PLASTIC ZONES IN AN INFINITELY LONG TRANSVERSELY ISOTROPIC SOLID CYLINDER CONTAINING A RING-SHAPED CRACK

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We certify that we have read this thesis and that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Doctor of Philosophy.

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ABSTRACT

In this study, a problem of an infinitely long solid cylinder containing a ring shaped-crack is considered. The ring-shaped crack is embedded in the mid-plane of the infinite cylinder. The problem is formulated for a transversely isotropic material by using integral transform technique and solved under the effect of uniform load. Due to the geometry of the configuration, Hankel and Fourier integral transform techniques are chosen and the problem is reduced to a singular integral equation. This integral equation is solved numerically by using Gaussian Quadrature Formulae and the values are evaluated for discrete points. The plastic zone widths are obtained by using the plastic strip model. They are plotted for various ring-shaped crack sizes and transversely isotropic materials.

Keywords: Transversely isotropic material, ring-shaped crack, singular integral equations, plastic zone.

ÖZET

Bu çalışmada yüzük şeklinde çatlağa haiz sonsuz uzun bir silindir ele alınmıştır. Yüzük şeklindeki çatlak sonsuz silindirin orta düzleminde bulunmaktadır. Problem enine izotrop bir malzeme için integral dönüşüm tekniği kullanılarak formüle edilmiş ve düzgün yayılı yük etkisi altında çözülmüştür. Problemin geometrisi gereğince Hankel ve Fourier integral dönüşüm teknikleri seçilmiş ve problem bir tekil integral denklemine indirgenmiştir. Bu integral denkleminin belli noktalardaki değerleri Gauss Quadrature formülü kullanılarak sayısal olarak elde edilmiştir. Plastik bölge genişlikleri, plastik bant modeli kullanılarak elde edilmiştir. Çeşitli yüzük çatlak boyutları ve değişik enine izotrop malzemeler için plastik bölge genişlikleri grafik olarak verilmiştir.

Anahtar Kelimeler : Enine izotrop malzeme, yüzük şeklinde çatlak, tekil integral denklemi, plastik bölge.

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LIST OF SYMBOLS

Symbol	
ε_i	Strain vector
a_{ij}	Strain coefficient matrix
σ_j	Stress vector
c_{ij}	Modulii of material
R	Radius of long solid cylinder
a	Crack inner radius
b	Crack outer radius
a_p	Plastic zone radius at inner crack tip
b_p	Plastic zone radius at outer crack tip
E	Young's modulus for tension and compression in plane of isotropy
E'	Young's modulus for tension and compression in a direction to perpendicular plane of isotropy
ν	Poisson's ratio characterizing contraction in a plane of isotropy for tension
ν'	Poisson's ratio characterizing contraction in a direction normal to plane of isotropy
G	Shear modulus for plane of isotropy
G'	Shear modulus for plane perpendicular to plane of isotropy
u_r, u_θ, ω	Displacement components along r, θ, and z directions
$\phi(r, z)$	Stress function
a, b, c, and d	Constants for stress function for a transversely isotropic body
λ, α	Dummy variables of infinite integral transforms
J_0, J_1	Bessel functions of the first kind order of zero and one

I_0, I_1	Modified Bessel functions of the first kind order of zero and one
K_0, K_1	Modified Bessel functions of the second kind order of zero and one
m_2, m_4, A, C	Unknown functions determined from boundary conditions
p(r)	External load applied on solid cylinder
p_0	Constant pressure prescribed on crack faces
Y	Flow stress of material
p_0/Y	Load ratio
G(r)	Unknown function
K(x)	Complete elliptic integral of first kind
E(x)	Complete elliptic integral of second kind
$k(r, \rho)$	Kernel of singular integral equation
η, τ	Normalization variables
k(a)	Stress intensity factor at inner crack tip
k(b)	Stress intensity factor at outer crack tip

CHAPTER ONE
INTRODUCTION

1. Introduction

Composite materials have a long history of usage. Their beginning are unknown, but all recorded history contains references to some form of composite material. For instance straw was used to strengthen mud bricks. More recently, fiber reinforced resin composites that have high specific strength and specific modulus have become important in weight sensitive applications.

Composite materials are conceptually simple. They offer unique opportunities in design beyond being lighter substituties of conventional materials. The structural performance offered by composite materials is much more versatile than can be realized with conventional materials.

Fiber reinforced composite materials with equal number of fibers in two perpendicular directions have been characterized as a transversely isotropic medium having five elastic constants [1,2]. Graphite-epoxy, barium titanate, E-glass can be treated as transversely isotropic materials. Hexagonal materials such as magnesium, cadmium and zinc are also transversely isotropic.

Axially symmetric deformation of a transversely isotropic body of revolution has been studied by Lekhnitskii [3].

The distribution of stress in a transversely isotropic cylinder containing penny-shaped crack has been investigated by Parhi & Atsumi [4]. Dahan [5,6] has searched both stress intensity factor and stress distribution in a transversely isotropic solid containing a penny shaped crack. Singular stresses in a transversely isotropic circular cylinder with circumferential edge crack have been examined by Atsumi & Shindo [7]. Konishi [8,9] has studied crack problems in transversely isotropic strip and medium. Fildiş has studied stress intensity factors for an infinitely long transversely isotropic solid cylinder containing a ring shaped cavity [10].

In addition to the numerous studies stated above, plastic deformations have also been considered by some researches. Notably among them Olesiak & Shadley [11] determined the plastic zone in a thick layer with a disk shaped crack. Crack opening displacements in an orthotropic strip have been found by using the plastic strip model [12]. Plastic deformations in a transversely isotropic layer and cylinder have been studied by Danyluk et al. [13,14]. All the above mentioned work related to plastic studies are based on the Dugdale's hypothesis [15]. The Dugdale model of a crack in a ductile material was introduced to investigate the inelastic zone at the ends of a stationary slit in steel sheets under static tension. The Dugdale model predictions agree closely with the experimental results.

In this study, the governing elasticity equation for the transversely isotropic axisymmetric problem in cylindrical coordinates, is obtained in terms of a Love type stress function. Hankel and Fourier sine transforms are applied on the stress function because of the geometry of the configuration and boundary conditions. The stress function is expressed as the summation of two solutions of the governing equation. Using the boundary conditions, the problem is reduced to a singular integral equation. This singular integral equation is solved by using the Gaussian Quadrature. Then the stress intensity factors at the crack tips are determined. Kaya & Erdogan's approach was modified to obtain the plastic zone widths at the crack tips.

The numerical results have been obtained for various ring-shaped crack sizes. The plastic zone widths are obtained for axial loading. The results are illustrated by graphs.

The stress intensity factors and plastic zone lengths may be used in any calculations to predict fracture, for example, in cylindrical machine parts and the general mathematical methods and techniques of this analysis may also be used in the solutions of any other axisymmetric elasticity problems.

CHAPTER TWO
DEFINITION OF THE PROBLEM

2. Basic formulation

The generalized Hooke's law relating strains to stresses can be written in contracted notation as

$$\varepsilon_i = a_{ij}\sigma_j \qquad i, j = 1..6 \tag{2.1}$$

where ε_i are the strain components, a_{ij} is the strain coefficient matrix, and the σ_j are the stress components.

Comparison between tensor and contracted notation for stresses and strains is available in [1].

The strain coefficient matrix, a_{ij}, has 36 constants in Eq. 2.1. It is symmetric and hence has 21 independent constants.

The Eq. 2.1 can be written in matrix notation

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{12} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{13} & a_{23} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{14} & a_{24} & a_{34} & a_{44} & a_{45} & a_{46} \\ a_{15} & a_{25} & a_{35} & a_{45} & a_{55} & a_{56} \\ a_{16} & a_{26} & a_{36} & a_{46} & a_{56} & a_{66} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{Bmatrix} \tag{2.2}$$

If at every point of a material there is one plane in which the mechanical properties are equal in all directions, then the material is termed transversely isotropic. If, for instance, the 1-2 plane is the special plane of isotropy, then the 1 and 2 subscripts on the strain coefficients are interchangeable. The strain-stress relation then has only five independent elastic constants and these are

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 & 0 & 0 \\ a_{12} & a_{11} & a_{13} & 0 & 0 & 0 \\ a_{13} & a_{13} & a_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(a_{11} - a_{12}) \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{Bmatrix} \tag{2.3}$$

2.1. Axisymmetric deformation

Let us consider the axisymmetric elasticity problem for a transversely long cylinder shown in Fig. 2.1. In this problem a single ring-shaped crack lies at z=0 plane. Elasto-plastic long cylinder is subjected to an axial load.

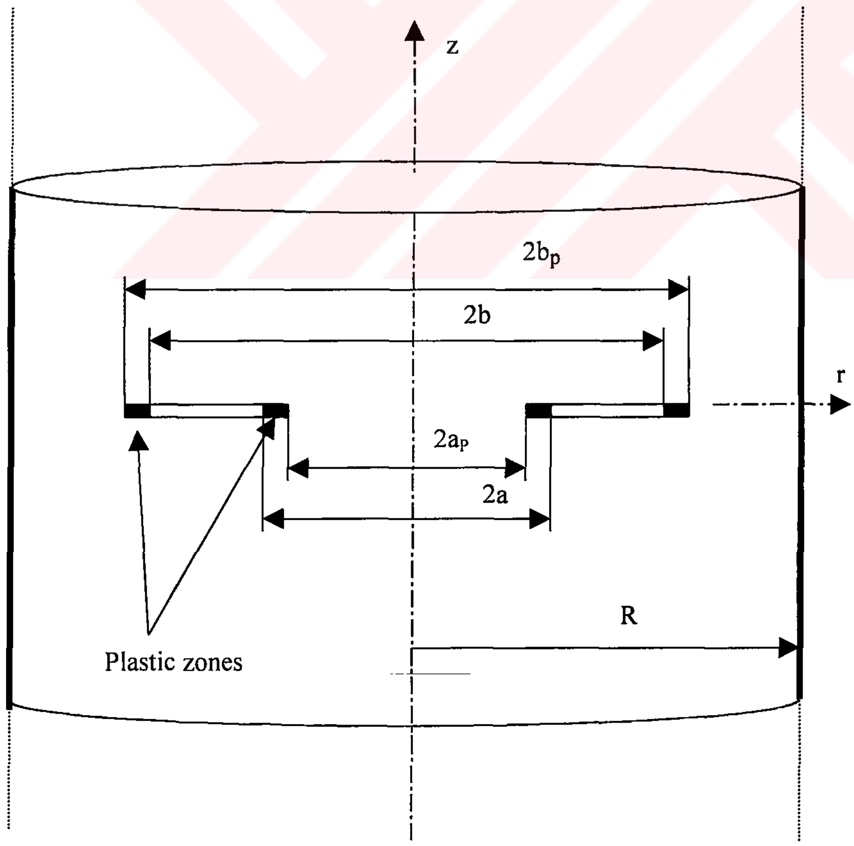

Figure 2.1. Geometry of the problem

It is convenient to select the cylindrical coordinate system for solving the problem. Let a body of an elastic homogeneous transversely isotropic material be referred to a cylindrical co-ordinate system r, θ, z with the origin placed at some point on the geometrical axis, the z axis directed along this axis, and the polar r axis directed arbitrarily in the plane of cross section. The generalized Hooke's law equations are written as in a Cartesian system, namely

$$\varepsilon_r = a_{11}\sigma_r + a_{12}\sigma_\theta + a_{13}\sigma_z$$

$$\varepsilon_\theta = a_{12}\sigma_r + a_{11}\sigma_\theta + a_{13}\sigma_z$$

$$\varepsilon_z = a_{13}\sigma_r + a_{13}\sigma_\theta + a_{33}\sigma_z \qquad (2.4)$$

$$\gamma_{\theta z} = a_{44}\sigma_{\theta z}$$

$$\gamma_{rz} = a_{44}\sigma_{rz}$$

$$\gamma_{r\theta} = 2(a_{11} - a_{12})\sigma_{r\theta}$$

For the materials considered in this study, the numerical values of the modulii c_{ij} (i,j=1..4) are taken from Huntington [16] and they are tabulated below.

Table 2.1. Values of Elastic Constants (in GPa)

Material	c_{11}	c_{12}	c_{13}	c_{33}	c_{44}
Magnesium	59.7	26.2	21.7	61.7	16.4
Cadmium	115.2	39.7	40.5	51.2	20.3
Barium-titanate	168.0	78.0	71.0	189.0	5.46

$$\sigma_j = c_{ij}\varepsilon_i \quad i,j = 1..6 \qquad (2.1)$$

Substituting Eq. (2.1) into $\sigma_j = c_{ij}\varepsilon_i$ $(i,j = 1..6)$ the strain coefficients a_{ij} (i,j=1..4) can be easily evaluated in terms of c_{ij} by using matrix operation rules. These are

$$a_{11} = \frac{c_{11}c_{33} - c_{13}^2}{O.P.} \qquad a_{12} = \frac{-(c_{12}c_{33} - c_{13}^2)}{O.P.}$$

$$a_{13} = \frac{c_{13}(c_{12} - c_{11})}{O.P.} \qquad a_{33} = \frac{c_{11}^2 - c_{12}^2}{O.P.} \qquad a_{44} = \frac{1}{c_{44}}. \tag{2.5}$$

here

$$O.P. = (c_{11} - c_{12})(c_{11}c_{33} - 2c_{13}^2 + c_{12}c_{33}) \tag{2.6}$$

The strain coefficients a_{ij} may also be expressed in terms of the engineering constants

$$a_{11} = \frac{1}{E} \qquad a_{12} = -\frac{\upsilon}{E} \qquad a_{33} = \frac{1}{E'} \qquad a_{13} = -\frac{\upsilon'}{E'}$$

$$a_{44} = \frac{1}{G'} \qquad 2(a_{11} - a_{12}) = \frac{2(1+\upsilon)}{E} = \frac{1}{G} \tag{2.7}$$

Where E and E' are Young's modulii for tension and compression in the plane of isotropy and in a direction to perpendicular it, υ is Poisson's ratio characterizing contraction in the plane of isotropy for tension in the same plane, υ' is Poisson's ratio characterizing contraction in a direction normal to the plane of isotropy for tension in this plane. $G = E/2(1+\upsilon)$ and G' are the shear modulii for the plane of isotropy and perpendicular (radial) planes.

The elastic properties of isotropic materials become different in any direction due to certain technological processes such as rolling and the condition of anisotropy must be considered. Steel, copper, and aluminum may be considered as a slightly anisotropic material. Only steel is considered in this study. The engineering constants of steel are obtained experimentally and given in Table 2.2.

Table 2.2. Values of engineering constants of steel (modulii are in GPa)

Material	E	E'	υ	υ'	G'
Steel	220	210	0.3	0.3	80.77

The calculated a_{ij} values for the materials are give in appendix C.

Because of the symmetry of the force distribution and the elastic symmetry the radial sections remain plane and the body remains a body of revolution in the strained condition, i.e.,

$$u_r = u_r(r, z) \qquad u_\theta = 0 \qquad \omega = \omega(r, z) \tag{2.8}$$

It follows that $\gamma_{\theta z} = \gamma_{r\theta} = 0$ and $\tau_{\theta z} = \tau_{r\theta} = 0$, the other components of strain are independent of θ,

$$\varepsilon_r = \frac{\partial u_r}{\partial r} \qquad \varepsilon_\theta = \frac{u_r}{r} \qquad \varepsilon_z = \frac{\partial \omega}{\partial z} \qquad \gamma_{rz} = \frac{\partial u_r}{\partial z} + \frac{\partial \omega}{\partial r} \tag{2.9}$$

In the absence of body forces, the four non-trivial components of stress satisfy two equilibrium equations,

$$\frac{\partial \sigma_r}{\partial r} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{\sigma_r - \sigma_\theta}{r} = 0 \tag{2.10}$$

$$\frac{\partial \sigma_{rz}}{\partial r} + \frac{\partial \sigma_z}{\partial z} + \frac{\sigma_{rz}}{r} = 0 \tag{2.11}$$

The compatibility equations are,

$$\varepsilon_r - \varepsilon_\theta - r\frac{\partial \varepsilon_\theta}{\partial r} = 0 \tag{2.12}$$

$$\frac{\partial^2 \varepsilon_r}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial r^2} - \frac{\partial^2 \gamma_{rz}}{\partial z \partial r} = 0 \tag{2.13}$$

Substituting the expressions for the strains from Eq. 2.4, we obtain,

$$a_{11}\sigma_r + a_{12}\sigma_\theta + a_{13}\sigma_z - \frac{\partial}{\partial r}\left[r\left(a_{12}\sigma_r + a_{11}\sigma_\theta + a_{13}\sigma_z\right)\right] = 0 \tag{2.14}$$

$$\frac{\partial^2}{\partial z^2}\left(a_{11}\sigma_r + a_{12}\sigma_\theta + a_{13}\sigma_z\right) + \frac{\partial^2}{\partial r^2}\left(a_{12}\sigma_r + a_{11}\sigma_\theta + a_{13}\sigma_z\right) - a_{44}\frac{\partial^2 \sigma_{rz}}{\partial r \partial z} = 0 \tag{2.15}$$

2.2. Stress function

The equation of equilibrium (Eqs. 2.10 and 2.11) can be satisfied by a stress function $\phi(r, z)$ which is a generalization of the stress function for a transversely isotropic body are the form [3],

$$\sigma_r = -\frac{\partial}{\partial z}\left(\frac{\partial^2 \phi}{\partial r^2} + \frac{b}{r}\frac{\partial \phi}{\partial r} + a\frac{\partial^2 \phi}{\partial z^2}\right)$$

$$\sigma_\theta = -\frac{\partial}{\partial z}\left(b\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + a\frac{\partial^2 \phi}{\partial z^2}\right)$$

$$\sigma_z = \frac{\partial}{\partial z}\left(c\frac{\partial^2 \phi}{\partial r^2} + \frac{c}{r}\frac{\partial \phi}{\partial r} + d\frac{\partial^2 \phi}{\partial z^2}\right) \tag{2.16}$$

$$\sigma_{rz} = \frac{\partial}{\partial r}\left(\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + a\frac{\partial^2 \phi}{\partial z^2}\right)$$

$$u_r = -(1-b)(a_{11} - a_{12})\left(\frac{\partial^2 \phi}{\partial r \partial z}\right)$$

$$u_\theta = 0$$

$$w = a_{44}\left(\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r}\right) + (a_{33}d - 2a_{13}a)\frac{\partial^2 \phi}{\partial z^2}$$

where the constants a, b, c, and d are,

$$a = \frac{a_{13}(a_{11} - a_{12})}{a_{11}a_{33} - a_{13}^{\ 2}}$$

$$b = \frac{a_{13}(a_{13} + a_{44}) - a_{12}a_{33}}{a_{11}a_{33} - a_{13}^2}$$

$$c = \frac{a_{13}(a_{11} - a_{12}) + a_{44}a_{11}}{a_{11}a_{33} - a_{13}^2} \qquad (2.17)$$

$$d = \frac{a_{11}^2 - a_{12}^2}{a_{11}a_{33} - a_{13}^2}$$

To be able to find $\phi(r, z)$ let us substitute Eq. 2.16 into the equilibrium equation 2.11. We obtain the following homogeneous partial differential equation for the stress function.

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)\left(\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + a\frac{\partial^2\phi}{\partial z^2}\right) + \frac{\partial^2}{\partial r^2}\left(c\frac{\partial^2\phi}{\partial r^2} + \frac{c}{r}\frac{\partial\phi}{\partial r} + d\frac{\partial^2\phi}{\partial z^2}\right) = 0 \qquad (2.18)$$

Equation (2.18) can be written in another form,

$$\frac{\partial^4\phi}{\partial r^4} + \frac{2}{r}\frac{\partial^3\phi}{\partial r^3} - \frac{1}{r^2}\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r^3}\frac{\partial\phi}{\partial r}$$

$$+ (a + c)\frac{\partial^4\phi}{\partial r^2\partial z^2} + \frac{a + c}{r}\frac{\partial^3\phi}{\partial r\partial z^2} + d\frac{\partial^4\phi}{\partial z^4} = 0 \qquad (2.19)$$

The solution of (2.19) can be obtained by adding two functions. $\phi_1(r, z)$ is a function to satisfy the boundary conditions at $z = 0$ and $\phi_2(r, z)$ is another function to satisfy the boundary conditions at $r = R$.

2.2.1. Evaluating the stress function

The problem is an axisymmetric one and the stress function $\phi_1(r, z)$ has to be an even function of r. Therefore we shall define the Hankel transform pair of order zero of the function $\phi_1(r, z)$ as follows,

$$\phi_1(r, z) = \int_0^\infty \lambda.P(\lambda, z).J_0(\lambda r).d\lambda \qquad (2.20)$$

$$P(\lambda, z) = \int_0^\infty r.\phi_1(r, z).J_0(\lambda r).dr \tag{2.21}$$

By substituting (2.20) into Eq. (2.18), the following can be obtained,

$$\int_0^\infty \lambda \left[\begin{array}{l} \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)\left(\begin{array}{l} P(\lambda, z)\frac{\partial^2 J_0(\lambda r)}{\partial r^2} + \frac{P(\lambda, z)}{r}\frac{\partial J_0(\lambda r)}{\partial r} + \\ a.J_0(\lambda r)\frac{\partial^2 P(\lambda, z)}{\partial z^2} \end{array} \right) \\ \\ + \frac{\partial^2}{\partial r^2}\left(\begin{array}{l} c.P(\lambda, z)\frac{\partial^2 J_0(\lambda r)}{\partial r^2} + \frac{c}{r}P(\lambda, z)\frac{\partial J_0(\lambda r)}{\partial r} + \\ dJ_0(\lambda r)\frac{\partial^2 P(\lambda, z)}{\partial z^2} \end{array} \right) \end{array} \right] d\lambda = 0 \tag{2.22}$$

By differentiating the first order Bessel function with respect to r and arranging the terms, Eq. (2.22) can be reduced to the following homogeneous ordinary differential equation of $P(\lambda, z)$,

$$d\frac{d^4P(\lambda, z)}{\partial z^4} - \lambda^2(a + c)\frac{d^2P(\lambda, z)}{\partial z^2} + \lambda^4P(\lambda, z) = 0 \tag{2.23}$$

Associated with the fourth order differential equation (2.23) is the algebraic equation,

$$d.s^4 - (a + c)s^2 + 1 = 0 \tag{2.24}$$

whose roots are,

$$s_{1,2} = \pm\sqrt{\frac{(a + c) + \sqrt{(a + c)^2 - 4d}}{2d}} \tag{2.25.a}$$

$$s_{3,4} = \pm\sqrt{\frac{(a + c) - \sqrt{(a + c)^2 - 4d}}{2d}} \tag{2.25.b}$$

The solution of differential equation (2.23) is in the following form,

$$P(\lambda, z) = m_1 e^{s_1 \lambda z} + m_2 e^{s_2 \lambda z} + m_3 e^{s_3 \lambda z} + m_4 e^{s_4 \lambda z} \tag{2.26}$$

From the analysis of s_i (i=1..4), in order to have a convergent solution when z approaches to infinity, the constants m_1 and m_3 should be equal to zero. Thus $P(\lambda, z)$ is reduced to,

$$P(\lambda, z) = m_2 e^{s_2 \lambda z} + m_4 e^{s_4 \lambda z} \tag{2.27}$$

Substituting (2.27) into Eq. (2.20), the stress function $\phi_1(r, z)$ is expressed as follows,

$$\phi_1(r, z) = \int_0^\infty \lambda \left(m_2 e^{s_2 \lambda z} + m_4 e^{s_4 \lambda z} \right) J_0(\lambda r).d\lambda \tag{2.28}$$

where,

$s_2, s_4 < 0$

Since the problem is symmetric about $z = 0$ plane the second solution of Eq. (2.19), $\phi_2(r, z)$ is an odd function of z. It will be determined by defining the following Fourier sine transform pair,

$$\phi_2(r, z) = \frac{2}{\pi} \int_0^\infty L(\alpha, r) \sin(\alpha r).d\alpha \tag{2.29}$$

$$L(\alpha, r) = \int_0^\infty \phi_2(r, z) \sin(\alpha z) dz \tag{2.30}$$

By substituting (2.29) into Eq. (2.18), the following is obtained,

$$\frac{2}{\pi} \int_0^\infty \left[\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \right) \left(\frac{\partial^2 L(\alpha, r)}{\partial r^2} + \frac{1}{r} \frac{\partial L(\alpha, r)}{\partial r} - a\alpha^2 L(\alpha, r) \right) - \alpha^2 \left(c \frac{\partial^2 L(\alpha, r)}{\partial r^2} + \frac{c}{r} \frac{\partial L(\alpha, r)}{\partial r} - d\alpha^2 L(\alpha, r) \right) \right] \sin(\alpha z) d\alpha = 0 \tag{2.31}$$

To be able to satisfy equation (2.31) for every z value, the bracketed expression in Eq. (2.31) should be equal to zero. Thus,

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\right)\left(\frac{\partial^2 L(\alpha,r)}{\partial r^2}+\frac{1}{r}\frac{\partial L(\alpha,r)}{\partial r}-a\alpha^2 L(\alpha,r)\right)$$
$$-\alpha^2\left(c\frac{\partial^2 L(\alpha,r)}{\partial r^2}+\frac{c}{r}\frac{\partial L(\alpha,r)}{\partial r}-d\alpha^2 L(\alpha,r)\right)=0 \tag{2.32}$$

Eq. (2.32) can be written as follows,

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-c_1^2\alpha^2\right)\left(\frac{\partial^2 L(\alpha,r)}{\partial r^2}+\frac{1}{r}\frac{\partial L(\alpha,r)}{\partial r}-c_2^2\alpha^2 L(\alpha,r)\right)=0 \tag{2.33}$$

where,

$$c_1=\sqrt{\frac{2d}{(a+c)\pm\sqrt{(a+c)^2-4d}}} \tag{2.34.a}$$

$$c_2=\sqrt{\frac{(a+c)\pm\sqrt{(a+c)^2-4d}}{2}} \tag{2.34.b}$$

Since the values of c_1 and c_2 are expressed in terms of the material constants, the positive values of c_1 and c_2 are used throughout this study. Actually to be able to satisfy the boundary conditions in the following sections. This assumption is necessary.

Eq. (2.33) can also be expressed as,

$$D_1(D_2 L(\alpha,r))=0 \tag{2.35}$$

where D_1 and D_2 are the operators of Modified Bessel's differential equation of order zero and they are defined as,

$$D_1=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-c_1^2\alpha^2 \tag{2.36.a}$$

$$D_2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - c_2^2\alpha^2 \tag{2.36.b}$$

by defining,

$$\Omega(r,\alpha) = D_2 L(\alpha, r) \tag{2.37}$$

then equation (2.35) can be written as follows,

$$D_1\Omega(r,\alpha) = 0 \tag{2.38}$$

the solution of (2.38) is of the form,

$$\Omega(r,\alpha) = A.I_0(c_1\alpha r) + B.K_0(c_1\alpha r) \tag{2.39}$$

where $I_0(c_1\alpha r)$ and $K_0(c_1\alpha r)$ are modified Bessel functions of the first and second kind of order zero, respectively. By substituting Eq. (2.39) into Eq. (2.37), the following expression is obtained,

$$D_2 L(\alpha, r) = A.I_0(c_1\alpha r) + B.K_0(c_1\alpha r) \tag{2.40}$$

Homogeneous solution of the differential equation given in Eq. (2.40) can be expressed as follows,

$$L_h(\alpha, r) = C.I_0(c_2\alpha r) + D.K_0(c_2\alpha r) \tag{2.41.a}$$

its particular solution may be obtained by using the method of variation of parameters as ,

$$L_p(\alpha, r) = \frac{1}{(c_1^2 - c_2^2)\alpha^2}\left[A.I_0(c_1\alpha r) + B.K_0(c_1\alpha r)\right] \tag{2.41.b}$$

Then the general solution of nonhomogeneous differential equation (2.32) can be expressed as follows,

$$L_g(\alpha, r) = \frac{1}{(c_1^2 - c_2^2)\alpha^2}\left[A.I_0(c_1\alpha r) + B.K_0(c_1\alpha r)\right] + C.I_0(c_2\alpha r) + D.K_0(c_2\alpha r) \qquad (2.42)$$

When r goes to zero, one should get a convergent solution. Therefore B and D must be equal to zero and finally, the general solution of Eq. (2.32) can be expressed as follows,

$$L(\alpha, r) = \frac{A'}{\alpha^2} I_0(c_1\alpha r) + C.I_0(c_2\alpha r) \qquad (2.43)$$

where,

$$A' = \frac{A}{c_1^2 - c_2^2}$$

A and C are unknown functions of α to be determined from the boundary conditions.

The stress function $\phi_2(r, z)$ is obtained by substituting Eq. (2.43) into Eq. (2.29),

$$\phi_2(r, z) = \frac{2}{\pi}\int_0^\infty \left[\frac{A'}{\alpha^2} I_0(c_1\alpha r) + C.I_0(c_2\alpha r)\right]\sin(\alpha r).d\alpha \qquad (2.44)$$

Then the complete Love type stress function $\phi(r, z)$ can be obtained by adding two solution $\phi_1(r, z)$ and $\phi_2(r, z)$ as follows,

$$\phi(r, z) = \int_0^\infty \lambda\left(m_2 e^{s_2\lambda z} + m_4 e^{s_4\lambda z}\right)J_0(\lambda r)d\lambda + \frac{2}{\pi}\int_0^\infty \left(\frac{A'}{\alpha^2} I_0(c_1\alpha r) + C.I_0(c_2\alpha r)\right)\sin(\alpha z)d\alpha \qquad (2.45)$$

2.3. Derivatives of the stress function

In order to find the stresses and displacements, the stress function $\phi(r, z)$ is differentiated with respect to r and z and the following expressions are found,

$$\frac{\partial\phi(r,z)}{\partial r} = -\int_0^\infty \lambda^2\left(m_2 e^{s_2\lambda z} + m_4 e^{s_4\lambda z}\right)J_1(\lambda r)d\lambda$$

$$+\frac{2}{\pi}\int_0^\infty\left(\frac{A'}{\alpha}c_1 I_1(c_1\alpha r) + C\alpha c_2 I_1(c_2\alpha r)\right)\sin(\alpha z)d\alpha \tag{2.46.a}$$

$$\frac{\partial^2\phi(r,z)}{\partial r^2} = -\int_0^\infty \lambda^3\left(m_2 e^{s_2\lambda z} + m_4 e^{s_4\lambda z}\right)\left(J_0(\lambda r) - \frac{J_1(\lambda r)}{\lambda r}\right)d\lambda$$

$$+\frac{2}{\pi}\int_0^\infty\left\{A'c_1^2\left[I_0(c_1\alpha r) - \frac{I_1(c_1\alpha r)}{c_1\alpha r}\right]\right. \tag{2.46.b}$$

$$\left. + Cc_2^2\alpha^2\left[I_0(c_2\alpha r) - \frac{I_1(c_2\alpha r)}{c_2\alpha r}\right]\right\}\sin(\alpha z)d\alpha$$

$$\frac{\partial^3\phi(r,z)}{\partial r^3} = -\int_0^\infty \lambda^2\left(m_2 e^{s_2\lambda z} + m_4 e^{s_4\lambda z}\right)\left[\left(-\lambda^2 + \frac{2}{r^2}\right)J_1(\lambda r) - \frac{\lambda}{r}J_0(\lambda r)\right]d\lambda$$

$$+\frac{2}{\pi}\int_0^\infty\left\{\frac{A'}{\alpha}c_1\left[\left(c_1^2\alpha^2 + \frac{2}{r^2}\right)I_1(c_1\alpha r) - \frac{c_1\alpha}{r}I_0(c_1\alpha r)\right]\right. \tag{2.46.c}$$

$$\left. + C\alpha c_2\left[\left(c_2^2\alpha^2 + \frac{2}{r^2}\right)I_1(c_2\alpha r) - \frac{c_2\alpha}{r}I_0(c_2\alpha r)\right]\right\}\sin(\alpha z)d\alpha$$

$$\frac{\partial\phi(r,z)}{\partial z} = \int_0^\infty \lambda^2\left(m_2 s_2 e^{s_2\lambda z} + m_4 s_4 e^{s_4\lambda z}\right)J_0(\lambda r)d\lambda$$

$$+\frac{2}{\pi}\int_0^\infty\left(\frac{A'}{\alpha^2}I_0(c_1\alpha r) + C.I_0(c_2\alpha r)\right)\alpha\cos(\alpha z)d\alpha \tag{2.46.d}$$

$$\frac{\partial^2 \phi(r,z)}{\partial z^2} = \int_0^\infty \lambda^3 \left(m_2 s_2^2 e^{s_2\lambda z} + m_4 s_4^2 e^{s_4\lambda z}\right) J_0(\lambda r) d\lambda$$

$$-\frac{2}{\pi}\int_0^\infty \left(\frac{A'}{\alpha^2} I_0(c_1\alpha r) + C.I_0(c_2\alpha r)\right)\alpha^2 \sin(\alpha z) d\alpha \quad (2.46.e)$$

$$\frac{\partial^3 \phi(r,z)}{\partial z^3} = \int_0^\infty \lambda^4 \left(m_2 s_2^3 e^{s_2\lambda z} + m_4 s_4^3 e^{s_4\lambda z}\right) J_0(\lambda r) d\lambda$$

$$-\frac{2}{\pi}\int_0^\infty \left(\frac{A'}{\alpha^2} I_0(c_1\alpha r) + C.I_0(c_2\alpha r)\right)\alpha^3 \cos(\alpha z) d\alpha \quad (2.46.f)$$

$$\frac{\partial^2 \phi(r,z)}{\partial r \partial z} = -\int_0^\infty \lambda^3 \left(m_2 s_2 e^{s_2\lambda z} + m_4 s_4 e^{s_4\lambda z}\right) J_1(\lambda r) d\lambda$$

$$+\frac{2}{\pi}\int_0^\infty \left(\frac{A'}{\alpha} c_1 I_1(c_1\alpha r) + C\alpha c_2 I_1(c_2\alpha r)\right)\alpha \cos(\alpha z) d\alpha \quad (2.46.g)$$

$$\frac{\partial^3 \phi(r,z)}{\partial r^2 \partial z} = -\int_0^\infty \lambda^4 \left(m_2 s_2 e^{s_2\lambda z} + m_4 s_4 e^{s_4\lambda z}\right)\left(J_0(\lambda r) - \frac{J_1(\lambda r)}{\lambda r}\right) d\lambda$$

$$+\frac{2}{\pi}\int_0^\infty \left\{ A' c_1^2 \left[I_0(c_1\alpha r) - \frac{I_1(c_1\alpha r)}{c_1\alpha r}\right]\right. \quad (2.46.h)$$

$$\left. + C c_2^2 \alpha^2 \left[I_0(c_2\alpha r) - \frac{I_1(c_2\alpha r)}{c_2\alpha r}\right]\right\} \alpha \cos(\alpha z) d\alpha$$

$$\frac{\partial^3\phi(r,z)}{\partial r\partial z^2} = -\int_0^\infty \lambda^4\left(m_2 s_2^2 e^{s_2\lambda z} + m_4 s_4^2 e^{s_4\lambda z}\right)J_1(\lambda r)d\lambda$$

$$-\frac{2}{\pi}\int_0^\infty\left(\frac{A'}{\alpha}c_1 I_1(c_1\alpha r) + C\alpha c_2 I_1(c_2\alpha r)\right)\alpha^2\sin(\alpha z)d\alpha \qquad (2.46.i)$$

2.4. Stresses and strains

By substituting (2.46.a-i) into Eqs. (2.16), the following stress and displacement expressions are obtained,

$$\sigma_r = \int_0^\infty \lambda^4\left(m_2 s_2 e^{s_2\lambda z} + m_4 s_4 e^{s_4\lambda z}\right)\left(J_0(\lambda r) - \frac{J_1(\lambda r)}{\lambda r}\right)d\lambda$$

$$-\frac{2}{\pi}\int_0^\infty\left\{\begin{array}{l} A'c_1^2\left(I_0(c_1\alpha r) - \dfrac{I_1(c_1\alpha r)}{c_1\alpha r}\right) + \\ C\alpha^2 c_2^2\left(I_0(c_2\alpha r) - \dfrac{I_1(c_2\alpha r)}{c_2\alpha r}\right)\end{array}\right\}\alpha\cos(\alpha z)d\alpha$$

$$+\frac{b}{r}\left[\begin{array}{l}\left(\displaystyle\int_0^\infty \lambda^3\left(m_2 s_2 e^{s_2\lambda z} + m_4 s_4 e^{s_4\lambda z}\right)J_1(\lambda r)d\lambda\right) \\ -\dfrac{2}{\pi}\displaystyle\int_0^\infty\left(\frac{A'}{\alpha}c_1 I_1(c_1\alpha r) + C\alpha c_2 I_1(c_2\alpha r)\right)\alpha\cos(\alpha z)d\alpha\end{array}\right]$$

$$+a\left[\begin{array}{l}-\displaystyle\int_0^\infty \lambda^4\left(m_2 s_2^3 e^{s_2\lambda z} + m_4 s_4^3 e^{s_4\lambda z}\right)J_0(\lambda r)d\lambda \\ +\dfrac{2}{\pi}\displaystyle\int_0^\infty\left(\frac{A'}{\alpha^2}I_0(c_1\alpha r) + CI_0(c_2\alpha r)\right)\alpha^3\cos(\alpha z)d\alpha\end{array}\right] \qquad (2.47.a)$$

$$\sigma_\theta = b\left[\begin{array}{l}\displaystyle\int_0^\infty \lambda^4\left(m_2 s_2 e^{s_2\lambda z} + m_4 s_4 e^{s_4\lambda z}\right)\left(J_0(\lambda r) - \frac{J_1(\lambda r)}{\lambda r}\right)d\lambda \\ \displaystyle -\frac{2}{\pi}\int_0^\infty\left[\begin{array}{l} A' c_1^2\left(I_0(c_1\alpha r) - \dfrac{I_1(c_1\alpha r)}{c_1\alpha r}\right) + \\ C\alpha^2 c_2^2\left(I_0(c_2\alpha r) - \dfrac{I_0(c_2\alpha r)}{c_2\alpha r}\right)\end{array}\right]\alpha\cos(\alpha z)d\alpha\end{array}\right]$$

$$+\frac{1}{r}\left[\begin{array}{l}\displaystyle\int_0^\infty \lambda^3\left(m_2 s_2 e^{s_2\lambda z} + m_4 s_4 e^{s_4\lambda z}\right)J_1(\lambda r)d\lambda \\ \displaystyle -\frac{2}{\pi}\int_0^\infty\left(\frac{A'}{\alpha}c_1 I_1(c_1\alpha r) + C\alpha c_2 I_1(c_2\alpha r)\right)\alpha\cos(\alpha z)d\alpha\end{array}\right]$$

$$+a\left[\begin{array}{l}\displaystyle -\int_0^\infty \lambda^4\left(m_2 s_2^3 e^{s_2\lambda z} + m_4 s_4^3 e^{s_4\lambda z}\right)J_0(\lambda r)d\lambda \\ \displaystyle +\frac{2}{\pi}\int_0^\infty\left(\frac{A'}{\alpha^2}I_0(c_1\alpha r) + C.I_0(c_2\alpha r)\right)\alpha^3\cos(\alpha z)d\alpha\end{array}\right] \qquad (2.47.b)$$

$$\sigma_z = c\left[\begin{array}{l}\displaystyle -\int_0^\infty \lambda^4\left(m_2 s_2 e^{s_2\lambda z} + m_4 s_4 e^{s_4\lambda z}\right)\left(J_0(\lambda r) - \frac{J_1(\lambda r)}{\lambda r}\right)d\lambda \\ \displaystyle +\frac{2}{\pi}\int_0^\infty\left[\begin{array}{l} A' c_1^2\left(I_0(c_1\alpha r) - \dfrac{I_1(c_1\alpha r)}{c_1\alpha r}\right) \\ + C\alpha^2 c_2^2\left(I_0(c_2\alpha r) - \dfrac{I_1(c_2\alpha r)}{c_2\alpha r}\right)\end{array}\right]\alpha\cos(\alpha z)d\alpha\end{array}\right]$$

$$+\frac{c}{r}\left[\begin{array}{l}\displaystyle -\int_0^\infty \lambda^3\left(m_2 s_2 e^{s_2\lambda z} + m_4 s_4 e^{s_4\lambda z}\right)J_1(\lambda r)d\lambda \\ \displaystyle +\frac{2}{\pi}\int_0^\infty\left(\frac{A'}{\alpha}c_1 I_1(c_1\alpha r) + C\alpha c_2 I_1(c_2\alpha r)\right)\alpha\cos(\alpha z)d\alpha\end{array}\right]$$

$$+d\left[\begin{array}{l}\displaystyle \int_0^\infty \lambda^4\left(m_2 s_2^3 e^{s_2\lambda z} + m_4 s_4^3 e^{s_4\lambda z}\right)J_0(\lambda r)d\lambda \\ \displaystyle -\frac{2}{\pi}\int_0^\infty\left(\frac{A'}{\alpha^2}I_0(c_1\alpha r) + C.I_0(c_2\alpha r)\right)\alpha^3\cos(\alpha z)d\alpha\end{array}\right] \qquad (2.47.c)$$

$$\sigma_{rz} = -\int_0^\infty \lambda^3\left(m_2 e^{s_2\lambda z} + m_4 e^{s_4\lambda z}\right)\left[\left(-\lambda + \frac{2}{\lambda r^2}\right)J_1(\lambda r) - \frac{J_0(\lambda r)}{r}\right]d\lambda$$

$$+\frac{2}{\pi}\int_0^\infty \left[\begin{array}{l} A' c_1^2\left[\left(c_1\alpha + \frac{2}{c_1\alpha r^2}\right)I_1(c_1\alpha r) - \frac{I_0(c_1\alpha r)}{r}\right] \\ + C\alpha^2 c_2^2\left[\left(c_2\alpha + \frac{2}{c_2\alpha r^2}\right)I_1(c_2\alpha r) - \frac{I_0(c_2\alpha r)}{r}\right]\end{array}\right]\sin(\alpha z)d\alpha$$

$$+\frac{1}{r^2}\left[\begin{array}{l} \int_0^\infty \lambda^2\left(m_2 e^{s_2\lambda z} + m_4 e^{s_4\lambda z}\right)J_1(\lambda r)d\lambda \\ -\frac{2}{\pi}\int_0^\infty\left(\frac{A'}{\alpha}c_1 I_1(c_1\alpha r) + C\alpha c_2 I_1(c_2\alpha r)d\alpha\right)\sin(\alpha z)d\alpha\end{array}\right]$$

$$+\frac{1}{r}\left[\begin{array}{l} -\int_0^\infty \lambda^3\left(m_2 e^{s_2\lambda z} + m_4 e^{s_4\lambda z}\right)\left(J_0(\lambda r) - \frac{J_1(\lambda r)}{\lambda r}\right)d\lambda \\ + \int_0^\infty \frac{2}{\pi}\left(\begin{array}{l} A' c_1^2\left(I_0(c_1\alpha r) - \frac{I_1(c_1\alpha r)}{c_1\alpha r}\right) + \\ C\alpha^2 c_2^2\left(I_0(c_2\alpha r) - \frac{I_1(c_2\alpha r)}{c_2\alpha r}\right)\end{array}\right)\sin(\alpha z)d\alpha\end{array}\right]$$

$$-a\left[\begin{array}{l} \int_0^\infty \lambda^4\left(m_2 s_2^2 e^{s_2\lambda z} + m_4 s_4^2 e^{s_4\lambda z}\right)J_1(\lambda r)d\lambda \\ +\frac{2}{\pi}\int_0^\infty\left(\frac{A'}{\alpha}c_1 I_1(c_1\alpha r) + C\alpha c_2 I_1(c_2\alpha r)\right)\alpha^2\sin(\alpha z)d\alpha\end{array}\right] \tag{2.47.d}$$

$$
w = a_{44}\left[\begin{array}{l}
-\int_0^\infty \lambda^3\left(m_2 e^{s_2\lambda z} + m_4 e^{s_4\lambda z}\right)\left(J_0(\lambda r) - \frac{J_1(\lambda r)}{\lambda r}\right)d\lambda \\[2ex]
+\frac{2}{\pi}\int_0^\infty \left(\begin{array}{l} A' c_1^2\left(I_0(c_1\alpha r) - \frac{I_1(c_1\alpha r)}{c_1\alpha r}\right) + \\[2ex] C\alpha^2 c_2^2\left(I_0(c_2\alpha r) - \frac{I_1(c_2\alpha r)}{c_2\alpha r}\right)\end{array}\right)\sin(\alpha z)d\alpha \\[4ex]
+\frac{1}{r}\left(\begin{array}{l} -\int_0^\infty \lambda^2\left(m_2 e^{s_2\lambda z} + m_4 e^{s_4\lambda z}\right)J_1(\lambda r)d\lambda \\[2ex] +\frac{2}{\pi}\int_0^\infty \left(\begin{array}{l} \frac{A'}{\alpha} c_1 I_1(c_1\alpha r) \\[2ex] + C\alpha c_2 I_1(c_2\alpha r)\end{array}\right)\sin(\alpha z)d\alpha \end{array}\right)
\end{array}\right]
$$

$$
+\left(a_{33}d - 2a_{13}a\right)\left(\begin{array}{l}
\int_0^\infty \lambda^3\left(m_2 s_2^2 e^{s_2\lambda z} + m_4 s_4^2 e^{s_4\lambda z}\right)J_0(\lambda r)d\lambda \\[4ex]
-\frac{2}{\pi}\int_0^\infty \left(\begin{array}{l} \frac{A'}{\alpha^2} I_0(c_1\alpha r) \\[2ex] + C.I_0(c_2\alpha r)\end{array}\right)\alpha^2\sin(\alpha z)d\alpha
\end{array}\right) \qquad (2.47.e)
$$

$$
\frac{\partial w}{\partial r}=a_{44}\left[\begin{array}{l}
-\displaystyle\int_0^\infty \lambda^3\left(m_2e^{s_2\lambda z}+m_4e^{s_4\lambda z}\right)\left(\left(-\lambda+\frac{2}{\lambda r^2}\right)J_1(\lambda r)-\frac{J_0(\lambda r)}{r}\right)d\lambda \\
+\dfrac{2}{\pi}\displaystyle\int_0^\infty\left[\begin{array}{l}
A'c_1^2\left[\left(c_1\alpha+\dfrac{2}{c_1\alpha r^2}\right)I_1(c_1\alpha r)-\dfrac{I_0(c_1\alpha r)}{r}\right] \\
+C\alpha^2c_2^2\left[\begin{array}{l}\left(c_2\alpha+\dfrac{2}{c_2\alpha r^2}\right)I_1(c_2\alpha r) \\ -\dfrac{I_0(c_2\alpha r)}{r}\end{array}\right]
\end{array}\right]\sin(\alpha z)d\alpha
\end{array}\right]
$$

$$
+a_{44}\left[\begin{array}{l}
+\dfrac{1}{r}\left[\begin{array}{l}
-\displaystyle\int_0^\infty \lambda^3\left(m_2e^{s_2\lambda z}+m_4e^{s_4\lambda z}\right)\left(J_0(\lambda r)-\frac{J_1(\lambda r)}{\lambda r}\right)d\lambda \\
+\dfrac{2}{\pi}\displaystyle\int_0^\infty\left(\begin{array}{l}
A'c_1^2\left(I_0(c_1\alpha r)-\dfrac{I_1(c_1\alpha r)}{c_1\alpha r}\right) \\
+C\alpha^2c_2^2\left(I_0(c_2\alpha r)-\dfrac{I_1(c_2\alpha r)}{c_2\alpha r}\right)
\end{array}\right)\sin(\alpha z)d\alpha
\end{array}\right] \\
+\dfrac{1}{r^2}\left[\begin{array}{l}
\displaystyle\int_0^\infty \lambda^2\left(m_2e^{s_2\lambda z}+m_4e^{s_4\lambda z}\right)J_1(\lambda r)d\lambda \\
-\dfrac{2}{\pi}\displaystyle\int_0^\infty\left(\begin{array}{l}
\dfrac{A'}{\alpha}c_1I_1(c_1\alpha r) \\
+C\alpha c_2I_1(c_2\alpha r)
\end{array}\right)\sin(\alpha z)d\alpha
\end{array}\right]
\end{array}\right] \tag{2.47.f}
$$

$$
-\left(a_{33}d-2a_{13}a\right)\left[\begin{array}{l}
\displaystyle\int_0^\infty \lambda^4\left(m_2s_2^2e^{s_2\lambda z}+m_4s_4^2e^{s_4\lambda z}\right)J_1(\lambda r)d\lambda \\
+\dfrac{2}{\pi}\displaystyle\int_0^\infty\left(\begin{array}{l}
\dfrac{A'}{\alpha}c_1I_1(c_1\alpha r) \\
+C\alpha c_2I_1(c_2\alpha r)
\end{array}\right)\alpha^2\sin(\alpha z)d\alpha
\end{array}\right]
$$

After some straight forward manipulation the Eqs. 2.47 can be reduced to

$$\sigma_r(r,z) = \int_0^\infty \lambda^4 \left[(1-as_2^2)m_2 s_2 e^{s_2\lambda z} + (1-as_4^2)m_4 s_4 e^{s_4\lambda z}\right] J_0(\lambda r)d\lambda$$

$$-\frac{1-b}{r}\int_0^\infty \lambda^3 \left(m_2 s_2 e^{s_2\lambda z} + m_4 s_4 e^{s_4\lambda z}\right) J_1(\lambda r)d\lambda$$

$$-\frac{2}{\pi}\int_0^\infty \left[\begin{array}{l} \dfrac{A'}{\alpha^2}\left(\begin{array}{l}(c_1{}^2 - a)\alpha^2 I_0(c_1\alpha r) \\ + \dfrac{c_1}{r}(b-1)\alpha I_1(c_1\alpha r)\end{array}\right) \\ + C\left(\begin{array}{l}(c_2{}^2 - a)\alpha^2 I_0(c_1\alpha r) \\ + \dfrac{c_2}{r}(b-1)\alpha I_1(c_2\alpha r)\end{array}\right)\end{array}\right] \alpha\cos(\alpha z)d\alpha \tag{2.48}$$

$$\sigma_\theta(r,z) = \int_0^\infty \lambda^4 \left[\left(s_2 b - s_2^3 a\right)m_2 e^{s_2\lambda z} + \left(s_4 b - s_4^3 a\right)m_4 e^{s_4\lambda z}\right] J_0(\lambda r)d\lambda$$

$$+\frac{2}{\pi}\int_0^\infty \left[\begin{array}{l}\left(a - c_1^2 b\right)A'\alpha I_0(c_1\alpha r) \\ + \left(a - c_2^2 b\right)C\alpha^3 I_0(c_2\alpha r)\end{array}\right]\cos(\alpha z)d\alpha$$

$$+\frac{1-b}{r}\left[\begin{array}{l}\displaystyle\int_0^\infty \lambda^3\left(m_2 s_2 e^{s_2\lambda z} + m_4 s_4 e^{s_4\lambda z}\right)J_1(\lambda r)d\lambda \\ \displaystyle -\frac{2}{\pi}\int_0^\infty \left(A' I_1(c_1\alpha r) + C\alpha^2 I_1(c_2\alpha r)\right)\cos(\alpha z)d\alpha\end{array}\right] \tag{2.49}$$

$$\sigma_z(r,z)=\int_0^\infty \lambda^4\left[\left(s_2^3 d - s_2 c\right)m_2 e^{s_2\lambda z}+\left(s_4^3 d - s_4 c\right)m_4 e^{s_4\lambda z}\right]J_0(\lambda r)d\lambda$$

$$+\frac{2}{\pi}\int_0^\infty\begin{bmatrix}\left(c_1{}^2 c - d\right)A'\alpha I_0(c_1\alpha r)\\ +\left(c_2{}^2 c - d\right)C\alpha^3 I_0(c_2\alpha r)\end{bmatrix}\cos(\alpha z)d\alpha \qquad (2.50)$$

$$\sigma_{rz}(r,z)=\int_0^\infty \lambda^4\left[\left(1-as_2^2\right)m_2 e^{s_2\lambda z}+\left(1-as_4^2\right)m_4 e^{s_4\lambda z}\right]J_1(\lambda r)d\lambda$$

$$+\frac{2}{\pi}\int_0^\infty\begin{bmatrix}\left(c_1^3 - c_1 a\right)A'\alpha I_1(c_1\alpha r)\\ +\left(c_2^3 - c_2 a\right)C\alpha^3 I_1(c_2\alpha r)\end{bmatrix}\sin(\alpha z)d\alpha \qquad (2.51)$$

$$\omega(r,z)=\int_0^\infty \lambda^3\begin{bmatrix}\left(\left(a_{33}d-2a_{13}a\right)s_2^2-a_{44}\right)m_2 e^{s_2\lambda z}\\ +\left(\left(a_{33}d-2a_{13}a\right)s_4^2-a_{44}\right)m_4 e^{s_4\lambda z}\end{bmatrix}J_0(\lambda r)d\lambda$$

$$+\frac{2}{\pi}\int_0^\infty\begin{bmatrix}\left(a_{44}c_1^2-a_{33}d+2a_{13}a\right)A' I_0(c_1\alpha r)\\ +\left(a_{44}c_2^2-a_{33}d+2a_{13}a\right)C\alpha^2 I_0(c_2\alpha r)\end{bmatrix}\sin(\alpha z)d\alpha \qquad (2.52)$$

$$\frac{\partial\omega(r,z)}{\partial r}=\int_0^\infty \lambda^4\begin{bmatrix}(a_{44}-(a_{33}d-2a_{13}a)s_2^2 m_2 e^{s_2\lambda z}\\ +(a_{44}-(a_{33}d-2a_{13}a)s_4^2 m_4 e^{s_4\lambda z}\end{bmatrix}J_1(\lambda r)d\lambda$$

$$+\frac{2}{\pi}\int_0^\infty\begin{bmatrix}\left(a_{44}c_1^2-a_{33}d+2a_{13}a\right)c_1 A'\alpha I_1(c_1\alpha r)\\ +\left(a_{44}c_2^2-a_{33}d+2a_{13}a\right)c_2 C\alpha^3 I_1(c_2\alpha r)\end{bmatrix}\sin(\alpha z)d\alpha \qquad (2.53)$$

CHAPTER THREE
SOLUTION OF THE PROBLEM

3. Evaluating the unknown functions

Let the ring-shaped crack be embedded in the mid-plane of an infinite cylinder. The material of the cylinder is a transversely isotropic and elastic-plastic material. In practice, the curved surface is stress free. Therefore, on the plane z=0, it is required that

$$\sigma_r(R, z) = 0 \qquad 0 < |z| < \infty \qquad (3.1.a)$$

$$\sigma_{rz}(R, z) = 0 \qquad 0 < |z| < \infty \qquad (3.1.b)$$

$$\sigma_{rz}(r, 0) = 0 \qquad 0 \le r \le R \qquad (3.1.c)$$

$$\sigma_z(r, z) = -p(r) \qquad a_p < r < b_p \qquad (3.1.d)$$

$$\omega(r, 0) = 0 \qquad 0 \le r \le a_p \text{ and } b_p \le r \le R \qquad (3.1.e)$$

The cylindrical surface at r = R is free from normal and shear tractions.

We are considering an axially-symmetric deformation of the material and under the Dugdale assumption there are thin annular regions of inelastic deformation surrounding the ring-shaped crack (see Fig. 2.1). The inelastic zones at inner and outer crack tips are described by inner radii a_p and b, and outer radii a and b_p, respectively. A tensile stress Y is uniformly distributed in the inelastic regions. Therefore, we find that

$$p(r) = \begin{cases} p_0 & a < r < b \\ -Y & a_p < r < a \text{ and } b < r < b_p \end{cases} \qquad (3.2)$$

where the pressure p_0 (constant) is prescribed on the crack faces.

3.1. New unknown function

In order to help to find plastic solution, let us solve the problem regardless the plastic strip. The plastic strip will be introduced later.

Boundary conditions in Eq. 3.1(a-c) may be used to eliminate three of the four unknowns. The mixed boundary conditions in Eq. 3.1(d-e) may be used to obtain a system of dual integral equations for the fourth unknown function. It is convenient to reduce the mixed boundary condition to an integral equation. The integral equation will be singular. In order to avoid strong singularity in the resulting equation, it is necessary to introduce a new function as the derivative of the displacement $\omega(r, z)$, rather than the displacement. The new unknown function will be defined as follows

$$G(r) = \frac{\partial \omega}{\partial r}(r,0) \tag{3.3}$$

with the help of Eq. (2.53), boundary condition (3.1.e) and Eq. (2.53) are equivalent to

$$G(r) = 0 \qquad 0 \le r \le a \qquad b \le r \le R \tag{3.4}$$

$$\int_a^b G(r)dr = 0 \tag{3.5}$$

Substituting Eq. (2.53) into Eq (3.3) and by using Eq. (3.4), the following equation can be obtained

$$\int_0^\infty \lambda^4 \left[\left(a_{44} - (a_{33}d + 2a_{13}a)s_2^2\right)m_2 + \left(a_{44} - (a_{33}d + 2a_{13}a)s_4^2\right)m_4 \right] J_1(\lambda r) d\lambda = G(r) \tag{3.6}$$

Taking inverse Hankel transform and with the help of Eq. (3.4), Eq. (3.6) is expressed as follows

$$\left(a_{44} - (a_{33}d + 2a_{13}a)s_2^2\right)m_2 + \left(a_{44} - (a_{33}d + 2a_{13}a)s_4^2\right)m_4 = H(\lambda) \tag{3.7}$$

where

$$H(\lambda) = \frac{1}{\lambda^3}\int_a^b \rho G(\rho) J_1(\rho\lambda) d\rho \tag{3.8}$$

by substituting Eq. (2.51) into Eq. (3.1.c), we get

$$(1 - as_2^2)m_2 + (1 - as_4^2)m_4 = 0 \tag{3.9}$$

m_2 and m_4 can be easily obtained from Eqs. (3.7) and (3.9)

$$m_2(\lambda) = -C_{m_2} H(\lambda) \tag{3.10}$$

$$m_4(\lambda) = C_{m_4} H(\lambda) \tag{3.11}$$

C_{m_2} and C_{m_4} are given in appendix A.

Now by substituting Eq. (2.48) into boundary condition (3.1.a) and by taking inverse Fourier-cosine transform,

$$\begin{aligned}
&\int_0^\infty \lambda^4 J_0(\lambda r) d\lambda \int_0^\infty \left[s_2\left(1 - as_2^2\right)m_2 e^{s_2\lambda z} + s_4\left(1 - as_4^2\right)m_4 e^{s_4\lambda z}\right]\cos\alpha z dz \\
&-\frac{1-b}{R}\int_0^\infty \lambda^3 J_1(\lambda r) d\lambda \int_0^\infty \left[s_2 m_2 e^{s_2\lambda z} + s_4 m_4 e^{s_4\lambda z}\right]\cos\alpha z dz \\
&= \frac{A'(\alpha)}{\alpha} C_{11} + C(\alpha)\alpha C_{12}
\end{aligned} \tag{3.12}$$

By using the closed form integral formula B.1 [17], to Eq. (3.12)

$$-\int_0^{\infty}\lambda^5\left[\frac{1-as_2^2}{\lambda^2+\left(\frac{\alpha}{s_2}\right)^2}m_2+\frac{1-as_4^2}{\lambda^2+\left(\frac{\alpha}{s_4}\right)^2}m_4\right]J_0(\lambda R)d\lambda$$

$$+\frac{1-b}{R}\int_0^{\infty}\lambda^4\left[\frac{1}{\lambda^2+\left(\frac{\alpha}{s_2}\right)^2}m_2+\frac{1}{\lambda^2+\left(\frac{\alpha}{s_4}\right)^2}m_4\right]J_1(\lambda R)d\lambda \qquad (3.13)$$

$$=\frac{A'(\alpha)}{\alpha}C_{11}+C(\alpha).\alpha C_{12}$$

Substituting Eqs. (3.10) and (3.11) into Eq. (3.13) and recalling that $G(\rho)=0$ in the intervals $0\le r\le a$ and $b\le r\le\infty$

$$\int_a^b\rho G(\rho)d\rho\int_0^{\infty}\lambda^2\left[\frac{1-as_2^2}{\lambda^2+\left(\frac{\alpha}{s_2}\right)^2}C_{m_2}-\frac{1-as_4^2}{\lambda^2+\left(\frac{\alpha}{s_4}\right)^2}C_{m_4}\right]J_0(\lambda R)J_1(\rho\lambda)d\lambda$$

$$+\frac{1-b}{R}\int_a^b\rho G(\rho)d\rho\int_0^{\infty}\lambda\left[\frac{-1}{\lambda^2+\left(\frac{\alpha}{s_2}\right)^2}C_{m_2}+\frac{1}{\lambda^2+\left(\frac{\alpha}{s_4}\right)^2}C_{m_4}\right]J_1(\lambda R)J_1(\rho\lambda)d\lambda \qquad (3.14)$$

$$=\frac{A'(\alpha)}{\alpha}C_{11}+C(\alpha).\alpha C_{12}$$

By using closed form integral formula B.3

$$F_1\int_a^b\rho G(\rho)d\rho=\frac{A'(\alpha)}{\alpha}C_{11}+\alpha C(\alpha)C_{12} \qquad (3.15)$$

Finally by substituting Eq. (2.51) into boundary condition (3.1.b), and by taking inverse Fourier-sine transform,

$$\int_0^\infty \lambda^4 J_1(\lambda R)d\lambda \int_0^\infty [(1-as_2^2)m_2 e^{s_2\lambda z} + (1-as_4^2)m_4 e^{s_4\lambda z}]\sin\alpha z dz$$
$$= -\left[\frac{A'(\alpha)}{\alpha^2}C_{13} + C(\alpha)C_{14}\right] \tag{3.16}$$

Applying the closed form integral formula B.2 to Eq. (3.16)

$$\int_a^b \rho G(\rho)d\rho \int_0^\infty \lambda \begin{bmatrix} -C_{m_2}\dfrac{1-as_2^2}{s_2^2}\dfrac{\alpha}{\lambda^2+\left(\dfrac{\alpha}{s_2}\right)^2} \\ +C_{m_4}\dfrac{1-as_4^2}{s_4^2}\dfrac{\alpha}{\lambda^2+\left(\dfrac{\alpha}{s_4}\right)^2} \end{bmatrix} J_1(\lambda R)J_1(\rho\lambda)d\lambda \tag{3.17}$$
$$= \frac{A'(\alpha)}{\alpha^2}C_{13} + C(\alpha)C_{14}$$

Again by using the closed form integral B.3, Eq. (3.13) takes the following form

$$F_2\int_a^b \rho G(\rho)d\rho = \frac{A'(\alpha)}{\alpha^2}C_{13} + C(\alpha)C_{14} \tag{3.18}$$

A'(α) and C(α) are obtained from Eq. (3.15) and Eq.(3.18)

$$A'(\alpha) = \frac{1}{C_2}(C_{14}F_1\alpha - C_{12}F_2\alpha^3)\int_a^b \rho G(\rho)d\rho \tag{3.19}$$

$$C(\alpha) = \frac{1}{C_2}(C_{11}F_2\alpha - \frac{1}{\alpha}C_{13}F_1)\int_a^b \rho G(\rho)d\rho \tag{3.20}$$

where C_{11}, C_{12} , C_{13}, C_{14} F_1 , F_1 and C_2 are given in appendix A.

All the unknown functions of the problem $m_2(\lambda)$, $m_4(\lambda)$, $A'(\alpha)$, and $C(\alpha)$ are expressed in terms of the new unknown function $G(\rho)$.

3.2. Singular integral equation

In order to find $G(\rho)$, let us substitute $m_2(\lambda)$, $m_4(\lambda)$, $A'(\alpha)$, and $C(\alpha)$ into Eq. (2.50).

$$\sigma_z = \left\{ \begin{array}{l} \int_0^\infty C_3 \rho\lambda J_0(\lambda r) J_1(\lambda\rho) d\lambda \\ + \dfrac{2}{\pi}\rho \int_0^\infty \dfrac{1}{C_2} \left[\begin{array}{l} (c_1^2 c - d)\alpha I_0(c_1 \alpha r) \begin{pmatrix} C_{14} F_1 \alpha \\ - C_{12} F_2 \alpha^3 \end{pmatrix} \\ - (c_2^2 c - d)\alpha^2 I_0(c_2 \alpha r) \begin{pmatrix} C_{13} F_1 \alpha \\ - C_{11} F_2 \alpha^3 \end{pmatrix} \end{array} \right] \cos\alpha z d\alpha \end{array} \right\} \int_a^b G(\rho) d\rho \tag{3.21}$$

where

$$C_3 = C_{m_4}(s_4^3 d - s_4 c) - C_{m_2}(s_2^3 d - s_2 c)$$

By substituting Eq. (3.21) into boundary condition (3.1.d)

$$\left\{ \begin{array}{l} \int_0^\infty C_3 \rho\lambda J_0(\lambda r) J_1(\lambda\rho) d\lambda \\ + \dfrac{2}{\pi}\rho \int_0^\infty \dfrac{1}{C_2} \left[\begin{array}{l} (c_1^2 c - d)\alpha I_0(c_1 \alpha r) \begin{pmatrix} C_{14} F_1 \alpha \\ - C_{12} F_2 \alpha^3 \end{pmatrix} \\ - (c_2^2 c - d)\alpha^2 I_0(c_2 \alpha r) \begin{pmatrix} C_{13} F_1 \alpha \\ - C_{11} F_2 \alpha^3 \end{pmatrix} \end{array} \right] d\alpha \end{array} \right\} \int_a^b \rho G(\rho) d\rho = -p(r) \tag{3.22}$$

In Eq. (3.22) one can evaluate the first infinite integral in the closed form that is,

$$I = \int_0^\infty \rho\lambda J_0(\lambda r) J_1(\lambda\rho) d\lambda = -\rho \frac{\partial}{\partial r}\left[\int_0^\infty J_0(\lambda r) J_0(\lambda\rho) d\lambda \right] \tag{3.23}$$

$$\int_0^\infty J_0(\lambda r) J_0(\lambda\rho) d\lambda = \begin{cases} \dfrac{2}{\pi r} K\left(\dfrac{\rho}{r}\right), & \rho < r \\ \dfrac{2}{\pi\rho} K\left(\dfrac{r}{\rho}\right), & \rho > r \end{cases} \tag{3.24}$$

where K is the complete elliptic integral of the first kind. Differentiating Eq. (3.24) with respect to x by using following formula (see appendix D),

$$\frac{dK(x)}{dx} = D[K(x)] = \frac{E(x)}{x(1-x^2)} - \frac{K(x)}{x} \tag{3.25}$$

where E is the complete elliptic integral of the second kind.

$$A = -\rho\frac{\partial}{\partial r}\left[\frac{2}{\pi r}K\left(\frac{\rho}{r}\right)\right] = -\frac{2\rho}{\partial r^2}D\left[K\left(\frac{\rho}{r}\right)\right]$$

$$A = \frac{2}{\pi}\frac{r}{\rho^2 - r^2}E\left(\frac{\rho}{r}\right) + \frac{2}{\pi r}K\left(\frac{\rho}{r}\right)$$

$$A = \frac{1}{\pi}\left(\frac{1}{\rho - r} + \frac{1}{\rho + r}\right)\left[\frac{r}{\rho}E\left(\frac{\rho}{r}\right) + \frac{\rho^2 - r^2}{r.\rho}K\left(\frac{\rho}{r}\right)\right] \tag{3.26}$$

$$B = -\rho\frac{\partial}{\partial r}\left[\frac{2}{\pi\rho}K\left(\frac{r}{\rho}\right)\right] = \frac{2}{\pi\rho}\left\{\frac{r}{\rho}D\left[K\left(\frac{r}{\rho}\right)\right] + K\left(\frac{r}{\rho}\right)\right\}$$

$$B = \frac{2}{\pi}\frac{\rho}{\rho^2 - r^2}E\left(\frac{r}{\rho}\right) = \frac{1}{\pi}\left(\frac{1}{\rho - r} + \frac{1}{\rho + r}\right)E\left(\frac{r}{\rho}\right) \tag{3.27}$$

$$I = \begin{cases} A, & \rho < r \\ B, & \rho > r \end{cases}$$

$$I = \frac{1}{\pi}m(r,\rho)\left(\frac{1}{\rho - r} + \frac{1}{\rho + r}\right) \tag{3.28}$$

where

$$m(r,\rho) = \begin{cases} \dfrac{r}{\rho}E\left(\dfrac{\rho}{r}\right) + \dfrac{\rho^2 - r^2}{r.\rho}K\left(\dfrac{\rho}{r}\right), & \rho < r \\ E\left(\dfrac{r}{\rho}\right), & \rho > r \end{cases} \tag{3.29.a}$$

$$m(r, r) = 1 \tag{3.29.b}$$

Let us define the kernel of Eq. (3.22)

$$k(r,\rho) = C_3 k_1(r,\rho) + 2\rho k_2(r,\rho) \tag{3.30}$$

$$k_1(r,\rho) = \frac{m(r,\rho)-1}{\rho - r} + \frac{m(r,\rho)}{\rho + r} \tag{3.31}$$

$$k_2(r,\rho) = \int_0^\infty \frac{1}{C_2} \begin{bmatrix} (c_1^2 c - d)\alpha I_0(c_1 \alpha r)(C_{14} F_1 \alpha - C_{12} F_2 \alpha^3) \\ -(c_2^2 c - d)\alpha^2 I_0(c_2 \alpha r)(C_{13} F_1 \alpha - C_{11} F_2 \alpha^3) \end{bmatrix} d\alpha \tag{3.32}$$

The unknown function G(ρ) can be found as follows,

$$\frac{1}{\pi}\int_a^b \left[\frac{C_3}{\rho - r} + k(r,\rho)\right] G(\rho)d\rho = -p(r) \tag{3.33}$$

From the boundary condition (3.1.e) and Eq. (3.3), it is clear that the integral equation must be solved under the following single valuedness condition.

$$\int_a^b G(\rho)d\rho = 0 \tag{3.34}$$

CHAPTER FOUR
NUMERICAL SOLUTION

4.1. Type of the singular equation

Examining the kernel in Eq. (3.33), when $r = \rho$ it is obvious that the first part of the kernel, $k_1(r, \rho)$ has a simple logarithmic singularity in the form of $\log|\rho - r|$ (Cauchy type singularity). The second part of the kernel, $k_2(r, \rho)$ is bounded in the closed interval $a \leq (r, \rho) \leq b$. This condition means that the crack is a fully embedded internal crack. In this case Cauchy kernel is the dominant kernel.

A standard numerical technique can be used to find out the unknown function $G(\rho)$ [18]. To be able to apply the numerical solution technique to the singular integral equation, it should be normalized. Normalization is carried out by the following quantities:

$$r = \frac{b-a}{2}\eta + \frac{b+a}{2} \tag{4.1.a}$$

$$\rho = \frac{b-a}{2}\tau + \frac{b+a}{2} \tag{4.1.b}$$

Eqs. (3.33) and (3.34) become,

$$\frac{1}{\pi}\int_{-1}^{1}\left[\frac{C_3}{\tau-\eta} + K(\tau,\eta)\right]G(\tau)d\tau = -p(\eta) \tag{4.2}$$

$$\int_{-1}^{1} G(\eta)d\eta = 0 \tag{4.3}$$

where,

$$K(\tau,\eta) = \frac{b-a}{2} k(\tau,\eta) \tag{4.4}$$

The function $G(\tau)$ in Eq. (4.3) is a flux-type quantity. The end points ± 1 are points of geometric singularity. At this points $G(\tau)$ has an integrable singularity.

The singular behavior of the function $G(\tau)$ around $\tau = \pm 1$ may be obtained by analyzing the dominant part of integral equation (4.2) through the use of function–theoretic methods [19]. Following the procedure of [19], it can easily be shown that the fundamental functions of (4.2) which characterize the singular behavior of $G(\tau)$ are given by

$$R(\tau) = (1+\tau)^{(-1/2)+\alpha} (1-\tau)^{(1/2)+\beta} \tag{4.5}$$

where $(\alpha, \beta = 0, \pm 1)$; $(-1 < -1/2+\alpha < 1)$, $(-1 < 1/2+\beta < 1)$ and the index of the integral equation is $\kappa = -(\alpha+\beta) = \pm 1$. Since $G(\tau)$ has an integrable singularity, $\alpha = 0$, $\beta = -1$ and

$$R(\tau) = \left(1-\tau^2\right)^{-1/2}, \qquad \kappa = 1 \tag{4.6}$$

For $\kappa = 1$, the solution of Eq. (4.2) is determined by using single valuedness condition in Eq. (4.4) [18].

4.2. Evaluating the new unknown function

By expressing the unknown function as

$$G(\tau) = R(\tau).F(\tau) \tag{4.7}$$

This method leads to a system of Fredholm integral equations in the new set of unknown functions $F(\tau)$ which is bounded and continuous in the interval $-1 \le \tau \le 1$. However, from the viewpoint of numerical analysis, the method is rather cumbersome and very laborious.

An effective approximate method preserving the correct nature of singularities of the function $G(\tau)$ is described in [20]. Here noting that the fundamental function $R(\tau)$ given by Eq. (4.6) is the weight of Chebyshev polynomials $T_k(\tau)$.

In this study a more direct numerical method of solving the dual integral equations (4.2) and (4.3) is used. The method is based on the notion that by selecting the nodal points τ_k and η_k in the interval(-1, 1) properly, the system can be treated as if it were a system of Fredholm equations and the unknown function $G(\tau)$ may be determined by using the conventional collocation technique.

4.3. Solution of singular integral equation

Substituting Eqs. (4.6) and(4.7) into Eq. (4.2) we obtain

$$\frac{1}{\pi}\int_{-1}^{1}\left[\frac{C_3}{\tau-\eta}+K(\tau,\eta)\right]\frac{F(\tau)}{(1-\tau^2)^{1/2}}d\tau=-p(\eta) \tag{4.8}$$

$F(\tau)$ has to be obtained from Eq. (4.8) subjected to the single-valuedness condition,

$$\int_{-1}^{1}\frac{F(\eta)}{(1-\eta^2)^{1/2}}d\eta=0 \tag{4.9}$$

Note that since $F(\tau)$ and $K(\eta,\tau)$ are bounded, the integral equations (4.8) and (4.9) can be evaluated by using the Gauss-Chebyshev integration formula [21].

$$\frac{1}{\pi}\int_{-1}^{1}\frac{f(t)}{(1-t^2)^{1/2}}dt\cong\sum_{1}^{n}\frac{f(t_k)}{n}, \qquad T_n(t_k)=0 \tag{4.10}$$

Thus from Eqs. (4.8), (4.9) and (4.10) we obtain

$$\sum_{k=1}^{n}\frac{1}{n}F(\tau_k)\left[\frac{C_3}{\tau_k-\eta_r}+K(\tau_k,\eta_r)\right]=-p(\eta_r) \qquad (r=1,..,n-1) \tag{4.11}$$

$$\sum_{r=1}^{n}\frac{\pi}{n}F(\eta_r)=0 \tag{4.12}$$

$$\tau_k = \cos\left(\frac{\pi}{2n}(2k-1)\right) \qquad (k = 1,..,n) \tag{4.13}$$

$$\eta_r = \cos\left(\frac{\pi r}{n}\right) \qquad (r = 1,..,n-1) \tag{4.14}$$

The set of n simultaneous algebraic equations of (4.11) and (4.12) is solved and one can find n values for $F(\tau_i)$ (i=1..n). In order to determine the stress intensity factors at the inner and outer crack tips, the values of F(+1) and F(-1) must be evaluated from the set of $F(\tau_i)$. Evaluation is performed by means of the interpolation technique [22].

4.4. Stress intensity factors

The Mode I stress intensity factors at the crack tips are defined as

$$k(a) = \lim_{r \to a} \sqrt{2(a-r)}.\sigma_z(r,0) \tag{4.15}$$

$$k(b) = \lim_{r \to b} \sqrt{2(r-b)}.\sigma_z(r,0) \tag{4.16}$$

k(a) and k(b) can also be expressed in terms of unknown function G(r) [23],

$$k(a) = \lim_{r \to a} C_3\sqrt{2(a-r)}.G(r) = C_3\sqrt{(b-a)/2}.F(-1) \tag{4.17}$$

$$k(b) = \lim_{r \to b} C_3\sqrt{2(r-b)}.G(r) = -C_3\sqrt{(b-a)/2}.F(+1) \tag{4.18}$$

Under uniform axial tension $\sigma_z=\sigma_0$, the normalized forms of above expressions are [23],

$$k'(a) = \frac{k(a)}{\sigma_0\sqrt{\dfrac{b-a}{2}}} \tag{4.19}$$

$$k'(b) = \frac{k(b)}{\sigma_0\sqrt{\frac{b-a}{2}}} \tag{4.20}$$

4.5. Plastic zone length

It is considered an axially-symmetric deformation of the material and that under the Dugdale assumption, there are thin annular regions of inelastic deformation surrounding the ring-shaped crack tips. A tensile stress **Y** is uniformly distributed in the inelastic regions.

Under the given external loading (represented by σ) let the plastic zones spread to $r = a_p < a$ and $r = b_p > b$, a and b being the actual crack lengths. Solving now the integral equation 3.18 with b_p replacing b, for the given external loads one may obtain a stress intensity factor at b_p. Also a stress intensity factor at a_p may be obtained by solving singular integral equation with a_p replacing a. These stress intensity factors would be linearly dependent on the magnitude of external load σ and would be functions of b_p/R and a_p/R respectively. Repeating the solutions with only external load $\sigma(0,r) = p(r) = -\mathbf{Y}$ $(b < r < b_p)$ and $(a_p < r < a)$, may again obtain stress intensity factors and these would be linearly dependent on **Y** and would be functions of b/R and b_p/R, a/R and a_p/R, respectively. Here Y is the yield stress and represents the yield behavior of the material. Since the stress state at the fictitious crack tips $r = b_p$ and $r = a_p$ must be bounded, the sum of these two stress intensity factors must be zero satisfying the following conditions [12].

$$\sigma k_{1b}(b_p) + Y k_{2b}(b_p, b) = 0 \tag{4.21.a}$$

$$\sigma k_{1a}(a_p) + Y k_{2a}(a_p, a) = 0 \tag{4.21.b}$$

Noting that k_{1a}, k_{1b}, k_{2a}, and k_{2b} correspond to the stress intensity factors calculated from the respective "unit loads". The term $\sigma.k_1$ gives the stress intensity factor under the external load σ and $Y.k_2$ gives the stress intensity factor under the yield stress **Y**. Eqs.4.17.a-b provide a simple means for calculating the plastic zone lengths (b_p-b) and (a-a_p) for a given "load ratio" **σ/Y** in an inverse manner.

4.6. Investigation of the relation between a_p and k

From Eqs. (4.21), load ratios at inner and outer crack tips are obtained as

$$(P_0 / Y)_1 = -\frac{k_{2b}}{k_{1b}} \tag{4.22.a}$$

$$(P_0 / Y)_2 = -\frac{k_{2a}}{k_{1a}} \tag{4.22.b}$$

Let us define Diff, which will be used determining iteration tolerance in programming

$$\text{Diff} = (P_0 / Y)_1 - (P_0 / Y)_2 \tag{4.23}$$

The sample results obtained in the investigation of the plastic zone, made with cadmium material, are given in Table 4.1.

Table 4.1. Iterative approximation results for determining plastic zone width

Cadmium a=0.4 b=0.5 b_p=0.6				
a_p	**Diff**	**k_{1a}**	**k_{1b}**	**k_{2a}**
0.3000000	8.65E-3	-0.3604728	0.3796277	0.2062239
0.2946862	-5.72E-3	-0.3637689	0.3830249	0.2114639
0.2973431	1.39E-3	-0.3621250	0.3813244	0.2088606
0.2960146	-2.20E-3	-0.3629463	0.3821805	0.2101682
0.2966788	-4.09E-4	-0.3625370	0.3817533	0.2095158
0.2970110	4.75E-4	-0.3623282	0.3815430	0.2091901
0.2968449	3.52E-5	-0.3624321	0.3816478	0.2093520

The variations of k_{1a} and k_{2a} with a_p have been given in Figs. (4.1) and (4.2). It is clearly seen from these graphics that the stress intensity factors are linearly dependent on a_p. It can also be shown with a similar investigation that stress intensity factors are linearly dependent on b_p.

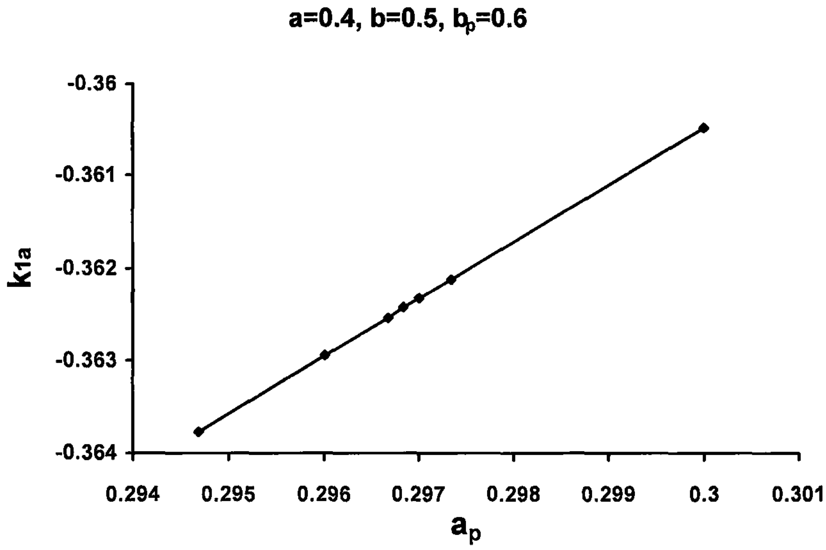

Figure 4.1. Variation of k_{1a} versus a_p in cadmium cylinder

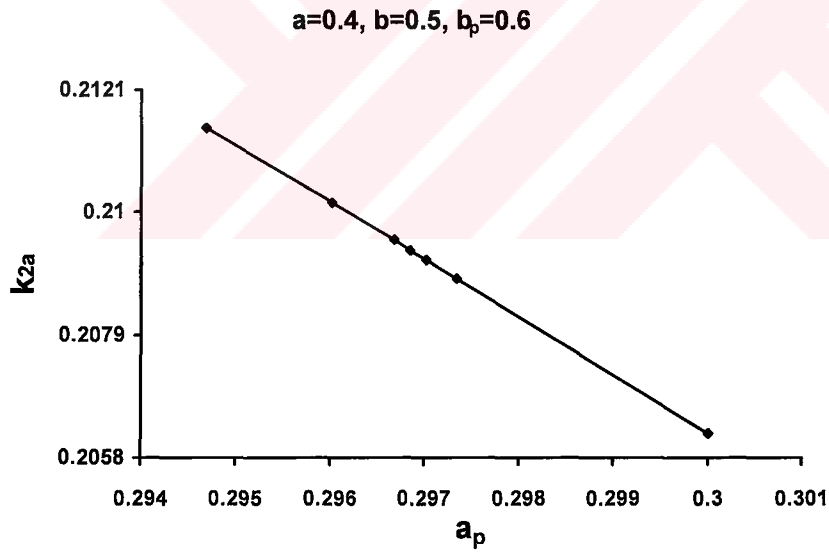

Figure 4.2. Variation of k_{2a} versus a_p in cadmium cylinder

In the determination of the plastic zone at the crack tip, the load ratio (P_0/Y), corresponding to the plastic region (b_p-b) given beforehand has been determined in this study. Afterwards, the plastic zone (a-ap) which will give this (P_0/Y) load ratio has been controllably calculated by iteration. At the beginning of iteration, a second load ratio $(P_0/Y)_2$ has been calculated by giving an initial (a-a_p) plastic zone once more. The $(b_p\text{-}b)$ and $(a\text{-}a_p)$ plastic zones and the load ratio (P_0/Y) in the Diff where the difference between two calculated load ratios is less than $1\text{x}10^{-4}$ have been recorded. In Table 4.1 the determination of the plastic zones by changing the ap values until the Diff is less than $1\text{x}10^{-4}$ has been exemplified.

It is concluded from the variation of Diff with a_p in Fig. (4.3) that there is a linear relationship between a_p and Diff. This property provides us much more ease in programming. That is, by the calculation of Diff corresponding to any two a_p values, the value of a_p which will make Diff zero can be easily determined by interpolation. This way, the plastic zone sizes and the load ratio can be determined in three steps. Due to the high accuracy adopted in programming and the precision of data from the numerical solution, the results are obtained in three or four steps.

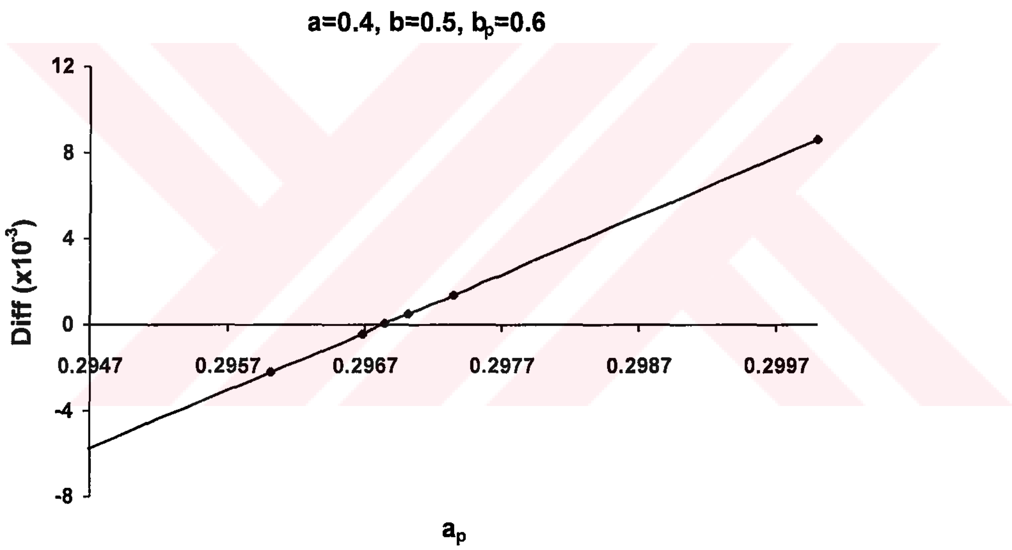

Figure 4.3. Variation of Diff versus a_p in cadmium cylinder

CHAPTER FIVE
RESULTS AND DISCUSSION

An infinitely long cylinder containing a ring-shaped crack in its mid-plane, subjected to uniform loading, is formulated by using the integral transform technique for a transversely isotropic material. After being reduced to a singular integral equation the problem is numerically analyzed and the data obtained is used to determine the stress intensity factors and the length of the plastic zones.

The stress intensity factors are examined and the results are presented in the form of graphics in section 5.1. The obtained lengths of the plastic zone are analyzed and the relation between the stress intensity factor and the length of the plastic zone is put forward in section 5.2.

Magnesium, cadmium, barium-titanate and steel are used in this work. Mg, Cd, and barium-titanate are chosen for their characteristic of transverse isotropy while the steel is picked to be able to make a comparison by using its transversely isotropic properties.

The term $(c_1^2 - c_2^2)$ in the denominator of the Eq.(2.43) becomes equal to zero for perfectly isotropic materials. Hence the perfectly isotropic materials can not be analyzed by making use of the formulation given in this problem.

5.1. Mode I normalized stress intensity factor

Stress intensity factors are determined for almost each position of the crack inside cylinder. The dimensionless inner radius of the crack starts from a/R=0.1 and is increased to 0.8 by a 0.1 increment. Subsequently, in return for a/R values, the dimensionless outer radius b/R starts from a/R+0.1 and increased to 0.9 again by a 0.1 increment.

In Fig. 5.1, the variation of stress intensity factor at the inner tip of the crack in Mg cylinder subjected to unit load is seen for the case where the a/R ratio is chosen constant and b/R ratio is increased; which means an increase of the crack length. k(a) values determined by changing the b/R ratios for each different a/R ratio, can be seen as different series in the same graphic. The maximum change in k(a) is seen in a/R=0.1 series. For this series, the value 1.655 which is the greatest K(a) value reached, corresponds to a/R=0.1 and b/R=0.9 which indicate the greatest crack length. The increase of k with the increase in crack length is an expected outcome. As the inner radius a, where the crack initiates, increases, that is, the initiation of the crack moves farther away from r=0, a slight decrease in k(a) is observed. For example, the value of k(a) is 0.2538 for the series a/R=0.1, while k(a)=0.22976 for a/R=0.6. This decrease is even greater for the cases where the crack length increases. For example, for (b-a)=0.6 and a/R=0.1, k(a)=1.03742, k(a)=0.8894 for a/R=0.2 and k(a)=0.872 for a/R=0.3. In summary, for investigations carried out with the constant crack length (b-a), it can be said that the initiation of the crack effect the k(a) value.

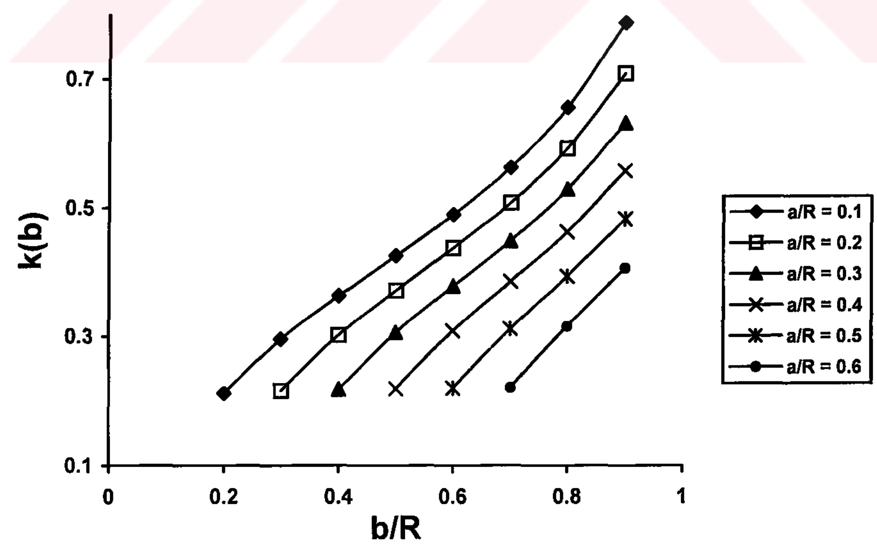

Figure 5.1. Mode I S.I.F.' s at inner crack tip versus b/R in a magnesium cylinder subjected to unit load

In Fig. 5.2, the variation of the stress intensity factor k(b) at the outer tip of the crack with the increase in b/R ratio keeping a/R constant in Mg cylinder subjected to unit load is seen. The variation of k(b) displays similar characteristics with the variation of the stress intensity factor k(a) at the inner tip of the crack shown in Fig. 5.1.

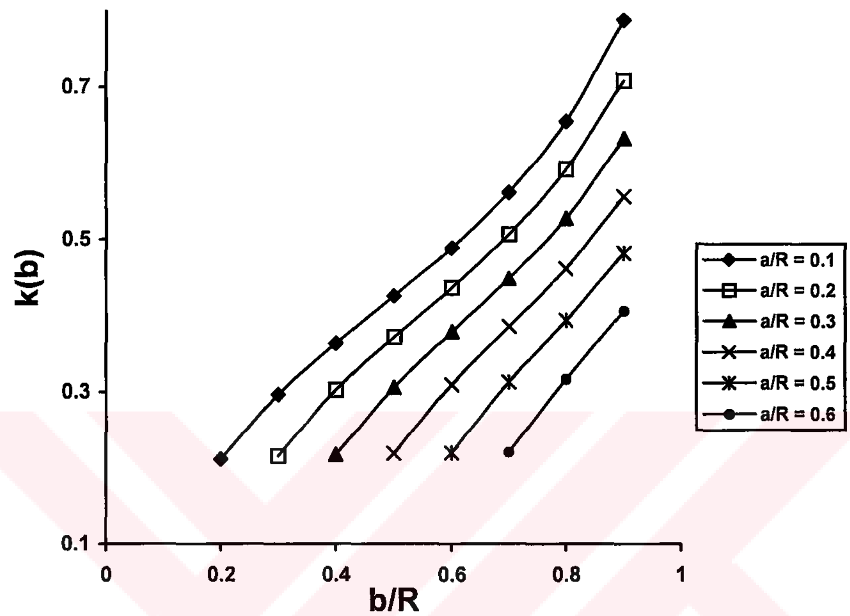

Figure 5.2. Mode I S.I.F.' s at outer crack tip versus b/R in a magnesium cylinder subjected to unit load

As seen from Figs. 5.1 and 5.2, the stress intensity factor at the inner tip of the crack is always greater than the stress intensity factor at the outer tip of the crack. This situation in log cylinders generally occurs as a result of the spreading of the inner defects inward[10]. Although the stress intensity factors at the inner and outer tips of the crack increase as the crack length increases, the increase at the outer tip of the crack is smaller. This increase accelerates as the value of b/R goes to 1.

The variation of stress intensity factor at the inner tip of the crack with b/R in a Cd cylinder under the effect a unit load is shown in Fig. 5.3. The characteristic of the variation is similar to that the Mg cylinder given in Fig.5.1. The only difference is the more accelerated increase in k(a) as b/R goes to 1 in the series a/R=0.1, 0.2, and 0.3.

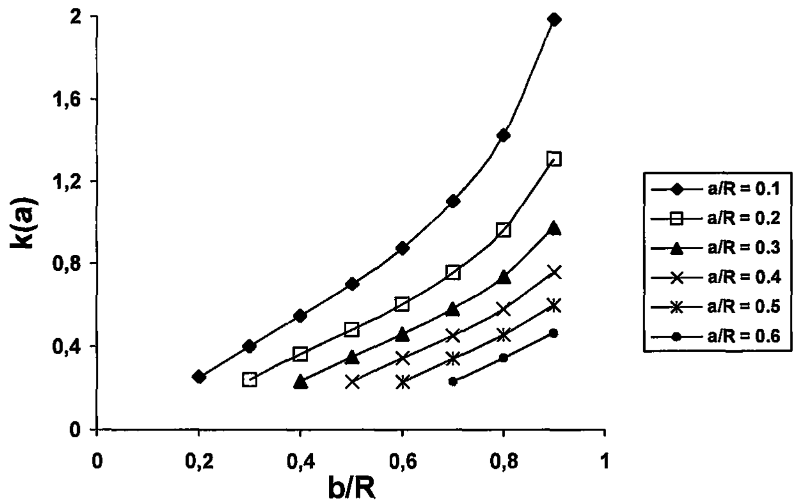

Figure 5.3. Mode I S.I.F.' s at inner crack tip versus b/R in a cadmium cylinder subjected to unit load

In Fig. 5.4 the variation of stress intensity factor at the outer tip of the crack with b/R in a Cd cylinder under the effect a unit load is shown. The characteristic of the variation is similar to that drawn in Fig.5.2 for the Mg cylinder.

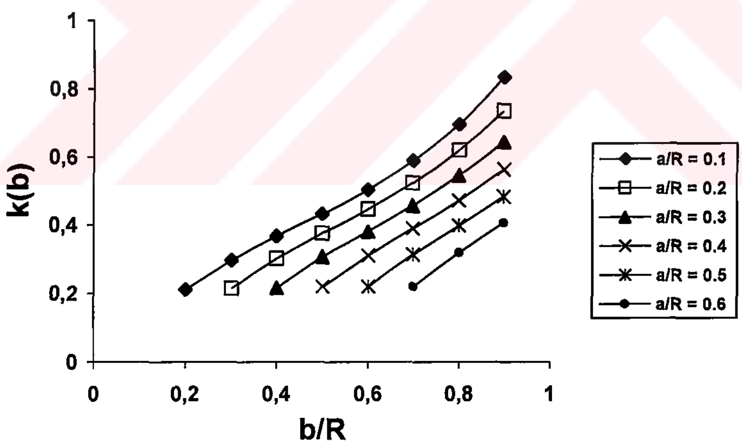

Figure 5.4. Mode I S.I.F.' s at outer crack tip versus b/R in a cadmium cylinder subjected to unit load

The variation of Mode I stress intensity factor at the inner and outer tips of the crack subjected to a unit load with b/R in a Barium-titanate cylinder are seen in Figs. 5.5 and 5.6. The variations display similarity to those given for the Mg cylinder in Figs.5.1 and 5.2.

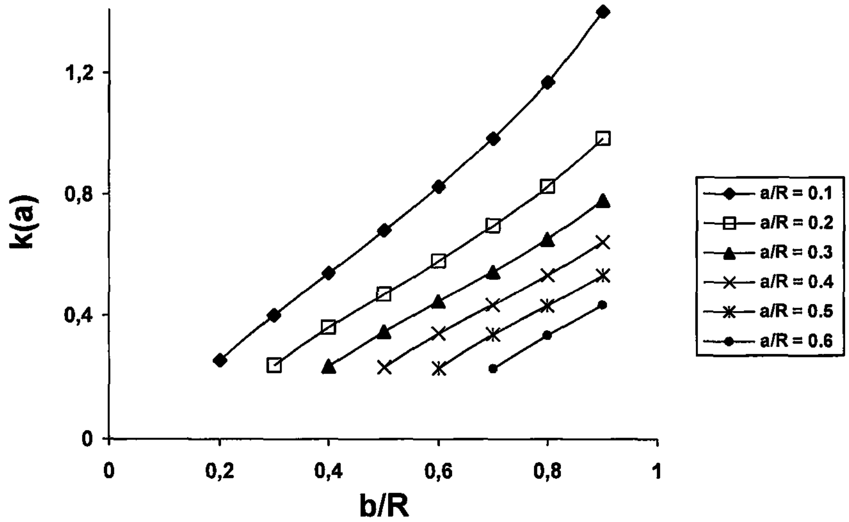

Figure 5.5. Mode I S.I.F.' s at inner crack tip versus b/R in a barium-titanate cylinder subjected to unit load

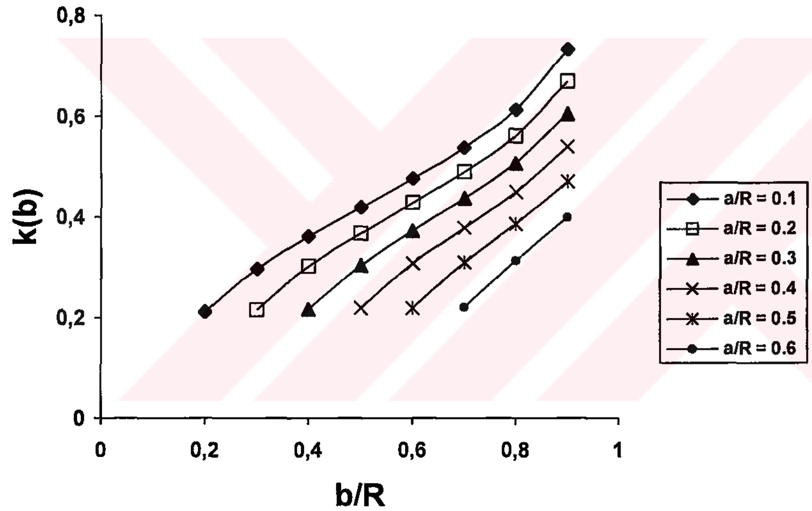

Figure 5.6. Mode I S.I.F.' s at outer crack tip versus b/R in a barium-titanate cylinder subjected to unit load

In Figs. 5.7 and 5.8, the variation of Mode I stress intensity factor at the inner and outer tips of the crack in a steel cylinder subjected to a uniform unit load are given. The variations are similar to those given for the Cd cylinder in Figs.5.3 and 5.4.

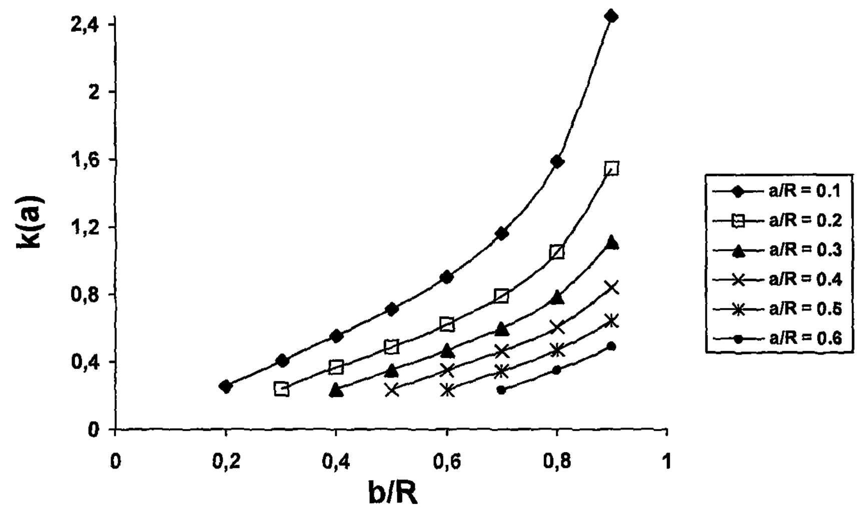

Figure 5.7. Mode I S.I.F.' s at inner crack tip versus b/R in a steel cylinder subjected to unit load

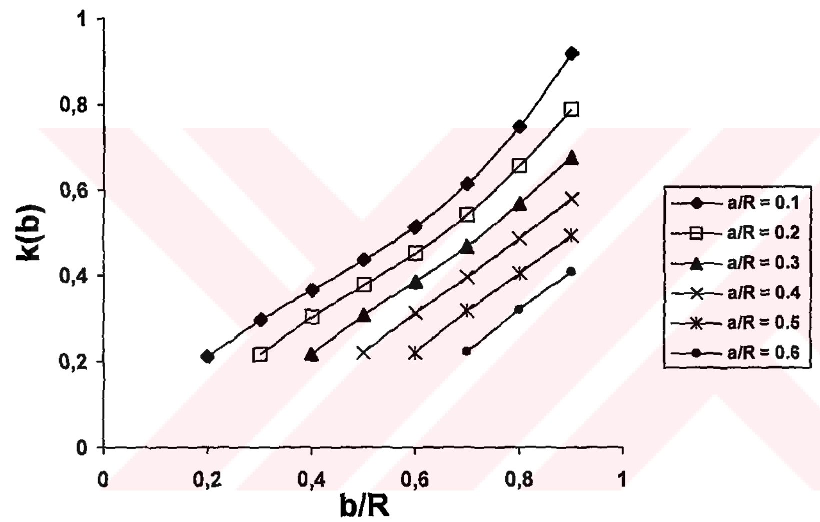

Figure 5.8. Mode I S.I.F.' s at outer crack tip versus b/R in a steel cylinder

Among the graphics given in Figs. 5.1 to 5.8, the series where the crack length is greatest is the one a/R=0.1 for b/R=0.2 at the crack inner tip in Mg, where the first investigation is carried out, the value of k(a), as b/R reaches to 0.9, increases approximately 6.52 times. This increase at the outer tip of the crack is 3.72 times. Increases in other materials and the engineering constants are given in Table 5.1.

Table 5.1 Increments of Mode I S.I.F. in the series of a/R=0.1 and engineering constants

Material	Increment at inner crack tip	Increment at outer crack tip	E (GPa)	E' (GPa)	ν	G' (GPa)
Mg	6.52	3.72	45.24	50.76	0.35	16.58
Cd	7.82	3.92	77.52	27.1	0.12	15.63
Ba-titanate	5.53	3.45	122.7	147.9	0.36	5.46
Steel	9.63	4.33	220.0	210.0	0.30	80.77

The increase in stress intensity factors with the increase in crack length is greatest in steel while it is lowest in barium-titanate. The difference in increase for these two materials is 42.6%.

5.2. Plastic zone length

The work for determining the length of developed plastic zone in various cylinders necessitated solving the singular integral equation (3.33) using a appropriate values of the elastic constants. The results are tabulated in Table (E.1) to (E.12) and given in appendix E. Each table for a different material and different crack configuration. When the values from Table E.1 to E.12 are examined, it is seen that P_0/Y ratio in which a plastic zone is formed, is smallest in steel while it is greatest in barium-titanate. The difference between the values obtained for each material is maximum 4.5% and minimum 0.006%. Therefore, it has been considered adequate only to include the plastic zone variations for magnesium and these variations have been given in the form of graphics in Figs. 5.9 to 5.14. In plastic zone examinations, the dimensions of the plastic zone have been determined by equation 4.21 without calculating the yield stress of the materials. Since the load ratio P_0/Y has been considered instead of yield stress Y, the difference among the plastic zone dimensions obtained for each material is very little.

In magnesium cylinder, the variation of the plastic zone length at the crack outer tip with the load ratio P_0/Y for a/R=0.2 is given in Fig. 5.9. In this graphic, plastic zone dimensions for seven different crack positions have been shown. A constant plastic zone is formed at lower load ratios as the crack length increases. For example, while $(b_p-b)/R=0.1$ is met in $b/R=0.3$ series at $P_0/Y=0.721$, in $b/R=0.5$ and $b/R=0.8$ series, where the crack length increases, it is met at $P_0/Y=0.504$ and $P_0/Y=0.317$, respectively. Since the stress intensity factor will increase with the increase of the crack, P_0/Y ratio will decrease. This result is as expected in connection with the variation of the stress intensity factor along the crack length.

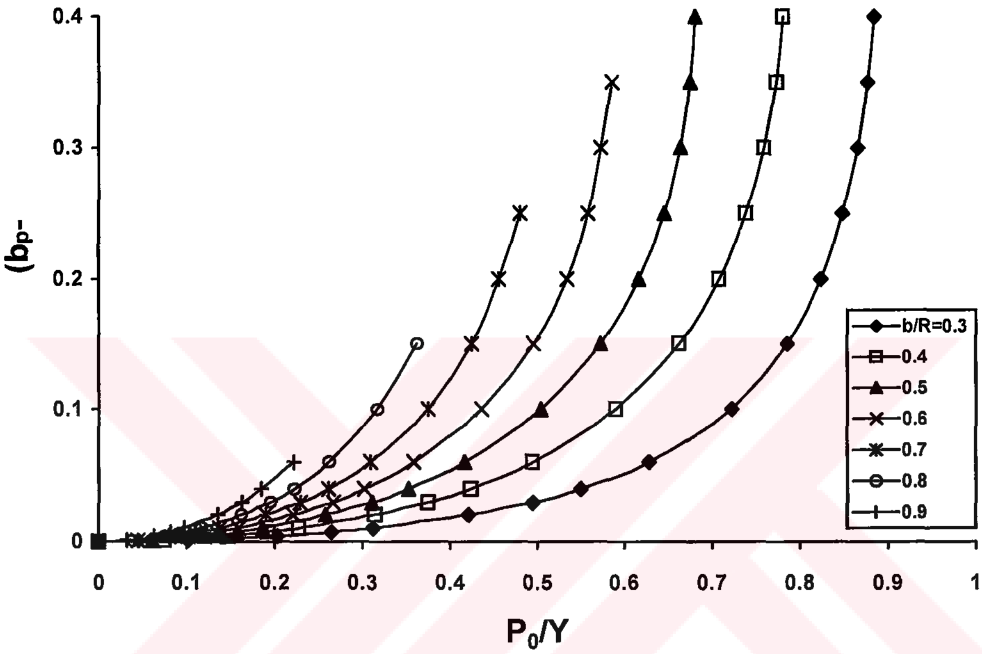

Figure 5.9 Variation of the plastic zone at outer crack tip in a cylinder of magnesium for $a/R=0.2$

For all series in Fig. 5.9 prior to plastic zone reaching the cylinder radius R, relatively small increase in the plastic zone length is observed against comparatively large increase in P_0/Y ratio. As it reaches the cylinder radius R, although the P_0/Y increases slightly the plastic zone increases rapidly. This cases arises from the fact that, the stress intensity factor increases rapidly as the crack approaches the cylinder outer surface. The dimension of the final plastic zone for the series given in graphics, is the length (R-b) between the crack outer radius and the cylinder outer radius in which the plastic zone will be greatest for the crack outer tip. Therefore, all of the series has ended at a point approaching this value.

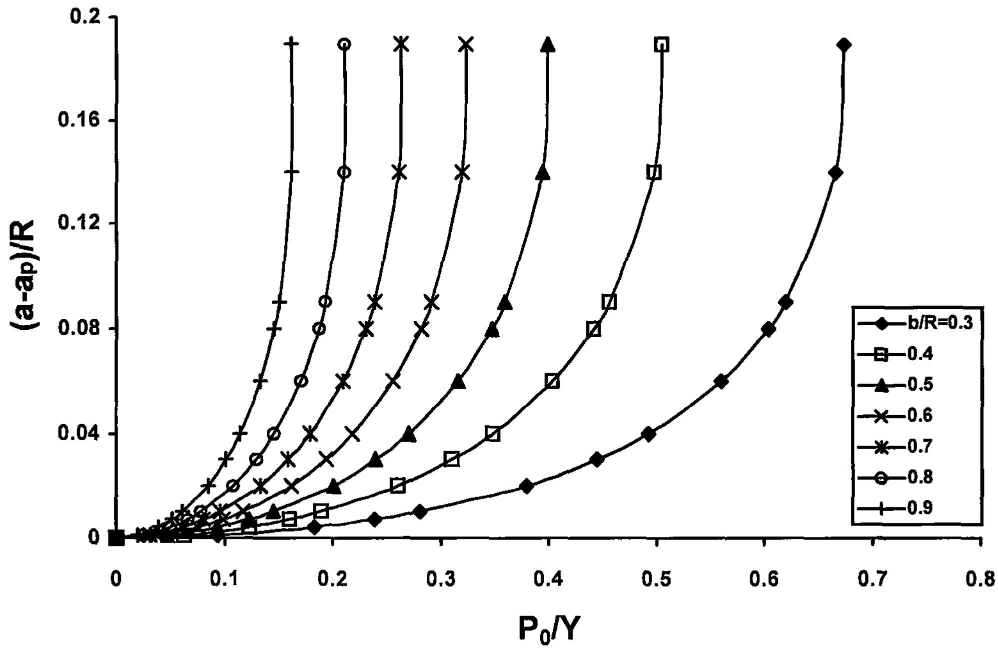

Figure 5.10 Variation of the plastic zone at inner crack tip in a cylinder of magnesium for a/R=0.2

In Fig. 5.10, the variation of the plastic zone length at inner tip of the crack in magnesium cylinder by the load ratio P_0/Y for a/R=0.2 is given. Plastic zone dimensions for seven different crack positions are given in this graphic. As observed at the crack outer tip in Fig. 5.9, a constant plastic zone is formed at lower load ratios as the crack length increases. In all series, the plastic zone expands gradually with the increase in P_0/Y ratio and its increase accelerates as the plastic zone approaches r=0.In the numerical solution of stress intensity factors, the solution goes to infinity for a=0. Therefore, as the plastic zone approaches r=0, the stress intensity factor calculated at the end of the plastic zone increases rapidly. This effect can be observed in plastic zone dimensions. In the graphics, each of the series has been terminated as it approaches a, where the plastic zone will be greatest.

Another solution observed in the examination of the plastic zone carried out for a/R=0.2 is the following: the load ratio P_0/Y corresponding to any plastic zone at the inner tip of the crack for each b/R series, is lower than the load ratio P_0/Y corresponding to the same plastic zone at outer tip of the crack. For any crack position, the stress intensity factor at the inner tip of the crack is always greater than that at the outer tip of the crack(see section 5.1). Therefore, since the stress intensity factor at the inner tip of the crack is large, the P_0/Y ratio is also lower compared to the outer tip of the crack. Greater stress intensity factor causes wider plastic zone. On the contrary, the plastic zone occurring at the inner tip of the crack for any load ratio is greater than that occurring at the outer tip of the crack. For example, as seen from Table E.1, while the plastic zone length occurring at the crack outer tip for the load ratios P_0/Y=0.627874 at b/R series is 0.06, the plastic zone at the crack inner tip is 0.09566. This case is seen for each crack position and load ratios.

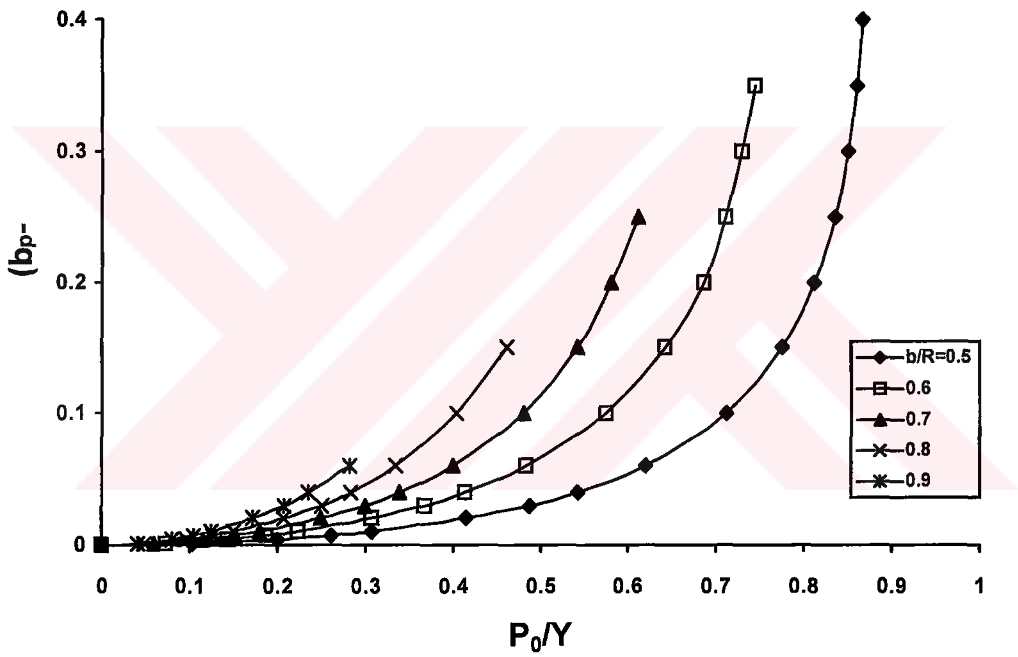

Figure 5.11 Variation of the plastic zone at outer crack tip in a cylinder of magnesium for a/R=0.4

The variation of the plastic zone length at the crack outer tip with the load ratio P_0/Y in magnesium cylinder for a/R=0.4 is drawn for five different crack positions, Fig. 5.11. The variation of each series in this graphic is similar to that drawn in Fig. 5.9.

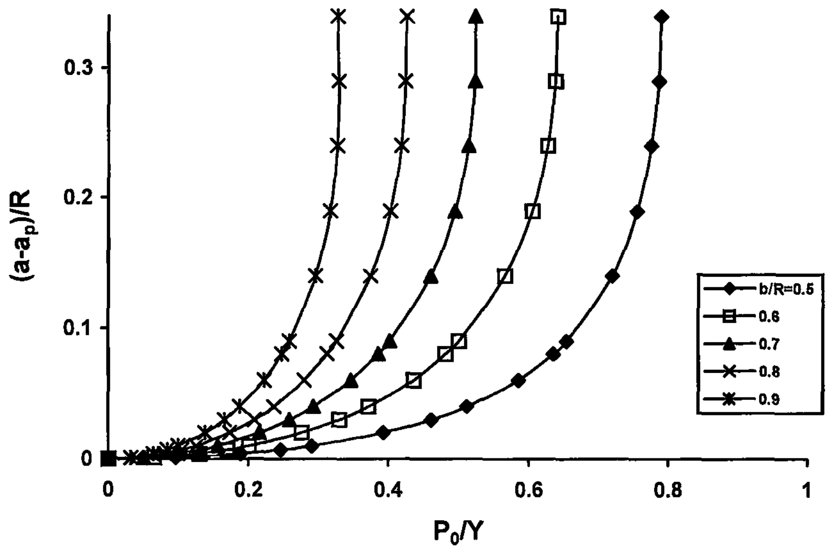

Figure 5.12 Variation of the plastic zone at inner crack tip in a cylinder of magnesium for a/R=0.4

The variation of the plastic zone length at the crack inner tip with the load ratio P_0/Y in Mg cylinder for a/R=0.4 is drawn for five different crack positions, Fig. 5.12. The variation of each series in this graphic is similar to that drawn in Fig. 5.10.

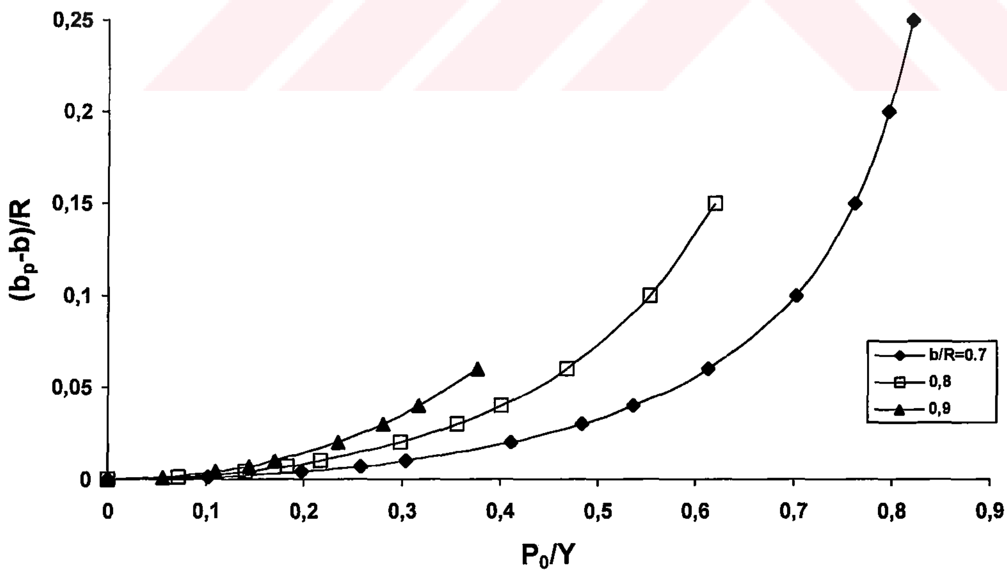

Figure 5.13 Variation of the plastic zone at outer crack tip in a cylinder of magnesium for a/R=0.6

The plastic zone lengths at the crack outer and inner tips for a/R=0.6 in magnesium cylinder are given in Figs. 5.13 and 5.14 for three crack positions, respectively. The plastic zone variations in these graphics are similar to those given in Figs. 5.9 and 5.10.

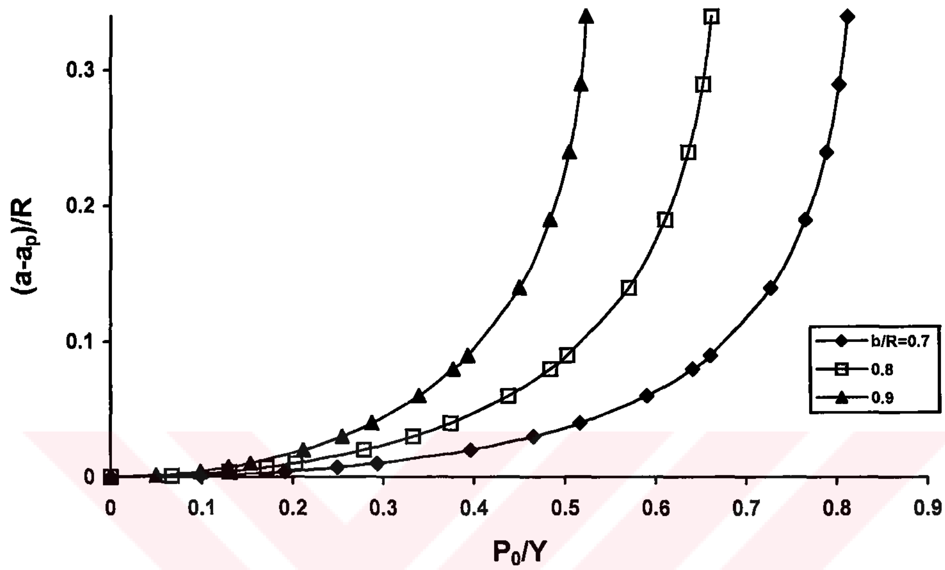

Figure 5.14 Variation of the plastic zone at inner crack tip in a cylinder of magnesium for a/R=0.6

The plastic zone investigation for a penny-shaped crack in an infinite transversely isotropic cylinder is given in [14]. In this study graphite-epoxy, e-glass, magnesium, and zinc have been used as materials. The plastic zone length obtained from this study has been made dimensionless by dividing it to crack radius and its variation with the load ratio λ has been presented graphically. Whether the plastic zone expands up to the outer radius of cylinder is not clear from these graphics. In our study, though, this phenomenon can be easily observed. The methods followed in the solution of the problem given in [14] are entirely dissimilar to the methods employed in our study. Therefore, it is not possible to carry out a through comparison due to the differences both in crack types and the methods employed.

CONCLUSIONS

In this study, an infinitely long solid cylinder containing a ring shaped crack embedded at its mid plane is investigated under the effect of uniform load. The problem is formulated for a transversely isotropic material by using integral transform technique and then is reduced to a singular integral equation. This integral equation is solved numerically. Plastic zone lengths at the outer and the inner crack tips are determined for various crack configurations. All the numerical values obtained are presented as tables and graphs.

Depending on the crack geometries, the greatest stress intensity factor occurs in the steel cylinder followed by cadmium, magnesium, and barium-titanate cylinders, respectively. On the contrary, the plastic zone lengths decrease in the reverse order for the above cylinders. The largest increase in stress intensity factors is observed in steel, the difference between the steel and barium-titanate in which the smallest increase is observed, being 43%. This huge difference between the stress intensity factors is not met however in plastic zone lengths where the largest value in approximately 4.5%. The small difference between the plastic zones compared to the relatively large difference in stress intensity factors, arises from the fact that plastic zones are determined by the load ratio P_0/Y instead of P_0.

Finally, we expect that the results of this study will be useful in practice.

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APPENDIX A

A. ABBREVIATIONS

$$C_{m_2} = \frac{1 - as_4^2}{(s_2^2 - s_4^2)(a_{33}d - 2a_{13}a - a_{44}a)} \tag{A.1}$$

$$C_{m_4} = \frac{1 - as_2^2}{(s_2^2 - s_4^2)(a_{33}d - 2a_{13}a - a_{44}a)} \tag{A.2}$$

$$C_{11} = (c_1^2 - a)\alpha^2 I_0(c_1\alpha R) + \frac{c_1\alpha}{R}(b-1)I_1(c_1\alpha R) \tag{A.3}$$

$$C_{12} = (c_2^2 - a)\alpha^2 I_0(c_2\alpha R) + \frac{c_2\alpha}{R}(b-1)I_1(c_2\alpha R) \tag{A.4}$$

$$C_{13} = (c_1^3 - c_1 a)\alpha^3 I_1(c_1\alpha R) \tag{A.5}$$

$$C_{14} = (c_2^3 - c_2 a)\alpha^3 I_1(c_2\alpha R) \tag{A.6}$$

$$C_2 = C_{11}C_{14} - C_{12}C_{13} \tag{A.7}$$

$$F_1 = C_{m_2}\frac{\alpha}{s_2}(1 - as_2^2)I_1\left(\frac{\rho\alpha}{s_2}\right)K_0\left(\frac{R\alpha}{|s_2|}\right) + C_{m_4}\frac{\alpha}{s_4}(1 - as_4^2)I_1\left(\frac{\rho\alpha}{s_4}\right)K_0\left(\frac{R\alpha}{|s_4|}\right)$$
$$+\frac{1-b}{R}\left[C_{m_2}I_1\left(\frac{\rho\alpha}{s_2}\right)K_1\left(\frac{R\alpha}{|s_2|}\right) + C_{B_4}I_1\left(\frac{\rho\alpha}{s_4}\right)K_1\left(\frac{R\alpha}{|s_4|}\right)\right] \tag{A.8}$$

$$F_2 = C_{m_2}\frac{(1 - as_2^2)}{s_2^2}I_1\left(\frac{\rho\alpha}{s_2}\right)K_1\left(\frac{R\alpha}{|s_2|}\right) - C_{m_4}\frac{(1 - as_4^2)}{s_4^2}I_1\left(\frac{\rho\alpha}{s_4}\right)K_1\left(\frac{R\alpha}{|s_4|}\right) \tag{A.9}$$

APPENDIX B

B. CLOSED FORM INTEGRALS

$$\int_0^\infty e^{-ax}\cos(mx)dx = \frac{a}{a^2+m^2}, \qquad (a>0) \tag{B.1}$$

$$\int_0^\infty e^{-ax}\sin(mx)dx = \frac{m}{a^2+m^2}, \qquad (a>0) \tag{B.2}$$

$$\int_0^\infty x^{\mu-\upsilon+1+2n} J_\mu(ax) J_\upsilon(bx)\frac{dx}{x^2+c^2} = (-1)^n c^{\mu-\upsilon+2n} I_\upsilon(bc) K_\mu(ac) \tag{B.3}$$

$(a>b>0,\ R_\nu - 2n + 2 > R_\mu > -n-1,\ n \geq 0$ and an integer)

$$\int_0^\infty x^{\upsilon-\mu+1+2n} J_\mu(ax) J_\upsilon(bx)\frac{dx}{x^2+c^2} = (-1)^n c^{\upsilon-\mu+2n} I_\mu(ac) K_\nu(bc) \tag{B.4}$$

$(b>a>0,\ R_\mu - 2n + 2 > R_\nu > -n-1,\ n \geq 0$ and an integer)

APPENDIX C

C. STRAIN COEFFICIENTS (in 10^{-12}xPa^{-1})

Material	a_{11}	a_{12}	a_{13}	a_{33}	a_{44}
Magnesium	22.1	-7.7	-4.9	19.7	60.3
Cadmium	12.9	-1.5	-9.3	36.9	64.0
Barium-titanate	8.15	-2.96	-1.95	6.76	183.1
Steel	4.5	-1.35	-1.43	4.76	12.38

APPENDIX D

D. COMPLETE ELLIPTIC INTEGRALS OF FIRST AND SECOND KIND

$$K(x) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - x^2 \sin^2 \theta}} \tag{D.1}$$

$$E(x) = \int_0^{\pi/2} \sqrt{1 - x^2 \sin^2 \theta}\, d\theta \tag{D.2}$$

Differentiation rules for elliptic functions

$$\frac{dK(x)}{dx} = \frac{E(x)}{x(1 - x^2)} - \frac{K(x)}{x} \tag{D.3}$$

$$\frac{dE(x)}{dx} = \frac{E(x) - K(x)}{x} \tag{D.4}$$

$$\frac{d[E(x) - K(x)]}{dx} = \frac{x.E(x)}{x^2 - 1} \tag{D.5}$$

$$\int_0^{\infty} J_i(r\alpha) J_j(\rho\alpha)\, d\alpha = m_{ij}(r, \rho) \tag{D.6}$$

$$m_{00}(r, \rho) = \begin{cases} \dfrac{2}{\pi r} K\left(\dfrac{\rho}{r}\right), & \rho < r \\ \dfrac{2}{\pi \rho} K\left(\dfrac{r}{\rho}\right), & \rho > r \end{cases} \tag{D.7}$$

$$m_{01}(r,\rho)=\begin{cases}0, & \rho<r\\ \dfrac{1}{\rho}, & \rho>r\end{cases} \tag{D.7}$$

$$m_{10}(r,\rho)=\begin{cases}\dfrac{1}{r}, & \rho<r\\ 0, & \rho>r\end{cases} \tag{D.8}$$

$$m_{11}(r,\rho)=\begin{cases}\dfrac{2}{\pi\rho}\left[K\left(\dfrac{\rho}{r}\right)-E\left(\dfrac{\rho}{r}\right)\right], & \rho<r\\ \dfrac{2}{\pi r}\left[K\left(\dfrac{r}{\rho}\right)-E\left(\dfrac{r}{\rho}\right)\right], & \rho>r\end{cases} \tag{D.7}$$

where K(x) and E(x) are the complete elliptic integral of the first and second kind respectively.

APPENDIX E

E. NUMERICAL VALUES OBTAINED

P_0/Y: Load ratio
a/R: Dimensionless radius of the inner tip of the crack
b/R: Dimensionless radius of the outer tip of the crack
$(a - a_p)/R$: Dimensionless plastic zone length occurs at the inner tip of the crack
$(b_p - b)/R$: Dimensionless plastic zone length occurs at the outer tip of the crack

Table E.1. Numerical values obtained for magnesium cylinder for a/R=0.2

	B/R=0.3			b/R=0.4	
P_0/Y	$(b_p - b)/R$	$(a - a_p)/R$	P_0/Y	$(b_p - b)/R$	$(a - a_p)/R$
0	0	0	0	0	0
0.1023244	0.001	0.001209125	0.0732856	0.001	0.00141178
0.2028467	0.004	0.004928857	0.1463066	0.004	0.00580488
0.2644555	0.007	0.008719742	0.1919953	0.007	0.01025046
0.3116852	0.01	0.01257317	0.2277483	0.01	0.01481847
0.4212313	0.02	0.02597898	0.3138496	0.02	0.03072809
0.4947286	0.03	0.04041603	0.3749325	0.03	0.04860936
0.5496547	0.04	0.05612780	0.4228313	0.04	0.06892537
0.6278740	0.06	0.09566031	0.4952478	0.06	0.13291010
0.7211655	0.10		0.5899754	0.10	
0.7855355	0.15		0.6620325	0.15	
0.8238092	0.20		0.7075495	0.20	
0.8484581	0.25		0.7380536	0.25	
0.8649913	0.30		0.7585250	0.30	
0.8764596	0.35		0.7721577	0.35	
0.8841472	0.40		0.7805061	0.40	
0.8916823	0.50		0.7859236	0.50	
	b/R=0.5			b/R=0.6	
P_0/Y	$(b_p - b)/R$	$(a - a_p)/R$	P_0/Y	$(b_p - b)/R$	$(a - a_p)/R$
0	0	0	0	0	0
0.0610325	0.001	0.00168528	0.0518762	0.001	0.00187098
0.1196941	0.004	0.00661922	0.1018266	0.004	0.00748615
0.1572689	0.007	0.01170649	0.1339051	0.007	0.01329103

Table E.1. a/R=0.2 magnesium ......(continued)

0.1868040	0.01	0.01705872	0.1591821	0.01	0.01928486
0.2587121	0.02	0.03581861	0.2209948	0.02	0.04091883
0.3110926	0.03	0.05736813	0.2663744	0.03	0.06729360
0.3524964	0.04	0.08397510	0.3024514	0.04	0.10272240
0.4166913	0.06		0.3587795	0.06	
0.5036890	0.10		0.4357533	0.10	
0.5721122	0.15		0.4962314	0.15	
0.6159338	0.20		0.5340907	0.20	
0.6448751	0.25		0.5579553	0.25	
0.6634195	0.30		0.5727306	0.30	
0.6746476	0.35		0.5850902	0.35	
0.6806952	0.40				
	b/R=0.7			b/R=0.8	
P_0/Y	$(b_p - b)/R$	$(a - a_p)/R$	P_0/Y	$(b_p - b)/R$	$(a - a_p)/R$
0	0	0	0	0	0
0.0446886	0.001	0.00209902	0.0382549	0.001	0.00231724
0.0877334	0.004	0.00833066	0.0750727	0.004	0.00921056
0.1153894	0.007	0.01480466	0.0987021	0.007	0.01646852
0.1371905	0.01	0.02156423	0.1173019	0.01	0.02404276
0.1905334	0.02	0.04632130	0.1626730	0.02	0.05297081
0.2297038	0.03	0.07858466	0.1958123	0.03	0.09340102
0.2608341	0.04	0.13640980	0.2219950	0.04	
0.3093269	0.06		0.2624174	0.06	
0.3749732	0.10		0.3165011	0.10	
0.4252693	0.15		0.3624126	0.15	
0.4562114	0.20				
0.4804285	0.25				
	b/R=0.9				
P_0/Y	$(b_p - b)/R$	$(a - a_p)/R$			
0	0	0			
0.0319639	0.001	0.00255819			
0.0626758	0.004	0.01047400			
0.0823348	0.007	0.01878963			
0.0977796	0.01	0.02762093			
0.1353533	0.02	0.06327848			
0.1628059	0.03	0.14861840			
0.1848122	0.04				
0.2215870	0.06				

Table E.2. Numerical values obtained for magnesium cylinder for a/R=0.4

	b/R=0.5			b/R=0.6	
P_0/Y	$(b_p - b)/R$	$(a - a_p)/R$	P_0/Y	$(b_p - b)/R$	$(a - a_p)/R$
0	0	0	0	0	0
0.1031129	0.001	0.00115299	0.0732328	0.001	0.00125817
0.2004135	0.004	0.00449672	0.143312	0.004	0.00493267
0.2610416	0.007	0.00788668	0.1878932	0.007	0.00872773
0.3074915	0.01	0.01133272	0.2227169	0.01	0.01245791
0.4151667	0.02	0.02285159	0.3064327	0.02	0.02513373
0.4880573	0.03	0.03478286	0.3662498	0.03	0.03831020
0.5421673	0.04	0.04700834	0.4127277	0.04	0.05189011
0.6194953	0.06	0.07277620	0.4830348	0.06	0.08059663
0.7118781	0.10	0.13127420	0.5744787	0.10	0.14765220
0.7751567	0.15	0.23419100	0.6426556	0.15	0.35037650
0.8120152	0.20		0.6426556	0.20	
0.8351599	0.25		0.7110094	0.25	
0.8499863	0.30		0.7287301	0.30	
0.8597118	0.35		0.7442759	0.35	
0.8661086	0.40				
	b/R=0.7			b/R=0.8	
P_0/Y	$(b_p - b)/R$	$(a - a_p)/R$	P_0/Y	$(b_p - b)/R$	$(a - a_p)/R$
0	0	0	0	0	0
0.0588847	0.001	0.00138539	0.0490096	0.001	0.00149611
0.1154532	0.004	0.00539503	0.0961336	0.004	0.00584844
0.1516566	0.007	0.00948465	0.1263309	0.007	0.01023716
0.1800888	0.01	0.01359111	0.1500726	0.01	0.01469111
0.2491730	0.02	0.02738661	0.2078542	0.02	0.02973792
0.2993142	0.03	0.04177228	0.2499314	0.03	0.04535770
0.3387807	0.04	0.05680662	0.2830934	0.04	0.06148681
0.3995355	0.06	0.08815500	0.3342089	0.06	0.09629166
0.4804231	0.10	0.16496510	0.4027093	0.10	0.18853110
0.5419256	0.15		0.4610169	0.15	
0.5804781	0.20				
0.6111421	0.25				
	b/R=0.9				
P0/Y	(bp - b)/R	(a - ap)/R			
0	0	0			
0.0406730	0.001	0.00164601			
0.0797520	0.004	0.00641725			
0.1047679	0.007	0.01128188			
0.1244207	0.01	0.01604828			
0.1722284	0.02	0.03258851			
0.2071763	0.03	0.05045965			
0.2351511	0.04	0.06920418			
0.2816513	0.06	0.11608160			

Table E.3. Numerical values obtained for magnesium cylinder for a/R=0.6

	b/R=0.7			b/R=0.8	
P_0/Y	$(b_p - b)/R$	$(a - a_p)/R$	P_0/Y	$(b_p - b)/R$	$(a - a_p)/R$
0	0	0	0	0	0
0.1021747	0.001	0.00114405	0.0715441	0.001	0.00117445
0.1977019	0.004	0.00435370	0.1399442	0.004	0.00466359
0.2586243	0.007	0.00760615	0.1833935	0.007	0.00817240
0.3046093	0.01	0.01093107	0.2172824	0.01	0.01160604
0.4111190	0.02	0.02191800	0.2984835	0.02	0.02341163
0.4830995	0.03	0.03318393	0.3561699	0.03	0.03538084
0.5364218	0.04	0.04455298	0.4007052	0.04	0.04732698
0.6123308	0.06	0.06778723	0.4674016	0.06	0.07193536
0.7020825	0.10	0.11691680	0.5526178	0.10	0.12369720
0.7621922	0.15	0.18439110	0.6194153	0.15	0.20371920
0.7965547	0.20	0.26309270			
0.8211141	0.25				
	b/R=0.9				
P_0/Y	$(b_p - b)/R$	$(a - a_p)/R$			
0	0	0			
0.0557856	0.001	0.00129801			
0.1092486	0.004	0.00495887			
0.1433416	0.007	0.00871766			
0.1700239	0.01	0.01242340			
0.2344382	0.02	0.02495700			
0.2808501	0.03	0.03764915			
0.3174199	0.04	0.05061245			
0.3763297	0.06	0.07939214			

Table E.4. Numerical values obtained for cadmium cylinder for a/R=0.2

	b/R=0.3			b/R=0.4	
P_0/Y	$(b_p - b)/R$	$(a - a_p)/R$	P_0/Y	$(b_p - b)/R$	$(a - a_p)/R$
0	0	0	0	0	0
0.102250	0.001	0.00120917	0.073032	0.001	0.00141244
0.202690	0.004	0.00493012	0.145783	0.004	0.00580780
0.264239	0.007	0.00871916	0.191285	0.007	0.01025605
0.311415	0.01	0.01257244	0.226878	0.01	0.01482725
0.420798	0.02	0.02598748	0.312522	0.02	0.03088492
0.494139	0.03	0.04041776	0.373192	0.03	0.04866226
0.548909	0.04	0.05614099	0.420691	0.04	0.06904015
0.626808	0.06	0.09576435	0.492319	0.06	0.13443470
0.719423	0.10		0.585445	0.10	
0.782879	0.15		0.655407	0.15	
0.820160	0.20		0.698720	0.20	
0.843737	0.25		0.726913	0.25	
0.859114	0.30		0.744985	0.30	
0.869330	0.35		0.756161	0.35	
0.875661	0.40		0.762102	0.40	
0.880181	0.50		0.765132	0.50	
0.877859	0.60				
	b/R=0.5			b/R=0.6	
P_0/Y	$(b_p - b)/R$	$(a - a_p)/R$	P_0/Y	$(b_p - b)/R$	$(a - a_p)/R$
0	0	0	0	0	0
0.060480	0.001	0.00168909	0.050931	0.001	0.00188538
0.118584	0.004	0.00663486	0.099938	0.004	0.00755253
0.155776	0.007	0.01181966	0.131377	0.007	0.01341845
0.184990	0.01	0.01711392	0.156124	0.01	0.01949614
0.256009	0.02	0.03597039	0.216505	0.02	0.04151344
0.307613	0.03	0.05768000	0.260669	0.03	0.06870839
0.348291	0.04	0.08479864	0.295640	0.04	0.10741860
0.411092	0.06		0.349912	0.06	
0.495372	0.10		0.423102	0.10	
0.560411	0.15		0.479303	0.15	
0.600868	0.20		0.513609	0.20	
0.626550	0.25		0.535395	0.25	
0.642159	0.30		0.551920	0.30	
0.651342	0.35		0.580814	0.35	
0.658088	0.40				

Table E.4. a/R=0.2 cadmium ......(continued)

	b/R=0.7			b/R=0.8	
P_0/Y	$(b_p - b)/R$	$(a - a_p)/R$	P_0/Y	$(b_p - b)/R$	$(a - a_p)/R$
0	0	0	0	0	0
0.043304	0.001	0.00216094	0.036563	0.001	0.00249434
0.084978	0.004	0.00855863	0.07173	0.004	0.00998005
0.111718	0.007	0.01525687	0.094282	0.007	0.01791123
0.132768	0.01	0.02225703	0.112019	0.01	0.02631301
0.184133	0.02	0.04867041	0.155236	0.02	0.06039535
0.221683	0.03	0.08397209	0.186779	0.03	0.12836070
0.251392	0.04		0.211731	0.04	
0.297386	0.06		0.250549	0.06	
0.359040	0.10		0.305397	0.10	
0.406522	0.15		0.370316	0.15	
0.439539	0.20				
0.482972	0.25				
	b/R=0.9				
P_0/Y	$(b_p - b)/R$	$(a - a_p)/R$			
0	0	0			
0.030889	0.001	0.00329129			
0.060681	0.004	0.01334469			
0.079878	0.007	0.02481881			
0.095078	0.01	0.03776287			
0.132850	0.02	0.12686180			
0.161866	0.03				
0.187055	0.04				
0.238606	0.06				

Table E.5. Numerical values obtained for cadmium cylinder for a/R=0.4

	b/R=0.5			b/R=0.6	
P_0/Y	$(b_p - b)/R$	$(a - a_p)/R$	P_0/Y	$(b_p - b)/R$	$(a - a_p)/R$
0	0	0	0	0	0
0.102974	0.001	0.00115231	0.072811	0.001	0.00125960
0.200128	0.004	0.00449479	0.142460	0.004	0.00498116
0.260648	0.007	0.00789955	0.186743	0.007	0.00874808
0.307002	0.01	0.01133198	0.221313	0.01	0.01247755
0.414391	0.02	0.02287748	0.304318	0.02	0.02523509
0.487010	0.03	0.03481710	0.363505	0.03	0.03849295
0.540854	0.04	0.04703864	0.409393	0.04	0.05213431
0.617651	0.06	0.07282028	0.478578	0.06	0.08147994
0.708972	0.10	0.13206600	0.567937	0.10	0.15044650
0.770939	0.15	0.23934250	0.633864	0.15	
0.806551	0.20		0.673938	0.20	
0.828593	0.25		0.700490	0.25	
0.842615	0.30		0.721333	0.30	
0.852191	0.35		0.751164	0.35	
0.860097	0.40				
	b/R=0.7			b/R=0.8	
P_0/Y	$(b_p - b)/R$	$(a - a_p)/R$	P_0/Y	$(b_p - b)/R$	$(a - a_p)/R$
0	0	0	0	0	0
0.058101	0.001	0.00139874	0.047976	0.001	0.00153717
0.113886	0.004	0.00545141	0.094093	0.004	0.00607836
0.149560	0.007	0.00951493	0.123636	0.007	0.01065654
0.177554	0.01	0.01374906	0.146857	0.01	0.01537171
0.245463	0.02	0.0279052	0.203366	0.02	0.03123432
0.294626	0.03	0.04264402	0.244552	0.03	0.04839057
0.333228	0.04	0.05804637	0.277102	0.04	0.06658930
0.392470	0.06	0.09121329	0.327735	0.06	0.10877080
0.471118	0.10	0.1800886	0.399140	0.10	
0.531877	0.15		0.478931	0.15	
0.574462	0.20				
0.624629	0.25				
	b/R=0.9				
P0/Y	(bp - b)/R	(a - ap)/R			
0	0	0			
0.040371	0.001	0.00187364			
0.079296	0.004	0.00737903			
0.104365	0.007	0.01326787			
0.124197	0.01	0.01924193			
0.173352	0.02	0.04095101			
0.210870	0.03	0.06762967			
0.243000	0.04	0.10375060			
0.306257	0.06				

Table E.6. Numerical values obtained for cadmium cylinder for a/R=0.6

	b/R=0.7			b/R=0.8	
P_0/Y	$(b_p - b)/R$	$(a - a_p)/R$	P_0/Y	$(b_p - b)/R$	$(a - a_p)/R$
0	0	0	0	0	0
0.101981	0.001	0.00110745	0.071111	0.001	0.001194
0.198178	0.004	0.00435078	0.139944	0.004	0.00466359
0.258079	0.007	0.00764936	0.182255	0.007	0.00827301
0.303935	0.01	0.01087373	0.215919	0.01	0.01180124
0.410071	0.02	0.02188915	0.296572	0.02	0.02373260
0.481715	0.03	0.03319794	0.353898	0.03	0.03607863
0.534726	0.04	0.04464549	0.398231	0.04	0.04863793
0.610078	0.06	0.06820142	0.465020	0.06	0.07509595
0.699057	0.10	0.11887610	0.553337	0.10	0.13733750
0.759293	0.15	0.19481050	0.637584	0.15	
0.796383	0.20	0.32931320			
0.831539	0.25				
	b/R=0.9				
P_0/Y	$(b_p - b)/R$	$(a - a_p)/R$			
0	0	0			
0.055930	0.001	0.00136358			
0.109656	0.004	0.00539863			
0.144058	0.007	0.00943887			
0.171111	0.01	0.01365167			
0.237273	0.02	0.02806652			
0.286431	0.03	0.04373437			
0.327127	0.04	0.06134403			
0.401690	0.06	0.11461210			

Table E.7. Numerical values obtained for barium-titanate cylinder for a/R=0.2

	b/R=0.3			b/R=0.4	
P_0/Y	$(b_p - b)/R$	$(a - a_p)/R$	P_0/Y	$(b_p - b)/R$	$(a - a_p)/R$
0	0	0	0	0	0
0.102394	0.001	0.00120910	0.073528	0.001	0.00141142
0.202993	0.004	0.00493227	0.146809	0.004	0.00580335
0.264658	0.007	0.00871806	0.192677	0.007	0.01024756
0.311938	0.01	0.01256657	0.228584	0.01	0.01481391
0.421637	0.02	0.02596843	0.315128	0.02	0.03084016
0.495282	0.03	0.04040220	0.376612	0.03	0.04857123
0.550358	0.04	0.05612083	0.424902	0.04	0.06883056
0.628882	0.06	0.09558994	0.498096	0.06	0.13190590
0.722824	0.10		0.594428	0.10	
0.788088	0.15		0.668626	0.15	
0.827344	0.20		0.716432	0.20	
0.853065	0.25		0.749360	0.25	
0.870758	0.30		0.772333	0.30	
0.883471	0.35		0.788455	0.35	
0.892468	0.40		0.799081	0.40	
0.902639	0.50		0.806243	0.50	
0.904178	0.60				
	b/R=0.5			b/R=0.6	
P_0/Y	$(b_p - b)/R$	$(a - a_p)/R$	P_0/Y	$(b_p - b)/R$	$(a - a_p)/R$
0	0	0	0	0	0
0.061580	0.001	0.00168322	0.052857	0.001	0.00186300
0.120795	0.004	0.00661089	0.103791	0.004	0.00743330
0.158751	0.007	0.01169097	0.136539	0.007	0.01319636
0.188607	0.01	0.01703788	0.162373	0.01	0.01916483
0.261408	0.02	0.03575253	0.225704	0.02	0.04062590
0.314578	0.03	0.05717926	0.272391	0.03	0.06655145
0.356724	0.04	0.08352658	0.309671	0.04	0.10071060
0.422362	0.06		0.368275	0.06	
0.512233	0.10		0.449564	0.10	
0.584331	0.15		0.515114	0.15	
0.631880	0.20		0.557339	0.20	
0.664451	0.25		0.583937	0.25	
0.686187	0.30		0.597609	0.30	
0.699493	0.35		0.598613	0.35	
0.704942	0.40				

Table E.7. a/R=0.2 barium-titanate......(continued)

	b/R=0.7			b/R=0.8	
P_0/Y	$(b_p - b)/R$	$(a - a_p)/R$	P_0/Y	$(b_p - b)/R$	$(a - a_p)/R$
0	0	0	0	0	0
0.046224	0.001	0.00202252	0.040332	0.001	0.002171203
0.090795	0.004	0.00818396	0.079193	0.004	0.008764103
0.119479	0.007	0.01455754	0.104174	0.007	0.01562263
0.142128	0.01	0.02116638	0.123869	0.01	0.02273571
0.197735	0.02	0.04526919	0.172054	0.02	0.04886220
0.238798	0.03	0.07583482	0.207386	0.03	0.08309518
0.271620	0.04	0.12379710	0.235367	0.04	0.17066200
0.323173	0.06		0.278501	0.06	
0.394004	0.10		0.334029	0.10	
0.448606	0.15		0.366614	0.15	
0.478913	0.20				
0.489259	0.25				
	b/R=0.9				
P_0/Y	$(b_p - b)/R$	$(a - a_p)/R$			
0	0	0			
0.033856	0.001	0.00223729			
0.066321	0.004	0.00886142			
0.087025	0.007	0.01567157			
0.103214	0.01	0.02265285			
0.142042	0.02	0.04761000			
0.169363	0.03	0.07751138			
0.189818	0.04	0.12170500			
0.217436	0.06				

Table E.8. Numerical values obtained for barium-titanate cylinder for a/R=0.4

	b/R=0.5			b/R=0.6	
P_0/Y	$(b_p - b)/R$	$(a - a_p)/R$	P_0/Y	$(b_p - b)/R$	$(a - a_p)/R$
0	0	0	0	0	0
0.103254	0.001	0.001152188	0.073689	0.001	0.001256108
0.200705	0.004	0.004495412	0.144235	0.004	0.00496462
0.261443	0.007	0.00788343	0.189143	0.007	0.008650005
0.307991	0.01	0.01131779	0.224245	0.01	0.01239845
0.415963	0.02	0.02289343	0.308751	0.02	0.02508092
0.489137	0.03	0.03483015	0.369280	0.03	0.03827375
0.543527	0.04	0.04700688	0.416436	0.04	0.05179131
0.621424	0.06	0.07270026	0.488064	0.06	0.08038935
0.714978	0.10	0.1311412	0.582091	0.10	0.14661650
0.779767	0.15	0.2321277	0.653321	0.15	0.28918140
0.818145	0.20		0.697475	0.20	
0.842725	0.25		0.725415	0.25	
0.858720	0.30		0.741221	0.30	
0.868969	0.35		0.746338	0.35	
0.874355	0.40				
	b/R=0.7			b/R=0.8	
P_0/Y	$(b_p - b)/R$	$(a - a_p)/R$	P_0/Y	$(b_p - b)/R$	$(a - a_p)/R$
0	0	0	0	0	0
0.059815	0.001	0.001377672	0.050467	0.001	0.001462162
0.117319	0.004	0.005362958	0.099033	0.004	0.005706221
0.154163	0.007	0.009426057	0.130192	0.007	0.009980887
0.18313	0.01	0.01342091	0.154719	0.01	0.01420546
0.253677	0.02	0.02716631	0.214537	0.02	0.02865747
0.305077	0.03	0.04139671	0.258215	0.03	0.04354057
0.345692	0.04	0.05603394	0.292687	0.04	0.05872923
0.408565	0.06	0.08706954	0.345701	0.06	0.09061447
0.493072	0.10	0.15968990	0.414301	0.10	0.15887810
0.557296	0.15		0.457089	0.15	0.25765660
0.594006	0.20				
0.610001	0.25				
	b/R=0.9				
P0/Y	(bp - b)/R	(a - ap)/R			
0	0	0			
0.041996	0.001	0.00149030			
0.082276	0.004	0.00568149			
0.107975	0.007	0.00986475			
0.128081	0.01	0.01410908			
0.176379	0.02	0.02789795			
0.210529	0.03	0.04133198			
0.236256	0.04	0.05408657			
0.271635	0.06	0.07663962			

Table E.9. Numerical values obtained for barium-titanate cylinder for a/R=0.6

	b/R=0.7			b/R=0.8	
P_0/Y	$(b_p - b)/R$	$(a - a_p)/R$	P_0/Y	$(b_p - b)/R$	$(a - a_p)/R$
0	0	0	0	0	0
0.102425	0.001	0.00108278	0.072276	0.001	0.00120163
0.198178	0.004	0.00435078	0.141417	0.004	0.00462991
0.259335	0.007	0.00763637	0.185376	0.007	0.00811052
0.305492	0.01	0.01092410	0.219692	0.01	0.01154172
0.412515	0.02	0.02187914	0.302052	0.02	0.02316123
0.484978	0.03	0.03315705	0.360699	0.03	0.03490222
0.538767	0.04	0.04441237	0.406049	0.04	0.04674649
0.615578	0.06	0.06762129	0.473953	0.06	0.07044601
0.706943	0.10	0.11621390	0.558943	0.10	0.11701980
0.768244	0.15	0.18053810	0.612441	0.15	0.16488860
0.801382	0.20	0.24473540			
0.817318	0.25	0.29446180			
	b/R=0.9				
P_0/Y	$(b_p - b)/R$	$(a - a_p)/R$			
0	0	0			
0.056637	0.001	0.001230597			
0.110869	0.004	0.004707396			
0.145390	0.007	0.008286595			
0.172342	0.01	0.01165789			
0.236888	0.02	0.02302384			
0.282339	0.03	0.03409737			
0.316590	0.04	0.04448378			
0.364139	0.06	0.06276536			

Table E.10. Numerical values obtained for steel cylinder for a/R=0.2

	b/R=0.3			b/R=0.4	
P_0/Y	$(b_p - b)/R$	$(a - a_p)/R$	P_0/Y	$(b_p - b)/R$	$(a - a_p)/R$
0	0	0	0	0	0
0.1021897	0.001	0.00120920	0.0728203	0.001	0.00141287
0.2025624	0.004	0.00492999	0.1453449	0.004	0.00580964
0.2640621	0.007	0.00871891	0.1906899	0.007	0.01025948
0.3111942	0.01	0.01257189	0.2261483	0.01	0.01483254
0.4204446	0.02	0.02596822	0.3114066	0.02	0.03089364
0.4936565	0.03	0.04039468	0.3717247	0.03	0.04870096
0.5482965	0.04	0.05614823	0.4188827	0.04	0.06915337
0.6259289	0.06	0.09582806	0.4898297	0.06	0.13556990
0.7179745	0.10		0.5815390	0.10	
0.7806396	0.15		0.6495717	0.15	
0.8170307	0.20		0.6907352	0.20	
0.8395978	0.25		0.7164939	0.25	
0.8538130	0.30		0.7317581	0.30	
0.8626666	0.35		0.7396220	0.35	
0.8673613	0.40		0.7415954	0.40	
0.8673002	0.50		0.7357788	0.50	
0.8582995	0.60				
	b/R=0.5			b/R=0.6	
P_0/Y	$(b_p - b)/R$	$(a - a_p)/R$	P_0/Y	$(b_p - b)/R$	$(a - a_p)/R$
0	0	0	0	0	0
0.0600006	0.001	0.00169157	0.0500653	0.001	0.00189485
0.1176204	0.004	0.00664483	0.0982040	0.004	0.00761128
0.1544782	0.007	0.01184067	0.1290515	0.007	0.01350470
0.1834107	0.01	0.01715240	0.1533052	0.01	0.01961564
0.2536470	0.02	0.03605734	0.2123382	0.02	0.04187781
0.3045574	0.03	0.05790077	0.2553356	0.03	0.06964636
0.3445809	0.04	0.08531465	0.2892255	0.04	0.11060920
0.4061048	0.06		0.3414295	0.06	
0.4878021	0.10		0.4105583	0.10	
0.5494004	0.15		0.4615352	0.15	
0.5860771	0.20		0.4903798	0.20	
0.6075237	0.25		0.5067700	0.25	
0.6183678	0.30		0.5203226	0.30	
0.6223888	0.35		0.5696589	0.35	
0.6254157	0.40				

Table E.10. a/R=0.2 steel ......(continued)

	b/R=0.7			b/R=0.8	
P_0/Y	$(b_p - b)/R$	$(a - a_p)/R$	P_0/Y	$(b_p - b)/R$	$(a - a_p)/R$
0	0	0	0	0	0
0.0419192	0.001	0.00218977	0.0345733	0.001	0.00263062
0.0822132	0.004	0.00871727	0.0677735	0.004	0.01055947
0.1080199	0.007	0.01556636	0.0890099	0.007	0.01902029
0.1282988	0.01	0.02274813	0.1056726	0.01	0.02808987
0.1775852	0.02	0.05014159	0.1460679	0.02	0.06708696
0.2133755	0.03	0.08813476	0.1753231	0.03	
0.241484	0.04		0.1983004	0.04	
0.2844944	0.06		0.2338143	0.06	
0.3406104	0.10		0.2853079	0.10	
0.381958	0.15		0.3717864	0.15	
0.4121855	0.20				
0.4786541	0.25				
	b/R=0.9				
P_0/Y	$(b_p - b)/R$	$(a - a_p)/R$			
0	0	0			
0.0287862	0.001	0.00395787			
0.0566083	0.004	0.01684564			
0.0746149	0.007	0.03218003			
0.0889553	0.010	0.05159523			
0.1252798	0.020				
0.1547182	0.030				
0.1828039	0.040				
0.2579062	0.060				

Table E.11. Numerical values obtained for steel cylinder for a/R=0.4

	b/R=0.5			b/R=0.6	
P_0/Y	$(b_p - b)/R$	$(a - a_p)/R$	P_0/Y	$(b_p - b)/R$	$(a - a_p)/R$
0	0	0	0	0	0
0.1028492	0.001	0.001153111	0.0724045	0.001	0.00125951
0.1998690	0.004	0.004499674	0.1416384	0.004	0.00498816
0.2602906	0.007	0.007898778	0.1856294	0.007	0.00876093
0.3065577	0.01	0.01134777	0.2199512	0.01	0.01249927
0.4136837	0.02	0.02291283	0.3022506	0.02	0.02525532
0.4860513	0.03	0.03485233	0.3608011	0.03	0.03854361
0.5396449	0.04	0.04704142	0.4060810	0.04	0.05227894
0.6159365	0.06	0.0728634	0.4740746	0.06	0.08174142
0.7062070	0.10	0.132066	0.5610614	0.10	0.15188710
0.7667834	0.15	0.2424998	0.6240150	0.15	
0.8009165	0.20		0.6612497	0.20	
0.8213976	0.25		0.6856161	0.25	
0.8338344	0.30		0.7071669	0.30	
0.8420383	0.35		0.7556103	0.35	
0.8500051	0.40				
	b/R=0.7			b/R=0.8	
P_0/Y	$(b_p - b)/R$	$(a - a_p)/R$	P_0/Y	$(b_p - b)/R$	$(a - a_p)/R$
0	0	0	0	0	0
0.0572636	0.001	0.00140798	0.0466251	0.001	0.00159121
0.1122064	0.004	0.00548983	0.0914030	0.004	0.00625032
0.1473028	0.007	0.00965980	0.1200476	0.007	0.01096755
0.1748133	0.010	0.01385343	0.1425337	0.010	0.01578972
0.2413949	0.020	0.02814341	0.1971129	0.020	0.03257760
0.2894102	0.030	0.04305425	0.2367553	0.030	0.05075714
0.3269539	0.040	0.05851126	0.2680114	0.040	0.07059300
0.3842164	0.060	0.09284908	0.3166742	0.060	0.12112530
0.4592941	0.100	0.19162920	0.3877600	0.100	
0.5167560	0.150		0.4931509	0.150	
0.5602031	0.200				
0.6349040	0.250				
	b/R=0.9				
P0/Y	(bp - b)/R	(a - ap)/R			
0	0	0			
0.0391265	0.001	0.00214437			
0.0769426	0.004	0.00860563			
0.1014100	0.007	0.01541090			
0.1208780	0.010	0.02262792			
0.1699894	0.020	0.05151561			
0.2092230	0.030	0.09621584			
0.2455492	0.040				
0.3339868	0.060				

Table E.12. Numerical values obtained for steel cylinder for a/R=0.6

	b/R=0.7			b/R=0.8	
P_0/Y	$(b_p - b)/R$	$(a - a_p)/R$	P_0/Y	$(b_p - b)/R$	$(a - a_p)/R$
0	0	0	0	0	0
0.1017502	0.001	0.001108468	0.07044	0.001	0.001229465
0.1977019	0.004	0.004353702	0.1377356	0.004	0.004759669
0.2574234	0.007	0.007655025	0.1804383	0.007	0.008351207
0.3031212	0.010	0.01089501	0.2137130	0.010	0.01198834
0.4087847	0.020	0.02197665	0.2933139	0.020	0.02404445
0.4799861	0.030	0.03325462	0.3497775	0.030	0.036672
0.5325689	0.040	0.04464865	0.3933901	0.040	0.04960287
0.6070935	0.060	0.06839746	0.4591669	0.060	0.07747161
0.6945902	0.100	0.11983500	0.5483289	0.100	0.1514364
0.7537106	0.150	0.20108700	0.6526501	0.150	
0.7921729	0.200				
0.8402433	0.250				
	b/R=0.9				
P_0/Y	$(b_p - b)/R$	$(a - a_p)/R$			
0	0	0			
0.0552832	0.001	0.00145781			
0.1084692	0.004	0.00577724			
0.1426229	0.007	0.01020467			
0.1695811	0.010	0.01473719			
0.2362802	0.020	0.03114617			
0.2873946	0.030	0.05051363			
0.3320854	0.040	0.07599807			
0.4277665	0.060	0.14526000			

APPENDIX F

F. MAIN PART OF THE DEVELOPED PROGRAM (CODED IN QBasic)

```
'      IIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIII
'      IIIII   THIS PROGRAM    CALCULATES THE PLASTIC ZONES  IIIII
'      IIIII       FOR A MATL. HAS A RING-SHAPED CRACK       IIIII
'      IIIII     REVISED BY MESUT UYANER ON 07.02.1999       IIIII
'      IIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIII
'      Po/Y and ap/R are determined under the effect of uniform unit
'      load.
DEFDBL L, V

DECLARE SUB Kernel1 (ro, sr, k1)
DECLARE SUB Kernel2 (al, sr, ro, AX, BX, CX, DX, S2, S4, CM2, CM4,
C1, C2, k2)
DECLARE SUB GaussElim (b(), a(), N)
DECLARE FUNCTION I0# (x)
DECLARE FUNCTION In# (x, m, ZI#)
DECLARE FUNCTION BesselK (x, m)
DECLARE FUNCTION Eliptik1 (RATIO)
DECLARE FUNCTION Eliptik2 (RATIO)
DECLARE FUNCTION LagEntpl (x(), y(), m, xv)

      OPEN "Rplast.txt" FOR OUTPUT AS #2
      OPEN "DataKaKb.TXT" FOR INPUT AS #3
      OPEN "Rplstexc.TXT" FOR OUTPUT AS #4

CONST N = 20
CONST R = 1
CONST Pi = 3.141592654#
CONST Po = 1     'unit load [MPa]
CONST TOL = .00001
CONST TRUE = -1
CONST FALSE = NOT TRUE
CONST intv = 4
CONST intviii = 8
CONST zero = 0

      DIM COEF(N, N), rhs(N), xr(N - 1), tk(N), root(6), weight(6)
      DIM WE(N), CO(N, N), ap(2), Diff(2), Ktablo(18, 9, intviii)
      FOR k = 1 TO intviii
      FOR i = 1 TO 18
      FOR j = 1 TO 8
      INPUT #3, Ktablo(i, j, k), a
```

```
      NEXT j
      INPUT #3, Ktablo(i, j, k)
      NEXT i
      NEXT k
      CLOSE #3

      FOR iii = 1 TO intviii
      SELECT CASE iii
      CASE 1
      '.... Anisotropic Steel ....
      A11 = 4.504504E-12
      A12 = -1.351351E-12
      A13 = -1.428571E-12
      A33 = 4.761905E-12
      A44 = 1.238095E-11
      MATL$ = " Anisotropic Steel"
      CASE 2
' ........ Magnesium .......
      A11 = 2.21E-11
      A12 = -7.7E-12
      A13 = -4.9E-12
      A33 = 1.97E-11
      A44 = 6.03E-11
      MATL$ = " Magnesium "
      CASE 3
' ........ Cadmium .........
      A11 = 1.29E-11
      A12 = -1.5E-12
      A13 = -9.3E-12
      A33 = 3.69E-11
      A44 = 6.4E-11
      MATL$ = " Cadmium "
      CASE 4
' ........ Barium Titanate .........
        A11 = 8.150854E-12
        A12 = -2.960257E-12
        A13 = -1.949906E-12
        A33 = 6.756014E-12
        A44 = 1.831502E-10
      MATL$ = " Barium-Titanate"
      CASE 5
'........  Graphite -Epoxy  .........
      A11 = 1.359636E-10
      A12 = -4.54259E-11
      A13 = -2.972723E-13
      A33 = 1.152269E-11
      A44 = 2.411382E-10
      MATL$ = " Graphite - Epoxy"
      CASE 6
'........  E -Glass   ..............
      A11 = 8.396149E-11
      A12 = -3.490218E-11
      A13 = -5.442501E-12
      A33 = 2.236262E-11
      A44 = 2.107482E-10
      MATL$ = " E - Glass"
      CASE 7
' ........ Anisotropic Copper .........
      A11 = 5.232862E-12
```

```
      A12 = -1.622187E-12
      A13 = -1.722222E-12
      A33 = 5.555556E-12
      A44 = 1.455556E-11
      MATL$ = " Anisotropic Copper"
      CASE 8
' ........ Anisotropic Aluminium  .......
      A11 = 1.31406E-11
      A12 = -4.204993E-12
      A13 = -4.571428E-12
      A33 = 1.428571E-11
      A44 = 3.771428E-11
      MATL$ = " Anisotropic Aluminium"
      END SELECT

'     ------------------------------------------
'     - Gauss-Legendre integration formula. n=6 -
'     ------------------------------------------
'     -   ROOTS & WEIGHTS  -
'     ---------------------
      root(1) = -.9324695142#
      root(2) = -.6612093865000001#
      root(3) = -.2386191861#
      weight(1) = .1713244924#
      weight(2) = .360761573#
      weight(3) = .4679139346#
'     ------------------------------
'     -   REST OF ROOTS & WEIGHTS  -
'     ------------------------------
      FOR i = 1 TO 3
      root(7 - i) = -root(i)
      weight(7 - i) = weight(i)
      NEXT i

      XE(1) = .001
      XE(2) = 6
      XE(3) = 20
      XE(4) = 40
      XE(5) = 60 'since k2 is convergent, upper limit of the k2 is
                 'chosen 60 instead of infinity. It is adequate...
'     -------------------------------
'     -   CALCULATION OF CONSTATNTS :
'     -------------------------------

'  AX , BX , CX , DX

      DN1 = A11 * A33 - A13 ^ 2
      AX = A13 * (A11 - A12) / DN1
      BX = (A13 * (A13 + A44) - A12 * A33) / DN1
      CX = (A13 * (A11 - A12) + A11 * A44) / DN1
      DX = (A11 ^ 2 - A12 ^ 2) / DN1

'  S2 , S4 , C1 , C2

      AC = AX + CX
      DET = SQR(AC ^ 2 - 4 * DX)
      S2 = -SQR((AC - DET) / (2 * DX))
      S4 = -SQR((AC + DET) / (2 * DX))
```

```
       PRINT #2, "MATERIAL : "; MATL$
       PRINT #4, "MATERIAL : "; MATL$

       C1 = SQR(2 * DX / (AC + DET))
       C2 = SQR((AC + DET) / 2)

'  CM2 , CM4

       DENOM = (S2 ^ 2 - S4 ^ 2) * (A33 * DX - 2 * A13 * AX - A44*AX)
       CM2 = (1 - AX * S4 ^ 2) / DENOM
       CM4 = (1 - AX * S2 ^ 2) / DENOM
       C3 = CM4 * S4 *(S4 * S4 * DX - CX) - CM2 * S2 *(S2 * S2*DX-CX)

'      -----------------------------------------------------------------
'--
'      -  CALCULATION OF WEIGHTS, ROOTS & NORMALIZATION OF LHS ARRAY
'-
'      -----------------------------------------------------------------
'--

'   WEIGHTS:
       FOR i = 2 TO N - 1
       WE(i) = 1 / (N - 1)
       NEXT i
       WE(1) = 1 / (2 * (N - 1))
       WE(N) = WE(1)
'  NORMALIZATION
       FOR i = 1 TO N
       tk(i) = COS(Pi * (i - 1) / (N - 1))
       NEXT i
       FOR i = 1 TO N - 1
       xr(i) = COS(Pi * (2 * i - 1) / (2 * N - 2))
       NEXT i

       FOR ia = 1 TO 8
       a = .1 * ia
       FOR ib = 1 TO 7 'STEP 3
       b = a + .1 * ib
       IF b > .95 THEN EXIT FOR 'ib dongusunden cIk.
       PRINT "a/R ="; a, "b/R ="; b
       PRINT #2, "a/R ="; a, "b/R ="; b
       PRINT #4, "a/R ="; a, "b/R ="; b
       PRINT #4, "Po/Y"; CHR$(9); "(bp - b)/R"; CHR$(9); "(a - ap)/R"
       PRINT #4, zero; CHR$(9); zero; CHR$(9); zero

       ka = Ktablo(ia + 1, ib + ia, intviii)
       kb = Ktablo(ia + 10, ib + ia, intviii)
       ARTIS = ABS(ka / kb) * 1.25
       PRINT "k(a)="; ka; "k(b)="; kb, "k(a)/k(b)="; ka / kb

' .................Crack starts at r=a
       pldurum = FALSE
       alt = 1: ust = 4
       'lbp Plastik bolge genisligi %verilen ilk deger%
       FOR mm = 1 TO 5
       SELECT CASE mm
       CASE 1
       baslangIc = .001: bitis = .008: adIm = .003
       CASE 2
```

```
baslangIc = .01: bitis = .03: adIm = .01
CASE 3
baslangIc = .04: bitis = .06: adIm = .02
CASE 4
baslangIc = .1: bitis = .4: adIm = .05
CASE 5
baslangIc = .5: bitis = .9: adIm = .1
END SELECT
FOR lbp = baslangIc TO bitis STEP adIm
lap = lbp  'Plastik bolge genisligi %verilen ilk TAHMiN%
bp = b + lbp
IF bp >= .99 THEN EXIT FOR ' lbp dongusunden cIkIs.
ap = a - lap
IF ap = 0 THEN
alt = 3: ust = 4
END IF
IF pldurum = TRUE THEN ap = 0

FOR ilap = 1 TO 8

FOR m = alt TO ust

SELECT CASE m
CASE 1
  AltSINIR = ap
  UstSINIR = b
  YUK = Po
CASE 2
  AltSINIR = ap
  UstSINIR = a
  YUK = -Po  'represents the flow stress Y
CASE 3
  AltSINIR = a
  UstSINIR = bp
  YUK = Po
CASE 4
  AltSINIR = b
  UstSINIR = bp
  YUK = -Po  'represents the flow stress Y
END SELECT

PRINT "alt SINIR ="; AltSINIR, "Ust SINIR ="; UstSINIR
PRINT #2, "alt SINIR ="; AltSINIR, "Ust SINIR ="; UstSINIR

dTodro = (UstSINIR - AltSINIR) / 2


'  MATRIX FORMATION:
   FOR j = 1 TO N - 1
   sr = ((UstSINIR - AltSINIR) * xr(j) + UstSINIR + AltSINIR) / 2
   FOR i = 1 TO N
   ro = ((UstSINIR - AltSINIR) * tk(i) + UstSINIR + AltSINIR) / 2

   Kernel1 ro, sr, k1

   TOT = 0
   FOR LL = 1 TO intv
   aa = XE(LL)
   bb = XE(LL + 1)
```

```
        Differ = (bb - aa) / 2

        FOR L = 1 TO 6
        al = ((bb - aa) * root(L) + bb + aa) / 2 'al: alpha

        Kernel2 al, sr, ro, AX, BX, CX, DX, S2, S4, CM2, CM4,C1, C2,k2

        TOT = TOT + weight(L) * k2 * Differ

        NEXT L, LL

        KrRo = C3 * k1 + 2 * ro * TOT
        COEF(j, i) = WE(i) * (C3 / (ro - sr) + KrRo) * dTodro
        CO(j, i) = COEF(j, i)
        NEXT i
        NEXT j

  ' SINGLE VALUEDNESS CONDITION %last row equals to weights %

        FOR i = 1 TO N: COEF(N, i) = WE(i): NEXT i

  '..........RHS OF EQNS................
        FOR i = 1 TO N - 1
        rhs(i) = -YUK
        NEXT i
        rhs(N) = 0

        GaussElim rhs(), COEF(), N

  ' ------   STRESS INTENSITY FACTORS : RING-SHAPED     -----

        SIFconst = C3 * SQR((UstSINIR - AltSINIR) / 2)
        SELECT CASE m
        CASE 1
          k1a = SIFconst * rhs(N)
        CASE 2
          k2a = SIFconst * rhs(N)
        CASE 3
          k1b = -SIFconst * rhs(1)
        CASE 4
          k2b = -SIFconst * rhs(1)
        END SELECT
        NEXT m

  ' CALCULATING THE RATIOS OF Po/Y WITH RESPECT TO THE CONDITION *.a
 'AND *.b
  ' condition *.a: Po*k1b(ap,bp) + Y*k2b(b,bp) = 0
  ' condition *.b: Po*k1a(ap,bp) + Y*k2a(ap,a) = 0

        PoOverYb = -k2b / k1b
        PoOverYa = -k2a / k1a
        Diff = PoOverYb - PoOverYa
        PRINT "(Po/Y)1 = "; PoOverYb, "(Po/Y)2 = "; PoOverYa
        PRINT "Difference = "; Diff

        IF ABS(Diff) < TOL THEN
           alt = 1: ust = 4
           EXIT FOR 'ilap dongusunden cIk.
```

```
END IF

IF ap <= 0 THEN

  pldurum = TRUE
  alt = 3: ust = 4
  EXIT FOR 'ilap dongusunden cIkIs. ic bolge tamamen plastik!
END IF

alt = 1: ust = 2

SELECT CASE ilap
CASE 1
      ap(1) = ap
      Diff(1) = Diff
      IF Diff > 0 THEN
          ap = a - (lap * ARTIS)
          ELSE
          ap = a + (lap * ARTIS)
      END IF
          ' IF ap >= a THEN ap = a * .5
CASE 2
      ap(2) = ap
      Diff(2) = Diff
      ap = LagEntpl(Diff(), ap(), 2, 0)
CASE ELSE
      ap(1) = ap(2)
      ap(2) = ap
      Diff(1) = Diff(2)
      Diff(2) = Diff
      ap = LagEntpl(Diff(), ap(), 2, 0)
END SELECT
IF ap <= 0 OR ap > a THEN
alt = 1: ust = 4
EXIT FOR
END IF
NEXT ilap

PRINT #2, "bp="; bp, "ap="; ap
PRINT #2, "(Po/Y)1 = "; PoOverYb; "(Po/Y)2 = "; PoOverYa
IF ap >= 0 AND alt = 1 THEN
 PRINT #4, PoOverYb; CHR$(9); bp - b; CHR$(9); a - ap; CHR$(9)
 ELSE
 PRINT #4, PoOverYb; CHR$(9); bp - b
END IF

NEXT lbp
NEXT mm
NEXT ib
NEXT ia
NEXT iii
CLOSE #2
CLOSE #4

SYSTEM
```