TAYLOR POLYNOMIAL SOLUTIONS OF NONLINEAR FREDHOLM INTEGRAL EQUATIONS OF THE SECOND KIND

Lineer Olmayan İkinci Tür Fredholm İntegral Denklemlerin Taylor Polinom Çözümleri

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ABSTRACT

In this paper, the methods presented in [1] and [2] for the solution of linear Fredholm integral equations are applied to certain nonlinear Fredholm integral equations of the second kind. Two examples, which had been examined by [3], are solved in terms of Taylor polynomials and the results are compaired.

ÖZET

Bu makalede, lineer integral denklemlerin çözümü için [1] ve [2] de verilen yöntemler, ikinci tür lineer olmayan bazı integral denklemlere uygulanmıştır. [3]'de incelenmiş olan iki örnek, Taylor polinomları cinsinden çözülüp sonuçlar karşılaştırılmıştır.

1. INTRODUCTION

A Taylor expansion approach for solving linear Fredholm integral equations in the form

$$g(x) = f(x) + \lambda \int_a^b K(x,y)\, g(y)\, dy$$

has been presented by Kanwal and Liu [1], where g is the unknown function, while the functions f(x) and K(x,y) are the known functions, and a,b and λ are constants. The method (Taylor I) is based on first differentiating both sides of the integral equation n times and then substituting the Taylor series for the unknown function in the resulting equation. In addition, a matrix method (Taylor II) has been given by

Sezer and Doğan [2], which is based on first taking the Taylor expansions of the functions in the integral equation and then substituting their matrix forms in the equation. In both cases, it has been obtained a linear algebraic system with the unknown coefficients or a matrix equation and has been solved approximately by a suitable truncation scheme.

In this study, the basic ideas of the mentioned works (Taylor I and Taylor II methods) are applied to the nonlinear Fredholm integral equations of the second kind

$$g(x) = f(x) + \lambda \int_a^b K(x,y)\, [g(y)]^2\, dy \qquad (1)$$

and the solution is expressed in the form

$$g(x) = \sum_{k=0}^{N} \frac{g^{(k)}(0)}{k!} x^k \qquad (2)$$

or

$$g(x) = \sum_{k=0}^{N} g_k x^k, \quad g_k = \frac{g^{(k)}(0)}{k!} \qquad (3)$$

which is a Taylor polynomial of degree N at x=o. Here $g^{(k)}_{(o)}$ or g_k, $k=0,1,\ldots,N$ are the Taylor coefficients to be determined.

2.THE METHODS OF SOLUTION

Taylor I method: To obtain the solution of the equation (1) in the form (2), we follow the similar way as [1]. In this case, we first differentiate the both sides of (1) n times with respect to x and have

$$g^{(n)}(x) = f^{(n)}(x) + \lambda \int_a^b \frac{\partial^{(n)} K(x,y)}{\partial x^n} [g(y)]^2\, dy.$$

When we put x=o in this equation, we get

$$g^{(n)}(0) = f^{(n)}(0) + \lambda \int_a^b \frac{\partial^{(n)} K(x,y)}{\partial x^n}\Bigg|_{x=0} G(y)\,dy \quad (4)$$

where $G(y) = [g(y)]^2$. Next, we expand $G(y)$ in Taylor series at $y=o$, i.e.

$$G(y) = \sum_{m=0}^{\infty} \frac{1}{m!} G^{(m)}(0)\, y^m, \quad (5)$$

and substitute it in (4); where the coefficients $G^{(m)}(o), m=0,1,\ldots$ can be computed by Leibniz's rule as

$$G^{(m)}(0) = \sum_{i=0}^{m} \binom{m}{i} g^{(m-i)}(0)\, g^{(i)}(0) \quad (6)$$

The result is

$$g^{(n)}(0) = f^{(n)}(0) + \lambda \int_a^b \frac{\partial^{(n)} K(x,y)}{\partial x^n}\Bigg|_{x=0} \left[\sum_{m=0}^{\infty} \frac{1}{m!} G^{(m)}(0)\, y^m\right] dy$$

or

$$g^{(n)}(0) = f^{(n)}(0) + \lambda \sum_{m=0}^{\infty} T_{nm} \cdot G^{(m)}(0), \quad (7)$$

where, for $n,m = 0,1,2,\ldots$

$$T_{nm} = \frac{1}{m!} \int_a^b \frac{\partial^{(n)} K(x,y)}{\partial x^n}\Bigg|_{x=0} y^m\,dy \quad (8)$$

The relations (7) for $n,m=0,1,\ldots$ occur an infinite nonlinear algebraic system with the unknown $g^{(n)}(o)$. This system can be solved approximately by taking $n,m=0,1,\ldots,N$ so that N is a sufficiently large number. In this case, the quantities $g^{(m)}(o)$ obtained by (7) for $n=0,1,\ldots,N$ correspond to the Taylor coefficients of the function $g(x)$ at $x=o$. Thus the solution of (1) becomes a Taylor polynomial in the form (2).

Taylor II Method: This is a matrix method, the similar to [2], which makes use of Taylor series about origin and can be used to obtain a polynomial solution of (1) in the form (3). For this purpose, we assume that the functions f(x), K(x,y) and g(x) are approximated by the truncated Taylor series (Taylor polynomials) of the forms, respectirely,

$$f(x) = \sum_{n=0}^{N} f_n x^n, \quad f_n = \frac{f^{(n)}(0)}{n!} \tag{9}$$

$$K(x,y) = \sum_{n=0}^{N} \sum_{m=0}^{N} k_{nm} x^n y^m, \quad k_{nm} = \frac{1}{n!\,m!} \left. \frac{\partial^{(n+m)} K(x,y)}{\partial x^n \, \partial y^m} \right|_{\substack{x=0,\\ y=0}} \tag{10}$$

and

$$g(x) = \sum_{n=0}^{N} g_n x^n, \quad g_n = \frac{g^{(n)}(0)}{n!} \tag{11}$$

In addition, using (11) it is possible to express the term $[g(y)]^2 = G(y)$ of Eq.(1) as

$$G(y) = \sum_{n=0}^{N} \sum_{m=0}^{N} g_n g_m y^{n+m} \tag{12}$$

For n,m= 0,1, ...,N, the functions in equation (1) can be written in the matrix forms

$$f(x) = XF \qquad g(x) = XG \tag{13}$$
$$K(x,y) = X\,K\,Y^t \qquad G(y) = Y^*B$$

so that

$$X = [1\; x\; x^2 \ldots x^N] \qquad G = [g_o\; g_1 \ldots g_N]^t$$
$$Y = [1\; y\; y^2 \ldots y^N] \qquad F = [f_o\; f_1 \ldots f_N]^t$$
$$Y^* = [1\; y\; y^2 \ldots y^{2N}] \qquad K = [k_{nm}], n;m=0,1, \ldots, N$$
$$B = [b_o\; b_1 \ldots b_{2N}]^t$$

where F, K and B are matrices with elements

$$f_n = \frac{f^{(n)}(0)}{n!} \tag{14}$$

$$k_{nm} = \frac{1}{n!m!} \left. \frac{\partial^{(n+m)} K(x,y)}{\partial x^n \partial y^m} \right|_{\substack{x=0\\y=0}} \qquad (15)$$

and

$$b_j = \begin{cases} \sum_{k=0}^{j} g_k\, g_{k-j} & ; \quad j = 0, 1, \ldots, N \\ \sum_{k=j-N}^{N} g_k\, g_{j-k} & , \quad j = N+1,\ N+2, \ldots, 2N \end{cases} \qquad (16)$$

Next, we substitute the matrix forms (13) in the integral equation (1) and have the matrix equation

$$G = F + \lambda\, KHB \qquad (17)$$

where H is matrix with elements

$$h_{mj} = \int_a^b y^{m+j}\, dy = \frac{b^{m+j+1} - a^{m+j+1}}{m+j+1} \qquad (18)$$

$$m = 0, 1, \ldots, N \ ; \quad j = 0, 1, \ldots, 2N$$

Also, the equation (17) corresponds to a nonlinear algebraic system with the unknown coefficients g_n, n=0,1, ..., N. This system can be solved approximately by the known methods [4].

3. ILLUSTRATIONS

In this paper some methods of approximation solution, with Taylor polynomial terms, and at the some time, if exist, analytical solutions as polynomial form, of the integral equation (1) have been evaluated. Now we illustrate it by the following examples.

Example 1. Let us find the solution of the integral equation.

$$g(x) = \frac{1}{2} - \frac{1}{8}x + \int_{-1}^{1} \mathrm{Sin}\left[\frac{1}{4}x(y+1)\right][g(y)]^2\,dy \qquad (19)$$

in terms of the Taylor polynomial of third degree, so that

$$\left\langle f(x)=\frac{1}{2}-\frac{1}{8}x, K(x,y)=\sin\left[\frac{1}{4}x(y+1)\right], a=-1,\ b=1,\ d=1,\ N=3\right\rangle$$

a) **Taylor I Method:** For this purpose we evaluate the quantities $f^{(n)}(o)$ and T_{nm} for, n,m=0,1,2,3:

$f^{(0)}(0)=1/2$ $\quad f^{(1)}(0) = -1/8$ $\quad f^{(2)}(0) = 0$ $\quad f^{(3)}(0) = 0$

$T_{00}=0$	$T_{01} = 0$	$T_{02} = 0$	$T_{03} = 0$
$T_{10}=1/2$	$T_{11} = 1/6$	$T_{12} = 1/12$	$T_{13} = 1/60$
$T_{20}=0$	$T_{21} = 0$	$T_{22} = 0$	$T_{23} = 0$
$T_{30}=-\frac{1}{16}$	$T_{31}=-\frac{3}{80}$	$T_{32}=-\frac{7}{480}$	$T_{33}=-\frac{13}{3360}$

Next, we substitute these values in (7) and get the nonlinear system

$$g^{(0)}(0) = 1/2$$

$$g^{(1)}(0) = \frac{1}{2}G^{(0)}(0) + \frac{1}{6}G^{(1)}(0) + \frac{1}{12}G^{(2)}(0) + \frac{1}{60}G^{(3)}(0) - \frac{1}{8}$$

$$g^{(2)}(0)=0$$

$$g^{(3)}(0) = -\frac{1}{16}G^{(0)}(0) - \frac{3}{80}G^{(1)}(0) - \frac{7}{480}G^{(2)}(0) - \frac{13}{3360}G^{(3)}(0)$$

where

$G^{(0)}(0)= [g^{(0)}(0)]^2$ $\qquad G^{(1)}(0) = 2\,g^{(0)}(0)\ g^{(1)}(0)$

$$G^{(2)}(0) = 2\, g^{(0)}(0)\, g^{(2)}(0) + 2\, [g^{(1)}(0)]^2$$

$$G^{(3)}(0) = 2\, g^{(0)}(0)\, g^{(3)}(0) + 6\, g^{(1)}(0)\, g^{(2)}(0)$$

This nonlinear system has two solutions:

$g^{(0)}(0) = 0{,}5$	$g^{(0)}(0) = 0{,}5$
$g^{(1)}(0) = -0{,}000311$	$g^{(1)}(0) = 5{,}018627$
$g^{(2)}(0) = 0$	$g^{(2)}(0) = 0$
$g^{(3)}(0) = -0{,}015551$	$g^{(3)}(0) = -0{,}934815$

Consequently, substituting these values in (2),we obtain two solutions of the integral equation (19) as

$$g_1(x) = 0{,}5 - 0{,}000311\, x - 0{,}002592\, x^3$$

and

$$g_2(x) = 0{,}5 + 5.018627\, x - 0{,}155803\, x^3$$

b) Taylor II Method. Using the relations (14), (15) and (18) for n,m=0,1,2,3, j=0,1,...6 we obtain the matrices

$$F = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{8} \\ 0 \\ 0 \end{bmatrix} \qquad K = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{1}{384} & -\frac{1}{128} & -\frac{1}{128} & -\frac{1}{384} \end{bmatrix}$$

$$H = \begin{bmatrix} 2 & 0 & 2/3 & 0 & 2/5 & 0 & 2/7 \\ 0 & 2/3 & 0 & 2/5 & 0 & 2/7 & 0 \\ 2/3 & 0 & 2/5 & 0 & 2/7 & 0 & 2/9 \\ 0 & 2/5 & 0 & 2/7 & 0 & 2/9 & 0 \end{bmatrix}$$

Substituting these matrices in Eg. (17) and then using the result matrix equation, we have the nonlinear algebraic system

$$g_0 = \frac{1}{2} \qquad g_2 = 0$$

$$g_1 = -\frac{1}{8} + \frac{1}{2} b_0 + \frac{1}{6} b_1 + \frac{1}{6} b_2 + \frac{1}{10} b_3 + \frac{1}{10} b_4 + \frac{1}{14} b_5 + \frac{1}{14} b_6$$

$$g_3 = -\frac{1}{96} b_0 - \frac{1}{160} b_1 - \frac{7}{1440} b_2 - \frac{13}{3360} b_3 - \frac{11}{3360} b_4 - \frac{17}{6048} b_5 - \frac{5}{2016} b_6$$

where b_j, $j = 0, 1, \ldots, 6$ are defined by (16). The solutions of this system are

$g_0 = 0{,}5$	$g_0 = 0{,}5$
$g_1 = -\,0{,}0003075$	$g_1 = 5{,}2096699$
$g_2 = 0$	$g_2 = 0$
$g_3 = -\,0{,}0025923$	$g_3 = -\,0{,}16104615$

and thereby the solutions of the given integral equation become

$$g_1(x) = 0{,}5 - 0{,}0003075\, x - 0{,}0025923\, x^3$$

$$g_2(x) = 0{,}5 + 5{,}2096699\, x - 0{,}16104615\, x^3$$

The numerical solution of the integral equation (19) in Chebyshev series was given by Shimasaki and Kiyono [3]. The final solutions are tabulated together with Chebyshev solution in the Tables 1 and 2.

Table 1. The Solution $g_1(x)$ for Example 1

x	Chebyshev 5th degree	Taylor I 3th degree	Taylor II 3th degree
-1,0	0,502879	0,502903	0,502899
-0,5	0,500478	0,500479	0,500478
0	0,499999	0,5	0,5
0,5	0,499522	0,499520	0,499522
1,0	0,497119	0,497097	0,497100

Table 2. The Solution g_2 (x) for Example 1

x	Chebyshew 8th degree	Taylor I 3th degree	Taylor II 3th degree
-1,0	-4,548743	-4,362825	-4,548624
-0,5	-2,083991	-1,989838	-2,084704
0	0,499999	0,5	0,5
0,5	3,083992	2,989838	3,084704
1,0	5,548745	5,362825	5,548624

Example 2. Let us solve the integral equation

$$g(x) = \frac{5}{6} x2 - \frac{8}{105} x - 1 + \int_0^1 (x^2y+xy^2) [g(y)]^2 dy \qquad (20)$$

so that

$$f(x) = \frac{5}{6} x2 - \frac{8}{105} x-1, \quad K(x,y) = x^2y+xy^2, \; a=o, \; b=1, \; \lambda = 1$$

Let us apply the first method to (20) in the case N=4. Then we evaluate the quantities $f^{(n)}(0)$ and T_{nm}, n,m = 0,1,2,3,4;

$$f^{(0)}(0) = -1 \quad f^{(1)}(0) = -\frac{8}{105} \quad f^{(2)}(0) = \frac{5}{3} \quad f^{(3)}(0)=0 \quad f^{(4)}(0)=0$$

$$T_{0m}=0 \quad T_{1m} = \frac{1}{m!(m+3)} \quad T_{2m} = \frac{2}{m!(m+2)} \quad T_{3m}=0 \quad T_{4m}=0$$

We substitute these values in (7) for n,m=0,1,2,3,4 and get the nonlinear algebraic system

$$g^{(0)}(0) = -1 \quad g^{(3)}(0) = 0 \quad g^{(4)}(0) = 0$$

$$g^{(1)}(0) = \frac{1}{3} G^{(0)}(0) + \frac{1}{4} G^{(1)}(0) + \frac{1}{10} G^{(2)}(0) + \frac{1}{36} G^{(3)}(0) + \frac{1}{168} G^{(4)}(0) - \frac{8}{105}$$

$$g^{(2)}(0) = G^{(0)}(0) + \frac{2}{3} G^{(1)}(0) + \frac{1}{4} G^{(2)}(0) + \frac{1}{15} G^{(3)}(0) + \frac{1}{72} G^{(4)}(0) + \frac{5}{3}$$

where $G^{(m)}(0)$, m=0,1,2,3,4, are defined by (6). From this system, the coefficients $g^{(n)}(0)$, n=0,1,2,3,4 are computed as

$$g^{(0)}(0)= -1 \quad g^{(1)}(0)= 0 \quad g^{(2)}(0)=2 \quad g^{(3)}(0)= 0 \quad g^{(4)}(0)= 0$$

and thus the solution of the integral equation (20) becomes $g(x) = x^2-1$ which is the exact solution.

By taking N=2 and using the second method, we have the same result easily.

4-CONCLUSIONS

Taylor Methods have been used to obtain a polynomial solution of nonlinear Fredholm integral equations of the second kind, which is expressed in the form

$$g(x) = \sum_{n=0}^{N} \frac{g^{(n)}(0)}{n!} x^n \qquad \text{or} \qquad g(x) = \sum_{n=0}^{N} g_k x^n ,$$

provided that the functions f(x) and K(x,y) can be expanded to Taylor series at origin. Here the number N, the degree of the approximating polynomial to g(x), must be chosen sufficiently large.

Generally, the first method depends on the differentiating and integrating operations; we can be encountered with difficulties. In the second method, the elements of matrices are based on the algebraic relations and may be computed easily. Therefore, in many cases, this method is rather useful and requires less computational labour than the first one.

The mentioned methods, for small values of N, can be used in finding a good starting solution for the Newton method [3]. On the other hand, for the appropriate values of N, they give results of nearly the same accuracy by Newton method; as shown in Table 1 and 2. Here we observe that the value of N in the first method must be taken greater than the second one.

An important property of these methods is that we get exact solutions in many cases, as demostrated in Example 2. In this example, we obtained the exact solution by taking n=4 for the first method and N=2 for the second method. So it is appropriate to consider a priori estimate for N.

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