DOKUZ EYLÜL UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

WAVE PROPAGATION IN COMPOSITE MATERIALS

by
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İZMİR

WAVE PROPAGATION IN COMPOSITE MATERIALS

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by

Demet ERSOY

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İZMİR

M.Sc. THESIS EXAMINATION RESULT FORM

We have read the thesis entitled **"WAVE PROPAGATION IN COMPOSITE MATERIALS"** completed by **Demet ERSOY** under supervision of **PROF. DR. VALERY G. YAKHNO** and we certify that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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Demet ERSOY

WAVE PROPAGATION IN COMPOSITE MATERIALS

ABSTRACT

The system of anisotropic elasticity with piecewise constant coefficients is considered in this thesis. The main object of the thesis is to model an initial value problem (IVP) and an initial boundary value problem (IBVP) for the considered system. The main results are explicit formulae for solutions of initial value problem and initial boundary value problem. Using these formulae the simulation of elastic waves have been obtained. Results of the simulations have clear physical interpretation of wave propagation in layered medium from the point source.

The method of characteristics has been used for constructing explicit formulae and MATLAB codes has been successfully applied for the simulation of the waves.

Keywords: anisotropic elastic system, elastic layered medium, initial value problem, initial boundary value problem, modeling, simulation, wave propagation.

BİLEŞİK MATERYALLERDE DALGA YAYILIMI

ÖZ

Bu tezde parçalı sabit katsayılı, anizotropik elastik sistem çalışıldı. Bu tezdeki ana hedef çalışılan sistemin başlangıç değer problemine (BDP) ve başlangıç sınır değer problemine (BSDP) modellenmesidir. Bu başlangıç değer ve başlangıç sınır değer problemlerinin temel sonucu formüllerle belirtilen çözümleridir. Bu formüller kullanılarak elastik dalgaların simulasyonları elde edilmiş ve sonuçları katmanlı elastik ortamlarda oluşan dalga yayılımının fiziksel yorumlarıyla uyum göstermiştir.

Çözümleri elde edebilmek için karakteristikler metodu kullanılmış ve dalgaların simulasyonları için MATLAB kodları başarılı bir şekilde uygulanmıştır.

Anahtar Sözcükler: Anizotropik elastik sistem, elastik katmanlı ortam, başlangıç değer problemi, başlangıç sınır değer problemi, modelleme, simulasyon, dalga yayılımı.

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CHAPTER ONE
INTRODUCTION

Anisotropic elasticity has been mostly studied in different applied sciences such as engineering sciences, geophysics, solids and structures sciences etc. for the last thirty years due to its applications to composite materials. [(Ting, 2000), (Yahkno & Akmaz, 2005)]

The propagation of elastic waves in anisotropic media is governed by a system of second order partial differential equations.[see, for example, (Dieulesaint and Royer, 1980), (Fedorov, 1968), (Ting, 1996), (Ting & Barnet & Wu, 1990)] Here, we formulate shortly the problems which are considered in this thesis.

1.1 Equations of Anisotropic Elasticity

Let $x = (x_1, x_2, x_3) \in \mathbf{R}^2 \times [0, \infty)$ and $t \in \mathbf{R}$ be variables. The displacement of the point x is the vector $u(x,t) = (u_1, u_2, u_3)$ with components

$$u(x,t) = u_j(x,t), \quad \text{for each } j = 1,2,3.$$

Initial value problem (IVP) of anisotropic elastic layered medium is described by the following differential equations,

$$\rho(x_3)\frac{\partial^2 u_j}{\partial t^2} = \sum_{k=1}^{3}\sum_{\ell=1}^{3}\sum_{m=1}^{3}\frac{\partial}{\partial x_k}\left(c_{jk\ell m}(x_3)\frac{\partial u_j}{\partial x_m}\right),$$
$$0 < x_3 < \ell, \qquad \ell < x_3 < \infty, \qquad t \in R, \quad j = 1,2,3, \tag{1.1.1}$$

with initial data

$$u_j(x,0) = \varphi_j(x), \qquad \frac{\partial u_j}{\partial t}(x,t)\Big|_{t=0} = \psi_j(x),$$
$$0 < x_3 < \ell, \qquad \ell < x_3 < \infty, \qquad j = 1,2,3, \tag{1.1.2}$$

and matching conditions

$$u_j(x_3,t)\Big|_{x_3=\ell-0} = u_j(x_3,t)\Big|_{x_3=\ell+0}, \tag{1.1.3}$$

$$\sum_{\ell=1}^{3}\sum_{m=1}^{3} c_{j3\ell m}(x_3)\frac{\partial u_j}{\partial x_m}\Big|_{x_3=\ell-0} = \sum_{\ell=1}^{3}\sum_{m=1}^{3} c_{j3\ell m}(x_3)\frac{\partial u_j}{\partial x_m}\Big|_{x_3=\ell+0}, \tag{1.1.4}$$

where ℓ is given number, $\left\{c_{jk\ell m}(x_3)\right\}_{jk\ell m=1}^{3}$ are the elastic moduli of the medium; $\rho(x_3) > 0$

is the density of the elastic medium; φ_j, ψ_j and F_j are smooth functions for each $j = 1,2,3$.

For initial boundary value problem (IBVP) of anisotropic elastic layered medium, we add the following boundary condition to the system $(1.1.1) - (1.1.4)$, the boundary condition

$$\sum_{\ell=1}^{3}\sum_{m=1}^{3} c_{j3\ell m}\frac{\partial u_\ell}{\partial x_m}\Big|_{x_3=0} = F_j(t), \quad t \in \mathbf{R}. \tag{1.1.5}$$

The elastic moduli of the medium is positive definite and satisfy the symmetry property

$$c_{jk\ell m}(x_3) = c_{\ell m jk}(x_3) = c_{kj\ell m}(x_3)$$

so that the system of anisotropic elasticity can be written as Cauchy problem of second order partial differential equations (Yahkno & Akmaz, 2005). The assumptions and detailed explanations can be found in the Chapter 2.

1.2 Problems and Methods for Equations of Anisotropic Elasticity

In the recent years, there exists substantially modern methods for solving initial and boundary value problems [(Boyce & DiPrima, 1992), (Dieulesaint & Royer, 1980), (Courant & Hilbert, 1989), (Cohen & Heikkola & Joly & Neittaan, 2003)] so that many researchers get a great chance to study more about the phenomena of the elastic wave propagation. And the developments of computer facilities-applications of analytical methods [(Rand & Rovenski, 2005), (Pavlovic, 2003)], special softwares such as Mathematica, Maple, Matlab etc.-provide better understanding of invisible elastic waves.

In this section, we mention some approaches for constructing solutions of IVPs and IBVPS.

1.2.1 Plane Wave Formalism-Stroh Formalism

Stroh formalism (Stroh, 1958) is a well-known approach for the system of elasticity in material sciences, applied mathematics and Physics community (Ting, 2000). In the method of plane wave approach, the system of elasticity is considered in a unbounded domain and the solution of the systems have the form

$$\mathbf{u}(x,t) = \mathbf{a}f(x.\mathbf{n} - ct). \tag{1.2.1}$$

where $\mathbf{n}$, $\mathbf{a}$, c are values to be determined. Substitution of (1.2.1) into the system, gives us

$$(\Lambda - \lambda I)\mathbf{a} = 0, \tag{1.2.2}$$

where $\lambda = c^2$ and Λ is second-order tensor with components

$$\Lambda_{jl} = \sum_{k,m=1}^{3} c_{jklm} n_k n_m$$

for all n_k and n_m. The construction of a solution is reduced to eigenvalues and eigenfunctions problem for Λ.

1.2.2 Green's Functions Method

A different method to obtain the solution of the system is Green's functions method. The main idea of applying this method is Fourier transforms. The system is firstly solved in the Fourier-transformed domain. Then the solution of the system is derived by using Fourier-inverse transform (Yang, 2004). In the article of Yang (2004), after applying 2-D Fourier transform with the variables (k_1, k_2), the solution in Fourier-transformed domain is the following

$$\tilde{u}_i(k_1, k_2, y_3) = \int\int u(y_1, y_2, y_3) e^{ik_\alpha y_\alpha} dy_1 dy_2,$$

where e stands for *exponential* function, i is the imaginary number for both variables y_1, y_2. Fourier-inverse transform yield the solution of the system in the domain.

1.2.3 Finite Element Method

Besides the analytical approaches, the numerical methods can be applied to solve the systems. Finite element and finite difference methods are mostly used for some problems described by partial differential equations including system of elasticity. This approach is based on converting partial differential equations into an approximating system of ordinary differential equations.

1.2.4 Polynomial Solution Method

Polynomial Solution method (PS-method) is an analytical method for constructing solution of partial differential problems with the special form of initial data and inhomogeneous term [(Yakhno & Akmaz, 2005), (Yakhno & Akmaz, 2007)]. In the article of Yakhno & Akmaz (2005), it is proved that if the initial data are polynomials with respect to the lateral variables (x_1, x_2), then the solution of the problem which has coefficient functions depending on the other variable x_3, is in the form of polynomials depending of the same variables. The system in the article (Yakhno & Akmaz, 2005) can be written as follows

$$\rho \frac{\partial^2 u_j^\gamma}{\partial t^2} = \sum_{k=1}^{3} \frac{\partial \sigma_{jk}^\gamma}{\partial x_k}, \quad j = 1,2,3, \; x \in \mathbf{R}^3, \; t > 0$$

$$u_j^\gamma(x,0) = \varphi^\gamma(x), \quad j = 1,2,3, \; x \in \mathbf{R}^3$$

$$\frac{\partial u_j^\gamma}{\partial t}(x,t)\Big|_{t=0} = \psi^\gamma(x), \quad j = 1,2,3, \; x \in \mathbf{R}^3$$

where

$$u_j^\gamma = D^\gamma u_j, \; \varphi_j^\gamma = D^\gamma \varphi_j, \; \psi_j^\gamma = D^\gamma \psi_j, \; \sigma_{jk}^\gamma = \sum_{\ell,m=1} C_{jk\ell m} \varepsilon_{\ell m}^\gamma, \; \varepsilon_{\ell m}^\gamma = \frac{1}{2}\left(\frac{\partial u_\ell^\gamma}{\partial x_m} + \frac{\partial u_m^\gamma}{\partial x_\ell}\right).$$

By applying Polynomial Solution method (PS-method), the solution can be written in the form

$$u_j(x_1,x_2,x_3,t) = \sum_{k=0}^{\infty}\sum_{s=0}^{\infty} U_j^{s,k}(x_3,t) x_1^s x_2^k$$

where

$$U_j^{s,k}(x_3,t) = \frac{1}{s!k!} \frac{\partial^{s+k}}{\partial x_1^s x_2^k} u_j(x_1,x_2,x_3,t)\Big|_{x_1=x_2=0}, \quad j = 1,2,3; \quad s,k = 0,1,2.$$

1.3 Plan of the Thesis

The system of anisotropic elasticity with piecewise constant coefficients is a mathematical model of elastic wave propagation in layered media (composite elastic materials). The main goal of the thesis is *to construct explicit formulae for the solutions of the considered problems and using these formulae to obtain the simulation of the elastic waves*. The thesis is organized as follows.

In Chapter 1, we describe initial value problem (IVP) and initial boundary value problem (IBVP) of anisotropic elastic layered medium. We mention about other studies and approaches for solving the system of anisotropic elasticity and the way of finding solutions. In addition, the main goal of this thesis is given.

In Chapter 2, we reformulate initial boundary value problem of anisotropic elasticity in two layered half space. The following section deals with the reduction of the system to the Cauchy problem of the wave equation. For solving this problem, we separate the half space into different subregions. By using *the method of characteristics*, the solution of IBVP is investigated in these subregions. The explicit formula of a solution is constructed. The simulations of wave propagation are obtained and analyzed.

Chapter 3 starts with the formulations of initial value problem (IVP) of the wave equation with piecewise constant coefficients. IBVP in Chapter 2 is reformulated as IVP in three layered medium. Similarly, we separate the space into different subregions and the solution of the problem is investigated independently. By using the explicit formula of the solution, the simulations of wave propagation are obtained and analyzed.

Chapter 4 starts with initial value problem (IVP) that is formulated in Chapter 3 with two layered space. The techniques of finding solution is described in detail. Analysis of the formulations and the results of the simulations are dealed extensively. In addition, the Matlab codes of IVP in two layered medium are given.

Chapter 5 is related with the conclusion of the thesis.

CHAPTER TWO

INITIAL BOUNDARY VALUE PROBLEM OF ANISOTROPIC LAYERED ELASTIC HALF SPACE

Let $x = (x_1, x_2, x_3) \in \mathbf{R}^3$, $\quad t \in \mathbf{R}$ $\quad$ and let

- $\varphi = (\varphi_1, \varphi_2, \varphi_3)$ $\quad$ and $\quad$ $\psi = (\psi_1, \psi_2, \psi_3)$ $\quad$ be given vector functions depending on x;
- $F = (F_1, F_2, F_3)$ be given vector function depending on t;
- $u = (u_1, u_2, u_3)$ be unknown vector function depending on x and t.

2.1 Statement of the Problem

Initial boundary value problem of anisotropic elastic half space is to find unknown function $u = (u_1, u_2, u_3)$ satisfying the following system of differential equations

$$\rho(x_3)\frac{\partial^2 u_j}{\partial t^2} = \sum_{k=1}^{3}\sum_{\ell=1}^{3}\sum_{m=1}^{3}\frac{\partial}{\partial x_k}\Big(c_{jk\ell m}(x_3)\frac{\partial u_j}{\partial x_m}\Big), \;\; 0 < x_3 < \ell, \;\; \ell < x_3 < \infty, \;\; t \in R \tag{2.1.1}$$

with initial data

$$u_j(x,0) = \varphi_j(x)\,, \qquad \frac{\partial u_j}{\partial t}(x,t)\Big|_{t=0} = \psi_j(x), \;\; 0 < x_3 < \ell, \;\; \ell < x_3 < \infty, \tag{2.1.2}$$

the boundary condition

$$\sum_{\ell=1}^{3}\sum_{m=1}^{3} c_{j3\ell m}\frac{\partial u_\ell}{\partial x_m}\Big|_{x_3=0} = F_j(t), \quad t \in \mathbf{R} \tag{2.1.3}$$

and matching conditions

$$u_j(x_3,t)\Big|_{x_3=\ell-0} = u_j(x_3,t)\Big|_{x_3=\ell+0} \tag{2.1.4}$$

$$\sum_{\ell=1}^{3}\sum_{m=1}^{3} c_{j3\ell m}(x_3)\frac{\partial u_j}{\partial x_m}\Big|_{x_3=\ell-0} = \sum_{\ell=1}^{3}\sum_{m=1}^{3} c_{j3\ell m}(x_3)\frac{\partial u_j}{\partial x_m}\Big|_{x_3=\ell+0} \tag{2.1.5}$$

where ℓ is given number, $\;(x_1, x_2) \in \mathbf{R}^2\;$ and for each $j = 1, 2, 3\;\; u_j(x,t)\;$ is jth component of the displacement vector $\;u(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t))\,;\;\; \rho(x_3)$ is the density of the elastic medium and $\left\{c_{jk\ell m}(x_3)\right\}_{jk\ell m=1}^{3}$ are the elastic moduli of the medium.

2.2 Assumptions

The elastic moduli $c_{jk\ell m}(x_3)$ satisfy the symmetry properties

$$c_{jk\ell m}(x_3) = c_{\ell m jk}(x_3) = c_{kj\ell m}(x_3)$$

and also $c_{jk\ell m}(x_3)$ is positive definite for each $j,k,\ell,m = 1,2,3$ i.e. there exists a positive constant M such that

$$\sum_{j,k,\ell,m=1}^{3} c_{jk\ell m}(x_3)\varepsilon_{jk}\varepsilon_{\ell m} \geqslant M \cdot \sum_{j,k,\ell,m=1}^{3} \varepsilon_{jk}^2$$

for all ε_{jk} such that $\varepsilon_{jk} = \varepsilon_{kj}$.

There exists a real, symmetric, positive definite 6x6 matrix $C = (c_{\gamma\sigma}(x_3))_{6x6}$ which includes $c_{jk\ell m}(x_3)$ as its entries by relating the pair (j,k) of indices $j,k = 1,2,3$ to a single index $\gamma = 1,2,\ldots,6$ and the pair (ℓ,m) of indices $\ell,m = 1,2,3$ to a single index $\sigma = 1,2,\ldots,6$.

$$\begin{aligned} &(1,1) \leftrightarrow 1\ , \qquad &(2,3),(3,2) \leftrightarrow 4\ , \\ &(2,2) \leftrightarrow 2\ , \qquad &(1,3),(3,1) \leftrightarrow 5\ , \\ &(3,3) \leftrightarrow 3\ , \qquad &(1,2),(2,1) \leftrightarrow 6\ . \end{aligned} \tag{2.2.1}$$

due to the symmetry properties. Then the matrix C is the following,

$$C(x_3) = \begin{pmatrix} c_{11}(x_3) & c_{12}(x_3) & c_{13}(x_3) & c_{14}(x_3) & c_{15}(x_3) & c_{16}(x_3) \\ c_{21}(x_3) & c_{22}(x_3) & c_{23}(x_3) & c_{24}(x_3) & c_{25}(x_3) & c_{26}(x_3) \\ c_{31}(x_3) & c_{32}(x_3) & c_{33}(x_3) & c_{34}(x_3) & c_{35}(x_3) & c_{36}(x_3) \\ c_{41}(x_3) & c_{42}(x_3) & c_{43}(x_3) & c_{44}(x_3) & c_{45}(x_3) & c_{46}(x_3) \\ c_{51}(x_3) & c_{52}(x_3) & c_{53}(x_3) & c_{54}(x_3) & c_{55}(x_3) & c_{56}(x_3) \\ c_{61}(x_3) & c_{62}(x_3) & c_{63}(x_3) & c_{64}(x_3) & c_{65}(x_3) & c_{66}(x_3) \end{pmatrix} = (c_{\gamma\sigma}(x_3))_{6\times 6}$$

In this work, we assume that

$$c_{\gamma\sigma}(x_3) = \begin{cases} c^1_{\gamma\sigma}, & 0 < x_3 < \ell; \\ c^2_{\gamma\sigma}, & \ell < x_3 < \infty. \end{cases} \qquad \rho(x_3) = \begin{cases} \rho^1, & 0 < x_3 < \ell; \\ \rho^2, & \ell < x_3 < \infty. \end{cases} \tag{2.2.2}$$

where $c^1_{\gamma\sigma},\ c^2_{\gamma\sigma},\ \rho^1 > 0$ and $\rho^2 > 0$ are given constants.

2.3 Reduction to IBVP for Wave Equations in Two Layered Half Space

Under these assumptions, the equations $(2.1.1) - (2.1.3)$ can be written as follows

$$\rho \frac{\partial^2 u}{\partial t^2} = A_{33} \frac{\partial^2 u}{\partial {x_3}^2} + \sum_{i=j\neq 3\,;\, i,j=1}^{3} A_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j}, \quad 0 < x_3 < \ell, \quad \ell < x_3 < \infty \tag{2.3.1}$$

$$u(x,0) = \varphi(x)\ , \qquad \frac{\partial u}{\partial t}(x,t)\Big|_{t=0} = \psi(x)\ , \quad 0 < x_3 < \ell, \quad \ell < x_3 < \infty \tag{2.3.2}$$

$$A_{33} \frac{\partial u}{\partial x_3}\Big|_{x_3=0} + \sum_{i=1}^{2} A_i \frac{\partial u}{\partial x_i}\Big|_{x_3=0} = F(t), \quad t \in \mathbf{R} \tag{2.3.3}$$

where $\mathbf{u}$ is the vector $u(x,t) = \Big(u_1(x,t), u_2(x,t), u_3(x,t)\Big)$ under the assumption that $\mathbf{u}$ does not depend on the variables x_1 and x_2 i.e. $u(x,t) = u(x_3,t)$. And where the matrices are as follow,

$$A_{11}(x_3) = \begin{pmatrix} c_{11}(x_3) & c_{16}(x_3) & c_{15}(x_3) \\ c_{16}(x_3) & c_{66}(x_3) & c_{56}(x_3) \\ c_{15}(x_3) & c_{56}(x_3) & c_{55}(x_3) \end{pmatrix},$$

$$A_{12}(x_3) = \frac{1}{2} \begin{pmatrix} 2c_{16}(x_3) & c_{12}(x_3) + c_{66}(x_3) & c_{14}(x_3) + c_{56}(x_3) \\ c_{66}(x_3) + c_{12}(x_3) & 2c_{26}(x_3) & c_{46}(x_3) + c_{25}(x_3) \\ c_{56}(x_3) + c_{14}(x_3) & c_{25}(x_3) + c_{46}(x_3) & 2c_{45}(x_3) \end{pmatrix},$$

$$A_{22}(x_3) = \begin{pmatrix} c_{66}(x_3) & c_{26}(x_3) & c_{46}(x_3) \\ c_{26}(x_3) & c_{22}(x_3) & c_{24}(x_3) \\ c_{46}(x_3) & c_{24}(x_3) & c_{44}(x_3) \end{pmatrix},$$

$$A_{13}(x_3) = \begin{pmatrix} 2c_{15}(x_3) & c_{14}(x_3)+c_{56}(x_3) & c_{13}(x_3)+c_{55}(x_3) \\ c_{56}(x_3)+c_{14}(x_3) & 2c_{46}(x_3) & c_{36}(x_3)+c_{45}(x_3) \\ c_{55}(x_3)+c_{13}(x_3) & c_{45}(x_3)+c_{36}(x_3) & 2c_{35}(x_3) \end{pmatrix},$$

$$A_{33}(x_3) = \begin{pmatrix} c_{55}(x_3) & c_{45}(x_3) & c_{35}(x_3) \\ c_{45}(x_3) & c_{44}(x_3) & c_{34}(x_3) \\ c_{35}(x_3) & c_{34}(x_3) & c_{33}(x_3) \end{pmatrix},$$

$$A_{23}(x_3) = \frac{1}{2}\begin{pmatrix} 2c_{56}(x_3) & c_{46}(x_3)+c_{25}(x_3) & c_{36}(x_3)+c_{45}(x_3) \\ c_{25}(x_3)+c_{46}(x_3) & 2c_{24}(x_3) & c_{23}(x_3)+c_{44}(x_3) \\ c_{45}(x_3)+c_{36}(x_3) & c_{44}(x_3)+c_{23}(x_3) & 2c_{34}(x_3) \end{pmatrix},$$

$$A_1(x_3) = \begin{pmatrix} c_{15}(x_3) & c_{56}(x_3) & c_{55}(x_3) \\ c_{14}(x_3) & c_{46}(x_3) & c_{45}(x_3) \\ c_{13}(x_3) & c_{36}(x_3) & c_{35}(x_3) \end{pmatrix}, \; A_2(x_3) = \begin{pmatrix} c_{56}(x_3) & c_{25}(x_3) & c_{45}(x_3) \\ c_{46}(x_3) & c_{24}(x_3) & c_{44}(x_3) \\ c_{36}(x_3) & c_{23}(x_3) & c_{34}(x_3) \end{pmatrix}.$$

We assume that

$$c_{45}(x_3) = 0, \quad c_{35}(x_3) = 0, \quad c_{34}(x_3) = 0,$$
$$c_{54}(x_3) = 0, \quad c_{53}(x_3) = 0, \quad c_{43}(x_3) = 0.$$

Under these assumptions, A_{33} has diagonal form,

$$A_{33}(x_3) = \begin{pmatrix} c_{55}(x_3) & 0 & 0 \\ 0 & c_{44}(x_3) & 0 \\ 0 & 0 & c_{33}(x_3) \end{pmatrix} \tag{2.3.4}$$

Then the equations (2.3.1) – (2.3.3) can be written as

$$\frac{\partial^2 u}{\partial t^2} = \Lambda(x_3)\frac{\partial^2 u}{\partial {x_3}^2}, \qquad 0 < x_3 < \ell, \quad \ell < x_3 < \infty, \, t \in \mathbf{R},$$

$$u(x,0) = \varphi(x), \qquad \frac{\partial u}{\partial t}(x,t)\Big|_{t=0} = \psi(x), \quad 0 < x_3 < \ell, \quad \ell < x_3 < \infty$$

$$\Lambda(x_3)\frac{\partial u}{\partial x_3}\Big|_{x_3=0} = F(t), \quad t \in \mathbf{R}$$

where $\Lambda(x_3) = \dfrac{1}{\rho(x_3)} A_{33}(x_3),\ \ \rho(x_3) > 0.$

Consider the matching conditions $(2.1.4) - (2.1.5)$, The equation $(2.1.4)$ is obvious. Under the above assumptions and notations in $(2.2.1)$, the equation $(2.1.5)$ has the form,

$$A_{33}(x_3)\frac{\partial u}{\partial x_3}\Big|_{x_3=\ell-0} + \sum_{i=1}^{2} A_i(x_3)\frac{\partial u}{\partial x_i}\Big|_{x_3=\ell-0} = A_{33}(x_3)\frac{\partial u}{\partial x_3}\Big|_{x_3=\ell+0} + \sum_{i=1}^{2} A_i(x_3)\frac{\partial u}{\partial x_i}\Big|_{x_3=\ell+0}$$

Since there is no dependence on x_1 and x_2 . So the equation $(2.2.1)$ has the following form,

$$\Lambda(x_3)\frac{\partial u}{\partial x_3}\Big|_{x_3=\ell-0} = \Lambda(x_3)\frac{\partial u}{\partial x_3}\Big|_{x_3=\ell+0}$$

where $\Lambda(x_3) = \dfrac{1}{\rho(x_3)} A_{33}(x_3)$, $\rho(x_3) > 0$ where $A_{33}(x_3)$ is defined in (2.3.4).

Notice that the matrix Λ is diagonal. Since the matrix C is positive definite and $\rho(x_3) > 0$, then

$$\Lambda = \begin{pmatrix} d_{11}^2(x_3) & 0 & 0 \\ 0 & d_{22}^2(x_3) & 0 \\ 0 & 0 & d_{33}^2(x_3) \end{pmatrix} \tag{2.3.5}$$

where $d_{11}^2 = \dfrac{c_{55}(x_3)}{\rho(x_3)},\ d_{22}^2 = \dfrac{c_{44}(x_3)}{\rho(x_3)},\ d_{33}^2 = \dfrac{c_{33}(x_3)}{\rho(x_3)}$.

The initial and boundary value problem of anisotropic elastic half space is for each $k = 1,2,3$,

$$\frac{\partial^2 U_k}{\partial t^2} = d_{kk}^2(x_3)\frac{\partial^2 U_k}{\partial {x_3}^2}\ , \quad 0 < x_3 < \ell\ ,\ \ \ell < x_3 < \infty, \quad t \in \mathbf{R}, \tag{2.3.6}$$

with initial and boundary conditions,

$$U_k(x,0) = \Phi_k(x)\ , \qquad \frac{\partial U_k}{\partial t}(x,t)\Big|_{t=0} = \Psi_k(x) \quad 0 < x_3 < \ell\ ,\ \ \ell < x_3 < \infty \tag{2.3.7}$$

$$d_{kk}^2(x_3)\frac{\partial U_k}{\partial x_3}\Big|_{x_3=0} = F_k(t)\ , \quad t \in \mathbf{R} \tag{2.3.8}$$

and the matching conditions,

$$U_k(x_3,t)\Big|_{x_3=\ell-0} = U_k(x_3,t)\Big|_{x_3=\ell+0} \tag{2.3.9}$$

$$d_{kk}^2(x_3)\frac{\partial U_k}{\partial x_3}(x_3,t)\Big|_{x_3=\ell-0} = d_{kk}^2(x_3)\frac{\partial U_k}{\partial x_3}(x_3,t)\Big|_{x_3=\ell+0} \tag{2.3.10}$$

2.4 IBVP Of Isotropic Elastic Half Space

Let $\Phi_k(x_3),\ \Psi_k(x_3)$ and $d_{kk}(x_3)$ for $k = 1,2,3$ are in the form,

$$d_{kk}(x_3) = \begin{cases} \alpha_k, & 0 < x_3 < \ell; \\ \beta_k, & \ell < x_3 < \infty. \end{cases} \tag{2.4.1}$$

$$\Phi_k(x_3) = \begin{cases} \varphi_k(x_3), & 0 < x_3 < \ell; \\ w_k(x_3), & \ell < x_3 < \infty. \end{cases} \quad \Psi_k(x_3) = \begin{cases} \psi_k(x_3), & 0 < x_3 < \ell; \\ \phi_k(x_3), & \ell < x_3 < \infty. \end{cases} \tag{2.4.2}$$

In our further consideration, we consider the scalar equation with fixed k together with initial data and boundary condition. We will omit the index k for simplicity writing.

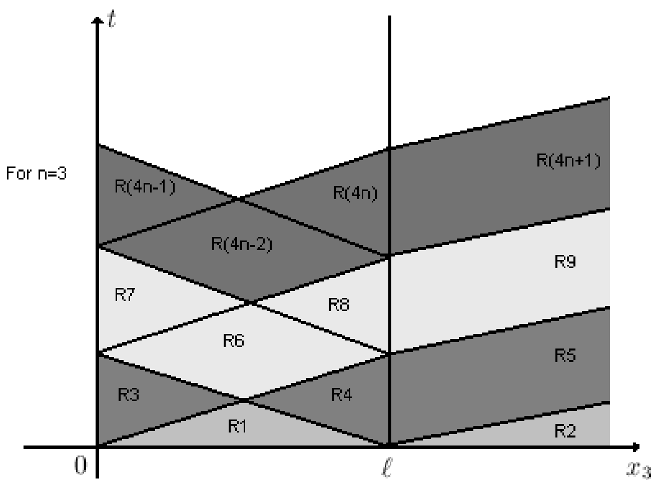


Figure 2.1 The Regions for n=2,3,4,...

Initial boundary value problem (2.3.6) – (2.3.10) may be written in the form of

$$U_k(x_3,t) = \begin{cases} u_k(x_3,t), & 0 < x_3 < \ell; \\ v_k(x_3,t), & \ell < x_3 < \infty. \end{cases}$$

as follows,

$$\frac{\partial^2 u_k}{\partial t^2} = \alpha_k^2 \frac{\partial^2 u_k}{\partial {x_3}^2}, \qquad 0 < x_3 < \ell, \quad t \in \mathbf{R}, \tag{2.4.3}$$

$$\frac{\partial^2 v_k}{\partial t^2} = \beta_k^2 \frac{\partial^2 v_k}{\partial {x_3}^2}, \qquad \ell < x_3 < \infty, \quad t \in \mathbf{R}, \tag{2.4.4}$$

with initial and boundary data,

$$u_k(x_3,0)=\varphi_k(x_3), \qquad \frac{\partial u_k}{\partial t}(x_3,t)\Big|_{t=0}=\psi_k(x_3), \quad 0<x_3<\ell, \tag{2.4.5}$$

$$v_k(x_3,0)=w_k(x_3), \qquad \frac{\partial v_k}{\partial t}(x_3,t)\Big|_{t=0}=\phi_k(x_3), \quad \ell<x_3<\infty, \tag{2.4.6}$$

$$\alpha_k^2\frac{\partial u_k}{\partial x_3}\Big|_{x_3=0}=F_k(t)\ , \quad \text{for } k=1,2,3. \tag{2.4.7}$$

and the matching conditions,

$$u_k\Big|_{x_3=\ell-0}=v_k\Big|_{x_3=\ell+0} \tag{2.4.8}$$

$$\alpha_k^2\frac{\partial u_k}{\partial x_3}\Big|_{x_3=\ell-0}=\beta_k^2\frac{\partial v_k}{\partial x_3}\Big|_{x_3=\ell+0} \tag{2.4.9}$$

2.5 Construction of the Solution

To find the solution, we separate half space into subregions and the formulation of the solution of the problem (2.4.3) – (2.4.9) is constructed for each subregions, independently by using *the method of characteristics.*

$$u_k(x_3,t)=\Big\{\ u_{km}(x_3,t), \quad \text{if } (x_3,t)\in R_m \tag{2.5.1}$$

Here, k denotes the the component of the matrix $u(x_3,t)$ and m denotes the index of subregion.

2.6 Zero Step

Zero step includes the regions R1 and R2 (see, Figure 2.1) Let us consider the problem (2.4.3) – (2.4.9) for zero step. Notice that in this step there is no
boundary, so we use only initial conditions.

Theorem 2.6.1. *Let* $\varphi_k(x_3)$, $\psi_k(x_3)$, w_k *and* $\phi_k(x_3)$ *be given continuous functions depending on* x_3*;* $u_k(x_3,t)$ *is unknown function in the form* (2.5.1)*. Then the solution of the problem* (2.4.3) – (2.4.9) *for zero step is the following,*

$$U_k(x_3,t) = \begin{cases} \frac{1}{2}\left[\varphi_k(x_3+\alpha_k t)+\varphi_k(x_3-\alpha_k t)\right] \\ +\frac{1}{2\alpha_k}\int_{x_3-\alpha_k t}^{x_3+\alpha_k t}\psi_k(\gamma)d\gamma, & \textit{if } (x_3,t)\in R1; \\ \\ \frac{1}{2}\left[w_k(x_3+\beta_k t)-w(x_3-\beta_k t)\right] \\ +\frac{1}{2\beta_k}\int_{x_3-\beta_k t}^{x_3+\beta_k t}\phi_k(\nu)d\nu\ , & \textit{if } (x_3,t)\in R2. \end{cases} \tag{2.6.1}$$

where

$$R1 = \left\{(x_3,t)\middle|\ 0 < x_3 < \ell\ , \quad t < \frac{x_3}{\alpha_k}\ \wedge\ t < \frac{\ell - x_3}{\alpha_k}\right\}$$

$$R2 = \left\{(x_3,t)\middle|\ \ell < x_3 < \infty\ , \quad t < \frac{x_3 - \ell}{\beta_k}\right\}$$

for each $k = 1,2,3$.

Proof. Let us consider the problem (2.4.3) – (2.4.4) with initial conditions (2.4.5) – (2.4.6) in the regions R1 and R2, respectively.

2.6.1 The Region R1

Let us consider the problem (2.4.3) – (2.4.9) in the region R1,

$$R1 = \left\{(x_3,t)\middle|\ 0 < x_3 < \ell\ , \quad t < \frac{x_3}{\alpha_k}\ \wedge\ t < \frac{\ell - x_3}{\alpha_k}\right\}$$

for $k = 1,2,3$.

The equation (2.4.3) can be written

$$\frac{\partial q_k}{\partial t} - \alpha_k\frac{\partial q_k}{\partial x_3} = 0, \quad (x_3,t)\in R1, \tag{2.6.2}$$

$$\frac{\partial u_k}{\partial t} + \alpha_k\frac{\partial u_k}{\partial x_3} = q_k(x_3,t), \quad (x_3,t)\in R1. \tag{2.6.3}$$

For the solution of the problem, we use the method of characteristics. So, the characteristics of the equations (2.6.2) – (2.6.3) are respectively,

$$\frac{d\xi}{d\tau} = -\alpha_k\,, \quad \xi(t) = x_3 \quad ; \quad \xi = -\alpha_k\tau + x_3 + \alpha_k t,$$

$$\frac{d\xi}{d\tau} = \alpha_k\,, \quad \xi(t) = x_3 \quad ; \quad \xi = \alpha_k\tau + x_3 - \alpha_k t.$$

By integrating along the characteristics, we get the following

$$q_k(x_3,t) = \psi_k(x_3+\alpha_k t) + \alpha_k\varphi_k'(x_3+\alpha_k t)$$

and

$$\int_0^t \frac{\partial}{\partial\tau}\Big[u_k(x_3-\alpha_k(t-\tau),\tau)\Big]d\tau = \int_0^t \psi_k(x_3-\alpha_k t+2\alpha_k\tau)d\tau$$
$$+\alpha_k\int_0^t \varphi_k'(x_3-\alpha_k t+2\alpha_k\tau)d\tau$$

Let

$$x_3-\alpha_k t+2\alpha_k\tau = \gamma\,, \qquad 2\alpha_k d\tau = d\gamma$$
$$\gamma_{low} = x_3-\alpha_k t\,, \qquad \gamma_{up} = x_3+\alpha_k t$$

So, we get

$$u_k(x_3,t) - u_k(x_3-\alpha_k t,0) = \frac{1}{2}\left[\varphi_k(x_3+\alpha_k t) - \varphi_k(x_3-\alpha_k t)\right]$$
$$+\frac{1}{2\alpha_k}\int_{x_3-\alpha_k t}^{x_3+\alpha_k t}\psi_k(\gamma)d\gamma$$

By substituting the initial conditions (2.4.5), we have the solution

$$u_k(x_3,t) = \frac{1}{2}\left[\varphi_k(x_3+\alpha_k t)+\varphi_k(x_3-\alpha_k t)\right] + \frac{1}{2\alpha_k}\int_{x_3-\alpha_k t}^{x_3+\alpha_k t}\psi_k(\gamma)d\gamma,\ \ (x_3,t)\in R1.$$

2.6.2 The Region R2

Let us consider the problem (2.4.3) – (2.4.9) in the region R2, for each $k = 1,2,3$. The equation (2.4.4) can be written

$$\frac{\partial q_k}{\partial t} - \beta_k\frac{\partial q_k}{\partial x_3} = 0\,, \quad (x_3,t)\in R2, \tag{2.6.4}$$

$$\frac{\partial v_k}{\partial t} + \beta_k\frac{\partial v_k}{\partial x} = q_k(x_3,t)\,, \quad (x_3,t)\in R2. \tag{2.6.5}$$

The characteristic of the equation (2.6.4) – (2.6.5) are respectively,

$$\frac{d\xi}{d\tau} = -\beta_k\ , \quad \xi(t) = x_3 \quad ; \quad \xi = -\beta_k\tau + x_3 + \beta_k t,$$

$$\frac{d\xi}{d\tau} = \beta_k\ , \quad \xi(x_3) = t \quad ; \quad \xi = \beta_k\tau + x_3 - \beta_k t.$$

Then by the same argument, we integrate along the characteristics so we get,

$$v_k(x_3,t) = \frac{1}{2}\left[w_k(x_3+\beta_k t) - w(x_3-\beta_k t)\right] + \frac{1}{2\beta_k}\int_{x_3-\beta_k t}^{x_3+\beta_k t} \phi_k(\nu)d\nu, \ \ (x_3,t) \in R2.$$

□

2.7 The First Step

The first step includes the regions R3, R4 and R5 (see, Figure 2.1). In this step, we consider initial boundary data and also matching conditions defined on the boundary $x = \ell$.

Before finding the solution for the first step, we must define the following functions,

$$u_k(0,t) = g_k(t), \quad u_k(\ell,t) = f_k(t) \quad \text{and} \quad \frac{\partial u_k}{\partial x_3}\Big|_{x_3=\ell} = G_k(t)\ . \tag{2.7.1}$$

We must construct these functions by initial and boundary data and also by the matching conditions.

Theorem 2.7.1. *Let* $\varphi_k(x_3)$, $\psi_k(x_3)$, w_k *and* $\phi_k(x_3)$ *be given continuous functions depending on* x_3*;* $F_k(t)$ *be given continuous function depending on t;* $u_k(x_3,t)$ *is unknown function in the form* (2.5.1)*. Then the solution of the problem* (2.4.3) – (2.4.9) *for the first step is the following,*

$$U_k(x_3,t)=\begin{cases} g_k\left(t-\dfrac{x_3}{\alpha_k}\right)+\dfrac{1}{2}\left[\varphi_k(x_3+\alpha_k t)-\varphi_k(-x_3+\alpha_k t)\right] \\ +\dfrac{1}{2\alpha_k}\displaystyle\int_{-x_3+\alpha_k t}^{x_3+\alpha_k t}\psi_k(\mu)d\mu, \qquad \textit{if } (x_3,t)\in R3; \\ \\ f_k\left(t+\dfrac{x_3-\ell}{\alpha_k}\right)+\dfrac{1}{2}\left[\varphi_k(x_3-\alpha_k t)-\varphi_k(-x_3-\alpha_k t+2\ell)\right] \\ -\dfrac{1}{2\alpha_k}\displaystyle\int_{-x_3-\alpha_k t+2\ell}^{x_3-\alpha_k t}\psi_k(\mu)d\mu, \qquad \textit{if } (x_3,t)\in R4; \\ \\ f_k\left(t-\dfrac{x_3-\ell}{\beta_k}\right)+\dfrac{1}{2}\left[w_k(x_3+\beta_k t)-w_k(-x_3+\beta_k t+2\ell)\right] \\ +\dfrac{1}{2\beta_k}\displaystyle\int_{-x_3+\beta_k t+2\ell}^{x_3+\beta_k t}\phi_k(\nu)d\nu, \qquad \textit{if } (x_3,t)\in R5. \end{cases} \tag{2.7.2}$$

where

$$R3=\left\{(x_3,t)\middle|\ 0<x_3<\ell\,,\quad \frac{x_3}{\alpha_k}<t<\frac{\ell-x_3}{\alpha_k}\right\},$$

$$R4=\left\{(x_3,t)\middle|\ 0<x_3<\ell\,,\quad \frac{\ell-x_3}{\alpha_k}<t<\frac{x_3}{\alpha_k}\right\},$$

$$R5=\left\{(x_3,t)\middle|\ \ell<x_3<\infty\,,\quad \frac{x_3-\ell}{\beta_k}<t<\frac{x_3-\ell}{\beta_k}+\frac{\ell}{\alpha_k}\right\},$$

and the functions defined in (2.7.1) *are constructed by initial-boundary data and the matching conditions as follows*

$$g_k(t)=(\varphi_k(\alpha_k t)-\varphi_k(0))+\int_0^t\psi_k(\alpha_k\tau)d\tau-\frac{1}{\alpha_k}\int_0^t F_k(\tau)d\tau \tag{2.7.3}$$

$$G_k(t)=\frac{1}{\alpha_k}f_k'(t)+\varphi_k'(\ell-\alpha_k t)-\frac{1}{\alpha_k}\psi_k(\ell-\alpha_k t) \tag{2.7.4}$$

$$\begin{aligned} f_k(t)=&\frac{\alpha_k}{\alpha_k+\beta_k}[\varphi_k(\ell-\alpha_k t)-\varphi_k(\ell)]-\frac{1}{\alpha_k+\beta_k}\int_\ell^{\ell-\alpha_k t}\psi_k(s)ds \\ &+\frac{\beta_k}{\alpha_k+\beta_k}[w_k(\ell+\beta_k t)-w(\ell)]+\frac{1}{\alpha_k+\beta_k}\int_\ell^{\ell+\beta_k t}\phi_k(z)dz \end{aligned} \tag{2.7.5}$$

for each $k=1,2,3$.

Proof. Let us consider the problem (2.4.3) – (2.4.4) with initial-boundary data (2.4.5) – (2.4.7) and the matching conditions (2.4.8) – (2.4.9) in the regions R3, R4 and R5 respectively.

Now, we analyze the regions, independently.

2.7.1 The Region R3

Let us consider the problem (2.4.3) – (2.4.9) in the region R3 (see, Figure 2.1), for $k = 1,2,3$.

$$R3 = \left\{ (x_3,t) \Big| \, 0 < x_3 < \ell \, , \quad \frac{x_3}{\alpha_k} < t < \frac{\ell - x_3}{\alpha_k} \right\}$$

The equation (2.4.3) can be written as in the form,

$$\frac{\partial q_k}{\partial t} - \alpha_k \frac{\partial q_k}{\partial x_3} = 0 \, , \quad (x_3,t) \in R3, \tag{2.7.6}$$

$$\frac{\partial u_k}{\partial t} + \alpha_k \frac{\partial u_k}{\partial x_3} = q_k(x_3,t) \, , \quad (x_3,t) \in R3. \tag{2.7.7}$$

The characteristic of the equation (2.7.6) – (2.7.7) are respectively,

$$\frac{d\xi}{d\tau} = -\alpha_k \, , \quad \xi(t) = x_3 \quad ; \quad \xi = -\alpha_k \tau + x_3 + \alpha_k t \, ,$$

$$\frac{d\xi}{d\tau} = \alpha_k \, , \quad \xi(t) = x_3 \quad ; \quad \xi = \alpha_k \tau + x_3 - \alpha_k t \quad \text{and if} \quad \xi = 0 \, ; \quad \tau = t - \frac{x_3}{\alpha_k} \, .$$

By integrating along the characteristics,

$$q_k(x_3,t) = \psi_k(x_3 + \alpha_k t) + \alpha_k \varphi_k'(x_3 + \alpha_k t)$$

Then by integrating along the characteristic,

$$u_k(x_3,t) - u_k\left(0, t - \frac{x_3}{\alpha_k}\right) = \int_{t-\frac{x_3}{\alpha_k}}^{t} \psi_k(x_3 - \alpha_k t + 2\alpha_k \tau) d\tau$$

$$+ \alpha_k \int_{t-\frac{x_3}{\alpha_k}}^{t} \varphi_k'(x_3 - \alpha_k t + 2\alpha_k \tau) d\tau \, ,$$

Let

$$x_3 - \alpha_k t + 2\alpha_k \tau = \nu \, , \qquad 2\alpha_k d\tau = d\nu$$

$$\nu_{low} = -x_3 + \alpha_k t \, , \qquad \nu_{up} = x_3 + \alpha_k t$$

By substituting the initial conditions (2.4.5), we have the solution

$$u_k(x_3,t) = g_k\left(t - \frac{x_3}{\alpha_k}\right) + \frac{1}{2}\left[\varphi_k(x_3+\alpha_k t) - \varphi_k(-x_3+\alpha_k t)\right]$$

$$+\frac{1}{2\alpha_k}\int_{-x_3+\alpha_k t}^{x_3+\alpha_k t}\psi_k(\mu)d\mu\ , \quad (x_3,t)\in R3,$$

and the function $g_k(t)$ defined in (2.7.1) is the following,

$$g_k(t) = (\varphi_k(\alpha_k t) - \varphi_k(0)) + \int_0^t \psi_k(\alpha_k\tau)d\tau - \frac{1}{\alpha_k}\int_0^t F_k(\tau)d\tau\ .$$

2.7.2 The Region R4

Let us consider the problem (2.4.3) – (2.4.9) in the region R4 (see, Figure 2.1), for $k = 1,2,3$.

$$R4 = \left\{(x_3,t)\middle|\ 0 < x_3 < \ell\ , \quad \frac{\ell - x_3}{\alpha_k} < t < \frac{x_3}{\alpha_k}\right\}$$

The equation (2.4.3) can be written as in the form,

$$\frac{\partial q_k}{\partial t} + \alpha_k\frac{\partial q_k}{\partial x_3} = 0\ , \quad (x_3,t)\in R4\ , \tag{2.7.8}$$

$$\frac{\partial u_k}{\partial t} - \alpha_k\frac{\partial u_k}{\partial x_3} = q_k(x_3,t)\ , \quad (x_3,t)\in R4\ . \tag{2.7.9}$$

The characteristics of the equations (2.7.8) – ((2.7.9)) are respectively,

$$\frac{d\xi}{d\tau} = \alpha_k\ , \quad \xi(t) = x_3 \quad ; \quad \xi = \alpha_k\tau + x_3 - \alpha_k t\ ,$$

$$\frac{d\xi}{d\tau} = -\alpha_k\ , \quad \xi = -\alpha_k\tau + x_3 + \alpha_k t\ , \text{ when } \xi = \ell\ ; \quad \tau = t + \frac{x_3 - \ell}{\alpha_k}\ .$$

By integrating along the characteristics, we get

$$q_k(x_3,t) = \psi_k(x_3 - \alpha_k t) - \alpha_k\varphi_k'(x_3 - \alpha_k t)$$

Similarly, we integrate along the characteristic and by using the boundary data (2.4.7), we get the following formula

$$u_k(x_3,t) = f_k\left(t + \frac{x_3-\ell}{\alpha_k}\right) + \frac{1}{2}\left[\varphi_k(x_3-\alpha_k t) - \varphi_k(-x_3-\alpha_k t + 2\ell)\right]$$

$$-\frac{1}{2\alpha_k}\int_{-x_3-\alpha_k t+2\ell}^{x_3-\alpha_k t} \psi_k(\mu)d\mu\ , \quad (x_3,t) \in R4.$$

2.7.3 The Region R5

Let us consider the problem (2.4.3) – (2.4.9) in the region R5 (see, Figure 2.1), for $k = 1,2,3$.

$$R5 = \left\{(x_3,t)\middle|\ \ell < x_3 < \infty\ , \quad \frac{x_3-\ell}{\beta_k} < t < \frac{x_3-\ell}{\beta_k} + \frac{\ell}{\alpha_k}\right\}$$

The equation (2.4.4) can be written as in the form,

$$\frac{\partial q_k}{\partial t} - \beta_k \frac{\partial q_k}{\partial x_3} = 0\ , \quad (x_3,t) \in R5, \tag{2.7.10}$$

$$\frac{\partial v_k}{\partial t} + \beta_k \frac{\partial v_k}{\partial x_3} = q_k(x_3,t)\ , \quad (x_3,t) \in R5. \tag{2.7.11}$$

The characteristics of the equations (2.7.10) – ((2.7.11)) are respectively,

$$\frac{d\xi}{d\tau} = -\beta_k\ , \quad \xi(t) = x_3 \quad ; \quad \xi = -\beta_k\tau + x_3 + \beta_k t,$$

$$\frac{d\xi}{d\tau} = \beta_k\ , \quad \xi(t) = x_3\ ; \quad \xi = \beta_k\tau + x_3 - \beta_k t, \text{ when } \xi = \ell; \quad \tau = t - \frac{x_3-\ell}{\beta_k}\ .$$

So

$$q_k(x_3,t) = \phi_k(x_3+\beta_k t) + \beta_k w_k'(x_3+\beta_k t)\ ,$$

Similarly, by integrating along the characteristics and by using initial conditions, we get the following formula

$$v_k(x_3,t) = h_k\left(t - \frac{x_3-\ell}{\beta_k}\right) + \frac{1}{2}\left[w_k(x_3+\beta_k t) - w_k(-x_3+\beta_k t + 2\ell)\right]$$

$$+\frac{1}{2\beta_k}\int_{-x_3+\beta_k t+2\ell}^{x_3+\beta_k t}\phi_k(\nu)d\nu\ , \quad (x_3,t)\in R5.$$

To find the functions $f_k(t)$ and $G_k(t)$ defined in (2.7.1), we must use the matching conditions in (2.4.8) – (2.4.9).

2.7.4 Matching Conditions Between R4 and R5

The formula for the region R4 is in the form,

$$u_k(x_3,t)=f_k\left(t-\frac{\ell-x_3}{\alpha_k}\right)+\frac{1}{2}[\varphi_k(x_3-\alpha_k t)-\varphi_k(-x_3-\alpha_k t+2\ell)]$$

$$-\frac{1}{2\alpha_k}\int_{-x_3-\alpha_k t+2\ell}^{x_3-\alpha_k t}\psi_k(\nu)d\nu\ ,$$

and the formula for the region R5 is in the form,

$$v_k(x_3,t)=h_k\left(t+\frac{\ell-x_3}{\beta_k}\right)+\frac{1}{2}[w_k(x_3+\beta_k t)-w_k(-x_3+\beta_k t+2\ell)]$$

$$+\frac{1}{2\beta_k}\int_{-x_3+\beta_k t+2\ell}^{x_3+\beta_k t}\phi_k(\nu)d\nu\ .$$

By the first matching condition (2.4.8), we have,

$$u_k(\ell-0,t)=v_k(\ell+0,t)=f_k(t)$$

To use the second matching condition (2.4.9), we must differentiate the formulas for the regions R4 and R5, and substitute $x=\ell$. Then we get the function $G_k(t)$ defined in (2.7.1),

$$G_k(t)=\frac{\partial u_k}{\partial x_3}\Big|_{x_3=\ell-0}=\frac{1}{\alpha_k}f_k'(t)+\varphi_k'(\ell-\alpha_k t)-\frac{1}{\alpha_k}\psi_k(\ell-\alpha_k t)$$

By using the second matching condition (2.4.9) and by integrating the resulting formula from 0 to t, we get the function $f_k(t)$ defined in (2.7.1) as follows,

$$f_k(t)=\frac{\alpha_k}{\alpha_k+\beta_k}[\varphi_k(\ell-\alpha_k t)-\varphi_k(\ell)]-\frac{1}{\alpha_k+\beta_k}\int_{\ell}^{\ell-\alpha_k t}\psi_k(s)ds$$

$$+\frac{\beta_k}{\alpha_k+\beta_k}[w_k(\ell+\beta_k t)-w(\ell)]+\frac{1}{\alpha_k+\beta_k}\int_{\ell}^{\ell+\beta_k t}\phi_k(z)dz$$

□

2.8 General Case

In zero and the first step, we have constructed the formulations of $u_k(x_3,t),\ v_k(x_3,t)$ and the functions $g_k(t),\ f_k(t),\ G_k(t)$ defined in $(2.7.1)$ for $n=0$ and $n=1$. After the first step, we generalize the number of the step with index n, for $n=2,3,\ldots$ So, we reformulate the initial boundary value problem.

Initial boundary value problem is to find $u_{nk}(x_3,t)$ in the form

$$U_{nk}(x_3,t)=\begin{cases} u_{nk}(x_3,t), & 0<x_3<\ell; \\ v_{nk}(x_3,t), & \ell<x_3<\infty. \end{cases}$$

for each $k=1,2,3$ and $n=2,3,\ldots$ satisfying

$$\frac{\partial^2 u_{nk}}{\partial t^2}=\alpha_k^2\frac{\partial^2 u_{nk}}{\partial {x_3}^2}\ , \qquad 0<x_3<\ell\ , \quad t\in\mathbf{R}, \tag{2.8.1}$$

$$\frac{\partial^2 v_{nk}}{\partial t^2}=\beta_k^2\frac{\partial^2 v_{nk}}{\partial {x_3}^2}\ , \qquad \ell<x_3<\infty\ , \quad t\in\mathbf{R}, \tag{2.8.2}$$

with initial and boundary data,

$$u_{nk}(x,0)=\varphi_k(x_3)\ , \qquad \frac{\partial u_{nk}}{\partial t}(x,t)\Big|_{t=0}=\psi_k(x_3)\ , \quad 0<x_3<\ell\ , \tag{2.8.3}$$

$$v_{nk}(x,0)=w_k(x_3)\ , \qquad \frac{\partial v_{nk}}{\partial t}(x,t)\Big|_{t=0}=\phi_k(x_3)\ , \quad \ell<x_3<\infty\ , \tag{2.8.4}$$

$$\alpha_k^2\frac{\partial u_{nk}}{\partial x_3}\Big|_{x_3=0}=F_k(t)\ , \quad t\in\mathbf{R} \tag{2.8.5}$$

and the matching conditions,

$$u_{nk}\Big|_{x_3=\ell-0}=v_{nk}\Big|_{x_3=\ell+0} \tag{2.8.6}$$

$$\alpha_k^2\frac{\partial u_{nk}}{\partial x_3}\Big|_{x_3=\ell-0}=\beta_k^2\frac{\partial v_{nk}}{\partial x_3}\Big|_{x_3=\ell+0} \tag{2.8.7}$$

The General case includes the regions $R(4n-2), R(4n-1), R(4n)$ and $R(4n+1)$ (see, Figure 2.1). Notice that, unlike in the first step, in the general case we have an additional subregion, namely the region R(4n-2).

However, similar to the first step, in the general case we consider initial boundary data and also matching conditions defined on the boundary $x = \ell$.

Before finding the solution for the general case, we must define the following functions,

$$u_{nk}(0,t) = g_{nk}(t), \quad u_{nk}(\ell,t) = f_{nk}(t) \quad \text{and} \quad \frac{\partial u_{nk}}{\partial x_3}\Big|_{x_3=\ell} = G_{nk}(t) \tag{2.8.8}$$

We must construct these functions by initial-boundary data and also by the matching conditions. Similar to (2.5.1), the solution of the problem (2.8.1) – (2.8.7) for the general case will be found in the following form by using *the method of characteristics*.

$$u_{kn}(x_3,t) = \left\{ \; u_{knm}(x_3,t), \quad \text{if } (x_3,t) \in Rm \right. \tag{2.8.9}$$

Here, the index k denotes the component of the vector function $u(x_3,t)$, the index n denotes the number of the step and the index m denotes the number of subregion.

Theorem 2.8.1. *Let* $\varphi_k(x_3)$, $\psi_k(x_3)$, w_k *and* $\phi_k(x_3)$ *be given continuous functions depending on* x_3*;* $F_k(t)$ *be given continuous function depending on t;* $u_k(x_3,t)$ *is unknown function in the form* (2.5.1)*. Then the solution of the problem* (2.8.1) – (2.8.7) *for the general case is the following,*

$$U_k(x_3,t) = \begin{cases} g_{(n-1)k}\left(t - \dfrac{x_3}{\alpha_k}\right) + \dfrac{1}{2} f_{(n-1)k}\left(t + \dfrac{x_3-\ell}{\alpha_k}\right) - \dfrac{1}{2} f_{(n-1)k}\left(t - \dfrac{x_3+\ell}{\alpha_k}\right) \\ + \dfrac{\alpha_k}{2}\displaystyle\int_{t-\frac{x_3+\ell}{\alpha_k}}^{t+\frac{x_3-\ell}{\alpha_k}} G_{(n-1)k}(\mu)d\mu, \qquad \textit{if } (x_3,t) \in R(4n-2); \\ \\ g_{nk}\left(t - \dfrac{x_3}{\alpha_k}\right) + \dfrac{1}{2}\left[f_{(n-1)k}\left(t + \dfrac{x_3-\ell}{\alpha_k}\right) - f_{(n-1)k}\left(t - \dfrac{x_3+\ell}{\alpha_k}\right)\right] \\ + \dfrac{\alpha_k}{2}\displaystyle\int_{t-\frac{x_3+\ell}{\alpha_k}}^{t+\frac{x_3-\ell}{\alpha_k}} G_{(n-1)k}(\mu)d\mu, \qquad \textit{if } (x_3,t) \in R(4n-1); \\ \\ f_{nk}\left(t + \dfrac{x_3-\ell}{\alpha_k}\right) + \dfrac{1}{2}\left[g_{(n-1)k}\left(t - \dfrac{x_3}{\alpha_k}\right) - g_{(n-1)k}\left(t + \dfrac{x_3-2\ell}{\alpha_k}\right)\right] \\ - \dfrac{\alpha_k}{2}\displaystyle\int_{t+\frac{x_3-2\ell}{\alpha_k}}^{t-\frac{x_3}{\alpha_k}} F_{(n-1)k}(\gamma)d\gamma, \qquad \textit{if } (x_3,t) \in R(4n); \\ \\ f_{nk}\left(t - \dfrac{x_3-\ell}{\beta_k}\right) + \dfrac{1}{2}\left[w_k(x_3+\beta_k t) - w_k(-x_3+\beta_k t+2\ell)\right] \\ + \dfrac{1}{2\beta_k}\displaystyle\int_{-x_3+\beta_k t+2\ell}^{x_3+\beta_k t} \phi_k(\nu)d\nu, \qquad \textit{if } (x_3,t) \in R(4n+1). \end{cases} \tag{2.8.10}$$

where

$$R(4n-2)=\left\{(x_3,t)\Big|\ 0<x_3<\ell\ ,\quad \frac{(n-2)\ell}{\alpha_k}<t-\frac{x_3}{\alpha_k}<\frac{(n-1)\ell}{\alpha_k}\quad and\right.$$

$$\left.\frac{(n-1)\ell}{\alpha_k}<t+\frac{x_3}{\alpha_k}<\frac{n\ell}{\alpha_k}\right\}$$

$$R(4n-1)=\left\{(x_3,t)\Big|\ 0<x_3<\ell\ ,\quad \frac{x_3+(n-1)\ell}{\alpha_k}<t<\frac{n\ell-x_3}{\alpha_k}\right\}$$

$$R(4n)=\left\{(x_3,t)\Big|\ 0<x_3<\ell\ ,\quad \frac{n\ell-x_3}{\alpha_k}<t<\frac{x_3+(n-1)\ell}{\alpha_k}\right\}$$

$$R(4n+1)=\left\{(x_3,t)\Big|\ \ell<x_3<\infty\ ,\quad \frac{(n-1)\ell}{\alpha_k}<t-\frac{x_3-\ell}{\beta_k}<\frac{n\ell}{\alpha_k}\right\}$$

for each $n=2,3,\ldots$ *and the functions defined in* (2.8.8) *are constructed by initial-boundary data and the matching conditions as follows*

$$G_{nk}(t)=\frac{1}{\alpha_k}f'_{nk}(t)-\frac{1}{\alpha_k}g'_{(n-1)k}\left(t-\frac{\ell}{\alpha_k}\right)+F_{(n-1)k}\left(t-\frac{\ell}{\alpha_k}\right), \tag{2.8.11}$$

$$g_{nk}(t)=\left[f_{(n-1)k}\left(t-\frac{\ell}{\alpha_k}\right)-f_{(n-1)k}\left(-\frac{\ell}{\alpha_k}\right)\right]+\alpha_k\int_{-\frac{\ell}{\alpha_k}}^{t-\frac{\ell}{\alpha_k}}G_{(n-1)k}(\gamma)d\gamma$$
$$-\frac{1}{\alpha_k}\int_0^t F_{nk}(\tau)d\tau, \tag{2.8.12}$$

$$f_{nk}(t)=\frac{\alpha_k}{\alpha_k+\beta_k}[g_{(n-1)k}\left(t-\frac{\ell}{\alpha_k}\right)-g_{(n-1)k}\left(-\frac{\ell}{\alpha_k}\right)]+\frac{\beta_k}{\alpha_k+\beta_k}w_k(\ell+\beta_k t)$$
$$-\frac{\beta_k}{\alpha_k+\beta_k}w_k(\ell)-\frac{\alpha_k^2}{\alpha_k+\beta_k}\int_{-\frac{\ell}{\alpha_k}}^{t-\frac{\ell}{\alpha_k}}F_{(n-1)k}(s)ds$$
$$+\frac{1}{\alpha_k+\beta_k}\int_{\ell}^{\ell+\beta_k t}\phi_k(z)dz \tag{2.8.13}$$

for each $k=1,2,3$ *and* $n=2,3,\ldots$

Proof. If we notice the subregions in $0<x_3<\ell$, namely the regions $R(4n-2)$, $R(4n-1)$ and $R(4n)$ (see, Figure 2.1), we do not use the initial conditions. Instead, we use the functions, defined in (2.8.8). As a result of this situation, the formulation of the defined functions (2.8.11) – (2.8.13) is in the form of recurrence relations.

Now, we analyze the regions, independently.

2.8.1 The Region R(4n-2)

The region $R(4n-2)$ has a different form (see, Figure 2.2). We use the boundary condition $F_k(t)$ and the functions $f_{(n-1)k}(t)$, $g_{(n-1)k}$ and $G_{(n-1)k}$ which we must find in the previous step.

In this region, we assume that there is a jump at $x = \dfrac{\ell}{2}$. We will apply the following matching conditions when the speeds are the same.

$$u_{nk}(\frac{\ell}{2}-0,t) = u_{nk}(\frac{\ell}{2}+0,t) \tag{2.8.14}$$

$$(\alpha_k^2)\frac{\partial u_{nk}}{\partial x_3}\Big|_{x_3=\frac{\ell}{2}-0} = (\alpha_k^2)\frac{\partial u_{nk}}{\partial x_3}\Big|_{x_3=\frac{\ell}{2}+0} \tag{2.8.15}$$

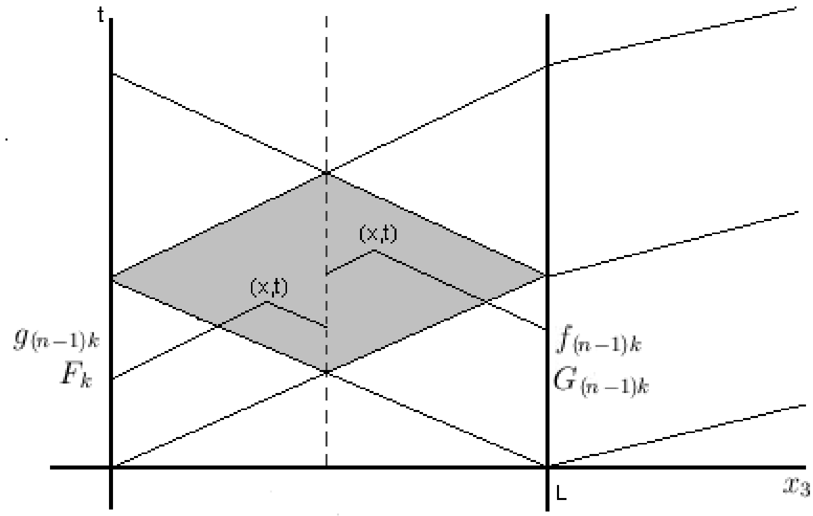


Figure 2.2 The Region R(4n-2)

Let us consider the problem $(2.8.1)-(2.8.7)$ in the region R(4n-2), for $k = 1,2,3.$ and $n = 2,3,\ldots$

$$R(4n-2) = \left\{ (x_3,t) \Big|\, 0 < x_3 < \ell\,, \quad \frac{(n-2)\ell}{\alpha_k} < t - \frac{x_3}{\alpha_k} < \frac{(n-1)\ell}{\alpha_k} \quad \text{and} \right.$$

$$\left. \frac{(n-1)\ell}{\alpha_k} < t + \frac{x_3}{\alpha_k} < \frac{n\ell}{\alpha_k} \right\}$$

The equation (2.8.1) can be written as in the form,

$$\frac{\partial q_{nk}}{\partial t} + \alpha_k \frac{\partial q_{nk}}{\partial x_3} = 0\ , \quad (x_3,t) \in R(4n-2), \tag{2.8.16}$$

$$\frac{\partial u_{nk}}{\partial t} - \alpha_k \frac{\partial u_{nk}}{\partial x_3} = q_{nk}(x_3,t)\ , \quad (x_3,t) \in R(4n-2). \tag{2.8.17}$$

The characteristics of the equation (2.8.16) – (2.8.17) are the following,

$$\frac{d\xi}{d\tau} = \alpha_k\ ,\ \ \xi(t) = x_3\ ; \quad \xi = \alpha_k \tau + x_3 - \alpha_k t\ ,\ \text{when}\ \xi = 0\ ; \quad \tau = t - \frac{x_3}{\alpha_k}\ ,$$

$$\frac{d\xi}{d\tau} = -\alpha_k\ ,\ \ \xi(t) = x_3\ ; \quad \xi = -\alpha_k \tau + x_3 + \alpha_k t\ ,\text{when}\ \xi = \frac{\ell}{2}\ ; \quad \tau = t + \frac{2x_3 - \ell}{2\alpha_k}\ .$$

By integrating along the characteristic $\xi = x_3 - \alpha_k(t-\tau)$, from $t - \frac{x_3}{\alpha_k}$ to t,

$$q_{nk}(x_3,t) = g_{(n-1)k}(t - \frac{x_3}{\alpha_k})$$

Then by integrating along the characteristic $\xi = x_3 + \alpha_k(t-\tau)$, from $t + \frac{2x_3 - \ell}{2\alpha_k}$ to t,

$$\int_{t+\frac{2x_3-\ell}{2\alpha_k}}^{t} \frac{\partial}{\partial \tau}[u_{nk}(x_3 + \alpha_k(t-\tau),\tau)]d\tau = \int_{t+\frac{2x_3-\ell}{2\alpha_k}}^{t} g'\left(2\tau - t - \frac{x_3}{\alpha_k}\right)d\tau$$

Let

$$2\tau - t - \frac{x_3}{\alpha_k} = \mu\ , \qquad 2d\tau = d\mu$$

$$\mu_{low} = t + \frac{x_3 - \ell}{\alpha_k}\ , \qquad \mu_{up} = t - \frac{x_3}{\alpha_k}$$

By letting $u_{nk}(\frac{\ell}{2},t) = m_{nk}(t)$, we get

$$u_{nk}(x_3,t) = m_{nk}\left(t + \frac{2x_3 - \ell}{2\alpha_k}\right) + \frac{1}{2}\left[g_{(n-1)k}\left(t - \frac{x_3}{\alpha_k}\right) - g_{(n-1)k}\left(t + \frac{x_3 - \ell}{\alpha_k}\right)\right]\ . \tag{2.8.18}$$

Similarly, the equation (2.8.1) can be written as in the form,

$$\frac{\partial q_{nk}}{\partial t} - \alpha_k \frac{\partial q_{nk}}{\partial x_3} = 0\ , \quad (x_3,t) \in R(4n-2)\ , \tag{2.8.19}$$

$$\frac{\partial u_{nk}}{\partial t} + \alpha_k \frac{\partial u_{nk}}{\partial x_3} = q_{nk}(x_3,t)\ , \quad (x_3,t) \in R(4n-2)\ . \tag{2.8.20}$$

The characteristics of the equation (2.8.18) – (2.8.19) are the following,

$$\frac{d\xi}{d\tau} = -\alpha_k\ ,\ \ \xi(t) = x_3\ ; \quad \xi = -\alpha_k\tau + x_3 + \alpha_k t\ ,\ \text{when}\ \xi = \ell\ ; \quad \tau = t + \frac{x_3 - \ell}{\alpha_k}\ ,$$

$$\frac{d\xi}{d\tau} = \alpha_k\ ,\ \ \xi(t) = x_3\ ; \quad \xi = \alpha_k\tau + x_3 - \alpha_k t,\ \text{when}\ \xi = \frac{\ell}{2}\ ; \quad \tau = t + \frac{\ell - 2x_3}{2\alpha_k}\ .$$

By integrating along the characteristic $\xi = x_3 + \alpha_k(t-\tau)$, from $t + \dfrac{x_3 - \ell}{\alpha_k}$ to t,

$$q_{nk}(x_3,t) = f'_{(n-1)k}\left(t + \frac{x_3-\ell}{\alpha_k}\right) + \alpha_k G_{(n-1)k}\left(t + \frac{x_3-\ell}{\alpha_k}\right)$$

Similarly, by letting $u_{nk}(\frac{\ell}{2}+0,t) = r(t)$ and integrating along the characteristic $\xi = x_3 - \alpha_k(t-\tau)$, from $t + \dfrac{\ell - 2x_3}{2\alpha_k}$ to t, we get

$$u_{nk}(x_3,t) = r\left(t + \frac{\ell-2x_3}{2\alpha_k}\right) + \frac{1}{2}\left[f_{(n-1)k}\left(t + \frac{x_3-\ell}{\alpha_k}\right) - f_{(n-1)k}\left(t - \frac{x_3}{\alpha_k}\right)\right]$$

$$+\frac{\alpha_k}{2}\int_{t-\frac{x_3}{\alpha_k}}^{t+\frac{x_3-\ell}{\alpha_k}} G_{(n-1)k}(z)dz \qquad (2.8.21)$$

If we use the first matching condition (2.8.14), we get

$$m(t) = r(t)$$

By using the second matching condition (2.8.15), we get

$$m(t) = \frac{1}{2}\left[g\left(t - \frac{\ell}{2\alpha_k}\right) - g\left(-\frac{\ell}{2\alpha_k}\right)\right] + \frac{\alpha_k}{2}\int_{-\frac{\ell}{2\alpha_k}}^{t-\frac{\ell}{2\alpha_k}} G_{(n-1)k}(\mu)d\mu$$

$$+\frac{1}{2}\left[f\left(t - \frac{\ell}{2\alpha_k}\right) - f\left(-\frac{\ell}{2\alpha_k}\right)\right]\ ,$$

If we substitute the function $m(t)$ into the formulation (2.8.21), we get

$$u_{nk}(x_3,t) = g_{(n-1)k}\left(t - \frac{x_3}{\alpha_k}\right) + \frac{1}{2}f_{(n-1)k}\left(t + \frac{x_3-\ell}{\alpha_k}\right) - \frac{1}{2}f_{(n-1)k}\left(t - \frac{x_3+\ell}{\alpha_k}\right)$$

$$+\frac{\alpha_k}{2}\int_{t-\frac{x_3+\ell}{\alpha_k}}^{t+\frac{x_3-\ell}{\alpha_k}} G_{(n-1)k}(\mu)d\mu, \qquad (x_3,t) \in R(4n-2).$$

2.8.2 The Region R(4n-1)

Let us consider the problem $(2.8.1)-(2.8.7)$ in the region R(4n-1), for $k=1,2,3.$ and $n=2,3,\ldots$

$$R(4n-1)=\left\{(x_3,t)\Big|\, 0<x_3<\ell\,,\quad \frac{x_3+(n-1)\ell}{\alpha_k}<t<\frac{n\ell-x_3}{\alpha_k}\right\}$$

The equation $(2.8.1)$ can be written in the form,

$$\frac{\partial q_{nk}}{\partial t}-\alpha_k\frac{\partial q_{nk}}{\partial x_3}=0\,,\quad (x_3,t)\in R(4n-1), \tag{2.8.22}$$

$$\frac{\partial u_{nk}}{\partial t}+\alpha_k\frac{\partial u_{nk}}{\partial x_3}=q_{nk}(x_3,t)\,,\quad (x_3,t)\in R(4n-1). \tag{2.8.23}$$

The characteristics of the equation $(2.8.22)-(2.8.23)$ are the following,

$$\frac{d\xi}{d\tau}=-\alpha_k\,,\ \xi(t)=x_3\,;\quad \xi=-\alpha_k\tau+x_3+\alpha_k t\,,\text{ when } \xi=\ell\,;\quad \tau=t+\frac{x_3-\ell}{\alpha_k}\,,$$

$$\frac{d\xi}{d\tau}=\alpha_k\,,\quad \xi(t)=x_3\quad ;\quad \xi=\alpha_k\tau+x_3-\alpha_k t\quad\text{and if}\quad \xi=0\,;\quad \tau=t-\frac{x_3}{\alpha_k}\,.$$

So, by integrating along the characteristic $\xi=x_3+\alpha_k(t-\tau)$ from $t+\dfrac{x_3-\ell}{\alpha_k}$, to t,

$$q_{nk}(x_3,t)=f'_{(n-1)k}\left(t+\frac{x_3-\ell}{\alpha_k}\right)+\alpha_k G_{(n-1)k}\left(t+\frac{x_3-\ell}{\alpha_k}\right)\,,$$

By integrating along the characteristic, $\xi=x_3-\alpha_k(t-\tau)$ we get the solution

$$u_{nk}(x_3,t)=g_{nk}\left(t-\frac{x_3}{\alpha_k}\right)=\frac{1}{2}\left[f_{(n-1)k}\left(t+\frac{x_3-\ell}{\alpha_k}\right)-f_{(n-1)k}\left(t-\frac{x_3+\ell}{\alpha_k}\right)\right]$$

$$+\frac{\alpha_k}{2}\int_{t-\frac{x_3+\ell}{\alpha_k}}^{t+\frac{x_3-\ell}{\alpha_k}}G_{(n-1)k}(\mu)d\mu\,,\quad (x_3,t)\in R(4n-1).$$

And the function g_{nk} defined in (2.8.8) is in the form,

$$g_{nk}(t)=\left[f_{(n-1)k}\left(t-\frac{\ell}{\alpha_k}\right)-f_{(n-1)k}\left(-\frac{\ell}{\alpha_k}\right)\right]+\alpha_k\int_{-\frac{\ell}{\alpha_k}}^{t-\frac{\ell}{\alpha_k}}G_{(n-1)k}(\gamma)d\gamma$$

$$-\frac{1}{\alpha_k}\int_0^t F_{nk}(\tau)d\tau\,.$$

2.8.3 The Region R(4n)

Let us consider the problem $(2.8.1)-(2.8.7)$ in the region R(4n), for $k=1,2,3.$ and $n=2,3,\ldots$

$$R(4n)=\left\{(x_3,t)\Big|\, 0<x_3<\ell\,,\quad \frac{n\ell-x_3}{\alpha_k}<t<\frac{x_3+(n-1)\ell}{\alpha_k}\right\}$$

The equation $(2.8.1)$ can be written as in the form,

$$\frac{\partial q_{nk}}{\partial t}+\alpha_k\frac{\partial q_{nk}}{\partial x_3}=0\,,\quad (x_3,t)\in R(4n), \tag{2.8.24}$$

$$\frac{\partial u_{nk}}{\partial t}-\alpha_k\frac{\partial u_{nk}}{\partial x_3}=q_{nk}(x_3,t)\,,\quad (x_3,t)\in R(4n). \tag{2.8.25}$$

The characteristics of the equations $(2.8.24)-(2.8.25)$ are the following

$$\frac{d\xi}{d\tau}=\alpha_k\,,\quad \xi(t)=x_3\quad ;\quad \xi=\alpha_k\tau+x_3-\alpha_k t\quad \text{and if}\quad \xi=0\,;\quad \tau=t-\frac{x_3}{\alpha_k}\,,$$

$$\frac{d\xi}{d\tau}=-\alpha_k\,,\ \xi(t)=x_3\quad \xi=-\alpha_k\tau+x_3+\alpha_k t\,,\quad \text{when } \xi=\ell\,;\quad \tau=t+\frac{x_3-\ell}{\alpha_k}\,.$$

So, by integrating along the characteristic $\xi=x_3-\alpha_k(t-\tau)$ from $t-\dfrac{x_3}{\alpha_k}$ to t,

$$q_{nk}(x_3,t)=g'_{(n-1)k}\left(t-\frac{x_3}{\alpha_k}\right)-\alpha_k F_{(n-1)k}\left(t-\frac{x_3}{\alpha_k}\right)\,,$$

Similarly, by integrating along the characteristic $\xi=-\alpha_k\tau+x_3+\alpha_k t$ from $t+\dfrac{x_3-\ell}{\alpha_k}$ to t, we get

$$u_{nk}(x_3,t)=f_{nk}\left(t+\frac{x_3-\ell}{\alpha_k}\right)+\frac{1}{2}\left[g_{(n-1)k}\Big(t-\frac{x_3}{\alpha_k}\Big)-g_{(n-1)k}\Big(t+\frac{x_3-2\ell}{\alpha_k}\Big)\right]$$
$$-\frac{\alpha_k}{2}\int_{t+\frac{x_3-2\ell}{\alpha_k}}^{t-\frac{x_3}{\alpha_k}}F_{(n-1)k}(\gamma)d\gamma\,,\quad (x_3,t)\in R(4n). \tag{2.8.26}$$

2.8.4 The Region R(4n+1)

Let us consider the problem $(2.8.1)-(2.8.7)$ in the region R(4n+1), for $k=1,2,3.$ and $n=2,3,\ldots$

$$R(4n+1)=\left\{(x_3,t)\Big|\ \ell<x_3<\infty\,,\quad \frac{(n-1)\ell}{\alpha_k}<t-\frac{x_3-\ell}{\beta_k}<\frac{n\ell}{\alpha_k}\right\}$$

The equation $(2.8.2)$ can be written as in the form,

$$\frac{\partial q_{nk}}{\partial t}-\beta_k\frac{\partial q_{nk}}{\partial x_3}=0\,,\quad (x_3,t)\in R(4n+1), \tag{2.8.27}$$

$$\frac{\partial v_{nk}}{\partial t}+\beta_k\frac{\partial v_{nk}}{\partial x_3}=q_{nk}(x_3,t)\,,\quad (x_3,t)\in R(4n+1). \tag{2.8.28}$$

The characteristics of the equation $(2.8.27)-(2.8.28)$ are the following,

$$\frac{d\xi}{d\tau}=-\beta_k\,,\quad \xi(x_3)=t\quad ;\quad \xi=-\beta_k\tau+x_3+\beta_k t\,,$$

$$\frac{d\xi}{d\tau}=\beta_k\,,\quad \xi(x_3)=t\quad ;\quad \xi=\beta_k\tau+x_3-\beta_k t\,,\quad \text{when } \xi=\ell\,;\quad \tau=t-\frac{x_3-\ell}{\beta_k}\,.$$

So, by integrating along $\xi=\beta_k\tau+x_3-\beta_k t$ from 0 to t,

$$q_{nk}(x_3,t)=\phi_k(x_3+\beta_k t)+\beta_k w_k'(x_3+\beta_k t),$$

Similarly, by integrating along the characteristic $\xi=\beta_k\tau+x_3-\beta_k t$ from $\tau=t-\dfrac{x_3-\ell}{\beta_k}$ to t, we get

$$v_{nk}(x_3,t)=f_{nk}\left(t-\frac{x_3-\ell}{\beta_k}\right)+\frac{1}{2}\left[w_k(x_3+\beta_k t)-w_k(-x_3+\beta_k t+2\ell)\right]$$
$$+\frac{1}{2\beta_k}\int_{-x_3+\beta_k t+2\ell}^{x_3+\beta_k t}\phi_k(\nu)d\nu\ ,(x_3,t)\in R(4n+1). \tag{2.8.29}$$

2.8.5 Matching Conditions Between R(4n) and R(4n+1)

Consider the formulations in $(2.8.26)-(2.8.29)$ for the regions $R(4n)$ and $R(4n+1)$. By the first matching condition (2.8.6), we get the relation

$$u_{nk}(\ell-0,t)=u_{nk}(\ell+0,t)=f_{nk}(t) \tag{2.8.30}$$

To apply the second matching condition (2.8.7), we must differentiate the formulations in (2.8.26) – (2.8.29), then by substituting $x_3 = \ell$, we get the function $G_{nk}(t)$ defined in (2.8.8)

$$G_{nk}(t) = \frac{\partial u_{nk}}{\partial x_3}\Big|_{x_3=\ell-0} = \frac{1}{\alpha_k} f'_{nk}(t) - \frac{1}{\alpha_k} g'_{(n-1)k}\left(t - \frac{\ell}{\alpha_k}\right) + F_{(n-1)k}\left(t - \frac{\ell}{\alpha_k}\right)$$

and by the second matching condition (2.8.7), we get the function $f_{nk}(t)$,

$$f_{nk}(t) = \frac{\alpha_k}{\alpha_k+\beta_k}[g_{(n-1)k}\left(t - \frac{\ell}{\alpha_k}\right) - g_{(n-1)k}\left(-\frac{\ell}{\alpha_k}\right)]$$

$$+\frac{\beta_k}{\alpha_k+\beta_k}[w_k(\ell+\beta_k t) - w_k(\ell)] + \frac{1}{\alpha_k+\beta_k}\int_{\ell}^{\ell+\beta_k t} \phi_k(z)dz$$

$$-\frac{\alpha_k^2}{\alpha_k+\beta_k}\int_{-\frac{\ell}{\alpha_k}}^{t-\frac{\ell}{\alpha_k}} F_{(n-1)k}(s)ds\ .$$

□

2.9 Examples of Simulations of Wave Propagation in Two Layered Medium

In this section, we deal with examples of simulations of wave propagation in two layered elastic half space. As the mathematical model of wave propagation, we study IBVP of wave equations in two layered medium.,

We took a pulse point source in different positions in half space: Between the boundaries $x_3 = 0$ and $x_3 = \ell$, outside the boundary $x_3 = \ell$. In each case, the half space has two layers with different speed. The speed of the first layer is $\alpha = 1$ and the speed of the second layer is $\beta = 2$. We considered the matching conditions (2.4.8) – (2.4.9) only on the boundary $x_3 = \ell$.

For all examples, we omit the index k for simplicity writing.

2.9.1 Example 1 - The Pulse Point Source is Between the Boundaries $x_3 = 0$ *and* $x_3 = \ell$

Let us consider initial boundary value problem (2.4.3) – (2.4.9) with its general form (2.8.1) – (2.8.7) for $k = 1$ and $n = 2, 3, \ldots$ The initial conditions $\varphi(x_3)$, $\psi(x_3)$, $w(x_3)$, $\phi(x_3)$ have the following form

$$\varphi(x_3) = \delta(x_3 - x_3^0), \quad \psi(x_3) = 0,$$

$$w(x_3) = 0, \quad \phi(x_3) = 0.$$

where $\delta(x_3)$ is Dirac delta function, the boundary $\ell = 40,$ the point source is located at $x_3^0 = 10$ and the boundary condition

$$F(x_3) = 0.$$

By the properties of Dirac delta function and the assumptions, the solution of IBVP can be written as follows:

$$U(x_3,t) = \begin{cases} \frac{1}{2}\left[\delta(x_3+\alpha t - x_3^0) + \delta(x_3 - \alpha t - x_3^0)\right], & \text{if } (x_3,t) \in R1; \\ \\ 0, & \text{if } (x_3,t) \in R2. \end{cases}$$

$$U(x_3,t) = \begin{cases} g\left(t - \frac{x_3}{\alpha}\right) + \frac{1}{2}\cdot\delta(x_3+\alpha t - x_3^0) \\ -\frac{1}{2}\cdot\delta(-x_3+\alpha t - x_3^0), & \text{if } (x_3,t) \in R3; \\ \\ f\left(t + \frac{x_3-\ell}{\alpha}\right) + \frac{1}{2}\cdot\delta(x_3 - \alpha t - x_3^0) \\ -\frac{1}{2}\cdot\delta(-x_3 - \alpha t + 2\ell - x_3^0), & \text{if } (x_3,t) \in R4; \\ \\ f\left(t - \frac{x_3-\ell}{\beta}\right), & \text{if } (x_3,t) \in R5. \end{cases}$$

Here, the function $g(t),\ f(t)$ and $G(t),$ constructed in Theorem 2.7.1, can be also written as

$$g(t) = \delta(\alpha t - x_3^0),$$

$$G(t) = -\frac{\beta}{\alpha(\alpha+\beta)}\cdot\frac{\partial}{\partial t}\left[\delta(\ell - \alpha t - x_3^0)\right]$$

$$f(t) = \frac{\alpha}{\alpha+\beta}\cdot\delta(\ell - \alpha t - x_3^0)$$

For $n = 2, 3, \ldots$

$$U(x_3,t)=\begin{cases} g_{(n-1)}\left(t-\frac{x_3}{\alpha}\right)+\frac{1}{2}\cdot f_{(n-1)}\left(t+\frac{x_3-\ell}{\alpha}\right)-\frac{1}{2}f_{(n-1)}\left(t-\frac{x_3+\ell}{\alpha}\right) \\ +\frac{\alpha}{2}\int_{t-\frac{x_3+\ell}{\alpha}}^{t+\frac{x_3-\ell}{\alpha}} G_{(n-1)}(\mu)d\mu, \qquad \text{if } (x_3,t)\in R(4n-2); \\ \\ g_n\left(t-\frac{x_3}{\alpha}\right)+\frac{1}{2}\left[f_{(n-1)}\left(t+\frac{x_3-\ell}{\alpha}\right)-f_{(n-1)}\left(t-\frac{x_3+\ell}{\alpha}\right)\right] \\ +\frac{\alpha}{2}\int_{t-\frac{x_3+\ell}{\alpha}}^{t+\frac{x_3-\ell}{\alpha}} G_{(n-1)}(\mu)d\mu, \qquad \text{if } (x_3,t)\in R(4n-1); \\ \\ f_n\left(t+\frac{x_3-\ell}{\alpha}\right)+\frac{1}{2}\cdot g_{(n-1)}\left(t-\frac{x_3}{\alpha}\right) \\ -\frac{1}{2}\cdot g_{(n-1)}\left(t+\frac{x_3-2\ell}{\alpha}\right), \qquad \text{if } (x_3,t)\in R(4n); \\ \\ f_n\left(t-\frac{x_3-\ell}{\beta}\right), \qquad \text{if } (x_3,t)\in R(4n+1). \end{cases}$$

Here, the function $g_n(t)$, $f_n(t)$ and $G_n(t)$ are constructed in Theorem 2.8.1, can be also written as for $n=2,3,\dots$

$$G_n(t)=\frac{1}{\alpha}\cdot f_n'(t)-\frac{1}{\alpha}\cdot g_{(n-1)}'\left(t-\frac{\ell}{\alpha}\right),$$

$$g_n(t)=\left[f_{(n-1)}\left(t-\frac{\ell}{\alpha}\right)-f_{(n-1)}\left(-\frac{\ell}{\alpha}\right)\right]+\alpha\int_{-\frac{\ell}{\alpha}}^{t-\frac{\ell}{\alpha}} G_{(n-1)}(\gamma)d\gamma$$

$$f_n(t)=\frac{\alpha}{\alpha+\beta}[g_{(n-1)}\left(t-\frac{\ell}{\alpha}\right)-g_{(n-1)}\left(-\frac{\ell}{\alpha}\right)]$$

with

$$G_1(t)=-\frac{\beta}{\alpha(\alpha+\beta)}\cdot\frac{\partial}{\partial t}\left[\delta(\ell-\alpha t-x_3^0)\right]$$

$$G_{(n-1)}(t)=\frac{1}{\alpha}\cdot f_{(n-1)}'(t)-\frac{1}{\alpha}\cdot g_{(n-2)}'\left(t-\frac{\ell}{\alpha}\right),$$

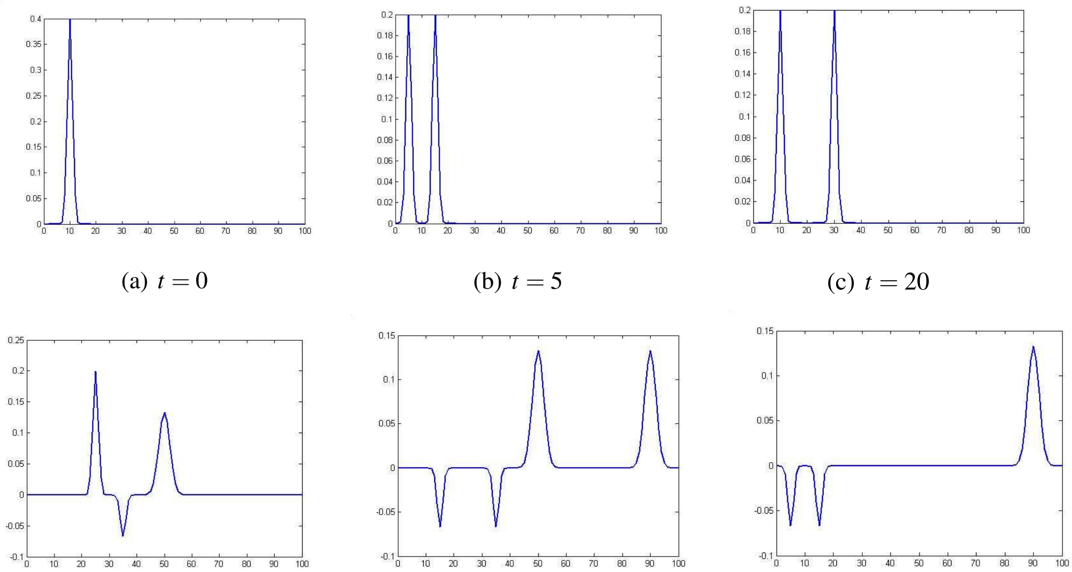
$$g_1(t)=\delta(\alpha t-x_3^0),$$

$$g_{(n-1)}(t)=\left[f_{(n-2)}\left(t-\frac{\ell}{\alpha}\right)-f_{(n-2)}\left(-\frac{\ell}{\alpha}\right)\right]+\alpha\int_{-\frac{\ell}{\alpha}}^{t-\frac{\ell}{\alpha}} G_{(n-2)}(\gamma)d\gamma$$

$$f_1(t)=\frac{\alpha}{\alpha+\beta}\cdot\delta(\ell-\alpha t-x_3^0)$$

$$f_{(n-1)}(t)=\frac{\alpha}{\alpha+\beta}[g_{(n-2)}\left(t-\frac{\ell}{\alpha}\right)-g_{(n-2)}\left(-\frac{\ell}{\alpha}\right)]$$

By using Matlab codes, we simulate the solution of IBVP $(2.8.1)-(2.8.7)$



(a) $t=0$ (b) $t=5$ (c) $t=20$

(d) $t=35$ (e) $t=55$ (f) $t=75$

Figure 2.3 $U_k(x_3,t)$ in two layered medium

In these figures, we simulate the wave propagation in two layered elastic half space that is the first layer is located $0<x_3<40$, while the second layer is located $40<x_3<\infty$ (the boundary $\ell=40$).

In the figures, the horizontal axes x and the vertical axes y show the location and the magnitude of the wave front, respectively. In figure (a), we can see the fluctuation arising from the pulse point source $x_3^0=10$, described by the function $\varphi(x_3)=\delta(x_3-x_3^0)$. In the figure (b), the separated waves began to move to the opposite sides along the characteristics. In the figure (c), The wave front that is moving to the left, touches the boundary $x_3=0$, while time is passing. Then it turns back and starts to move to the right. This time, they both move to the right. In the figure (d), the reflected and transmitted waves can be seen after the wave front touched the boundary $x_3=40$. Notice the magnitudes of the reflected and transmitted waves, The substraction of reflected wave form the transmitted wave, gives us the previous magnitude of wave front in. And the magnitude of the reflected wave in the figure(d) has the negative sign, this is the result of that the speed of the second layer is bigger than the first one. (For more details, chapter 4)

In the figure (e), similarly the other wave front is separated into the reflected and the transmitted waves. When the reflected waves are moving to the left, the transmitted waves are moving to the right. In the figure (f), the reflected wave touches the boundary $x = 0$, it turns back and starts to move to the right similar to the figure (c). And one of the transmitted waves disappears by the time is passing.

2.9.2 *Example 2 - The Pulse Point Source is between ℓ and ∞*

Let us consider initial boundary value problem $(2.4.3)-(2.4.9)$ with its general form $(2.8.1)-(2.8.7)$ for $k=1$ and $n=2,3,\ldots$ The initial conditions $\varphi(x_3),\ \psi(x_3),\ w(x_3),\ \phi(x_3)$ have the following form

$$\varphi(x_3) = 0,$$

$$\psi(x_3) = 0,$$

$$w(x_3) = \delta(x_3 - x_3^0),$$

$$\phi(x_3) = 0.$$

where $\delta(x_3)$ is Dirac delta function, the boundary $\ell = 40$, the point source is located at $x_3^0 = 60$ and the boundary condition

$$F(x_3) = 0.$$

From the properties of Dirac delta function and the assumptions, the solution $U(x_3,t)$ of IBVP can be written as follows:

$$U(x_3,t) = \begin{cases} 0, & \text{if } (x_3,t) \in R1; \\ \frac{1}{2}\left[\delta(x_3+\beta t - x_3^0) - \delta(x_3 - \beta t - x_3^0)\right], & \text{if } (x_3,t) \in R2. \end{cases}$$

$$U(x_3,t) = \begin{cases} 0, & \text{if } (x_3,t) \in R3; \\ f\left(t + \dfrac{x_3-\ell}{\alpha}\right), & \text{if } (x_3,t) \in R4; \\ h\left(t - \dfrac{x_3-\ell}{\beta}\right) + \dfrac{1}{2}\cdot\delta(x_3+\beta t - x_3^0) & \\ -\dfrac{1}{2}\cdot\delta(-x_3+\beta t + 2\ell - x_3^0), & \text{if } (x_3,t) \in R5. \end{cases}$$

Here, the function $f(t)$ and $G(t)$ are constructed in Theorem 2.7.1, can be also written as

$$G(t) = \frac{\beta}{\alpha(\alpha+\beta)} \frac{\partial}{\partial t} \left[\delta(\ell+\beta t - x_3^0)\right] ,$$

$$f(t) = \frac{\beta}{\alpha+\beta} \cdot \delta(\ell+\beta t - x_3^0) .$$

For $n = 2, 3, \ldots$

$$U(x_3,t) = \begin{cases} g_{(n-1)}\left(t - \frac{x_3}{\alpha}\right) + \frac{1}{2} f_{(n-1)}\left(t + \frac{x_3-\ell}{\alpha}\right) - \frac{1}{2} f_{(n-1)}\left(t - \frac{x_3+\ell}{\alpha}\right) \\ + \frac{\alpha}{2} \int_{t-\frac{x_3+\ell}{\alpha}}^{t+\frac{x_3-\ell}{\alpha}} G_{(n-1)}(\mu) d\mu, \qquad \text{if } (x_3,t) \in R(4n-2); \\ \\ g_n\left(t - \frac{x_3}{\alpha}\right) + \frac{1}{2}\left[f_{(n-1)}\left(t + \frac{x_3-\ell}{\alpha}\right) - f_{(n-1)}\left(t - \frac{x_3+\ell}{\alpha}\right)\right] \\ + \frac{\alpha}{2} \int_{t-\frac{x_3+\ell}{\alpha}}^{t+\frac{x_3-\ell}{\alpha}} G_{(n-1)}(\mu) d\mu, \qquad \text{if } (x_3,t) \in R(4n-1); \\ \\ f_n\left(t + \frac{x_3-\ell}{\alpha}\right) + \frac{1}{2} \cdot g_{(n-1)}\left(t - \frac{x_3}{\alpha}\right) \\ - \frac{1}{2} \cdot g_{(n-1)}\left(t + \frac{x_3-2\ell}{\alpha}\right), \qquad \text{if } (x_3,t) \in R(4n); \\ \\ f_n\left(t - \frac{x_3-\ell}{\beta}\right) + \frac{1}{2} \cdot \delta(x_3+\beta t - x_3^0) \\ -\frac{1}{2} \cdot \delta(-x_3+\beta t + 2\ell - x_3^0), \qquad \text{if } (x_3,t) \in R(4n+1). \end{cases}$$

Here, the function $g_n(t)$, $f_n(t)$ and $G_n(t)$ are constructed in Theorem 2.8.1, can be also written as

$$G_n(t) = \frac{1}{\alpha} \cdot f_n'(t) - \frac{1}{\alpha} \cdot g_{(n-1)}'\left(t - \frac{\ell}{\alpha}\right),$$

$$g_n(t) = \left[f_{(n-1)}\left(t - \frac{\ell}{\alpha}\right) - f_{(n-1)}\left(-\frac{\ell}{\alpha}\right)\right] + \alpha \int_{-\frac{\ell}{\alpha}}^{t-\frac{\ell}{\alpha}} G_{(n-1)}(\gamma) d\gamma,$$

$$f_n(t) = \frac{\alpha}{\alpha+\beta}[g_{(n-1)}\left(t - \frac{\ell}{\alpha}\right) - g_{(n-1)}\left(-\frac{\ell}{\alpha}\right)] + \frac{\beta}{\alpha+\beta}\delta(\ell+\beta t - x_3^0) .$$

with

$$g_1(t) = 0 ,$$

$$g_{(n-1)}(t) = \left[f_{(n-2)}\left(t - \frac{\ell}{\alpha}\right) - f_{(n-2)}\left(-\frac{\ell}{\alpha}\right) \right] + \alpha \int_{-\frac{\ell}{\alpha}}^{t-\frac{\ell}{\alpha}} G_{(n-2)}(\gamma) d\gamma,$$

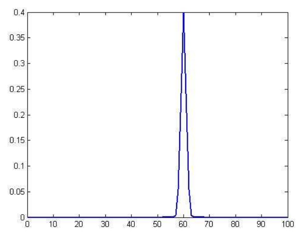
$$G_1(t) = \frac{\beta}{\alpha(\alpha+\beta)} \frac{\partial}{\partial t} \left[\delta(\ell + \beta t - x_3^0) \right] ,$$

$$G_{(n-1)}(t) = \frac{1}{\alpha} \cdot f'_{(n-1)}(t) - \frac{1}{\alpha} \cdot g'_{(n-2)}\left(t - \frac{\ell}{\alpha}\right),$$

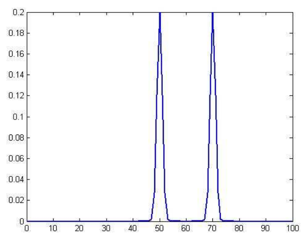
$$f_1(t) = \frac{\beta}{\alpha+\beta} \cdot \delta(\ell + \beta t - x_3^0) \ .$$

$$f_{(n-1)}(t) = \frac{\alpha}{\alpha+\beta} [g_{(n-2)}\left(t - \frac{\ell}{\alpha}\right) - g_{(n-2)}\left(-\frac{\ell}{\alpha}\right)] + \frac{\beta}{\alpha+\beta} \delta(\ell + \beta t - x_3^0) \ .$$

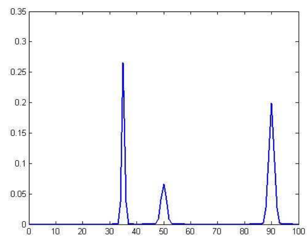
By using Matlab codes, we simulate the solution of IBVP $(2.8.1) - (2.8.7)$.



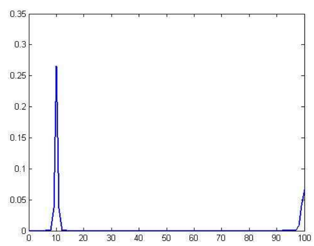
(a) $t = 0$



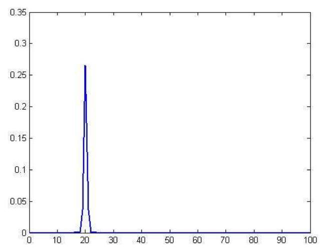
(b) $t = 5$



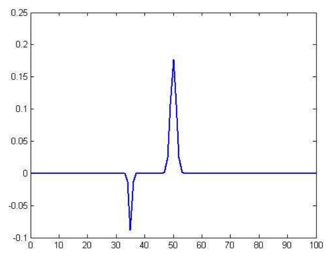
(c) $t = 15$



(d) $t = 40$



(e) $t = 70$



(f) $t = 95$

Figure 2.4 $U_k(x_3, t)$ in two layered medium

Similarly, in these figures, the boundary is located at $\ell = 40$ and the pulse point source is located outside the boundary $\ell = 40$, at $x_3^0 = 60$. In figure (a), we can see the fluctuation arising from the pulse point source $x_3^0 = 60$ described by the function $w(x_3) = \delta(x_3 - x_3^0)$. In the figure (b), the separated waves begin to move along the characteristics. In the figure (c), the reflected and transmitted waves can be seen after the wave front touches the boundary $x_3 = 40$.

In the figure(d), the reflected wave continues its movement to the right while the transmitted wave moves to the left. By the time is passing, the transmitted wave touches the boundary of the half space, then starts to move to the left in the figure(e). In the figure(f), the wave front are separated into the reflected and the transmitted waves because of the boundary located at $x_3 = 40$.

Notice that, the magnitude of the reflected wave in the figure(f) has the negative sign, this is the result of that the speed of the second layer is bigger than the first one. (For more details, chapter 4)

2.10 Conclusion of Chapter Two

- The system of elastic waves is reduced to IBVP of anisotropic layered elastic half space.
- Explicit formulae for the solution of IBVP with matching conditions has been constructed.
- Using this formulae, the simulation of wave propagation has been obtained.
- Results of the simulations have clear physical interpretation of wave propagation in two layered media from the point source.

CHAPTER THREE

INITIAL VALUE PROBLEM IN THREE LAYERED MEDIUM

Let us consider the problem (2.3.6) – (2.3.10). In this work, we omit the index k for simplicity writing. Let $(x,t) \in \mathbf{R}^2$, $\Phi(x), \Psi(x)$ and $d(x)$ have the following form,

$$d(x) = \begin{cases} d_0, & -\infty < x < 0; \\ d_1, & 0 < x < \ell; \\ d_1, & \ell < x < \infty. \end{cases} \tag{3.0.1}$$

$$\Phi(x) = \begin{cases} \varphi_0, & -\infty < x < 0; \\ \varphi_1, & 0 < x < \ell; \\ \varphi_2, & \ell < x < \infty. \end{cases} \qquad \Psi(x) = \begin{cases} \psi_0, & -\infty < x < 0; \\ \psi_1, & 0 < x < \ell; \\ \psi_2, & \ell < x < \infty. \end{cases} \tag{3.0.2}$$

where d_0, d_1, d_2 are given constants; $\varphi_0(x)$, $\varphi_1(x)$, $\varphi_2(x)$, $\psi_0(x)$, $\psi_1(x)$ and $\psi_2(x)$ are given functions depending on x.

In addition, we assume that there is no boundary condition and we have the matching conditions not only on the boundary $x = \ell$, but also on the boundary $x = 0$, as the following differential problem,

$$\frac{\partial^2 u}{\partial t^2} - d^2(x)\frac{\partial^2 u}{\partial x^2} = 0, \quad -\infty < x < 0,\ 0 < x < \ell,\ \ell < x < \infty,\ t \in \mathbf{R}, \tag{3.0.3}$$

with initial data,

$$u(x,0) = \varphi(x), \quad \frac{\partial u}{\partial t}\Big|_{t=0} = \psi(x), \quad -\infty < x < 0, 0 < x < \ell, \ell < x < \infty, \tag{3.0.4}$$

and the matching conditions,

$$u_0(x,t)\Big|_{x=-0} = u_1(x,t)\Big|_{x=+0} \tag{3.0.5}$$

$$\frac{\partial u_0}{\partial x}(x,t)\Big|_{x=-0} = \frac{\partial u_1}{\partial x}(x,t)\Big|_{x=+0} \tag{3.0.6}$$

$$u_1(x,t)\Big|_{x=\ell-0} = u_2(x,t)\Big|_{x=\ell+0} \tag{3.0.7}$$

$$\frac{\partial u_1}{\partial x}(x,t)\Big|_{x=\ell-0} = \frac{\partial u_2}{\partial x}(x,t)\Big|_{x=\ell+0} \tag{3.0.8}$$

3.1 IVP of Wave Equations in Three Layered Medium

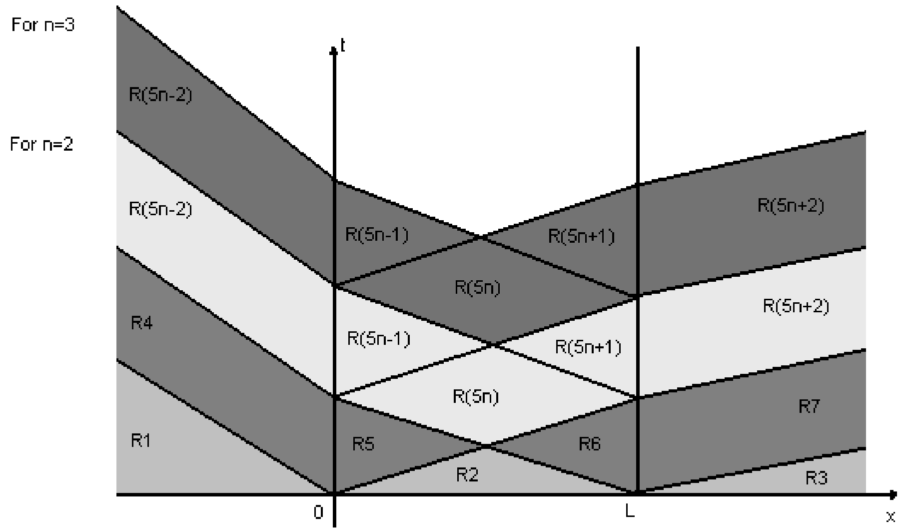


Figure 3.1 The Regions with index $n = 2, 3, 4, \ldots$

Initial value problem $(3.0.3)-(3.0.8)$ may be written in the term of

$$u(x,t) = \begin{cases} u_0(x,t), & -\infty < x < 0; \\ u_1(x,t), & 0 < x < \ell; \\ u_2(x,t), & \ell < x < \infty. \end{cases} \tag{3.1.1}$$

as follows

$$\frac{\partial^2 u_0}{\partial t^2} - d_0^2 \frac{\partial^2 u_0}{\partial x^2} = 0, \qquad -\infty < x < 0, \quad t \in \mathbf{R}, \tag{3.1.2}$$

$$\frac{\partial^2 u_1}{\partial t^2} - d_1^2 \frac{\partial^2 u_1}{\partial x^2} = 0, \qquad 0 < x < \ell, \qquad t \in \mathbf{R}, \tag{3.1.3}$$

$$\frac{\partial^2 u_2}{\partial t^2} - d_2^2 \frac{\partial^2 u_2}{\partial x^2} = 0, \qquad \ell < x < \infty, \quad t \in \mathbf{R}, \tag{3.1.4}$$

with initial data,

$$u_0(x,0) = \varphi_0(x), \quad \frac{\partial u_0}{\partial t}\Big|_{t=0} = \psi_0(x), \qquad -\infty < x < 0, \tag{3.1.5}$$

$$u_1(x,0) = \varphi_1(x), \quad \frac{\partial u_1}{\partial t}\Big|_{t=0} = \psi_1(x), \qquad 0 < x < \ell, \tag{3.1.6}$$

$$u_2(x,0)=\varphi_2(x), \quad \frac{\partial u_2}{\partial t}\Big|_{t=0}=\psi_2(x), \qquad \ell<x<\infty, \tag{3.1.7}$$

the matching conditions firstly defined on the boundary $x=0$,

$$u_0(x,t)\Big|_{x=-0}=u_1(x,t)\Big|_{x=+0} \tag{3.1.8}$$

$$d_0^2\frac{\partial u_0}{\partial x}(x,t)\Big|_{x=0-}=d_1^2\frac{\partial u_1}{\partial x}(x,t)\Big|_{x=0+} \tag{3.1.9}$$

and also defined on the boundary $x=\ell$,

$$u_1(x,t)\Big|_{x=\ell-0}=u_2(x,t)\Big|_{x=\ell+0} \tag{3.1.10}$$

$$d_1^2\frac{\partial u_1}{\partial x}(x,t)\Big|_{x=\ell-0}=d_2^2\frac{\partial u_2}{\partial x}(x,t)\Big|_{x=\ell+0} \tag{3.1.11}$$

3.2 Construction of the Solution

Similar to the previous chapter, to find the solution, we separate half space into subregions and the solution of the problem $(3.1.2)-(3.1.11)$ is investigated in these subregions, independently by using *the method of characteristics.*

$$u(x,t)=\left\{\; u_m(x,t), \quad \text{if } (x,t)\in R_m\right. \tag{3.2.1}$$

Here, the index m denotes the number of subregion.

3.3 Zero Step

Zero step includes the regions R1, R2 and R3 (see, Figure 3.4) Let us consider the problem $(3.1.2)-(3.1.11)$ for zero step. Notice that in this step we use only initial conditions.

Theorem 3.3.1. *Let* $\Phi(x)$ *and* $\Psi(x)$ *are given continuous functions as in the form* $(3.0.2)$ *depending on x;* $u(x,t)$ *is unknown function as in the form* $(3.1.1)$. *Then the solution of the problem* $(3.1.2)-(3.1.11)$ *for zero step is the following,*

$$u(x,t) = \begin{cases} \frac{1}{2}\Big[\varphi_0(x+d_0t)+\varphi_0(x-d_0t)\Big] \\ +\frac{1}{2d_0}\int_{x-d_0t}^{x+d_0t}\psi_0(\gamma)d\gamma, & \mathit{if}\ (x,t)\in R1; \\ \\ \frac{1}{2}\Big[\varphi_1(x+d_1t)+\varphi_1(x-d_1t)\Big] \\ +\frac{1}{2d_1}\int_{x-d_1t}^{x+d_1t}\psi_1(\gamma)d\gamma\,, & \mathit{if}\ (x,t)\in R2; \\ \\ \frac{1}{2}\Big[\varphi_2(x+d_2t)+\varphi_2(x-d_2t)\Big] \\ +\frac{1}{2d_2}\int_{x-d_2t}^{x+d_2t}\psi_2(\gamma)d\gamma\,, & \mathit{if}\ (x,t)\in R3. \end{cases} \tag{3.3.1}$$

where

$$R1 = \left\{(x,t)\Big|\ -\infty < x < 0\,,\quad t < \frac{-x}{d_0}\right\}$$

$$R2 = \left\{(x,t)\Big|\ 0 < x < \ell\,,\quad t < -\frac{x}{d_1}\ \wedge\ t < \frac{\ell-x}{d_1}\right\}$$

$$R3 = \left\{(x,t)\Big|\ \ell < x < \infty\,,\quad t < \frac{x-\ell}{d_2}\right\}$$

Proof. Let us consider the problem (3.1.2) – (3.1.4) with the initial data (3.1.5) – (3.1.7) in the form, for each of the regions R1, R2 and R3,

$$\frac{\partial^2 u_i}{\partial t^2} - c_i^2\frac{\partial^2 u_i}{\partial x^2} = 0\,,\qquad i=0,1,2$$

$$u_i(x,0)=\varphi_i(x),\qquad \frac{\partial u_i}{\partial t}(x,0)=\psi_i(x)\qquad i=0,1,2\,.$$

If we rewrite the first equation as the following

$$\frac{\partial q_i}{\partial t} - d_i\frac{\partial q_i}{\partial x} = 0\,,\quad (x,t)\in R(i),\quad i=0,1,2.$$

$$\frac{\partial u_i}{\partial t} + d_i\frac{\partial u_i}{\partial x} = q_i(x,t)\quad (x,t)\in R(i),\quad i=0,1,2.$$

The characteristics of the equations are respectively,

$$\frac{d\xi}{d\tau} = -d_i\,,\quad \xi(t)=x\quad ;\quad \xi=-d_i\tau+x+d_it,\ \ i=0,1,2.$$

$$\frac{d\xi}{d\tau} = d_i\,,\quad \xi(t)=x\quad ;\quad \xi=d_i\tau+x-d_it,,\ \ i=0,1,2.$$

By integrating along the characteristics, we get the following

$$q_i(x,t) = \psi_i(x+d_it) + d_i\varphi_i'(x+d_it), \quad i = 0,1,2.$$

and

$$\int_0^t \frac{\partial}{\partial\tau}\Big[u_i(x-d_i(t-\tau),\tau)\Big]d\tau = d_i\int_0^t \varphi_i'(x-d_it+2d_i\tau)d\tau$$
$$+\int_0^t \psi_i(x-d_it+2d_i\tau)d\tau, \quad i=0,1,2.$$

Let

$$x-d_it+2d_i\tau = \gamma\,, \qquad 2d_id\tau = d\gamma$$
$$\gamma_{low} = x-d_it\,, \qquad \gamma_{up} = x+d_it$$

By substituting the initial conditions, we get

$$u_i(x,t) = \frac{1}{2}\Big[\varphi_i(x+d_it)+\varphi_i(x-d_it)\Big] + \frac{1}{2d_2}\int_{x-d_it}^{x+d_it}\psi_i(\gamma)d\gamma \tag{3.3.2}$$

where i=0,1,2. Hence,

$$u_0(x,t) = \frac{1}{2}\Big[\varphi_0(x+d_0t)+\varphi_0(x-d_0t)\Big]$$
$$+\frac{1}{2d_0}\int_{x-d_0t}^{x+d_0t}\psi_0(\gamma)d\gamma, \qquad (x,t)\in R1\,,$$

$$u_1(x,t) = \frac{1}{2}\Big[\varphi_1(x+d_1t)+\varphi_1(x-d_1t)\Big]$$
$$+\frac{1}{2d_1}\int_{x-d_1t}^{x+d_1t}\psi_1(\gamma)d\gamma, \qquad (x,t)\in R2,$$

$$u_2(x,t) = \frac{1}{2}\Big[\varphi_2(x+d_2t)+\varphi_2(x-d_2t)\Big]$$
$$+\frac{1}{2d_2}\int_{x-d_2t}^{x+d_2t}\psi_2(\gamma)d\gamma, \quad (x,t)\in R3.$$

□

3.4 The First Step

The first step includes the regions R4, R5, R6 and R7 (see, Figure 3.4). In this step, we consider initial data and also matching conditions defined on the boundaries $x = 0$ and $x = \ell$. Before finding the solution for the first step, we must define the following functions,

$$u(0,t) = g(t), \quad u(\ell,t) = h(t), \quad \frac{\partial u}{\partial x}\bigg|_{x=0} = G(t) \quad \text{and} \quad \frac{\partial u}{\partial x}\bigg|_{x=\ell} = H(t)\,. \tag{3.4.1}$$

We must construct these functions by initial data and the matching conditions.

Theorem 3.4.1. *Let* $\Phi(x)$, $\Psi(x)$ *be given continuous functions depending on x in the form* (3.0.2)*;* $u(x,t)$ *is unknown function in the form* (3.1.1). *Then the solution of the problem* (3.1.2) – (3.1.11) *for the first step is the following,*

$$u(x,t) = \begin{cases} g\left(t + \dfrac{x}{d_0}\right) + \dfrac{1}{2}\left[\varphi_0(x - d_0 t) - \varphi_0(-x + d_0 t)\right] \\ -\dfrac{1}{2d_0}\displaystyle\int_{-x-d_0t}^{x-d_0t} \psi_0(\mu)d\mu, \qquad \textit{if } (x,t) \in R4; \\ \\ g\left(t - \dfrac{x}{d_1}\right) + \dfrac{1}{2}\left[\varphi_1(x + d_1 t) - \varphi_1(-x + d_1 t)\right] \\ +\dfrac{1}{2d_1}\displaystyle\int_{-x+d_1t}^{x+d_1t} \psi_1(\nu)d\nu, \qquad \textit{if } (x,t) \in R5; \\ \\ h\left(t - \dfrac{\ell - x}{d_1}\right) + \dfrac{1}{2}\left[\varphi_1(x - d_1 t) - \varphi_1(-x - d_1 t + 2\ell)\right] \\ -\dfrac{1}{2d_1}\displaystyle\int_{-x-d_1t+2\ell}^{x-d_1t} \psi_1(\eta)d\eta, \qquad \textit{if } (x,t) \in R6; \\ \\ h\left(t + \dfrac{\ell - x}{d_2}\right) + \dfrac{1}{2}\left[\varphi_2(x + d_2 t) - \varphi_2(-x + d_2 t + 2\ell)\right] \\ +\dfrac{1}{2d_2}\displaystyle\int_{-x+d_2t+2\ell}^{x+d_2t} \psi_2(\nu)d\nu, \qquad \textit{if } (x,t) \in R7. \end{cases} \tag{3.4.2}$$

where

$$R4 = \left\{ (x,t) \Big| -\infty < x < 0\,, \quad -\frac{x}{d_0} < t < \frac{\ell}{d_1} - \frac{x}{d_0} \right\}$$

$$R5 = \left\{ (x,t) \Big| \quad 0 < x < \ell \quad \textit{and} \quad \frac{x}{d_1} < t < \frac{\ell - x}{d_1} \right\}$$

$$R6 = \left\{ (x,t) \Big| \quad 0 < x < \ell \quad and \quad \frac{\ell - x}{d_1} < t < \frac{x}{d_1} \right\}$$

$$R7 = \left\{ (x,t) \Big| \quad \ell < x < \infty \quad and \quad \frac{x-\ell}{d_2} < t < \frac{x-\ell}{d_2} + \frac{\ell}{d_1} \right\}$$

and the functions defined in $(3.4.1)$ *are constructed by initial data and the matching conditions as follows*

$$G(t) = -\frac{1}{d_1} g'(t) + \varphi_1'(d_1 t) + \frac{1}{d_1} \psi_1(d_1 t) \tag{3.4.3}$$

$$H(t) = \frac{1}{d_1} h'(t) + \varphi_1'(\ell - d_1 t) - \frac{1}{d_1} \psi_1(\ell - d_1 t) \tag{3.4.4}$$

$$\begin{aligned} g(t) = & \frac{d_0}{d_0+d_1}(\varphi_0(-d_0 t) - \varphi_0(0)) + \frac{d_1}{d_0+d_1}(\varphi_1(d_1 t) - \varphi_1(0)) \\ & - \frac{1}{d_0+d_1}\int_0^{-d_0 t} \psi_0(s)ds + \frac{1}{d_0+d_1}\int_0^{d_1 t} \psi_1(z)dz \end{aligned} \tag{3.4.5}$$

$$\begin{aligned} h(t) = & \frac{d_1}{d_1+d_2}[\varphi_1(\ell - d_1 t) - \varphi_1(\ell)] + \frac{d_2}{d_1+d_2}[\varphi_2(\ell + d_2 t) - \varphi_2(\ell)] \\ & - \frac{1}{d_1+d_2}\int_\ell^{\ell - d_1 t} \psi_1(s)ds + \frac{1}{d_1+d_2}\int_\ell^{\ell + d_2 t} \psi_2(z)dz \end{aligned} \tag{3.4.6}$$

Proof. Let us consider the problem $(3.1.2) - (3.1.4)$ with initial data $(3.1.5) - (3.1.7)$ and the matching conditions $(3.1.8) - (3.1.11)$ in the regions R4, R5, R6 and R7 respectively.

Notice that, since $u(x,t)$ is defined as in $(3.1.1)$, then the functions, defined in $(3.4.1)$, can be written as follows Now, we analyze the regions, independently.

3.4.1 The Region R4

Let us consider the problem $(3.1.2) - (3.1.11)$ in the region R4 (see, Figure 3.4),

$$R4 = \left\{ (x,t) \Big| -\infty < x < 0\,, \quad -\frac{x}{d_0} < t < \frac{\ell}{d_1} - \frac{x}{d_0} \right\}$$

The equation $(3.1.2)$ can be written as in the form,

$$\frac{\partial q_0}{\partial t} + d_0 \frac{\partial q_0}{\partial x} = 0\,, \quad (x,t) \in R4, \tag{3.4.7}$$

$$\frac{\partial u_0}{\partial t} - d_0 \frac{\partial u_0}{\partial x} = q_0(x,t)\,, \quad (x,t) \in R4. \tag{3.4.8}$$

The characteristic of the equation (3.4.7) – (3.4.8) are respectively,

$$\frac{d\xi}{d\tau} = d_0 \ , \quad \xi(t) = x \quad ; \quad \xi = d_0\tau + x - d_0 t \ ,$$

$$\frac{d\xi}{d\tau} = -d_0 \ , \quad \xi(t) = x \quad ; \quad \xi = -d_0\tau + x + d_0 t \quad \text{and if} \quad \xi = 0 \ ; \quad \tau = t + \frac{x}{d_0} \ .$$

By integrating along the characteristics,

$$q_0(x,t) = \psi_0(x - d_0 t) - d_0 \varphi_0'(x - d_0 t)$$

Then by integrating along the characteristic,

$$u_0(x,t) - u_0\left(0, t + \frac{x}{d_0}\right) = \int_{t+\frac{x}{d_0}}^{t} \psi_0(x + d_0 t - 2d_0\tau) d\tau$$

$$-d_0 \int_{t+\frac{x}{d_0}}^{t} \varphi_0'(x + d_0 t - 2d_0\tau) d\tau \ ,$$

Let

$$x + d_0 t - 2d_0\tau = \mu \ , \qquad -2d_0 d\tau = d\mu$$

$$\mu_{low} = -x - d_0 t \ , \qquad \mu_{up} = x - d_0 t$$

By substituting the initial conditions (3.1.5), we have the solution and by the function $g(t)$ defined in (3.4.1)

$$u_0(x,t) = g\left(t + \frac{x}{d_0}\right) + \frac{1}{2}\left[\varphi_0(x - d_0 t) - \varphi_0(-x - d_0 t)\right]$$

$$-\frac{1}{2d_0} \int_{-x-d_0 t}^{x-d_0 t} \psi_0(\mu) d\mu \ , \quad (x,t) \in R4,$$

3.4.2 The Region R5

Let us consider the problem (3.1.2) – (3.1.11) in the region R5 (see, Figure 3.4),

$$R5 = \left\{ (x,t) \Big| \quad 0 < x < \ell \quad \text{and} \quad \frac{x}{d_1} < t < \frac{\ell - x}{d_1} \right\}$$

The equation (3.1.3) can be written as in the form,

$$\frac{\partial q_1}{\partial t} - d_1 \frac{\partial q_1}{\partial x} = 0 \ , \quad (x,t) \in R5, \tag{3.4.9}$$

$$\frac{\partial u_1}{\partial t} + d_1 \frac{\partial u_1}{\partial x} = q_1(x,t) \ , \quad (x,t) \in R5. \tag{3.4.10}$$

The characteristic of the equation (3.4.9) – (3.4.10) are respectively,

$$\frac{d\xi}{d\tau} = -d_1 \ , \quad \xi(t) = x \quad ; \quad \xi = -d_1\tau + x + d_1 t \ ,$$

$$\frac{d\xi}{d\tau} = d_1 \ , \quad \xi(t) = x \quad ; \quad \xi = d_1\tau + x - d_1 t \quad \text{and if} \quad \xi = 0\,; \quad \tau = t - \frac{x}{d_1} \ .$$

Similarly, by integrating along the characteristics,

$$q_1(x,t) = \psi_1(x + d_1 t) + d_1 \varphi_1'(x + d_1 t)$$

By the same way in the region R4 and the function $g(t)$ defined in (3.4.1), we get

$$u_1(x,t) = g\left(t - \frac{x}{d_1}\right) + \frac{1}{2}\left[\varphi_1(x + d_1 t) - \varphi_1(-x + d_1 t)\right]$$

$$+ \frac{1}{2d_1}\int_{-x+d_1 t}^{x+d_1 t} \psi_1(\nu)d\nu \ , \quad (x,t) \in R5,$$

To find the functions defined on (3.4.1), we must apply the matching conditions between R4 and R5.

3.4.3 Matching Conditions Between R4 and R5

The formula for the region R4 is in the form,

$$u_0(x,t) = g\left(t + \frac{x}{d_0}\right) + \frac{1}{2}\left[\varphi_0(x - d_0 t) - \varphi_0(-x - d_0 t)\right]$$

$$- \frac{1}{2d_0}\int_{-x-d_0 t}^{x-d_0 t} \psi_0(\mu)d\mu \ , \quad (x,t) \in R4,$$

and the formula for the region R5 is in the form,

$$u_1(x,t) = g\left(t - \frac{x}{d_1}\right) + \frac{1}{2}\left[\varphi_1(x + d_1 t) - \varphi_1(-x + d_1 t)\right]$$

$$+ \frac{1}{2d_1}\int_{-x+d_1 t}^{x+d_1 t} \psi_1(\nu)d\nu \ , \quad (x,t) \in R5,$$

By the first matching condition (3.1.8), we have,

$$u(-0,t) = u(+0,t) = g(t)$$

To use the second matching condition (3.1.9), we must differentiate the formulas for the regions R4 and R5, and substitute $x = 0$. Then we get the function $G(t)$ defined in (3.4.1),

$$G(t) = -\frac{1}{d_1}g'(t) + \varphi_1'(d_1 t) + \frac{1}{d_1}\psi_1(d_1 t)$$

By using the second matching condition (3.1.9) And we get the function $g(t)$ as follows,

$$g(t) = \frac{d_0}{d_0+d_1}(\varphi_0(-d_0 t) - \varphi_0(0)) + \frac{d_1}{d_0+d_1}(\varphi_1(d_1 t) - \varphi_1(0))$$

$$-\frac{1}{d_0+d_1}\int_0^{-d_0 t}\psi_0(s)ds + \frac{1}{d_0+d_1}\int_0^{d_1 t}\psi_1(z)dz$$

3.4.4 The Region R6

Let us consider the problem (3.1.2) – (3.1.11) in the region R6 (see, Figure 3.4),

$$R6 = \left\{ (x,t) \Big| \quad 0 < x < \ell \quad \text{and} \quad \frac{\ell - x}{d_1} < t < \frac{x}{d_1} \right\}$$

The equation (3.1.3) can be written as in the form,

$$\frac{\partial q_1}{\partial t} + d_1\frac{\partial q_1}{\partial x} = 0\,, \quad (x,t) \in R6, \tag{3.4.11}$$

$$\frac{\partial u_1}{\partial t} - d_1\frac{\partial u_1}{\partial x} = q_1(x,t)\,, \quad (x,t) \in R6. \tag{3.4.12}$$

The characteristic of the equation (3.4.11) – (3.4.12) are respectively,

$$\frac{d\xi}{d\tau} = d_1\,, \quad \xi(t) = x \quad ; \quad \xi = d_1\tau + x - d_1 t\,,$$

$$\frac{d\xi}{d\tau} = -d_1\,, \quad \xi(t) = x \quad ; \quad \xi = -d_1\tau + x + d_1 t \quad \text{and if} \quad \xi = \ell; \quad \tau = t - \frac{\ell - x}{d_1}\,.$$

Similarly, by integrating along the characteristics,

$$q_1(x,t) = \psi_1(x - d_1 t) - d_1\varphi_1'(x - d_1 t)$$

By the same way in the region R4 and the function $h(t)$ defined in $(3.4.1)$, we get

$$u_1(x,t) = h\left(t - \frac{\ell - x}{d_1}\right) + \frac{1}{2}\left[\varphi_1(x - d_1 t) - \varphi_1(-x - d_1 t + 2\ell)\right]$$

$$-\frac{1}{2d_1}\int_{-x-d_1t+2\ell}^{x-d_1t} \psi_1(\eta)d\eta \ , \quad (x,t) \in R6,$$

3.4.5 The Region R7

Let us consider the problem $(3.1.2) - (3.1.11)$ in the region R7 (see, Figure 3.4),

$$R7 = \left\{ (x,t) \middle| \quad \ell < x < \infty \quad \text{and} \quad \frac{x-\ell}{d_2} < t < \frac{x-\ell}{d_2} + \frac{\ell}{d_1} \right\}$$

The equation $(3.1.4)$ can be written as in the form,

$$\frac{\partial q_2}{\partial t} - d_2\frac{\partial q_2}{\partial x} = 0 \ , \quad (x,t) \in R7, \tag{3.4.13}$$

$$\frac{\partial u_2}{\partial t} + d_2\frac{\partial u_2}{\partial x} = q_2(x,t) \ , \quad (x,t) \in R7. \tag{3.4.14}$$

The characteristic of the equation $(3.4.13) - (3.4.14)$ are respectively,

$$\frac{d\xi}{d\tau} = -d_2 \ , \quad \xi(t) = x \quad ; \quad \xi = -d_2\tau + x + d_2 t \ ,$$

$$\frac{d\xi}{d\tau} = d_2 \ , \quad \xi(t) = x \quad ; \quad \xi = d_2\tau + x - d_2 t \quad \text{and if} \quad \xi = \ell \ ; \quad \tau = t + \frac{\ell - x}{d_2} \ .$$

Similarly, by integrating along the characteristics,

$$q_2(x,t) = \psi_2(x + d_2 t) + d_2\varphi_2'(x + d_2 t)$$

By the same way in the region R4 and the function $h(t)$ defined in $(3.4.1)$, we get

$$u_2(x,t) = h\left(t + \frac{\ell - x}{d_2}\right) + \frac{1}{2}\left[\varphi_2(x + d_2 t) - \varphi_2(-x + d_2 t + 2\ell)\right]$$

$$+\frac{1}{2d_2}\int_{-x+d_2t+2\ell}^{x+d_2t} \psi_2(\nu)d\nu \ , \quad (x,t) \in R7,$$

To find the functions defined on $(3.4.1)$, we must apply the matching conditions between R6 and R7.

3.4.6 Matching Conditions Between R6 and R7

The formula for the region R6 is in the form,

$$u_1(x,t) = h\left(t - \frac{\ell - x}{d_1}\right) + \frac{1}{2}\left[\varphi_1(x - d_1 t) - \varphi_1(-x - d_1 t + 2\ell)\right]$$

$$-\frac{1}{2d_1}\int_{-x-d_1t+2\ell}^{x-d_1t} \psi_1(\eta)d\eta \ , \quad (x,t) \in R6,$$

and the formula for the region R7 is in the form,

$$u_2(x,t) = h\left(t + \frac{\ell - x}{d_2}\right) + \frac{1}{2}\left[\varphi_2(x + d_2 t) - \varphi_2(-x + d_2 t + 2\ell)\right]$$

$$+\frac{1}{2d_2}\int_{-x+d_2t+2\ell}^{x+d_2t} \psi_2(\nu)d\nu \ , \quad (x,t) \in R7,$$

By the first matching condition (3.1.10), we have,

$$u(\ell - 0, t) = u(\ell + 0, t) = h(t)$$

To use the second matching condition (3.1.11), we must differentiate the formulas for the regions R6 and R7, and substitute $x = \ell$. Then we get the function $H(t)$ defined in (3.4.1),

$$H(t) = \frac{1}{d_1}h'(t) + \varphi_1'(\ell - d_1 t) - \frac{1}{d_1}\psi_1(\ell - d_1 t)$$

By using the second matching condition (3.1.11) And we get the function $h(t)$ as follows,

$$h(t) = \frac{d_1}{d_1 + d_2}\left[\varphi_1(\ell - d_1 t) - \varphi_1(\ell)\right] + \frac{d_2}{d_1 + d_2}\left[\varphi_2(\ell + d_2 t) - \varphi_2(\ell)\right]$$

$$-\frac{1}{d_1 + d_2}\int_{\ell}^{\ell - d_1 t} \psi_1(s)ds + \frac{1}{d_1 + d_2}\int_{\ell}^{\ell + d_2 t} \psi_2(z)dz$$

□

3.5 General Case

In zero and the first step, we have constructed the formulations of $u(x,t)$ and the functions $g(t)$, $h(t)$, $G(t)$, $H(t)$ defined in (3.4.1) for $n = 0$ and $n = 1$, respectively.

After the first step, we generalize the number of the step with index n, for $n = 2,3,\ldots$ So, we reformulate the initial value problem.

Initial value problem is to find $u_n(x,t)$ in the form

$$u_n(x,t) = \begin{cases} u_{0n}(x,t), & -\infty < x < 0; \\ u_{1n}(x,t), & 0 < x < \ell. \\ u_{2n}(x,t), & \ell < x < \infty. \end{cases} \tag{3.5.1}$$

for each $n = 2,3,\ldots$ satisfying

$$\frac{\partial^2 u_{0n}}{\partial t^2} = d_0^2 \frac{\partial^2 u_{0n}}{\partial x^2}, \qquad -\infty < x < 0, \quad t \in \mathbf{R}, \tag{3.5.2}$$

$$\frac{\partial^2 u_{1n}}{\partial t^2} = d_1^2 \frac{\partial^2 u_{1n}}{\partial x^2}, \qquad 0 < x < \ell, \quad t \in \mathbf{R}, \tag{3.5.3}$$

$$\frac{\partial^2 u_{2n}}{\partial t^2} = d_2^2 \frac{\partial^2 u_{2n}}{\partial x^2}, \qquad \ell < x < \infty, \quad t \in \mathbf{R}, \tag{3.5.4}$$

with initial data,

$$u_{0n}(x,0) = \varphi_0(x), \qquad \frac{\partial u_{0n}}{\partial t}(x,t)\Big|_{t=0} = \psi_0(x), \quad -\infty < x < 0, \tag{3.5.5}$$

$$u_{1n}(x,0) = \varphi_1(x), \qquad \frac{\partial u_{1n}}{\partial t}(x,t)\Big|_{t=0} = \psi_1(x), \quad 0 < x < \ell, \tag{3.5.6}$$

$$u_{2n}(x,0) = \varphi_2(x), \qquad \frac{\partial u_{2n}}{\partial t}(x,t)\Big|_{t=0} = \psi_2(x), \quad \ell < x < \infty, \tag{3.5.7}$$

with the matching conditions defined on the boundary $x = 0$,

$$u_{0n}\Big|_{x=-0} = u_{0n}\Big|_{x=+0} \tag{3.5.8}$$

$$d_0^2 \frac{\partial u_{0n}}{\partial x}\Big|_{x=-0} = d_1^2 \frac{\partial u_{1n}}{\partial x}\Big|_{x=+0} \tag{3.5.9}$$

and also the matching conditions defined on the boundary $x = \ell$,

$$u_{1n}\Big|_{x=\ell-0} = u_{2n}\Big|_{x=\ell+0} \tag{3.5.10}$$

$$d_1^2 \frac{\partial u_{1n}}{\partial x}\Big|_{x=\ell-0} = d_2^2 \frac{\partial u_{2n}}{\partial x}\Big|_{x=\ell+0} \tag{3.5.11}$$

The General case includes the regions $R(5n-2)$, $R(5n-1)$, $R(5n)$, $R(5n+1)$ and $R(5n+2)$ (see, Figure 3.4). Notice that, unlike in the first step, in the general case we have an additional subregion, namely the region R(5n).

However, similar to the first step, in the general case we consider initial data and matching conditions defined on the boundaries $x=0,\ x=\ell$.

Before finding the solution for the general case, we must define the following
functions,

$$\begin{array}{ll} u_n(0,t)=g_n(t), & \left.\frac{\partial u_n}{\partial x}\right|_{x=0}=G_n(t) \\ u_n(\ell,t)=h_n(t), & \left.\frac{\partial u_n}{\partial x}\right|_{x=\ell}=H_n(t) \end{array} \tag{3.5.12}$$

We must construct these functions by initial data and also by the matching conditions. Similar to $(3.2.1)$, the solution of the problem $(3.5.2)-(3.5.11)$ will be found in the following form by using *the method of characteristics*.

$$u_n(x,t)=\left\{\ u_{nm}(x,t),\quad \text{if } (x,t)\in Rm\right. \tag{3.5.13}$$

Here, the index n denotes the number of the step and the index m denotes the number of subregion.

Theorem 3.5.1. *Let* $\Phi(x)$, $\Psi(x)$ *be given continuous functions depending on x in the form* $(3.0.2)$*;* $u_n(x,t)$ *is unknown function in the form* $(3.5.1)$*. Then the solution of the problem*

$(3.5.2)-(3.5.11)$ *for the general case is the following,*

$$u(x,t)=\begin{cases} g_n\left(t+\dfrac{x}{d_0}\right)+\dfrac{1}{2}\Big[\varphi_0(x-d_0t)-\varphi_0(-x-d_0t)\Big] \\ +\dfrac{1}{2d_0}\displaystyle\int_{x-d_0t}^{-x-d_0t}\psi_0(\mu)d\mu \qquad \textit{if } (x,t)\in R(5n-2); \\ \\ g_n\Big(t-\dfrac{x}{d_1}\Big)+\dfrac{1}{2}\Big[h_{(n-1)}\Big(t-\dfrac{\ell-x}{d_1}\Big)-h_{(n-1)}\Big(t-\dfrac{\ell+x}{d_1}\Big)\Big] \\ +\dfrac{d_1}{2}\displaystyle\int_{t-\frac{\ell+x}{d_1}}^{t-\frac{\ell-x}{d_1}}H_{(n-1)}(\eta)d\eta, \qquad \textit{if } (x,t)\in R(5n-1); \\ \\ \dfrac{1}{2}\left[g_{(n-1)}\left(t-\dfrac{x}{d_1}\right)-g_{(n-1)}\left(-\dfrac{\ell}{2d_1}\right)\right]+\dfrac{1}{2}h_{(n-1)}\left(t-\dfrac{\ell-x}{d_1}\right) \\ -\dfrac{1}{2}h_{(n-1)}\left(-\dfrac{\ell}{2d_1}\right)+\dfrac{d_1}{2}\displaystyle\int_{-\frac{\ell}{2d_1}}^{t-\frac{\ell-x}{d_1}}H_{(n-1)}(\nu)d\nu \\ -\dfrac{d_1}{2}\displaystyle\int_{-\frac{\ell}{2d_1}}^{t-\frac{x}{d_1}}G_{(n-1)}(\gamma)d\gamma, \qquad \textit{if } (x,t)\in R(5n); \\ \\ h_n(t-\dfrac{\ell-x}{d_1})+\dfrac{1}{2}\Big[g_{(n-1)}(t-\dfrac{x}{d_1})-g_{(n-1)}(t-\dfrac{2\ell-x}{d_1})\Big] \\ -\dfrac{d_1}{2}\displaystyle\int_{t-\frac{2\ell-x}{d_1}}^{t-\frac{x}{d_1}}G_{(n-1)}(\gamma)d\gamma, \qquad \textit{if } (x,t)\in R(5n+1); \\ \\ h_n\left(t+\dfrac{\ell-x}{d_2}\right)+\dfrac{1}{2}\Big[\varphi_2(x+d_2t)-\varphi_2(-x+d_2t+2\ell)\Big] \\ +\dfrac{1}{2d_2}\displaystyle\int_{-x+d_2t+2\ell}^{x+d_2t}\psi_2(\xi)d\xi, \qquad \textit{if } (x,t)\in R(5n+2). \end{cases} \tag{3.5.14}$$

where

$$R(5n+2)=\left\{(x,t)\Big|\ \ell<x<\infty\,, \quad \frac{(n-1)\ell}{d_1}<\Big(t-\frac{x-\ell}{d_2}\Big)<\frac{n\ell}{d_1}\right\}$$

$$R(5n+1)=\left\{(x,t)\Big| \quad 0<x<\ell\,, \quad \frac{n\ell-x}{d_1}<t<\frac{(n-1)\ell+x}{d_1}\right\}$$

$$R(5n)=\Big\{(x,t)\Big|\ 0<x<\ell\,,$$

$$\left.\frac{(n-2)\ell+x}{d_1}<t<\frac{n\ell-x}{d_1}\ \wedge\ \frac{(n-1)\ell-x}{d_1}<t<\frac{(n-1)\ell+x}{d_1}\right\}$$

$$R(5n-1)=\left\{(x,t)\Big| \quad 0<x<\ell\,, \quad \frac{(n-1)\ell+x}{d_1}<t<\frac{n\ell-x}{d_1}\right\}$$

$$R(5n-2)=\left\{(x,t)\Big| \quad -\infty<x<0\,, \quad \frac{(n-1)\ell}{d_1}-\frac{x}{d_0}<t<\frac{n\ell}{d_1}-\frac{x}{d_0}\right\}$$

and the functions defined in (3.5.12) *are constructed by initial data and the matching conditions as follows*

$$G_n(t) = -\frac{1}{d_1}g_n'(t) + \frac{1}{d_1}h_{(n-1)}'\left(t - \frac{\ell}{d_1}\right) + H_{(n-1)}\left(t - \frac{\ell}{d_1}\right) \tag{3.5.15}$$

$$H_n(t) = \frac{1}{d_1}h_n'(t) - \frac{1}{d_1}g_{(n-1)}'\left(t - \frac{\ell}{d_1}\right) - \frac{1}{d_1}G_{(n-1)}\left(t - \frac{\ell}{d_1}\right) \tag{3.5.16}$$

$$\begin{aligned} g_n(t) = {} & \frac{d_0}{d_0+d_1}(\varphi_0(-d_0 t) - \varphi_0(0)) + \frac{d_1}{d_0+d_1}\left[h_{(n-1)}\left(t - \frac{\ell}{d_1}\right) - h_{(n-1)}\left(-\frac{\ell}{d_1}\right)\right] \\ & - \frac{1}{d_0+d_1}\int_0^{-d_0 t}\psi_0(s)ds + \frac{d_1^2}{d_0+d_1}\int_{-\frac{\ell}{d_1}}^{t-\frac{\ell}{d_1}} H_{(n-1)}(z)dz \end{aligned} \tag{3.5.17}$$

$$\begin{aligned} h_n(t) = {} & \frac{d_1}{d_1+d_2}[g_{(n-1)}\left(t - \frac{\ell}{d_1}\right) - g_{(n-1)}\left(-\frac{\ell}{d_1}\right)] + \frac{d_2}{d_1+d_2}[\varphi_2(\ell+d_2 t) - \varphi_2(\ell)] \\ & - \frac{d_1^2}{d_1+d_2}\int_{-\frac{\ell}{d_1}}^{t-\frac{\ell}{d_1}} G_{(n-1)}(z)dz + \frac{1}{d_1+d_2}\int_{\ell}^{\ell+d_2 t}\psi_2(s)ds \end{aligned} \tag{3.5.18}$$

Proof. In zero and the first step, we constructed the formulations of $u(x,t)$ and the functions, defined in (3.4.1), for $n=0$ and $n=1$, respectively. In the general step, we reformulate initial value problem $(3.1.2)-(3.1.11)$ with the index n, for $n = 2, 3, \dots$

In some regions, namely in $R(5n-1)$, $R(5n)$, $R(5n+1)$, we do not use the initial conditions. Instead we use the functions defined in (3.5.12), in the form of recurrence relations.

3.5.1 The Region $R(5n-2)$

Let us consider the problem $(3.5.2)-(3.5.11)$ in the region $R(5n-2)$ (see, Figure 3.4),

$$R(5n-2) = \left\{(x,t)\Big| \quad -\infty < x < 0\,, \quad \frac{(n-1)\ell}{d_1} - \frac{x}{d_0} < t < \frac{n\ell}{d_1} - \frac{x}{d_0}\right\}$$

The equation (3.5.2) can be written as in the form,

$$\frac{\partial q_{0n}}{\partial t} + d_0\frac{\partial q_{0n}}{\partial x} = 0\,, \quad (x,t) \in R(5n-2), \tag{3.5.19}$$

$$\frac{\partial u_{0n}}{\partial t} - d_0\frac{\partial u_{0n}}{\partial x} = q_{0n}(x,t)\,, \quad (x,t) \in R(5n-2). \tag{3.5.20}$$

The characteristic of the equation (3.5.19) – (3.5.20) are respectively,

$$\frac{d\xi}{d\tau} = d_0\ , \quad \xi(t) = x \quad ; \quad \xi = d_0\tau + x - d_0t\ ,$$

$$\frac{d\xi}{d\tau} = -d_0\ , \quad \xi(t) = x \quad ; \quad \xi = -d_0\tau + x + d_0t \quad \text{and if} \quad \xi = 0\ ; \quad \tau = t + \frac{x}{d_0}\ .$$

By integrating along the characteristics,

$$q_{0n}(x,t) = \psi_0(x - d_0t) - d_0\varphi_0'(x - d_0t)$$

Then by integrating along the characteristic,

$$u_{0n}(x,t) - u_{0n}\left(0, t + \frac{x}{d_0}\right) = \int_{t+\frac{x}{d_0}}^{t} \psi_0(x + d_0t - 2d_0\tau)d\tau$$

$$-d_0 \int_{t+\frac{x}{d_0}}^{t} \varphi_0'(x + d_0t - 2d_0\tau)d\tau\ ,$$

Let

$$x + d_0t - 2d_0\tau = \mu\ , \qquad -2d_0d\tau = d\mu$$

$$\mu_{low} = -x - d_0t\ , \qquad \mu_{up} = x - d_0t$$

By substituting the initial conditions (3.5.5), we have the solution and by the function $g_n(t)$ defined in (3.5.12)

$$u_{0n}(x,t) = g_n\left(t + \frac{x}{d_0}\right) + \frac{1}{2}\left[\varphi_0(x - d_0t) - \varphi_0(-x - d_0t)\right]$$

$$-\frac{1}{2d_0}\int_{-x-d_0t}^{x-d_0t} \psi_0(\mu)d\mu\ , \quad (x,t) \in R(5n-2).$$

3.5.2 The Region $R(5n-1)$

Let us consider the problem (3.5.2) – (3.5.11) in the region $R(5n-1)$ (see, Figure 3.4),

$$R(5n-1) = \left\{(x,t)\middle| \quad 0 < x < \ell\ , \quad \frac{(n-1)\ell + x}{d_1} < t < \frac{n\ell - x}{d_1}\right\}$$

The equation (3.5.3) can be written as in the form,

$$\frac{\partial q_{1n}}{\partial t} - d_1\frac{\partial q_{1n}}{\partial x} = 0\ , \quad (x,t) \in R(5n-1), \tag{3.5.21}$$

$$\frac{\partial u_{1n}}{\partial t}+d_1\frac{\partial u_{1n}}{\partial x}=q_{1n}(x,t)\ ,\quad (x,t)\in R(5n-1). \tag{3.5.22}$$

The characteristic of the equation $(3.5.21)-(3.5.22)$ are respectively,

$$\frac{d\xi}{d\tau}=-d_1\ ,\quad \xi(t)=x\quad ;\quad \xi=-d_1\tau+x+d_1t\quad \text{and if}\quad \xi=\ell\,;\quad \tau=t-\frac{\ell-x}{d_1}\ ,$$

$$\frac{d\xi}{d\tau}=d_1\ ,\quad \xi(t)=x\quad ;\quad \xi=d_1\tau+x-d_1t\quad \text{and if}\quad \xi=0\,;\quad \tau=t-\frac{x}{d_1}\ .$$

Similarly, by integrating along the characteristics,

$$q_{1n}(x,t)=h'_{(n-1)}\Big(t-\frac{\ell-x}{d_1}\Big)+d_1H_{(n-1)}\Big(t-\frac{\ell-x}{d_1}\Big)$$

By the same way in the region $R(5n-2)$ and the function $g_n(t)$ defined in (**??**), we get

$$u_{1n}(x,t)=g_n\Big(t-\frac{x}{d_1}\Big)+\frac{1}{2}\Big[h_{(n-1)}\Big(t-\frac{\ell-x}{d_1}\Big)-h_{(n-1)}\Big(t-\frac{\ell+x}{d_1}\Big)\Big]$$

$$+\frac{d_1}{2}\int_{t-\frac{\ell+x}{d_1}}^{t-\frac{\ell-x}{d_1}}H_{(n-1)}(\eta)d\eta\ ,\quad (x,t)\in R(5n-1)$$

To find the functions defined on (3.5.12), we must apply the matching conditions between $R(5n-2)$ and $R(5n-1)$.

3.5.3 Matching Conditions Between $R(5n-2)$ *and* $R(5n-1)$

The formula for the region $R(5n-2)$ is in the form,

$$u_{0n}(x,t)=g_n\left(t+\frac{x}{d_0}\right)+\frac{1}{2}\left[\varphi_0(x-d_0t)-\varphi_0(-x-d_0t)\right]$$

$$-\frac{1}{2d_0}\int_{-x-d_0t}^{x-d_0t}\psi_0(\mu)d\mu\ ,\quad (x,t)\in R(5n-2),$$

and the formula for the region $R(5n-1)$ is in the form,

$$u_{1n}(x,t)=g_n\Big(t-\frac{x}{d_1}\Big)+\frac{1}{2}\Big[h_{(n-1)}\Big(t-\frac{\ell-x}{d_1}\Big)-h_{(n-1)}\Big(t-\frac{\ell+x}{d_1}\Big)\Big]$$

$$+\frac{d_1}{2}\int_{t-\frac{\ell+x}{d_1}}^{t-\frac{\ell-x}{d_1}}H_{(n-1)}(\eta)d\eta\ ,\quad (x,t)\in R(5n-1).$$

By the first matching condition (3.5.8), we have,

$$u_{0n}(-0,t) = u_{1n}(+0,t) = g_n(t)$$

To use the second matching condition (3.5.9), we must differentiate the formulas for the regions $R(5n-2)$ and $R(5n-1)$, and substitute $x=0$. Then we get the function $G_n(t)$ defined in (3.5.12),

$$G_n(t) = -\frac{1}{d_1}g_n'(t) + \frac{1}{d_1}h'_{(n-1)}\left(t - \frac{\ell}{d_1}\right) + H_{(n-1)}\left(t - \frac{\ell}{d_1}\right)$$

By using the second matching condition (3.5.9) And we get the function $g_n(t)$ as follows,

$$g_n(t) = \frac{d_0}{d_0+d_1}(\varphi_0(-d_0t) - \varphi_0(0)) + \frac{d_1}{d_0+d_1}\left[h_1\left(t - \frac{\ell}{d_1}\right) - h_1\left(-\frac{\ell}{d_1}\right)\right]$$

$$-\frac{1}{d_0+d_1}\int_0^{-d_0t}\psi_0(s)ds + \frac{d_1^2}{d_0+d_1}\int_{-\frac{\ell}{d_1}}^{t-\frac{\ell}{d_1}} H_{(n-1)}(z)dz$$

3.5.4 The Region $R(5n)$

Similar to the previous chapter, there is a region, namely the region $R(5n)$, in the general case has a different form. (see Figure 3.5)

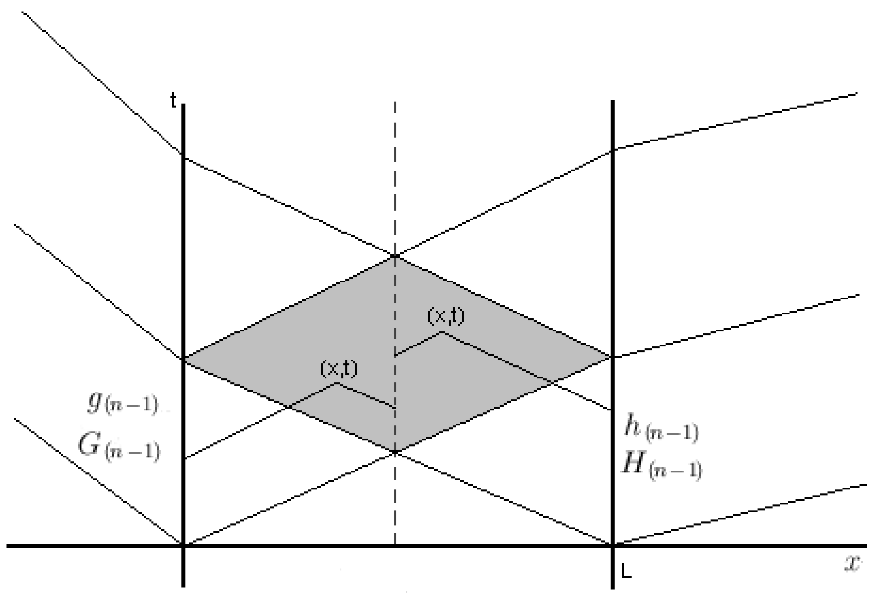


Figure 3.2 The Region R(5n)

In this region, we use the functions $g_{(n-1)}$, $h_{(n-1)}$, $G_{(n-1)}$ and $H_{(n-1)}$ which we must found in the previous step.

We assume that there is a jump at $x = \frac{\ell}{2}$. We will apply the following matching conditions when the speeds are the same.

$$u_{1n}(x,t)\Big|_{x=\frac{\ell}{2}-0} = u_{1n}(x,t)\Big|_{x=\frac{\ell}{2}+0} \tag{3.5.23}$$

$$d_1^2 \frac{\partial u_{1n}}{\partial x}\Big|_{x=\frac{\ell}{2}-0} = d_1^2 \frac{\partial u_{1n}}{\partial x}\Big|_{x=\frac{\ell}{2}+0} \tag{3.5.24}$$

Let us consider the problem $(3.5.2)-(3.5.11)$ in the region $R(5n)$, for $n = 2,3,\dots$

$$R(5n) = \Big\{(x,t)\Big|\ 0 < x < \ell\ ,$$

$$\frac{(n-2)\ell + x}{d_1} < t < \frac{n\ell - x}{d_1}\ \wedge\ \frac{(n-1)\ell - x}{d_1} < t < \frac{(n-1)\ell + x}{d_1}\Big\}$$

The equation $(3.5.3)$ can be written as in the form,

$$\frac{\partial q_{1n}}{\partial t} + d_1 \frac{\partial q_{1n}}{\partial x} = 0\ , \quad (x,t) \in R(5n), \tag{3.5.25}$$

$$\frac{\partial u_{1n}}{\partial t} - d_1 \frac{\partial u_{1n}}{\partial x} = q_{1n}(x,t)\ , \quad (x,t) \in R(5n). \tag{3.5.26}$$

The characteristics of the equation $(3.5.25)-(3.5.26)$ are the following,

$$\frac{d\xi}{d\tau} = d_1\ ,\ \ \xi(t) = x\ ; \quad \xi = d_1\tau + x - d_1 t\ ,\ \text{when}\ \xi = 0\ ; \quad \tau = t - \frac{x}{d_1}\ ,$$

$$\frac{d\xi}{d\tau} = -d_1\ ,\ \ \xi(t) = x\ ; \quad \xi = -d_1\tau + x + d_1 t\ ,\text{when}\ \xi = \frac{\ell}{2}\ ; \quad \tau = t - \frac{\ell - 2x}{2d_1}\ .$$

By integrating along the characteristic $\xi = x - d_1(t-\tau)$, from $t - \frac{x}{d_1}$ to t,

$$q_{1n}(x,t) = g'_{(n-1)}\left(t - \frac{x}{d_1}\right) - d_1 G_{(n-1)}\left(t - \frac{x}{d_1}\right)$$

Then by integrating along the characteristic $\xi = x + d_1(t-\tau)$, from $t - \frac{\ell - 2x}{2d_1}$ to t,

$$\int_{t-\frac{\ell-2x}{2d_1}}^{t} \frac{\partial}{\partial \tau}[u_{1n}(x + d_1(t-\tau), \tau)]d\tau = \int_{t-\frac{\ell-2x}{2d_1}}^{t} g'\left(2\tau - t - \frac{x}{d_1}\right)d\tau$$

Let

$$2\tau - t - \frac{x}{d_1} = \mu \ , \qquad 2d\tau = d\mu$$
$$\mu_{low} = t - \frac{\ell - x}{d_1} \ , \qquad \mu_{up} = t - \frac{x}{d_1}$$

By letting $u_{1n}(\frac{\ell}{2}, t) = m_{1n}(t)$, we get

$$u_{1n}(x,t) = m_{1n}\left(t - \frac{\ell - 2x}{2d_1}\right) + \frac{1}{2}\left[g_{(n-1)}\left(t - \frac{x}{d_1}\right) - g_{(n-1)}\left(t - \frac{\ell - x}{d_1}\right)\right] \ . \qquad (3.5.27)$$

Similarly, the equation (3.5.3) can be written as in the form,

$$\frac{\partial q_{1n}}{\partial t} - d_1 \frac{\partial q_{1n}}{\partial x} = 0 \ , \quad (x,t) \in R(5n), \qquad (3.5.28)$$

$$\frac{\partial u_{1n}}{\partial t} + d_1 \frac{\partial u_{1n}}{\partial x} = q_{1n}(x,t) \ , \quad (x,t) \in R(5n). \qquad (3.5.29)$$

The characteristics of the equation (3.5.28) – (3.5.29) are the following,

$$\frac{d\xi}{d\tau} = -d_1 \ , \ \ \xi(t) = x \, ; \quad \xi = -d_1\tau + x + d_1 t \ , \text{ when } \xi = \ell \, ; \quad \tau = t - \frac{\ell - x}{d_1} \ ,$$

$$\frac{d\xi}{d\tau} = d_1 \ , \ \ \xi(t) = x \, ; \quad \xi = d_1\tau + x - d_1 t, \text{ when } \xi = \frac{\ell}{2} \, ; \quad \tau = t + \frac{\ell - 2x}{2d_1} \ .$$

By integrating along the characteristic $\xi = x + d_1(t - \tau)$, from $t - \frac{\ell - x_3}{d_1}$ to t,

$$q_{1n}(x,t) = h'_{(n-1)}\left(t - \frac{\ell - x}{d_1}\right) + d_1 H_{(n-1)}\left(t - \frac{\ell - x}{d_1}\right)$$

Similarly, by letting $u_{1n}(\frac{\ell}{2} + 0, t) = r(t)$ and integrating along the characteristic $\xi = x - d_1(t - \tau)$, from $t + \frac{\ell - 2x}{2d_1}$ to t, we get

$$u_{1n}(x,t) = r\left(t + \frac{\ell - 2x}{2d_1}\right) + \frac{1}{2}\left[h_{(n-1)}\left(t - \frac{\ell - x}{d_1}\right) - h_{(n-1)}\left(t - \frac{x}{d_1}\right)\right]$$
$$+ \frac{d_1}{2}\int_{t - \frac{x}{d_1}}^{t - \frac{\ell - x}{d_1}} H_{(n-1)}(z)dz \qquad (3.5.30)$$

If we use the first matching condition (3.5.23), we get

$$m(t) = r(t)$$

By using the second matching condition (3.5.24), we get

$$m(t)=\frac{1}{2}\left[g\left(t-\frac{\ell}{2d_1}\right)-g\left(-\frac{\ell}{2d_1}\right)\right]-\frac{d_1}{2}\int_{-\frac{\ell}{2d_1}}^{t-\frac{\ell}{2d_1}}G_{(n-1)}(\nu)d\nu$$

$$+\frac{1}{2}\left[h\left(t-\frac{\ell}{2d_1}\right)-h\left(-\frac{\ell}{2d_1}\right)\right]+\frac{d_1}{2}\int_{-\frac{\ell}{2d_1}}^{t-\frac{\ell}{2d_1}}H_{(n-1)}(\nu)d\nu$$

If we substitute the function $m(t)$ into the formulation (2.8.21), we get

$$u_{1n}(x,t)=\frac{1}{2}\left[g_{(n-1)}\left(t-\frac{x}{d_1}\right)-g\left(-\frac{\ell}{2d_1}\right)\right]$$

$$+\frac{1}{2}\left[h_{(n-1)}\left(t-\frac{\ell-x}{d_1}\right)-h\left(-\frac{\ell}{2d_1}\right)\right]$$

$$-\frac{d_1}{2}\int_{-\frac{\ell}{2d_1}}^{t-\frac{x}{d_1}}G_{(n-1)}(\nu)d\nu+\frac{d_1}{2}\int_{-\frac{\ell}{2d_1}}^{t-\frac{\ell-x}{d_1}}H_{(n-1)}(\nu)d\nu\ ,\quad (x,t)\in R(5n).$$

3.5.5 The Region R(5n+1)

Let us consider the problem (3.5.2) – (3.5.11) in the region $R(5n+1)$ (see, Figure 3.4),

$$R(5n+1)=\left\{(x,t)\Big|\quad 0<x<\ell\ ,\quad \frac{n\ell-x}{d_1}<t<\frac{(n-1)\ell+x}{d_1}\right\}$$

The equation (3.5.3) can be written as in the form,

$$\frac{\partial q_{1n}}{\partial t}+d_1\frac{\partial q_{1n}}{\partial x}=0\ ,\quad (x,t)\in R(5n+1), \tag{3.5.31}$$

$$\frac{\partial u_{1n}}{\partial t}-d_1\frac{\partial u_{1n}}{\partial x}=q_{1n}(x,t)\ ,\quad (x,t)\in R(5n+1). \tag{3.5.32}$$

The characteristic of the equation (3.5.31) – (3.5.32) are respectively,

$$\frac{d\xi}{d\tau}=d_1\ ,\quad \xi(t)=x\quad ;\quad \xi=d_1\tau+x-d_1t\quad \text{and if}\quad \xi=0\ ;\quad \tau=t-\frac{x}{d_1}$$

$$\frac{d\xi}{d\tau}=-d_1\ ,\quad \xi(t)=x\quad ;\quad \xi=-d_1\tau+x+d_1t\quad \text{and if}\quad \xi=\ell\ ;\quad \tau=t-\frac{\ell-x}{d_1}\ .$$

Similarly, by integrating along the characteristics,

$$q_{1n}(x,t)=g'_{(n-1)}\left(t-\frac{x}{d_1}\right)-d_1G_{(n-1)}\left(t-\frac{x}{d_1}\right)$$

$$u_{1n}(x,t) = h_n\left(t - \frac{\ell - x}{d_1}\right) + \frac{1}{2}\left[g_{(n-1)}(t - \frac{x}{d_1}) - g_{(n-1)}(t - \frac{2\ell - x}{d_1})\right]$$

$$-\frac{d_1}{2}\int_{t-\frac{2\ell-x}{d_1}}^{t-\frac{x}{d_1}} G_{(n-1)}(\gamma)d\gamma, \quad (x,t) \in R(5n+1);$$

3.5.6 The Region $R(5n+2)$

Let us consider the problem $(3.5.2)-(3.5.11)$ in the region $R(5n+2)$ (see, Figure 3.4),

$$R(5n+2) = \left\{(x,t)\Big|\ \ell < x < \infty\,, \quad \frac{(n-1)\ell}{d_1} < \left(t - \frac{x-\ell}{d_2}\right) < \frac{n\ell}{d_1}\right\}$$

The equation $(3.5.4)$ can be written as in the form,

$$\frac{\partial q_{2n}}{\partial t} - d_2\frac{\partial q_{2n}}{\partial x} = 0\,, \quad (x,t) \in R(5n+2), \tag{3.5.33}$$

$$\frac{\partial u_{2n}}{\partial t} + d_2\frac{\partial u_{2n}}{\partial x} = q_{2n}(x,t)\,, \quad (x,t) \in R(5n+2). \tag{3.5.34}$$

The characteristic of the equation $(3.5.33)-(3.5.34)$ are respectively,

$$\frac{d\xi}{d\tau} = -d_2\,, \quad \xi(t) = x \quad ; \quad \xi = -d_2\tau + x + d_2t\,,$$

$$\frac{d\xi}{d\tau} = d_2\,, \quad \xi(t) = x \quad ; \quad \xi = d_2\tau + x - d_2t \quad \text{and if} \quad \xi = \ell\,; \quad \tau = t + \frac{\ell - x}{d_2}\,.$$

Similarly, by integrating along the characteristics,

$$q_{2n}(x,t) = \psi_2(x + d_2t) + d_2\varphi_2'(x + d_2t)$$

then

$$u_{2n}(x,t) = h_n\left(t + \frac{\ell - x}{d_2}\right) + \frac{1}{2}\left[\varphi_2(x + d_2t) - \varphi_2(-x + d_2t + 2\ell)\right]$$

$$+\frac{1}{2d_2}\int_{-x+d_2t+2\ell}^{x+d_2t} \psi_2(\nu)d\nu\,, \quad (x,t) \in R(5n+2),$$

To find the functions defined on $(3.5.12)$, we must apply the matching conditions between $R(5n+1)$ and $R(5n+2)$.

3.5.7 Matching Conditions Between $R(5n+1)$ *and* $R(5n+2)$

The formula for the region $R(5n+1)$ is in the form,

$$u_{1n}(x,t) = h_n\left(t - \frac{\ell - x}{d_1}\right) + \frac{1}{2}\left[g_{(n-1)}(t - \frac{x}{d_1}) - g_{(n-1)}(t - \frac{2\ell - x}{d_1})\right]$$

$$-\frac{d_1}{2}\int_{t-\frac{2\ell-x}{d_1}}^{t-\frac{x}{d_1}} G_{(n-1)}(\gamma)d\gamma, \quad (x,t) \in R(5n+1);$$

and the formula for the region R(5n+2) is in the form,

$$u_{2n}(x,t) = h_n\left(t + \frac{\ell - x}{d_2}\right) + \frac{1}{2}\left[\varphi_2(x + d_2 t) - \varphi_2(-x + d_2 t + 2\ell)\right]$$

$$+\frac{1}{2d_2}\int_{-x+d_2t+2\ell}^{x+d_2t} \psi_2(\nu)d\nu\ , \quad (x,t) \in R(5n+2),$$

By the first matching condition (3.5.10), we have,

$$u_{1n}(\ell - 0, t) = u_{2n}(\ell + 0, t) = h(t)$$

To use the second matching condition (3.5.11), we must differentiate the formulas for the regions $R(5n+1)$ and $R(5n+2)$, and substitute $x = \ell$. Then we get the function $H_n(t)$ defined in (3.5.12),

$$H_n(t) = \frac{1}{d_1}h_n'(t) - \frac{1}{d_1}g_{(n-1)}'\left(t - \frac{\ell}{d_1}\right) - \frac{1}{d_1}G_{(n-1)}\left(t - \frac{\ell}{d_1}\right)$$

By using the second matching condition (3.5.11) And we get the function $h_n(t)$ as follows,

$$h_n(t) = \frac{d_1}{d_1 + d_2}[g_{(n-1)}\left(t - \frac{\ell}{d_1}\right) - g_{(n-1)}\left(-\frac{\ell}{d_1}\right)] + \frac{d_2}{d_1 + d_2}[\varphi_2(\ell + d_2 t) - \varphi_2(\ell)]$$

$$-\frac{d_1^2}{d_1 + d_2}\int_{-\frac{\ell}{d_1}}^{t-\frac{\ell}{d_1}} G_{(n-1)}(z)dz + \frac{1}{d_1 + d_2}\int_{\ell}^{\ell + d_2 t} \psi_2(s)ds$$

□

3.6 Examples of Simulations of Wave Propagation in Three Layered Medium

In this section, we deal with examples of simulations of wave propagation in three layered medium. IVP of wave equations is studied as the mathematical model of wave propagation.

In this work, the space has three layers that are separated with two boundaries $x=0$ and $x=\ell$. Each layer has different speed. We defined the matching conditions not only on the boundary $x=\ell$ but also on $x=0$.

A pulse point source was taken in different positions in the space: Between $-\infty$ and $x=0$; between the boundaries $x=0$ and $x=\ell$; between $x=\ell$ and ∞.

3.6.1 Example 1 - The Pulse Point Source is Between $-\infty$ and $x=0$

Let us consider initial value problem (3.1.2) – (3.1.11). The initial conditions (3.1.5) – (3.1.7) have the following form

$$\begin{aligned} \varphi_0(x) &= \delta(x-x^0), & \psi_0(x) &= 0, \\ \varphi_1(x) &= 0, & \psi_1(x) &= 0, \\ \varphi_2(x) &= 0, & \psi_2(x) &= 0, \end{aligned}$$

where $\delta(x)$ is Dirac delta function, the boundary $\ell=40$, the point source $x^0=-20$. By the properties of Dirac delta function and the assumptions, the solution $u(x,t)$ of IVP can be written as follows:

$$u(x,t)=\begin{cases} \dfrac{1}{2}\Big[\delta(x+d_0t-x^0)+\delta(x-d_0t-x^0)\Big] & \text{if } (x,t)\in R1; \\ 0, & \text{if } (x,t)\in R2; \\ 0, & \text{if } (x,t)\in R3. \end{cases}$$

$$u(x,t)=\begin{cases} g\left(t+\dfrac{x}{d_0}\right)+\dfrac{1}{2}\cdot\delta(x-d_0t-x^0) \\ -\dfrac{1}{2}\cdot\delta(-x+d_0t-x^0), & \text{if } (x,t)\in R4; \\ g\left(t-\dfrac{x}{d_1}\right), & \text{if } (x,t)\in R5; \\ 0, & \text{if } (x,t)\in R6; \\ 0, & \text{if } (x,t)\in R7. \end{cases}$$

Here, the function $g(t)$, constructed in Theorem 3.4.1, can be also written as

$$g(t) = \frac{d_0}{(d_0+d_1)} \cdot \delta(-d_0 t - x^0) \ ,$$

For $n = 2, 3, \ldots$

$$u(x,t) = \begin{cases} g_n\left(t + \dfrac{x}{d_0}\right) + \dfrac{1}{2} \cdot \delta(x - d_0 t - x^0) \\ -\dfrac{1}{2} \cdot \delta(-x - d_0 t - x^0), \qquad \text{if } (x,t) \in R(5n-2); \\[2ex] g_n\left(t - \dfrac{x}{d_1}\right) + \dfrac{1}{2}\left[h_{(n-1)}\left(t - \dfrac{\ell - x}{d_1}\right) - h_{(n-1)}\left(t - \dfrac{\ell + x}{d_1}\right)\right] \\ +\dfrac{d_1}{2} \displaystyle\int_{t-\frac{\ell+x}{d_1}}^{t-\frac{\ell-x}{d_1}} H_{(n-1)}(\eta) d\eta, \qquad \text{if } (x,t) \in R(5n-1); \\[2ex] \dfrac{1}{2}\left[g_{(n-1)}\left(t - \dfrac{x}{d_1}\right) - g_{(n-1)}\left(-\dfrac{\ell}{2d_1}\right)\right] + \dfrac{1}{2} h_{(n-1)}\left(t - \dfrac{\ell - x}{d_1}\right) \\ -\dfrac{1}{2} h_{(n-1)}\left(-\dfrac{\ell}{2d_1}\right) + \dfrac{d_1}{2} \displaystyle\int_{-\frac{\ell}{2d_1}}^{t-\frac{\ell-x}{d_1}} H_{(n-1)}(\nu) d\nu \\ -\dfrac{d_1}{2} \displaystyle\int_{-\frac{\ell}{2d_1}}^{t-\frac{x}{d_1}} G_{(n-1)}(\gamma) d\gamma \, , \qquad \text{if } (x,t) \in R(5n); \\[2ex] h_n(t - \dfrac{\ell - x}{d_1}) + \dfrac{1}{2}\left[g_{(n-1)}(t - \dfrac{x}{d_1}) - g_{(n-1)}(t - \dfrac{2\ell - x}{d_1})\right] \\ -\dfrac{d_1}{2} \displaystyle\int_{t-\frac{2\ell-x}{d_1}}^{t-\frac{x}{d_1}} G_{(n-1)}(\gamma) d\gamma, \qquad \text{if } (x,t) \in R(5n+1); \\[2ex] h_n\left(t + \dfrac{\ell - x}{d_2}\right), \qquad \text{if } (x,t) \in R(5n+2). \end{cases}$$

Here, the functions $g_n(t)$, $h_n(t)$, $G_n(t)$, and $H_n(t)$, constructed in Theorem 3.5.1, can be also written as for $n = 2, 3, \ldots$

$$G_n(t) = \frac{-d_0}{d_1(d_0+d_1)} \cdot \frac{\partial}{\partial t}\left[\delta(-d_0 t - x^0)\right] + \frac{d_0}{d_1(d_0+d_1)} \cdot h'_{(n-1)}\left(t - \frac{\ell}{d_1}\right)$$

$$+\frac{d_0}{(d_0+d_1)} \cdot H_{(n-1)}\left(t - \frac{\ell}{d_1}\right)$$

$$H_n(t) = \frac{-d_2}{d_1(d_1+d_2)} \cdot g'_{(n-1)}\left(t - \frac{\ell}{d_1}\right) + \frac{d_2}{(d_1+d_2)} \cdot G_{(n-1)}\left(t - \frac{\ell}{d_1}\right)$$

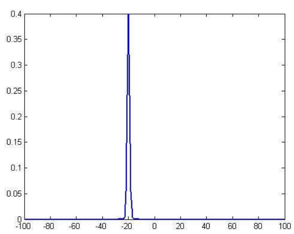
$$g_n(t) = \frac{d_0}{d_0+d_1} \cdot \delta(-d_0 t - x^0) + \frac{d_1}{d_0+d_1}\left[h_{(n-1)}\left(t-\frac{\ell}{d_1}\right) - h_{(n-1)}\left(-\frac{\ell}{d_1}\right)\right]$$
$$+\frac{d_1^2}{d_0+d_1}\int_{-\frac{\ell}{d_1}}^{t-\frac{\ell}{d_1}} H_{(n-1)}(z)dz$$
$$h_n(t) = \frac{d_1}{d_1+d_2}[g_{(n-1)}\left(t-\frac{\ell}{d_1}\right) - g_{(n-1)}\left(-\frac{\ell}{d_1}\right)] - \frac{d_1^2}{d_1+d_2}\int_{-\frac{\ell}{d_1}}^{t-\frac{\ell}{d_1}} G_{(n-1)}(z)dz$$

with

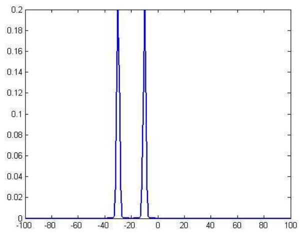
$$h_1(t) = 0\ .$$
$$h_{(n-1)}(t) = \frac{d_1}{d_1+d_2}[g_{(n-2)}\left(t-\frac{\ell}{d_1}\right) - g_{(n-2)}\left(-\frac{\ell}{d_1}\right)] - \frac{d_1^2}{d_1+d_2}\int_{-\frac{\ell}{d_1}}^{t-\frac{\ell}{d_1}} G_{(n-1)}(z)dz$$
$$g_1(t) = \frac{d_0}{(d_0+d_1)} \cdot \delta(-d_0 t - x^0)\ ,$$
$$g_{(n-1)}(t) = \frac{d_0}{d_0+d_1} \cdot \delta(-d_0 t - x^0) + \frac{d_1}{d_0+d_1}\left[h_{(n-2)}\left(t-\frac{\ell}{d_1}\right) - h_{(n-2)}\left(-\frac{\ell}{d_1}\right)\right]$$
$$+\frac{d_1^2}{d_0+d_1}\int_{-\frac{\ell}{d_1}}^{t-\frac{\ell}{d_1}} H_{(n-2)}(z)dz$$
$$H_1(t) = 0\ ,$$
$$H_{(n-1)}(t) = \frac{-d_2}{d_1(d_1+d_2)} \cdot g'_{(n-2)}\left(t-\frac{\ell}{d_1}\right) + \frac{d_2}{(d_1+d_2)} \cdot G_{(n-2)}\left(t-\frac{\ell}{d_1}\right)$$
$$G_1(t) = \frac{-d_0}{d_1(d_0+d_1)} \cdot \frac{\partial}{\partial t}\left[\delta(-d_0 t - x^0)\right]\ ,$$
$$G_{(n-1)}(t) = \frac{-d_0}{d_1(d_0+d_1)} \cdot \frac{\partial}{\partial t}\left[\delta(-d_0 t - x^0)\right] + \frac{d_0}{d_1(d_0+d_1)} \cdot h'_{(n-2)}\left(t-\frac{\ell}{d_1}\right)$$
$$+\frac{d_0}{(d_0+d_1)} \cdot H_{(n-2)}\left(t-\frac{\ell}{d_1}\right)$$

By using Matlab codes, we simulate the solution $u(x,t)$ of IVP $(3.1.2)-(3.1.11)$.

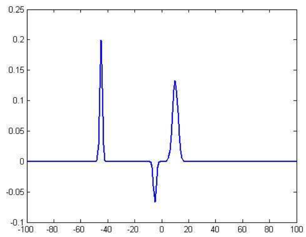
In these figures, we simulate the wave propagation in three layered medium that is separated with two boundaries; the first boundary is $x=0$ and the second boundary is $x=\ell$.(In this example, $\ell = 40$.)



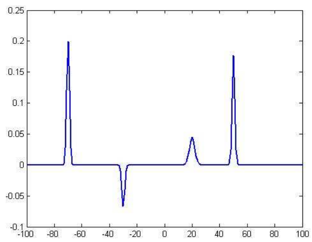
(a) $t = 0$



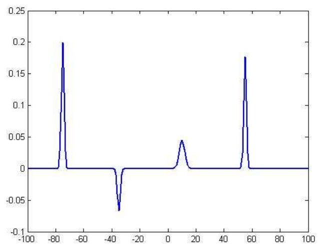
(b) $t = 10$



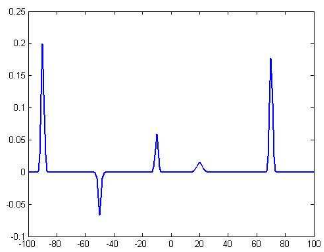
(c) $t = 25$



(d) $t = 50$



(e) $t = 55$



(f) $t = 70$

Figure 3.3 Pulse Point Source is between $-\infty$ and 0

In the figures, the horizontal axes x and the vertical axes y show the location and the magnitude of the wave front, respectively. In figure (a), we can see the fluctuation arising from the pulse point source $x^0 = -20$ described by the function $\varphi_0(x) = \delta(x - x^0)$. In the figure (b), the separated waves began to move along the characteristics. In the figure (c), the reflected and transmitted waves can be seen after the wave front touched the boundary $x = 0$.

Notice that in the figure(c), the reflected wave has the negative sign. This the result of that the speed of the second layer is bigger than the first layer.(For more detail, chapter 4.)

After the transmitted wave touched the second boundary $(x = \ell)$, it is separated into transmitted and reflected waves in the figure(d). In the figure(e), the movement of the reflected and transmitted waves can be seen. Especially, the reflected wave is moving between two boundaries, so in the figure(f), the reflected wave touches the boundary and is separated into reflected and the transmitted waves.

3.6.2 Example 2 - The Pulse Point Source is Between the Boundaries $x = 0$ and $x = \ell$

Let us consider initial value problem (3.1.2) – (3.1.11). The initial conditions (3.1.5) – (3.1.7) have the following form

$$\begin{aligned} \varphi_0(x) &= 0, & \psi_0(x) &= 0, \\ \varphi_1(x) &= \delta(x - x^0), & \psi_1(x) &= 0, \\ \varphi_2(x) &= 0, & \psi_2(x) &= 0, \end{aligned}$$

where $\delta(x)$ is Dirac delta function, the boundary $\ell = 40$, the point source $x^0 = 10$. By the properties of Dirac delta function and the assumptions, the solution $u(x,t)$ of IVP can be written as follows:

$$u(x,t) = \begin{cases} 0, & \text{if } (x,t) \in R1; \\ \frac{1}{2}\Big[\delta(x + d_1 t - x^0) + \delta(x - d_1 t - x^0)\Big], & \text{if } (x,t) \in R2; \\ 0, & \text{if } (x,t) \in R3. \end{cases}$$

$$u(x,t) = \begin{cases} g\left(t + \frac{x}{d_0}\right), & \text{if } (x,t) \in R4; \\ g\left(t - \frac{x}{d_1}\right) + \frac{1}{2} \cdot \delta(x + d_1 t - x^0) \\ \quad - \frac{1}{2} \cdot \delta(-x + d_1 t - x^0), & \text{if } (x,t) \in R5; \\ h\left(t - \frac{\ell - x}{d_1}\right) + \frac{1}{2} \cdot \delta(x - d_1 t - x^0) \\ \quad - \frac{1}{2} \cdot \delta(-x - d_1 t + 2\ell - x^0), & \text{if } (x,t) \in R6; \\ h\left(t + \frac{\ell - x}{d_2}\right), & \text{if } (x,t) \in R7. \end{cases}$$

Here, the functions $g(t)$, $h(t)$, $G(t)$, and $H(t)$, constructed in Theorem 3.4.1, can be also written as

$$g(t) = \frac{d_1}{d_0+d_1} \cdot \delta(d_1 t - x^0)$$

$$h(t) = \frac{d_1}{d_1+d_2} \cdot \delta(\ell - d_1 t - x^0)$$

$$G(t) = \frac{d_0}{d_1(d_0+d_1)} \frac{\partial}{\partial t} \left[\delta(d_1 t - x^0)\right]$$

$$H(t) = \frac{-d_2}{d_1(d_1+d_2)} \frac{\partial}{\partial t} \left[\delta(\ell - d_1 t - x^0)\right]$$

For $n = 2, 3, \ldots$

$$u(x,t) = \begin{cases} g_n\left(t + \dfrac{x}{d_0}\right), & \text{if } (x,t) \in R(5n-2); \\[2ex] g_n\left(t - \dfrac{x}{d_1}\right) + \dfrac{1}{2}\left[h_{(n-1)}\left(t - \dfrac{\ell - x}{d_1}\right) - h_{(n-1)}\left(t - \dfrac{\ell + x}{d_1}\right)\right] \\ + \dfrac{d_1}{2} \displaystyle\int_{t-\frac{\ell+x}{d_1}}^{t-\frac{\ell-x}{d_1}} H_{(n-1)}(\eta)d\eta, & \text{if } (x,t) \in R(5n-1); \\[2ex] \dfrac{1}{2}\left[g_{(n-1)}\left(t - \dfrac{x}{d_1}\right) - g_{(n-1)}\left(-\dfrac{\ell}{2d_1}\right)\right] + \dfrac{1}{2} h_{(n-1)}\left(t - \dfrac{\ell - x}{d_1}\right) \\ -\dfrac{1}{2} h_{(n-1)}\left(-\dfrac{\ell}{2d_1}\right) + \dfrac{d_1}{2} \displaystyle\int_{-\frac{\ell}{2d_1}}^{t-\frac{\ell-x}{d_1}} H_{(n-1)}(\nu)d\nu \\ -\dfrac{d_1}{2} \displaystyle\int_{-\frac{\ell}{2d_1}}^{t-\frac{x}{d_1}} G_{(n-1)}(\gamma)d\gamma, & \text{if } (x,t) \in R(5n); \\[2ex] h_n(t - \dfrac{\ell - x}{d_1}) + \dfrac{1}{2}\left[g_{(n-1)}(t - \dfrac{x}{d_1}) - g_{(n-1)}(t - \dfrac{2\ell - x}{d_1})\right] \\ -\dfrac{d_1}{2} \displaystyle\int_{t-\frac{2\ell-x}{d_1}}^{t-\frac{x}{d_1}} G_{(n-1)}(\gamma)d\gamma, & \text{if } (x,t) \in R(5n+1); \\[2ex] h_n\left(t + \dfrac{\ell - x}{d_2}\right), & \text{if } (x,t) \in R(5n+2). \end{cases}$$

Here, the functions $g_n(t)$, $h_n(t)$, $G_n(t)$, and $H_n(t)$, constructed in Theorem 3.5.1, can be also written as for $n = 2, 3, \ldots$

$$G_n(t) = \frac{d_0}{d_1(d_0+d_1)} \cdot h'_{(n-1)}\left(t - \frac{\ell}{d_1}\right) + \frac{d_0}{(d_0+d_1)} \cdot H_{(n-1)}\left(t - \frac{\ell}{d_1}\right)$$

$$H_n(t) = \frac{-d_2}{d_1(d_1+d_2)} \cdot g'_{(n-1)}\left(t - \frac{\ell}{d_1}\right) + \frac{d_2}{(d_1+d_2)} \cdot G_{(n-1)}\left(t - \frac{\ell}{d_1}\right)$$

$$h_n(t) = \frac{d_1}{d_1+d_2}[g_{(n-1)}\left(t-\frac{\ell}{d_1}\right) - g_{(n-1)}\left(-\frac{\ell}{d_1}\right)] - \frac{d_1^2}{d_1+d_2}\int_{-\frac{\ell}{d_1}}^{t-\frac{\ell}{d_1}} G_{(n-1)}(z)dz$$

$$g_n(t) = \frac{d_1}{d_0+d_1}\left[h_{(n-1)}\left(t-\frac{\ell}{d_1}\right) - h_{(n-1)}\left(-\frac{\ell}{d_1}\right)\right]$$

$$+\frac{d_1^2}{d_0+d_1}\int_{-\frac{\ell}{d_1}}^{t-\frac{\ell}{d_1}} H_{(n-1)}(z)dz$$

with

$$g_1(t) = \frac{d_1}{d_0+d_1}\cdot\delta(d_1t-x^0)$$

$$g_{(n-1)}(t) = \frac{d_1}{d_0+d_1}\left[h_{(n-2)}\left(t-\frac{\ell}{d_1}\right) - h_{(n-2)}\left(-\frac{\ell}{d_1}\right)\right]$$

$$h_1(t) = \frac{d_1}{d_1+d_2}\cdot\delta(\ell-d_1t-x^0)$$

$$h_{(n-1)}(t) = \frac{d_1}{d_1+d_2}[g_{(n-2)}\left(t-\frac{\ell}{d_1}\right) - g_{(n-2)}\left(-\frac{\ell}{d_1}\right)] - \frac{d_1^2}{d_1+d_2}\int_{-\frac{\ell}{d_1}}^{t-\frac{\ell}{d_1}} G_{(n-2)}(z)dz$$

$$G_1(t) = \frac{d_0}{d_1(d_0+d_1)}\frac{\partial}{\partial t}\left[\delta(d_1t-x^0)\right]$$

$$G_{(n-1)}(t) = \frac{d_0}{d_1(d_0+d_1)}\cdot h'_{(n-2)}\left(t-\frac{\ell}{d_1}\right) + \frac{d_0}{(d_0+d_1)}\cdot H_{(n-2)}\left(t-\frac{\ell}{d_1}\right)$$

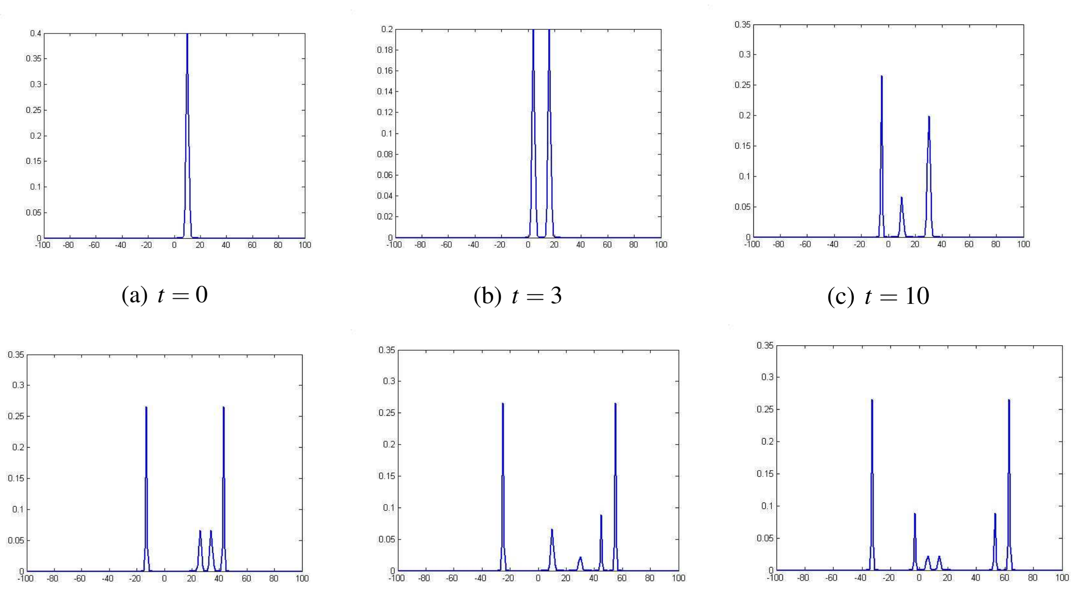
$$H_1(t) = \frac{-d_2}{d_1(d_1+d_2)}\frac{\partial}{\partial t}\left[\delta(\ell-d_1t-x^0)\right]$$

$$H_{(n-1)}(t) = \frac{-d_2}{d_1(d_1+d_2)}\cdot g'_{(n-2)}\left(t-\frac{\ell}{d_1}\right) + \frac{d_2}{(d_1+d_2)}\cdot G_{(n-2)}\left(t-\frac{\ell}{d_1}\right)$$

By using Matlab codes, we simulate the solution $u(x,t)$of IVP $(3.1.2)-(3.1.11)$ for $n = 2,3,\ldots$

Similarly, in these figures, we simulate the wave propagation in three layered medium in which a pulse point source is located at $x^0 = 10$. And the horizontal axes x and the vertical axes y show the location and the magnitude of the wave front, respectively.

In figure (a), we can see the fluctuation arising from the pulse point source $x^0 = 10$ described by the function $\varphi_1(x) = \delta(x-x^0)$. In the figure (b), the separated waves began to move along the characteristics. In the figure (c), the reflected and transmitted waves can be seen after the wave front touched the boundary $x = 0$.



(a) $t = 0$ (b) $t = 3$ (c) $t = 10$

(d) $t = 18$ (e) $t = 30$ (f) $t = 38$

Figure 3.4 Pulse Point Source is between $x = 0$ and $x = \ell$

On the other hand, the separated wave front in the figure (c), touches the boundary $x = \ell$. So in the figure (d), it is separated into transmitted and reflected waves.

Notice that, in these figures the movements of reflected waves occur between the boundaries $x = 0$ and $x = \ell$. In a small time period, they are separated over and over again. In the figures (e) and (f), we can see the separation of the reflected waves into transmitted and reflected waves.

3.6.3 Example 3 - The Pulse Point Source is Between 0 *and* ∞

Let us consider initial value problem $(3.1.2)-(3.1.11)$. The initial conditions $(3.1.5)-(3.1.7)$ have the following form

$$\begin{aligned} \varphi_0(x) &= 0, & \psi_0(x) &= 0, \\ \varphi_1(x) &= 0, & \psi_1(x) &= 0, \\ \varphi_2(x) &= \delta(x - x^0), & \psi_2(x) &= 0, \end{aligned}$$

where $\delta(x)$ is Dirac delta function, the boundary $\ell = 40$, the point source is located at $x^0 = 60$. By the properties of Dirac delta function and the assumptions, the solution $u(x,t)$ of

IVP can be written as follows:

$$u(x,t)=\begin{cases} 0, & \text{if } (x,t)\in R1; \\ \\ 0, & \text{if } (x,t)\in R2; \\ \\ \dfrac{1}{2}\Big[\delta(x+d_2t-x^0)+\delta(x-d_2t-x^0)\Big], & \text{if } (x,t)\in R3. \end{cases}$$

$$u(x,t)=\begin{cases} 0, & \text{if } (x,t)\in R4; \\ \\ 0, & \text{if } (x,t)\in R5; \\ \\ h\left(t-\dfrac{\ell-x}{d_1}\right), & \text{if } (x,t)\in R6; \\ \\ h\left(t+\dfrac{\ell-x}{d_2}\right)+\dfrac{1}{2}\cdot\delta(x+d_2t-x^0) & \\ -\dfrac{1}{2}\cdot\delta(-x+d_2t+2\ell-x^0), & \text{if } (x,t)\in R7. \end{cases}$$

Here, the function $h(t)$, constructed in Theorem 3.4.1, can be also written as

$$h(t)=\frac{d_2}{d_1+d_2}\cdot\delta(\ell+d_2t-x^0)$$

For $n = 2, 3, \ldots$ *in the general case;*

$$u(x,t) = \begin{cases} g_n\left(t + \dfrac{x}{d_0}\right), & \text{if } (x,t) \in R(5n-2); \\[2ex] g_n\left(t - \dfrac{x}{d_1}\right) + \dfrac{1}{2}\left[h_{(n-1)}\left(t - \dfrac{\ell - x}{d_1}\right) - h_{(n-1)}\left(t - \dfrac{\ell + x}{d_1}\right)\right] \\ +\dfrac{d_1}{2}\displaystyle\int_{t-\frac{\ell+x}{d_1}}^{t-\frac{\ell-x}{d_1}} H_{(n-1)}(\eta)d\eta, & \text{if } (x,t) \in R(5n-1); \\[2ex] \dfrac{1}{2}\left[g_{(n-1)}\left(t - \dfrac{x}{d_1}\right) - g_{(n-1)}\left(-\dfrac{\ell}{2d_1}\right)\right] + \dfrac{1}{2}h_{(n-1)}\left(t - \dfrac{\ell - x}{d_1}\right) \\ -\dfrac{1}{2}h_{(n-1)}\left(-\dfrac{\ell}{2d_1}\right) + \dfrac{d_1}{2}\displaystyle\int_{-\frac{\ell}{2d_1}}^{t-\frac{\ell-x}{d_1}} H_{(n-1)}(\nu)d\nu \\ -\dfrac{d_1}{2}\displaystyle\int_{-\frac{\ell}{2d_1}}^{t-\frac{x}{d_1}} G_{(n-1)}(\gamma)d\gamma\,, & \text{if } (x,t) \in R(5n); \\[2ex] h_n(t - \dfrac{\ell - x}{d_1}) + \dfrac{1}{2}\left[g_{(n-1)}(t - \dfrac{x}{d_1}) - g_{(n-1)}(t - \dfrac{2\ell - x}{d_1})\right] \\ -\dfrac{d_1}{2}\displaystyle\int_{t-\frac{2\ell-x}{d_1}}^{t-\frac{x}{d_1}} G_{(n-1)}(\gamma)d\gamma, & \text{if } (x,t) \in R(5n+1); \\[2ex] h_n\left(t + \dfrac{\ell - x}{d_2}\right) + \dfrac{1}{2}\delta(x + d_2 t - x^0) \\ -\dfrac{1}{2}\delta(-x + d_2 t + 2\ell - x^0), & \text{if } (x,t) \in R(5n+2). \end{cases}$$

Here, the functions $g_n(t)$, $h_n(t)$, $G_n(t)$, and $H_n(t)$, constructed in Theorem 3.5.1, can be also written as for $n = 2, 3, \ldots$

$$g_n(t) = \frac{d_1}{d_0 + d_1}\left[h_{(n-1)}\left(t - \frac{\ell}{d_1}\right) - h_{(n-1)}\left(-\frac{\ell}{d_1}\right)\right] + \frac{d_1^2}{d_0 + d_1}\int_{-\frac{\ell}{d_1}}^{t-\frac{\ell}{d_1}} H_{(n-1)}(z)dz$$

$$h_n(t) = \frac{d_2}{d_1 + d_2}\cdot\delta(\ell + d_2 t - x^0) + \frac{d_1}{d_1 + d_2}g_{(n-1)}\left(t - \frac{\ell}{d_1}\right)$$

$$-\frac{d_1}{d_1 + d_2}g_{(n-1)}\left(-\frac{\ell}{d_1}\right) - \frac{d_1^2}{d_1 + d_2}\int_{-\frac{\ell}{d_1}}^{t-\frac{\ell}{d_1}} G_{(n-1)}(z)dz$$

$$G_n(t) = \frac{d_0}{d_1(d_0 + d_1)}h'_{(n-1)}\left(t - \frac{\ell}{d_1}\right) + \frac{d_0}{(d_0 + d_1)}H_{(n-1)}\left(t - \frac{\ell}{d_1}\right)$$

$$H_n(t) = \frac{d_2}{d_1(d_1 + d_2)}\frac{\partial}{\partial t}[\delta(\ell + d_2 t - x^0)]$$

$$-\frac{d_2}{d_1(d_1 + d_2)}g'_{(n-1)}\left(t - \frac{\ell}{d_1}\right) + \frac{d_2}{(d_1 + d_2)}G_{(n-1)}\left(t - \frac{\ell}{d_1}\right)$$

with

$$g_1(t) = 0,$$

$$g_{(n-1)}(t) = \frac{d_1}{d_0+d_1}\left[h_{(n-2)}\left(t-\frac{\ell}{d_1}\right) - h_{(n-2)}\left(-\frac{\ell}{d_1}\right)\right] + \frac{d_1^2}{d_0+d_1}\int_{-\frac{\ell}{d_1}}^{t-\frac{\ell}{d_1}} H_{(n-2)}(z)dz$$

$$G_1(t) = 0,$$

$$G_{(n-1)}(t) = \frac{d_0}{d_1(d_0+d_1)}h'_{(n-2)}\left(t-\frac{\ell}{d_1}\right) + \frac{d_0}{(d_0+d_1)}H_{(n-2)}\left(t-\frac{\ell}{d_1}\right)$$

$$h_1(t) = \frac{d_2}{d_1+d_2}\cdot\delta(\ell+d_2t-x^0)$$

$$h_{(n-1)}(t) = \frac{d_2}{d_1+d_2}\cdot\delta(\ell+d_2t-x^0) + \frac{d_1}{d_1+d_2}g_{(n-2)}\left(t-\frac{\ell}{d_1}\right)$$

$$-\frac{d_1}{d_1+d_2}g_{(n-2)}\left(-\frac{\ell}{d_1}\right) - \frac{d_1^2}{d_1+d_2}\int_{-\frac{\ell}{d_1}}^{t-\frac{\ell}{d_1}} G_{(n-2)}(z)dz$$

$$H_1(t) = \frac{d_2}{d_1(d_1+d_2)}\frac{\partial}{\partial t}[\delta(\ell+d_2t-x^0)]$$

$$H_{(n-1)}(t) = \frac{d_2}{d_1(d_1+d_2)}\frac{\partial}{\partial t}[\delta(\ell+d_2t-x^0)]$$

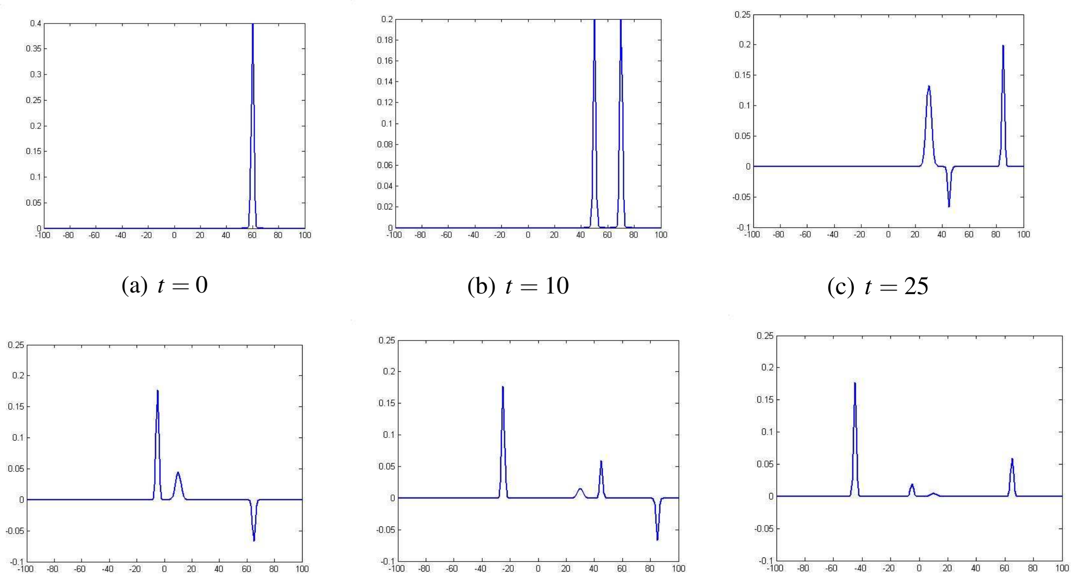
$$-\frac{d_2}{d_1(d_1+d_2)}g'_{(n-2)}\left(t-\frac{\ell}{d_1}\right) + \frac{d_2}{(d_1+d_2)}G_{(n-2)}\left(t-\frac{\ell}{d_1}\right)$$

By using Matlab codes, we simulate the solution of $u(x,t)$ IVP (3.1.2) – (3.1.11) for $n = 2,3,\ldots$

In this example, the pulse point source is located on the right of the boundary $x=\ell$, at $x^0 = 60$. Similarly, in figure (a), we can see the fluctuation arising from the pulse point source described by the function $\varphi_2(x) = \delta(x-x^0)$. In the figure (b), the separated waves began to move along the characteristics. In the figure (c), the reflected and transmitted waves can be seen after the wave front touched the boundary $x=\ell$.

Notice that in the figure(c), the reflected wave has the negative sign. This the result of that the speed of the second layer is bigger than the third layer.(For more detail, chapter 4.)

In the figure (d), the separated wave front, that is on the right in the figure (c), disappear by



(a) $t = 0$ (b) $t = 10$ (c) $t = 25$

(d) $t = 45$ (e) $t = 65$ (f) $t = 85$

Figure 3.5 Pulse Point Source is between ℓ and ∞

the time is passing. And on the other hand, the reflected wave is separated into transmitted and reflected waves again, after it touches the other boundary $(x = 0)$.

In the figures(e) and (f), the reflected waves are separated over and over again, as the result of touching the boundaries $x = \ell$ and $x = 0$, respectively.

3.7 Conclusion of Chapter Three

- Explicit formulae for the solution of IVP with matching conditions has been constructed.
- Using this formulae, the simulation of wave propagation has been obtained.
- Results of the simulations have clear physical interpretation of wave propagation in three layered media from the point source.

CHAPTER FOUR

INITIAL VALUE PROBLEMS WITH ONE BOUNDARY

4.1 IVP-I

Let us consider the problem (2.3.6) – (2.3.10). In this work, we omit the index k. Let $(x,t) \in \mathbf{R}^2$, $\Phi(x), \Psi(x)$ and $d(x)$ have the following form,

$$d(x) = \begin{cases} d_0, & -\infty < x < 0; \\ d_1, & 0 < x < \infty; \end{cases} \tag{4.1.1}$$

$$\Phi(x) = \begin{cases} \varphi_0, & -\infty < x < 0; \\ \varphi_1, & 0 < x < \infty; \end{cases} \qquad \Psi(x) = \begin{cases} \psi_0, & -\infty < x < 0; \\ \psi_1, & 0 < x < \infty; \end{cases} \tag{4.1.2}$$

where d_0, d_1, are given constants; $\varphi_0(x)$, $\varphi_1(x)$, $\psi_0(x)$ and $\psi_1(x)$ are given functions depending on x.

In addition, we assume that there is no boundary condition and we have the matching conditions defined on one boundary $x = 0$.

Initial value problem (2.3.6) – (2.3.10) may be written in the term of

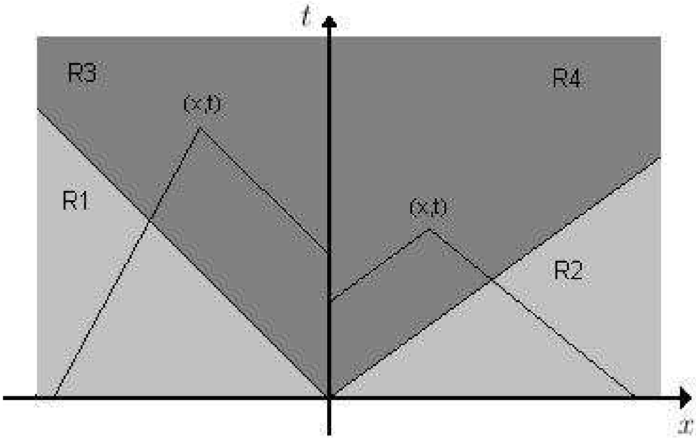


Figure 4.1 Initial value problems with one boundary $x = 0$

$$u(x,t) = \begin{cases} u_0(x,t), & -\infty < x < 0; \\ u_1(x,t), & 0 < x < \infty; \end{cases} \tag{4.1.3}$$

as follows

$$\frac{\partial^2 u_0}{\partial t^2} - d_0^2 \frac{\partial^2 u_0}{\partial x^2} = 0, \quad -\infty < x < 0, \ t \in \mathbf{R}, \tag{4.1.4}$$

$$\frac{\partial^2 u_1}{\partial t^2} - d_1^2 \frac{\partial^2 u_1}{\partial x^2} = 0, \quad 0 < x < \infty, \ t \in \mathbf{R}, \tag{4.1.5}$$

with initial data,

$$\begin{aligned} u_0(x,0) = \varphi_0(x), \qquad \frac{\partial u_0}{\partial t}\Big|_{t=0} = \psi_0(x), \quad -\infty < x < 0, \\ u_1(x,0) = \varphi_1(x), \qquad \frac{\partial u_1}{\partial t}\Big|_{t=0} = \psi_1(x), \quad 0 < x < \infty, \end{aligned} \tag{4.1.6}$$

and the matching conditions,

$$u_0(x,t)\Big|_{x=-0} = u_1(x,t)\Big|_{x=+0} \tag{4.1.7}$$

$$\frac{\partial u_0}{\partial x}(x,t)\Big|_{x=-0} = \frac{\partial u_1}{\partial x}(x,t)\Big|_{x=+0} \tag{4.1.8}$$

Before finding solution of initial value problem $(4.1.4)-(4.1.8)$, we must define the following function.

$$u(0,t) = g(t) \tag{4.1.9}$$

We must construct the function $g(t)$ by initial data and the matching conditions.

Theorem 4.1.1. *Let* $\Phi(x)$ *and* $\Psi(x)$ *be given functions in the form* $(4.1.2)$; $u(x,t)$ *be unknown functions in the form* $(4.1.3)$ *then the solution* $u(x,t)$ *of IVP* $(4.1.4)-(4.1.8)$ *is the following,*

$$\begin{cases} \frac{1}{2}\left[\varphi_0(x+d_0t)+\varphi_0(x-d_0t)\right] \\ +\frac{1}{2d_0}\int_{x-d_0t}^{x+d_0t}\psi_0(\gamma)d\gamma, & (x,t)\in R1\ , \\ \\ \frac{1}{2}\Big[\varphi_0(x-d_0t)-\varphi_0(-x-d_0t)\Big] \\ +\int_{x-d_0t}^{-x-d_0t}\psi_0(\xi)d\xi+g\left(t+\frac{x}{d_0}\right)\ , & (x,t)\in R3; \\ \\ \frac{1}{2}\Big[\varphi_1(x+d_1t)+\varphi_1(x-d_1t)\Big] \\ +\frac{1}{2d_1}\int_{x-d_1t}^{x+d_1t}\psi_1(\xi)d\xi\quad , & (x,t)\in R2; \\ \\ \frac{1}{2}\Big[\varphi_1(x+d_1t)-\varphi_1(-x+d_1t)\Big] \\ +\frac{1}{2d_1}\int_{-x+d_1t}^{x+d_1t}\psi_1(\xi)d\xi+g\Big(t-\frac{x}{d_1}\Big)\ , & (x,t)\in R4. \end{cases} \tag{4.1.10}$$

where the regions $R1$, $R2$, $R3$ *and* $R4$ *are the following,*

$$R1=\left\{(x,t)\Big|\ -\infty<x<0\ ,\quad t<\frac{-x}{d_0}\right\}$$

$$R2=\left\{(x,t)\Big|\ 0<x<\infty\ ,\quad t<\frac{x}{d_1}\right\}$$

$$R3=\left\{(x,t)\Big|\ -\infty<x<0\ ,\quad t>\frac{-x}{d_0}\right\}$$

$$R4=\left\{(x,t)\Big|\ 0<x<\infty\ ,\quad t>\frac{x}{d_1}\right\}$$

and the function $g(t)$ *defined in* $(4.1.9)$ *is the following,*

$$g(t)=\frac{d_0}{d_0+d_1}(\varphi_0(-d_0t)-\varphi_0(0))+\frac{d_1}{d_0+d_1}(\varphi_1(d_1t)-\varphi_1(0))$$

$$-\frac{1}{d_0+d_1}\int_0^{-d_0t}\psi_0(s)ds+\frac{1}{d_0+d_1}\int_0^{d_1t}\psi_1(z)dz \tag{4.1.11}$$

Proof. In this work, we have four subregions, namely the regions R1, R2,R3 and R4. Let us investigate these subregions, independently.

4.1.1 The Region R1 and R2

Let us consider the problem $(4.1.4)-(4.1.8)$ in the region R1,

$$R1 = \left\{ (x,t) \Big| -\infty < x < 0\ , \quad t < \frac{-x}{d_0} \right\}$$

$$R2 = \left\{ (x,t) \Big| \ 0 < x < \infty\ , \quad t < \frac{x}{d_1} \right\}$$

The equation $(4.1.4)$ can be written

$$\frac{\partial q_i}{\partial t} - d_i \frac{\partial q_i}{\partial x} = 0, \quad (x,t) \in R(i), \quad \text{for} i = 0,1. \tag{4.1.12}$$

$$\frac{\partial u_i}{\partial t} + d_i \frac{\partial u_i}{\partial x} = q_i(x,t), \quad (x,t) \in R(i) \quad \text{for} i = 0,1. \tag{4.1.13}$$

For the solution of the problem, we use the method of characteristics. So, the characteristics of the equations $(4.1.12)-(4.1.13)$ are respectively,

$$\frac{d\xi}{d\tau} = -d_i\ , \quad \xi(t) = x \quad ; \quad \xi = -d_i\tau + x + d_i t, \quad \text{for} i = 0,1.$$

$$\frac{d\xi}{d\tau} = d_i\ , \quad \xi(t) = x \quad ; \quad \xi = d_i\tau + x - d_i t \quad \text{for} i = 0,1.$$

By integrating along the characteristics, we get the following

$$q_i(x,t) = \psi_i(x + d_i t) + d_i \varphi_i'(x + d_i t), \quad \text{for} i = 0,1.$$

and

$$\int_0^t \frac{\partial}{\partial \tau} \Big[u_i(x - d_i(t-\tau), \tau) d\tau \Big] = \int_0^t \psi_i(x - d_i t + 2d_i\tau) d\tau$$

$$+ d_i \int_0^t d_i \varphi_i'(x - d_i t + 2d_i\tau) d\tau, \quad \text{for} i = 0,1.$$

Then let, for $i = 0, 1$.

$$x - d_i t + 2d_i\tau = \gamma \,, \qquad 2d_i d\tau = d\gamma$$

$$\gamma_{low} = x - d_i t \,, \qquad \gamma_{up} = x + d_i t$$

So, we get

$$u_i(x,t) - u_i(x - d_i t, 0) = \frac{1}{2}\left[\varphi_i(x + d_i t) - \varphi_i(x - d_i t)\right]$$

$$+ \frac{1}{2d_i}\int_{x-d_i t}^{x+d_i t} \psi_i(\gamma)d\gamma, \quad \text{for } i = 0, 1.$$

By substituting the initial conditions (4.1.6), we have the solution

$$u_0(x,t) = \frac{1}{2}\left[\varphi_0(x + d_0 t) + \varphi_0(x - d_0 t)\right] + \frac{1}{2d_0}\int_{x-d_0 t}^{x+d_0 t} \psi_0(\gamma)d\gamma, \quad (x,t) \in R1.$$

$$u_1(x,t) = \frac{1}{2}\Big[\varphi_1(x + d_1 t) + \varphi_1(x - d_1 t)\Big] + \frac{1}{2d_1}\int_{x-d_1 t}^{x+d_1 t} \psi_1(\xi)d\xi, \quad (x,t) \in R2$$

4.1.2 The Region R3

Let us consider the problem (4.1.4) – (4.1.8) in the region R3 (see, Figure 3.4),

$$R4 = \left\{ (x,t) \Big| \; 0 < x < \infty \,, \quad t > \frac{x}{d_1} \right\}$$

The equation (4.1.4) can be written as in the form,

$$\frac{\partial q_1}{\partial t} - d_1 \frac{\partial q_1}{\partial x} = 0 \,, \quad (x,t) \in R4, \tag{4.1.14}$$

$$\frac{\partial u_1}{\partial t} + d_1 \frac{\partial u_1}{\partial x} = q_1(x,t) \,, \quad (x,t) \in R4. \tag{4.1.15}$$

The characteristic of the equation (4.1.14) – (4.1.15) are respectively,

$$\frac{d\xi}{d\tau} = -d_1 \,, \quad \xi(t) = x \quad ; \quad \xi = -d_1\tau + x + d_1 t \,,$$

$$\frac{d\xi}{d\tau} = d_1 \,, \quad \xi(t) = x \quad ; \quad \xi = d_1\tau + x - d_1 t \quad \text{and if} \quad \xi = 0 \,; \quad \tau = t - \frac{x}{d_1} \,.$$

By integrating along the characteristics,

$$q_1(x,t) = \psi_1(x+d_1t) + d_1\varphi_1'(x+d_1t)$$

Then by integrating along the characteristic,

$$u_1(x,t) - u_1\left(0, t - \frac{x}{d_1}\right) = \int_{t-\frac{x}{d_1}}^{t} \psi_1(x - d_1t + 2d_1\tau)d\tau$$

$$+d_1\int_{t-\frac{x}{d_1}}^{t} \varphi_1'(x - d_1t + 2d_1\tau)d\tau ,$$

Let

$$x - d_1t + 2d_1\tau = \mu , \qquad 2d_1d\tau = d\mu$$

$$\mu_{low} = -x - d_0t , \qquad \mu_{up} = x - d_0t$$

By substituting the initial conditions (4.1.6), we have the solution and by the function $g(t)$ defined in (4.1.9)

$$u_0(x,t) = g\left(t + \frac{x}{d_0}\right) + \frac{1}{2}\left[\varphi_0(x - d_0t) - \varphi_0(-x - d_0t)\right]$$

$$+\frac{1}{2d_0}\int_{x-d_0t}^{-x-d_0t} \psi_0(\mu)d\mu , \quad (x,t) \in R3,$$

4.1.3 The Region R4

Let us consider the problem (4.1.4) – (4.1.8) in the region R4 (see, Figure 3.4),

$$R3 = \left\{(x,t)\Big| -\infty < x < 0 , \quad t > \frac{-x}{d_0}\right\}$$

The equation (4.1.4) can be written as in the form,

$$\frac{\partial q_0}{\partial t} + d_0\frac{\partial q_0}{\partial x} = 0 , \quad (x,t) \in R3, \tag{4.1.16}$$

$$\frac{\partial u_0}{\partial t} - d_0\frac{\partial u_0}{\partial x} = q_0(x,t) , \quad (x,t) \in R3. \tag{4.1.17}$$

The characteristic of the equation (4.1.16) – (4.1.17) are respectively,

$$\frac{d\xi}{d\tau} = d_0 , \quad \xi(t) = x \quad ; \quad \xi = d_0\tau + x - d_0t ,$$

$$\frac{d\xi}{d\tau} = -d_0 \ , \quad \xi(t) = x \quad ; \quad \xi = -d_0\tau + x + d_0 t \quad \text{and if} \quad \xi = 0 \, ; \quad \tau = t + \frac{x}{d_0} \ .$$

By integrating along the characteristics,

$$q_0(x,t) = \psi_0(x - d_0 t) - d_0 \varphi_0'(x - d_0 t)$$

Then by integrating along the characteristic,

$$u_0(x,t) - u_0\left(0, t + \frac{x}{d_0}\right) = \int_{t+\frac{x}{d_0}}^{t} \psi_0(x + d_0 t - 2d_0\tau) d\tau$$

$$-d_0 \int_{t+\frac{x}{d_0}}^{t} \varphi_0'(x + d_0 t - 2d_0\tau) d\tau \ ,$$

Let

$$x + d_0 t - 2d_0\tau = \mu \ , \qquad -2d_0 d\tau = d\mu$$

$$\mu_{low} = -x + d_1 t \ , \qquad \mu_{up} = x + d_1 t$$

By substituting the initial conditions (4.1.6), we have the solution and by the function $g(t)$ defined in (4.1.9)

$$u_1(x,t) = g\Big(t - \frac{x}{d_1}\Big) + \frac{1}{2}\Big[\varphi_1(x + d_1 t) - \varphi_1(-x + d_1 t)\Big]$$

$$+\frac{1}{2d_1}\int_{-x+d_1 t}^{x+d_1 t} \psi_1(\xi) d\xi \ , \quad (x,t) \in R4.$$

4.1.4 Matching Conditions Between R3 and R4

The formula for the region R3 is in the form,

$$u_0(x,t) = g\left(t + \frac{x}{d_0}\right) + \frac{1}{2}\left[\varphi_0(x - d_0 t) - \varphi_0(-x - d_0 t)\right]$$

$$+\frac{1}{2d_0}\int_{x-d_0 t}^{-x-d_0 t} \psi_0(\mu) d\mu \ , \quad (x,t) \in R3,$$

and the formula for the region R4 is in the form,

$$u_1(x,t) = g\Big(t - \frac{x}{d_1}\Big) + \frac{1}{2}\Big[\varphi_1(x + d_1 t) - \varphi_1(-x + d_1 t)\Big]$$

$$+\frac{1}{2d_1}\int_{-x+d_1 t}^{x+d_1 t} \psi_1(\xi) d\xi \ , \quad (x,t) \in R4$$

By the first matching condition (4.1.7), we have,

$$u(-0,t) = u(+0,t)(t) = g(t)$$

To get an explicit formula for the function $g(t)$, we must differentiate the formulas for the regions R3 and R4, and substitute $x = 0$.Then by using the second matching condition (4.1.8) And we get the function $g(t)$ as follows,

$$g(t) = \frac{d_0}{d_0+d_1}(\varphi_0(-d_0t) - \varphi_0(0)) + \frac{d_1}{d_0+d_1}(\varphi_1(d_1t) - \varphi_1(0))$$
$$-\frac{1}{d_0+d_1}\int_0^{-d_0t}\psi_0(s)ds + \frac{1}{d_0+d_1}\int_0^{d_1t}\psi_1(z)dz$$

□

Lemma 4.1.2. *Let $u(x,t)$ be the solution of initial value problem* $(4.1.4)-(4.1.8)$ *in the form* $(4.1.18)$. *And if the function $g(t)$ is in the form*

$$g(t) = \frac{d_0}{d_0+d_1}(\varphi_0(-d_0t) - \varphi_0(0)) + \frac{d_1}{d_0+d_1}(\varphi_1(d_1t) - \varphi_1(0))$$
$$-\frac{1}{d_0+d_1}\int_0^{-d_0t}\psi_0(s)ds + \frac{1}{d_0+d_1}\int_0^{d_1t}\psi_1(z)dz$$

Then the solution $u(x,t)$ have the form,

$$
\begin{cases}
\frac{1}{2}\left[\varphi_0(x+d_0t)+\varphi_0(x-d_0t)\right] \\
+\frac{1}{2d_0}\int_{x-d_0t}^{x+d_0t}\psi_0(\gamma)d\gamma, & (x,t)\in R1\ , \\[2ex]
\frac{d_0-d_1}{2(d_0+d_1)}\varphi_0(-x-d_0t)+\frac{1}{2}\varphi_0(x-d_0t) \\
-\frac{d_0}{d_0+d_1}\varphi_0(0)+\frac{1}{d_0+d_1}\int_0^{d_1t+\frac{d_1}{d_0}}\psi_1(z)dz \\
+\frac{1}{2d_0}\int_{x-d_0t}^{-x-d_0t}\psi_0(\xi)d\xi+\frac{d_1}{d_0+d_1}\varphi_1\left(\frac{d_1}{d_0}x+d_1t\right) \\
-\frac{d_1}{d_0+d_1}\varphi_1(0)-\frac{1}{d_0+d_1}\int_0^{-x-d_0t}\psi_0(s)ds, & (x,t)\in R3; \\[2ex]
\frac{1}{2}\Big[\varphi_1(x+d_1t)+\varphi_1(x-d_1t)\Big] \\
+\frac{1}{2d_1}\int_{x-d_1t}^{x+d_1t}\psi_1(\xi)d\xi\quad , & (x,t)\in R2; \\[2ex]
\frac{d_1-d_0}{2(d_0+d_1)}\varphi_1(-x+d_1t)+\frac{1}{2}\varphi_1(x+d_1t) \\
-\frac{d_0}{d_0+d_1}\varphi_0(0)+\frac{1}{2d_1}\int_{-x+d_1t}^{x+d_1t}\psi_1(\xi)d\xi \\
+\frac{d_0}{d_0+d_1}\varphi_0\left(\frac{d_0}{d_1}x-d_0t\right)-\frac{1}{d_0+d_1}\int_0^{-d_0t+\frac{d_0}{d_1}x}\psi_0(s)ds \\
-\frac{d_1}{d_0+d_1}\varphi_1(0)+\frac{1}{d_0+d_1}\int_0^{-x+d_1t}\psi_1(z)dz, & (x,t)\in R4;
\end{cases}
\tag{4.1.18}
$$

where

$$R1=\left\{(x,t)\Big|-\infty<x<0\ ,\quad t<\frac{-x}{d_0}\right\}$$

$$R2=\left\{(x,t)\Big|\ 0<x<\infty\ ,\quad t<\frac{x}{d_1}\right\}$$

$$R3=\left\{(x,t)\Big|-\infty<x<0\ ,\quad t>\frac{-x}{d_0}\right\}$$

$$R4=\left\{(x,t)\Big|\ 0<x<\infty\ ,\quad t>\frac{x}{d_1}\right\}$$

Proof. The solution of initial value problem (4.1.4) – (4.1.8) is in the form (4.1.18). And if the function $g(t)$ is in the form

$$g(t)=\frac{d_0}{d_0+d_1}(\varphi_0(-d_0t)-\varphi_0(0))+\frac{d_1}{d_0+d_1}(\varphi_1(d_1t)-\varphi_1(0))$$

$$-\frac{1}{d_0+d_1}\int_0^{-d_0t}\psi_0(s)ds+\frac{1}{d_0+d_1}\int_0^{d_1t}\psi_1(z)dz$$

Then by substituting the formula of $g(t)$ into the solution, we get the following formula for the region R3,

$$u_0(x,t)=\frac{d_0-d_1}{2(d_0+d_1)}\varphi_0(-x-d_0t)+\frac{1}{2}\varphi_0(x-d_0t)-\frac{d_0}{d_0+d_1}\varphi_0(0)$$

$$+\frac{1}{d_0+d_1}\int_0^{d_1t+\frac{d_1}{d_0}}\psi_1(z)dz+\frac{1}{2d_0}\int_{x-d_0t}^{-x-d_0t}\psi_0(\xi)d\xi+\frac{d_1}{d_0+d_1}\varphi_1\left(\frac{d_1}{d_0}x+d_1t\right)$$

$$-\frac{d_1}{d_0+d_1}\varphi_1(0)-\frac{1}{d_0+d_1}\int_0^{-x-d_0t}\psi_0(s)ds,\quad (x,t)\in R3;$$

And the formula for the region R4 is the following,

$$u_1(x,t)=\frac{d_1-d_0}{2(d_0+d_1)}\varphi_1(-x+d_1t)+\frac{1}{2}\varphi_1(x+d_1t)-\frac{d_0}{d_0+d_1}\varphi_0(0)$$

$$+\frac{1}{2d_1}\int_{-x+d_1t}^{x+d_1t}\psi_1(\xi)d\xi-\frac{1}{d_0+d_1}\int_0^{-d_0t+\frac{d_0}{d_1}x}\psi_0(s)ds+\frac{d_0}{d_0+d_1}\varphi_0\left(\frac{d_0}{d_0}x-d_0t\right)$$

$$-\frac{d_1}{d_0+d_1}\varphi_1(0)+\frac{1}{d_0+d_1}\int_0^{-x+d_1t}\psi_1(z)dz,\quad (x,t)\in R4;$$

□

Corollary 4.1.3. *Let us consider initial value problem* (4.1.4) – (4.1.8) *in the term of,*

$$u(x,t)=\begin{cases} u_0(x,t), & -\infty<x<0 \\ u_1(x,t), & 0<x<\infty \end{cases}$$

as the following differential equations

$$\frac{\partial^2 u_0}{\partial t^2}-d_0^2\frac{\partial^2 u_0}{\partial x^2}=0\,,\qquad -\infty<x<0,\quad t\in\mathbb{R}\,,\tag{4.1.19}$$

$$\frac{\partial^2 u_1}{\partial t^2}-d_1^2\frac{\partial^2 u_1}{\partial x^2}=0\,,\qquad 0<x<\infty,\quad t\in\mathbb{R}\,,\tag{4.1.20}$$

with the special initial data,

$$u_0(x,0)=f\Big(\frac{-x}{d_0}\Big),\quad \frac{\partial u_0}{\partial t}(x,0)=f'\Big(\frac{-x}{d_0}\Big)\,,\quad -\infty<x<0,\quad t\in\mathbb{R}\,,\tag{4.1.21}$$

$$u_1(x,0)=0\,,\quad \frac{\partial u_1}{\partial t}(x,0)=0\,,\quad 0<x<\infty\,,\ t\in\mathbb{R}\,,\tag{4.1.22}$$

and matching conditions,

$$u_0(0,t) = u_1(0,t) \ , \tag{4.1.23}$$

$$c_0^2 \frac{\partial u_0}{\partial x}(0,t) = c_1^2 \frac{\partial u_1}{\partial x}(0,t) \ . \tag{4.1.24}$$

where $f(x)$ *is given in* $C^2(-\infty,0]$. *Then a solution of the problem* (4.1.19)*-*(4.1.24) *is the following,*

$$u(x,t) = \begin{cases} f\left(t - \dfrac{x}{d_0}\right) \quad , & (x,t) \in R1, \\[2ex] f\left(t - \dfrac{x}{d_0}\right) + \dfrac{d_0 - d_1}{d_0 + d_1} f\left(t + \dfrac{x}{d_0}\right), & (x,t) \in R3, \\[2ex] 0 \quad , & (x,t) \in R2, \\[2ex] \dfrac{2d_0}{d_0 + d_1} f\left(t - \dfrac{x}{d_1}\right), & (x,t) \in R4. \end{cases} \tag{4.1.25}$$

where the coefficient $\dfrac{d_0 - d_1}{d_0 + d_1}$ *of* $f\left(t + \dfrac{x}{d_0}\right)$ *in equation* (4.1.25) *is called 'Reflection Coefficient' donated by* R *and the coefficient* $\dfrac{2d_0}{d_0 + d_1}$ *of* $f\left(t - \dfrac{x}{c_1}\right)$ *in equation* (4.1.25) *is called 'Transmission Coefficient' donated by* T *.(Zauderer, E., 1998. Partial Differential Equations of Applied Mathematics. John Wiley & Sons, New York.)*

Proof. Let us consider the initial value problem $(4.1.4) - (4.1.8)$ and let the functions $\varphi_0(x)$, $\varphi_1(x)$, $\psi_0(x)$ and $\psi_1(x)$ be in the following form,

$$\varphi_0(x) = f\left(-\frac{x}{d_0}\right),$$

$$\psi_0(x) = f'\left(-\frac{x}{d_0}\right),$$

$$\varphi_1(x) = 0,$$

$$\psi_1(x) = 0.$$

Then the solution of initial value problem in (4.1.18) in Lemma 4.1.2, has the form,

$$u(x,t) = \begin{cases} f\Big(t - \dfrac{x}{d_0}\Big) \quad , & (x,t) \in R1, \\ \\ f\Big(t - \dfrac{x}{d_0}\Big) + \dfrac{d_0 - d_1}{d_0 + d_1} f\Big(t + \dfrac{x}{d_0}\Big), & (x,t) \in R3, \\ \\ 0 \quad , & (x,t) \in R2, \\ \\ \dfrac{2d_0}{d_0 + d_1} f\Big(t - \dfrac{x}{d_1}\Big), & (x,t) \in R4. \end{cases}$$

□

4.2 Examples of Simulations of Wave Propagation

In this section, we deal with simulation examples of wave propagations in two layered space that is separated with one boundary $x = 0$. Each layer has different speed. The speed of the first layer is $d_0 = 1$, and the speed of the second layer is $d_1 = 2$. We defined the matching conditions on the boundary $x = 0$.

A pulse point source was located in different positions: Between $-\infty$ and 0; between 0 and ∞.

4.2.1 Example 1 - The Pulse Point Source is Between $-\infty$ and 0

Let us consider initial value problem $(4.1.4)-(4.1.8)$. The initial conditions $(4.1.6)$ have the following form

$$\begin{aligned} \varphi_0 &= \delta(x - x^0), & \psi_0 &= 0, \\ \varphi_1 &= 0, & \psi_1 &= 0. \end{aligned} \tag{4.2.1}$$

where $\delta(x)$ is Dirac delta function, the point source is located $x^0 = -20$. By the properties of Dirac delta function and the assumptions, the solution $u(x,t)$ of IVP can be written as follows:

$$\begin{cases} \frac{1}{2}\left[\delta(x+d_0t-x^0)+\delta(x-d_0t-x^0)\right], & (x,t)\in R1\ , \\ \frac{1}{2}\left[\delta(x-d_0t-x^0)-\delta(-x-d_0t-x^0)\right]+g\left(t+\frac{x}{d_0}\right)\ , & (x,t)\in R3; \\ 0, & (x,t)\in R2; \\ g\left(t-\frac{x}{d_1}\right)\ , & (x,t)\in R4. \end{cases} \tag{4.2.2}$$

Here the function $g(t)$, constructed in Theorem 4.1.1 can be also written as follows

$$g(t)=\frac{d_0}{d_0+d_1}\cdot\delta(-d_0t-x^0) \tag{4.2.3}$$

Lemma 4.2.1. *Let $\varphi_0(x)$, $\varphi_1(x)$, $\psi_0(x)$, $psi_1(x)$ be given in the form (4.2.1); $u(x,t)$ be the solution of initial value problem (4.1.4) − (4.1.8) in the form (4.2.2). And if the function $g(t)$ has the form*

$$g(t)=\frac{d_0}{d_0+d_1}\cdot\delta(-d_0t-x^0)$$

Then the solution $u(x,t)$ have the form,

$$\begin{cases} \frac{1}{2}\left[\delta(x+d_0t-x^0)+\delta(x-d_0t-x^0)\right], & (x,t)\in R1\ , \\ \frac{d_0-d_1}{2(d_0+d_1)}\cdot\delta(-x-d_0t-x^0)+\frac{1}{2}\cdot\delta(x-d_0t-x^0), & (x,t)\in R3; \\ 0, & (x,t)\in R2; \\ \frac{d_0}{d_0+d_1}\cdot\delta\left(\frac{d_0}{d_1}x-d_0t-x^0\right), & (x,t)\in R4; \end{cases} \tag{4.2.4}$$

where the regions $R1$, $R2$, $R3$ and $R4$ are the following,

$$R1=\left\{(x,t)\middle|\ -\infty<x<0\ ,\quad t<\frac{-x}{d_0}\right\}$$

$$R2=\left\{(x,t)\middle|\ 0<x<\infty\ ,\quad t<\frac{x}{d_1}\right\}$$

$$R3 = \left\{ (x,t) \Big| -\infty < x < 0 \,, \quad t > \frac{-x}{d_0} \right\}$$

$$R4 = \left\{ (x,t) \Big| \, 0 < x < \infty \,, \quad t > \frac{x}{d_1} \right\}$$

Proof. By substituting the formulation (4.2.3) of the function $g(t)$ into the equation (4.2.2), we get the resulting formulation (4.2.4). □

By using Matlab codes, we simulate the solution $u(x,t)$ of IVP (4.1.4) – (4.1.8).

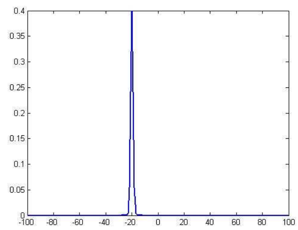
4.2.1.1 Commands of Matlab for Example 1

To run the program in Matlab successfully, we define some functions such as the function $g(t)$, constructed for example 1 in (4.2.3), and Dirac delta function. These functions are the tools which the program uses while running. To define Dirac delta function to the program, we use the regularization of Dirac delta function.

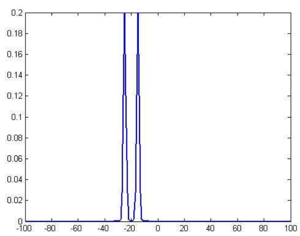
```
% Defining Dirac Delta Function:

function S=dirac(e,j,x);

% S:output value
% e:epsilon
% j=x^{0} (pulse point source)
% x: variable
% Regularization of Dirac delta function

S=(1/(2*sqrt(pi*e)))*exp(-(((x-j)^2)/(4*e)));



% Defining g-function:

function g=gfunction(t,j,a,b,e)

% t:variable
```

```
% e:epsilon
% a=d_{0} (speed of the first layer x<0.)
% b=d_{1} (speed of the second layer x>0.)
% j=x^{0} (pulse point source)
% gfunction is the function defined on the boundary x=0.
% gfunction calls Dirac delta function.

g=((a/(a+b))*dirac(e,j,((-a)*t)));

%Algorithm:

x=[-100:1:100];
t=50;%---------time-----------------
a=1; %-------d0------
b=2; %---------------d1------------
e=.5;%-------------epsilon------------
j=-20; for i=1:length(x) if x(i)<0 if t<-x(i)/a
m(i)=dirac(e,j,(x(i)+(a*t)));
z(i)=(1/2)*(m(i)+dirac(e,j,(x(i)-(a*t))));%-R1-
else
d(i)=gfunction((t+(x(i)/a)),j,a,b,e);
q(i)=(1/2)*dirac(e,j,(x(i)-(a*t)));
z(i)=(q(i)-(1/2)*dirac(e,j,(-x(i)-(a*t))))+d(i);%-R3-
end
elseif 0<x(i)
if t<(x(i)/b)
z(i)=0;%-R2-
else t>(x(i)/b)
z(i)=gfunction((t-(x(i)/b)),j,a,b,e);%-R4-
end
end
end
plot(x,z);
```

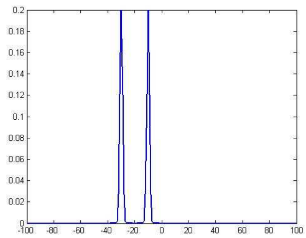
4.2.1.2 Results of Simulations by the Formula (4.2.2)



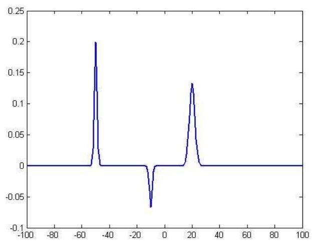
(a) $t = 0$



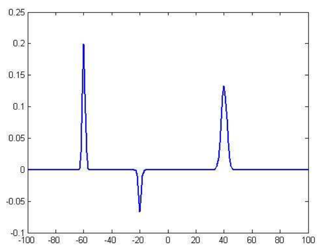
(b) $t = 5$



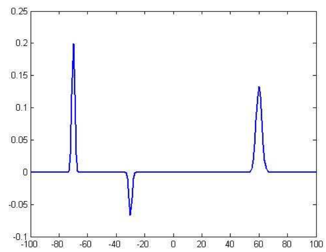
(c) $t = 10$



(d) $t = 30$



(e) $t = 40$



(f) $t = 50$

Figure 4.2 Pulse Point Source is between $-\infty$ and $x = 0$

In these figures, we simulate the wave propagation in two layered medium that is separated with one boundaries. The horizontal axes x and the vertical axes y show the location and the magnitude of the wave front, respectively. In figure (a), we can see the fluctuation arising from the pulse point source $x^0 = -20$ described by the function $\varphi_0(x) = \delta(x - x^0)$. In the figure (b), the separated waves began to move along the characteristics. Notice that, in the figure (c), the location of the wave fronts are respectively at $x = -30$ and $x = -10$ as a result of the value of the speed $d_0 = 1$ in the first layer. In the figure (d), the reflected and transmitted waves can be seen after the wave front touched the boundary $x = 0$.

Notice that in the figure(c), the reflected wave has the negative sign. This the result of that the speed of the second layer is bigger than the first layer.

In the figures (e) and (f), the movement of reflected and transmitted waves can be seen. In the figure (e), the transmitted wave front reaches the point $x = 40$ with the speed $d_1 = 2$, while the reflected wave front reaches the point $x = -20$ with the speed $d_0 = 1$. Since there is no other boundary, both of the wave fronts move along their characteristics without any changing in their magnitudes.

Remark 4.2.2. Let us consider Lemma 4.2.1. In the formulations for the region R3, the coefficient of Dirac delta function is $\frac{d_0 - d_1}{2(d_0 + d_1)}$. In this example, the speed of the second layer $d_1 = 2$ is bigger than the speed of the second layer $d_0 = 1$. As a result, the reflected wave has the negative sign.

Let M denote the magnitude of the fluctuation. In the figure (a), the magnitude of Dirac delta arising from the pulse point source is

$$\boxed{M \approx 0.4,}$$

in the figure (b), after separation, the fluctuation has the magnitude of

$$\boxed{M \approx 0.2,}$$

in this example, the coefficient is $\frac{d_0 - d_1}{2(d_0 + d_1)} = -\frac{1}{6}$ so, in the figure (d), the magnitude of reflected wave front M_r and transmitted wave front M_t are respectively

$$\boxed{M_r \approx -0.07,} \qquad \boxed{M_t \approx 0.13,}$$

$$\boxed{M = M_t - M_r \approx 0.2.}$$

Hence, the substraction of the reflected wave from the transmitted wave gives us the previous magnitude of dirac delta.

4.2.2 *Example 2 - The Pulse Point Source is Between* 0 *and* ∞

Let us consider initial value problem $(4.1.4) - (4.1.8)$. The initial conditions $(4.1.6)$ have the following form

$$\begin{aligned} \varphi_0 &= 0, & \psi_0 &= 0, \\ \varphi_1 &= \delta(x - x^0), & \psi_1 &= 0. \end{aligned} \tag{4.2.5}$$

where $\delta(x)$ is Dirac delta function, the point source is located $x^0 = 20$. By the properties of Dirac delta function and the assumptions, the solution $u(x,t)$ of IVP can be written as follows:

$$\begin{cases} 0, & (x,t) \in R1 \,, \\ g\left(t+\dfrac{x}{d_0}\right), & (x,t) \in R3; \\ \dfrac{1}{2}\left[\delta(x+d_1t-x^0)+\delta(x-d_1t-x^0)\right], & (x,t) \in R2; \\ \dfrac{1}{2}\left[\delta(x+d_1t-x^0)-\delta(-x+d_0t-x^0)\right]+g\left(t-\dfrac{x}{d_1}\right), & (x,t) \in R4. \end{cases} \tag{4.2.6}$$

Here the function $g(t)$, constructed in Theorem 4.1.1 can be also written as follows

$$g(t) = \frac{d_1}{d_0+d_1} \cdot \delta(d_1t-x^0) \tag{4.2.7}$$

Lemma 4.2.3. *Let $\varphi_0(x)$, $\varphi_1(x)$, $\psi_0(x)$, $psi_1(x)$ be given in the form (4.2.5); $u(x,t)$ be the solution of initial value problem (4.1.4) – (4.1.8) in the form (4.2.6). And if the function $g(t)$ has the form*

$$g(t) = \frac{d_1}{d_0+d_1} \cdot \delta(d_1t-x^0)$$

Then the solution $u(x,t)$ have the form,

$$\begin{cases} 0, & (x,t) \in R1 \,, \\ \dfrac{d_1}{d_0+d_1} \cdot \delta\left(\dfrac{d_1}{d_0}x+d_1t-x^0\right), & (x,t) \in R3; \\ \dfrac{1}{2}\left[\delta(x+d_1t-x^0)+\delta(x-d_1t-x^0)\right], & (x,t) \in R2; \\ \dfrac{d_1-d_0}{2(d_0+d_1)} \cdot \delta(-x+d_1t-x^0)+\dfrac{1}{2} \cdot \delta(x+d_1t-x^0), & (x,t) \in R4; \end{cases} \tag{4.2.8}$$

where the regions $R1$, $R2$, $R3$ and $R4$ are the following,

$$R1 = \left\{(x,t)\Big| -\infty < x < 0\,, \quad t < \frac{-x}{d_0}\right\}$$

$$R2 = \left\{(x,t)\Big|\, 0 < x < \infty\,, \quad t < \frac{x}{d_1}\right\}$$

$$R3 = \left\{ (x,t) \middle| -\infty < x < 0 \,, \quad t > \frac{-x}{d_0} \right\}$$

$$R4 = \left\{ (x,t) \middle| \, 0 < x < \infty \,, \quad t > \frac{x}{d_1} \right\}$$

Proof. By substituting the formulation (4.2.7) of the function $g(t)$ into the equation (4.2.6), we get the resulting formulation (4.2.8). □

By using Matlab codes, we simulate the solution $u(x,t)$ of IVP (4.1.4) – (4.1.8).

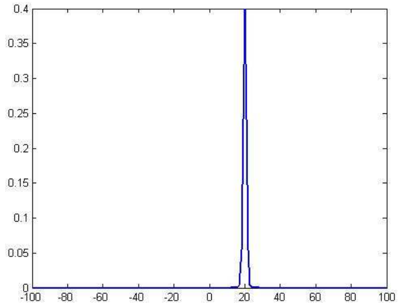
4.2.2.1 *Commands of Matlab for Example 2*

Similarly to the previous example, we must define some functions such as the function $g(t)$, constructed for example 1 in (4.2.3), and Dirac delta function to run the program in Matlab. To define Dirac delta function to the program, we use the regularization of Dirac delta function.

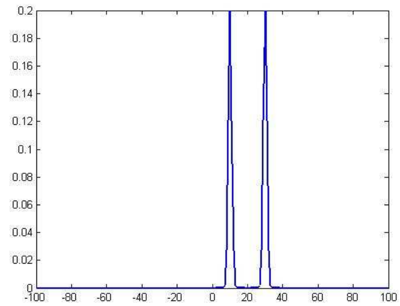
```
% Defining Dirac Delta Function:

function S=dirac(e,j,x);

% S:output value
% e:epsilon
% j=x^{0} (pulse point source)
% x: variable
% Regularization of Dirac delta function

S=(1/(2*sqrt(pi*e)))*exp(-(((x-j)^2)/(4*e)));



% Defining g-function:

function g=gfunction(t,j,a,b,e)

% t:variable
% e:epsilon
```

```
% a=d_{0} (speed of the first layer x<0.)
% b=d_{1} (speed of the second layer x>0.)
% j=x^{0} (pulse point source)
% gfunction is the function defined on the boundary x=0.
% gfunction calls Dirac delta function.

g=((b/(a+b))*dirac(e,j,(b*t)));

%Algorithm:

x=[-100:1:100];
t=50;%---------time-----------------
a=1; %-------d0------
b=2; %---------------d1------------
e=.5;%-------------epsilon------------
j=-20; for i=1:length(x) if x(i)<0 if t<-x(i)/a
z(i)=0;%-R1-
else
z(i)=gfunction((t+(x(i)/a)),j,a,b,e);%-R3-
end
elseif 0<x(i)
if t<(x(i)/b)
m(i)=dirac(e,j,(x(i)+(b*t)));
z(i)=((1/2)*(m(i)+dirac(e,j,(x(i)-(b*t)))));%-R2-
else t>(x(i)/b)
d(i)=gfunction((t-(x(i)/b)),j,a,b,e);
q(i)=(1/2)*dirac(e,j,(x(i)+(b*t)));
z(i)=(q(i)-(1/2)*dirac(e,j,(-x(i)+(b*t))))+d(i);%-R3-
end
end
end
plot(x,z);
```

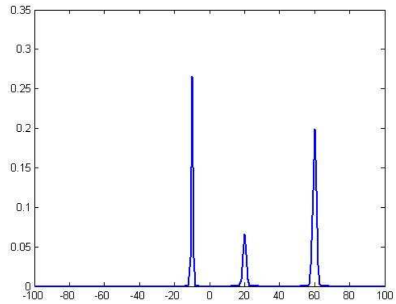
4.2.2.2 Results of Simulations by the Formula (4.2.2)



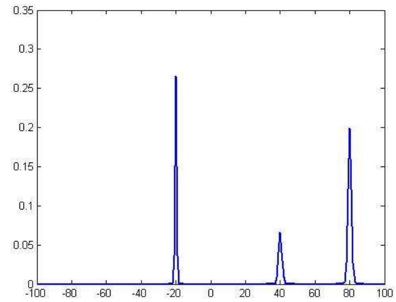
(a) $t = 0$



(b) $t = 5$



(c) $t = 10$



(d) $t = 30$

Figure 4.3 Pulse Point Source is between 0 and ∞

Similarly, in figure (a), we can see the fluctuation arising from the pulse point source $x^0 = 20$ described by the function $\varphi_1(x) = \delta(x - x^0)$. In the figure (b), the separated waves began to move along the characteristics. Notice that, the location of the wave fronts are respectively at $x = 10$ and $x = 30$ with the speed $d_1 = 2$ in the second layer. In the figure (c), the reflected and transmitted waves can be seen after the wave front touched the boundary $x = 0$.

In the figure (d), the movement of reflected and transmitted waves can be seen. The transmitted wave front reaches the point $x = -20$ with the speed $d_0 = 1$ in the first layer, while the reflected wave front reaches the point $x = 40$ with the speed $d_1 = 2$ in the second layer. Since there is no other boundary, both of the wave fronts move along their characteristics without any changing in their magnitudes.

Remark 4.2.4. Let us consider Lemma 4.2.3. In the formulations for the region R4, the coefficient of Dirac delta function is $\dfrac{d_1 - d_0}{2(d_0 + d_1)}$. Since $d_1 = 2$ is bigger than $d_0 = 1$, the reflected wave has positive sign. Let M denote the magnitude of the fluctuation. Similarly, In the figure (a), the magnitude of Dirac delta arising from the pulse point source is

$$\boxed{M \approx 0.4,}$$

in the figure (b), after separation, the fluctuation has the magnitude of

$$\boxed{M \approx 0.2,}$$

in this example, the coefficient is $\dfrac{d_1 - d_0}{2(d_0 + d_1)} = \dfrac{1}{6}$ so, in the figure (d), the magnitude of reflected wave front M_r and transmitted wave front M_t are respectively

$$\boxed{M_r \approx 0.07,} \qquad \boxed{M_t \approx 0.27,}$$

$$\boxed{M = M_t - M_r \approx 0.2.}$$

Hence, the substraction of the reflected wave from the transmitted wave gives us the previous magnitude of dirac delta.

4.3 Conclusion of Chapter Four

- Explicit formulae for the solution of IVP with matching conditions has been constructed.
- Using this formulae, the simulation of wave propagation has been obtained.
- Results of the simulations have clear physical interpretation of wave propagation in two layered media from the point source.

CHAPTER FIVE
CONCLUSION

The main results of this thesis are the following;

- The system of anisotropic elasticity is reduced to one-dimensional initial value problem (IVP) and initial boundary value problem (IBVP).
- Explicit formulae for the solutions of IVP and IBVP with boundary and matching condition has been constructed.
- Using these formulae, the simulations of wave propagation have been obtained.
- Results of the simulations have clear physical interpretation of wave propagation in two and three layered media from the point source.

We note that the method of characteristics has been used for constructing explicit formulae and MATLAB codes has been successfully applied for the simulation of the waves.

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