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A TAYLOR POLYNOMIAL APPROXIMATION FOR SOLVING LINEAR FREDHOLM INTEGRAL EQUATIONS

Lineer Fredholm Integral Denklemlerin Çözümü İçin Bir Taylor Polinom Yaklaşımı

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ABSTRACT

In this paper, a matrix method is introduced for numerically solving linear Fredholm integral equations. This method in based on first taking the Taylor expansions of the functions in the integral equation and then substituting their matrix forms in the equation. Hence, the matrix equation obtained by a suitable truncation of the Taylor series can be solved and the unknown coefficients can be found approximately. The illustration of the method by numerical examples are given and the results are compaired with the results in the references.

Key Words : Integral Equations

ÖZET

Bu makelede, Fredholm integral denklemlerin sayısal çözümü için bir matris yöntemi sunulmuştur. Yöntem, önce integral denklemdeki fonksiyonların Taylor açılımlarını almaya ve sonra bunların matris formlarını denklemde yerine koymaya dayandırılır. Böylece, Taylor serilerinin uygun bir kesimi ile elde edilen matris denklemi çözülebilir ve bilinmeyen katsayılar yaklaşık olarak bulunabilir. Yöntemin sayısal örnekle açıklaması verilir ve sonuçlar yayınlanmış olan sonuçlarla karşılaştırılır.

1. INTRODUCTION

The Fredholm, Neumann, Hilbert-Schmidt and Chebyshev expansion approaches to solving integral equations are well known. Recently, Taylor polynomial or series solutions of certain classes of ordinary differentia equations [1] and integral equations [2] have been studied. In this paper, in the wiev of the mentioned studies, we discuss a Taylor polynomial method, depended on the matrix equation, for solving the linear Fredholm integral equation of the second kind, which is defined by

$$g(x) = f(x) + \lambda \int_a^b K(x,y)\, g(y)dy \qquad (1)$$

Here g is the unknown function, while the functions f(x) and K(x,y) are the known functions, and the limits of integration a and b are constants; λ is a nonzero, reel or complex parameter.

2.METHOD OF SOLUTION

We assume that the integral equation (1) has a solution in the form

$$g(x) = \sum_{n=0}^{\infty} g_n \, (x-c)^n \; . \qquad (2)$$

This is the power series (Taylor series about x = c) for the unknown function g(x) with the unknown coefficients g_n. The Taylor series of the known function f(x) about x = c can be computed as

$$f(x) = \sum_{n=0}^{\infty} f_n \, (x-c)^n \; ; \quad f_n = \frac{f^{(n)}(c)}{n!} \, . \qquad (3)$$

Also the kernel function K(x,y) can be expanded in a Taylor series about x=c, y=c as follows

$$K(x,y) = \sum_{n=0}^{\infty} \frac{1}{n!} \left[(x-c) \frac{\partial}{\partial x} + (y-c) \frac{\partial}{\partial y} \right]^n K(c,c)$$

or

$$K(x,y) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} k_{nm} \, (x-c)^n \, (y-c)^m \qquad (4)$$

where

$$k_{nm} = \frac{1}{n!\, m!} \; \frac{\partial^{n+m} K(c,c)}{\partial x^n \, \partial y^n} \; ; \; n,m = 0,1,2,\ldots$$

For the approximate solution of the integral equation (1), the infinite series (2), (3) and (4) have to be truncated after a finite number of terms, say N+1 (n, m = 0,1,2,...,N). In this case, the functions g(x), f(x) and K(x,y) are approximated by the polynomials of degree N in the forms, respectively,

$$g(x) = \sum_{n=0}^{N} g_n \, (x-c)^n \qquad (5)$$

$$f(x) = \sum_{n=0}^{N} f_n \, (x-c)^n \; . \qquad (6)$$

and

$$K(x,y) = \sum_{n=0}^{N} \sum_{m=0}^{N} k_{nm} \, (x-c)^n \, (y-c)^m . \qquad (7)$$

where

$$f_n = \frac{f^{(n)}(c)}{n!} \; , \quad k_{nm} = \frac{1}{n!\,m!} \, \frac{\partial^{n+m} K(c,c)}{\partial x^n \, \partial y^m} \qquad (8)$$

Each of the functions (5), (6) and (7) can be written in the matrix form as follows:

$$g(x) = XG \qquad f(x) = XF$$

$$K(x,y) = XKY^t \qquad g(y) = YG$$

where

$$X = [\,(x-c)^0 \; (x-c)^1 \; \ldots \; (x-c)^N\,]$$

$$Y = [\,(y-c)^0 \; (y-c)^1 \; \ldots \; (y-c)^N\,]$$

$$G = \begin{bmatrix} g_0 \\ g_1 \\ \vdots \\ g_N \end{bmatrix} , \quad K = \begin{bmatrix} k_{00} & k_{01} & & k_{0N} \\ k_{10} & k_{11} & \cdots & k_{1N} \\ \cdot & \cdot & \cdots & \cdot \\ k_{N0} & k_{N1} & \cdots & k_{NN} \end{bmatrix} , \quad F = \begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_n \end{bmatrix}$$

Substituting these matrix forms of the functions g(x), f(x) and K(x,y) into the integral equation (1), we have

$$XG = XF + \lambda \int_a^b X K Y^t Y G \, dy$$

or

$$G = F + \lambda KHG \qquad (9)$$

where

$$H = [h_{nm}] = \int_a^b Y^t . Y \, dy$$

$$h_{nm} = \frac{(b-c)^{n+m+1} - (a-c)^{n+m+1}}{n+m+1} \quad (10)$$

The matrix equation (9) may be written in the form

$$(I - \lambda KH) G = F \quad (11)$$

where I is the unit matrix. In the equation (11), if

$$D(\lambda) = | I - \lambda KH | = 0, \quad (12)$$

we obtain the matrix equation

$$G = (I - \lambda KH)^{-1} F. \quad (13)$$

Thus the unknown coefficients g_n, $(n = 0,1,2,...,N)$ are uniquely determined by the equation (13).

The matrix equation (11) corresponds to a system of (N+1) algebraic equations for the unknowns g_n. The determinant $D(\lambda) = |I - \lambda KH|$ of this system is a polynomial in λ of degree at most (N+1). Morever, it is not identically zero, since, when $\lambda = 0$, it reduces to unity. For all values of λ for which $D(\lambda) = 0$, the matrix equation (11) and thereby the integral equation (1), has a unique solution. This solution is given by the Taylor polynomial

$$g(x) = \sum_{n=0}^{N} g_n (x-c)^n. \quad (14)$$

On the other hand, for all values of λ for which $D(\lambda)$ becomes equal to zero, and with it the integral equation (1), either is insoluble or has an infinite number of solutions. In some cases, the matrix equation can be solved only for some particular values of the quantities f_n.

3.NUMERICAL EXAMPLES

The method of this paper is useful in finding approximate and exact solutions of certain integral equations. We illustrate it by the following exampes:

Exampe 1. Let us first illustrate the method with the help of the integral equation

$$g(x) = \operatorname{ch} x - \frac{x}{2} + \frac{1}{3} \int_0^1 xy \, g(y) \, dy \quad (15)$$

so that

$$f(x) = chx - \frac{x}{2}, \quad K(x,y) = xy, \quad \lambda = \frac{1}{3}, \quad a = 0, \quad b = 1.$$

Let c=0 and N=6. Then we evaluate the quantities f_n, k_{nm} and h_{nm}, from the relations (8) and (10), as

$$f_0 = 1, \quad f_1 = -\frac{1}{2}, \quad f_2 = \frac{1}{2}, \quad f_3 = 0, \quad f_4 = \frac{1}{24}, \quad f_5 = 0, \quad f_6 = \frac{1}{720}$$

$$k_{nm} = \begin{cases} 1, & \text{if } n = m = 1 \\ 0, & \text{otherwise} \end{cases}$$

$$h_{nm} = \frac{1}{n+m+1} \quad ; \; n,m = 0,1,\ldots, 6$$

Substituting these values in the matrix equation (11), we get

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{6} & \frac{8}{9} & -\frac{1}{12} & -\frac{1}{15} & -\frac{1}{18} & -\frac{1}{21} & -\frac{1}{24} \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} g_0 \\ g_1 \\ g_2 \\ g_3 \\ g_4 \\ g_5 \\ g_6 \end{bmatrix} = \begin{bmatrix} 1 \\ -\frac{1}{2} \\ \frac{1}{2} \\ 0 \\ \frac{1}{24} \\ 0 \\ \frac{1}{720} \end{bmatrix}$$

we solve it for the quantities g_n and get

$$g_0 = 1 \quad g_1 = -\frac{4999}{15360} \quad g_2 = \frac{1}{2} \quad g_3 = 0 \quad g_4 = \frac{1}{24} \quad g_5 = 0 \quad g_6 = \frac{1}{720}$$

Substituting these values in (5) we obtain

$$g(x) = 1 - \frac{4999}{15360} x + \frac{1}{2} x^2 + \frac{1}{24} x^4 + \frac{1}{720} x^6. \qquad (16)$$

The exact solution of the integral equation (14) is

$$g(x) = chx - \frac{3}{16}\left(1 + \frac{2}{e}\right)x \qquad (17)$$

which is given by [4]. The comparison of the solutions (16) and (17) as follows:

x	Exact solution g (x)	Taylor Polynomial Appr. g(x)
0.0	1	1
0.1	0.972459	0.922458
0.2	0.954976	0.954976
0.3	0.947702	0.947701
0.4	0.950890	0.950890
0.5	0.964899	0.964898
0.6	0.990192	0.990191
0.7	1.027351	1.027349
0.8	1.077071	1.077066
0.9	1.140177	1.140165
1.0	1.217626	1.217600

Example 2. Let us now study the integral equation

$$g(x) = e^{x} - 2\sin x + \int_{-1}^{1} (\sin x)e^{-y}\, g(y)\, dy \qquad (18)$$

where

$f(x) = e^{x} - 2\sin x$, $K(x,y) = (\sin x)\, e^{-y}$, $\lambda = 1$, $a=-1$, $b=1$.

If we choose $c=0$ and $N=5$, by means of the relations (8) and (10) we obtain the matrices F, K and
in the forms

$$F = \begin{bmatrix} 1 & -1 & 1/2 & 1/2 & 1/24 & -1/120 \end{bmatrix}^t$$

$$K = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 1/2 & -1/6 & 1/24 & -1/120 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1/6 & 1/6 & -1/12 & 1/36 & -1/144 & 1/720 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1/120 & -1/120 & 1/240 & -1/720 & 1/2880 & -1/14400 \end{bmatrix}$$

$$H = \begin{bmatrix} 2 & 0 & 2/3 & 0 & 2/5 & 0 \\ 0 & 2/3 & 0 & 2/5 & 0 & 2/7 \\ 2/3 & 0 & 2/5 & 0 & 2/7 & 0 \\ 0 & 2/5 & 0 & 2/7 & 0 & 2/9 \\ 2/5 & 0 & 2/7 & 0 & 2/9 & 0 \\ 0 & 2/7 & 0 & 2/9 & 0 & 2/11 \end{bmatrix}$$

We substitute them in the matrix equation (11) and solve it for the quantities g_n and get

$g_0=1$, $g_1=0.999376$, $g_2=0.5$, $g_3=0.16677$, $g_4=0.041667$, $g_5=0.008327$

Thus the Taylor polynomial solution of (18) is

$$g(x)= 1+ 0.999376x + 0.5x^2 + 0.16677x^3+0.0411667x^4 + 0.008327x^5.$$

By means of the known methods, we can find the exact solution to be $g(x) = e^x$. The comparison of these solutions is left to the reader.

Example 3. Let us solve the integral equation

$$g(x) = 2x + \lambda \int_0^1 (x + y)\, g(y)\, dy \qquad (19)$$

by the method of the Taylor polynomial approximation.

By taking c=0 and N = 2, we obtain the matrices F, K and H as

$$F = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \qquad K = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad H = \begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{bmatrix}$$

Substituting these martices in (11) and then solving it for the quantities g_0, g_1, and g_2, we get.

$$g_0 = \frac{8\lambda}{12-12\lambda-\lambda^2}, \quad g_1 = \frac{12\,(2-\lambda)}{12-12\lambda-\lambda^2}, \quad g_2 = 0$$

Next we substitute these values in (5) and obtain

$$g(x) = \frac{8\lambda + 12(2-\lambda)x}{12-12\lambda-\lambda^2} \qquad (20)$$

which is the exact solution [3]. On the other hand, note [3] that the solution obtained by the method of successive approximations to the tird order is

$$g_3(x) = 2x + \lambda\left[x + \frac{2}{3}\right] + \lambda^2\left[\frac{7}{6}x + \frac{2}{3}\right] + \lambda^3\left[\frac{13}{12}x + \frac{5}{8}\right].$$

The determinant $D(\lambda) = |I - \lambda KH| = 0$ gives $\lambda^2 + 12\lambda - 12 = 0$. Thus, the eigenvalues are

$$\lambda 1 = -6 + 4\sqrt{3} \qquad \lambda 2 = -6 - 4\sqrt{3}.$$

For these two values of λ, the homogenous part of (19) has a nontrivial solution, while the integral equation (19) is, in general, not soluble. When λ differs from these values, the solution of (19) becomes (20)

Eample 4. Solve the integral equation

$$g(x) = e^x - x - \int_0^1 x\,(e^{xy}-1)\, g(y)\, dy$$

so that

$f(x) = e^x - x$, $\lambda=-1$, $K(x,y) = x\,(e^{xy}-1)$, $a = 0$, $b = 1$

ι us c = υ anu N = 2. Then we have

$$F = \begin{bmatrix} 1 \\ 0 \\ 1/2 \end{bmatrix} \qquad K = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 2 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad H = \begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{bmatrix}$$

and therefore, from (11),

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 7/6 & 1/6 \\ 1/2 & 1/3 & 5/4 \end{bmatrix} \begin{bmatrix} g_0 \\ g_1 \\ g_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1/2 \end{bmatrix} \Rightarrow \begin{bmatrix} g_0 \\ g_1 \\ g_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Thus we obtain $g(x) = 1$ which is the exact solution [2].

Note that the approximate solution of (21) by means of the Taylor series is given by [3] in the form

$$g(x) = e^x - x - 0.5010x^2 - 0.1671x^3 - 0.0423x^4$$

Eample 5. Let us consider the equation

$$g(x) = 4x-2 + 6 \quad (x-y)\, g(y)\, dy.$$

In this case

$f(x) = 4x-2$, $K(x,y) = x-y$, $\lambda = 6$, $a = 1$, $b = 2$.

If we take $c = 1$ and $N = 2$, we find

$$F = \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}, \quad K = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad H = \begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{bmatrix}$$

and

$$\begin{bmatrix} g_0 \\ g_1 \\ g_2 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 3/2 \\ -6 & -2 & -2 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ 7 \\ 0 \end{bmatrix}$$

Thereby we find that

$$g(x) = -3 + 7(x-1) \text{ or } g(x) = 7x - 10$$

which is the exact solution. In the cases of $c = 0$ and $N > 1$, the same result is obtained.

4. CONCLUSIONS

In this paper, the usefulness of Taylor polynomial expansions for the solution of Fredholm integral equations of the second kind is demostrated. The nonsingular Fredholm integral equations of the second kind are chosen to show the accuracy of the method. Also the method can be applied to the integral equations of the first kind and homogenous. A considerable advantage of the method is that the solution is expressed as a truncated Taylor series. This means that, after calculation of the series coefficients, the solution g(x) of the equation can be evaluated for arbitrary values of x at low computation effort. This implies that our method is very efficient. A disadvantage is that the method is not generally applicable, since it requires a matrix equation which depends on the Taylor series expansions of the function f and K about $x = c$, $y = c$. However it would appear that the Taylor polynomial method shows to best advantage when the kernel K(x,y) and inhomogenous part f(x) are both infinitely differentiable functions. In this case, g(x) is also infinitely differentiable, and the results of Section 2 and 3 indicate that the Taylor series expansion of such a function will converge fairly rapidly.

For computational efficiency, some estimate for N, the degree of the approximating polynomial to g (x), should be available. Because the choice of N determines the precision of the solution g(x). If N is chosen too large, unnecessary labour may be done; but If N is taken a small value, the solution will not be sufficiently accurate. Therefore N must be chosen sufficiently large to get a reasonable approximation. In this case computer may be used.

An interesting feature of the Taylor polynomial method is that the method is used in finding exact solutions in many cases, as demostrated by examples in Section 3. In the case of integral equations as well as in the case of ordinary differential equations, the method gives higher convergence speed than that of the picard method of successive approximations, provided that the truncation limit N is appropriate (Ex. 3).

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