & & \downarrow \beta' & & \\ \mathbb{E} : 0 & \longrightarrow & A & \xrightarrow{g \circ f} & B & \xrightarrow{\sigma} & B/\text{Im}(g \circ f) & \longrightarrow & 0 \end{array}$$

where σ' and σ are canonical epimorphisms and $\beta' : D/\text{Im}(f) \rightarrow B/\text{Im}(g \circ f)$ is defined by $\beta'(d + \text{Im}(f)) = \beta(d) + \text{Im}(g \circ f)$ for all $d + \text{Im}(f) \in D/\text{Im}(f)$ (where $d \in D$). Then \mathbb{E}' is the pull back of \mathbb{E} and by the property (PB), \mathbb{E}' is also in the class \mathcal{A} so f is an \mathcal{A} -monomorphism. Similarly if $g \circ f$ is an \mathcal{A} -epimorphism where $g : D \rightarrow C$ and $f : B \rightarrow D$ are R -module homomorphisms, we can construct the following commutative diagram:

$$\begin{array}{ccccccccc} \mathbb{E} : 0 & \longrightarrow & \text{Ker}(\sigma) & \longrightarrow & B & \xrightarrow{\sigma = g \circ f} & C & \longrightarrow & 0 \\ & & \downarrow f' & & \downarrow f & & \parallel & & \\ \mathbb{E}' : 0 & \longrightarrow & \text{Ker}(g) & \longrightarrow & D & \xrightarrow{g} & C & \longrightarrow & 0 \end{array}$$

where $f' : \text{Ker}(\sigma) \rightarrow \text{Ker}(g)$ is defined by $f'(x) = f(x)$ for all $x \in \text{Ker}(\sigma)$. This is a push out diagram, that is \mathbb{E}' is a push out of \mathbb{E} . Then by the property (PO), \mathbb{E}' is in the class of \mathcal{A} , and so g is an \mathcal{A} -epimorphism.

(3) \Rightarrow (1) is trivial since (P5') implies (P5) and (P6') implies (P6) clearly. \square

An important example for proper classes in abelian groups is $\mathbb{Z}\text{Pure}$: The proper class of all short exact sequences of abelian groups and abelian group homomorphisms such that $\text{Im}(f)$ is a pure subgroup of B , where a subgroup A of a group B is **pure** in B if $A \cap nB = nA$ for all integers n (see Fuchs (1970, §26-30) for the important notion

(2) The sequence

$$\mathrm{Hom}_R(M, \mathbb{E}) : 0 \longrightarrow \mathrm{Hom}_R(M, A) \xrightarrow{f_*} \mathrm{Hom}_R(M, B) \xrightarrow{g_*} \mathrm{Hom}_R(M, C) \longrightarrow 0$$

is exact for all $\mathbb{E} \in \mathcal{A}$.

The class of all \mathcal{A} -projective R -modules is denoted by $\pi(\mathcal{A})$:

$$\pi(\mathcal{A}) = \{M \mid M \text{ is an } R\text{-module and } \mathrm{Hom}_R(M, \mathbb{E}) \text{ is exact for all } \mathbb{E} \in \mathcal{A}\}$$

Dually, an R -module M is said to be \mathcal{A} -**injective** (or relative injective with respect to the proper class \mathcal{A}) if any of the following equivalent conditions holds:

(1) Every diagram

$$\mathbb{E} : \begin{array}{ccccccc} 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C \longrightarrow 0 \\ & & \alpha \downarrow & & \nearrow \tilde{\alpha} & & \\ & & M & & & & \end{array}$$

where \mathbb{E} is any short exact sequence of R -modules in \mathcal{A} and $\alpha : A \longrightarrow M$ is an R -module homomorphism can be embedded in a commutative diagram by choosing an R -module homomorphism $\tilde{\alpha} : B \longrightarrow M$; that is, for every R -module homomorphism $\alpha : A \longrightarrow M$, there exists an R -module homomorphism $\tilde{\alpha} : B \longrightarrow M$ such that $\tilde{\alpha} \circ f = \alpha$. In this case we say that the module M is projective relative to the short exact sequence \mathbb{E} .

(2) The sequence

$$\mathrm{Hom}_R(\mathbb{E}, M) : 0 \longrightarrow \mathrm{Hom}_R(C, M) \xrightarrow{g^*} \mathrm{Hom}_R(B, M) \xrightarrow{f^*} \mathrm{Hom}_R(A, M) \longrightarrow 0$$

is exact for all $\mathbb{E} \in \mathcal{A}$.

The class of all \mathcal{A} -injective modules is denoted by

$$\iota(\mathcal{A}) = \{M \mid M \text{ is a } R\text{-module and } \mathrm{Hom}_R(\mathbb{E}, M) \text{ is exact for all } \mathbb{E} \in \mathcal{A}\}$$

Also a *right* R -module M is said to be \mathcal{A} -**flat** (or relative flat with respect to the proper class \mathcal{A}) if the sequence

$$M \otimes_R \mathbb{E} : 0 \longrightarrow M \otimes_R A \xrightarrow{1_M \otimes f} M \otimes_R B \xrightarrow{1_M \otimes g} M \otimes_R C \longrightarrow 0$$

is exact for all $\mathbb{E} \in \mathcal{A}$. The class of all \mathcal{A} -flat R -modules is denoted by

$$\tau(\mathcal{A}) = \{M \mid M \text{ is a right } R\text{-module and } M \otimes_R \mathbb{E} \text{ is exact for all } \mathbb{E} \in \mathcal{A}\}$$

Note also the following elementary property that we shall use:

Proposition 2.3.1. *Let \mathcal{A} be a proper class of short exact sequences of R -modules. An R -module P is \mathcal{A} -projective if and only if every short exact sequence in \mathcal{A} which ends with P splits.*

Proof. Suppose P is an \mathcal{A} -projective R -module and take any short exact sequence \mathbb{E} in \mathcal{A} which ends with P , that is,

$$\mathbb{E} : 0 \longrightarrow A \longrightarrow B \xrightarrow{g} P \longrightarrow 0$$

Let $1_P : P \longrightarrow P$ be the identity R -module homomorphism. Since P is an \mathcal{A} -projective R -module, there exists an R -module homomorphism $h : P \longrightarrow B$ such that $g \circ h = 1_P$, that is, the following diagram

$$\begin{array}{ccccccc} & & & & P & & \\ & & & & \uparrow & & \\ & & & & h & & \\ & & & & \swarrow & & \\ & & & & P & & \\ & & & & \downarrow & & \\ & & & & 1_P & & \\ & & & & \downarrow & & \\ 0 & \longrightarrow & A & \longrightarrow & B & \xrightarrow{g} & P & \longrightarrow & 0 \end{array}$$

commutes. So \mathbb{E} is a splitting short exact sequence.

For the converse, suppose that every short exact sequence in \mathcal{A} which ends with P splits. We want to show that P is \mathcal{A} -projective, that is, $\text{Hom}_R(P, \mathbb{E})$ is exact for all \mathbb{E} in \mathcal{A} , or equivalently, if

$$\mathbb{E} : 0 \longrightarrow A \xrightarrow{\chi} B \xrightarrow{\sigma} C \longrightarrow 0$$

is any short exact sequence in \mathcal{A} and f is any R -homomorphism from P to C , then there exists an $\tilde{f} : P \longrightarrow B$ such that $\sigma \circ \tilde{f} = f$. We have a short exact sequence \mathbb{E} and

a R -module homomorphism f . By the construction of pull back, we can obtain the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} \mathbb{E}f : 0 & \longrightarrow & A & \xrightarrow{\chi'} & D & \xrightarrow{\sigma'} & P & \longrightarrow & 0 \\ & & \parallel & & \downarrow \beta & & \downarrow f & & \\ \mathbb{E} : 0 & \longrightarrow & A & \xrightarrow{\chi} & B & \xrightarrow{\sigma} & C & \longrightarrow & 0 \end{array}$$

By Lemma 2.2.1, we can say that the pull back $\mathbb{E}f$ of the short exact sequence \mathbb{E} in \mathcal{A} is also in \mathcal{A} . Then by hypothesis $\mathbb{E}f$ splits, that is, there exists a R -module homomorphism $j : P \longrightarrow D$ such that $\sigma' \circ j = 1_P$. If we choose $\tilde{f} = \beta \circ j$, then $\sigma \circ \tilde{f} = \sigma \circ (\beta \circ j) = (\sigma \circ \beta) \circ j = (f \circ \sigma') \circ j = f \circ (\sigma' \circ j) = f \circ 1_P = f$. Hence we obtain $\sigma \circ \tilde{f} = f$, so P is \mathcal{A} -projective. \square

2.4 Projectively, Injectively and Flatly Generated Proper Classes

For a given class \mathcal{M} of R -modules, denote by $\pi^{-1}(\mathcal{M})$ the class of all short exact sequences \mathbb{E} of R -modules and R -module homomorphisms such that $\text{Hom}_R(M, \mathbb{E})$ is exact for all $M \in \mathcal{M}$, that is,

$$\pi^{-1}(\mathcal{M}) = \{\mathbb{E} \in {}_R\mathcal{A}bs \mid \text{Hom}_R(M, \mathbb{E}) \text{ is exact for all } M \in \mathcal{M}\}.$$

$\pi^{-1}(\mathcal{M})$ is the largest proper class \mathcal{A} for which each $M \in \mathcal{M}$ is \mathcal{A} -projective. It is called the proper class **projectively generated** by \mathcal{M} . Of course, all these can be done for short exact sequences of *right* R -modules. If \mathcal{M} is a class of *right* R -modules, we denote by $\pi^{-1}(\mathcal{M})$ the class of all short exact sequences \mathbb{E} of *right* R -modules for which $\text{Hom}_R(M, \mathbb{E})$ is exact for every $M \in \mathcal{M}$.

For a given class \mathcal{M} of R -modules, denote by $\iota^{-1}(\mathcal{M})$ the class of all short exact sequences \mathbb{E} of R -modules and R -module homomorphisms such that $\text{Hom}_R(\mathbb{E}, M)$ is exact for all $M \in \mathcal{M}$, that is,

$$\iota^{-1}(\mathcal{M}) = \{\mathbb{E} \in {}_R\mathcal{A}bs \mid \text{Hom}_R(\mathbb{E}, M) \text{ is exact for all } M \in \mathcal{M}\}.$$

$\iota^{-1}(\mathcal{M})$ is the largest proper class \mathcal{A} for which each $M \in \mathcal{M}$ is \mathcal{A} -injective. It is called the proper class **injectively generated** by \mathcal{M} . Of course, all these can be done

for short exact sequences of *right* R -modules. If \mathcal{M} is a class of *right* R -modules, we denote by $\iota^{-1}(\mathcal{M})$ the class of all short exact sequences \mathbb{E} of *right* R -modules for which $\text{Hom}_R(\mathbb{E}, M)$ is exact for every $M \in \mathcal{M}$.

For a given class \mathcal{M} of *right* R -modules, denote by $\tau^{-1}(\mathcal{M})$ the class of all short exact sequences \mathbb{E} of R -modules and R -module homomorphisms such that $M \otimes_R \mathbb{E}$ is exact for all $M \in \mathcal{M}$:

$$\tau^{-1}(\mathcal{M}) = \{\mathbb{E} \in {}_R\mathcal{A}bs \mid M \otimes \mathbb{E} \text{ is exact for all } M \in \mathcal{M}\}.$$

$\tau^{-1}(\mathcal{M})$ is the largest proper class \mathcal{A} of R -modules for which each $M \in \mathcal{M}$ is \mathcal{A} -flat. It is called the proper class **flatly generated** by the class \mathcal{M} of *right* R -modules. Of course, all these can be done for short exact sequences of R -modules. If \mathcal{M} is a class of R -modules, we denote by $\pi^{-1}(\mathcal{M})$ the class of all short exact sequences \mathbb{E} of *right* R -modules such that $\mathbb{E} \otimes_R M$ is exact for every $M \in \mathcal{M}$.

For each R -module M , let $T(M, \cdot) : R\text{-Mod} \rightarrow \mathcal{A}b$ be an additive functor (covariant or contravariant), that is left or right exact. If \mathcal{M} is given class of R -modules, we denote by $t^{-1}(\mathcal{M})$ the class of short exact sequences \mathbb{E} of R -modules such that $T(M, \mathbb{E})$ is exact for all $M \in \mathcal{M}$. By the below theorem, it follows that the above three classes $\pi^{-1}(\mathcal{M})$, $\iota^{-1}(\mathcal{M})$ and $\tau^{-1}(\mathcal{M})$ are proper classes.

Theorem 2.4.1. (Sklyarenko (1978, Lemma 0.1)) $t^{-1}(\mathcal{M})$ is a proper class for every class \mathcal{M} of R -modules.

For completeness, in the following three propositions, we shall give the proof of the above theorem for special functors which are important for us: the functors $\text{Hom}_R(M, -)$, $\text{Hom}_R(-, M)$ and $M \otimes_R -$.

Proposition 2.4.2. (by Sklyarenko (1978, Lemma 0.1)) $\pi^{-1}(\mathcal{M})$ is a proper class for every class \mathcal{M} of R -modules.

Proof. We know that $\text{Hom}_R(M, -)$ is an additive covariant left exact functor for every R -module M .

Proof of (P1): Let $\mathbb{E} : 0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ be a short exact sequence in

$\pi^{-1}(\mathcal{M})$ and let \mathbb{E}' be an **isomorphic short exact sequence**, that is, the following diagram is commutative with vertical R -module homomorphisms being isomorphisms:

$$\begin{array}{ccccccccc} \mathbb{E} : 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \\ \mathbb{E}' : 0 & \longrightarrow & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \longrightarrow & 0 \end{array}$$

so α, β, γ are isomorphisms and $\beta \circ f = f' \circ \alpha, \gamma \circ g = g' \circ \beta$. Let $M \in \mathcal{M}$. Since $\mathbb{E} \in \pi^{-1}(\mathcal{M})$ and $\text{Hom}_R(M, -)$ is a covariant left exact functor, we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \text{Hom}_R(M, A) & \xrightarrow{\text{Hom}_R(M, f)} & \text{Hom}_R(M, B) & \xrightarrow{\text{Hom}_R(M, g)} & \text{Hom}_R(M, C) & \longrightarrow & 0 \\ & & \downarrow \text{Hom}_R(M, \alpha) & & \downarrow \text{Hom}_R(M, \beta) & & \downarrow \text{Hom}_R(M, \gamma) & & \\ 0 & \longrightarrow & \text{Hom}_R(M, A') & \xrightarrow{\text{Hom}_R(M, f')} & \text{Hom}_R(M, B') & \xrightarrow{\text{Hom}_R(M, g')} & \text{Hom}_R(M, C') & \longrightarrow & 0 \end{array}$$

It suffices to show that $\text{Hom}_R(M, g')$ is an epimorphism. We have $\text{Hom}_R(M, g') \circ \text{Hom}_R(M, \beta) = \text{Hom}_R(M, g' \circ \beta) = \text{Hom}_R(M, \gamma \circ g) = \text{Hom}_R(M, \gamma) \circ \text{Hom}_R(M, g)$. Since γ is an isomorphism, the homomorphism $\text{Hom}_R(M, \gamma)$ is also an isomorphism. By exactness of the first row, $\text{Hom}_R(M, g)$ is an epimorphism. Thus $\text{Hom}_R(M, \gamma) \circ \text{Hom}_R(M, g)$ is an epimorphism. So necessarily $\text{Hom}_R(M, g')$ is an epimorphism.

Proof of (P2): Let $\mathbb{E} : 0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ be a splitting short exact sequence. So there exists an R -module homomorphisms f' and g' such that $g \circ g' = 1_C$ and $f' \circ f = 1_A$. Let $M \in \mathcal{M}$. Since $\text{Hom}_R(M, -)$ is a covariant left exact functor, it suffices to show that $\text{Hom}_R(M, g)$ is an epimorphism. We have $\text{Hom}_R(M, g) \circ \text{Hom}_R(M, g') = \text{Hom}_R(M, g \circ g') = \text{Hom}_R(M, 1_C) = 1_{\text{Hom}_R(M, C)}$, and so $\text{Hom}_R(M, g)$ is an epimorphism. Hence $\mathbb{E} \in \pi^{-1}(\mathcal{M})$, that is, all splitting short exact sequences are in $\pi^{-1}(\mathcal{M})$.

Proof of (P3): Let $f : B \longrightarrow C$ and $g : C \longrightarrow D$ be $\pi^{-1}(\mathcal{M})$ -monomorphisms. We want to show that $g \circ f$ is also a $\pi^{-1}(\mathcal{M})$ -monomorphism. Consider the following short exact sequences

$$0 \longrightarrow B \xrightarrow{g \circ f} D \xrightarrow{\sigma} \text{Coker}(g \circ f) \longrightarrow 0$$

where the σ is canonical epimorphism. Let $M \in \mathcal{M}$. Since $\text{Hom}_R(M, -)$ is a covariant left exact functor, it suffices to show that $\text{Hom}_R(M, \sigma)$ is an epimorphism. We have

short exact sequences

$$0 \longrightarrow B \xrightarrow{f} C \xrightarrow{\sigma_1} \text{Coker}(f) \longrightarrow 0$$

and

$$0 \longrightarrow C \xrightarrow{g} D \xrightarrow{\sigma_2} \text{Coker}(g) \longrightarrow 0$$

where σ_1 and σ_2 are canonical epimorphisms. We then obtain long exact sequences by using the first long exact sequence for Ext:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Hom}_R(M, B) & \xrightarrow{f_*} & \text{Hom}_R(M, C) & \xrightarrow{\sigma_{1*}} & \text{Hom}_R(M, \text{Coker}(f)) \\ & & & & \delta & & \\ & \searrow & & & & & \\ & & \text{Ext}_R^1(M, B) & \xrightarrow{\text{Ext}_R^1(M, f)} & \text{Ext}_R^1(M, C) & \longrightarrow & \text{Ext}_R^1(M, \text{Coker}(f)) \longrightarrow \dots \end{array}$$

where ψ_* denotes $\text{Hom}_R(M, \psi)$ for every R -module homomorphism ψ in the above and below diagrams

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Hom}_R(M, C) & \xrightarrow{g_*} & \text{Hom}_R(M, D) & \xrightarrow{\sigma_{2*}} & \text{Hom}_R(M, \text{Coker}(g)) \\ & & & & \tilde{\delta} & & \\ & \searrow & & & & & \\ & & \text{Ext}_R^1(M, C) & \xrightarrow{\text{Ext}_R^1(M, g)} & \text{Ext}_R^1(M, D) & \longrightarrow & \text{Ext}_R^1(M, \text{Coker}(g)) \longrightarrow \dots \end{array}$$

By the assumption $\text{Hom}_R(M, \sigma_1) = \sigma_{1*}$ is an epimorphism and so $\text{Ker}(\delta) = \text{Hom}_R(M, \text{Coker}(f))$. Thus $\delta = 0$. From exactness $\text{Ker}(\text{Ext}_R^1(M, f)) = \text{Im}(\delta) = 0$, so $\text{Ext}_R^1(M, f)$ is a monomorphism. Similarly we obtain that easily $\text{Ext}_R^1(M, g)$ is a monomorphism. Since $\text{Ext}_R^1(M, -)$ is a functor, the homomorphism $\text{Ext}_R^1(M, g \circ f) = \text{Ext}_R^1(M, g) \circ \text{Ext}_R^1(M, f)$ is also a monomorphism. We then use the following long exact sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Hom}_R(M, B) & \xrightarrow{(g \circ f)_*} & \text{Hom}_R(M, D) & \xrightarrow{\sigma_*} & \text{Hom}_R(M, \text{Coker}(g \circ f)) \\ & & & & \delta' & & \\ & \searrow & & & & & \\ & & \text{Ext}_R^1(M, B) & \xrightarrow{h} & \text{Ext}_R^1(M, D) & \longrightarrow & \text{Ext}_R^1(M, \text{Coker}(g \circ f)) \longrightarrow \dots \end{array}$$

where $h = \text{Ext}_R^1(M, g \circ f)$. Since $\text{Ext}_R^1(M, g \circ f)$ is a monomorphism, $\text{Im}(\delta') = \text{Ker}(\text{Ext}_R^1(M, g \circ f)) = 0$. Thus $\delta' = 0$, and so $\text{Ker}(\delta') = \text{Hom}_R(M, \text{Coker}(g \circ f))$. Then $\text{Im}(\sigma_*) = \text{Ker}(\delta') = \text{Hom}_R(M, \text{Coker}(g \circ f))$. Hence $\sigma_* = \text{Hom}_R(M, \sigma)$ is an epimorphism. This shows that $g \circ f$ is also a $\pi^{-1}(\mathcal{M})$ -monomorphism.

Proof of (P4): Let $f : A \longrightarrow B$ and $g : B \longrightarrow C$ be $\pi^{-1}(\mathcal{M})$ -epimorphisms. We shall

show that $g \circ f : A \longrightarrow C$ is also a $\pi^{-1}(\mathcal{M})$ -epimorphism. Consider the following short exact sequence where $\text{Ker}(g \circ f) \longrightarrow A$ is the inclusion homomorphism:

$$0 \longrightarrow \text{Ker}(g \circ f) \longrightarrow A \xrightarrow{g \circ f} C \longrightarrow 0$$

Let $M \in \mathcal{M}$. Since $\text{Hom}_R(M, -)$ is a covariant left exact functor, it suffices to show that $\text{Hom}_R(M, g \circ f)$ is an epimorphism. We have $\text{Hom}_R(M, g \circ f) = \text{Hom}_R(M, g) \circ \text{Hom}_R(M, f)$. By the hypothesis $\text{Hom}_R(M, f)$ and $\text{Hom}_R(M, g)$ are epimorphisms since f and g are $\pi^{-1}(\mathcal{M})$ -epimorphisms. So their composition is also an epimorphism. This $g \circ f$ is a $\pi^{-1}(\mathcal{M})$ -epimorphism.

Proof of (P5): Let $\alpha : A \longrightarrow B$ and $\beta : B \longrightarrow C$ be monomorphisms. Suppose that $\beta \circ \alpha$ is a $\pi^{-1}(\mathcal{M})$ -monomorphism. We can construct the following commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \xrightarrow{\alpha} & B & \xrightarrow{\sigma_1} & \text{Coker}(\alpha) \longrightarrow 0 \\ & & \parallel & & \downarrow \beta & & \downarrow \tilde{\beta} \\ 0 & \longrightarrow & A & \xrightarrow{\beta \circ \alpha} & C & \xrightarrow{\sigma_2} & \text{Coker}(\beta \circ \alpha) \longrightarrow 0 \end{array}$$

where σ_1 and σ_2 are canonical epimorphisms and $\tilde{\beta}$ is the R -module homomorphism of the R -module homomorphism induced by β : $\tilde{\beta}(b + \text{Im}(\alpha)) = \beta(b) + \text{Im}(\beta \circ \alpha)$ for all $b \in B$. It is easily checked $\tilde{\beta}$ is a monomorphism since β is a monomorphism. By the properties of pull backs, (B, β, σ_1) is a pull back of $\tilde{\beta}$ and σ_2 . Let $M \in \mathcal{M}$. If we apply the covariant left exact functor $\text{Hom}_R(M, -)$, we obtain the following commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Hom}_R(M, A) & \xrightarrow{\text{Hom}_R(M, \alpha)} & \text{Hom}_R(M, B) & \xrightarrow{\text{Hom}_R(M, \sigma_1)} & \text{Hom}_R(M, \text{Coker}(\alpha)) \\ & & \parallel & & \downarrow \text{Hom}_R(M, \beta) & & \downarrow \text{Hom}_R(M, \tilde{\beta}) \\ 0 & \longrightarrow & \text{Hom}_R(M, A) & \xrightarrow{\text{Hom}_R(M, \beta \circ \alpha)} & \text{Hom}_R(M, C) & \xrightarrow{\text{Hom}_R(M, \sigma_2)} & \text{Hom}_R(M, \text{Coker}(\beta \circ \alpha)) \longrightarrow 0 \end{array}$$

The second row is exact because $\beta \circ \alpha$ is a $\pi^{-1}(\mathcal{M})$ -monomorphism. Since we want to show that α is a $\pi^{-1}(\mathcal{M})$ -monomorphism, so it suffices to show that the map $\text{Hom}_R(M, \sigma_1)$ is an epimorphism which means that if we have an R -module homomorphism $f : M \longrightarrow \text{Coker}(\alpha)$, then there exists a R -module $\tilde{f} : M \longrightarrow B$ such that $\sigma_1 \circ \tilde{f} = f$. Since the bottom row is exact, the R -module homomorphism $\text{Hom}_R(M, \sigma_2)$ is an epimorphism. So for $\tilde{\beta} \circ f : M \longrightarrow \text{Coker}(\beta \circ \alpha)$, there exists a R -module homomorphism $g : M \longrightarrow C$ such that $\sigma_2 \circ g = \tilde{\beta} \circ f$. Since (B, β, σ_1) is a

pull back of $\tilde{\beta}$ and σ_2 and $\tilde{\beta} \circ f = \sigma_2 \circ g$, by the definition of pull back there exists a unique R -module homomorphism $\theta : M \rightarrow B$ such that $\beta \circ \theta = g$ and $\sigma_1 \circ \theta = f$, that is, we have the following commutative diagram:

$$\begin{array}{ccc}
 M & \xrightarrow{f} & \text{Coker}(\alpha) \\
 \searrow \theta & \swarrow & \downarrow \tilde{\beta} \\
 B & \xrightarrow{\sigma_1} & \text{Coker}(\alpha) \\
 \downarrow \beta & & \downarrow \tilde{\beta} \\
 C & \xrightarrow{\sigma_2} & \text{Coker}(\beta \circ \alpha) \\
 \uparrow g & & \\
 M & &
 \end{array}$$

So for $\tilde{f} = \theta$, we have $\sigma_1 \circ \tilde{f} = f$ and this ends the proof of (P5).

Proof of (P6): Let $f : A \rightarrow B$ and $g : B \rightarrow C$ be epimorphisms. Suppose that $g \circ f$ is a $\pi^{-1}(\mathcal{M})$ -epimorphism. We can then construct the following commutative diagram:

$$\begin{array}{ccccccc}
 & & & & 0 & & \\
 & & & & \downarrow & & \\
 & & & & \text{Ker}(g) & & \\
 & & & & \downarrow & & \\
 A & \xrightarrow{f} & B & \longrightarrow & 0 & & \\
 \parallel & & \downarrow g & & & & \\
 A & \xrightarrow{g \circ f} & C & \longrightarrow & 0 & & \\
 & & \downarrow & & & & \\
 & & 0 & & & &
 \end{array}$$

where the R -module homomorphism $\text{Ker}(g) \rightarrow B$ is the inclusion homomorphism. We need to show that the short exact sequence in the last column is in $\pi^{-1}(\mathcal{M})$. Let $M \in \mathcal{M}$. By applying the covariant left exact functor $\text{Hom}_R(M, -)$, we obtain the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccc}
 & & & & 0 & & \\
 & & & & \downarrow & & \\
 & & & & \text{Hom}_R(M, \text{Ker}(g)) & & \\
 & & & & \downarrow & & \\
 \text{Hom}_R(M, A) & \xrightarrow{\text{Hom}_R(M, f)} & \text{Hom}_R(M, B) & & \text{Hom}_R(M, B) & & \\
 \parallel & & \downarrow \text{Hom}_R(M, g) & & \downarrow \text{Hom}_R(M, g) & & \\
 \text{Hom}_R(M, A) & \xrightarrow{\text{Hom}_R(M, g \circ f)} & \text{Hom}_R(M, C) & \longrightarrow & 0 & &
 \end{array}$$

The last row is exact because $g \circ f$ is a $\pi^{-1}(\mathcal{M})$ -epimorphism. It suffices to show that $\text{Hom}_R(M, g)$ is an epimorphism. Take any $x \in \text{Hom}_R(M, C)$. Since

$\text{Hom}_R(M, g \circ f)$ is an epimorphism, there exists an element $a \in \text{Hom}_R(M, A)$ such that $\text{Hom}_R(M, g \circ f)(a) = x$. So $\text{Hom}_R(M, g)(\text{Hom}_R(M, f)(a)) = \text{Hom}_R(M, g \circ f)(a) = x$. Let $y = \text{Hom}_R(M, f)(a) \in \text{Hom}_R(M, B)$. Then $\text{Hom}_R(M, g)(y) = x$. So $\text{Hom}_R(M, g)$ is an epimorphism. This ends the proof of (P6). \square

Proposition 2.4.3. (by Sklyarenko (1978, Lemma 0.1)) $\iota^{-1}(\mathcal{M})$ is a proper class for every class \mathcal{M} of R -modules.

Proof. We know that $\text{Hom}_R(-, M)$ is an additive contravariant left exact functor for every R -module M .

Proof of (P1): Let $\mathbb{E} : 0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ be a short exact sequence in $\iota^{-1}(\mathcal{M})$. Let \mathbb{E}' be an isomorphic short exact sequence to the short exact sequence \mathbb{E} , that is,

$$\begin{array}{ccccccccc} \mathbb{E} : 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \\ \mathbb{E}' : 0 & \longrightarrow & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \longrightarrow & 0 \end{array}$$

where α, β and γ are R -module isomorphisms and $\beta \circ f = f' \circ \alpha, \gamma \circ g = g' \circ \beta$. Since $\mathbb{E} \in \iota^{-1}(\mathcal{M})$ and $\text{Hom}_R(-, M)$ is a contravariant left exact functor, we obtain the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \text{Hom}_R(C', M) & \xrightarrow{\text{Hom}_R(g', M)} & \text{Hom}_R(B', M) & \xrightarrow{\text{Hom}_R(f', M)} & \text{Hom}_R(A', M) & & \\ & & \downarrow \text{Hom}_R(\gamma, M) & & \downarrow \text{Hom}_R(\beta, M) & & \downarrow \text{Hom}_R(\alpha, M) & & \\ 0 & \longrightarrow & \text{Hom}_R(C, M) & \xrightarrow{\text{Hom}_R(g, M)} & \text{Hom}_R(B, M) & \xrightarrow{\text{Hom}_R(f, M)} & \text{Hom}_R(A, M) & \longrightarrow & 0 \end{array}$$

It suffices to show that $\text{Hom}_R(f', M)$ is an epimorphism. From the commutativity we have $\text{Hom}_R(\alpha, M) \circ \text{Hom}_R(f', M) = \text{Hom}_R(f' \circ \alpha, M) = \text{Hom}_R(\beta \circ f, M) = \text{Hom}_R(f, M) \circ \text{Hom}_R(\beta, M)$. Since α, β are isomorphisms, so $\text{Hom}_R(\alpha, M)$ and $\text{Hom}_R(\beta, M)$ are also isomorphisms. By the hypothesis $\text{Hom}_R(f, M)$ is an epimorphism, so $\text{Hom}_R(f, M) \circ \text{Hom}_R(\beta, M)$ is an epimorphism. Since $\text{Hom}_R(\alpha, M)$ is an isomorphism, thus $\text{Hom}_R(f', M)$ is an epimorphism.

Proof of (P2): Let $\mathbb{E} : 0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ be a splitting short exact sequence. So there exists an R -module homomorphisms f' and g' such that $g \circ g' = 1_C$ and $f' \circ f = 1_A$. Let $M \in \mathcal{M}$. Since $\text{Hom}_R(-, M)$ is a contravariant left exact functor, it suffices to show that $\text{Hom}_R(f, M)$ is an epimorphism. We have $\text{Hom}_R(f, M) \circ$

$\text{Hom}_R(f', M) = \text{Hom}_R(f' \circ f, M) = \text{Hom}_R(1_A, M) = 1_{\text{Hom}_R(A, M)}$, and so $\text{Hom}_R(f, M)$ is an epimorphism. Hence $\mathbb{E} \in \iota^{-1}(\mathcal{M})$, that is, all splitting short exact sequences are in $\iota^{-1}(\mathcal{M})$.

Proof of (P3): Let $f : B \rightarrow C$ and $g : C \rightarrow D$ be $\iota^{-1}(\mathcal{M})$ -monomorphisms. We can construct the following short exact sequences

$$0 \longrightarrow B \xrightarrow{f} C \xrightarrow{\sigma_1} \text{Coker } f \longrightarrow 0$$

$$0 \longrightarrow C \xrightarrow{g} D \xrightarrow{\sigma_2} \text{Coker } g \longrightarrow 0$$

and since f, g are $\iota^{-1}(\mathcal{M})$ -monomorphisms, the maps $\text{Hom}_R(f, M)$ and $\text{Hom}_R(g, M)$ are epimorphisms. We want to show that $g \circ f$ is a $\iota^{-1}(\mathcal{M})$ -monomorphism, that is, we want to obtain a short exact sequence if we apply the contravariant left exact functor $\text{Hom}_R(-, M)$ to the following short exact sequence

$$0 \longrightarrow B \xrightarrow{g \circ f} D \xrightarrow{\sigma} \text{Coker } g \circ f \longrightarrow 0$$

So it suffices to show that $\text{Hom}_R(g \circ f, M)$ is an epimorphism. The homomorphism $\text{Hom}_R(f, M) \circ \text{Hom}_R(g, M) = \text{Hom}_R(g \circ f, M)$ is an epimorphism, since $\text{Hom}_R(f, M)$ and $\text{Hom}_R(g, M)$ are epimorphisms.

Proof of (P4): Let $f : A \rightarrow B$ and $g : B \rightarrow C$ be $\iota^{-1}(\mathcal{M})$ -epimorphisms. We shall show that $g \circ f : A \rightarrow C$ is also a $\iota^{-1}(\mathcal{M})$ -epimorphism. We can construct the following short exact sequences of R -modules and R -module homomorphisms:

$$0 \longrightarrow \text{Ker}(f) \xrightarrow{i_1} A \xrightarrow{f} B \longrightarrow 0$$

$$0 \longrightarrow \text{Ker}(g) \xrightarrow{i_2} B \xrightarrow{g} C \longrightarrow 0$$

$$0 \longrightarrow \text{Ker}(g \circ f) \xrightarrow{i} A \xrightarrow{g \circ f} C \longrightarrow 0$$

If we apply the contravariant left exact functor $\text{Hom}_R(-, M)$ to these short exact sequences, then by hypothesis the first two short exact sequences are also short exact, that is $\text{Hom}_R(i_1, M)$ and $\text{Hom}_R(i_2, M)$ are epimorphisms. So it suffices to show that $\text{Hom}_R(i, M)$ is an epimorphism. We have

$$0 \longrightarrow \text{Ker}(f) \xrightarrow{i_1} A \xrightarrow{f} B \longrightarrow 0$$

and

$$0 \longrightarrow \text{Ker}(g) \xrightarrow{i_2} B \xrightarrow{g} C \longrightarrow 0$$

short exact sequences and so we can obtain long exact sequences by using the second long exact sequence for Ext, that is,

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Hom}_R(B, M) & \xrightarrow{f^*} & \text{Hom}_R(A, M) & \xrightarrow{i_1^*} & \text{Hom}_R(M, \text{Ker}(f)) \\ & & & & \delta & & \downarrow \\ & & & & & & \text{Ext}_R^1(\text{Ker}(f), M) \\ & & & & & & \longrightarrow \dots \\ & & & & & & \text{Ext}_R^1(B, M) \xrightarrow{\text{Ext}_R^1(f, M)} \text{Ext}_R^1(A, M) \longrightarrow \text{Ext}_R^1(\text{Ker}(f), M) \longrightarrow \dots \end{array}$$

where ψ^* denotes $\text{Hom}_R(\psi, M)$ for every R -module homomorphism ψ in the above diagram. Since $\text{Hom}_R(i_1, M)$ epimorphism, so $\text{Ext}_R^1(f, M)$ (and also same way $\text{Ext}_R^1(g, M)$) is a monomorphism. By the same way if we use the short exact sequence

$$0 \longrightarrow \text{Ker}(g \circ f) \xrightarrow{i} A \xrightarrow{g \circ f} C \longrightarrow 0$$

we obtain the following long exact sequence:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Hom}_R(C, M) & \xrightarrow{(g \circ f)^*} & \text{Hom}_R(A, M) & \xrightarrow{i^*} & \text{Hom}_R(\text{Ker}(g \circ f), M) \\ & & & & \delta' & & \downarrow \\ & & & & & & \text{Ext}_R^1(\text{Ker}(g \circ f), M) \\ & & & & & & \longrightarrow \dots \\ & & & & & & \text{Ext}_R^1(C, M) \xrightarrow{h} \text{Ext}_R^1(A, M) \longrightarrow \text{Ext}_R^1(\text{Ker}(g \circ f), M) \longrightarrow \dots \end{array}$$

where $h = \text{Ext}_R^1(g \circ f, M)$. Since $\text{Ext}_R^1(f, M)$ and $\text{Ext}_R^1(g, M)$ are monomorphisms and Ext is a functor so their union $\text{Ext}_R^1(g \circ f, M)$ is a monomorphism. So $\text{Ker}(\text{Ext}_R^1(g \circ f, M)) = 0 = \text{Im}(\delta')$. Thus $\delta' = 0$ and so $\text{Ker}(\delta') = \text{Hom}_R(\text{Ker}(g \circ f), M)$ and from exactness $\text{Ker}(\delta') = \text{Hom}_R(i, M)$. Hence $\text{Hom}_R(i, M)$ is an epimorphism.

Proof of (P5): Let $\alpha : A \longrightarrow B$ and $\beta : B \longrightarrow C$ be monomorphisms and $\beta \circ \alpha$ be a $\iota^{-1}(\mathcal{M})$ -monomorphism. We can construct the following commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \xrightarrow{\alpha} & B & \xrightarrow{\sigma_1} & \text{Coker}(\alpha) \longrightarrow 0 \\ & & \parallel & & \downarrow \beta & & \downarrow \tilde{\beta} \\ 0 & \longrightarrow & A & \xrightarrow{\beta \circ \alpha} & C & \xrightarrow{\sigma_2} & \text{Coker}(\beta \circ \alpha) \longrightarrow 0 \end{array}$$

where σ_1 and σ_2 are canonical epimorphisms and $\tilde{\beta}$ is the R -module homomorphism induced by β where $\tilde{\beta}(b + \text{Im}(\alpha)) = \beta(b) + \text{Im}(\beta \circ \alpha)$ for all $b \in B$. It is easily checked that $\tilde{\beta}$ is a monomorphism. If we apply the contravariant left exact functor

$\text{Hom}_R(-, M)$ to the above diagram, we obtain the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \text{Hom}_R(\text{Coker}(\beta \circ \alpha), M) & \xrightarrow{\text{Hom}_R(\sigma_2, M)} & \text{Hom}_R(C, M) & \xrightarrow{\text{Hom}_R(\beta \circ \alpha, M)} & \text{Hom}_R(A, M) \longrightarrow 0 \\
& & \downarrow \text{Hom}_R(\tilde{\beta}, M) & & \downarrow \text{Hom}_R(\beta, M) & & \parallel \\
0 & \longrightarrow & \text{Hom}_R(\text{Coker}(\alpha), M) & \xrightarrow{\text{Hom}_R(\sigma_1, M)} & \text{Hom}_R(B, M) & \xrightarrow{\text{Hom}_R(\alpha, M)} & \text{Hom}_R(A, M)
\end{array}$$

It suffices to show that $\text{Hom}_R(\alpha, M)$ is an epimorphism. Take any $x \in \text{Hom}_R(A, M)$. Since $\text{Hom}_R(\beta \circ \alpha, M)$ is an epimorphism, so there exists an element $c \in \text{Hom}_R(C, M)$ such that $\text{Hom}_R(\beta \circ \alpha)(c) = x$. So $\text{Hom}_R(\alpha, M)(\text{Hom}_R(\beta, M)(c)) = x$. Let $y = \text{Hom}_R(\beta, M)(c) \in \text{Hom}_R(B, M)$. Then $\text{Hom}_R(\alpha, M)(y) = x$. So $\text{Hom}_R(\alpha, M)$ is an epimorphism.

Proof of (P6): Let $f : A \longrightarrow B$ and $g : B \longrightarrow C$ be epimorphisms and $g \circ f$ is a $\iota^{-1}(\mathcal{M})$ -epimorphism. We can construct the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \text{Ker}(f) & \xrightarrow{i_2} & A & \xrightarrow{g \circ f} & C \longrightarrow 0 \\
& & \downarrow \tilde{f} & & \downarrow f & & \parallel 1_C \\
0 & \longrightarrow & \text{Ker}(g) & \xrightarrow{i_1} & B & \xrightarrow{g} & C \longrightarrow 0
\end{array}$$

where \tilde{f} induced R -module homomorphism by the R -module homomorphism f . So B together with the R -module homomorphisms f and i_1 is a push out of the pair i_2, \tilde{f} . We want to show that $\text{Hom}_R(i_1, M)$ is an epimorphism which means if we have an R -module homomorphism $h : \text{Ker}(g) \longrightarrow M$, then there exists an R -module homomorphism $\tilde{h} : B \longrightarrow M$ such that $\tilde{h} \circ i_1 = h$. Since $\text{Hom}_R(i_2, M)$ is an epimorphism so for $h \circ \tilde{f} : \text{Ker}(g \circ f) \longrightarrow M$ there exists an R -module homomorphism $\theta : A \longrightarrow M$ such that $\theta \circ i_2 = h \circ \tilde{f}$, that is,

$$\begin{array}{ccccc}
& & M & & \\
& & \uparrow h & & \\
& & \text{Ker}(g) & \xleftarrow{i_1} & B \\
& & \uparrow \tilde{f} & & \uparrow f \\
& & \text{Ker}(g \circ f) & \xleftarrow{i_2} & A \\
& & \uparrow \theta & & \\
& & M & &
\end{array}$$

So B together with the R -module homomorphisms f and i_1 is a push out of the pair i_2, \tilde{f} . By the push out property there exists a unique R -module homomorphism $\psi : B \longrightarrow$

M such that $\psi \circ i_1 = h$ and $\psi \circ f = \theta$, that is,

$$\begin{array}{ccccc}
 & & M & & \\
 & & \swarrow \psi & & \nwarrow h \\
 & & B & \xleftarrow{i_1} & \text{Ker}(g) \\
 & \uparrow f & & & \uparrow \tilde{f} \\
 & A & \xleftarrow{i_2} & \text{Ker}(g \circ f) & \\
 & \swarrow \theta & & & \nwarrow
 \end{array}$$

If we choose $\tilde{h} = \psi$ then we complete the proof. \square

Proposition 2.4.4. (by Sklyarenko (1978, Lemma 0.1)) $\tau^{-1}(\mathcal{M})$ is a proper class for every class \mathcal{M} of right R -modules.

Proof. We shall follow the proof by (Demirci, 2008, p. 13). We know that $M \otimes_R -$ is an additive covariant right exact functor for every right R -module M .

Proof of (P1): Let $\mathbb{E} : 0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$ be a short exact sequence in $\tau^{-1}(\mathcal{M})$ and let \mathbb{E}' be an isomorphic short exact sequence, that is,

$$\begin{array}{ccccccc}
 \mathbb{E} : 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C \longrightarrow 0 \\
 & & \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma \\
 \mathbb{E}' : 0 & \longrightarrow & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' \longrightarrow 0
 \end{array}$$

where α, β, γ are R -module isomorphisms and $\beta \circ f = f' \circ \alpha$, $\gamma \circ g = g' \circ \beta$. Since $\mathbb{E} \in \tau^{-1}(\mathcal{M})$ and $M \otimes_R -$ is a covariant right exact functor, so we can construct the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & M \otimes_R A & \xrightarrow{1_M \otimes f} & M \otimes_R B & \xrightarrow{1_M \otimes g} & M \otimes_R C \longrightarrow 0 \\
 & & \downarrow 1_M \otimes \alpha & & \downarrow 1_M \otimes \beta & & \downarrow 1_M \otimes \gamma \\
 & & M \otimes_R A' & \xrightarrow{1_M \otimes f'} & M \otimes_R B' & \xrightarrow{1_M \otimes g'} & M \otimes_R C' \longrightarrow 0
 \end{array}$$

It suffices to show that $1_M \otimes f'$ is a monomorphism. $(1_M \otimes f') \circ (1_M \otimes \alpha) = 1_M \otimes (f' \circ \alpha) = 1_M \otimes (\beta \circ f) = (1_M \otimes \beta) \circ (1_M \otimes f)$ since β is an isomorphism then $1_M \otimes \beta$ is also an isomorphism and by assumption $1_M \otimes f$ is an epimorphism then $(1_M \otimes \beta) \circ (1_M \otimes f)$ is an epimorphism. Hence $1_M \otimes f'$ is an epimorphism.

Proof of (P2): Let $\mathbb{E} : 0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$ be a splitting short exact sequence. So there exist R -module homomorphisms f' and g' such that $g \circ g' = 1_C$ and $f' \circ f = 1_A$. Let $M \in \mathcal{M}$. Since $M \otimes_R -$ is a covariant right exact functor, it suffices to show that

$1_M \otimes f$ is a monomorphism. We have $(1_M \otimes f') \circ (1_M \otimes f) = 1_M \otimes (f' \circ f) = 1_M \otimes 1_A = 1_{M \otimes_R A}$ so $1_M \otimes f$ is a monomorphism. Hence $\mathbb{E} \in \tau^{-1}(\mathcal{M})$, that is, all splitting short exact sequences are in $\tau^{-1}(\mathcal{M})$.

Proof of (P3): Let $\alpha : A \rightarrow B$ and $\beta : B \rightarrow C$ be \mathcal{A} -monomorphisms. So $1_M \otimes \alpha$ and $1_M \otimes \beta$ are monomorphisms and $1_M \otimes (\beta \circ \alpha) = (1_M \otimes \beta) \circ (1_M \otimes \alpha)$ is a monomorphism. So $\beta \circ \alpha$ is an \mathcal{A} -monomorphisms.

Proof of (P4): Let $h : B \rightarrow C$ and $g : C \rightarrow D$ be $\tau^{-1}(\mathcal{M})$ -epimorphisms and $A' = \text{Ker}(g \circ h)$. Then the mapping derived functors

$$\text{Tor}_1^R(M, B) \longrightarrow \text{Tor}_1^R(M, C) \longrightarrow \text{Tor}_1^R(M, D)$$

is epimorphic, therefore $\text{Tor}_1^R(M, A') \longrightarrow \text{Tor}_1^R(M, B)$ is a monomorphism hence $g \circ h$ is a $\tau^{-1}(\mathcal{M})$ -epimorphism.

Proof of (P5): Let $\alpha : A \rightarrow B$ and $\beta : B \rightarrow C$ be monomorphisms and $\beta \circ \alpha$ be a $\tau^{-1}(\mathcal{M})$ -monomorphism. We can construct the following commutative diagram with an exact row:

$$\begin{array}{ccccccc} & & M \otimes_R A & \xrightarrow{1_M \otimes \alpha} & M \otimes_R B & & \\ & & \parallel & & \downarrow \beta & & \\ 0 & \longrightarrow & M \otimes_R A & \xrightarrow{1_M \otimes \beta \circ \alpha} & M \otimes_R C & \longrightarrow & 0 \end{array}$$

If $x \in \text{Ker}(1_M \otimes \alpha)$, then $1_M \otimes \beta \circ \alpha(x) = 1_M \otimes \beta \circ 1_M \otimes \alpha(x) = 0$. Then $x \in \text{Ker}(1_M \otimes \beta \circ \alpha) = 0$. So $\text{Ker}(1_M \otimes \alpha) = 0$ which means $1_M \otimes \alpha$ is a monomorphism.

Proof of (P6): Let $\mu : B \rightarrow C$ and $v : C \rightarrow D$ be epimorphisms and $v \circ \mu$ is a $\tau^{-1}(\mathcal{M})$ -epimorphism. We can construct the following commutative diagram with exact rows where h, u, f and w are R -module homomorphisms:

$$\begin{array}{ccccccc} & & & & 0 & & 0 \\ & & & & \downarrow & & \downarrow \\ 0 & \longrightarrow & A & \xrightarrow{h} & X & \xrightarrow{u} & N \longrightarrow 0 \\ & & \parallel & & \downarrow g & & \downarrow f \\ 0 & \longrightarrow & A & \xrightarrow{w} & B & \xrightarrow{\mu} & C \longrightarrow 0 \\ & & & & \downarrow v \circ \mu & & \downarrow v \\ & & & & D & = & D \\ & & & & \downarrow & & \downarrow \\ & & & & 0 & & 0 \end{array}$$

Applying the functor $M \otimes_R -$ to this diagram, we see that the second column of the

diagram

$$\begin{array}{ccccccc}
M \otimes_R A & \xrightarrow{1_M \otimes h} & M \otimes_R X & \xrightarrow{1_M \otimes u} & M \otimes_R N & \longrightarrow & 0 \\
\parallel & & \downarrow 1_M \otimes g & & \downarrow 1_M \otimes f & & \\
M \otimes_R A & \xrightarrow{1_M \otimes w} & M \otimes_R B & \xrightarrow{1_M \otimes \mu} & M \otimes_R C & \longrightarrow & 0 \\
& & \downarrow 1_M \otimes v \circ \mu & & \downarrow 1_M \otimes v & & \\
& & M \otimes_R D & \xlongequal{\quad} & M \otimes_R D & & \\
& & \downarrow & & \downarrow & & \\
& & 0 & & 0 & &
\end{array}$$

is exact, since $v \circ \mu$ is a $\tau^{-1}(\mathcal{M})$ -epimorphism. In order to show that v is a $\tau^{-1}(\mathcal{M})$ -epimorphism, we have to show that $1_M \otimes f$ is a monomorphism. Let $n \in \text{Ker}(1_M \otimes f)$. $n = (1_M \otimes u)(x)$ for some $x \in M \otimes_R X$ since $1_M \otimes u$ is an epimorphism. $((1_M \otimes \mu) \circ (1_M \otimes g))(x) = ((1_M \otimes f) \circ (1_M \otimes u))(x) = 0$. Then $(1_M \otimes g)(x) \in \text{Ker}(1_M \otimes \mu) = \text{Im}(1_M \otimes w)$, that is, $(1_M \otimes g)(x) = (1_M \otimes w)(a)$ for some $a \in M \otimes_R A$. $(1_M \otimes g)(x) = (1_M \otimes w)(a) = ((1_M \otimes g) \circ (1_M \otimes h))(a)$ implies $x - (1_M \otimes h)(a) \in \text{Ker}(1_M \otimes g) = 0$. So $\text{Ker}(1_M \otimes f) = 0$ and v is a $\tau^{-1}(\mathcal{M})$ -epimorphism. \square

2.5 Inductively Closed Proper Classes

For the definitions and properties in this section, see for example Sklyarenko (1978, §6), Vermani (2003, §1.6), Rotman (2009, §5.2) and Lam (1999, §4J).

Definition 2.5.1. A set S is called a **directed set** if there is a relation \leq defined on S such that;

- (i) \leq is reflexive.
- (ii) \leq is transitive.
- (iii) for every pair $\alpha, \beta \in S$, there exist $\gamma \in S$ such that, $\alpha \leq \gamma$ and $\beta \leq \gamma$.

Definition 2.5.2. A **direct system** of sets $\{X, \pi\}$ over a directed set S is a function which attaches to each $\alpha \in S$, a set X^α , and, to each pair α, β with $\alpha \leq \beta$ in S , a map $\pi_\alpha^\beta : X^\alpha \longrightarrow X^\beta$ such that for each $\alpha \in S$, π_α^α is the identity map from X^α to X^α , and for all $\alpha \leq \beta \leq \gamma$ in S , $\pi_\beta^\gamma \pi_\alpha^\beta = \pi_\alpha^\gamma$.

Definition 2.5.3. Let $\{M, \pi\}_S$ be a direct system over a directed set S , such that for each $\alpha \in S$, M^α is an R -module and for every $\alpha \leq \beta$ in S , $\pi_\alpha^\beta : M^\alpha \rightarrow M^\beta$ is an R -module homomorphism. Let Q be the submodule of $\bigoplus_{\alpha \in S} M^\alpha$ generated by all elements of the type $\pi_\alpha^\beta(x) - x$, $x \in M^\alpha$, $\alpha, \beta \in S$ and $\alpha < \beta$. The quotient module $(\bigoplus_{\alpha \in S} M^\alpha)/Q$ is called **direct limit** of the direct system $\{M, \pi\}_S$ and is denoted by $\varinjlim \{M, \pi\}_S$ or $\varinjlim M^\alpha$, $\alpha \in S$.

Observe that we are here identifying M^α with its canonical image in the direct sum $\bigoplus_{\alpha \in S} M^\alpha$. The natural projection $\bigoplus_{\alpha \in S} M^\alpha \rightarrow \varinjlim \{M, \pi\}_S$ restricted to the submodule M^α of $\bigoplus_{\alpha \in S} M^\alpha$ defines homomorphism $\pi_\alpha : M^\alpha \rightarrow \varinjlim \{M, \pi\}_S$ called projection and given by $\pi_\alpha(x) = x + Q$, $x \in M^\alpha$.

Next consider an axiomatic description of direct limit.

Definition 2.5.4. Given a direct family $\{M, \pi\}_S$ of R -modules, an R -module M together with homomorphisms $\pi_\alpha : M^\alpha \rightarrow M$ is called direct limit of the family if

- (i) $\pi_\alpha = \pi_\beta \pi_\alpha^\beta$ for every $\alpha \leq \beta$.
- (ii) when N is another R -module with a family of R -module homomorphisms $\lambda_\alpha : M^\alpha \rightarrow N$ such that $\lambda_\alpha = \lambda_\beta \pi_\alpha^\beta$ for every $\alpha \leq \beta$, then there exist a unique R -module homomorphism $\lambda : M \rightarrow N$ such that $\lambda \pi_\alpha = \lambda_\alpha$ for every $\alpha \in S$.

We know that direct limit of any direct system $\{M, \pi\}_S$ over a directed set S exists. Let M with R -module homomorphisms $\pi_\alpha : M^\alpha \rightarrow M$, $\alpha \in S$ and N with R -module homomorphisms $\lambda_\alpha : M^\alpha \rightarrow N$, $\alpha \in S$, be two direct limits of the given direct family. Then there exist an isomorphism $\theta : M \rightarrow N$ such that $\theta \pi_\alpha = \lambda_\alpha$ for every $\alpha \in S$.

A directed set S , when viewed as a category, has as its objects the elements of S and as its morphisms exactly one morphism π_α^β when $\alpha \leq \beta$. It is easy to see that direct systems in $R\text{-Mod}$ over S are merely covariant functors $M : S \rightarrow R\text{-Mod}$; in our original notation $M(\alpha) = M^\alpha$ and $M(\pi_\alpha^\beta) : M^\alpha \rightarrow M^\beta$.

Definition 2.5.5. Let $\{A, \pi\}_S$ and $\{B, \lambda\}_S$ be direct systems of R -modules over the same directed set S . A **morphism of direct systems** of R -modules is a natural

transformation $r : A \longrightarrow B$.

In more detail, r is an indexed family of R -module homomorphisms

$$r = (r_\alpha = A^\alpha \longrightarrow B^\alpha), \alpha \in S$$

making the following diagrams commute for all $\alpha < \beta$:

$$\begin{array}{ccc} A^\alpha & \xrightarrow{r_\alpha} & B^\alpha \\ \downarrow \pi_\beta^\alpha & & \downarrow \lambda_\beta^\alpha \\ A^\beta & \xrightarrow{r_\beta} & B^\beta \end{array}$$

A morphism of direct systems $r : \{A, \pi\}_S \longrightarrow \{B, \lambda\}_S$ over a same directed set S determines a homomorphism $\vec{r} : \varinjlim A^\alpha \longrightarrow \varinjlim B^\alpha$ by $\vec{r} : (\sum \gamma_\alpha(a_\alpha) + Q_1) = \sum \mu_\alpha(r_\alpha(a_\alpha)) + Q_2$ where $Q_1 \leq \oplus A^\alpha$ and $Q_2 \leq \oplus B^\alpha$ are the relation submodules in the construction of $\varinjlim A^\alpha$ and $\varinjlim B^\alpha$, respectively, and γ_α and μ_α are the injection of A^α and B^α , respectively, into their direct sums.

Let us note that some properties which we will use later about direct limits.

- (1) If A is a right R -module, then the functor $A \otimes_R -$ preserve direct limits. Thus if $\{B, \pi\}_S$ is a direct system of R -modules over a directed set S , then there is a natural isomorphism

$$A \otimes_R \varinjlim B^\alpha \cong \varinjlim (A \otimes_R B^\alpha)$$

- (2) Let S be a directed set. Let $\{A, \pi\}_S$, $\{B, \lambda\}_S$ and $\{C, \psi\}_S$ be direct systems of R -modules. If $r : \{A, \pi\}_S \longrightarrow \{B, \lambda\}_S$ and $t : \{B, \lambda\}_S \longrightarrow \{C, \psi\}_S$ are morphisms of direct systems, and if

$$\mathbb{E}^\alpha : 0 \longrightarrow A^\alpha \xrightarrow{r_\alpha} B^\alpha \xrightarrow{t_\alpha} C^\alpha \longrightarrow 0$$

is exact for each $\alpha \in S$ then there is an exact sequence

$$\varinjlim \mathbb{E}^\alpha : 0 \longrightarrow \varinjlim A^\alpha \xrightarrow{\vec{r}} \varinjlim B^\alpha \xrightarrow{\vec{t}} \varinjlim C^\alpha \longrightarrow 0$$

- (3) A flatly generated proper class is always inductively closed since the tensor product and a direct limit of a direct system commute.

A proper class \mathcal{A} is said to be **inductively closed proper class** if for every direct system $\{\mathbb{E}, \pi\}_S$ over a directed set S in \mathcal{A} , the direct limit $\varinjlim \mathbb{E}^\alpha$ is also in \mathcal{A} (see (Sklyarenko, 1978, §8)).

Definition 2.5.6. A short exact sequence

$$\mathbb{E} : 0 \longrightarrow A \xrightarrow{f} B \longrightarrow C \longrightarrow 0$$

of R -modules is said to be **pure-exact** if $M \otimes_R \mathbb{E}$ is exact for every *right* R -module M . If this is the case, we say that $\text{Im}(f)$ is a **pure submodule** of B . We denote all pure short exact sequences by ${}_{\mathcal{R}Pure} = \tau^{-1}(\{\text{all right } R\text{-modules}\})$. A submodule A of an R -module B is said to be a **pure submodule** of B if the inclusion monomorphism $i_A : A \longrightarrow B$ is a ${}_{\mathcal{R}Pure}$ -monomorphism.

Let us note some properties which we will use later about purity:

- (1) Any split short exact sequence is pure-exact.
- (2) Any pure short exact sequence is a direct limit of splitting short exact sequences.
- (3) For every module M , there exists a pure exact sequence, ends with M ; more precisely, for each R -module M , there exists a short exact sequence

$$0 \longrightarrow K \longrightarrow H \longrightarrow M \longrightarrow 0$$

that is in ${}_{\mathcal{R}Pure}$, where H is a direct sum of finitely presented modules. So by Proposition 2.3.1 a ${}_{\mathcal{R}Pure}$ -projective R -module is a direct summand of a direct sum of finitely presented R -modules. For the definition and properties of finitely presented modules, see Section 4.1.

- (4) ${}_{\mathcal{R}Pure}$ is the smallest inductively closed proper class. Since any proper class contains all splitting short exact sequences and any pure short exact sequence is a direct limit of splitting short exact sequences.

Two functors that we shall use frequently are the R -dual functor

$$(-)^* = \text{Hom}_R(-, R) : R\text{-Mod} \longrightarrow \text{Mod-}R$$

and the character module functor

$$(-)^{\flat} = \text{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z}) : R\text{-Mod} \longrightarrow \text{Mod-}R.$$

For an R -module M , its R -dual $M^* = \text{Hom}_R(M, R)$ is a *right* R -module. The character module functor $(-)^{\flat} : R\text{-Mod} \longrightarrow \text{Mod-}R$ uses the injective cogenerator \mathbb{Q}/\mathbb{Z} for $\mathbb{Z}\text{-Mod}$: For a R -module M , $M^{\flat} = \text{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ is a *right* R -module.

For a functor T from a category \mathcal{C} of left or right R -modules to a category \mathcal{B} of left or right S -modules (where R, S are rings), and for a given class \mathcal{F} of short exact sequences in \mathcal{B} , let $T^{-1}(\mathcal{F})$ be the class of those short exact sequences of \mathcal{C} which are carried into \mathcal{F} by the functor T . If the functor T is left or right exact, then $T^{-1}(\mathcal{F})$ is a proper class; see Stenström (1967b, Proposition 2.1).

Example 2.5.7. The third purity example below (generalized from pure subgroups of abelian groups) is the main motivation for relative homological algebra; this is the reason why proper classes are also called purities.

- (1) ${}^R\mathcal{S}plit$ is the smallest proper class consisting of all splitting short exact sequences of R -modules.
- (2) ${}^R\mathcal{A}bs$ is the largest proper class consisting of all short exact sequences of R -modules
- (3) ${}^R\mathcal{P}ure$ is the classical **Cohn's purity**:

$$\begin{aligned} {}^R\mathcal{P}ure &= \pi^{-1}(\{\text{all finitely presented } R\text{-modules}\}) \\ &= \tau^{-1}(\{\text{all finitely presented } \textit{right } R\text{-modules}\}) \\ &= \tau^{-1}(\{\text{all } \textit{right } R\text{-modules}\}) \\ &= [(-)^{\flat}]^{-1}({}^R\mathcal{S}plit). \\ &= \iota^{-1}(\{M^{\flat} \mid M \text{ is a finitely presented } \textit{right } R\text{-module}\}) \end{aligned}$$

See for example (Facchini, 1998, §1.4) for the proof of the first four of these equalities. See (Sklyarenko, 1978, Proposition 6.2) for the last equality. The second equality above that allows us to pass from a proper class projectively generated by a class of finitely presented R -modules to a flatly generated proper

class is a general idea; what is being used in this passage is the Auslander-Bridger transpose of finitely presented R -modules. See Section 4.2.

CHAPTER THREE

THE PROPER CLASSES *Neat* and *P-Pure*

László Fuchs has characterized the commutative domains for which neatness and ${}_R\mathcal{P}$ -purity coincide; see Fuchs (2012). Fuchs calls a ring R to be an N -domain if R is a commutative domain such that neatness and ${}_R\mathcal{P}$ -purity coincide. He proved that a commutative domain R is an N -domain if and only if all the maximal ideals of the commutative domain R are (finitely generated) projective R -modules. Motivated by Fuchs' result for commutative domains, we wish to extend this result to a class of commutative rings larger than commutative domains. In this chapter, we will give the definitions of our main objects which are neatness and ${}_R\mathcal{P}$ -purity. These give us the proper classes ${}_R\mathcal{N}$ and ${}_R\mathcal{P}$ -Pure of short exact sequences of R -modules. We will see some properties of these proper classes.

3.1 Proper Classes Generated by Simple Modules

A submodule A of a module B is said to be a **complement in B** or is said to be a **complement submodule** of B if A is a complement of some submodule K of B , that is, $K \cap A = 0$ and A is *maximal* with respect to this property. A submodule A of a module B is said to be **closed in B** if A has *no* proper essential extension in B , that is, there exists *no* submodule \tilde{A} of B such that $A \subsetneq \tilde{A}$ and $A \trianglelefteq \tilde{A}$ ($A \trianglelefteq \tilde{A}$ means that A is **essential** in \tilde{A} , that is, for every *non-zero* submodule X of \tilde{A} , we have $A \cap X \neq 0$). We also say in this case that A is a **closed submodule** and it is known that closed submodules and complement submodules in a module coincide. See the monograph Dung, N. V. and Huynh, D.V. and Smith, P. F. and Wisbauer, R. (1994) for a survey of results in the related concepts. Dually, a submodule A of a module B is said to be a **supplement in B** or A is said to be a **supplement submodule of B** if A is a supplement of some submodule K of B , that is, $B = K + A$ and A is *minimal* with respect to this property; equivalently, $K + A = B$ and $K \cap A \ll A$ ($K \cap A \ll A$ means that $K \cap A$ is *small* (=superfluous) in A , that is, for *no* proper submodule X of A , $K \cap A + X = A$).

For the definitions and related properties, see (Wisbauer, 1991, §41); the monograph Clark et al. (2006) focuses on the concepts related with supplements.

Mermut (2004) deals with Complements (closed submodules) and supplements in R -modules using relative homological algebra via the known two dual proper classes ${}_{R}\mathcal{C}ompl$ and ${}_{R}\mathcal{S}uppl$ of short exact sequences in $R\text{-Mod}$, and related other proper classes like ${}_{R}\mathcal{N}eat$ and ${}_{R}\mathcal{C}o\text{-}\mathcal{N}eat$. The proper class ${}_{R}\mathcal{C}ompl$ [${}_{R}\mathcal{S}uppl$] consists of all short exact sequences

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

in $R\text{-Mod}$ such that $\text{Im}(f)$ is a complement [resp. supplement] in B . The proper class ${}_{R}\mathcal{N}eat$ [${}_{R}\mathcal{C}o\text{-}\mathcal{N}eat$] consists of all short exact sequences in $R\text{-Mod}$ with respect to which every simple module is projective [resp. every module with zero radical is injective].

The notations of the proper classes related with complements and supplements are the following:

- (1) ${}_{R}\mathcal{C} = {}_{R}\mathcal{C}ompl$
- (2) ${}_{R}\mathcal{S} = {}_{R}\mathcal{S}uppl$
- (3) ${}_{R}\mathcal{N}eat^{\pi} = {}_{R}\mathcal{N}eat = \pi^{-1}(\{R/P \mid P \text{ is a maximal left ideal of } R\})$ is the proper class projectively generated by all simple R -modules
- (4) ${}_{R}\mathcal{N}eat^{\tau} = \tau^{-1}(\{R/P \mid P \text{ is a maximal right ideal of } R\})$ is the proper class flatly generated by all simple *right* R -modules.
- (5) ${}_{R}\mathcal{N}eat^{\iota} = \iota^{-1}(\{R/P \mid P \text{ is a maximal left ideal of } R\})$ is the proper class injectively generated by all simple R -modules.
- (6) ${}_{R}\mathcal{C}o\text{-}\mathcal{N}eat = \iota^{-1}(\{M \in R\text{-Mod} \mid \text{Rad}(M) = 0\})$ is the proper class injectively generated by all R -modules with zero radical.
- (7) ${}_{R}\mathcal{P}\text{-}\mathcal{P}ure = \tau^{-1}(\{R/P \mid P \in \mathcal{P}\})$, where \mathcal{P} is the collection of all left primitive ideals of R .

Note that when R is a commutative ring, the proper classes in (4), (5) and (7) coincide, that is,

$${}_R\mathcal{N}eat^l = {}_R\mathcal{N}eat^r = {}_R\mathcal{P}\text{-Pure}.$$

The last equality is obvious since the collection of all left primitive ideals of a commutative ring R coincide with the collection of all maximal ideals of a commutative ring R . For the first equality, see Proposition 3.3.2 and Corollary 3.3.3.

With this terminology of proper classes, László Fuchs' result for N -domains is that:

Theorem 3.1.1. Fuchs' characterization of N -domains. (*Fuchs (2012, Theorem 5.2)*)
For a commutative domain R , ${}_R\mathcal{N}eat = {}_R\mathcal{P}\text{-Pure}$ if and only if all the maximal ideals of the commutative domain R are (finitely generated) projective modules (that is, they are invertible ideals).

For a commutative domain R , Fuchs has proved that ${}_R\mathcal{N}eat = {}_R\mathcal{P}\text{-Pure}$ if and only if the projective dimension of every simple module is ≤ 1 . We always have ${}_{\mathcal{R}}\mathcal{C}ompl \subseteq {}_R\mathcal{N}eat$ and ${}_R\mathcal{S}uppl \subseteq {}_{\mathcal{R}}\mathcal{C}o\text{-}Neat \subseteq {}_R\mathcal{N}eat^l$; see Stenström (1967a, Proposition 5), Mermut (2004, Ch. 3), Alizade & Mermut (2004), Al-Takhman et al. (2006) or Clark et al. (2006, §10 and 20.7). If the ring R is commutative, then we have;

$${}_{\mathcal{R}}\mathcal{C}o\text{-}Neat \subseteq {}_R\mathcal{N}eat^l = {}_R\mathcal{N}eat^r = {}_R\mathcal{P}\text{-Pure}$$

The proper classes in (3), (4) and (5) that are projectively, flatly or injectively generated by simple (left or right) modules are natural ways to extend the concept of neat subgroups to modules; so we have named all of them using 'neat'. Note that Fuchs (2012) calls the short exact sequences in ${}_{\mathcal{R}}\mathcal{P}\text{-Pure}$ co-neat but we reserve the word co-neat as defined in (6) above because it has also been used for its relation with supplements. Being a co-neat submodule looks like being a supplement; see Mermut (2004, Proposition 3.4.2) or Al-Takhman et al. (2006, 1.14) or Clark et al. (2006, 10.14) for the following characterization of co-neat submodules: A submodule A of a module B is a co-neat submodule of B if and only if A is a *Rad-supplement* of some submodule K of B , that is, $K + A = B$ and $K \cap A \subseteq \text{Rad}(A)$.

3.2 The Proper Class \mathcal{Neat}

A short exact sequence $\mathbb{E} : 0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ of R -modules is said to be a **neat-exact sequence** if for every simple R -module S , the sequence

$$\mathrm{Hom}_R(S, \mathbb{E}) : 0 \longrightarrow \mathrm{Hom}_R(S, A) \xrightarrow{f_*} \mathrm{Hom}_R(S, B) \xrightarrow{g_*} \mathrm{Hom}_R(S, C) \longrightarrow 0$$

is exact. Observe that \mathbb{E} is a neat-exact sequence if and only if $\mathrm{Im}(f)$ is a neat submodule of B . A submodule A of an R -module B is neat in B if and only if the short exact sequence $0 \longrightarrow A \xrightarrow{i_A} B \xrightarrow{\sigma} C = B/A \longrightarrow 0$ (where i_A is a inclusion monomorphism) is a neat-exact sequence. We denote the proper class of neat-exact sequences of R -modules by ${}_R\mathcal{Neat}$.

For completeness, we shall prove the following useful lemma.

Lemma 3.2.1. (see for example Fuchs & Salce (2001, Lemma I.8.4) or Sklyarenko (1978, Lemma 1.2, without proof)) Let

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{f_1} & B & \xrightarrow{g_1} & C & \longrightarrow & 0 \\ & & \alpha \downarrow & \circlearrowleft & \downarrow \beta & \tilde{\gamma} \circlearrowright & \downarrow \gamma & & \\ 0 & \longrightarrow & D & \xrightarrow{f_2} & E & \xrightarrow{g_2} & C' & \longrightarrow & 0 \end{array}$$

be a commutative diagram of R -modules and R -module homomorphisms with exact rows. There exists an R -module homomorphism $\tilde{\alpha} : B \longrightarrow D$ with the upper left triangle commutative if and only if there exist an R -module homomorphism $\tilde{\gamma} : C \longrightarrow E$ with the lower right triangle commutative.

Proof. Suppose we have a commutative diagram as above and there exist an R -module homomorphism $\tilde{\alpha} : B \longrightarrow D$ with the upper triangle commutative, that is, $\tilde{\alpha} \circ f_1 = \alpha$. We want to show that there exists an R -module homomorphism $\tilde{\gamma} : C \longrightarrow E$ where $g_2 \circ \tilde{\gamma} = \gamma$. Define $\tilde{\gamma}$ as follows: For $c \in C$, there exists an element $b \in B$ such that $g_1(b) = c$ (since g_1 is an epimorphism) and define $\tilde{\gamma}(c) = \beta(b) - f_2(\tilde{\alpha}(b))$. $\tilde{\gamma}$ is a well defined R -module homomorphism, because, if $c = g_1(b) = g_1(b')$ where b, b' are in B , then $b - b' \in \mathrm{Ker}(g_1) = \mathrm{Im}(f_1)$ implies $b - b' = f_1(a)$ for some $a \in A$. If we apply $\tilde{\alpha}$, then we obtain $\tilde{\alpha}(b - b') = (\tilde{\alpha} \circ f_1)(a) = \alpha(a)$. Applying f_2 , we obtain

$f_2(\tilde{\alpha}(b - b')) = f_2(\alpha(a)) = (\beta \circ f_1)(a) = \beta(f_1(a)) = \beta(b - b')$, that is, $f_2(\tilde{\alpha}(b - b')) = \beta(b - b')$. Hence $\beta(b) - f_2(\tilde{\alpha}(b)) = \beta(b') - f_2(\tilde{\alpha}(b'))$. This shows that $\tilde{\gamma}$ is a well defined R -module homomorphism. Since β , f_2 , $\tilde{\alpha}$ and g_1 are R -module homomorphisms, it is easily checked that $\tilde{\gamma}$ is also an R -module homomorphism. Let $c \in C$. Then $(g_2 \circ \tilde{\gamma})(c) = g_2(\beta(b) - f_2(\tilde{\alpha}(b)))$ for some $b \in B$ where $g_1(b) = c$, so $g_2(\beta(b)) - (g_2 \circ f_2)(\tilde{\alpha}(b)) = (g_2 \circ \beta)(b) = (\gamma \circ g_1)(b) = \gamma(g_1(b)) = \gamma(c)$ since $g_2 \circ f_2 = 0$ by exactness of the second row and $g_2 \circ \beta = \gamma \circ g_1$, by the commutativity of the diagram. Hence $g_2 \circ \tilde{\gamma} = \gamma$. Conversely, suppose we have a commutative diagram as above and there exists an R -module homomorphism $\tilde{\gamma}: C \rightarrow E$ with the lower right triangle commutative, that is, $g_2 \circ \tilde{\gamma} = \gamma$. We want to find a homomorphism $\tilde{\alpha}: B \rightarrow D$ such that $\tilde{\alpha} \circ f_1 = \alpha$. For $b \in B$ define $\tilde{\alpha}(b) = d$ where $d \in D$ is chosen such that $\tau(b) = f_2(d)$ for $\tau = \beta - \tilde{\gamma} \circ g_1$. To prove that $\tilde{\alpha}$ is well defined, we need to show that such a d is unique. τ is an R -module homomorphism since β , $\tilde{\gamma}$ and g_1 are R -module homomorphisms. Firstly $\text{Im}(\tau) \leq \text{Ker}(g_2) = \text{Im}(f_2)$ because $g_2 \circ \tau = g_2(\beta - \tilde{\gamma} \circ g_1) = g_2 \circ \beta - g_2 \circ (\tilde{\gamma} \circ g_1) = g_2 \circ \beta - \gamma \circ g_1 = 0$. So for $b \in B$ $\tau(b) = f_2(d)$ for some $d \in D$, and this d is unique since f_2 is a monomorphism. This shows that $\tilde{\alpha}$ is a well defined function. Let's check that $\tilde{\alpha}$ is an R -module homomorphism. For any $r \in R$ and $b, b' \in B$, $\tilde{\alpha}(rb + b') = d$ where $\tau(rb + b') = f_2(d)$. Let $r \in R$ and $b, b' \in B$. Say $\tilde{\alpha}(b) = d$ and $\tilde{\alpha}(b') = d'$, that is, by definition of $\tilde{\alpha}$, $\tau(b) = f_2(d)$ and $\tau(b') = f_2(d')$. Then $\tau(rb + b') = r\tau(b) + \tau(b') = rf_2(d) + f_2(d') = f_2(rd + d')$. So by the definition of $\tilde{\alpha}$, $\tilde{\alpha}(rb + b') = rd + d' = r\tilde{\alpha}(b) + \tilde{\alpha}(b')$. It remains to show that $\tilde{\alpha} \circ f_1 = \alpha$. Let $a \in A$. Let $b = f_1(a)$. We have $\tau(b) = (\beta - \tilde{\gamma} \circ g_1)(b) = \beta(b) - (\tilde{\gamma} \circ g_1)(b) = \beta(f_1(a)) - \tilde{\gamma} \circ g_1 \circ f_1(a) = (\beta \circ f_1)(a) = f_2(\alpha(a))$ since $g_1 \circ f_1 = 0$ and the diagram commutative. By the definition of $\tilde{\alpha}$, $\tau(b) = f_2(\alpha(a))$ implies that $\tilde{\alpha}(b) = \alpha(a)$, that is $\tilde{\alpha} \circ f_1 = \alpha$ as required. \square

We shall also give the detailed proof of the following:

Proposition 3.2.2. *Mermut (2004, Proposition 3.2.4). For a left ideal I in a ring R , the following are equivalent for a short exact sequence $\mathbb{E}: 0 \rightarrow A \xrightarrow{i_A} B \xrightarrow{g} C \rightarrow 0$ of R -modules and R -module homomorphisms where A is a submodule of B and i_A is the inclusion map:*

(1) $\text{Hom}_R(R/I, B) \xrightarrow{g_*} \text{Hom}_R(R/I, C)$ is an epimorphism, that is, R/I is projective relative to the short exact sequence \mathbb{E} .

(2) For every $b \in B$, if $Ib \leq A$, then there exists an element $a \in A$ such that $I(b - a) = 0$.

Proof. Suppose g_* is an epimorphism, that is, given an R -module homomorphism $h : R/I \rightarrow C$, there exist $\tilde{h} : R/I \rightarrow B$ that makes the following diagram commute:

$$\begin{array}{ccc} & R/I & \\ \tilde{h} \swarrow & \downarrow h & \\ B & \xrightarrow{g} & C \longrightarrow 0 \end{array}$$

Let $b \in B$ be such that $Ib \leq A$. Then we can define $\alpha : I \rightarrow A$ by $\alpha(r) = rb$ for all $r \in I$. Also define $\alpha' : R \rightarrow B$ by $\alpha'(r) = rb$ for all $r \in R$. Then $\alpha(r) = \alpha'(r)$ for all $r \in I$. Let $f_1 : I \rightarrow R$ be the inclusion map and $g_1 : R \rightarrow R/I$ be the canonical epimorphism. Define the R -module homomorphism $\beta : R/I \rightarrow C$ by $\beta(r + I) = g(\alpha'(r))$ for all $r + I \in R/I$ ($r \in R$). By our hypothesis, R/I is projective with respect to \mathbb{E} . So there exists an R -module homomorphism $\tilde{\beta} : R/I \rightarrow B$ such that $g \circ \tilde{\beta} = \beta$. By Lemma 3.2.1, there exists an R -module homomorphism $\tilde{\alpha} : R \rightarrow A$ such that $\tilde{\alpha} \circ f_1 = \alpha$, that is, the upper left triangle commutes in the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I & \xrightarrow{f_1} & R & \xrightarrow{g_1} & R/I & \longrightarrow & 0 \\ & & \alpha \downarrow & \circlearrowleft & \downarrow \alpha' & \circlearrowleft & \downarrow \tilde{\beta} & \circlearrowleft & \downarrow \beta \\ 0 & \longrightarrow & A & \longrightarrow & B & \xrightarrow{g} & C & \longrightarrow & 0 \end{array}$$

Let $a = \tilde{\alpha}(1_R) \in A$. Then for each $r \in I$, we have $rb = \alpha(r) = \tilde{\alpha}(f_1(r)) = \tilde{\alpha}(r) = r\tilde{\alpha}(1_R) = ra$. So $r(b - a) = 0$ for all $r \in I$, that is $I(b - a) = 0$. Conversely, suppose that for every $b \in B$, if $Ib \leq A$, then there exists an element $a \in A$ such that $I(b - a) = 0$. Let $\beta : R/I \rightarrow C$ be a given R -module homomorphism. Let $f_1 : I \rightarrow R$ be the inclusion map and $g_1 : R \rightarrow R/I$ the canonical epimorphism. Since R is projective, there exists an R -module homomorphism $\alpha' : R \rightarrow B$ such that $g \circ \alpha' = \beta \circ g_1$, that is we can construct the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I & \xrightarrow{f_1} & R & \xrightarrow{g_1} & R/I & \longrightarrow & 0 \\ & & & & \downarrow \alpha' & & \downarrow \beta & & \\ 0 & \longrightarrow & A & \xrightarrow{i_A} & B & \xrightarrow{g} & C & \longrightarrow & 0 \end{array}$$

By commutativity, $g(\alpha'(I)) = \beta(g_1 I) = \beta(0) = 0$, and so $\alpha'(I) \leq \text{Ker}(g) = \text{Im}(i_A) = A$. Hence, we can define $\alpha : I \rightarrow A$ by $\alpha(r) = \alpha'(r)$ for all $r \in I$ and then we obtain the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I & \xrightarrow{f_1} & R & \xrightarrow{g_1} & R/I & \longrightarrow & 0 \\ & & \downarrow \alpha & & \downarrow \alpha' & & \downarrow \beta & & \\ 0 & \longrightarrow & A & \xrightarrow{i_A} & B & \xrightarrow{g} & C & \longrightarrow & 0 \end{array}$$

Let $b = \alpha'(1)$. For every $r \in R$, $\alpha'(r) = rb$ and so $\alpha(I) = Ib \leq A$. Hence by our hypothesis, there exists an element $a \in A$ such that $I(b-a) = 0$, that is, for all $r \in I$ $rb = ra$. So define $\tilde{\alpha} : R \rightarrow A$ by $\tilde{\alpha}(r) = ra$ for all $r \in R$. Then $\tilde{\alpha} \circ f_1 = \alpha$ because for all $r \in I$, $(\tilde{\alpha} \circ f_1)(r) = \tilde{\alpha}(r) = ra = rb = \alpha'(r) = \alpha(r)$. Thus the upper left triangle commutes in the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I & \xrightarrow{f_1} & R & \xrightarrow{g_1} & R/I & \longrightarrow & 0 \\ & & \downarrow \alpha & \circlearrowleft \tilde{\alpha} & \downarrow \alpha' & & \downarrow \beta & & \\ 0 & \longrightarrow & A & \longrightarrow & B & \xrightarrow{g} & C & \longrightarrow & 0 \end{array}$$

By Lemma 3.2.1, there exists an R -module homomorphism $\tilde{\beta} : R/I \rightarrow B$ such that $g \circ \tilde{\beta} = \beta$. This proves that g_* is an epimorphism. \square

The proof of the following corollary follows easily from the above proposition.

Corollary 3.2.3. *Let R be a commutative ring. For a maximal ideal P in a ring R , the following are equivalent for a short exact sequence*

$$\mathbb{E} : 0 \longrightarrow A \xrightarrow{i_A} B \xrightarrow{g} C \longrightarrow 0$$

of R -modules and R -module homomorphisms where A is a submodule of B and i_A is the inclusion map:

(1) $\text{Hom}_R(R/P, B) \xrightarrow{g_*} \text{Hom}_R(R/P, C)$ is an epimorphism.

(2) For every $b \in B$, if $Pb \leq A$, then there exists an element $a \in A$ such that

$$P(b-a) = 0$$

For commutative rings, another equivalent formulation of neatness is given by using $M[S]$, the S -socle of M , where S is a simple R -module, M is an R -module and R is a commutative ring, and where P is the unique maximal ideal of the commutative ring R such that $S \cong R/P$ and so $P = \text{ann}(S)$:

$$\begin{aligned} M[S] &= \sum_{\substack{T \leq M \\ T \cong S}} T \\ &= \{x \in M \mid P \cdot x = 0\} \\ &= \{x \in M \mid \text{ann}(S) \cdot x = 0\} \\ &= \{x \in M \mid \text{ann}(x) = P\} \cup \{0\}. \end{aligned}$$

Let us see how can we show the last equation. It is clear that if $x \in M$ and $\text{ann}(x) = P$ or $x = 0$, then $Px = 0$. Conversely let $x \in M$ and $Px = 0$. Then $P \leq \text{ann}(x)$, and since P is a maximal ideal of R , we obtain $P = \text{ann}(x)$ or $\text{ann}(x) = R$. In other words $P = \text{ann}(x)$ or $x = 0$. By this equation if $0 \neq x \in M[S] = \sum_{\substack{T \leq M \\ T \cong S}} T$, then we can say $Rx \cong \mathbb{R}/\text{ann}(x) = R/P \cong S$. Observe also that if $f : A \rightarrow B$ is an R -module homomorphism of R -modules, then $f(A[S]) \leq B[S]$ because if $a \in A[S]$, then $P \cdot a = 0$ and so $P \cdot f(a) = f(P \cdot a) = f(0) = 0$ which implies that $f(a) \in B[S]$.

Proposition 3.2.4. (see p. 2 of Fuchs (2012)) Let R be a commutative ring. Let A be a submodule of an R -module B . Consider the short exact sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C = B/A \longrightarrow 0$$

where f is the inclusion monomorphism and g is the natural epimorphism. The submodule A is a neat submodule of B if and only if for every simple R -module S , the sequence

$$0 \longrightarrow A[S] \xrightarrow{f'} B[S] \xrightarrow{g'} C[S] \longrightarrow 0$$

is exact where $f'(a) = f(a)$ for all $a \in A[S]$ and $g'(b) = g(b)$ for all $b \in B[S]$.

Proof. Firstly observe that the map f' is well defined because $f(A[S]) \leq B[S]$ as pointed out before the proposition. Similarly g' is well defined. Suppose A is neat in B , that is, every simple R -module S is projective with respect to the short exact sequence $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C = B/A \longrightarrow 0$. Let S be a simple R -module. It is clear that f' is a

monomorphism since $f'(x) = f(x)$ for all $x \in A$ and f is the inclusion monomorphism. Let's show that g' is an epimorphism. Take $0 \neq c \in T$ where $T \leq C$ and $T \cong S$. Since $T \cong S$ is a simple R -module and $0 \neq c \in T$, we have $c \in Rc = T \cong S$, and for some $b \in B$, we have $g(b) = c$ since g is an epimorphism. Then $g(Rb) = R(g(b)) = Rc = T \cong S \cong R/P$ where P is a maximal ideal of the commutative ring R . Since always $Rc \cong R/\text{ann}(c)$ where $\text{ann}(c) = \{r \in R \mid rc = 0\}$ is an ideal of the commutative ring R , and since $Rc \cong R/P$, we must have $P = \text{ann}(R/P) = \text{ann}(R/\text{ann}(c)) = \text{ann}(c)$ since we are in a commutative ring. Thus $Pc = 0$. Then $g(Pb) = P(g(b)) = Pc = 0$, and so $Pb \leq \text{Ker}(g) = \text{Im}(f) = A$ since f is the inclusion homomorphism. By hypothesis, A is neat in B . By Corollary 3.2.3, $Pb \leq A$ then implies that there exists an element $a \in \text{Ker}(g)$ such that $P(b-a) = 0$. Thus $P \leq \text{ann}(b-a)$. Since $g(b) = c \neq 0$ we have $b \notin A$. But $a \in A$, and so $b \neq a$. Thus $\text{ann}(b-a) \neq R$ because $1 \notin \text{ann}(b-a)$ as $b \neq a$. Now $P \leq \text{ann}(b-a) \neq R$ implies $P = \text{ann}(b-a)$ since P is a maximal ideal of R . Then $R(b-a) \cong R/P \cong S$ which implies $R(b-a) \leq B[S]$. So $b-a \in B[S]$ and $g'(b-a) = g(b-a) = g(b) - g(a) = g(b) = c$ since $g(a) = 0$. Thus $T \leq \text{Im}(g)$ and so $C[S] = \sum_{\substack{T \leq C \\ T \cong S}} T \leq \text{Im}(g)$. Hence g' is an epimorphism. For ending the proof of this part, it suffices to show that $\text{Im}(f') = \text{Ker}(g')$. Take any $a \in A[S]$. Then $(g' \circ f')(a) = g'(f'(a)) = g'(f(a)) = g(f(a)) = 0$ from exactness and this shows $\text{Im}(f') \leq \text{Ker}(g')$. For the other part, take any $0 \neq b \in \text{Ker}(g') \leq B[S]$. We want to find an element $a \in A[S]$ such that $f'(a) = f(a) = b$. Since $0 \neq b \in \text{Ker}(g')$, $g'(b) = g(b) = 0$, so $b \in \text{Ker}(g) = \text{Im}(f)$. Then we have $f(a) = b$ for some $0 \neq a \in A$. Furthermore, $f(Pa) = Pf(a) = Pb = 0$ since $b \in B[S]$. So $Pa \leq \text{Ker}(f) = 0$ since f is a monomorphism, which implies that $P \leq \text{ann}(a) \neq R$ (since $a \neq 0$). Hence by the maximality of P , we have $\text{ann}(a) = P$ and so $a \in Ra \cong R/\text{ann}(a) \cong R/P \cong S$ which implies that $a \in Ra \leq A[S]$. Thus $\text{Ker}(g') \leq \text{Im}(f')$. Hence $\text{Im}(f') = \text{Ker}(g')$. We showed that $0 \longrightarrow A[S] \xrightarrow{f'} B[S] \xrightarrow{g'} C[S] \longrightarrow 0$ is exact. Conversely, suppose that $0 \longrightarrow A[S] \xrightarrow{f'} B[S] \xrightarrow{g'} C[S] \longrightarrow 0$ is exact. Then

$$B[S] \xrightarrow{g'} C[S] \longrightarrow 0$$

is onto, that is for each $0 \neq c \in C[S] = \sum_{\substack{T \leq C \\ T \cong S}} T$, $c \in T = Rc \cong S$ there exists an element $b \in B[S] = \sum_{\substack{T \leq B \\ T \cong S}} T$ where $b \in T' = Rb \cong S$ such that $g'(b) = g(b) = c$ and $g(T') = g(Rb) =$

$Rg(b) = Rc = T$. Let $0 \neq x \in S$. Since S is simple, $S = Rx$. Let $h : S = Rx \rightarrow C$ be an R -module homomorphism that is not the zero homomorphism. Then, since S is a simple R -module, $h(S) \cong S$. So $S \cong h(S) = h(Rx) = Rh(x)$. Thus $h(x) \in C[S]$. Since g' is an epimorphism there exists an element $b \in B[S]$ such that $g'(b) = g(b) = h(x)$. Then, $g'(T) = g(T) = g(Rb) = Rg(b) = Rh(x) = h(S)$. Define $\tilde{h} : S = Rx \rightarrow B$ by $\tilde{h}(rx) = rb$. \tilde{h} is well defined because $r, r' \in R$, $rx = r'x$ implies $r - r' \in \text{ann}(x) = \text{ann}(b)$ so $rb = r'b$. \tilde{h} is an R -module homomorphism because $\tilde{h}(r_1x + r_2x) = \tilde{h}((r_1 + r_2)x) = (r_1 + r_2)b = r_1b + r_2b = \tilde{h}(r_1x) + \tilde{h}(r_2x)$ and $\tilde{h}(r(r'x)) = \tilde{h}((rr')x) = (rr')b = r(r'b) = r\tilde{h}(r'x)$ for all $r_1, r_2, r, r' \in R$. The R -module homomorphism \tilde{h} satisfies $g \circ \tilde{h} = h$ because for all $r \in R$, $(g \circ \tilde{h})(rx) = g(rb) = rg(b) = rh(x) = h(rx)$. This shows that A is a neat submodule of B . \square

We can also rephrase the definition of neatness in terms of systems of equations. If the maximal ideal P is generated by the elements r_i ($i \in I$), then we consider the system of equations

$$r_i x = a_i \in A \quad (i \in I)$$

with the single unknown x and constants in A .

Proposition 3.2.5. (*Fuchs (2012, Lemma 2.2)*) *Let R be a commutative ring. A submodule A of an R -module B is neat in B if and only if such systems are solvable in A , whenever they are solvable in B .*

Proof. Suppose A is neat in B and the system of equations $r_i x = a_i$, $a_i \in A$ ($i \in I$) is solvable in B . This means that there exists $b \in B$ such that $r_i b = a_i$, $a_i \in A$. Since the elements r_i ($i \in I$) are generators for P , we obtain $Pb \leq A$. By Corollary 3.2.3, since A is neat in B there exists $a \in A$ such that $P(b - a) = 0$. So $rb = ra$ for every $r \in P$. Thus $r_i a = r_i b = a_i$, $a_i \in A$, and so this system is solvable in A . Conversely, suppose if such systems solvable in B , then they are also in A . So for every $b \in B$, if $Pb \leq A$, then $r_i b \in A$ for every $i \in I$. Let $a_i = r_i b$. Thus the system $r_i x = a_i$, ($i \in I$) is solvable in B . So it is solvable in A by hypothesis, that is, there exists $a \in A$ such that $r_i a = a_i$ for every $i \in I$. Then $r_i(b - a) = r_i b - r_i a = a_i - a_i = 0$ for every $i \in I$. So $P(b - a) = 0$

since P is generated by the elements r_i where $i \in I$. Thus by Corollary 3.2.3 A is neat in B . \square

We obtain a proper class of short exact sequences is closed under pull-back and push-out. For completeness, let us see this observation for neatness:

Proposition 3.2.6. (Fuchs (2012, Lemma 2.3)) *Let R be a commutative ring. If the middle sequence in the following commutative diagram of R -modules and R -homomorphisms with exact rows is neat-exact, then same holds for the top and the bottom exact sequences for every R -module homomorphism $\gamma : C' \rightarrow C$ and for every R -module homomorphism $\alpha : A \rightarrow A''$.*

$$\begin{array}{ccccccccc}
 \mathbb{E}' : 0 & \longrightarrow & A & \xrightarrow{\chi'} & B' & \xrightarrow{\sigma'} & C' & \longrightarrow & 0 \\
 & & \parallel & & \downarrow \beta' & & \downarrow \gamma & & \\
 \mathbb{E} : 0 & \longrightarrow & A & \xrightarrow{\chi} & B & \xrightarrow{\sigma} & C & \longrightarrow & 0 \\
 & & \downarrow \alpha & & \downarrow \beta'' & & \parallel & & \\
 \mathbb{E}'' : 0 & \longrightarrow & A'' & \xrightarrow{\chi''} & B'' & \xrightarrow{\sigma''} & C & \longrightarrow & 0
 \end{array}$$

Proof. By hypothesis $\mathbb{E} : 0 \rightarrow A \xrightarrow{\chi} B \xrightarrow{\sigma} C \rightarrow 0$ is a neat-exact sequence, that is, every simple R -module S is projective with respect to \mathbb{E} ; equivalently, the sequence

$$\mathrm{Hom}_R(S, \mathbb{E}) : 0 \rightarrow \mathrm{Hom}_R(S, A) \rightarrow \mathrm{Hom}_R(S, B) \rightarrow \mathrm{Hom}_R(S, C) \rightarrow 0$$

is exact. We want to show that $\mathrm{Hom}_R(S, \mathbb{E}')$ and $\mathrm{Hom}_R(S, \mathbb{E}'')$ are exact for every simple R -module S . Since $\mathrm{Hom}_R(S, -)$ is a left exact covariant functor, we have the following commutative diagram with the exact rows: where the middle row is exact since \mathbb{E} is neat-exact sequence.

$$\begin{array}{ccccccccc}
 0 & \longrightarrow & \mathrm{Hom}_R(S, A) & \xrightarrow{\mathrm{Hom}_R(S, \chi')} & \mathrm{Hom}_R(S, B') & \xrightarrow{\mathrm{Hom}_R(S, \sigma')} & \mathrm{Hom}_R(S, C') & & \\
 & & \parallel \mathrm{Hom}_R(S, \alpha) & & \downarrow & & \downarrow \mathrm{Hom}_R(S, \gamma) & & \\
 0 & \longrightarrow & \mathrm{Hom}_R(S, A) & \xrightarrow{\mathrm{Hom}_R(S, \chi)} & \mathrm{Hom}_R(S, B) & \xrightarrow{\mathrm{Hom}_R(S, \sigma)} & \mathrm{Hom}_R(S, C) & \longrightarrow & 0 \\
 & & \downarrow \mathrm{Hom}_R(S, \alpha) & & \downarrow & & \parallel & & \\
 0 & \longrightarrow & \mathrm{Hom}_R(S, A'') & \xrightarrow{\mathrm{Hom}_R(S, \chi'')} & \mathrm{Hom}_R(S, B'') & \xrightarrow{\mathrm{Hom}_R(S, \sigma'')} & \mathrm{Hom}_R(S, C) & &
 \end{array}$$

We will show that the top and the bottom rows are short exact. It suffices to show that $\mathrm{Hom}_R(S, \sigma')$ and $\mathrm{Hom}_R(S, \sigma'')$ are epimorphisms. So we want to show that if $\beta : S \rightarrow$

C' is an R -module homomorphism, then there exists an R -module homomorphism $\theta : S \longrightarrow \beta'$ such that $\beta = \sigma' \circ \theta$, that is the following diagram

$$\begin{array}{ccccc} & & S & & \\ & \nearrow \theta & \downarrow \beta & & \\ B' & \xrightarrow{\sigma'} & C' & \longrightarrow & 0 \end{array}$$

commutes. Let us try to find θ . Since $\gamma \circ \beta$ is an R -module homomorphism and by hypothesis $\text{Hom}_R(S, \sigma)$ is an epimorphism (since \mathbb{E} is a neat-exact sequence), there exists an R -module homomorphism $f : S \longrightarrow B$ such that $\sigma \circ f = \gamma \circ \beta$. So we have the following commutative diagram:

$$\begin{array}{ccccc} S & & & & \\ & \searrow \beta & & & \\ & & B' & \xrightarrow{\sigma'} & C' \\ & & \downarrow \beta' & & \downarrow \gamma \\ & \searrow f & B & \xrightarrow{\sigma} & C \end{array}$$

Since (B', β', σ') β' is a pull back of the pair σ and γ , there exists a unique $\theta : S \longrightarrow \beta'$ such that $\sigma' \circ \theta = \beta$ and $\beta' \circ \theta = f$. This shows that $\text{Hom}_R(S, \sigma')$ is an epimorphism. It remains to show that $\text{Hom}_R(S, \sigma'')$ is an epimorphism. Let $\beta : S \longrightarrow C$ be an R -module homomorphism. To satisfy the epimorphism property of $\text{Hom}_R(S, \sigma'')$, we have to find $\tilde{h} : S \longrightarrow B''$ such that $\sigma'' \circ \tilde{h} = \beta$. Since S is projective with respect to \mathbb{E} , there exists an R -module homomorphism $f : S \longrightarrow B$ such that $\sigma \circ f = \beta$. Let $\tilde{h} = \beta'' \circ f : S \longrightarrow B''$. Then $\sigma'' \circ \tilde{h} = \sigma'' \circ (\beta'' \circ f) = (\sigma'' \circ \beta'') \circ f = (1_C \circ \sigma) \circ f = \sigma \circ f = \beta$ as required.

Indeed that result always hold for a proper class of short exact sequences of R -modules. ${}_R\mathcal{Neat} = \pi^{-1}(\{\text{all simple } R\text{-modules}\})$ is a projectively generated proper class (generated by all simple R -modules). By Lemma 2.2.1, proper classes are closed under pull back and push out. Hence neat-exactness of \mathbb{E} implies that \mathbb{E}' and \mathbb{E}'' are also neat exact sequences. \square

By the terminology for proper classes, remember that by a ${}_R\mathcal{Neat}$ -projective R -module, we mean an R -module H which has the projective property with respect all neat-exact sequences of R -modules; equivalently, if

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is a neat-exact sequence of R -modules, then exactness holds for the sequence

$$0 \longrightarrow \text{Hom}_R(H, A) \longrightarrow \text{Hom}_R(H, B) \longrightarrow \text{Hom}_R(H, C) \longrightarrow 0.$$

Theorem 3.2.7. (Fuchs, 2012, Theorem 2.6) *Let R be a commutative ring.*

(1) *Every R -module M can be embedded in a neat-exact sequence*

$$0 \longrightarrow K \longrightarrow H \longrightarrow M \longrightarrow 0$$

of R -modules where H is a direct sum of a projective and a semisimple R -module.

(2) *Every R -neat-projective R -module is a direct summand of a projective and a semisimple R -module.*

Proof.

(1): Let $\{S_j \mid j \in J\}$ be a set of representatives of all simple R -modules, that is, each simple R -module is isomorphic to S_j for a unique $j \in J$. Given M , consider the R -module $H = (\bigoplus_{a \in M} R_a) \oplus [\bigoplus_{j \in J} (\bigoplus_{\phi_j} S_j)]$ where the modules R_a are copies of R , while ϕ_j runs over the nonzero elements of $\text{Hom}_R(S_j, M)$. The map $\phi : H \longrightarrow M$ is defined by mapping $1_a \in R_a$ to $a \in M$ and acting on S_j as ϕ_j . Let $K = \text{Ker}(\phi)$. It is obvious that ϕ is surjective. To show that the arising exact sequence

$$0 \longrightarrow K \longrightarrow H \longrightarrow M \longrightarrow 0$$

is neat-exact we must show that all simple R -modules have projective property with respect to it. So we must show that for each R -module homomorphism $h : S \longrightarrow M$, there exists an R -module homomorphism $g : S \longrightarrow H$ such that $\phi \circ g = h$. If $h = 0$, take $g = 0$. Suppose $h \neq 0$. Since $\{S_j \mid j \in J\}$ is a set of representatives of all simple R -modules, the simple R -module S isomorphic to S_{j_0} for some $j_0 \in J$ via say the isomorphism $f : S_{j_0} \longrightarrow S$. Thus $h \circ f : S_{j_0} \longrightarrow M$ is an element of $\text{Hom}_R(S_{j_0}, M)$. Let $\gamma : S_{j_0} \longrightarrow \bigoplus_{\phi_{j_0} \in \text{Hom}_R(S_{j_0}, M)} S_j$ be the inclusion monomorphism of direct sums corresponds to the term for $h \circ f \in \text{Hom}_R(S_{j_0}, M)$ defined for every $x \in S_{j_0}$ by $\gamma(x) = (y_{\phi_{j_0}})_{\phi_{j_0} \in \text{Hom}_R(S_{j_0}, M)}$ where

$$y_{\phi_{j_0}} = \begin{cases} 0, & \text{if } \phi_{j_0} \neq h \circ f \\ x, & \text{if } \phi_{j_0} = h \circ f \end{cases}$$

Let $\psi : \bigoplus_{\phi_{j_0}} S_{j_0} \longrightarrow \bigoplus_{j \in J} (\bigoplus_{\phi_j} S_j)$ be the inclusion monomorphism of the j_0 th component. Let $\delta : \bigoplus_{j \in J} (\bigoplus_{\phi_j} S_j) \longrightarrow (\bigoplus_{a \in M} R_a) \oplus [\bigoplus_{j \in J} (\bigoplus_{\phi_j} S_j)]$ be the inclusion monomorphism into the direct sum. Let $i = \delta \circ \psi \circ \gamma : S_{j_0} \longrightarrow H$. By the construction of H and by the definition of ϕ that we have $\phi \circ i = h \circ f$. So we have the following commutative diagram:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & K & \longrightarrow & H = (\bigoplus_{a \in M} R_a) \oplus [\bigoplus_{j \in J} (\bigoplus_{\phi_j} S_j)] & \xrightarrow{\phi} & M \longrightarrow 0 \\
 & & & & & & \uparrow h \\
 & & & & & & S \\
 & & & & & & \uparrow f \\
 & & & & & & S_{j_0} \\
 & & & & \swarrow i & & \uparrow \cong
 \end{array}$$

Let $g = i \circ f^{-1}$. Then $\phi \circ g = \phi \circ (i \circ f^{-1}) = \phi \circ i \circ f^{-1} = (\phi \circ i) \circ f^{-1} = (h \circ f) \circ f^{-1} = h \circ (f \circ f^{-1}) = h$. Hence S is projective relative to this exact sequence. Note that $\bigoplus_{a \in M} R_a \cong \bigoplus_{a \in M} R$ is a free R -module and so projective, and $\bigoplus_{j \in J} (\bigoplus_{\phi_j} S_j)$ is a semisimple R -module.

(2): Let M be a $\mathcal{R}\mathcal{N}eat$ -projective R -module. From (1) it follows that there exists a neat-exact sequence

$$0 \longrightarrow K \longrightarrow H \longrightarrow M \longrightarrow 0$$

where $H = F \oplus L$, F is a free R -module and L is a semisimple R -module. Since M is $\mathcal{R}\mathcal{N}eat$ -projective, by Proposition 2.3.1 this short exact sequence splits. Hence M is isomorphic to a direct summand of $H = F \oplus L$. Thus M is a direct summand of a direct sum of a projective and a semisimple R -module. \square

3.3 The Proper Class \mathcal{P} -Pure

An ideal P in R is said to be a **left primitive ideal** if P is the annihilator of a simple R -module. If S is a simple R -module and $P = \text{ann}(S)$, then we can write $S = R/M$ for some maximal left ideal M of R . It is easily seen that $\text{ann}(S) \leq M$, but the equality may not hold in the noncommutative case. If R is a commutative ring, then necessarily $M = P$.

Denote by \mathcal{P} the collection of all left primitive ideals of the ring R . A short exact

sequence $\mathbb{E} : 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ of R -modules is said to be ${}_R\mathcal{P}$ -**pure-exact** if for every left primitive ideal P of R , the sequence

$$(R/P) \otimes_R \mathbb{E} : 0 \longrightarrow (R/P) \otimes_R A \longrightarrow (R/P) \otimes_R B \longrightarrow (R/P) \otimes_R C \longrightarrow 0$$

is exact, that is, $\mathbb{E} \in \tau^{-1}(\{R/P \mid P \in \mathcal{P}\}) = {}_R\mathcal{P}\text{-Pure}$. By for example Osborne (2000, Proposition 2.2), for right ideal I of R , we have $(R/I) \otimes_R B \cong B/IB$ where IB is the submodule of B generated by all rb , $r \in I$, $b \in B$. So for each $P \in \mathcal{P}$, $(R/P) \otimes_R \mathbb{E}$ is exact if and only if $0 \longrightarrow A/PA \longrightarrow B/PB \longrightarrow C/PC \longrightarrow 0$ is exact if and only if $0 \longrightarrow A/PA \longrightarrow B/PB$ is a monomorphism (since tensor product is a right exact functor) if and only if $PA = (PB) \cap A$. We say that a submodule A of an R -module B is ${}_R\mathcal{P}$ -pure in B if $PA = (PB) \cap A$ for all $P \in \mathcal{P}$, or equivalently the short exact sequence $0 \longrightarrow A \xrightarrow{i_A} B \longrightarrow C \longrightarrow 0$ of R -modules is ${}_R\mathcal{P}$ -pure-exact (where i_A is the inclusion monomorphism).

If the ring R is commutative, then \mathcal{P} is the collection of all maximal ideals of R . In this case, a short exact sequence \mathbb{E} of R -modules is ${}_R\mathcal{P}$ -pure-exact if for every simple R -module S , the sequence $S \otimes_R \mathbb{E}$ is exact. A submodule A of an R -module B is ${}_R\mathcal{P}$ -pure in B if $PA = (PB) \cap A$ for all maximal ideals $P \in \mathcal{P}$. So in the commutative case ${}_R\mathcal{P}\text{-Pure} = \tau^{-1}(\{\text{all simple } R\text{-modules}\})$. Now let us see some further properties in the commutative case:

Proposition 3.3.1. *If R is a commutative ring, P is a maximal ideal of R , and M is an R -module annihilated by P , that is, $PM = 0$, then M is isomorphic (as an R -module) to a direct sum of copies of R/P : $M \cong \bigoplus_{\lambda \in \Gamma} R/P$ for some index set Γ .*

Proof. Since M is annihilated by P and P is a (two-sided) ideal of the commutative ring R , M can be considered as an (R/P) -module also. Since P is a maximal ideal of R , R/P is a field. So M is a vector space over the field R/P . Hence M is a free (R/P) -module (every vector space has a basis, so it is a free module). Thus $M \cong \bigoplus_{\lambda \in \Gamma} (R/P)$ for some index set Γ . This is an isomorphism of (R/P) -modules but the R -module and (R/P) -module structures of M and R/P are the same. So this is an isomorphism of R -modules also. \square

Proposition 3.3.2. *(Fuchs (2012, Proposition 3.1)) Let R be a commutative ring. For*

a short exact sequence

$$\mathbb{E} : 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

of R -modules, and for a simple R -module S , the sequence

$$S \otimes_R \mathbb{E} : 0 \longrightarrow S \otimes_R A \xrightarrow{\alpha} S \otimes_R B \longrightarrow S \otimes_R C \longrightarrow 0$$

is exact if and only if the sequence

$$\mathrm{Hom}_R(\mathbb{E}, S) : 0 \longrightarrow \mathrm{Hom}_R(C, S) \longrightarrow \mathrm{Hom}_R(B, S) \longrightarrow \mathrm{Hom}_R(A, S) \longrightarrow 0$$

is exact.

Proof. We shall give a detailed proof by following the proof in Fuchs (2012, Proposition 3.1). The simple R -module $S \cong R/P$ for some maximal ideal P of R . Observe that for every R -module M , we have a natural isomorphism $\mathrm{Hom}_R(M, S) \cong \mathrm{Hom}_R(M/PM, S)$ from which we obtain the isomorphism

$$\mathrm{Hom}_R(M, S) \cong \mathrm{Hom}_R(M/PM, S) \cong \mathrm{Hom}_R(S \otimes_R M, S)$$

Note also that since R is a commutative ring, $S \otimes_R M$ is a homogeneous semisimple R -module (with all simple submodules isomorphic to S) because it is annihilated by P : $P(S \otimes_R M) = (PS) \otimes_R M = 0 \otimes_R M = 0$. If $S \otimes_R \mathbb{E}$ is exact, then it is splitting since the modules in this short exact sequence are semisimple. Since proper classes contain all splitting short exact sequences, this sequence $S \otimes_R \mathbb{E}$ must be in the proper class $\iota^{-1}(\{\text{all simple } R\text{-modules}\})$. Thus

$$0 \longrightarrow \mathrm{Hom}_R(S \otimes_R C, S) \longrightarrow \mathrm{Hom}_R(S \otimes_R B, S) \longrightarrow \mathrm{Hom}_R(S \otimes_R A, S) \longrightarrow 0$$

is exact. So by the above natural isomorphism $\mathrm{Hom}_R(M, S) \cong \mathrm{Hom}_R(S \otimes_R M, S)$, the sequence

$$0 \longrightarrow \mathrm{Hom}_R(C, S) \longrightarrow \mathrm{Hom}_R(B, S) \longrightarrow \mathrm{Hom}_R(A, S) \longrightarrow 0$$

is also exact.

Conversely suppose that for the simple R -module S , the sequence

$$0 \longrightarrow \mathrm{Hom}_R(C, S) \longrightarrow \mathrm{Hom}_R(B, S) \longrightarrow \mathrm{Hom}_R(A, S) \longrightarrow 0$$

is exact. So by the above natural isomorphism $\text{Hom}_R(M, S) \cong \text{Hom}_R(S \otimes_R M, S)$ the sequence

$$0 \longrightarrow \text{Hom}_R(S \otimes_R C, S) \longrightarrow \text{Hom}_R(S \otimes_R B, S) \xrightarrow{\alpha^*} \text{Hom}_R(S \otimes_R A, S) \longrightarrow 0$$

is also exact. This implies that $\alpha^* = \text{Hom}_R(\alpha, S)$ is an epimorphism, that is, if we have an R -module homomorphism $f : S \otimes_R A \longrightarrow S$, then there exists an R -module homomorphism $g : S \otimes_R B \longrightarrow S$ such that $g \circ \alpha = f$, that is, the following diagram is commutative:

$$\begin{array}{ccccc} 0 & \longrightarrow & S \otimes_R A & \xrightarrow{\alpha} & S \otimes_R B \\ & & \downarrow f & \swarrow g & \\ & & S & & \end{array}$$

We want to show that $0 \longrightarrow S \otimes_R A \xrightarrow{\alpha} S \otimes_R B \longrightarrow S \otimes_R C \longrightarrow 0$ is exact. It suffices to show that α is a monomorphism since tensor product is a right exact functor. Suppose to the contrary that α is not a monomorphism, so $\text{Ker}(\alpha) \neq 0$. Firstly note that $S \otimes_R A$ is a homogeneous semisimple R -module with all simple submodules isomorphic to S . Since $S \otimes_R A$ is a semisimple R -module, $\text{Ker}(\alpha)$ is a direct summand of $S \otimes_R A$, that is, $S \otimes_R A = \text{Ker}(\alpha) \oplus U$ for some submodule U of $S \otimes_R A$. Also $\text{Ker}(\alpha) = T \oplus V$ where T is a simple submodule of $\text{Ker}(\alpha)$ isomorphic to S and V is a homogeneous semisimple submodule of $\text{Ker}(\alpha)$ since $\text{Ker}(\alpha) \neq 0$ is a homogeneous semisimple R -module with all simple submodules isomorphic to S . Consider the following R -module homomorphisms: $S \otimes_R A \xrightarrow{\pi'} \text{Ker}(\alpha) \xrightarrow{\pi} T \xrightarrow{h} S$ where π' and π are projections onto the direct summands, and h is an isomorphism. Let $f' = h \circ \pi \circ \pi' : S \otimes_R A \longrightarrow S$. Then f' does not vanish on T , that is, $f'(x) \neq 0$ if $0 \neq x \in T$. By the hypothesis, as observed above, there exists an R -module homomorphism $g' : S \otimes_R B \longrightarrow S$ such that $f' = g' \circ \alpha$, that is the following diagram commutes:

$$\begin{array}{ccccc} 0 & \longrightarrow & S \otimes_R A' & \xrightarrow{\alpha} & S \otimes_R B \\ & & \downarrow f & \swarrow g' & \\ & & S & & \end{array}$$

But for any $0 \neq x \in T \leq \text{Ker}(\alpha)$, we have $f'(x) = g'(\alpha(x)) = g'(0) = 0$ which contradicts with $f'(x) \neq 0$. □

This proposition gives us the following corollary:

Corollary 3.3.3. *If R is a commutative ring, then*

$$\tau^{-1}(\{\text{all simple } R\text{-modules}\}) = \iota^{-1}(\{\text{all simple } R\text{-modules}\}).$$

By Fuchs (2012, Examples 3.1 and 3.2), we see that the proper class ${}_{R}\mathcal{P}\text{-Pure}$ is not equal to the proper class ${}_{R}\mathcal{N}\text{eat}$; furthermore neither is contained in the other. Although for the commutative rings R , neither of the concepts ${}_{R}\mathcal{P}$ -purity and neatness implies the other, we have some connections between them. Recall that the character module of an R -module M is defined as the right module $M^{\flat} = \text{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$.

Theorem 3.3.4. *(Fuchs (2012, Theorem 3.4)) Let R be a commutative ring. The exact sequence*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is ${}_{R}\mathcal{P}$ -pure-exact if and only if the sequence

$$0 \longrightarrow C^{\flat} \longrightarrow B^{\flat} \longrightarrow A^{\flat} \longrightarrow 0$$

of character modules is neat-exact.

Proof. The short exact sequence

$$\mathbb{E} : 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

being ${}_{R}\mathcal{P}$ -pure-exact, means that for every simple R -module S , $S \otimes_R \mathbb{E}$ is exact (also splitting). So for the additive contravariant functor $\text{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z})$, $S \otimes_R \mathbb{E}$ is exact if and only if $\text{Hom}_{\mathbb{Z}}(S \otimes_R \mathbb{E}, \mathbb{Q}/\mathbb{Z})$ is exact. It is well known that there is a natural isomorphism $\text{Hom}_{\mathbb{Z}}(A \otimes_R B, G) \cong \text{Hom}_R(B, \text{Hom}_{\mathbb{Z}}(A, G))$ for each *right* R -module A , R -module B and an abelian group G . For the proof of this well known isomorphism; see Osborne (2000, Theorem 2.4). Thus the sequence $\text{Hom}_{\mathbb{Z}}(S \otimes_R \mathbb{E}, \mathbb{Q}/\mathbb{Z})$ is exact if and only if

$$\begin{array}{c} 0 \longrightarrow \text{Hom}_R(S, \text{Hom}_{\mathbb{Z}}(C, \mathbb{Q}/\mathbb{Z})) \longrightarrow \text{Hom}_R(S, \text{Hom}_{\mathbb{Z}}(B, \mathbb{Q}/\mathbb{Z})) \\ \longleftarrow \text{Hom}_R(S, \text{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})) \longrightarrow 0 \end{array}$$

is exact, that is,

$$0 \longrightarrow \text{Hom}_R(S, C^b) \longrightarrow \text{Hom}_R(S, B^b) \longrightarrow \text{Hom}_R(S, A^b) \longrightarrow 0$$

is exact. The exactness of this last sequence for every simple R -module S means that, $0 \longrightarrow C^b \longrightarrow B^b \longrightarrow A^b \longrightarrow 0$ is neat-exact. \square

Theorem 3.3.5. (Fuchs (2012, Theorem 3.5)) *Let R be commutative ring such that the simple R -modules are finitely presented. The sequence*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is neat-exact if and only if the sequence

$$0 \longrightarrow C^b \longrightarrow B^b \longrightarrow A^b \longrightarrow 0$$

of character modules is ${}_R\mathcal{P}$ -pure-exact.

Proof. The sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is neat-exact if and only if for every simple R -module S , the sequence

$$0 \longrightarrow \text{Hom}_R(S, A) \longrightarrow \text{Hom}_R(S, B) \longrightarrow \text{Hom}_R(S, C) \longrightarrow 0$$

is exact. By Theorem 3.3.4 the following sequence

$$0 \longrightarrow (\text{Hom}_R(S, C))^b \longrightarrow (\text{Hom}_R(S, B))^b \longrightarrow (\text{Hom}_R(S, A))^b \longrightarrow 0$$

${}_R\mathcal{P}$ -pure-exact, means that for every simple R -module S ,

$$0 \longrightarrow S \otimes_R (\text{Hom}_R(S, C))^b \longrightarrow S \otimes_R (\text{Hom}_R(S, B))^b \longrightarrow S \otimes_R (\text{Hom}_R(S, A))^b \longrightarrow 0$$

is exact (also splitting). So for the additive contravariant functor $\text{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z})$,

$$0 \longrightarrow S \otimes_R (\text{Hom}_R(S, C))^b \longrightarrow S \otimes_R (\text{Hom}_R(S, B))^b \longrightarrow S \otimes_R (\text{Hom}_R(S, A))^b \longrightarrow 0$$

is exact if and only if

$$\begin{array}{c} 0 \longrightarrow \text{Hom}_{\mathbb{Z}}(\text{Hom}_R(S, C), \mathbb{Q}/\mathbb{Z}) \longrightarrow \text{Hom}_{\mathbb{Z}}(\text{Hom}_R(S, B), \mathbb{Q}/\mathbb{Z}) \\ \hspace{10em} \uparrow \\ \text{Hom}_{\mathbb{Z}}(\text{Hom}_R(S, A), \mathbb{Q}/\mathbb{Z}) \longrightarrow 0 \end{array}$$

is exact. As S is finitely presented we can make use of the natural isomorphism $S \otimes_R \text{Hom}_{\mathbb{Z}}(C, D) \cong \text{Hom}_{\mathbb{Z}}(\text{Hom}_R(S, C), D)$ with injective abelian group D (see Rotman (1979, Lemma 3.59)) to obtain that the exactness of the above sequence is equivalent to exactness of the sequence

$$0 \longrightarrow S \otimes_R C^b \longrightarrow S \otimes_R B^b \longrightarrow S \otimes_R A^b \longrightarrow 0$$

The exactness of this last sequence for every simple R -module S means that

$$0 \longrightarrow C^b \longrightarrow B^b \longrightarrow A^b \longrightarrow 0$$

is ${}_{\mathcal{R}}\mathcal{P}$ -pure-exact. This completes the proof. \square

By Theorems 3.3.4 and 3.3.5 we can easily obtain the following corollary.

Corollary 3.3.6. (by Lemma 2.4, Theorem 3.4 and Theorem 3.5 of Fuchs (2012)) *Let R be a commutative ring.*

- (1) *If ${}_{\mathcal{R}}\mathcal{P}\text{-Pure} \subseteq {}_{\mathcal{R}}\mathcal{N}\text{eat}$, then every simple R -module is finitely presented and ${}_{\mathcal{R}}\mathcal{P}\text{-Pure} = {}_{\mathcal{R}}\mathcal{N}\text{eat}$.*
- (2) *If ${}_{\mathcal{R}}\mathcal{N}\text{eat} \subseteq {}_{\mathcal{R}}\mathcal{P}\text{-Pure}$ and every simple R -module is finitely presented, then ${}_{\mathcal{R}}\mathcal{P}\text{-Pure} = {}_{\mathcal{R}}\mathcal{N}\text{eat}$.*

Proof. (1): Suppose ${}_{\mathcal{R}}\mathcal{P}\text{-Pure} \subseteq {}_{\mathcal{R}}\mathcal{N}\text{eat}$. The proper class ${}_{\mathcal{R}}\mathcal{P}\text{-Pure}$ is a flatly generated proper class and so it is inductively closed. Thus ${}_{\mathcal{R}}\mathcal{P}\text{-Pure}$ contains ${}_{\mathcal{R}}\mathcal{P}\text{ure}$ which is the smallest inductively closed proper class (see the notes for purity at the end of Section 2.5). So we obtain ${}_{\mathcal{R}}\mathcal{P}\text{ure} \subseteq {}_{\mathcal{R}}\mathcal{P}\text{-Pure} \subseteq {}_{\mathcal{R}}\mathcal{N}\text{eat}$. But, by Proposition 4.5.2, ${}_{\mathcal{R}}\mathcal{P}\text{ure} \subseteq {}_{\mathcal{R}}\mathcal{N}\text{eat}$ implies that every maximal ideal of R is finitely generated. So every simple R -module is finitely presented. Furthermore if the sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is neat-exact, then by Theorem 3.3.5 and by hypothesis the sequence

$$0 \longrightarrow C^b \longrightarrow B^b \longrightarrow A^b \longrightarrow 0$$

is also neat-exact. So by Theorem, 3.3.4 the sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is ${}_R\mathcal{P}$ -pure-exact. We obtain ${}_R\mathcal{N}eat \subseteq {}_R\mathcal{P}\text{-Pure}$. Hence ${}_R\mathcal{P}\text{-Pure} = {}_R\mathcal{N}eat$.

(2): Suppose ${}_R\mathcal{N}eat \subseteq {}_R\mathcal{P}\text{-Pure}$ and every simple R -module is finitely presented. Take any ${}_R\mathcal{P}$ -pure-exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

By Theorem 3.3.4 and by our hypothesis, the short exact sequence

$$0 \longrightarrow C^b \longrightarrow B^b \longrightarrow A^b \longrightarrow 0$$

is also ${}_R\mathcal{P}$ -pure-exact. Then by Theorem 3.3.5, the sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is neat-exact, that is, ${}_R\mathcal{P}\text{-Pure} \subseteq {}_R\mathcal{N}eat$. Hence ${}_R\mathcal{P}\text{-Pure} = {}_R\mathcal{N}eat$. □

CHAPTER FOUR

WHEN DO NEATNESS AND \mathcal{P} -PURITY COINCIDE?

In the preceding chapter, we have seen some properties of neatness and ${}_R\mathcal{P}$ -purity over a commutative ring R . By Corollary 3.3.6, if neatness and ${}_R\mathcal{P}$ -purity coincide over a commutative ring R , then the simple R -modules must be finitely presented. In general, over a commutative ring, Fuchs pointed out that these two proper classes ${}_R\mathcal{N}eat$ and ${}_R\mathcal{P}\text{-}Pure$ are *not* equal. In the first section, we just remind the definition and some properties of finitely presented R -modules. In Section 4.2, we will give the definition of ‘the’ Auslander-Bridger transpose of a finitely presented R -module and see some of its properties that we shall use. The main known result that we shall use is that a proper class projectively generated by a set of finitely presented R -modules is flatly generated by ‘the’ Auslander-Bridger transpose of these finitely presented R -modules (see for example Sklyarenko (1978, Section 8)); for completeness we shall also give the proof of that in Section 4.3. We shall prove, in Section 4.4 that for a commutative ring R , an Auslander-Bridger transpose of a finitely presented simple R -module S of projective dimension 1 is isomorphic to S . This enables us to prove the main result of our thesis in Section 4.5: if R is a commutative ring such that every maximal ideal of R is finitely generated and projective, then neatness and ${}_R\mathcal{P}$ -purity coincide. We still do not know if the converse necessarily holds over a commutative ring. But we will show that the converse holds for commutative rings with zero socle, that is, if R is a commutative ring with zero socle and if ${}_R\mathcal{N}eat = {}_R\mathcal{P}\text{-}Pure$, then every maximal ideal of R is finitely generated and projective. In the last section, we will give some examples of these commutative rings with zero socle.

4.1 Finitely Presented Modules

Let us start with the definition of finitely presented R -module.

Definition 4.1.1. An R -module M is said to be **finitely presented** if there is an exact sequence $0 \longrightarrow K \longrightarrow F \longrightarrow M \longrightarrow 0$ of R -modules such that F is finitely generated

and free, and K is finitely generated.

Proposition 4.1.2. (Bland, 2011, Lemma 5.3.13) *The following are equivalent for an R -module M :*

- (1) *There exists an exact sequence $F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$ where F_1 and F_0 are finitely generated free R -modules.*
- (2) *There exist positive integers m and n such that the sequence*

$$R^m \longrightarrow R^n \longrightarrow M \longrightarrow 0$$

is exact.

- (3) *M is finitely presented.*

Proof. (1) \Rightarrow (2) : Suppose there exists an exact sequence $F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$ where F_1 and F_0 are finitely generated free R -modules. Since F_0 and F_1 are finitely generated free, they have a finite basis, say with n and m elements respectively. Then $F_0 \cong R^n$ and $F_1 \cong R^m$, so we obtain an exact sequence $R^m \longrightarrow R^n \longrightarrow M \longrightarrow 0$.

(2) \Rightarrow (3) : Suppose we have an exact sequence $R^m \xrightarrow{f} R^n \xrightarrow{g} M \longrightarrow 0$ where m, n are positive integers. Let $K = \text{Ker}(g)$. By the exactness, we have $K = \text{Ker}(g) = \text{Im}(f)$. The submodule K of R^n is finitely generated since K is a homomorphic image of the finitely generated R -module R^m . Hence $0 \longrightarrow K \longrightarrow R^n \longrightarrow M \longrightarrow 0$ is a short exact sequence, where $K \longrightarrow R^n$ is the inclusion map. By our definition, this means that M is finitely presented.

(3) \Rightarrow (1) : Suppose M is finitely presented. Then by the definition, there exists a short exact sequence $0 \longrightarrow K \xrightarrow{u} F \xrightarrow{v} M \longrightarrow 0$ such that F is finitely generated and free, and K is finitely generated. There exists a finitely generated free R -module F_1 and an epimorphism $\phi : F_1 \longrightarrow K$ since K is finitely generated. For the homomorphism $u \circ \phi : F_1 \longrightarrow F$, we have $\text{Im}(u \circ \phi) = \text{Im}(u)$ and by the exactness we have $\text{Im}(u) = \text{Ker}(v)$. Thus we can obtain the desired exact sequence:

$$F_1 \xrightarrow{u \circ \phi} F \xrightarrow{v} M \longrightarrow 0$$

□

Note also the following elementary characterization for finitely presented R -modules.

Proposition 4.1.3. *An R -module M is finitely presented if and only if there exists an exact sequence $P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$ such that P_0 and P_1 are finitely generated and projective R -modules.*

Proof. (\Rightarrow) : Suppose M is finitely presented. By Proposition 4.1.2-(1) there exists an exact sequence $F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$ where F_1 and F_0 are finitely generated free R -modules. Since free R -modules are projective, we are done.

(\Leftarrow) : Suppose there exists an exact sequence $P_1 \xrightarrow{g} P_0 \xrightarrow{f} M \longrightarrow 0$ where P_0 and P_1 are finitely generated and projective R -modules. We want to show that M is a finitely presented R -module. Since P_0 is a finitely generated projective R -module, it is a direct summand of a finitely generated free R -module F_0 ; say, $F_0 = P_0 \oplus P'_0$ for some submodule P'_0 of F_0 . Define $\alpha_0 : F_0 \longrightarrow P_0$ to be the projection onto the direct summand P_0 . Similarly, there exists a finitely generated free R -module F_1 with $F_1 = P_1 \oplus P'_1$ for some submodule P'_1 of F_1 and with the projection $\alpha_1 : F_1 \longrightarrow P_1$. Construct the following sequence:

$$F_1 \oplus F_0 = (P_1 \oplus P'_1) \oplus F_0 \xrightarrow{\tilde{g}} P_0 \oplus P'_0 \xrightarrow{\tilde{f}} M \longrightarrow 0$$

by defining $\tilde{g}(p_1 + p'_1, a) = (g(p_1), \pi'(a))$ for $p_1 \in P_1, p'_1 \in P'_1, a \in F_0$ where $\pi' : F_0 \longrightarrow P'_0$ is the projection onto the direct summand P'_0 of F_0 . Define $\tilde{f}(p_0, p'_0) = f(p_0)$ for $p_0 \in P_0, p'_0 \in P'_0$. This sequence is exact because $\text{Ker}(\tilde{f}) = \text{Ker}(f) \oplus P'_0$ and $\text{Im}(\tilde{g}) = \text{Im}(g) \oplus P'_0$ are equal since $\text{Ker}(f) = \text{Im}(g)$, and \tilde{f} is onto since f is. Let $\tilde{F}_1 = F_1 \oplus F_0$. Then we have the desired exact sequence $\tilde{F}_1 \xrightarrow{\tilde{g}} F_0 \xrightarrow{\tilde{f}} M \longrightarrow 0$ where F_0 and \tilde{F}_1 are free and finitely generated R -modules. So by Proposition 4.1.2, M is a finitely presented R -module. \square

The exact sequences which are given in the above propositions are called **presentations** of the finitely presented R -module M . More precisely, for a finitely presented R -module M , an exact sequence

$$F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

where F_0 and F_1 are finitely generated free R -modules is called a **free presentation** of M , and an exact sequence

$$P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

where P_0 and P_1 are finitely generated projective R -modules is called a **projective presentation** of M . Similarly one defines finitely presented right R -modules.

Note also the following well known properties of finitely presented R -modules:

Proposition 4.1.4. (Schnauel's Lemma, by for example Rotman (2009, Proposition 3.12))

Given exact sequences

$$0 \longrightarrow K \xrightarrow{i} P \xrightarrow{\sigma} M \longrightarrow 0$$

and

$$0 \longrightarrow K' \xrightarrow{i'} P' \xrightarrow{\sigma'} M \longrightarrow 0$$

where P and P' are projective R -modules, then there is an isomorphism

$$K \oplus P' \cong K' \oplus P.$$

Proof. Consider the diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \xrightarrow{i} & P & \xrightarrow{\sigma} & M & \longrightarrow & 0 \\ & & \downarrow \alpha & & \downarrow \beta & & \parallel 1_M & & \\ 0 & \longrightarrow & K' & \xrightarrow{i'} & P' & \xrightarrow{\sigma'} & M & \longrightarrow & 0 \end{array}$$

Since P is projective R -module, there is a map $\beta : P \longrightarrow P'$ with $\sigma' \circ \beta = \sigma$; that is, the right square in the diagram commutes. A diagram chase shows that there is a map $\alpha : K \longrightarrow K'$ making the other square commute (see Rotman (2009, Proposition 2.71)).

This commutative diagram with exact rows gives an exact sequence

$$0 \longrightarrow K \xrightarrow{\theta} P \oplus K' \xrightarrow{\psi} P' \longrightarrow 0,$$

where $\theta(x) = (i(x), \alpha(x))$ and $\psi(u, x') = \beta(u) - i'(x')$, for all $x \in K$, $u \in P$, and $x' \in K'$. Exactness of this sequence is straightforward calculation. So by Proposition 2.3.1, this sequence splits because P' is projective R -module. Thus $K \otimes_{P'} \cong P \otimes_R K'$ \square

Corollary 4.1.5. (Rotman (2009, Corollary 3.13)) *If M is finitely presented R -module and*

$$0 \longrightarrow K \longrightarrow F \longrightarrow M \longrightarrow 0$$

is an exact sequence, where F is a finitely generated free R -module, then K is a finitely generated R -module.

Proof. Since M is finitely presented R -module, there is an exact sequence

$$0 \longrightarrow K' \longrightarrow F' \longrightarrow M \longrightarrow 0$$

with F' free and with both F' and K' finitely generated R -modules. By Proposition 4.1.4, $K \oplus F' \cong K' \oplus F$. Now $K' \oplus F$ is finitely generated R -module because both summands are, so that the left side is also finitely generated. But K , being a direct summand, is a homomorphic image of $K \oplus F'$, and hence it is finitely generated. \square

Before the ending this section note that the following well known results;
 Direct sum of finitely many finitely presented modules is finitely presented.
 Direct summand of a finitely presented module is finitely presented.

4.2 The Auslander-Bridger Transpose of Finitely Presented Modules

Let us start this section with the definition of Auslander-Bridger transpose.

Definition 4.2.1. Let M be a finitely presented R -module. Take a projective presentation of it, that is, take an exact sequence

$$\gamma: \quad P_1 \xrightarrow{f} P_0 \xrightarrow{g} M \longrightarrow 0$$

where P_0 and P_1 are finitely generated projective R -modules. Apply the functor $(-)^* = \text{Hom}_R(-, R)$ to this projective presentation:

$$0 \longrightarrow \text{Hom}_R(M, R) \xrightarrow{g^*} \text{Hom}_R(P_0, R) \xrightarrow{f^*} \text{Hom}_R(P_1, R)$$

Fill the right side of this sequence of *right R*-modules by the module $\text{Tr}_\gamma(M) = \text{Coker}(f^*) = P_1^*/\text{Im}(f^*)$ to obtain the exact sequence

$$\gamma^* : \quad P_0^* \xrightarrow{f^*} P_1^* \xrightarrow{\sigma} \text{Tr}_\gamma(M) \longrightarrow 0, \quad (4.2.1)$$

where σ is the canonical epimorphism. Since P_0^* and P_1^* are finitely generated projective *right R*-modules, the exact sequence (4.2.1) is a projective presentation for the finitely presented *right R*-module $\text{Tr}_\gamma(M)$ which is called the **Auslander-Bridger transpose** of the finitely presented *R*-module M with respect to the projective presentation γ .

The *meaning of the transpose* is comes from a *free presentation* of a finitely presented *R*-module. Let we take a free presentation F of a finitely presented *R*-module M , that is,

$$F : \quad F_1 \xrightarrow{f} F_0 \xrightarrow{g} M \longrightarrow 0$$

where $F_1 \cong R^k$ and $F_0 \cong R^n$. We can denote the *R*-module homomorphism from R^k to R^n is given by an $n \times k$ rectangular matrix \mathbf{A} . It can be easily check that the *R*-module homomorphism f^* given by an $k \times n$ rectangular matrix \mathbf{A}^T . Hence the $\text{Coker}(\mathbf{A}^T) \cong \text{Coker}(f^*) = \text{Tr}_F(M)$.

See Auslander & Bridger (1969), Auslander et al. (1995, §IV.1) and Mašek (2000).

Definition 4.2.2. Two *R*-modules A and B are said to be **projectively equivalent** if there exist projective *R*-modules P and Q such that $A \oplus P \cong B \oplus Q$. Denote this by $A \approx B$.

Proposition 4.2.3. (Mašek (2000, p. 5786)) *The relation \approx is an equivalence relation on the class of (finitely generated) *R*-modules.*

Proof. Let A be an *R*-module. Then of course $A \oplus P \cong A \oplus P$ for any projective *R*-module P . So \approx is reflexive. Let A and B be *R*-modules. If $A \approx B$, then it is clear that $B \approx A$ with the same projective *R*-modules. Let A , B and C be *R*-modules. If $A \approx B$ and $B \approx C$, then there exist projective *R*-modules P_1 , P_2 , P_3 and P_4 such that $A \oplus P_1 \cong B \oplus P_2$ and $B \oplus P_3 \cong C \oplus P_4$. Then it is clear that $A \oplus P_1 \oplus P_3 \cong B \oplus P_2 \oplus P_3 \cong$

$B \oplus P_3 \oplus P_2 \cong C \oplus P_4 \oplus P_2$. Hence $A \approx C$. So \approx is transitive. This shows that \approx is an equivalence relation on the class of R -modules, and also on the class of finitely generated R -modules. \square

We shall give the detailed proof of the following result by following the proof given in Mašek (2000, Proposition 4):

Theorem 4.2.4. (Auslander & Bridger (1969), Auslander et al. (1995), Mašek (2000))
An Auslander-Bridger transpose of a finitely presented R -module M is unique up to projective equivalence, that is, if γ and ρ are two projective presentations of a finitely presented R -module M , then $\text{Tr}_\gamma(M) \approx \text{Tr}_\rho(M)$.

Proof. Let M be a finitely presented R -module and let

$$\gamma: \quad P_1 \xrightarrow{u} P_0 \xrightarrow{f} M \longrightarrow 0$$

and

$$\rho: \quad Q_1 \xrightarrow{v} Q_0 \xrightarrow{g} M \longrightarrow 0$$

be two projective presentations of M . We say that γ **strictly dominates** ρ if there are R -module homomorphisms $\phi_i: P_i \rightarrow Q_i$ for $i = 0, 1$ satisfying the following three conditions.

- (i) ϕ_i is surjective, for $i = 0, 1$.
- (ii) ϕ_0 is a lifting of 1_M and ϕ_1 is a lifting of ϕ_0 , that is, the following diagram commutes:

$$\begin{array}{ccccccc} P_1 & \xrightarrow{u} & P_0 & \xrightarrow{f} & M & \longrightarrow & 0 \\ \downarrow \phi_1 & & \downarrow \phi_0 & & \parallel & & \\ Q_1 & \xrightarrow{v} & Q_0 & \xrightarrow{g} & M & \longrightarrow & 0 \end{array}$$

- (iii) The R -module homomorphism $\tilde{u}: K_1 \rightarrow K_0$ induced by u is surjective where $K_i = \text{Ker}(\phi_i)$ for $i = 0, 1$.

In other words, we should have a commutative diagram with exact rows and columns:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & (4.2.2) \\
 & & \downarrow & & \downarrow & & \\
 & & K_1 & \xrightarrow{\tilde{u}} & K_0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \\
 & & P_1 & \xrightarrow{u} & P_0 & \xrightarrow{f} & M \longrightarrow 0 \\
 & & \downarrow \phi_1 & & \downarrow \phi_0 & & \parallel \\
 & & Q_1 & \xrightarrow{v} & Q_0 & \xrightarrow{g} & M \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
 \end{array}$$

We prove the proposition in two steps:

Step 1: If γ strictly dominates ρ , then $\text{Tr}_\gamma(M) \approx \text{Tr}_\rho(M)$

Step 2: If γ and ρ are given two projective presentations of M , then there exists a projective presentation η of M such that η strictly dominates γ and ρ , that is, $\text{Tr}_\eta(M) \approx \text{Tr}_\gamma(M)$ and $\text{Tr}_\eta(M) \approx \text{Tr}_\rho(M)$. Since \approx is an equivalence relation on R -modules, we obtain $\text{Tr}_\gamma(M) \approx \text{Tr}_\rho(M)$.

Proof of Step 1: Assume that γ strictly dominates ρ . Then we have the diagram (4.2.2). Since Q_i 's are projective, the columns are split exact, $P_i \cong Q_i \oplus K_i$, and so K_i 's are projective (and finitely generated since P_i 's are finitely generated). Therefore $K_1 \xrightarrow{\tilde{u}} K_0 \longrightarrow 0$ splits since K_0 projective. Dualizing the diagram (4.2.2), that is, applying the left exact functor $(-)^* = \text{Hom}_R(-, R)$ to the diagram (4.2.2) we obtain the following diagram with exact rows:

$$\begin{array}{ccccccc}
 & & & & 0 & & 0 \\
 & & & & \downarrow & & \downarrow \\
 0 & \longrightarrow & M^* & \xrightarrow{g^*} & Q_0^* & \xrightarrow{v^*} & Q_1^* \\
 & & \parallel & & \downarrow \phi_0^* & & \downarrow \phi_1^* \\
 0 & \longrightarrow & M^* & \xrightarrow{f^*} & P_0^* & \xrightarrow{u^*} & P_1^* \\
 & & & & \downarrow & & \downarrow \\
 & & & & 0 & \xrightarrow{\tilde{u}^*} & K_1^*
 \end{array}$$

Since the columns in the diagram (4.2.2) are split exact, they remain split exact after we apply the functor $(-)^* = \text{Hom}_R(-, R)$. So we have the following diagram with exact columns. Similarly since $K_1 \xrightarrow{\tilde{u}} K_0 \longrightarrow 0$ is splitting, $0 \longrightarrow K_0^* \xrightarrow{\tilde{u}^*} K_1^*$ is also

splitting, and K^* and K_1^* finitely generated projective R -modules.

$$\begin{array}{ccccccc}
& & & 0 & & 0 & \\
& & & \downarrow & & \downarrow & \\
0 & \longrightarrow & M^* & \xrightarrow{g^*} & Q_0^* & \xrightarrow{v^*} & Q_1^* \\
& & \parallel & & \downarrow \phi_0^* & & \downarrow \phi_1^* \\
0 & \longrightarrow & M^* & \xrightarrow{f^*} & P_0^* & \xrightarrow{u^*} & P_1^* \\
& & & & \downarrow & & \downarrow \\
& & & & K_0^* & \xrightarrow{\tilde{u}^*} & K_1^* \\
& & & & \downarrow & & \downarrow \\
& & & & 0 & & 0
\end{array}$$

Complete the right side with $\text{Coker}(v^*)$, $\text{Coker}(u^*)$ and $\text{Coker}(\tilde{u}^*)$ with natural epimorphism onto them to obtain the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
& & & 0 & & 0 & & 0 \\
& & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & M^* & \xrightarrow{g^*} & Q_0^* & \xrightarrow{v^*} & Q_1^* & \dashrightarrow \text{Coker}(v^*) \dashrightarrow 0 \\
& & \parallel & & \downarrow \phi_0^* & & \downarrow \phi_1^* & & \downarrow \\
0 & \longrightarrow & M^* & \xrightarrow{f^*} & P_0^* & \xrightarrow{u^*} & P_1^* & \dashrightarrow \text{Coker}(u^*) \dashrightarrow 0 \\
& & & & \downarrow & & \downarrow & & \downarrow \\
& & & & K_0^* & \xrightarrow{\tilde{u}^*} & K_1^* & \dashrightarrow \text{Coker}(\tilde{u}^*) \dashrightarrow 0 \\
& & & & \downarrow & & \downarrow & & \downarrow \\
& & & & 0 & & 0 & & 0
\end{array}$$

The exactness of the last column, follows by the Snake-Lemma (Ker-Coker sequence) applied to the columns of the above diagram. By the definition of the Auslander-Bridger transpose, $\text{Tr}_\rho(M) = \text{Coker}(v^*)$ and $\text{Tr}_\gamma(M) = \text{Coker}(u^*)$. Let $K = \text{Coker}(\tilde{u}^*)$. The last column above exact, that is, the following sequence is exact:

$$0 \longrightarrow \text{Coker}(v^*) = \text{Tr}_\rho(M) \longrightarrow \text{Coker}(u^*) = \text{Tr}_\gamma(M) \longrightarrow \text{Coker}(\tilde{u}^*) = K \longrightarrow 0$$

Since \tilde{u}^* splits, the last row in the above diagram is splitting and so $K_1^* \cong K_0^* \oplus K$. Then K is projective since K_1^* is projective. Thus the short exact sequence

$$0 \longrightarrow \text{Tr}_\rho(M) \longrightarrow \text{Tr}_\gamma(M) \longrightarrow K \longrightarrow 0$$

is splitting since K is projective. Then $\text{Tr}_\gamma(M) \cong \text{Tr}_\rho(M) \oplus K$, and so $\text{Tr}_\gamma(M) \oplus 0 \cong \text{Tr}_\rho(M) \oplus K$ where 0 and K are projective R -modules. Hence $\text{Tr}_\gamma(M) \approx \text{Tr}_\rho(M)$. This

ends the proof of the first step.

Proof of Step 2: Now let γ and ρ be two projective presentations of M , which are as given in the beginning of the proof. We shall construct a new projective presentation η which strictly dominates both γ and ρ . Define $h : P_0 \oplus Q_0 \rightarrow M$ by $h(p_0, q_0) = f(p_0) + g(q_0)$ for all $(p_0, q_0) \in P_0 \oplus Q_0$. Clearly h is surjective since f and g are. Let $\alpha : E \rightarrow P_0 \oplus Q_0$ be an R -module homomorphism from a finitely generated projective R -module E onto $\text{Ker}(h)$, that is, $\text{Im}(\alpha) = \text{Ker}(h)$. Such an α exists because M is finitely presented, so $\text{Ker}(h)$ must be finitely generated by Corollary 4.1.5. Since $\text{Ker}(h)$ is a submodule of $P_0 \oplus Q_0$, let us embed $\text{Ker}(h)$ to the R -module $P_0 \oplus Q_0$ with an inclusion homomorphism i . We have the following commutative diagram with exact rows:

$$\begin{array}{ccccccc} E & \xrightarrow{\alpha} & P_0 \oplus Q_0 & \xrightarrow{h} & M & \longrightarrow & 0 \\ & & \downarrow \tilde{\alpha} & \nearrow i & & & \\ & & \text{Ker}(h) & & & & \end{array}$$

Extend the projective presentations γ and ρ one more step in a projective resolution of M , that is, take homomorphisms $u' : P_2 \rightarrow P_1$, $v' : Q_2 \rightarrow Q_1$ such that $\text{Im}(u') = \text{Ker}(u)$ and $\text{Im}(v') = \text{Ker}(v)$ where P_2 and Q_2 are projective R -modules; these give the following exact sequences:

$$P_2 \xrightarrow{u'} P_1 \xrightarrow{u} P_0 \xrightarrow{f} M \longrightarrow 0$$

and

$$Q_2 \xrightarrow{v'} Q_1 \xrightarrow{v} Q_0 \xrightarrow{g} M \longrightarrow 0$$

Define the projective presentation η of M as follows:

$$\eta : \quad E \oplus P_2 \oplus Q_2 \xrightarrow{w} P_0 \oplus Q_0 \xrightarrow{h} M \longrightarrow 0$$

where $w(e, p_2, q_2) = \alpha(e)$ for all $(e, p_2, q_2) \in E \oplus P_2 \oplus Q_2$. It is clear that $\text{Im}(w) = \text{Im}(\alpha) = \text{Ker}(h)$, so η is exact. We claim that η strictly dominates γ and ρ . Since the construction of η is symmetric with respect to γ and ρ , we will only show that η strictly dominates ρ . Lift 1_M to $\phi_0 : P_0 \rightarrow Q_0$ (since P_0 is projective, for the R -module homomorphism $1_M \circ f : P_0 \rightarrow M$, there exists an R -module homomorphism $\phi_0 : P_0 \rightarrow Q_0$ such that $g \circ \phi_0 = f \circ 1_M$), that is, obtain the following commutative

diagram with exact rows:

$$\begin{array}{ccccccc} \gamma: P_1 & \xrightarrow{u} & P_0 & \xrightarrow{f} & M & \longrightarrow & 0 \\ & & \downarrow \phi_0 & & \parallel & & \\ \rho: Q_1 & \xrightarrow{v} & Q_0 & \xrightarrow{g} & M & \longrightarrow & 0 \end{array}$$

Define $\chi_0 : P_0 \oplus Q_0 \longrightarrow Q_0$ by $\chi_0(p_0, q_0) = \phi_0(p_0) + q_0$ for all $(p_0, q_0) \in P_0 \oplus Q_0$. For each $q_0 \in Q_0$, we have $\chi_0(0, q_0) = \phi_0(0) + q_0 = q_0$, so χ_0 is surjective, and also $g \circ \chi_0 = h$ since $g \circ \chi_0(p_0, q_0) = g(\phi_0(p_0) + q_0) = g(\phi_0(p_0)) + g(q_0) = f(p_0) + g(q_0) = h(p_0, q_0)$. Hence we have the following commutative diagram:

$$\begin{array}{ccccccc} E & \xrightarrow{\alpha} & P_0 \oplus Q_0 & \xrightarrow{h} & M & \longrightarrow & 0 \\ & & \downarrow \chi_0 & & \parallel & & \\ Q_1 & \xrightarrow{v} & Q_0 & \xrightarrow{g} & M & \longrightarrow & 0 \end{array}$$

Lift χ_0 to $\delta : E \longrightarrow Q_1$, that is, construct the following commutative diagram:

$$\begin{array}{ccccccc} E & \xrightarrow{\alpha} & P_0 \oplus Q_0 & \xrightarrow{h} & M & \longrightarrow & 0 \\ \downarrow \delta & & \downarrow \chi_0 & & \parallel & & \\ Q_1 & \xrightarrow{v} & Q_0 & \xrightarrow{g} & M & \longrightarrow & 0 \end{array}$$

We can do this as is done with projective resolutions in the following way. Let $\eta : E \longrightarrow \text{Im}(v) = \text{Ker}(g)$ be defined by $\eta(x) = (\chi_0 \circ \alpha)(x)$ for all $x \in E$; it is well defined because $(\chi_0 \circ \alpha)(x) \in \text{Im}(v)$ for all $x \in E$ since $g(\chi_0 \circ \alpha)(x) = (h \circ \alpha)(x) = 0$ as $g \circ \chi_0 = h$ and $h \circ \alpha = 0$. We have then the following diagram where $\tilde{v}(x) = v(x)$ for all $x \in Q_1$

$$\begin{array}{ccc} & E & \\ \delta \swarrow & \downarrow \eta & \\ Q_1 & \xrightarrow{\tilde{v}} & \text{Im}(v) \longrightarrow 0 \end{array}$$

Since E is projective, there exists an R -module homomorphism $\delta : E \longrightarrow Q_1$ such that $\tilde{v} \circ \delta = \eta$. So for all $x \in E$, $(\chi_0 \circ \alpha)(x) = \eta(x) = (\tilde{v} \circ \delta)(x) = \tilde{v}(\delta(x)) = v(\delta(x)) = (v \circ \delta)(x)$. This gives $\chi_0 \circ \alpha = v \circ \delta$ as required. Finally define $\chi_1 : E \oplus P_2 \oplus Q_2 \longrightarrow Q_1$ by $\chi_1(e, p_2, q_2) = \delta(e) + v'(q_2)$ for all $(e, p_2, q_2) \in E \oplus P_2 \oplus Q_2$. Then $v \circ \chi_1(e, p_2, q_2) = v(\delta(e) + v'(q_2)) = v(\delta(e)) + v \circ v'(q_2) = \chi_0 \circ \alpha(e) + 0(q_2) = \chi_0 \circ w(e, p_2, q_2)$ since $\alpha(e) = w(e, p_2, q_2)$, that is, η and ρ sit in the following commutative diagram:

$$\begin{array}{ccccccc} E \oplus P_2 \oplus Q_2 & \xrightarrow{w} & P_0 \oplus Q_0 & \xrightarrow{h} & M & \longrightarrow & 0 \\ \downarrow \chi_1 & & \downarrow \chi_0 & & \parallel & & \\ Q_1 & \xrightarrow{v} & Q_0 & \xrightarrow{g} & M & \longrightarrow & 0 \end{array}$$

We want to show that η strictly dominates ρ . By the above commutative diagram, the second condition of strictly domination holds. So we must show that the first and the third condition of strictly domination hold. For the first condition, we must show that χ_0 and χ_1 are surjective. For the third condition, we must show that the induced R -module homomorphism $\bar{w} : \text{Ker}(\chi_0) \longrightarrow \text{Ker}(\chi_1)$ is also surjective. Above we have seen that χ_0 is surjective. Let's show that χ_1 is also surjective. Take any $q_1 \in Q_1$. Then $v(q_1) \in Q_0, (0, v(q_1)) \in P_0 \oplus Q_0$ and $h(0, v(q_1)) = f(0) + g \circ v(q_1) = 0$, so $(0, v(q_1)) \in \text{Ker}(h) = \text{Im}(\alpha)$ which implies that there exists an element $e \in E$ such that $\alpha(e) = (0, v(q_1))$. Then $v \circ \delta(e) = \chi_0 \circ \alpha(e) = \chi_0(0, v(q_1)) = \phi_0(0) + v(q_1) = v(q_1)$, so $q_1 - \delta(e) \in \text{Ker}(v) = \text{Im}(v')$ which implies that there exists an element $q_2 \in Q_2$ such that $v'(q_2) = q_1 - \delta(e)$. Thus $q_1 = \delta(e) + v'(q_2) = \chi_1(e, 0, q_2) \in \text{Im}(\chi_1)$. This shows that χ_1 is surjective. Let's now show the last condition of strictly domination property to finish the proof. Let $K_i = \text{Ker}(\chi_i)$ for $i = 0, 1$. The homomorphism $\bar{w} : K_1 \longrightarrow K_0$ which is induced by w must be surjective, that is, we want to construct the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
& & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \\
& & K_1 & \xrightarrow{\bar{w}} & K_0 & \dashrightarrow & 0 \\
& & \downarrow & & \downarrow & & \\
E \oplus P_2 \oplus Q_2 & \xrightarrow{w} & P_0 \oplus Q_0 & \xrightarrow{h} & M & \longrightarrow & 0 \\
& & \downarrow \chi_1 & & \downarrow \chi_0 & & \parallel \\
Q_1 & \xrightarrow{v} & Q_0 & \xrightarrow{g} & M & \longrightarrow & 0
\end{array}$$

Let $(p_0, q_0) \in K_0$. By commutativity of the diagram $g \circ \chi_0 = h$ and so $K_0 = \text{Ker}(\chi_0) \leq \text{Ker}(h)$. We also have $\text{Ker}(h) = \text{Im}(\alpha)$ by the construction of α . Thus $(p_0, q_0) \in \text{Im}(\alpha)$ which means that there exists an element $e \in E$ such that $\alpha(e) = (p_0, q_0)$. We also have $v \circ \delta(e) = \chi_0 \circ \alpha(e) = \chi_0(p_0, q_0) = 0$ which means $\delta(e) \in \text{Ker}(v) = \text{Im}(v')$ and so there exists an element $q_2 \in Q_2$ such that $v'(q_2) = \delta(e)$. Thus $\chi_1(e, 0, -q_2) = \delta(e) + v'(-q_2) = \delta(e) - v'(q_2) = 0$. Hence $(e, 0, -q_2) \in \text{Ker}(\chi_1) = K_1$ and $\bar{w}(e, 0, -q_2) = w(e, 0, -q_2) = \alpha(e) = (p_0, q_0)$. We showed that \bar{w} is onto as required. Hence η strictly dominates ρ , and similarly η is strictly dominates γ . These imply by the first step of

the proof that

$$\mathrm{Tr}_\rho(M) \approx \mathrm{Tr}_\eta(M) \text{ and } \mathrm{Tr}_\gamma(M) \approx \mathrm{Tr}_\eta(M)$$

By Proposition 4.2.3, the relation \approx is an equivalence relation; so we obtain $\mathrm{Tr}_\rho(M) \approx \mathrm{Tr}_\gamma(M)$. \square

See Auslander & Bridger (1969) or Mašek (2000, §1) for some other properties that we shall use. We shall just write $\mathrm{Tr}(M)$ for an Auslander-Bridger transpose of the finitely presented R -module M keeping in mind that this is unique up to projective equivalence. Similarly, the Auslander-Bridger transpose of *right* R -modules are defined and with the above notation for γ and γ^* in the beginning of the section, we obtain $\mathrm{Tr}_{\gamma^*}(\mathrm{Tr}_\gamma(M)) \cong M$ (because $P_i \cong P_i^{**}$ and u^{**} is identified canonically with u). If we drop the subscript for the dependent projective presentations γ^* and γ in $\mathrm{Tr}_{\gamma^*}(\mathrm{Tr}_\gamma(M)) \cong M$, then we can only say that $\mathrm{Tr}(\mathrm{Tr}(M))$ is projectively equivalent to M . Note that $\mathrm{Tr}_{\gamma^*}(\mathrm{Tr}_\gamma(M)) = \mathrm{Coker}(f^{**})$ is defined by the exact sequence

$$\gamma^{**} : \quad P_1^{**} \xrightarrow{f^{**}} P_0^{**} \xrightarrow{\sigma'} \mathrm{Tr}_{\gamma^*}(\mathrm{Tr}_\gamma(M)) \longrightarrow 0,$$

where σ' is the canonical epimorphism. On the other hand, applying the functor $(-)^*$ to the exact sequence:

$$\gamma^* : \quad P_0^* \xrightarrow{f^*} P_1^* \xrightarrow{\sigma} \mathrm{Tr}_\gamma(M) \longrightarrow 0,$$

we obtain the following exact sequence:

$$0 \longrightarrow (\mathrm{Tr}_\gamma(M))^* \xrightarrow{\sigma^*} P_1^{**} \xrightarrow{f^{**}} P_0^{**}$$

We have natural isomorphisms $P \cong P^{**}$ for every finitely generated projective R -module P , see Theorem 4.2.7, so we obtain $(\mathrm{Tr}_\gamma(M))^* \cong \mathrm{Im}(\sigma^*) = \mathrm{Ker}(f^{**}) \cong \mathrm{Ker}(f)$. This proves:

Proposition 4.2.5. (*Angeleri Hügel & Bazzoni, 2010, Lemma 6.1-(2)*) *For a finitely presented R -module M , $\mathrm{pd}(M) \leq 1$ if and only if there exists a projective presentation*

$$\gamma : \quad P_1 \xrightarrow{f} P_0 \xrightarrow{g} M \longrightarrow 0$$

of M such that (f is monic and) $(\mathrm{Tr}_\gamma(M))^ = 0$.*

The properties of the Auslander-Bridger transpose that we shall use are the following:

Theorem 4.2.6. (Sklyarenko, 1978, Proposition 5.1, Remarks 5.1 and 5.2) *Let M be a finitely presented R -module and let γ be a projective presentation of M :*

$$\gamma: \quad P_1 \xrightarrow{f} P_0 \xrightarrow{g} M \longrightarrow 0$$

- (1) *For every R -module N , there is a monomorphism $\text{Ext}_R^1(M, N) \longrightarrow \text{Tr}_\gamma(M) \otimes_R N$ and an epimorphism $\text{Hom}_R(\text{Tr}_\gamma(M), N) \longrightarrow \text{Tor}_1^R(M, N)$. Both are natural in N .*
- (2) *If $\text{pd}(M) \leq 1$, then the R -module homomorphism $f : P_1 \longrightarrow P_0$ in the above projective presentation γ can be taken to be a monomorphism and in this case the monomorphism and epimorphism in the previous part become isomorphisms. Moreover by taking $N = R$, we obtain*

$$\text{Tr}_\gamma(M) \cong \text{Ext}_R^1(M, R) \quad \text{and} \quad (\text{Tr}_\gamma(M))^* = \text{Hom}_R(\text{Tr}_\gamma(M), R) = 0$$

for the projective presentation γ of M where the R -module homomorphism $f : P_1 \longrightarrow P_0$ is a monomorphism.

- (3) *If $\text{pd}(M) \leq 1$, then $\text{Tr}(M)$ is projectively equivalent to $\text{Ext}_R^1(M, R)$.*
- (4) *If $M^* = \text{Hom}_R(M, R) = 0$, then in the projective presentation*

$$\gamma^* : \quad P_0^* \xrightarrow{f^*} P_1^* \xrightarrow{\sigma} \text{Tr}_\gamma(M) \longrightarrow 0,$$

we necessarily have that $f^ : P_0^* \longrightarrow P_1^*$ is a monomorphism and so $\text{pd}(\text{Tr}_\gamma(M)) \leq 1$ which implies*

$$M \cong \text{Tr}_{\gamma^*}(\text{Tr}_\gamma(M)) \cong \text{Ext}_R^1(\text{Tr}_\gamma(M), R).$$

- (5) *If M is not projective, then $\text{Tr}_\gamma(M) \neq 0$.*

Note that the isomorphisms above containing Ext or Tor are abelian group isomorphisms but because of the naturality in (1), when the ring R is a commutative ring, then all these isomorphisms become R -module isomorphisms. Even when R is not commutative, $\text{Ext}_R^1(-, R)$ has a right or left R -module structure using the bimodule structure ${}_R R_R$ and the isomorphisms containing those are left or right R -module isomorphisms.

Proof. Let M be finitely presented R -module and let γ be a projective presentation of M :

$$\gamma: \quad P_1 \xrightarrow{f} P_0 \xrightarrow{g} M \longrightarrow 0$$

Proof of (1): Let N be a R -module. Let us extend this projective presentation γ to a projective resolution of M , that is, we obtain the following commutative diagram with exact row:

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & P_3 & \longrightarrow & P_2 & \xrightarrow{\tilde{\phi}} & P_1 & \xrightarrow{f} & P_0 & \xrightarrow{g} & M & \longrightarrow & 0 \\ & & & & \downarrow \phi & \nearrow & & & & & & & \\ & & & & \text{Ker}(f) & & & & & & & & \end{array}$$

For $\text{Ext}_R^1(M, N)$, apply $\text{Hom}_R(-, N)$ to this projective resolution of M , to obtain the sequence

$$0 \longrightarrow \text{Hom}_R(M, N) \xrightarrow{g^*} \text{Hom}_R(P_0, N) \xrightarrow{f^*} \text{Hom}_R(P_1, N) \xrightarrow{\tilde{\phi}^*} \text{Hom}_R(P_2, N) \longrightarrow \cdots$$

By the definition of $\text{Ext}_R^1(M, N)$ using a projective resolution of M , we have $\text{Ext}_R^1(M, N) = \text{Ker}(\tilde{\phi}^*) / \text{Im}(f^*) \leq \text{Hom}_R(P_1, N) / \text{Im}(f^*) \cong (P_1^* \otimes_R N) / (\text{Im}(f^* \otimes 1_N)) \cong \text{Tr}_\gamma(M) \otimes_R N$. The first isomorphism $\text{Hom}_R(P_1, N) / \text{Im}(f^*) \cong (P_1^* \otimes_R N) / (\text{Im}(f^* \otimes 1_N))$ is coming from the following commutative diagram and the five lemma

$$\begin{array}{ccc} \text{Hom}_R(P_0, N) & \xrightarrow{f^*} & \text{Hom}_R(P_1, N) \\ \cong \uparrow & & \cong \uparrow \\ P_0^* \otimes_R N & \xrightarrow{f^* \otimes 1_N} & P_1^* \otimes_R N \end{array}$$

where the isomorphisms in the above diagram is natural, that is, if we say the isomorphism ζ from $P^* \otimes_R M$ to $\text{Hom}_R(P, M)$, then it is defined by $[\zeta(f \otimes m)](a) = f(a)m$ for all $f \in P^*$, for all $m \in M$ and for all $a \in P$. Since the sequence

$$P_0^* \xrightarrow{f^*} P_1^* \xrightarrow{\sigma} \text{Tr}_\gamma(M) \longrightarrow 0$$

is exact, applying the right exact functor $- \otimes_R N$, we obtain the exact sequence

$$P_0^* \otimes_R N \xrightarrow{f^* \otimes 1_N} P_1^* \otimes_R N \xrightarrow{\sigma \otimes 1_N} \text{Tr}_\gamma(M) \otimes_R N \longrightarrow 0$$

So, $\text{Tr}_\gamma(M) \otimes_R N = \text{Im}(\sigma \otimes 1_N) \cong (P_1^* \otimes_R N) / \text{Im}(f^* \otimes 1_N)$. To obtain the epimorphism $\text{Hom}_R(\text{Tr}_\gamma(M), N) \longrightarrow \text{Tor}_1^R(M, N)$, apply the left exact functor $\text{Hom}_R(-, N)$ to the exact sequence

$$\gamma^* : P_0^* \xrightarrow{f^*} P_1^* \xrightarrow{\sigma} \text{Tr}_\gamma(M) \longrightarrow 0,$$

We obtain the following commutative diagram with exact row:

$$\begin{array}{ccccc}
0 & \longrightarrow & \text{Hom}_R(\text{Tr}_\gamma(M), N) & \xrightarrow{g^{**}} & \text{Hom}_R(P_1^*, N) & \xrightarrow{f^{**}} & \text{Hom}_R(P_0^*, N) \\
& & & & \downarrow \cong & & \downarrow \cong \\
& & & & P_1 \otimes_R N & \xrightarrow{f_*} & P_0 \otimes_R N
\end{array}$$

By the exactness of the first row, we have $\text{Hom}_R(\text{Tr}_\gamma(M), N) \cong \text{Im}(g^{**}) = \text{Ker}(f^{**})$, and $\text{Ker}(f^{**}) \cong \text{Ker}(f_*)$ by the commutative square of the above diagram and by the five lemma. Now use the natural epimorphism

$$\text{Ker}(f_*) \xrightarrow{\sigma'} \text{Ker}(f_*) / \text{Im}(\tilde{\phi}_*) = \text{Tor}_1^R(M, N)$$

to construct $\text{Hom}_R(\text{Tr}_\gamma(M), N) \longrightarrow \text{Tor}_1^R(M, N)$ as an epimorphism. Remember that $\text{Tor}_1^R(M, N)$ is obtained using the projective resolution of M given in the beginning of the proof. It can be checked from the construction of the monomorphism $\text{Ext}_R^1(M, N) \longrightarrow \text{Tr}_\gamma(M) \otimes_R N$ and the epimorphism $\text{Hom}_R(\text{Tr}_\gamma(M), N) \longrightarrow \text{Tor}_1^R(M, N)$ that these are natural in N .

Proof of (2): (See (Osborne, 2000, Proposition 4.4)) Since $\text{pd}(M) \leq 1$ we can choose a projective resolution of M such that for the projective presentation γ of M , f is a monomorphism. So we can choose $\tilde{\phi} = 0$ which gives $\tilde{\phi}^* = 0$ and so $\text{Ker}(\tilde{\phi}^*) = \text{Hom}_R(P_1, N)$. By part (1) and by this equality, we obtain $\text{Ext}_R^1(M, N) = \text{Ker}(\tilde{\phi}^*) / \text{Im}(f^*) = \text{Hom}_R(P_1, N) / \text{Im}(f^*) \cong (P_1^* \otimes_R N) / (\text{Im}(f^* \otimes 1_N)) \cong \text{Tr}_\gamma(M) \otimes_R N$. Thus $\text{Ext}_R^1(M, N) \cong \text{Tr}_\gamma(M) \otimes_R N$. Similarly from the proof of part (1) using $\text{Im } \tilde{\phi}^* = 0$, we obtain $\text{Hom}_R(\text{Tr}_\gamma(M), N) \cong \text{Tor}_1^R(M, N)$. Hence if we have $\text{pd}(M) \leq 1$, then we can assume that f is monic in the presentation γ of M , and $\text{Ext}_R^1(M, N) \cong \text{Tr}_\gamma(M) \otimes_R N$ and $\text{Hom}_R(\text{Tr}_\gamma(M), N) \cong \text{Tor}_1^R(M, N)$. Moreover by taking $N = R$, we obtain $\text{Ext}_R^1(M, R) \cong \text{Tr}_\gamma(M) \otimes_R R \cong \text{Tr}_\gamma(M)$ and $(\text{Tr}_\gamma(M))^* = \text{Hom}_R(\text{Tr}_\gamma(M), R) \cong \text{Tor}_1^R(M, R) = 0$ where the last equation holds since R is projective.

Proof of (3): It follows from (2) and Theorem 4.2.4: If $\text{pd}(M) \leq 1$, we have $\text{Ext}_R^1(M, R) \cong \text{Tr}_\gamma(M)$ and $\text{Tr}(M) \approx \text{Tr}_\gamma(M) \cong \text{Ext}_R^1(M, R)$.

Proof of (4): The first part is obtained by applying the left exact functor $(-)^* = \text{Hom}_R(-, R)$ to γ to obtain the exact sequence

$$0 \longrightarrow M^* \xrightarrow{g^*} P_0^* \xrightarrow{f^*} P_1^*$$

Since $M^* = 0$ by hypothesis, this gives us that f^* is monic and so the presentation γ^* implies that $\text{pd}(\text{Tr}_\gamma(M)) \leq 1$. Then by part (2), we obtain

$$\text{Tr}_{\gamma^*}(\text{Tr}_\gamma(M)) \cong \text{Ext}_R^1(\text{Tr}_\gamma(M), R).$$

Since $M \cong \text{Tr}_{\gamma^*}(\text{Tr}_\gamma(M))$, we obtain

$$M \cong \text{Tr}_{\gamma^*}(\text{Tr}_\gamma(M)) \cong \text{Ext}_R^1(\text{Tr}_\gamma(M), R).$$

Proof of (5): Suppose to the contrary that $\text{Tr}_\gamma(M) = 0$. Since $M \cong \text{Tr}_{\gamma^*}(\text{Tr}_\gamma(M)) = \text{Tr}_{\gamma^*}(0)$ is projectively equivalent to 0, it must be projective which contradicts with our hypothesis. So this contradiction shows that if M is not projective, then $\text{Tr}_\gamma(M) \neq 0$. □

Theorem 4.2.7. (by Mašek (2000, Proposition 5)) *Let M be a finitely presented R -module. Let $\sigma_M : M \rightarrow M^{**}$ be the natural R -module homomorphism into the double dual. Let $K_M = \text{Ker}(\sigma_M)$ and $C_M = \text{Coker}(\sigma_M)$. Then we have natural isomorphisms*

$$K_M \cong \text{Ext}_R^1(\text{Tr}(M), R) \quad \text{and} \quad C_M \cong \text{Ext}_R^2(\text{Tr}(M), R).$$

Note that the right sides do not depend on which presentation of M is used to obtain $\text{Tr}(M)$. That is, we have the following exact sequence:

$$0 \rightarrow \text{Ext}_R^1(\text{Tr}(M), R) \rightarrow M \xrightarrow{\sigma_M} M^{**} \rightarrow \text{Ext}_R^2(\text{Tr}(M), R) \rightarrow 0,$$

Note that the Ext groups here are left R -modules and the isomorphisms are R -module isomorphisms.

Proof. Consider the projective presentation γ of M as before

$$\gamma: \quad P_1 \xrightarrow{f} P_0 \xrightarrow{g} M \rightarrow 0$$

Dualizing γ and completing it with $\text{Coker}(f^*) = \text{Tr}_\gamma(M)$, we obtain the following exact sequence:

$$0 \rightarrow M^* \xrightarrow{g^*} P_0^* \xrightarrow{f^*} P_1^* \xrightarrow{\sigma} \text{Tr}_\gamma(M) \rightarrow 0,$$

Split this exact sequence into two short exact sequences γ_0^* and γ_1^* , that is,

$$\gamma_0^*: \quad 0 \rightarrow M^* \xrightarrow{g^*} P_0^* \xrightarrow{\beta_0} N = \text{Coker}(g^*) \rightarrow 0$$

and

$$\gamma_1^* : 0 \longrightarrow N = \text{Coker}(g^*) \xrightarrow{\beta_1} P_1^* \xrightarrow{f^*} \text{Tr}_\gamma(M) \longrightarrow 0$$

where $\beta_1 \circ \beta_0 = f^*$. The long exact sequence for Ext for the short exact sequence γ_0^* , gives the following long exact sequence:

$$\gamma_0^{**} : 0 \longrightarrow N^* \xrightarrow{\beta_0^*} P_0^{**} \xrightarrow{g^{**}} M^{**} \longrightarrow \text{Ext}_R^1(N, R) \longrightarrow \text{Ext}_R^1(P_0^*, R) = 0$$

where the last equality holds since P_0^* is projective. The long exact sequence for Ext for the short exact sequence γ_1^* gives the following exact sequence:

$$\gamma_1^{**} : 0 \longrightarrow (\text{Tr}_\gamma(M))^* \xrightarrow{f^{**}} P_1^{**} \xrightarrow{\beta_1^*} N^{**} \xrightarrow{\delta} \text{Ext}_R^1(\text{Tr}_\gamma(M), R) \longrightarrow \text{Ext}_R^1(P_1^*, R) = 0$$

Since P_1^* is projective. Consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccc} & & P_1 & \xrightarrow{f} & P_0 & \xrightarrow{g} & M & \longrightarrow & 0 \\ & & \downarrow \beta_1^* \circ \sigma_{P_1} & & \downarrow \sigma_{P_0} & & \downarrow \sigma_M & & \\ 0 & \longrightarrow & N^* & \xrightarrow{\beta_0^*} & P_0^{**} & \xrightarrow{g^{**}} & M^{**} & & \end{array}$$

Since P_1 is finitely generated projective, the natural R -module homomorphism $\sigma_{P_1} : P_1 \longrightarrow P_1^{**}$ in to the double dual is an isomorphism, see Theorem 4.2.7. So we have $\text{Coker}(\beta_1^* \circ \sigma_{P_1}) = \text{Coker}(\beta_1^*)$ and $\text{Coker}(\beta_1^*) = N^{**} / \text{Im}(\beta_1^*) = N^{**} / \text{Ker}(\delta) \cong \text{Ext}_R^1(\text{Tr}_\gamma(M), R)$ since $\text{Im}(\beta_1^*) = \text{Ker}(\delta)$ and $N^* / \text{Ker}(\delta) \cong \text{Ext}_R^1(\text{Tr}_\gamma(M), R)$ by the exactness of γ_1^{**} and the first isomorphism theorem. As σ_{P_0} is also an isomorphism, the Snake Lemma gives $K_M = \text{Ker}(\sigma_M) \cong \text{Coker}(\beta_1^* \circ \sigma_{P_1})$. Thus $K_M \cong \text{Ext}_R^1(\text{Tr}_\gamma(M), R)$. On the other hand, in the above commutative diagram, we have $\sigma_M \circ g = g^{**} \circ \sigma_{P_0}$, g is surjective and σ_{P_0} is isomorphism. Then we obtain $\text{Im}(\sigma_M) = \text{Im}(g^{**})$ and therefore $C_M = \text{Coker}(\sigma_M) = \text{Coker}(g^{**}) \cong \text{Ext}_R^1(N, R)$ where the last isomorphism is coming from the exact sequence γ_0^{**} . The isomorphism $\text{Ext}_R^1(N, R) \cong \text{Ext}_R^2(\text{Tr}_\gamma(M), R)$ follows from the long exact sequence for Ext for the short exact sequence γ_1^* . Hence we obtain the exact sequence

$$0 \longrightarrow \text{Ext}_R^1(\text{Tr}_\gamma(M), R) \longrightarrow M \xrightarrow{\sigma_M} M^{**} \longrightarrow \text{Ext}_R^2(\text{Tr}_\gamma(M), R)$$

since we have the following commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & K_M & \longrightarrow & M & \xrightarrow{\sigma_M} & M^{**} & \longrightarrow & C_M & \longrightarrow & 0 \\ & & \downarrow \cong & & \parallel & & \parallel & & \downarrow \cong & & \\ 0 & \longrightarrow & \text{Ext}_R^1(\text{Tr}_\gamma(M), R) & \longrightarrow & M & \xrightarrow{\sigma_M} & M^{**} & \longrightarrow & \text{Ext}_R^2(\text{Tr}_\gamma(M), R) & \longrightarrow & 0 \end{array}$$

where it can be checked that the isomorphisms are natural. \square

4.3 Proper Classes Generated by Finitely Presented Modules

Let us start this section with a useful lemma.

Lemma 4.3.1. (by for example Fuchs (2012, by Proposition 4.1) or Sklyarenko (1978, Theorem 6.1)) For any finitely presented right R -module M and any short exact sequence \mathbb{E} of R -modules, the sequence $M \otimes_R \mathbb{E}$ is exact if and only if the sequence $\text{Hom}_R(\text{Tr}_F(M), \mathbb{E})$ is exact where F is a free presentation of M .

Proof. We shall follow the proof in Fuchs (2012, by Proposition 4.1) Let M be a finitely presented right R -module and let F be any free presentation of M , say F is the following exact sequence where F_1 and F_0 are finitely generated free R -modules:

$$F : \quad F_1 \xrightarrow{v} F_0 \xrightarrow{u} M \longrightarrow 0$$

Since F_1 and F_0 are finitely generated free R -modules, there exist positive integers n, k such that $F_0 \cong R^n$, $F_1 \cong R^k$, and the R -module homomorphism from R^k to R^n is given by an $n \times k$ rectangular matrix A . We have then the following commutative diagram with exact rows:

$$\begin{array}{ccccccc} F : & & F_1 & \xrightarrow{v} & F_0 & \xrightarrow{u} & M \longrightarrow 0 \\ & & \downarrow \cong & & \downarrow \cong & & \parallel \\ F_A : & & R^k & \xrightarrow{A} & R^n & \longrightarrow & M \longrightarrow 0 \end{array} \quad (4.3.1)$$

By using the transpose matrix A^T , we obtain the following free presentation of $\text{Tr}_F(M)$:

$$F_A^T : \quad R^n \xrightarrow{A^T} R^k \longrightarrow \text{Tr}_F(M) \longrightarrow 0$$

Application of $\text{Hom}_R(*, R)$ functor to F_A^T yields the exact sequence

$$0 \longrightarrow \text{Hom}_R(\text{Tr}_F(M), R) \longrightarrow R^k \xrightarrow{A} R^n \quad (4.3.2)$$

Since $(R^k)^* \cong R^k$ and $(R^n)^* \cong R^n$. Noting that the $R^k \xrightarrow{A} R^n$ part is identical in the bottom row of the diagram (4.3.1) and in the exact sequence (4.3.2), by combining them, we obtain the following exact sequence:

$$0 \longrightarrow \text{Hom}_R(\text{Tr}_F(M), R) \longrightarrow R^k \xrightarrow{A} R^n \longrightarrow M \longrightarrow 0 \quad (4.3.3)$$

Applying the functor $- \otimes_R B$ to the exact sequence $R^k \xrightarrow{\mathbf{A}} R^n \longrightarrow M \longrightarrow 0$ and applying the functor $\text{Hom}_R(-, B)$ to the exact sequence $R^n \xrightarrow{\mathbf{A}^T} R^k \longrightarrow \text{Tr}_F(M) \longrightarrow 0$, we proceed to obtain the following commutative diagram with exact rows:

$$\begin{array}{ccccccc} R^k \otimes_R B & \longrightarrow & R^n \otimes_R B & \longrightarrow & M \otimes_R B & \longrightarrow & 0 \\ \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \\ B^k & \xrightarrow{\mathbf{A}} & B^n & \longrightarrow & M \otimes_R B & \longrightarrow & 0 \end{array}$$

and

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Hom}_R(\text{Tr}_F(M), B) & \longrightarrow & \text{Hom}_R(R^k, B) & \longrightarrow & \text{Hom}_R(R^n, B) \\ & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ 0 & \longrightarrow & \text{Hom}_R(\text{Tr}_F(M), B) & \longrightarrow & B^k & \xrightarrow{\mathbf{A}} & B^n \end{array}$$

The rows are exact since the functor $- \otimes_R B$ is right exact and the functor $\text{Hom}_R(-, B)$ is left exact. If we combine these, we obtain a new exact sequence:

$$0 \longrightarrow \text{Hom}_R(\text{Tr}_F(M), B) \longrightarrow B^k \xrightarrow{\mathbf{A}} B^n \longrightarrow M \otimes_R B \longrightarrow 0 \quad (4.3.4)$$

Let us now start with a short exact sequence

$$\mathbb{E}: \quad 0 \longrightarrow A \xrightarrow{\theta_1} B \xrightarrow{\theta_2} C \longrightarrow 0$$

Consider the following commutative diagram where the vertical arrows denote R -module homomorphisms defined by the matrix A , that is, the matrix notation of every vertical R -module homomorphism is A :

$$\begin{array}{ccccccc} 0 & \longrightarrow & A^k & \xrightarrow{\theta_1^k} & B^k & \xrightarrow{\theta_2^k} & C^k \longrightarrow 0 \\ & & \downarrow \mathbf{A}_\alpha & & \downarrow \mathbf{A}_\beta & & \downarrow \mathbf{A}_\gamma \\ 0 & \longrightarrow & A^n & \xrightarrow{\theta_1^n} & B^n & \xrightarrow{\theta_2^n} & C^n \longrightarrow 0 \end{array}$$

Extend this to the following commutative diagram:

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \text{Ker}(\mathbf{A}_\alpha) & \longrightarrow & \text{Ker}(\mathbf{A}_\beta) & \longrightarrow & \text{Ker}(\mathbf{A}_\gamma) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & A^k & \xrightarrow{\theta_1^k} & B^k & \xrightarrow{\theta_2^k} & C^k \longrightarrow 0 \\ & & \downarrow \mathbf{A}_\alpha & & \downarrow \mathbf{A}_\beta & & \downarrow \mathbf{A}_\gamma \\ 0 & \longrightarrow & A^n & \xrightarrow{\theta_1^n} & B^n & \xrightarrow{\theta_2^n} & C^n \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & \text{Coker}(\mathbf{A}_\alpha) & \longrightarrow & \text{Coker}(\mathbf{A}_\beta) & \longrightarrow & \text{Coker}(\mathbf{A}_\gamma) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0 \end{array}$$

Taking the exact sequence (4.3.4) into consideration, we obtain that $\text{Ker}(\mathbf{A}_\alpha) \cong \text{Hom}_R(\text{Tr}_F(M), A)$, $\text{Ker}(\mathbf{A}_\beta) \cong \text{Hom}_R(\text{Tr}_F(M), B)$, $\text{Ker}(\mathbf{A}_\gamma) \cong \text{Hom}_R(\text{Tr}_F(M), C)$, $\text{Coker}(\mathbf{A}_\alpha) \cong M \otimes_R A$, $\text{Coker}(\mathbf{A}_\beta) \cong M \otimes_R B$ and $\text{Coker}(\mathbf{A}_\gamma) \cong M \otimes_R C$. For example for B , we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \text{Hom}_R(\text{Tr}_F(M), B) & \longrightarrow & B^k & \xrightarrow{\mathbf{A}_\beta} & B^n & \longrightarrow & M \otimes_R B & \longrightarrow & 0 \\
& & \downarrow \cong & & \parallel & & \parallel & & \downarrow \cong & & \\
0 & \longrightarrow & \text{Ker}(\mathbf{A}_\beta) & \longrightarrow & B^k & \xrightarrow{\mathbf{A}_\beta} & B^n & \longrightarrow & \text{Coker}(\mathbf{A}_\beta) & \longrightarrow & 0
\end{array}$$

The well known Snake Lemma (Ker-Coker sequence) leads us to the following long exact sequence:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \text{Ker}(\mathbf{A}_\alpha) & \longrightarrow & \text{Ker}(\mathbf{A}_\beta) & \longrightarrow & \text{Ker}(\mathbf{A}_\gamma) \\
& & & & \delta & & \downarrow \\
& & & & & & \text{Coker}(\mathbf{A}_\alpha) & \longrightarrow & \text{Coker}(\mathbf{A}_\beta) & \longrightarrow & \text{Coker}(\mathbf{A}_\gamma) & \longrightarrow & 0
\end{array}$$

So by the above isomorphism we obtain the following long exact sequence:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \text{Hom}_R(\text{Tr}_F(M), A) & \longrightarrow & \text{Hom}_R(\text{Tr}_F(M), B) & \longrightarrow & \text{Hom}_R(\text{Tr}_F(M), C) \\
& & & & \delta & & \downarrow \\
& & & & & & M \otimes_R A & \longrightarrow & M \otimes_R B & \longrightarrow & M \otimes_R C & \longrightarrow & 0
\end{array}$$

From this long exact sequence we obtain that the sequence

$$0 \longrightarrow M \otimes_R A \longrightarrow M \otimes_R B \longrightarrow M \otimes_R C \longrightarrow 0$$

is exact if and only if the sequence

$$0 \longrightarrow \text{Hom}_R(\text{Tr}_F(M), A) \longrightarrow \text{Hom}_R(\text{Tr}_F(M), B) \longrightarrow \text{Hom}_R(\text{Tr}_F(M), C) \longrightarrow 0$$

is exact. □

Theorem 4.3.2. (by for example Sklyarenko (1978, Corollary 5.1)) *Let M be a finitely presented right R -module and \mathbb{E} a short exact sequence of R -modules. Then the sequence $M \otimes_R \mathbb{E}$ is exact if and only if $\text{Hom}_R(\text{Tr}(M), \mathbb{E})$ is exact.*

Proof. This proof follows by the lemma and projectively equivalence property of any two Auslander-Bridger transposes of a finitely presented R -module. Let M be a finitely

presented right R -module and γ be a projective presentation, that is, we have the following exact sequence where P_0 and P_1 are finitely generated projective R -modules.

$$\gamma: \quad P_1 \xrightarrow{f} P_0 \xrightarrow{g} M \longrightarrow 0$$

We always have also a free presentation F of M , that is, a short exact sequence

$$F: \quad F_1 \xrightarrow{v} F_0 \xrightarrow{u} M \longrightarrow 0$$

where F_0 and F_1 are finitely generated free R -modules. It is clear that this free presentation is also a projective presentation of M . By the projective equivalence property, $\text{Tr}_\gamma(M) \approx \text{Tr}_F(M)$, that is, there exist two projective R -module \tilde{P}_1 and \tilde{P}_2 such that $\text{Tr}_\gamma(M) \oplus \tilde{P}_1 \cong \text{Tr}_F(M) \oplus \tilde{P}_2$. By the above lemma, for any short exact sequence \mathbb{E} of R -modules, we also have $M \otimes_R \mathbb{E}$ is exact if and only if $\text{Hom}_R(\text{Tr}_F(M), \mathbb{E})$ is exact. Note that the following well known properties;

- (1) If P is a projective R -module, then $\text{Hom}_R(P, \mathbb{E})$ is exact.
- (2) $\text{Hom}_R(A_1 \oplus A_2, \mathbb{E}) \cong \text{Hom}_R(A_1, \mathbb{E}) \oplus \text{Hom}_R(A_2, \mathbb{E})$ for R -modules A_1 and A_2 .

The sequence $\text{Hom}_R(\text{Tr}_F(M), \mathbb{E})$ is exact if and only if $\text{Hom}_R(\text{Tr}_F(M) \oplus \tilde{P}_2, \mathbb{E})$ is exact if and only if $\text{Hom}_R(\text{Tr}_\gamma(M) \oplus \tilde{P}_1, \mathbb{E})$ is exact if and only if $\text{Hom}_R(\text{Tr}_\gamma(M), \mathbb{E})$ is exact. Since this is true for any projective presentation γ of M , we can say that $M \otimes_R \mathbb{E}$ is exact if and only if $\text{Hom}_R(\text{Tr}(M), \mathbb{E})$ is exact. \square

This gives:

Theorem 4.3.3. (Sklyarenko, 1978, Theorem 8.3) *Let \mathcal{M} be a set of finitely presented right R -modules.*

Let $\text{Tr}(\mathcal{M}) = \{\text{Tr}(M) \mid M \in \mathcal{M}\}$. We may assume that $\text{Tr}(\text{Tr}(\mathcal{M})) = \mathcal{M}$. Then we have

$$\tau^{-1}(\mathcal{M}) = \pi^{-1}(\text{Tr}(\mathcal{M})) \quad \text{and} \quad \pi^{-1}(\mathcal{M}) = \tau^{-1}(\text{Tr}(\mathcal{M}))$$

Proof. Let \mathcal{M} be a set of finitely presented right R -modules. Take any short exact sequence $\mathbb{E} \in \tau^{-1}(\mathcal{M})$. By the definition of $\tau^{-1}(\mathcal{M})$, $M \otimes_R \mathbb{E}$ is exact for all $M \in \mathcal{M}$ and so by Theorem 4.3.2, $\text{Hom}_R(\text{Tr}(M), \mathbb{E})$ is exact for all $\text{Tr}(M) \in \text{Tr}(\mathcal{M})$ which means that $\mathbb{E} \in \pi^{-1}(\text{Tr}(\mathcal{M}))$. Thus $\tau^{-1}(\mathcal{M}) \subseteq \pi^{-1}(\text{Tr}(\mathcal{M}))$. Conversely, take

any $\mathbb{E} \in \pi^{-1}(\text{Tr}(\mathcal{M}))$. Then $\text{Hom}_R(\text{Tr}(M), \mathbb{E})$ is exact for all $M \in \mathcal{M}$, and so by Theorem 4.3.2, $M \otimes_R \mathbb{E}$ is exact for all $M \in \mathcal{M}$ which means that $\mathbb{E} \in \tau^{-1}(\mathcal{M})$. Thus $\pi^{-1}(\text{Tr}(\mathcal{M})) \subseteq \tau^{-1}(\mathcal{M})$. These give the equality $\tau^{-1}(\mathcal{M}) = \pi^{-1}(\text{Tr}(\mathcal{M}))$. The other equality follows similarly. \square

4.4 The Auslander-Bridger Transpose of Finitely Presented Simple Modules

We need the Auslander-Bridger tranpose of *finitely presented simple R -modules* to understand the sufficiency condition in the characterization of N -domains. See Fuchs (2012) where free presentations are used for the Auslander-Bridger tranpose of finitely presented simple R -modules over *commutative domains*.

In Angeleri Hügel & Bazzoni (2010, Lemma 6.1-(1)), it is written that for a finitely presented R -module U , $\text{pd}(\text{Tr}(U)) \leq 1$ if and only if $U^* = 0$. For the ‘only if’ part, we need to assume that U has no non-zero projective direct summands, that is, what they tacitly assume for U .

Theorem 4.4.1. *Let M be a finitely presented R -module.*

- (1) *If $M^* = 0$, then $\text{pd}(\text{Tr}(M)) \leq 1$.*
- (2) *If $\text{pd}(\text{Tr}(M)) \leq 1$, then M^* is projective and finitely generated.*
- (3) *If $\text{pd}(\text{Tr}(M)) \leq 1$ and M has no non-zero projective direct summands, then $M^* = 0$.*

Proof. (1) is just Theorem 4.2.6-(4). See the proof of (Angeleri Hügel & Bazzoni, 2010, Lemma 6.1-(1)).

Proof of (2): Suppose $\text{Tr}(M)$ has projective dimension at most 1. Let γ be a presentation of M :

$$\gamma: \quad P_1 \xrightarrow{f} P_0 \xrightarrow{g} M \longrightarrow 0$$

We have the following exact sequence

$$0 \longrightarrow M^* \xrightarrow{g^*} P_0^* \xrightarrow{f^*} P_1^* \xrightarrow{\sigma} \text{Tr}_\gamma(M) \longrightarrow 0$$

$\text{Ker}(\sigma)$ is the first kernel of a projective resolution, so $\text{Ker}(\sigma) = \text{Im}(f^*)$ is projective since $\text{pd}(\text{Tr}_\gamma(M)) \leq 1$. So M^* is projective and finitely generated because $M^* \cong \text{Im}(g^*) = \text{Ker}(f^*)$. Indeed $M^* \cong \text{Im}(g^*) = \text{Ker}(f^*)$ is a direct summand of P_0^* since $\text{Im}(f^*)$ is projective and so the following exact sequence splits:

$$\mathbb{E} : 0 \longrightarrow \text{Ker}(f^*) \longrightarrow P_0^* \longrightarrow \text{Im}(f^*) = \text{Ker}(\sigma) \longrightarrow 0$$

where $\text{Ker}(f^*) \longrightarrow P_0^*$ is the inclusion homomorphism and $P_0^* \longrightarrow \text{Im}(f^*)$ is the R -module homomorphism given by f^* . We have that $\text{Im}(f^*) = \text{Ker}(\sigma)$ is projective as $\text{pd}(\text{Tr}_\gamma(M)) \leq 1$. So \mathbb{E} splits, that is, $P_0^* \cong \text{Ker}(f^*) \oplus \text{Im}(f^*)$. Since P_0^* is finitely generated, so is its direct summand $\text{Ker}(f^*)$. Hence $\text{Ker}(f^*) \cong M^*$ is projective and finitely generated.

Proof of (3): By Theorem 4.2.7, we have the following exact sequence:

$$0 \longrightarrow \text{Ext}_R^1(\text{Tr}(M), R) \longrightarrow M \xrightarrow{\sigma_M} M^{**} \longrightarrow \text{Ext}_R^2(\text{Tr}(M), R) \longrightarrow 0,$$

The last term $\text{Ext}_R^2(\text{Tr}(M), R) = 0$ since $\text{pd}(\text{Tr}(M)) \leq 1$. Since M^* is finitely generated and projective, M^{**} is also projective and so the exact sequence

$$0 \longrightarrow \text{Ext}_R^1(\text{Tr}(M), R) \longrightarrow M \xrightarrow{\sigma_M} M^{**} \longrightarrow 0,$$

splits which gives $M \cong M^{**} \oplus \text{Ext}_R^1(\text{Tr}(M), R)$. If we assume that M has no non-zero projective direct summands, the projective direct summand $M^{**} = 0$ must hold. This then gives $M^* \cong M^{***} = 0^* = 0$, that is, $M^* = 0$. \square

Since the only non-zero direct summand of a simple R -module is itself, we obtain:

Corollary 4.4.2. *If S is a finitely presented simple R -module that is not projective, then*

$$\text{pd}(\text{Tr}(S)) \leq 1 \quad \iff \quad S^* = 0$$

This corollary does not hold if S is projective: for a finitely generated non-zero projective R -module P , we have $\text{Tr}(P) = 0$ and so $\text{pd}(\text{Tr}(P)) = 0 \leq 1$ but $P^* \neq 0$ because otherwise $P \cong P^{**} = 0^* = 0$ would hold. Thus:

Proposition 4.4.3. *The following are equivalent for a finitely presented simple R -module S :*

(1) $\text{pd}(\text{Tr}(S)) \leq 1$.

(2) $S^* = 0$ or S is projective.

If the ring R is commutative, S^* is homogeneous semisimple with every simple submodule isomorphic to S and so these are equivalent to:

(3) S^* is projective (and finitely generated).

Proof. Equivalence of (1) and (2) follows from the above corollary. (2) implies (3) for any ring R .

Proof of (3) \Rightarrow (2) : When the ring R is commutative, $S \cong R/P$ for a maximal ideal P of R . So $S^* = \text{Hom}_R(S, R)$ is annihilated by P also and thus S^* is homogeneous semisimple with every simple submodule isomorphic to $S \cong R/P$. Thus $S^* = \bigoplus_{i \in I} S_i$ for some simple submodules $S_i \cong S$ of S^* for each $i \in I$ where I is some indexing set. So if S^* is projective, there are two cases: either $S^* = 0$ (index set $I = \emptyset$) or $S^* \neq 0$. In the second case, the projective R -module S^* has a direct summand isomorphic to S which must then be also projective. \square

Note also that for a commutative ring R and a simple R -module S , $S^* = 0$ if and only if R has no simple submodule isomorphic to S .

We can extend the results in Fuchs (2012) for the Auslander-Bridger transpose of finitely presented simple modules of projective dimension ≤ 1 over a commutative domain to commutative rings:

Theorem 4.4.4. *Let R be a commutative ring and P be a finitely generated maximal ideal of R that is projective. Take the following presentation of the simple R -module $S = R/P$ (where f is the inclusion monomorphism and g is the natural epimorphism):*

$$\gamma: \quad P \xrightarrow{f} R \xrightarrow{g} S \longrightarrow 0$$

(1) If S is projective, then $S^* \neq 0$ and $\text{Tr}_\gamma(S) = 0$.

(2) If S is not projective, then $S^* = 0$ and $\text{Tr}_\gamma(S) \cong \text{Ext}_R^1(S, R) \cong S$.

(3) S is projective if and only if $S^* \neq 0$.

Proof. For the finitely presented simple R -module S , $\text{pd}(S) \leq 1$ by hypothesis. So we obtain $\text{Tr}_\gamma(S) \cong \text{Ext}_R^1(S, R)$ by Theorem 4.2.6-(2). Since $\text{Ext}_R^1(S, R)$ is annihilated by P (as S is annihilated by P), it must be a homogeneous semisimple R -module with every simple submodule isomorphic to S . Since $\text{Tr}_\gamma(S)$ is finitely generated, $\text{Ext}_R^1(S, R)$ must be a finite direct sum of copies of S , that is, $\text{Tr}_\gamma(S) \cong \text{Ext}_R^1(S, R) \cong \bigoplus_{i=1}^m S$ for some $m \in \mathbb{Z}^+ \cup \{0\}$.

If S is projective, then $\text{Ext}_R^1(S, R) = 0$ and $S^* \neq 0$ (because otherwise for the finitely generated non-zero projective R -module S , $S \cong S^{**} = 0^* = 0$ would hold); this proves (1).

To prove (2), assume that S is not projective. By Theorem 4.2.6-(5), $\text{Tr}_\gamma(S) \neq 0$. So the m in $\text{Tr}_\gamma(S) \cong \text{Ext}_R^1(S, R) \cong \bigoplus_{i=1}^m S$ must be positive. Since $\text{Tr}_\gamma(S) \cong \bigoplus_{i=1}^m S$ and $\text{pd}(S) \leq 1$ by hypothesis, we have $\text{pd}(\text{Tr}_\gamma(S)) \leq 1$. Then by Corollary 4.4.2, $S^* = 0$. So we can use Theorem 4.2.6-(4). to obtain that $S \cong \text{Ext}_R^1(\text{Tr}_\gamma(S), R)$. Thus

$$S \cong \text{Ext}_R^1(\text{Tr}_\gamma(S), R) \cong \text{Ext}_R^1\left(\bigoplus_{i=1}^m S, R\right) \cong \bigoplus_{i=1}^m \text{Ext}_R^1(S, R) \cong \bigoplus_{i=1}^m \left[\bigoplus_{j=1}^m S\right]$$

This then implies that $m = 1$ by the results for the structure of semisimple R -modules. (3) follows just by (1) and (2). □

Nunke shows that $I^{-1}/R \cong R/I$ if I is a non-zero ideal of a Dedekind domain R (see Nunke (1959, Lemma 4.4)). For an invertible maximal ideal P of a commutative ring R , we show next that $P^{-1}/R \cong R/P$ where the invertibility is in the total quotient ring of R , that is, the localization of R with respect to the set of all regular elements of R , and P^{-1} consists of all q in the total quotient ring of R such that $qP \leq R$.

Proposition 4.4.5. *If R is a commutative ring and P is a maximal ideal of R that is invertible in the total ring of quotients of R , then for the simple R -module $S = R/P$ and for the presentation*

$$\gamma: \quad P \xrightarrow{f} R \xrightarrow{g} S \longrightarrow 0$$

of S (where f is the inclusion monomorphism and g is the natural epimorphism), we have

$$\text{Tr}_\gamma(S) \cong P^{-1}/R \cong S = R/P.$$

Proof. By Lam (1999, Theorem 2.17), P must be projective (finitely generated) and contains a regular element. So $S^* = 0$ because if $f : S \rightarrow R$ is in $S^* = \text{Hom}_R(S, R)$, then either $\text{Im}(f) = 0$ (and so $f = 0$) or $\text{Im}(f) \cong S$ is simple. If $\text{Im}(f) \cong S$, then $\text{Im}(f) = Ra \leq R$ for some $a \in R$ such that P is the annihilator of a . So $Pa = 0$ and $a \neq 0$ (because $Ra \cong R/P = S \neq 0$). But by our hypothesis, every maximal ideal of R contains a regular element, say $b \in P$ is a regular element. Then $Pa = 0$ implies that $ba = 0$ contradicting regularity of b since $a \neq 0$. This contradiction shows that $\text{Im}(f) \cong S$ is not possible. So for every $f \in S^*$, we must have $f = 0$. That is, $S^* = 0$ for every simple R -module S . Since $S = R/P$ and P is projective, we have $\text{pd}(S) \leq 1$. Then by Theorem 4.4.4, $\text{Tr}_\gamma(S) \cong S = R/P$. The following exact sequence defines $\text{Tr}_\gamma(S)$:

$$\text{Hom}_R(R, R) \xrightarrow{f^*} \text{Hom}_R(P, R) \xrightarrow{\sigma} \text{Tr}_\gamma(S) = \text{Hom}(P, R) / \text{Im}(f^*) \longrightarrow 0,$$

where σ is the canonical epimorphism. By Lam (1999, Theorem 2.14), we have $P^{-1} \cong \text{Hom}_R(P, R)$ by the isomorphism $\beta : P^{-1} \rightarrow \text{Hom}_R(P, R)$ given for each $q \in P^{-1}$ by $\beta(q)(p) = pq$ for every $p \in P$. Observe that under this isomorphism R goes onto $\text{Im}(f^*)$. So $\text{Tr}_\gamma(S) = \text{Hom}_R(P, R) / \text{Im}(f^*) \cong P^{-1}/R$. \square

4.5 Finitely Generated and Projective Maximal Ideals

Using Theorem 4.4.4 of the previous section, we obtain the following sufficient condition for ${}_R\mathcal{N}eat = {}_R\mathcal{P}\text{-Pure}$ over commutative rings as in Fuchs' characterization of N -domains:

Theorem 4.5.1. *If R is a commutative ring such that every maximal ideal of R is finitely generated and projective, then ${}_R\mathcal{N}eat = {}_R\mathcal{P}\text{-Pure}$.*

Proof. By hypothesis, for every simple R -module S , $\text{pd}(S) \leq 1$ and S is finitely presented. Note that in the projectively or flatly generated classes, there is no need to put the projective ones in the generating class. By Theorem 4.3.3 and Theorem

4.4.4, we obtain (since R is a commutative ring):

$$\begin{aligned}
{}_R\mathcal{P}\text{-Pure} &= \tau^{-1}(\{\text{all simple } R\text{-modules}\}) \\
&= \tau^{-1}(\{R/P \mid P \text{ is a maximal ideal of } R\}) \\
&= \tau^{-1}(\{R/P \mid P \text{ is a maximal ideal of } R \text{ and } R/P \text{ is not projective}\}) \\
&= \pi^{-1}(\{\text{Tr}(R/P) \mid P \text{ is a maximal ideal of } R \text{ and } R/P \text{ is not projective}\}) \\
&= \pi^{-1}(\{R/P \mid P \text{ is a maximal ideal of } R \text{ and } R/P \text{ is not projective}\}) \\
&= \pi^{-1}(\{R/P \mid P \text{ is a maximal ideal of } R\}) \\
&= {}_R\mathcal{N}eat
\end{aligned}$$

□

For completeness, let us give the following proposition with detailed proof:

Proposition 4.5.2. (*Fuchs, 2012, Lemma 2.4*) *Let R be a commutative ring. Neatness is an inductive property if and only if pure-exact sequences of R -modules are neat-exact if and only if the maximal ideals of R are finitely generated.*

Proof. Suppose neatness is an inductive property. We know that splitting short exact sequences are neat-exact. Then the direct limits of splitting short exact sequences are neat-exact since by the hypothesis neatness is an inductive property. So pure-exact sequences are neat-exact since every pure exact sequence is a direct limit of splitting short exact sequences (for this property of purity, see the notes at the end of Section 2.5).

Next, suppose pure-exact sequences are neat-exact. Let P be a maximal ideal of R . Let $S = R/P$. Then S is a simple R -module, so ${}_R\mathcal{N}eat$ -projective which implies S is a ${}_R\mathcal{P}ure$ -projective module since ${}_R\mathcal{P}ure \subseteq {}_R\mathcal{N}eat$. The ${}_R\mathcal{P}ure$ -projective modules are known to be direct summands of a direct sum of finitely presented modules (for this property of purity, see the notes at the end of Section 2.5). So S is a direct summand of direct sum of finitely presented modules. Since the simple module S cyclic (so finitely generated), we can assume that S is a direct summand of a direct sum of finitely many finitely presented modules is finitely presented and a direct summand of a finitely presented module is finitely presented. For these properties of finitely

presented modules, see Section 4.1. Thus S itself a finitely presented. Since $S = R/P$ for a maximal ideal P of R , P is finitely generated by Corollary 4.1.5.

Finally, if the maximal ideals are finitely generated, then the simple R -modules S are finitely presented. So we can obtain the following class equality by Theorem 4.3.3:

$${}_R\mathcal{Neat} = \pi^{-1}(\{R/P \mid P \text{ maximal ideal of } R\}) = \tau^{-1}(\{\text{Tr}(R/P) \mid P \text{ maximal ideal of } R\})$$

A flatly generated proper class is always inductively closed since the tensor product and a direct limit of a direct system commute. So the inductive property of neatness holds. \square

Theorem 4.5.3. *If R is a commutative ring such that ${}_R\mathcal{Neat} = {}_R\mathcal{P}\text{-Pure}$, then every simple R -module is finitely presented (that is, every maximal ideal of R is finitely generated) and the following are equivalent:*

- (1) *Every maximal ideal of R is projective.*
- (2) *$\text{pd}(S) \leq 1$ for every simple R -module S .*
- (3) *For each simple R -module S , there exists a presentation γ of S such that $(\text{Tr}_\gamma(S))^* = 0$.*
- (4) *$\text{pd}(\text{Tr}(S)) \leq 1$ for every simple R -module S .*
- (5) *S^* is projective for every simple R -module S .*
- (6) *For each simple R -module S , $S^* = 0$ or S is projective.*

Proof. When ${}_R\mathcal{Neat} = {}_R\mathcal{P}\text{-Pure}$, every simple R -module is finitely presented by Corollary 3.3.6-(1). The equivalence of (1) and (2) is clear. The equivalence of (2) and (3) is by Proposition 4.2.5. The equivalence of (4),(5),(6) is by Proposition 4.4.3. It suffices to prove the equivalence of (2) and (4). Let $\{P_i \mid i \in I\}$ be the set of all maximal ideals of the ring R (where I is some indexing set). Let $S_i = R/P_i$ for every $i \in I$. Then each simple R -module will be isomorphic to one of the simple R -modules in the set $\{S_i \mid i \in I\}$. Since every simple R -module is finitely presented, we obtain using Theorem 4.3.3 that

$${}_R\mathcal{P}\text{-Pure} = \tau^{-1}(\{S_i \mid i \in I\}) = \pi^{-1}(\{\text{Tr}(S_i) \mid i \in I\})$$

Since ${}_R\mathcal{N}eat = \pi^{-1}(\{S_i \mid i \in I\})$ by definition, ${}_R\mathcal{N}eat = {}_R\mathcal{P}\text{-Pure}$ implies that

$${}_R\mathcal{N}eat = \pi^{-1}(\{S_i \mid i \in I\}) = \pi^{-1}(\{\text{Tr}(S_i) \mid i \in I\}) = {}_R\mathcal{P}\text{-Pure}.$$

By considering projectives with respect to a proper class projectively generated by a set of R -modules, we obtain that each S_i is ${}_R\mathcal{N}eat$ -projective and so ${}_R\mathcal{P}\text{-Pure}$ -projective and so must be a direct summand of a direct sum of a projective R -module and R -modules in the collection $\{\text{Tr}(S_j) \mid j \in I\}$ (by (Sklyarenko, 1978, Proposition 2.1) for the projectives relative to a proper class projectively generated by a set of R -modules). So if $\text{pd}(\text{Tr}(S_j)) \leq 1$ for all $j \in I$, then we must have $\text{pd}(S_i) \leq 1$ for each $i \in I$. Conversely, if $\text{pd}(S_j) \leq 1$ for all $j \in I$, then we must have $\text{pd}(\text{Tr}(S_i)) \leq 1$ for each $i \in I$. □

Note that in the above theorem if the equivalent conditions hold then for each simple R -module S , (2) and (6) implies that $\text{Tr}(S)$ is either projective or projectively equivalent to S by Theorem 4.4.4.

Corollary 4.5.4. *The following are equivalent for a commutative ring R such that for each simple R -module S , $S^* = 0$ or S is projective:*

- (1) ${}_R\mathcal{N}eat = {}_R\mathcal{P}\text{-Pure}$.
- (2) Every maximal ideal P of R is projective and finitely generated.

Proof. (2) implies (1) by Theorem 4.5.1. Since $S^* = 0$ or S is projective for every simple R -module S by hypothesis, (1) implies (2) by Theorem 4.5.3. □

4.6 Commutative Rings with Zero Socle

Examples of commutative rings R such that $S^* = 0$ for every simple R -module S , that is, $\text{Soc}(R) = 0$ are given below:

Example 4.6.1.

- (1) Commutative domains are among such rings. So the above corollary gives also the characterization of N -domains by Fuchs (Theorem 3.1.1).

- (2) Commutative rings in which every maximal ideal contains a regular element are also among such rings. So $S^* = 0$ for all simple R -module S (by Proposition 4.4.5)
- (3) A finite product of commutative domains are commutative rings such that every maximal ideal contains a regular element. It suffices to prove this for $R \times S$ where R and S are commutative rings such that every maximal ideal contains a regular element. Since ideals of $R \times S$ are necessarily of the form $I \times J$ for some ideal I of R and some ideal J of S , the maximal ideals of $R \times S$ are of the form $P \times S$ or $R \times Q$ where P is a maximal ideal of R and Q is a maximal ideal of S . Since P contains a regular element a in R , the element $(a, 1) \in P \times S$ is a regular element of $R \times S$. Similarly $R \times Q$ also contains a regular element.
- (4) A finite product of commutative rings in each of which every maximal ideal contains a regular element is also a commutative ring such that every maximal ideal contains a regular element. This is what we proved in (3.) above.
- (5) Patrick F. Smith has showed that for any nontrivial ring R (even not necessarily commutative), the polynomial ring $R[x]$ and the formal power series ring $R[[x]]$ are rings in which every maximal (two-sided) ideal contains a regular element. Let R be any nontrivial ring (that is, $1 \neq 0$). Let P be a maximal ideal of the polynomial ring $S = R[x]$. If $x \in P$, then P contains the regular element x . Suppose that $x \notin P$. Note that Sx is a two-sided ideal of S and hence $S = Sx + P$. There exists a polynomial $f(x) \in S$ such that $1 - f(x)x \in P$. Note that $1 - f(x)x$ is a regular element of S for every polynomial $f(x) \in S$. Because if $(1 - f(x)x)g(x) = 0$ or $g(x)(1 - f(x)x) = 0$ for some non-zero $g(x) \in S$, then by taking a to be the non-zero coefficient of $g(x)$ corresponding to the lowest possible power of x , we obtain $1a = 0$, a contradiction. Similar proof can be adapted to show that every maximal ideal of the formal power series ring contains a regular element.

Remark 4.6.2. By Proposition 4.4.5, if R is a commutative ring such that every maximal ideal contains a regular element, then $\text{Soc}(R) = 0$. Conversely, if R is a commutative Noetherian local ring such that $\text{Soc}(R) = 0$, then the unique maximal ideal of R contains a regular element by for example (Northcott, 1960, §9.4, Proposition 6*)

Theorems 4.5.1 and 4.5.3, and Corollary 4.5.4 do not suffice to completely generalize Fuchs' characterization of N -domains to commutative rings. A question that we could not have answered is to determine if there exists a commutative ring R such that ${}_R\mathcal{N}eat = {}_R\mathcal{P}\text{-}\mathcal{P}ure$ (and so necessarily every maximal ideal of R is finitely generated) but not every maximal ideal of R is projective.

For a ring R , the condition $S^* = 0$ for every simple R -module S means $\text{Soc}({}_R R) = 0$. This is the dual of left small rings (where the radical of every injective module is itself) in the sense that such rings are the rings where every projective module has zero socle. These rings are near to domains. Over commutative domains, a direct summand of a direct sum of a projective and semisimple module is again a direct sum of a projective and semisimple module. This can be proved considering the torsion parts. The same result also holds for rings with zero socle:

Lemma 4.6.3. *If $\text{Soc}({}_R R) = 0$, then a direct summand of a direct sum of a projective and a semisimple R -module is again a direct sum of a projective and a semisimple R -module.*

Proof. Let P be a projective module and N be a semisimple module. Let A be a direct summand of $P \oplus N$. Thus $A \oplus B = P \oplus N$ for some submodule B of $P \oplus N$. Then $\text{Soc}(A) \oplus \text{Soc}(B) = \text{Soc}(P) \oplus \text{Soc}(N)$. Since $\text{Soc}(R) = 0$, we have $\text{Soc}(P) = 0$ as P is projective. Since N is semisimple, $\text{Soc}(N) = N$. So $\text{Soc}(A) \oplus \text{Soc}(B) = 0 \oplus N$. Then

$$(A/\text{Soc}(A)) \oplus (B/\text{Soc}(B)) \cong (A \oplus B)/(\text{Soc}(A) \oplus \text{Soc}(B)) = (P \oplus N)/(0 \oplus N) \cong P$$

is projective and so its direct summand $A/\text{Soc}(A)$ is also projective. Thus the natural epimorphism $A \rightarrow A/\text{Soc}(A)$ splits and so $\text{Soc}(A)$ is a direct summand of A . That is, $A = \text{Soc}(A) \oplus A'$ for some submodule A' of A such that $A' \cong A/\text{Soc}(A)$ is projective. Hence A is also a direct sum of a projective and a semisimple module. \square

The structure of ${}_R\mathcal{N}eat$ -projectives over commutative domains (given in Fuchs (2012)) also holds for rings with zero socle:

Proposition 4.6.4. *If $\text{Soc}({}_R R) = 0$, then ${}_R\mathcal{N}eat$ -projective R -modules are modules which are a direct sum of a projective R -module and a semisimple R -module.*

Proof. Since ${}_{R}\mathcal{N}eat = \pi^{-1}(\{R/P \mid P \text{ is a maximal left ideal of } R\})$, a ${}_{R}\mathcal{N}eat$ -projective module is isomorphic to a direct summand of a direct sum of a projective module and modules in the set $\{R/P \mid P \text{ is a maximal left ideal of } R\}$ (by Theorem 3.2.7 for the projectives relative to a proper class projectively generated by a set of modules). Thus every ${}_{R}\mathcal{N}eat$ -projective module is a direct summand of a direct sum of a projective and a semisimple module. Now the above lemma ends the proof. \square

CHAPTER FIVE

CONCLUSION

The natural question was asked for a commutative ring R when “neatness= \mathcal{P} -purity”. László Fuchs wished to explore when the “=” relation holds for commutative rings R . He characterized integral domains for which these two concepts coincide, that is:

“ For a commutative domain R , $Neat = \mathcal{P}\text{-Pure}$ if and only if all the maximal ideals of the commutative domain R are (finitely generated) projective modules (that is, they are invertible ideals). ”

In this article, we wanted to extend Fuch’s conclusion to the commutative rings. For the sufficiency part we gave an answer, that is, “neatness= \mathcal{P} -purity” holds for all commutative rings R where all the maximal ideals are finitely generated and projective; we proved this using the Auslander-Bridger transpose of simple modules. Furthermore we gave an answer for the necessary part, that is; The necessary condition holds for commutative rings with zero socle.

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NOTATION

| | |
|----------------------------------|---|
| R | an associative ring with unit unless otherwise stated |
| \mathbb{Z}, \mathbb{Z}^+ | the ring of integers, the set of all positive integers |
| \mathbb{Q} | the field of rational numbers |
| R -module | <i>left</i> R -module |
| R -Mod, Mod- R | the categories of <i>left</i> R -modules, <i>right</i> R -modules |
| $\mathcal{A}b = \mathbb{Z}$ -Mod | the category of abelian groups (\mathbb{Z} -modules) |
| $1_A : A \longrightarrow A$ | the identity R -module homomorphism from the R -module A to the R -module A defined by $1_A(x) = x$ for all $x \in A$ |
| \cong | isomorphic |
| \leq | submodule |
| \subseteq | subset or equal |
| $\leq_{\mathcal{A}}$ | \mathcal{A} -submodule |
| $\leq_{\mathcal{N}}$ | neat submodule |
| \ll | small (=superfluous) submodule |
| \trianglelefteq | essential submodule |
| $M \leq N$ | M is a submodule of the R -module N |
| $M \oplus N$ | the direct sum of the R -modules M and N |
| $M \otimes_R N$ | the tensor product of the <i>right</i> R -module M and the <i>left</i> R -module N |
| $\text{ann}(x)$ | the annihilator of an element x of an R -module M ; $\text{ann}(x) = \{r \in R \mid rx = 0\}$ is a left ideal of R , and $Rx \cong R/\text{ann}(x)$ |
| $\text{ann}(M)$ | the annihilator of the R -module M , that is, $\text{ann}(M) = \{r \in R \mid rm = 0 \text{ for all } m \in M\}$ |
| $\text{Ker}(f)$ | the kernel of the R -module homomorphism f |
| $\text{Im}(f)$ | the image of the R -module homomorphism f |
| $\text{Coker}(f)$ | the cokernel of the R -module homomorphism $f : M \longrightarrow N$ is $N/\text{Im}(f)$ |
| $\text{Soc}(M)$ | the socle of the R -module M |

| | |
|------------------------|---|
| $\text{Soc}({}_R R)$ | the socle of the ring R consider as a <i>left</i> R -module |
| $\text{Rad}(M)$ | the radical of the R -module M |
| $\text{Hom}_R(M, N)$ | all R -module homomorphisms from M to N |
| f^* | $\text{Hom}_R(f, R) = f^*$ |
| f_* | $\text{Hom}_R(R, f) = f_*$ |
| $\text{Ext}_R^1(C, A)$ | all equivalence classes of short exact sequences starting with the R -module A and ending with the R -module C |
| $\alpha \mathbb{E}$ | the pushout of a short exact sequence \mathbb{E} with an R -module homomorphism α |
| $\mathbb{E} \gamma$ | the pullback of a short exact sequence \mathbb{E} with an R -module homomorphism γ |
| $\text{Tor}_n^R(A, B)$ | for a right R -module A , apply $A \otimes_R -$ to any projective resolution of the R -module B , and drop the last $A \otimes_R B$ term to obtain the complex |

$$\cdots \longrightarrow A \otimes_R P_{n+1} \xrightarrow{1_A \otimes d_{n+1}} A \otimes_R P_n \xrightarrow{1_A \otimes d_n} \cdots$$

$$\cdots \quad A \otimes_R P_1 \xrightarrow{1_A \otimes d_1} A \otimes_R P_0 \xrightarrow{1_A \otimes d_0} 0$$

$\text{Ker}(1_A \otimes d_n) / \text{Im}(1_A \otimes d_{n+1})$ (the n th homology of this complex), will be isomorphic to $\text{Tor}_n^R(A, B)$

| | |
|-----------------------|--|
| $\text{Tr}_\gamma(M)$ | the Auslander-Bridger transpose of the finitely presented R -module M with respect to its projective presentation γ |
| $\text{Tr}(M)$ | an Auslander-Bridger transpose of the finitely presented R -module M with respect to a projective presentation of it |
| $M \approx N$ | the R -module M is projectively equivalent to the R -module N |
| $\text{pd}(M)$ | the projective dimension of the R -module M |
| \mathcal{A} | a proper class of R -modules |
| ${}^R\mathcal{S}plit$ | the smallest proper class of R -modules consisting of <i>only splitting</i> short exact sequences of R -modules |
| ${}^R\mathcal{A}bs$ | the largest proper class of R -modules consisting of <i>all</i> short exact sequences of R -modules (absolute purity) |
| ${}^R\mathcal{P}ure$ | the proper class projectively generated by all finitely presented R -modules |

| | |
|-------------------------------------|--|
| ${}_R\mathcal{Neat}$ | the proper class of neat-exact sequences of R -modules, that is, the class of all short exact sequences \mathbb{E} of R -modules such that $\text{Hom}_R(S, \mathbb{E})$ is exact for every simple R -module S |
| ${}_R\mathcal{P}\text{-Pure}$ | the proper class of ${}_R\mathcal{P}$ -pure-exact sequences of R -modules, that is, the class of all short exact sequences \mathbb{E} of R -modules such that $(R/P) \otimes_R \mathbb{E}$ is exact for every $P \in \mathcal{P}$ where \mathcal{P} is the set of all left primitive (two sided) ideals of R |
| ${}_{\mathcal{R}}\mathcal{Compl}$ | the proper class defined using complement submodules of R -modules |
| ${}_{\mathcal{R}}\mathcal{Suppl}$ | the proper class defined using supplement submodules of R -modules |
| ${}_{\mathcal{R}}\mathcal{Co-Neat}$ | the proper class injectively generated by all R -modules with zero-radical |
| ${}_{\mathbb{Z}}\mathcal{Pure}$ | the proper class of pure-exact sequences of abelian groups |
| ${}_{\mathbb{Z}}\mathcal{Neat}$ | the proper class of neat-exact sequences of abelian groups |
| ${}_{\mathbb{Z}}\mathcal{Compl}$ | the proper class defined using complement subgroups of abelian groups |
| $\pi(\mathcal{A})$ | all \mathcal{A} -projective R -module |
| $\iota(\mathcal{A})$ | all \mathcal{A} -injective R -modules |
| $\tau(\mathcal{A})$ | all \mathcal{A} -flat <i>right</i> R -modules |
| \mathcal{M} | a class of <i>left</i> R -modules or a class of <i>right</i> R -modules |
| $\pi^{-1}(\mathcal{M})$ | the proper class of R -modules projectively generated by a class \mathcal{M} of R -modules |
| $\iota^{-1}(\mathcal{M})$ | the proper class of R -modules injectively generated by a class \mathcal{M} of R -modules |
| $\tau^{-1}(\mathcal{M})$ | the proper class of R -modules flatly generated by a class \mathcal{M} of <i>right</i> R -modules |

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